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Construction and number of self-dual skew codes over \mathbb{F}_{p^2} .

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Abstract

The aim of this text is to construct and to enumerate self-dual θ -cyclic and θ -negacyclic codes over \mathbb{F}_{p^2} where p is a prime number and θ is the Frobenius automorphism.

1 Introduction

A linear code over a finite field \mathbb{F}_q is a k-dimensional subspace of \mathbb{F}_q^n . Cyclic codes over \mathbb{F}_q form a class of linear codes who are invariant under a cyclic shift of coordinates. This cyclicity condition enables to describe a cyclic code as an ideal of $\mathbb{F}_q[X]/(X^n - 1)$. A self-dual linear code is a code who is equal to its annihilator (with respect to the scalar product). One reason of the interest in self-dual codes is that they have strong connections with combinatorics.

In 1983, N. J. A. Sloane and J. G. Thompson investigated the construction and the enumeration of self-dual cyclic binary codes with a given length n ([19]). These codes are determined by a polynomial equation whose solutions can be described thanks to some factorization properties of $X^n + 1$ in $\mathbb{F}_2[X]$. Later this study was generalized to self-dual cyclic codes over finite fields of characteristic 2 ([11, 10]) and to self-dual negacyclic codes over finite fields of odd characteristic ([5], [17]).

For θ automorphism of a finite field \mathbb{F}_q , θ -cyclic codes (also called skew cyclic codes) of length *n* were defined in [2]. These codes are such that a right circular shift of each codeword gives another word who belongs to the code after application of θ to each of its *n* coordinates. If θ is the identity, θ -cyclic codes are cyclic codes; if *q* is the square of a prime number and θ is the Frobenius automorphism (who therefore has order 2), θ -cyclic codes form a subclass of the class of quasi-cyclic codes of index 2 ([18]). Self-dual quasi-cyclic codes have been also studied in [8], [13], [14].

Skew cyclic codes have an interpretation in the Ore ring $R = \mathbb{F}_q[X; \theta]$ of skew polynomials where multiplication is defined by the rule $X \cdot a = \theta(a)X$ for a in \mathbb{F}_q . Like self-dual cyclic codes, self-dual θ -cyclic codes over \mathbb{F}_q are characterized by an equation, called "self-dual skew equation" and defined in the Ore ring $\mathbb{F}_q[X; \theta]$. When q is the square of a prime number and θ is the Frobenius automorphism over \mathbb{F}_q , properties specific to the ring $\mathbb{F}_q[X; \theta]$ will enable to extend N. J. A. Sloane and J. G. Thompson original approach to solve the self-dual skew equation. The text is organized as follows. In Section 2, some definitions and facts about θ -cyclic codes, θ -negacyclic codes and self-dual codes are recalled. The self-dual skew equation characterizing self-dual θ -cyclic or θ -negacyclic codes is recalled. Its solutions are least common right multiples of skew polynomials who satisfy intermediate skew equations in $\mathbb{F}_q[X;\theta]$ ([3]). The main goal of this paper consists in constructing and enumerating the solutions of these intermediate skew equations when q is the square of a prime number p and θ is the Frobenius automorphism over \mathbb{F}_{p^2} .

In Section 3, self-dual θ -cyclic and θ -negacyclic codes whose dimension is a power of p are considered over \mathbb{F}_{p^2} . In this case, the self-dual skew equation splits into one single intermediate skew equation. When p is equal to 2, the complete description of its solutions was obtained in [3] thanks to some factorization properties (recalled in Proposition 3) specific to $\mathbb{F}_{p^2}[X;\theta]$. Using the same arguments, one can also describe the solutions of the self-dual skew equation when p is an odd prime number (Proposition 4). The results are summed up in Table 1.

In Section 4, self-dual θ -cyclic and θ -negacyclic codes whose dimension is prime to p are considered over \mathbb{F}_{p^2} (Proposition 8). A resolution of the intermediate skew equations based on Cauchy interpolations over \mathbb{F}_{p^2} (Propositions 6 and 7) enables to provide a parametrization of the solutions.

In Section 5, self-dual θ -cyclic and θ -negacyclic codes of any dimension over \mathbb{F}_{p^2} are constructed and enumerated (Theorem 1). The steps of the resolutions of the intermediate skew equations are summed up in Tables 4 and 5. Proposition 4 (Section 3) and Proposition 8 (Section 4) can be seen as particular cases of Theorem 1.

The text ends in Section 6 with some concluding remarks and perspectives.

2 Generalities on self-dual skew constacyclic codes

For a finite field \mathbb{F}_q and θ an automorphism of \mathbb{F}_q one considers the ring $R = \mathbb{F}_q[X; \theta]$ where addition is defined to be the usual addition of polynomials and where multiplication is defined by the rule : for a in \mathbb{F}_q

$$X \cdot a = \theta(a) X. \tag{1}$$

The ring R is called a skew polynomial ring or Ore ring (cf. [16]) and its elements are skew polynomials. When θ is not the identity, the ring R is not commutative, it is a left and right Euclidean ring whose left and right ideals are principal. Left and right gcd and lcm exist in R and can be computed using the left and right Euclidean algorithms. The center of R is the commutative polynomial ring $Z(R) = \mathbb{F}_q^{\theta}[X^m]$ where \mathbb{F}_q^{θ} is the fixed field of θ and m is the order of θ . The **bound** B(h) of a skew polynomial h with a nonzero constant term is the monic skew polynomial f with a nonzero constant term belonging to Z(R) of minimal degree such that h divides f on the right in R ([9]).

Definition 1 (definition 1 of [3]) Consider an element a of \mathbb{F}_q and two integers n, k such that $0 \leq k \leq n$. A (θ, a) -constacyclic code or skew constacyclic code C of length n is a left R-submodule $Rg/R(X^n - a) \subset R/R(X^n - a)$ in the basis $1, X, \ldots, X^{n-1}$ where g is a monic skew polynomial dividing $X^n - a$ on the right in R with degree n - k. If a = 1, the code is θ -cyclic and if a = -1, it is θ -negacyclic. The skew polynomial g is called skew generator polynomial of C.

If θ is the identity then θ -cyclic and θ -negacylic codes are respectively cyclic and negacyclic codes.

Example 1 Consider p a prime number, $\theta : x \mapsto x^p$ the Frobenius automorphism over \mathbb{F}_{p^2} and α in \mathbb{F}_{p^2} . The remainder in the right division of $X^2 - 1$ by $X + \alpha$ in $\mathbb{F}_{p^2}[X; \theta]$ is equal to $\alpha^{p+1} - 1$:

$$X^{2} - 1 = (X - \theta(\alpha)) \cdot (X + \alpha) + \alpha \theta(\alpha) - 1.$$

Therefore, there are p+1 θ -cyclic codes of length 2 and dimension 1 over \mathbb{F}_{p^2} ; their skew generator polynomials are the skew polynomials $X + \alpha$ where $\alpha^{p+1} = 1$.

Definition 2 ([3], Definition 2) Consider an integer d and $h = \sum_{i=0}^{d} h_i X^i$ in R of degree

d. The skew reciprocal polynomial of h is $h^* = \sum_{i=0}^{d} X^{d-i} \cdot h_i = \sum_{i=0}^{d} \theta^i(h_{d-i}) X^i$. If m is the degree of the trailing term of h, the left monic skew reciprocal polynomial of h is $h^{\natural} := \frac{1}{\theta^{d-m}(h_m)} \cdot h^*$. The skew polynomial h is self-reciprocal if $h = h^{\natural}$.

Remark 1 For f, g in R, $(f \cdot g)^* = \Theta^{\deg(f)}(g^*) \cdot f^*$ (Lemma 4 of [3]). In particular, for f, h in R if f divides h on the left then f^{\natural} divides h^{\natural} on the right.

The **(Euclidean) dual** of a linear code C of length n over \mathbb{F}_q is defined as $C^{\perp} = \{x \in \mathbb{F}_q^n \mid \forall y \in C, \langle x, y \rangle = 0\}$ where for x, y in $\mathbb{F}_q^n, \langle x, y \rangle := \sum_{i=1}^n x_i y_i$ is the (Euclidean) scalar product of x and y. The code C is **self-dual** if C is equal to C^{\perp} .

According to [3], self-dual θ -constacyclic codes are necessarily θ -cyclic or θ -negacyclic. They can be characterized by a skew polynomial equation who is recalled below.

Proposition 1 (Corollary 1 of [3]) Consider ε in $\{-1,1\}$, two integers k,n with $k \leq n$ and C a (θ, ε) -constacyclic code with length n, dimension k. Consider g the skew generator polynomial of C and h the **skew check polynomial** of C defined by $g \cdot h = X^n - \varepsilon$. The Euclidean dual C^{\perp} of C is a (θ, ε) -constacyclic code generated by h^{\natural} . The code C is Euclidean self-dual if, and only if,

$$h^{\natural} \cdot h = X^{2k} - \varepsilon. \tag{2}$$

The equation (2) is called self-dual skew equation.

When k is fixed, a first approach to solve the self-dual skew equation consists in constructing the polynomial system satisfied by the unknown coefficients of a solution :

Example 2 Consider p a prime number and $\theta : x \mapsto x^p$ the Frobenius automorphism over \mathbb{F}_{p^2} . The self-dual θ -cyclic codes of dimension 1 over \mathbb{F}_{p^2} are the θ -cyclic codes whose skew check polynomials h satisfy the self-dual skew equation

$$h^{\natural} \cdot h = X^2 - 1.$$

The monic skew solutions of the self-dual skew equation are the monic skew polynomials $h = X + \alpha$ where α is in \mathbb{F}_{p^2} and

$$\left(X + \frac{1}{\theta(\alpha)}\right) \cdot (X + \alpha) = X^2 - 1.$$

Developing the left hand side of this relation thanks to the commutation law (1) and equating the terms of both sides, one gets the conditions $\alpha^2 + 1 = 0$ and $\alpha^{p-1} = -1$. If p = 2 then $\alpha = 1$ and if p is an odd prime number then $\alpha^2 = -1$ and $(-1)^{\frac{p-1}{2}} = -1$. Therefore if p = 2 there is one self-dual θ -cyclic code of dimension 1 over \mathbb{F}_4 ; if $p \equiv 3 \pmod{4}$ there are two self-dual θ -cyclic codes of dimension 1 over \mathbb{F}_{p^2} ; if $p \equiv 1 \pmod{4}$ then there is no self-dual θ -cyclic code of dimension 1 over \mathbb{F}_{p^2} .

When k is not fixed, a second approach is based on the factorization properties of the monic solutions of the self-dual skew equation. The starting point of the study is inspired from Sloane and Thompson construction of self-dual binary cyclic codes ([19]) who is extended to finite fields with characteristic 2 in [10]. Let us recall their strategy (and therefore assume that \mathbb{F}_q has characteristic 2 and that θ is the identity). Consider two integers s and t such that $k = 2^s \times t$ with t odd. The polynomial $X^n + 1 = X^{2k} + 1$ is factorized in $\mathbb{F}_q[X]$ as the product of r polynomials $f_i(X)^{2^{s+1}}$ where $f_i(X)$ is a self-reciprocal polynomial which is either irreducible or product of two distinct irreducible polynomials $g_i(X)$ and $g_i^{\natural}(X)$ in $\mathbb{F}_q[X]$. Consider h in $\mathbb{F}_q[X]$ such that $h^{\natural}h = X^{2k} + 1$. Necessarily, h is the product of polynomials $f_i(X)^{\alpha_i}$, $g_i(X)^{\beta_i}$ and $g_i^{\natural}(X)^{\gamma_i}$, where α_i, β_i and γ_i are integers of $\{0, \ldots, 2^{s+1}\}$. The relation $h^{\natural}h = X^{2k} + 1$ is satisfied if and only if $\alpha_i = 2^s$ and $\beta_i + \gamma_i = 2^{s+1}$, therefore there are $(2^{s+1} + 1)^m$ self-dual cyclic codes of dimension k where m is the number of polynomials h_i who satisfy the relation $h^{\natural}h = X^{2k} + 1$ are least common multiples of polynomials h_i who are defined by the intermediate equations $h_i^{\natural}h_i = f_i(X)^{2^{s+1}}$:

$$h^{\natural}h = X^{2k} + 1 \Leftrightarrow h = \operatorname{lcm}(h_1, \dots, h_r), h_i^{\natural}h_i = f_i(X)^{2^{s+1}}$$

In [3], this lcm decomposition was generalized to a lcrm decomposition over $R = \mathbb{F}_q[X; \theta]$ in the particular case when q is the square of a prime number and θ is the Frobenius automorphism (Proposition 28 of [3]). This decomposition enables to derive a first formula for the number of (θ, ε) -constacyclic codes of dimension k (Proposition 2 below). First one introduces some notations that will be useful later :

Notation 1 For $F = F(X^2)$ in $\mathbb{F}_p[X^2]$, k in \mathbb{N}^* and ε in $\{-1, 1\}$,

 $\mathcal{H}_F := \{h \in R \mid h \text{ is monic and } h^{\natural} \cdot h = F(X^2)\}$

$$\begin{split} \overline{\mathcal{H}}_F &:= \{h \in \mathcal{H}_F \mid \text{ no non constant divisor of } F(X^2) \text{ in } \mathbb{F}_p[X^2] \text{ divides } h \text{ in } R \} \\ \mathcal{D}_F &:= \{f = f(X^2) \in \mathbb{F}_p[X^2] \mid f \text{ is monic and } f \text{ divides } F(X^2) \} \\ \mathcal{F} &:= \{f = f(X^2) \in \mathbb{F}_p[X^2] \mid f \text{ is irreducible in } \mathbb{F}_p[X^2] \text{ and } \deg_{X^2}(f) > 1 \} \\ \mathcal{G} &:= \{f = f(X^2) \in \mathbb{F}_p[X^2] \mid f = gg^{\natural} \text{ with } g \neq g^{\natural} \text{ irreducible in } \mathbb{F}_p[X^2] \} \\ \mathcal{F}_{k,\varepsilon} &:= \mathcal{D}_{X^{2k}-\varepsilon} \cap \mathcal{F} \\ \mathcal{G}_{k,\varepsilon} &:= \mathcal{D}_{X^{2k}-\varepsilon} \cap \mathcal{G} \end{split}$$

Following this notation, the monic solutions of the self-dual skew equation are the elements of $\mathcal{H}_{X^{2k}-\varepsilon}$.

Proposition 2 Consider p a prime number, θ the Frobenius automorphism over \mathbb{F}_{p^2} , $R = \mathbb{F}_{p^2}[X;\theta]$, k a positive integer, s,t two integers such that $k = p^s \times t$ and p does not divide t. The number of self-dual (θ, ε) -constacyclic codes of dimension k over \mathbb{F}_{p^2} is

$$#\mathcal{H}_{X^{2k}-\varepsilon} = N_{\varepsilon} \times \prod_{f \in \mathcal{F}_{k,\varepsilon}} #\mathcal{H}_{f^{p^s}} \times \prod_{f \in \mathcal{G}_{k,\varepsilon}} #\mathcal{H}_{f^{p^s}}$$

where

$$N_{1} = \begin{cases} \#\mathcal{H}_{(X^{2}+1)^{p^{s}}} & \text{if } p = 2 \\ \#\mathcal{H}_{(X^{2}-1)^{p^{s}}} & \text{if } k \equiv 1 \pmod{2} \text{ and } p \text{ odd} \\ \#\mathcal{H}_{(X^{2}-1)^{p^{s}}} \times \#\mathcal{H}_{(X^{2}+1)^{p^{s}}} & \text{if } k \equiv 0 \pmod{2} \text{ and } p \text{ odd} \end{cases}$$

and

$$N_{-1} = \begin{cases} \#\mathcal{H}_{(X^2+1)^{p^s}} & if \quad k \equiv 1 \pmod{2} \text{ and } p \text{ odd} \\ 1 & if \quad k \equiv 0 \pmod{2} \text{ and } p \text{ odd} \end{cases}$$

Proof. Consider the factorization of $X^{2t} - \varepsilon$ over $\mathbb{F}_p[X^2]$ into the product of distinct irreducible polynomials of $\mathbb{F}_p[X^2]$ and split this product into two sub-products, the product of self-reciprocal irreducible factors and the product of non self-reciprocal irreducible factors. In this second product, factors appear by pairs $(g, g^{\natural} \neq g)$ therefore $X^{2k} - \varepsilon = (X^{2t} - \varepsilon)^{p^s} = \prod_{i=1}^r f_i^{p^s}$ where $f_i = f_i(X^2)$ is self-reciprocal, either irreducible in $\mathbb{F}_p[X^2]$ or product of two distinct irreducible polynomials $g_i(X^2)$ and $g_i^{\natural}(X^2)$ of $\mathbb{F}_p[X^2]$. Following [3], one has

1.
$$\mathcal{H}_{X^{2k}-\varepsilon} = \{\operatorname{lcrm}(h_1, \dots, h_r) \mid h_i \in \mathcal{H}_{f_i^{p^s}}\}$$
 ([3], Proposition 28);

2. If h belongs to $\mathcal{H}_{X^{2k}-\varepsilon}$, then $h = \operatorname{lcrm}(h_1, \ldots, h_r)$ where $h_i^{\natural} = \operatorname{gcrd}(f_i^{p^s}, h^{\natural})$ and $h_i \in \mathcal{H}_{f_i^{p^s}}$ ([3], Proposition 28, point (2)).

Therefore, the following application ϕ is well defined and is injective :

$$\phi: \left\{ \begin{array}{ccc} \mathcal{H}_{X^{2k}-\varepsilon} & \to & \mathcal{H}_{f_1^{p^s}} \times \dots \times \mathcal{H}_{f_r^{p^s}} \\ h & \mapsto & (h_1,\dots,h_r), & h_i^{\natural} = \operatorname{gcrd}(f_i^{p^s},h^{\natural}). \end{array} \right.$$

Let us prove that ϕ is surjective. Consider (h_1, \ldots, h_r) in $\mathcal{H}_{f_1^{p^s}} \times \cdots \times \mathcal{H}_{f_r^{p^s}}$ and $h = \operatorname{lcrm}(h_1, \ldots, h_r)$. According to point 1., the skew polynomial h belongs to $\mathcal{H}_{X^{2k}-\varepsilon}$. It remains to prove that for all i in $\{1, \ldots, r\}$, $h_i^{\natural} = \operatorname{gcrd}(f_i^{p^s}, h^{\natural})$. According to point 2., $h = \operatorname{lcrm}(\tilde{h}_1, \ldots, \tilde{h}_r)$ where $\tilde{h}_i^{\natural} = \operatorname{gcrd}(f_i^{p^s}, h^{\natural})$ and $\tilde{h}_i \in \mathcal{H}_{f_i^{p^s}}$. Consider i in $\{1, \ldots, r\}$, one has $h_i^{\natural} \cdot h_i = f_i^{p^s}$ and f_i is central, therefore h_i^{\natural} divides $f_i^{p^s}$ on the right. Furthermore $h = \operatorname{lcrm}(h_1, \ldots, h_r)$ therefore h_i divides h on the left and h_i^{\natural} divides h^{\natural} on the right (see Remark 1). As \tilde{h}_i^{\natural} is the greatest common right divisor of $f_i^{p^s}$ and h^{\natural} , h_i^{\natural} divides \tilde{h}_i^{\natural} on the right. Furthermore $h_i^{\natural} \cdot h_i = \tilde{h}_i^{\natural} \cdot \tilde{h}_i = f_i^{p^s}$ so h_i and \tilde{h}_i have the same degree and $h_i^{\natural} = \operatorname{gcrd}(f_i^{p^s}, h^{\natural})$. To conclude ϕ is bijective and

$$#\mathcal{H}_{X^{2k}-\varepsilon} = \prod_{i=1}^r #\mathcal{H}_{f_i^{p^s}} = N_{\varepsilon} \times \prod_{f \in \mathcal{F}_{k,\varepsilon}} #\mathcal{H}_{f^{p^s}} \times \prod_{f \in \mathcal{G}_{k,\varepsilon}} #\mathcal{H}_{f^{p^s}}$$

where $N_{\varepsilon} = \prod_{\deg(f_i)=1} \# \mathcal{H}_{f_i^{p^s}}$.

Let us determine N_{ε} in the three following cases : $p = 2, \varepsilon = 1$; p odd prime, $\varepsilon = 1$ and p odd prime, $\varepsilon = -1$.

For p = 2, the self-reciprocal polynomial of degree 1 in X^2 dividing $X^{2k} - 1$ is $X^2 + 1$ therefore $N_1 = \#\mathcal{H}_{(X^2+1)^{p^s}}$.

For p odd prime, the self-reciprocal polynomials of degree 1 in X^2 dividing $X^{2k} - 1$ are $X^2 - 1$ if k is odd; $X^2 - 1$ and $X^2 + 1$ if k is even therefore,

$$N_1 = \begin{cases} \#\mathcal{H}_{(X^2-1)^{p^s}} & \text{if } k \equiv 1 \pmod{2} \\ \#\mathcal{H}_{(X^2-1)^{p^s}} \times \#\mathcal{H}_{(X^2+1)^{p^s}} & \text{if } k \equiv 0 \pmod{2}. \end{cases}$$

For p odd prime and k even number, $X^{2k} + 1$ has no self-reciprocal factor of degree 1 in X^2 . If k is odd, $X^2 + 1$ is the only self-reciprocal polynomial of degree 1 in X^2 dividing $X^{2k} + 1$. Therefore,

$$N_{-1} = \begin{cases} \# \mathcal{H}_{(X^2+1)^{p^s}} & \text{if } k \equiv 1 \pmod{2} \\ 1 & \text{if } k \equiv 0 \pmod{2}. \end{cases}$$

The rest of the paper will be devoted to the enumeration of the elements of the set $\mathcal{H}_{X^{2k}-\varepsilon}$ when k is a power of p (Section 3), k is coprime with p (Section 4) and k is any integer (Section 5). Following Proposition 2, the main task will consist in constructing $\mathcal{H}_{f^{p^s}}$ for $f = X^2 \pm 1$, f in \mathcal{F} and f in \mathcal{G} . The main difficulty comes from the non unicity of the factorization of skew polynomials in the Ore ring R.

In Section 3, one assumes that k is a power of p, therefore $X^{2k} - \varepsilon$ factorizes over $\mathbb{F}_p[X^2]$ as $X^{2k} - \varepsilon = (X^2 - \varepsilon)^{p^s}$ and the self-dual skew equation splits into one single intermediate skew equation. For s > 0, it is solved by using a partition and factorization properties specific to $\mathbb{F}_{p^2}[X; \theta]$.

3 Self-dual θ -cyclic and θ -negacyclic codes with dimension p^s over \mathbb{F}_{p^2} .

The aim of this section is to construct and to enumerate self-dual θ -cyclic and θ -negacyclic codes over \mathbb{F}_{p^2} whose dimension is p^s where θ is the Frobenius automorphism. Recall that over \mathbb{F}_4 , there is one single self-dual cyclic code of dimension 2^s . When p is an odd prime number there is no self-dual cyclic code over \mathbb{F}_{p^2} and there are $p^s + 1$ self-dual negacyclic codes of dimension p^s (Corollary 3.3 of [5]). Lastly, there are only three self-dual θ -cyclic codes of dimension $2^s > 1$ over \mathbb{F}_4 (Corollary 26 of [3]). In what follows one proves that the number of self-dual θ -cyclic and θ -negacyclic codes of dimension p^s over \mathbb{F}_{p^2} is exponential in the dimension p^s when p is an odd prime number (Proposition 4 and Table 1).

In order to construct the set $\mathcal{H}_{X^{2k}-\varepsilon} = \mathcal{H}_{(X^2-\varepsilon)p^s}$, factorization properties specific to $\mathbb{F}_{p^2}[X;\theta]$ will be useful. The following proposition enables to characterize the skew polynomials that have a unique factorization into the product of monic linear skew polynomials dividing $X^2 - \varepsilon$ (see also Proposition 16 of [3]).

Proposition 3 Consider p a prime number, θ the Frobenius automorphism over \mathbb{F}_{p^2} , $R = \mathbb{F}_{p^2}[X;\theta]$, m a nonnegative integer, $f(X^2)$ in $\mathbb{F}_p[X^2]$ irreducible and $h = h_1 \cdots h_m$ in R where h_i is irreducible in R, monic and divides $f(X^2)$. The following assertions are equivalent :

- (i) The above factorization of h is not unique.
- (ii) $f(X^2)$ divides h.
- (iii) There exists i in $\{1, \ldots, m-1\}$ such that $h_i \cdot h_{i+1} = f(X^2)$.

Proof. Consider $f(X^2) \in \mathbb{F}_p[X^2]$ irreducible with degree d > 1 such that $f^{\natural}(X^2) = f(X^2)$. According to [15], page 6 (or Lemma 1.4.11 of [4] with e = 2), as $f(X^2)$ is irreducible in the center of R, the skew polynomial $f(X^2)$ has $((p^2)^d - 1)/(p^d - 1) = p^d + 1$ irreducible monic right factors of degree d in R, in particular it is reducible in R. According to Proposition 16 of [3] the points (i), (ii) and (iii) are therefore equivalent.

Corollary 1 Consider p a prime number, θ the Frobenius automorphism over \mathbb{F}_{p^2} , $R = \mathbb{F}_{p^2}[X;\theta]$, m a nonnegative integer, ε in $\{-1,1\}$ and $h = (X + \lambda_1) \cdots (X + \lambda_m)$ in R where $\lambda_i^{p+1} = \varepsilon$. The following assertions are equivalent :

- (i) The above factorization of h is not unique.
- (ii) $X^2 \varepsilon$ divides h.

(iii) There exists i in $\{1, \ldots, m-1\}$ such that $(X + \lambda_i) \cdot (X + \lambda_{i+1}) = X^2 - \varepsilon$ i.e. $\lambda_i \lambda_{i+1} = -\varepsilon$. **Proof.** This a consequence of Proposition 3 with $f(X^2) = X^2 - \varepsilon$. It suffices to notice that $X + \lambda_i$ divides $X^2 - \varepsilon$ if and only if $\lambda_i^{p+1} = \varepsilon$. In this case $(X + \lambda_i) \cdot (X + \lambda_{i+1}) = X^2 + (\lambda_i + \frac{\varepsilon}{\lambda_{i+1}})X + \lambda_i\lambda_{i+1}$ and $(X + \lambda_i) \cdot (X + \lambda_{i+1}) = X^2 - \varepsilon \Leftrightarrow \lambda_i\lambda_{i+1} = -\varepsilon$.

The elements of $\mathcal{H}_{(X^2-\varepsilon)^{p^s}}$ who have a unique factorization in R into the product of monic irreducible skew polynomials are therefore not divisible by $X^2 - \varepsilon$. In what follows one constructs for m in \mathbb{N} the set of elements of $\mathcal{H}_{(X^2-\varepsilon)^m}$ who are not divisible by $X^2 - \varepsilon$. Recall that one denotes $\overline{\mathcal{H}}_{(X^2-\varepsilon)^m}$ this set of elements (see notations in Section 2) :

$$\overline{\mathcal{H}}_{(X^2-\varepsilon)^m} := \{h \in \mathcal{H}_{(X^2-\varepsilon)^m} \mid X^2 - \varepsilon \text{ does not divide } h\}$$

Lemma 1 Consider p a prime number, θ the Frobenius automorphism, $R = \mathbb{F}_{p^2}[X; \theta]$, m a nonnegative integer and ε in $\{-1, 1\}$. Assume that p is odd and m is odd, then the number of elements of $\overline{\mathcal{H}}_{(X^2-\varepsilon)^m}$ is

$$\#\overline{\mathcal{H}}_{(X^2-\varepsilon)^m} = \begin{cases} 0 & \text{if } \varepsilon \equiv 1, p \equiv 1 \pmod{4} \text{ or } \varepsilon = -1, p \equiv 3 \pmod{4} \\ 2p^{\frac{m-1}{2}} & \text{if } \varepsilon \equiv 1, p \equiv 3 \pmod{4} \text{ or } \varepsilon = -1, p \equiv 1 \pmod{4}. \end{cases}$$

Assume that p is equal to 2, then the number of elements of $\overline{\mathcal{H}}_{(X^2+1)^m}$ is

$$\#\overline{\mathcal{H}}_{(X^2+1)^m} = \begin{cases} 0 & if \quad m > 2\\ 2 & if \quad m = 2\\ 1 & if \quad m = 1 \end{cases}$$

Proof.

• One first proves that the elements h of $\overline{\mathcal{H}}_{(X^2-\varepsilon)^m}$ are

$$h = (X + \lambda_1) \cdots (X + \lambda_m)$$

where

$$\begin{cases} \forall i \in \{1, \dots, m\}, \lambda_i^{p+1} = \varepsilon \\ \forall i \in \{1, \dots, m-1\}, \lambda_i \lambda_{i+1} \neq -\varepsilon \\ \lambda_1^2 = -1 \\ \forall j \in \{1, \dots, \lfloor \frac{m-1}{2} \rfloor\}, (\lambda_{2j} \lambda_{2j+1})^2 = 1. \end{cases}$$
(3)

Namely, consider h in $\overline{\mathcal{H}}_{(X^2-\varepsilon)^m}$. As h divides $(X^2-\varepsilon)^m$ and as $X^2-\varepsilon$ is irreducible with degree 1 in $\mathbb{F}_p[X^2]$, h is a (non necessarily commutative) product of linear monic skew polynomials dividing $X^2 - \varepsilon$ (Lemma 13 (2) of [3] or [15] page 6). Furthermore, the degree of h is equal to m (because $\deg(h^{\natural} \cdot h) = 2m$) therefore one has :

$$h = (X + \lambda_1) \cdots (X + \lambda_m)$$
 where $\lambda_i \in \mathbb{F}_{p^2}, \lambda_i^{p+1} = \varepsilon$

In particular, the first relation of (3) is satisfied. As $X^2 - \varepsilon$ does not divide h, according to Corollary 1 :

$$\forall i \in \{1, \dots, m-1\}, (X+\lambda_i) \cdot (X+\lambda_{i+1}) \neq X^2 - \varepsilon$$
(4)

therefore

$$\forall i \in \{1, \ldots, m-1\}, \lambda_i \lambda_{i+1} \neq -\varepsilon$$

which is the second relation of (3). The following expression of h^{\natural} can be obtained using an induction argument (left to the reader) :

$$h^{\natural} = (X + \tilde{\lambda}_m) \cdots (X + \tilde{\lambda}_1)$$

where for i in $\{1, \ldots, m\}$, $\tilde{\lambda}_i$ is defined by :

$$\tilde{\lambda}_i := \begin{cases} 1/\lambda_i \times \varepsilon \times (\lambda_1 \cdots \lambda_i)^2 & if \quad i \equiv 1 \pmod{2} \\ 1/\lambda_i \times \frac{\varepsilon}{(\lambda_1 \cdots \lambda_{i-1})^2} & if \quad i \equiv 0 \pmod{2}. \end{cases}$$
(5)

Furthermore, $X^2 - \varepsilon$ does not divide h^{\natural} , otherwise $X^2 - \varepsilon$ would divide h, therefore

$$\forall i \in \{1, \dots, m-1\}, (X + \tilde{\lambda}_{i+1}) \cdot (X + \tilde{\lambda}_i) \neq X^2 - \varepsilon.$$
(6)

The relation $h^{\natural} \cdot h = (X^2 - \varepsilon)^m$ can be written

$$(X + \tilde{\lambda}_m) \cdots (X + \tilde{\lambda}_1) \cdot (X + \lambda_1) \cdots (X + \lambda_m) = (X^2 - \varepsilon)^m.$$
(7)

As $X^2 - \varepsilon$ is central, the factorization of the skew polynomial $(X^2 - \varepsilon)^m$ into the product of monic skew polynomials dividing $X^2 - \varepsilon$ is not unique, therefore, according

to Corollary 1, $X^2 - \varepsilon$ is necessarily the product of two consecutive monic linear factors of the left hand side of (7). According to (4) and (6), the only possibility is

$$(X + \lambda_1) \cdot (X + \lambda_1) = X^2 - \varepsilon.$$

As $X^2 - \varepsilon$ is central, the relation (7) can be simplified and one gets

$$(X + \tilde{\lambda}_m) \cdots (X + \tilde{\lambda}_2) \cdot (X + \lambda_2) \cdots (X + \lambda_m) = (X^2 - \varepsilon)^{m-1}.$$

Using the same argument as before, one gets

$$(X + \tilde{\lambda}_2) \cdot (X + \lambda_2) = X^2 - \varepsilon$$

$$\vdots$$

$$(X + \tilde{\lambda}_m) \cdot (X + \lambda_m) = X^2 - \varepsilon.$$

From the equalities above, one deduces that

$$\forall i \in \{1, \dots, m\}, \lambda_i \tilde{\lambda}_i = -\varepsilon$$

and using the definition of $\tilde{\lambda}_i$ given in (5), one gets $\lambda_1^2 = -1$ (third relation of (3)) and for *i* odd, $(\lambda_i \lambda_{i+1})^2 = 1$ (fourth relation of (3)).

Conversely, consider $h = (X + \lambda_1) \cdots (X + \lambda_m)$ where $\lambda_1, \ldots, \lambda_m$ are defined by (3). According to the first relation of (3), the monic skew polynomials $X + \lambda_i$ divide $X^2 - \varepsilon$. According to the second relation of (3) and to Corollary 1, $X^2 - \varepsilon$ does not divide h. Like previously the skew polynomial h^{\natural} is equal to $(X + \tilde{\lambda}_m) \cdots (X + \tilde{\lambda}_1)$ where $\tilde{\lambda}_i$ is defined by the relations (5). Furthermore, according to the third and fourth relations of (3), if i is odd, $(\lambda_1 \cdots \lambda_i)^2 = -1$, so for all i in $\{1, \ldots, m\}, \lambda_i \tilde{\lambda}_i = -\varepsilon$ and $X^2 - \varepsilon = (X + \tilde{\lambda}_i) \cdot (X + \lambda_i)$. The product $h^{\natural} \cdot h$ can be simplified as follows :

$$\begin{split} h^{\natural} \cdot h &= (X + \tilde{\lambda}_m) \cdots (X + \tilde{\lambda}_1) \cdot (X + \lambda_1) \cdots (X + \lambda_m) \\ &= (X^2 - \varepsilon) \cdot (X + \tilde{\lambda}_m) \cdots (X + \tilde{\lambda}_2) \cdot (X + \lambda_2) \cdots (X + \lambda_m) \\ &\quad \text{(because } X^2 - \varepsilon \text{ is central)} \\ &\vdots \\ &= (X^2 - \varepsilon)^{m-1} \cdot (X + \tilde{\lambda}_m) \cdot (X + \lambda_m) \\ &= (X^2 - \varepsilon)^m \end{split}$$

and one concludes that h belongs to $\overline{\mathcal{H}}_{(X^2-\varepsilon)^m}$.

• The relations (3) enable to count the number of elements of $\overline{\mathcal{H}}_{(X^2-\varepsilon)^m}$. Namely according to Corollary 1, the elements of $\overline{\mathcal{H}}_{(X^2-\varepsilon)^m}$ have a unique factorization into the product of monic skew linear polynomials dividing $X^2 - \varepsilon$. Therefore the number of elements of the set $\overline{\mathcal{H}}_{(X^2-\varepsilon)^m}$ is the number of *m*-tuples $(\lambda_1,\ldots,\lambda_m)$ of $(\mathbb{F}_{p^2})^m$ satisfying the conditions (3).

Assume that p = 2 and that m is an integer greater than 2. Then the conditions $\lambda_2 \lambda_3 \neq -1$ and $(\lambda_2 \lambda_3)^2 = 1$ are not compatible, therefore the set $\overline{\mathcal{H}}_{(X^2-1)^m}$ is empty. If m = 1, it is reduced to $\{X + 1\}$ (see Example 1). If m = 2, the set $\overline{\mathcal{H}}_{(X^2-1)^m}$ is equal

to $\{(X + \lambda_1) \cdot (X + \lambda_2) \mid \lambda_1 = 1, \lambda_2 \neq 1\} = \{(X + 1) \cdot (X + a), (X + 1) \cdot (X + a^2)\}$ where $a^2 + a + 1 = 0$.

Assume that p and m are odd, then the conditions (3) can be simplified as follows :

$$\begin{cases} \lambda_1^2 = -1\\ \lambda_2 \neq \varepsilon \lambda_1\\ \forall i \in \{1, \dots, m\}, \lambda_i^{p+1} = \varepsilon\\ \forall j \in \{1, \dots, (m-1)/2\}, \lambda_{2j+1} = \varepsilon/\lambda_{2j}\\ \forall j \in \{1, \dots, (m-3)/2\}, \lambda_{2j+2} \neq -\lambda_{2j} \end{cases}$$

First, the conditions $\lambda_1^2 = -1$ and $\lambda_1^{p+1} = \varepsilon$ imply $(-1)^{(p+1)/2} = \varepsilon$ so $\overline{\mathcal{H}}_{(X^2 - \varepsilon)^m}$ is empty if $p \equiv 3 \pmod{4}$ and $\varepsilon = -1$ or $p \equiv 1 \pmod{4}$ and $\varepsilon = 1$.

If $p \equiv 3 \pmod{4}$ and $\varepsilon = 1$ or $p \equiv 1 \pmod{4}$ and $\varepsilon = -1$, then there are two possibilities for λ_1 , p possibilities for λ_2 , one possibility for λ_3 , p possibilities for λ_4 , one for λ_5 , and so on, therefore $\overline{\mathcal{H}}_{(X^2-\varepsilon)^m}$ has $2p^{\frac{m-1}{2}}$ elements.

Remark 2 If m is odd, one can simplify the relations (3) by taking $\alpha_0 = \lambda_1$, $\alpha_1 = \lambda_2$ and for i in $\{2, \ldots, (m-1)/2\}$, $\alpha_i = \lambda_{2i}$. Therefore one gets :

$$\overline{\mathcal{H}}_{(X^2-\varepsilon)^m} = \{ (X+\alpha_0) \cdot (X^2+2\alpha_1X+\varepsilon) \cdots (X^2+2\alpha_{(m-1)/2}X+\varepsilon) \mid \alpha_0^2 = -1, \alpha_1 \neq \varepsilon \alpha_0, \\ \forall i \in \{0, \dots, (m-1)/2\}, \alpha_i^{p+1} = \varepsilon, \\ \forall i \in \{2, \dots, (m-1)/2\}, \alpha_i \neq -\alpha_{i-1} \}.$$

To describe the set $\mathcal{H}_{(X^2-\varepsilon)^{p^s}}$ one uses the following partition :

Lemma 2 Consider p a prime number, θ the Frobenius automorphism over \mathbb{F}_{p^2} , $R = \mathbb{F}_{p^2}[X; \theta]$, s in \mathbb{N} and $f = f(X^2) \in \{X^2 \pm 1\} \cup \mathcal{F}$. One has the following partition :

$$\mathcal{H}_{f^{p^{s}}} = \bigsqcup_{i=0}^{\lfloor \frac{p^{s}}{2} \rfloor} f^{i} \cdot \overline{\mathcal{H}}_{f^{p^{s}-2i}}.$$
(8)

Proof. Consider $M = \lfloor \frac{p^s}{2} \rfloor$, h = h(X) in $\mathcal{H}_{f^{p^s}}$ and *i* the biggest integer in $\{0, \ldots, M\}$ such that f^i divides *h*. Consider H = H(X) in *R* such that $h = f^i \cdot H$ and *f* does not divide *H*. As f^i is central, $h^{\natural} = f^i \cdot H^{\natural}$ therefore $H^{\natural} \cdot H = f^{p^s - 2i}$ and *H* belongs to $\overline{\mathcal{H}}_{f^{p^s - 2i}}$. Conversely, if *H* in $\overline{\mathcal{H}}_{f^{p^s - 2i}}$, then $f^i \cdot H$ belongs to $\mathcal{H}_{f^{p^s}}$.

Furthermore consider i > i', H in $\overline{\mathcal{H}}_{f^{p^s-2i}}$ and H' in $\overline{\mathcal{H}}_{f^{p^s-2i'}}$ such that $f^i \cdot H = f^{i'} \cdot H'$ then $f^{i-i'}$ divides H', which is impossible as f does not divide H'. Therefore, for $i \neq i'$, the sets $f^i \cdot \overline{\mathcal{H}}_{f^{p^s-2i}}$ and $f^{i'} \cdot \overline{\mathcal{H}}_{f^{p^s-2i'}}$ are disjoint.

Remark 3 If p = 2 and $f(X^2) = X^2 + 1$, according to Lemma 2, one gets the following partition :

$$\mathcal{H}_{(X^2+1)^{2^s}} = \bigsqcup_{i=0}^{2^{s-1}} (X^2+1)^i \cdot \overline{\mathcal{H}}_{(X^2+1)^{2^s-2i}}.$$

According to Lemma 1, the sets $\overline{\mathcal{H}}_{(X^2+1)^{2^s-2i}}$ are empty when $2^s - 2i > 2$ and $\overline{\mathcal{H}}_{(X^2+1)^2} = \{(X+1) \cdot (X+a), (X+1) \cdot (X+a^2)\}$ where $a^2 + a + 1 = 0$. Therefore :

$$\mathcal{H}_{(X^2+1)^{2^s}} = (X^2+1)^{2^{s-1}} \cdot \overline{\mathcal{H}}_{(X^2+1)^0} \bigsqcup (X^2+1)^{2^{s-1}-1} \cdot \overline{\mathcal{H}}_{(X^2+1)^2} = \{(X+1)^{2^s}, (X+1)^{2^s-1} \cdot (X+a), (X+1)^{2^s-1} \cdot (X+a^2)\}$$

One gets that for s > 0 there are only three self-dual θ -cyclic codes of dimension 2^s over \mathbb{F}_4 (see also Corollary 26 of [3]).

Proposition 4 below gives a formula for the number of self-dual θ -cyclic and θ -negacyclic codes whose dimension is a power of p when p is an odd prime number. The results are also summed up in Table 1.

Proposition 4 Consider p an odd prime number, s an integer, ε in $\{-1,1\}$ and θ the Frobenius automorphism over \mathbb{F}_{p^2} . The number of self-dual (θ, ε) -constacyclic codes of dimension p^s over \mathbb{F}_{p^2} is

$$\begin{cases} 0 & \text{if } \varepsilon \equiv 1, p \equiv 1 \pmod{4} \text{ or } \varepsilon = -1, p \equiv 3 \pmod{4} \\ 2\frac{p^{(p^s+1)/2} - 1}{p-1} & \text{if } \varepsilon \equiv 1, p \equiv 3 \pmod{4} \text{ or } \varepsilon = -1, p \equiv 1 \pmod{4}. \end{cases}$$

Proof. Consider $R = \mathbb{F}_{p^2}[X; \theta]$. The number of self-dual (θ, ε) -constacyclic codes of dimension p^s over \mathbb{F}_{p^2} is equal to $\#\mathcal{H}_{X^{2p^s}-\varepsilon}$. According to Lemma 2, one has the following partition :

$$\mathcal{H}_{X^{2p^s}-\varepsilon} = \bigsqcup_{i=0}^{M} (X^2 - \varepsilon)^i \cdot \overline{\mathcal{H}}_{(X^2-\varepsilon)^{p^s-2i}}$$

where $M = \frac{p^s - 1}{2}$. According to Lemma 1, each set $\overline{\mathcal{H}}_{(X^2 - \varepsilon)^{p^s - 2i}}$ is empty if $\varepsilon \neq (-1)^{\frac{p+1}{2}}$ and has $2p^{M-i}$ elements if $\varepsilon = (-1)^{\frac{p+1}{2}}$. Therefore, if $\varepsilon \neq (-1)^{\frac{p+1}{2}}$, $\mathcal{H}_{X^{2p^s} - \varepsilon}$ is empty and otherwise it has $\sum_{i=0}^{M} 2p^{M-i} = 2\frac{p^{M+1} - 1}{p-1} = 2\frac{p^{(p^s+1)/2} - 1}{p-1}$ elements.

Example 3 According to Corollary 3.3 of [5], there are 4 self-dual negacyclic codes of dimension 3 over \mathbb{F}_9 . The corresponding skew check polynomials are the polynomials $(X - \gamma)^i (X + \gamma)^{3-i} \in \mathbb{F}_9[X]$ where *i* is in $\{0, 1, 2, 3\}$ and $\gamma^2 = -1$.

According to Proposition 4, for $\theta : x \mapsto x^3$ Frobenius automorphism over \mathbb{F}_9 , there are $2 \times (3^{(3+1)/2} - 1)/(3 - 1) = 8$ self-dual θ -cyclic codes of dimension 3 over \mathbb{F}_9 . Their skew check polynomials are the elements of \mathcal{H}_{X^6-1} and according to Lemma 2, $\mathcal{H}_{X^6-1} = \overline{\mathcal{H}}_{(X^2-1)^3} \bigsqcup (X^2 - 1) \cdot \overline{\mathcal{H}}_{(X^2-1)}$. The sets $\overline{\mathcal{H}}_{(X^2-1)}$ (with cardinal 2) and $\overline{\mathcal{H}}_{(X^2-1)^3}$ (with cardinal 6) are constructed with Lemma 1 and Remark 2:

$$\overline{\mathcal{H}}_{X^2-1} = \{X + \alpha_0 \mid \alpha_0^2 = -1, \alpha_0^4 = 1\} = \{X + \gamma, X - \gamma\}$$

p	negacyclic	$ heta ext{-cyclic}$	θ -negacyclic
$p \equiv 3 \pmod{4}$	$p^s + 1$	$2\frac{p^{(p^s+1)/2-1}}{p-1}$	0
$p \equiv 1 \pmod{4}$	$p^s + 1$	0	$2\frac{p^{(p^s+1)2-1}}{p-1}$

Table 1: Numbers of self-dual negacyclic (Corollary 3.3 of [5]), θ -cyclic (Proposition 4) and θ -negacyclic (Proposition 4) codes over \mathbb{F}_{p^2} of dimension p^s with p odd prime number and $\theta: x \mapsto x^p$.

and $\overline{\mathcal{H}}_{(X^2-1)^3} = \{(X + \alpha_0) \cdot (X^2 + 2\alpha_1 X + 1) \mid \alpha_0 = \pm \gamma, \alpha_1 \neq \alpha_0, \alpha_1 \in \{\pm \gamma, \pm 1\}\}$. The $2 \times 3 = 6$ elements of $\overline{\mathcal{H}}_{(X^2-1)^3}$ are listed below :

$$\begin{cases} (X+\gamma) \cdot (X^2+2X+1) &= X^3 + (\gamma-1)X^2 + (1-\gamma)X + \gamma \\ (X+\gamma) \cdot (X^2+X+1) &= X^3 + (\gamma+1)X^2 + (\gamma+1)X + \gamma \\ (X+\gamma) \cdot (X^2-2\gamma X+1) &= X^3 + \gamma \\ (X-\gamma) \cdot (X^2+2X+1) &= X^3 + (-\gamma-1)X^2 + (1+\gamma)X - \gamma \\ (X-\gamma) \cdot (X^2+X+1) &= X^3 + (-\gamma+1)X^2 + (-\gamma+1)X - \gamma \\ (X-\gamma) \cdot (X^2+2\gamma X+1) &= X^3 - \gamma. \end{cases}$$

Proposition 4 enables also to simplify Proposition 2 as follows. It will be useful in the two next sections.

Proposition 5 Consider p a prime number, θ the Frobenius automorphism over \mathbb{F}_{p^2} , $R = \mathbb{F}_{p^2}[X;\theta]$, k a positive integer, s,t two integers such that $k = p^s \times t$ and p does not divide t. The number of self-dual (θ, ε) -constacyclic codes over \mathbb{F}_{p^2} with dimension k is

$$\#\mathcal{H}_{X^{2k}-\varepsilon} = N_{\varepsilon} \times \prod_{f \in \mathcal{F}_{k,\varepsilon}} \#\mathcal{H}_{f^{p^s}} \times \prod_{f \in \mathcal{G}_{k,\varepsilon}} \#\mathcal{H}_{f^{p^s}}$$

where

$$N_{1} = \begin{cases} 0 & if \quad k \equiv 1 \pmod{2} \text{ and } p \equiv 1 \pmod{4} \\ & or \quad k \equiv 0 \pmod{2} \text{ and } p \text{ odd} \\ 1 & if \quad s = 0 \text{ and } p = 2 \\ 3 & if \quad s > 0 \text{ and } p = 2 \\ 2\frac{p^{(p^{s}+1)/2} - 1}{p - 1} & if \quad k \equiv 1 \pmod{2} \text{ and } p \equiv 3 \pmod{4} \end{cases}$$

and

$$N_{-1} = \begin{cases} 0 & \text{if } k \equiv 1 \pmod{2} \text{ and } p \equiv 3 \pmod{4} \\ 1 & \text{if } k \equiv 0 \pmod{2} \text{ and } p \text{ odd} \\ 2\frac{p^{(p^s+1)/2} - 1}{p - 1} & \text{if } k \equiv 1 \pmod{2} \text{ and } p \equiv 1 \pmod{4}. \end{cases}$$

Proof. One starts with Proposition 2 where the expression of N_{ε} is given in function of $\#\mathcal{H}_{(X^2\pm 1)^{p^s}}$. One simplifies N_{ε} thanks to Proposition 4 (for p odd prime) and Remark 3 (for p=2).

4 Self-dual θ -cyclic and θ -negacyclic codes with dimension prime to p over \mathbb{F}_{p^2} .

Over \mathbb{F}_{p^2} with p odd prime number, there is no self-dual cyclic code and the number of selfdual negacyclic codes with dimension k prime to p is given in Theorem 2 of [17]. Self-dual cyclic codes over \mathbb{F}_4 with odd dimension are studied in [10],

The aim of this section is to construct and to enumerate self-dual θ -cyclic and θ -negacyclic codes over \mathbb{F}_{p^2} whose dimension is prime to p when p is a prime number and θ is the Frobenius automorphism (Proposition 8).

The starting point of the study is Proposition 5 applied in the particular case when the dimension k of the code is prime to p (i.e. $k = p^s \times t, s = 0$ and $p \not| t$). One wants to determine now the set \mathcal{H}_f for f in $\mathcal{F} \cup \mathcal{G}$.

Consider $f = f(X^2)$ in $\mathcal{F} \cup \mathcal{G}$. Note that if f is in \mathcal{F} then the degree d of f in X^2 is even (see exercise 3.14 page 141 of [12]). Consider δ in \mathbb{N} such that $d = 2\delta$ where δ is in \mathbb{N}^* . Let h in R monic with degree 2δ :

$$h = X^{2\delta} + \sum_{i=0}^{2\delta-1} h_i X^i$$

= $(X^{2\delta} + \sum_{i=0}^{\delta-1} h_{2i} X^{2i}) + X \cdot \left(\sum_{i=0}^{\delta-1} \theta(h_{2i+1}) X^{2i} \right).$

The skew reciprocal polynomial h^* of h is

$$h^* = 1 + \sum_{i=1}^{\delta} h_{2\delta-2i} X^{2i} + \left(\sum_{i=0}^{\delta-1} \theta(h_{2\delta-2i-1}) X^{2i} \right) \cdot X .$$

One can associate to h the two polynomials defined in $\mathbb{F}_{p^2}[Z]$ by

$$A(Z) := Z^{\delta} + \sum_{i=0}^{\delta-1} h_{2i} Z^i \text{ and } B(Z) := \sum_{i=0}^{\delta-1} \theta(h_{2i+1}) Z^i.$$
(9)

Using the commutation law (1), one gets that $h^{\natural} \cdot h = f(X^2)$ if and only if the following polynomial relations in $\mathbb{F}_{p^2}[Z]$ are satisfied :

$$\begin{cases} Z^{\delta}A\left(\frac{1}{Z}\right)A(Z) + Z^{\delta}B\left(\frac{1}{Z}\right)B(Z) - h_0f(Z) = 0\\ Z^{\delta}A\left(\frac{1}{Z}\right)\Theta(B)(Z) + Z^{\delta-1}B\left(\frac{1}{Z}\right)\Theta(A)(Z) = 0 \end{cases}$$
(10)

where $\Theta : \sum a_i Z^i \mapsto \sum a_i^p Z^i$.

In the rest of the section, the following notation will be useful :

Notation 2 Consider $P(X^2) = \sum P_i X^{2i}$ in $\mathbb{F}_p[X^2]$, one denotes P(Z) the polynomial in $\mathbb{F}_{p^2}[Z]$ defined by $P(Z) = \sum P_i Z^i$. For a in $\overline{\mathbb{F}_{p^2}}$ and $P(X^2)$ in $\mathbb{F}_p[X^2]$, P(a) is $\sum P_i a^i$. The Frobenius automorphism θ defined over \mathbb{F}_{p^2} is extended to $\overline{\mathbb{F}_{p^2}}$ and is denoted with the same letter θ .

Finding (A, B) in $\mathbb{F}_{p^2}[Z] \times \mathbb{F}_{p^2}[Z]$ satisfying (10) with A monic, deg $(A) = \delta$ and deg $(B) \leq \delta - 1$ enables to construct the elements h of \mathcal{H}_f . One first considers the resolution of (10) when B(Z) = 0. This amounts to find the elements of $\mathcal{H}_f \cap \mathbb{F}_{p^2}[X^2]$.

Lemma 3 1. Consider $f = f(X^2)$ in \mathcal{F} with degree 2δ in X^2 and $f(X^2) = \tilde{f}(X^2) \times \Theta(\tilde{f})(X^2)$ the factorization of $f(X^2)$ in $\mathbb{F}_{p^2}[X^2]$.

$$\mathcal{H}_{f} \cap \mathbb{F}_{p^{2}}[X^{2}] = \begin{cases} \emptyset & \text{if } \delta \equiv 0 \pmod{2} \\ \{\tilde{f}(X^{2}), \Theta(\tilde{f})(X^{2})\} & \text{if } \delta \equiv 1 \pmod{2} \end{cases}$$

2. Consider $f = f(X^2)$ in \mathcal{G} with degree 2δ in X^2 and $g(X^2)$ such that $f(X^2) = g(X^2)g^{\natural}(X^2)$. When δ is even, consider the factorization of $g(X^2)$ in $\mathbb{F}_{p^2}[X^2]$: $g(X^2) = \tilde{g}(X^2) \times \Theta(\tilde{g})(X^2)$. $\mathcal{H}_f \cap \mathbb{F}_{p^2}[X^2] =$

$$\begin{cases} \{g(X^2), g^{\natural}(X^2), \tilde{g}(X^2)\Theta(\tilde{g}^{\natural})(X^2), \tilde{g}^{\natural}(X^2)\Theta(\tilde{g})(X^2)\} & if \quad \delta \equiv 0 \pmod{2} \\ \{g(X^2), g^{\natural}(X^2)\} & if \quad \delta \equiv 1 \pmod{2} \end{cases}$$

Proof. Recall that h is in \mathcal{H}_f if and only if (A(Z), B(Z)) defined by (9) satisfies the relation (10). Furthermore h is in $\mathbb{F}_{p^2}[X^2]$ if and only if B(Z) = 0. The elements of $\mathcal{H}_f \cap \mathbb{F}_{p^2}[X^2]$ are therefore characterized by the relations B(Z) = 0 and $Z^{\delta}A\left(\frac{1}{Z}\right)A(Z) = h_0f(Z)$ where h_0 is the constant term of A.

Here are now necessary conditions for h belonging to $\mathcal{H}_f \setminus \mathbb{F}_{p^2}[X^2]$.

Lemma 4 Consider $f = f(X^2)$ in $\mathcal{F} \cup \mathcal{G}$ with degree 2δ in X^2 , h in R monic with degree 2δ and (A(Z), B(Z)) defined in (9). If $h \in \mathcal{H}_f \setminus \mathbb{F}_{p^2}[X^2]$ then

- $(i) \ \gcd(A(Z), B(Z)) = 1$
- (*ii*) gcd(B(Z), f(Z)) = 1.

Proof.

(i) Assume that A(Z) and B(Z) have a common factor in $\mathbb{F}_{p^2}[Z]$ then according to the first relation of (10), this factor must divide f(Z). Furthermore, $B(Z) \neq 0$ and the degree of B(Z) is $\leq \delta - 1$, therefore f(Z) must have a nontrivial factor in $\mathbb{F}_{p^2}[Z]$ with degree $\leq \delta - 1$. Necessarily δ is even, $f = gg^{\natural}$ with $g = \tilde{g}\Theta(\tilde{g})$ product of two irreducible polynomials of degree $\delta/2$ in $\mathbb{F}_{p^2}[Z]$. Without loss of generality one can assume that $\tilde{g}(Z)$ is the common factor of A(Z) and B(Z) in $\mathbb{F}_{p^2}[Z]$.

Consider β such that $\tilde{g}(\beta) = 0$, a(Z) and b(Z) in $\mathbb{F}_{p^2}[Z]$ such that $A(Z) = \tilde{g}(Z)a(Z)$ and $B(Z) = \tilde{g}(Z)b(Z)$. From relations (10), one gets that

$$\begin{cases} Z^{\delta/2}a(1/Z)a(Z) + Z^{\delta/2}b(1/Z)b(Z) &= \lambda\Theta(\tilde{g})(Z)\tilde{g}^{\natural}(Z) \\ Z^{\delta/2}a(1/Z)\Theta(b)(Z) + Z^{\delta/2-1}b(1/Z)\Theta(a)(Z) &= 0 \end{cases}$$
(11)

where λ is a nonzero constant. Consider u in $\mathbb{F}_{p^{\delta}} \setminus \{0\}$ such that $a(\gamma) = u \times b(\gamma)$. According to (11) evaluated at γ ,

$$\begin{cases} a(\gamma)a(1/\gamma) + b(\gamma)b(1/\gamma) &= 0\\ \gamma\Theta(b)(\gamma)a(1/\gamma) + \Theta(a)(\gamma)b(1/\gamma) &= 0. \end{cases}$$

From the first relation, one deduces that $a(1/\gamma) = -1/u \times b(1/\gamma)$ and from the second relation, one deduces $-\gamma/u\Theta(b)(\gamma) + \Theta(a)(\gamma) = 0$ so $(-\gamma/u)^p \times b(\gamma^p) + a(\gamma^p) = 0$. As $a(\gamma^p)a(1/\gamma^p) + b(\gamma^p)b(1/\gamma^p) = 0$, one gets $a(1/\gamma^p) = (u/\gamma)^p \times b(1/\gamma^p)$. Therefore

$$\begin{cases} a(\gamma) &= u \times b(\gamma) \\ a(1/\gamma) &= -1/u \times b(1/\gamma) \\ a(\gamma^p) &= \gamma^p/u^p \times b(\gamma^p) \\ a(1/\gamma^p) &= -u^p/\gamma^p \times b(1/\gamma^p) \end{cases}$$

In particular, the polynomial $Z^{\delta/2}a(1/Z)a(Z) + Z^{\delta/2}b(1/Z)b(Z)$ cancels at γ , $1/\gamma$, γ^p , $1/\gamma^p$, therefore it is divisible by f(Z), which is impossible because of the first relation of (11).

(ii) Assume that B(Z) and f(Z) are not coprime in $\mathbb{F}_{p^2}[Z]$. Necessarily δ is even, $f = gg^{\natural}$ with $g = \tilde{g}\Theta(\tilde{g})$ product of two irreducible polynomials of degree $\delta/2$ in $\mathbb{F}_{p^2}[Z]$. Without loss of generality one can assume that $\tilde{g}(Z)$ is the common factor of f(Z) and B(Z) in $\mathbb{F}_{p^2}[Z]$. Consider β such that $\tilde{g}(\beta) = 0$, one has $B(\beta) = 0$, $B(\beta^{-1})$, $B(\beta^p)$, $B(\beta^{-p}) \neq 0$. Furthermore $\Theta(B)(\beta^p) = 0$ so according to the second relation of (10), $\Theta(A)(\beta^p)B(1/\beta^p) =$ 0 and $A(\beta) = 0$. Therefore A(Z) and B(Z) have a common factor in $\mathbb{F}_{p^2}[Z]$, which is impossible according to (i).

To characterize the elements h of \mathcal{H}_f such that h does not belong to $\mathbb{F}_{p^2}[X^2]$, one will use the following rational interpolation problem or Cauchy interpolation problem (Section 5.8 of [20]): given 2δ distinct points $x_0, \ldots, x_{2\delta-1}$ in $\mathbb{F}_{p^{2\delta}}$ and 2δ values $y_0, \ldots, y_{2\delta-1}$ in $\mathbb{F}_{p^{2\delta}}$, find a rational function $r/t \in \mathbb{F}_{p^{2\delta}}(Z)$ such that

$$(RI) : t(x_i) \neq 0, \frac{r(x_i)}{t(x_i)} = y_i \text{ for } 0 \le i < 2\delta - 1, \ \deg(r) < \delta + 1, \ \deg(t) \le \delta - 1$$

Note that this problem can be rewritten as

$$gcd(t, f) = 1, r \equiv P \times t^{-1} \pmod{f}, \deg(r) < \delta + 1, \deg(t) \le \delta - 1$$

where $f = \prod_{i=0}^{2\delta-1} (Z - x_i)$, P has degree $\leq 2\delta - 1$ and $P(x_i) = y_i$ for $0 \leq i < 2\delta - 1$. This problem can be solved using extended Euclidean algorithm ([20]).

4.1 Construction of \mathcal{H}_f for f in \mathcal{F}

For f in \mathcal{F} , one first gives a characterization of the elements h of $\mathcal{H}_f \setminus \mathbb{F}_{p^2}[X^2]$.

Lemma 5 Consider $f = f(X^2)$ in \mathcal{F} with degree $d = 2\delta$ in X^2 and α in $\mathbb{F}_{p^{2\delta}}$ such that $f(\alpha) = 0$.

1. Consider h in R monic with degree $d = 2\delta$ and (A(Z), B(Z)) defined in (9). Then $h \in \mathcal{H}_f \setminus \mathbb{F}_{p^2}[X^2]$ if and only if there exists u in \mathbb{F}_{p^d} such that

$$\begin{cases}
A(\alpha) = u \times B(\alpha) \\
A(\alpha^p) = \alpha^p / u^p \times B(\alpha^p) \\
\gcd(A(Z), B(Z)) = 1 \\
\gcd(B(Z), f(Z)) = 1
\end{cases}$$
(12)

and

$$\begin{cases} u^{p^{\delta}+1} = -1 & \text{if } \delta \text{ even} \\ u^{p^{\delta}-1} = -1/\alpha & \text{if } \delta \text{ odd.} \end{cases}$$
(13)

- 2. Consider u in \mathbb{F}_{p^d} such that the condition (13) is satisfied. There exists a unique solution (A, B) in $\mathbb{F}_{p^2}[Z] \times \mathbb{F}_{p^2}[Z]$ to (12) with A monic, $\deg(A) = \delta$, $\deg(B) \leq \delta 1$.
- 3. The set $\mathcal{H}_f \setminus \mathbb{F}_{p^2}[X^2]$ has $p^{\delta} + 1$ elements if δ is even and $p^{\delta} 1$ elements if δ is odd.

Proof. As f belongs to \mathcal{F} , $f(\alpha^{-1}) = 0$ so there exists i in $\{0, \ldots, d-1\}$ such that $\alpha^{-1} = \alpha^{p^i}$. As $\alpha^{p^{2i}} = (\alpha^{-1})^{p^i} = \alpha$, and as $f(X^2)$ is irreducible in $\mathbb{F}_p[X^2]$ with degree $d = 2\delta$, necessarily $i = \delta$ and $\alpha^{-1} = \alpha^{p^{\delta}}$.

1. Consider h in R with degree $d = 2\delta$ and (A(Z), B(Z)) defined by (9). If h belongs to $\mathcal{H}_f \setminus \mathbb{F}_{p^2}[X^2]$ then the relations (10) are satisfied by (A(Z), B(Z)) with

 $B(Z) \neq 0$. Consider u, v in \mathbb{F}_{p^d} such that $\begin{cases} A(\alpha) = u \times B(\alpha) \\ A(\alpha^p) = v \times B(\alpha^p) \end{cases}$. According to Lemma 4, $\gcd(B, f) = 1$ so $B(\alpha), B(\alpha^p) \neq 0$. If δ is even, then $A(\alpha^{-1}) = A(\alpha)^{p^{\delta}}$ and $B(\alpha^{-1}) = B(\alpha)^{p^{\delta}}$. Evaluating (10) at α one gets :

$$\begin{cases} u^{p^{\delta}} \times u + 1 &= 0\\ \alpha \times u^{p^{\delta}} + \theta^{-1}(v) &= 0 \end{cases}$$

therefore $v = \alpha^p / u^p$ and $u^{p^{\delta}+1} = -1$. If δ is odd, then $A(\alpha^{-1}) = A(\alpha^p)^{p^{\delta^{-1}}}$ and $B(\alpha^{-1}) = B(\alpha^p)^{p^{\delta^{-1}}}$. Evaluating (10) at α , one gets :

$$\begin{cases} v^{p^{\delta-1}} \times u + 1 &= 0\\ \alpha \times v^{p^{\delta-1}} + \theta^{-1}(v) &= 0 \end{cases}$$

therefore $v = \alpha^p / u^p$ and $u^{p^{\delta} - 1} = -1/\alpha$.

Conversely, if there exists u in \mathbb{F}_{p^d} such that (12) and (13) are satisfied, then one can check that the polynomials $Z^{\delta}A\left(\frac{1}{Z}\right)A(Z) + Z^{\delta}B\left(\frac{1}{Z}\right)B(Z) - h_0f(Z)$ and $Z^{\delta}A\left(\frac{1}{Z}\right)\Theta(B)(Z) + Z^{\delta-1}B\left(\frac{1}{Z}\right)\Theta(A)(Z)$ cancel at α and α^p . Therefore the relations (10) are satisfied and h belongs to \mathcal{H}_f . As gcd(B, f) = 1, $B \neq 0$ so h belongs to

- $\mathcal{H}_f \setminus \mathbb{F}_{p^2}[X^2].$
- 2. Consider u in $\mathbb{F}_{p^{2\delta}}$ such that the condition (13) is satisfied. Consider the 2δ points $(x_i, y_i)_{0 \le i \le 2\delta 1}$ defined by

$$(x_i, y_i) = \begin{cases} (\theta^i(\alpha), \theta^i(u)) & if \quad i \equiv 0 \pmod{2} \\ (\theta^i(\alpha), \theta^i(\alpha/u)) & if \quad i \equiv 1 \pmod{2}. \end{cases}$$

According to Corollary 5.18 of [20] there exists two nonzero polynomials A and B in $\mathbb{F}_{p^{2\delta}}[Z]$ such that $\deg(A) < \delta + 1$, $\deg(B) \leq \delta - 1$ and $A(x_i) = y_i B(x_i)$. Without loss of

generality, one can assume that A and B are coprime and that A is monic. Furthermore, the set $(x_i, y_i)_{0 \le i \le 2\delta - 1}$ is stable under the action of θ^2 , therefore $(\Theta^2(A), \Theta^2(B))$ satisfies the relations $\Theta^2(A)(x_i) = y_i \Theta^2(B)(x_i)$. As A and B are coprime and A is monic, $\Theta^2(A) = A, \Theta^2(B) = B$. Therefore A and B are polynomials of $\mathbb{F}_{p^2}[Z]$.

Considering the two first relations $A(x_0) = y_0 B(x_0)$ and $A(x_1) = y_1 B(x_1)$ one gets the relation (12), so the relation (10) is satisfied and the skew polynomial h associated to A and B belongs to \mathcal{H}_f . As A and B are coprime, B and f are also coprime (see Lemma 4). Assume that $\deg(A) \neq \delta$, then $Z^{\delta}A(1/Z)A(Z) + Z^{\delta}B(1/Z)B(Z)$ would be the zero polynomial and h would satisfy $h^* \cdot h = 0$ which is impossible.

Therefore for u in $\mathbb{F}_{p^{2\delta}}$ such that the condition (13) is satisfied, there exists (A, B) in $\mathbb{F}_{p^2}[Z] \times \mathbb{F}_{p^2}[Z]$ satisfying (12) with A monic, $\deg(A) = \delta$ and $\deg(B) \leq \delta - 1$. The unicity of (A, B) follows from the fact that A/B is the unique solution to the

The unicity of (A, B) follows from the fact that A/B is the unique solution to the rational interpolation problem (RI) with A and B coprime (Corollary 5.18 of [20]).

3. According to 1., $\mathcal{H}_f = \bigsqcup_u \{h \in R \mid h \text{ monic, } \deg(h) = d, (A, B) \text{ defined in } (9)$

solution of (12)} where u satisfies (13). According to 2., for each u satisfying (13), there is a unique h in R monic of degree d such that (A, B) defined in (9) is solution of (12). Therefore, the number of elements of \mathcal{H}_f is the number of u in \mathbb{F}_{p^d} satisfying $u^{p^{\delta}+1} = -1$ if δ is even and $u^{p^{\delta}-1} = -1/\alpha$ is δ is odd.

Proposition 6 Consider p a prime number, m a positive integer, θ the Frobenius automorphism over \mathbb{F}_{p^2} and $R = \mathbb{F}_{p^2}[X;\theta]$. Let $f = f(X^2)$ in \mathcal{F} and $d = 2\delta$ its degree in X^2 , then the set $\overline{\mathcal{H}}_f = \mathcal{H}_f$ has $1 + p^{\delta}$ elements.

Proof. The elements of $\mathcal{H}_f \cap \mathbb{F}_{p^2}[X^2]$ are given in point 1. of Lemma 3: there are two elements if δ is odd and no element if δ is even. The elements of \mathcal{H}_f who do not belong to $\mathbb{F}_{p^2}[X^2]$ are given in point 3. of Lemma 5. There are $p^{\delta} - 1$ elements if δ is odd and $p^{\delta} + 1$ elements if δ is even.

Example 4 Consider p = 2, θ the Frobenius automorphism over $\mathbb{F}_4 = \mathbb{F}_2(a)$ where $a^2 + a + 1 = 0$, $R = \mathbb{F}_4[X; \theta]$ and $f(X^2) = X^4 + X^2 + 1$ in \mathcal{F} . Consider $h = X^2 + h_1X + h_0$ in R, $A(Z) = Z + h_0$ and $B(Z) = \theta(h_1)$ in $\mathbb{F}_4[Z]$. One has

$$h^{\natural} \cdot h = X^{4} + X^{2} + 1 \Leftrightarrow \begin{cases} ZA(1/Z)A(Z) + ZB(1/Z)B(Z) = h_{0}(Z^{2} + Z + 1) \\ ZA(1/Z)\Theta(B)(Z) + B(1/Z)\Theta(A)(Z) = 0. \end{cases}$$

If $h_1 = 0$, one gets $h^{\natural} \cdot h = X^4 + X^2 + 1$ if and only if $ZA\left(\frac{1}{Z}\right)A(Z) = h_0(Z^2 + Z + 1)$. As $Z^2 + Z + 1 = (Z + a)(Z + a^2)$ and $a^2 = 1/a$, one gets A(Z) = Z + a or $A(Z) = Z + a^2$ (see Lemma 3), therefore if $h_1 = 0$, $h = X^2 + a$ or $h = X^2 + a^2$.

Following Lemma 5, if $h_1 \neq 0$, then $h^{\natural} \cdot h = X^4 + X^2 + 1$ if and only if there exists u in \mathbb{F}_4 such that u = 1/a and

$$\left\{ \begin{array}{rrl} A(a) &=& u\times B(a)=1/a\times B(a)\\ A(a^2) &=& \frac{a^2}{u^2}\times B(a^2)=a\times B(a^2). \end{array} \right.$$

Therefore when $h_1 \neq 0$, one gets $h \in \mathcal{H}_{X^4+X^2+1}$ if and only $h = X^2 + X + 1$. As a conclusion the set $\mathcal{H}_{X^4+X^2+1}$ is

$$\mathcal{H}_{X^4+X^2+1} = \overline{\mathcal{H}}_{X^4+X^2+1} = \{X^2 + a, X^2 + a^2, X^2 + X + 1\}.$$

Example 5 Consider p = 2, θ the Frobenius automorphism over $\mathbb{F}_4 = \mathbb{F}_2(a)$ with $a^2 + a + 1 = 0$ and $R = \mathbb{F}_4[X; \theta]$. The skew polynomial $X^{12} + X^6 + 1$ belongs to \mathcal{F} and its degree in X^2 is 6. Consider α in \mathbb{F}_{2^6} such that $\alpha^6 + \alpha^3 + 1 = 0$. According to Lemma 3 the elements of $\mathcal{H}_{X^{12}+X^6+1}$ with no term of odd degree are $X^6 + a$ and $X^6 + a^2$. According to Lemma 5, the other elements of $\mathcal{H}_{X^{12}+X^6+1}$ are the monic skew polynomials h of degree 6 such that (A(Z), B(Z)) defined by relations (9) are solutions of (12) with $u^7 = 1/\alpha$. The table below gives the solutions corresponding to the seven problems (12).

u	h
$1 + \alpha$	$X^{6} + a^{2}X^{5} + aX^{4} + aX^{2} + a^{2}X + a^{2}$
$1 + \alpha + \alpha^5$	$X^6 + X^5 + a^2 X^4 + a X^2 + X + 1$
$\alpha + \alpha^3 + \alpha^4 + \alpha^5$	$X^6 + X^4 + a^2 X^3 + a^2 X^2 + a^2$
$lpha^5$	$X^6 + X^3 + 1$
$1 + \alpha^3 + \alpha^4$	$X^6 + X^4 + aX^3 + aX^2 + a$
$\alpha + \alpha^3 + \alpha^4$	$X^6 + X^5 + aX^4 + a^2X^2 + X + 1$
$1 + \alpha^3 + \alpha^4 + \alpha^5$	$X^6 + aX^5 + a^2X^4 + a^2X^2 + aX + a$

The number of elements of $\mathcal{H}_{X^{12}+X^6+1}$ is $9 = 1 + 2^3$.

4.2 Construction of \mathcal{H}_f for f in \mathcal{G}

For f in \mathcal{G} , one gives a characterization of the elements of $\mathcal{H}_f \setminus \mathbb{F}_{p^2}[X^2]$.

Lemma 6 Consider $f = f(X^2)$ in \mathcal{G} with degree 2δ in X^2 and g irreducible in $\mathbb{F}_p[X^2]$ such that $f(X^2) = g(X^2)g^{\natural}(X^2)$. Consider β in $\mathbb{F}_{p^{\delta}}$ such that $g(\beta) = 0$.

1. Consider h in R monic with degree 2δ and (A(Z), B(Z)) defined by relations (9). Then $h \in \mathcal{H}_f \setminus \mathbb{F}_{p^2}[X^2]$ if and only if there exists u in \mathbb{F}_{p^d} such that

$$\begin{cases}
A(\beta) = u \times B(\beta) \\
A(1/\beta) = -1/u \times B(1/\beta) \\
A(\beta^{p}) = \beta^{p}/u^{p} \times B(\beta^{p}) \\
A(1/\beta^{p}) = -u^{p}/\beta^{p} \times B(1/\beta^{p}) \\
gcd(A(Z), B(Z)) = 1 \\
gcd(B(Z), f(Z)) = 1
\end{cases}$$
(14)

and

$$\begin{cases} u^{p^{\delta}-1} = 1 & if \quad \delta even \\ u^{p^{\delta}+1} = \beta & if \quad \delta odd. \end{cases}$$
(15)

2. Consider u in \mathbb{F}_{p^d} such that the condition (15) is satisfied. There exists a unique solution (A, B) in $\mathbb{F}_{p^2}[Z] \times \mathbb{F}_{p^2}[Z]$ to (14) with A monic, $\deg(A) = \delta$, $\deg(B) \leq \delta - 1$.

3. The set $\mathcal{H}_f \setminus \mathbb{F}_{p^2}[X^2]$ has $p^{\delta} - 1$ elements if δ is even and $p^{\delta} + 1$ elements if δ is odd. **Proof.**

1. Consider h in R with degree $d = 2\delta$ and (A(Z), B(Z)) defined by (9).

If $h \in \mathcal{H}_f \setminus \mathbb{F}_{p^2}[X^2]$ then (A(Z), B(Z)) satisfies the relation (10) with $B(Z) \neq 0$. According to Lemma 4, B is coprime with f, therefore $B(\beta)$, $B(1/\beta)$, $B(\beta^p)$ and $B(1/\beta^p) \neq 0$. Consider u in $\mathbb{F}_{p^{\delta}}$ such that $A(\beta) = u \times B(\beta)$. According to (10) evaluated at β ,

$$\begin{cases} A(\beta)A(1/\beta) + B(\beta)B(1/\beta) = 0\\ \beta\Theta(B)(\beta)A(1/\beta) + \Theta(A)(\beta)B(1/\beta) = 0 \end{cases}$$

From the first relation, one deduces that $A(1/\beta) = -1/u \times B(1/\beta)$ and from the second relation, one deduces $-\beta/u\Theta(B)(\beta) + \Theta(A)(\beta) = 0$ so $(-\beta/u)^p \times B(\beta^p) + A(\beta^p) = 0$. As $A(\beta^p)A(1/\beta^p) + B(\beta^p)B(1/\beta^p) = 0$, one gets $A(1/\beta^p) = (u/\beta)^p \times B(1/\beta^p)$. Therefore the relations (14) are satisfied.

Furthermore, if δ is odd, one gets another constraint, namely as $\beta = \beta^{p^{\delta}}$ one gets $\Theta(A)(\beta) = (\Theta(A)(\beta^p))^{p^{\delta-1}} = (u^p B(\beta)^p)^{p^{\delta-1}} = u^{p^{\delta}} \Theta(B)(\beta)$. Furthermore, $-\beta/u\Theta(B)(\beta) + \Theta(A)(\beta) = 0$, so $-\beta/u + u^{p^{\delta}} = 0$ and $u^{p^{\delta}+1} = \beta$. The relations (15) are therefore satisfied.

Conversely, if there exists u in $\mathbb{F}_{p^{\delta}}$ such that (14) and (15) are satisfied, then one can check that the polynomials $Z^{\delta}A\left(\frac{1}{Z}\right)A(Z) + Z^{\delta}B\left(\frac{1}{Z}\right)B(Z) - h_0f(Z)$ and $Z^{\delta}A\left(\frac{1}{Z}\right)\Theta(B)(Z) + Z^{\delta-1}B\left(\frac{1}{Z}\right)\Theta(A)(Z)$ cancel at β , $1/\beta$, β^p and $1/\beta^p$. Therefore the relations (10) are satisfied and h belongs to \mathcal{H}_f . As gcd(B, f) = 1, B is nonzero so h belongs to $\mathcal{H} \setminus \mathbb{F}_{p^2}[X^2]$.

2. Consider u in $\mathbb{F}_{p^{2\delta}}$ such that the condition (15) is satisfied. Consider the 2δ points $(x_i, y_i)_{0 \le i \le 2\delta - 1}$ defined by

$$(x_i, y_i) = \begin{cases} (\theta^i(\beta), \theta^i(u)) & if \quad i \equiv 0 \pmod{2}, i < \delta\\ (\theta^i(\beta), \theta^i(\beta/u)) & if \quad i \equiv 1 \pmod{2}, i < \delta \end{cases}$$
$$(x_{i+\delta}, y_{i+\delta}) = \begin{cases} (\theta^i(1/\beta), -\theta^i(1/u)) & if \quad i \equiv 0 \pmod{2}, i < \delta\\ (\theta^i(1/\beta), -\theta^i(u/\beta)) & if \quad i \equiv 1 \pmod{2}, i < \delta \end{cases}$$

According to Corollary 5.18 of [20] there exists two nonzero polynomials A and Bin $\mathbb{F}_{p^{2\delta}}[Z]$ such that $\deg(A) < \delta + 1$, $\deg(B) \leq \delta - 1$ and for i in $\{0, \ldots, 2\delta - 1\}$, $A(x_i) = y_i B(x_i)$. Without loss of generality, one can assume that A and B are coprime and that A is monic. Furthermore, as the the set of points $\{(x_i, y_i)\}$ is stable under the application of θ^2 , A(Z) and B(Z) belong to $\mathbb{F}_{p^2}[Z]$.

Considering the four relations $A(x_0) = y_0 B(x_0)$, $A(x_1) = y_1 B(x_1)$, $A(x_\delta) = y_\delta B(x_\delta)$ and $A(x_{\delta+1}) = y_{\delta+1} B(x_{\delta+1})$ one gets the relation (14), so the relation (10) is satisfied and the skew polynomial h associated to A and B belongs to \mathcal{H}_f . As A and B are coprime, B and f are also coprime (see Lemma 4). Assume that deg $(A) \neq \delta$, then $Z^{\delta}A(1/Z)A(Z) + Z^{\delta}B(1/Z)B(Z)$ would be the zero polynomial and h would satisfy $h^* \cdot h = 0$ which is impossible.

Therefore for u in $\mathbb{F}_{p^{2\delta}}$ such that the condition (15) is satisfied, there exists (A, B) in $\mathbb{F}_{p^2}[Z] \times \mathbb{F}_{p^2}[Z]$ satisfying (14) with A monic, $\deg(A) = \delta$ and $\deg(B) \leq \delta - 1$ with A and B coprime.

The unicity follows form the fact that A/B is the unique solution to the rational interpolation problem (RI) with A and B coprime (Corollary 5.18 of [20]).

3. Like in Lemma 5, the number of elements of \mathcal{H}_f is deduced from points 1. and 2.

Proposition 7 Consider p a prime number, θ the Frobenius automorphism over \mathbb{F}_{p^2} and $R = \mathbb{F}_{p^2}[X;\theta]$. Let $f = f(X^2)$ in \mathcal{G} and $d = 2\delta$ its degree in X^2 , then the set \mathcal{H}_f has $3 + p^{\delta}$ elements and $\overline{\mathcal{H}}_f$ has $1 + p^{\delta}$ elements.

Proof. The result is deduced from point 2. of Lemma 3 and point 3. of Lemma 6. Furthermore if $f(X^2) = g(X^2)g^{\natural}(X^2)$ belongs to \mathcal{G} , then $\overline{\mathcal{H}}_{f(X^2)} = \mathcal{H}_{f(X^2)} \setminus \{g(X^2), g^{\natural}(X^2)\}$ has $1 + p^{\delta}$ elements.

Example 6 Consider $\mathbb{F}_4 = \mathbb{F}_2(a)$ where $a^2 + a + 1 = 0$, θ the Frobenius automorphism, $R = \mathbb{F}_4[X;\theta]$ and $f(X^2) = (X^6 + X^2 + 1)(X^6 + X^4 + 1)$ in \mathcal{G} with degree $6 = 2 \times 3$ in X^2 . Consider β in \mathbb{F}_{2^3} such that $\beta^3 + \beta^2 + 1 = 0$. According to Lemma 3, the elements of \mathcal{H}_f with no term of odd degree are $X^6 + X^2 + 1$ and $X^6 + X^4 + 1$. According to Lemma 6, the other elements of \mathcal{H}_f are deduced from the polynomials A(Z) and B(Z) of $\mathbb{F}_4[Z]$ satisfying (14) with $u^9 = \beta$, A(Z) monic of degree 3 and B(Z) of degree ≤ 2 .

For example take $u = v^3$ where $v^6 + v^4 + v^3 + v + 1 = 0$, then $u^9 = \beta$ and the unique solution (A, B) in $\mathbb{F}_4[Z] \times \mathbb{F}_4[Z]$ of (14) with A monic of degree 3 and B of degree ≤ 2 is $(A, B) = (Z^3 + a_3Z^2 + a^2Z + 1)$. Therefore, $h(X) = (X^6 + a) + X \cdot (aX^4 + a^2X^2 + 1) = X^6 + a^2X^5 + aX^3 + X + a$ is an element of \mathcal{H}_f with at least one non zero term of odd degree. The entire set \mathcal{H}_f is $\{X^6 + X^2 + 1, X^6 + X^4 + 1, X^6 + X^5 + aX^3 + a^2X + a, X^6 + a^2X^5 + X^4 + X^2 + aX + 1, X^6 + x^5 + aX^3 + a^2X + a, X^6 + a^2X^5 + X^4 + X^2 + aX + 1, X^6 + a^2X^3 + X^4 + X^2 + a^2X^3 + X^4 + x^3 + X^2 + X + 1, X^6 + aX^4 + aX^3 + X^2 + a, X^6 + a^2X^4 + a^2X^3 + X^2 + a^2, X^6 + aX^5 + a^2X^3 + X + a^2, X^6 + X^5 + a^2X^3 + aX + a^2, X^6 + a^2X^5 + aX^3 + X + a\}$. It has $2^{\delta} + 3 = 11$ elements (Proposition 7).

4.3 Conclusion

The proposition below gives a formula for the number of self-dual θ -cyclic and θ -negacyclic codes over \mathbb{F}_{p^2} whose dimension is prime to p. Tables 2 and 3 illustrate this proposition over \mathbb{F}_4 and \mathbb{F}_9 and give some elements of comparison with cyclic and negacyclic codes.

Proposition 8 Consider p a prime number, θ the Frobenius automorphism over \mathbb{F}_{p^2} , k a positive integer not divisible by p and ε in $\{-1,1\}$. The number of self-dual (θ, ε) -constacyclic codes with dimension k defined over \mathbb{F}_{p^2} is

$$N_{\varepsilon} \times \prod_{f \in \mathcal{F}_{k,\varepsilon}} (p^{\delta} + 1) \times \prod_{f \in \mathcal{G}_{k,\varepsilon}} (p^{\delta} + 3)$$

where 2δ is the degree of f in X^2 and N_{ε} is defined below :

$$N_{1} = \begin{cases} 0 & if \quad k \equiv 1 \pmod{2} \text{ and } p \equiv 1 \pmod{4} \\ & or \quad k \equiv 0 \pmod{2} \text{ and } p \text{ odd} \\ 1 & if \quad p = 2 \\ 2 & if \quad k \equiv 1 \pmod{2} \text{ and } p \equiv 3 \pmod{4} \end{cases}$$
$$N_{-1} = \begin{cases} 0 & if \quad k \equiv 1 \pmod{2} \text{ and } p \equiv 3 \pmod{4} \\ 1 & if \quad k \equiv 0 \pmod{2} \text{ and } p \equiv 3 \pmod{4} \\ 2 & if \quad k \equiv 1 \pmod{2} \text{ and } p \text{ odd} \\ 2 & if \quad k \equiv 1 \pmod{2} \text{ and } p \equiv 1 \pmod{4}. \end{cases}$$

Proof. According to Proposition 5, with s = 0, the number of self-dual (θ, ε) -constacyclic codes over \mathbb{F}_{p^2} with dimension k is

$$#\mathcal{H}_{X^{2k}-\varepsilon} = N_{\varepsilon} \times \prod_{f \in \mathcal{F}_{k,\varepsilon}} #\mathcal{H}_f \times \prod_{f \in \mathcal{G}_{k,\varepsilon}} #\mathcal{H}_f$$

where N_{ε} satisfies the above conditions. The final result follows from Proposition 6 and Proposition 7.

Example 7 Consider $\theta : x \mapsto x^2$ the Frobenius automorphism over $\mathbb{F}_4 = \mathbb{F}_2(a)$ where $a^2 + a + 1 = 0$. The self-dual θ -cyclic codes of dimension 9 over \mathbb{F}_4 are characterized by the monic solutions of the self-dual skew equation $h^{\natural} \cdot h = X^{18} + 1$. As $X^{18} + 1 = (X^2 + 1)(X^4 + X^2 + 1)(X^{12} + X^6 + 1)$ in $\mathbb{F}_2[X^2]$ and as the polynomials $X^4 + X^2 + 1$ and $X^{12} + X^6 + 1$ are self-reciprocal and irreducible in $\mathbb{F}_2[X^2]$, the set $\mathcal{F}_{9,1}$ is $\{X^4 + X^2 + 1, X^{12} + X^6 + 1\}$ and the set $\mathcal{G}_{9,1}$ is empty. According to Proposition 8, the number of self-dual θ -cyclic codes of dimension 9 over \mathbb{F}_4 is $1 \times (2^1 + 1) \times (2^3 + 1) = 27$. More precisely the set $\mathcal{H}_{X^{18}+1}$ is given by

$$\mathcal{H}_{X^{18}+1} = \{ \operatorname{lcrm}(h_1, h_2, h_3) \mid h_1 \in \mathcal{H}_{X^2+1}, h_2 \in \mathcal{H}_{X^4+X^2+1}, h_3 \in \mathcal{H}_{X^{12}+X^6+1} \}$$

and the sets \mathcal{H}_{X^2+1} , $\mathcal{H}_{X^4+X^2+1}$ and $\mathcal{H}_{X^{12}+X^6+1}$ with cardinalities 1, 3 and 9 were previously computed in Examples 2, 4 and 5.

5 Self-dual θ -cyclic and θ -negacyclic codes with any dimension over \mathbb{F}_{p^2} .

In this section, one constructs and enumerates all self-dual θ -cyclic and θ -negacyclic codes over \mathbb{F}_{p^2} where p is a prime number and θ is the Frobenius automorphism.

Like in Section 4, the starting point of the construction is Proposition 5, who enables to write the monic solutions of the self-dual skew equation as least common right multiples of skew polynomials satisfying intermediate skew equations. The main topic of this section is therefore to construct the intermediate sets $\mathcal{H}_{f^{p^s}}$ where s > 0 and $f = f(X^2)$ belongs to $\mathcal{F} \cup \mathcal{G}$.

First, one assumes that $f = f(X^2)$ belongs to \mathcal{F} .

k	C	θ_{-c}]	k	C	H-c] _			
1	1	1		25	0	225445	{	k	с	heta-c
	1	1	{	30	9	220440	[69	9	25807570971
3	3	3	Į	37	1	262145	ļŀ	71	3	34359738371
5	1	5		39	9	799305		72	2	70344300625
7	3	11		41	1	1050625	1	10	0	10344300023
9	9	27	ĺ	43	3	2146689	1	75	27	306316140375
11	3		ł	45	81	10513035		77	27	389768283201
19	1	65		40	2	0200611	{ [79	3	549755813891
15	1	00		41	3	0000011		81	81	1859049764379
15	9	285]	49	9	23068705]	83	3	2100023255553
17	1	289		51	9	58159227		00 95	- 0 - 9	650220200005
19	3	513	ĺ	53	1	67108865	1	80	3	0302298310043
21	g	2211		55	9	173015535		87	9	13194944987145
21	2	2051	l	57	0	405017001	{	89	3	17695491973201
23	ა 1	2001	{	57	9	403017091		91	9	49242466343785
25	1	5125		-59	3	536870913	.	93	9	139327459600875
27	27	13851		61	1	1073741825	-	05	0	176265457835535
29	1	16385	1	63	9	5984882937	1	90	9 1	1102034310353535
31	3	42875		65	1	5801453125	1	97	1	281475010265089
33	9	107811		67	2	858003/1503		99	81	1041914208570939
00	3	101011		101	J	0003334030	<u> </u>			

Table 2: Numbers of self-dual cyclic codes (c , Corollary 1 of [10]) and θ -cyclic codes (θ -c, prop. 8) over \mathbb{F}_4 in odd dimension k < 100 where $\theta : x \mapsto x^2$.

Ana
0-11C
143296
429888
0
0
8046724
2186896
0
0
9839424
32119424
0
0
45664448
524238448
0
0
559559744

Table 3: Numbers of self-dual negacyclic (nc, Theorem 2 of [17]), self-dual θ -cyclic (θ -c, prop. 8) and self-dual θ -negacyclic (θ -nc, prop. 8) codes over \mathbb{F}_9 in dimension $k \leq 50$ coprime with 3 where $\theta : x \mapsto x^3$.

5.1 Construction of $\mathcal{H}_{f^{p^s}}$ for f in \mathcal{F}

The aim of this subsection is to compute $\mathcal{H}_{f^{p^s}}$ for f in \mathcal{F} and to compute its number of elements. The final result is given in Proposition 9 and the main steps are summed up in Table 4.

Consider $f = f(X^2)$ is in \mathcal{F} . Recall that according to Lemma 2, one has the partition :

$$\mathcal{H}_{f^{p^s}} = igsqcup_{i=0}^{\lfloorrac{p^s}{2}
floor} f^i \cdot \overline{\mathcal{H}}_{f^{p^s-2i}}$$

where for m in \mathbb{N} , the set $\overline{\mathcal{H}}_{f^m}$ is defined by

$$\overline{\mathcal{H}}_{f^m} = \{h \in \mathcal{H}_{f^m} \mid f \text{ does not divide } h\}.$$

Lemma 7 below generalizes Lemma 1 and uses the same type of arguments linked to the factorization of skew polynomials.

Lemma 7 Consider p a prime number, θ the Frobenius automorphism over \mathbb{F}_{p^2} , $R = \mathbb{F}_{p^2}[X; \theta]$, m a nonnegative integer and $f = f(X^2)$ in \mathcal{F} with degree $d = 2\delta > 1$ in X^2 .

- 1. The constant coefficients of the elements of $\overline{\mathcal{H}}_{f}$ are squares in $\mathbb{F}_{n^{2}}$.
- 2. The set $\overline{\mathcal{H}}_{f^m}$ has $(1+p^{\delta})p^{\delta(m-1)}$ elements and is equal to

$$\left\{ \left(h_1 \cdot \frac{1}{\nu_1}\right) \cdots \left(h_m \cdot \frac{1}{\nu_m}\right) \cdot \left(\prod_{j=1}^m \nu_j\right) \mid h_j \in \overline{\mathcal{H}}_f, \nu_j^2 = (h_j)_0, h_j \neq \nu_{j-1} \cdot h_{j-1}^{\natural} \cdot \frac{1}{\nu_{j-1}} \right\}$$

Proof.

To simplify the presentation, the following notations will be used in this proof : h = h(X), $f = f(X^2)$.

1. Consider $h = X^d + \sum_{i=0}^{d-1} h_i X^i$ in $\overline{\mathcal{H}}_f = \mathcal{H}_f$. If p = 2, then h_0 is a nonzero element of \mathbb{F}_4 and therefore is a sugare in \mathbb{F}_4 . Assume that p is odd. According to Section 4, the polynomials $A(Z) = Z^{\delta} + \sum_{i=0}^{\delta-1} h_{2i} Z^i$ and $B(Z) = \sum_{i=0}^{\delta-1} \theta(h_{2i+1}) Z^i$ defined in (9) satisfy the relations (10). If f(Z) and B(Z) are coprime then $f(Z) = f^{\natural}(Z)$ and $Z^{\delta-1}B(1/Z)$ are also coprime. Therefore the relations (10) imply that $f(Z) = A(Z)\Theta(A)(Z) - ZB(Z)\Theta(B)(Z), Z^{\delta-1}B(1/Z) = -h_0\Theta(B)(Z)$ and $Z^{\delta}A(1/Z) = h_0\Theta(A)(Z)$. In particular, one has $1 - h_0h_0^p = 0$ so h_0 is a square. If f(Z) and B(Z) are not coprime, then according to Lemma 4, B(Z) = 0 and using Lemma 3, one gets $A(Z) = \tilde{f}(Z)$ or $\Theta(\tilde{f})(Z)$ where $f(Z) = \tilde{f}(Z)\Theta(\tilde{f})(Z)$ is the factorization of f(Z) into irreducible polynomials of $\mathbb{F}_{p^2}[Z]$. As $f = f^{\natural}$, the constant coefficient of f is equal to 1, so one gets $h_0^{p+1} = 1$ and h_0 is a square. 2. Consider h in $\overline{\mathcal{H}}_{f^m}$. As h divides f^m and f is irreducible in $\mathbb{F}_p[X^2]$, all the irreducible factors of h divide f and have the same degree d (Lemma 13 (2) of [3] or [15] page 6):

$$h = \prod_{i=1}^{m} H_i, H_i \text{ monic}, \deg(H_i) = d, H_i | f, H_i \text{ irreducible}.$$

Furthermore, f does not divide h, therefore according to Proposition 3, for all j in $\{1 \dots m-1\}, H_j \cdot H_{j+1}$ is distinct of f.

Using an induction argument (left to the reader), one gets the following expression of h^{\natural} :

$$h^{\natural} = \prod_{i=0}^{m-1} \frac{1}{\mu_{m-i}} H_{m-i}^{\natural} \cdot \mu_{m-i}$$

where $\mu_i = (H_1 \cdots H_{i-1})_0$ is defined as the constant coefficient of $H_1 \cdots H_{i-1}$. Furthermore, this factorization (into the product of irreducible monic polynomials of same degree d dividing f) is unique (because the factorization of h is unique).

As the factorization of f^m into the product of irreducible factors is not unique (because f^m is central), according to Proposition 3, $f^m = h^{\natural} \cdot h$ must have two consecutive irreducible monic factors whose product is f. As h and h^{\natural} do not possess two consecutive factors whose product is f, necessarily, $\frac{1}{\mu_1}H_1^{\natural} \cdot \mu_1 \cdot H_1 = f$ and proceeding by induction, one gets

$$\forall j \in \{1, \dots, m-1\}, \frac{1}{\mu_j} H_j^{\natural} \cdot \mu_j \cdot H_j = f \text{ and } H_{j+1} \neq \frac{1}{\mu_j} H_j^{\natural} \cdot \mu_j.$$
(16)

Conversely, consider $h = H_1 \cdots H_m$ with $\frac{1}{\mu_j} H_j^{\natural} \cdot \mu_j \cdot H_j = f, H_{j+1} \neq \frac{1}{\mu_j} H_j^{\natural} \cdot \mu_j$ and μ_j constant coefficient of $H_1 \cdots H_{j-1}$, then $h^{\natural} \cdot h = f^m$ and $H_j \cdot H_{j+1} \neq f$. Furthermore, the skew polynomials H_j are all irreducible because they are nontrivial factors of f and f is irreducible in $\mathbb{F}_p[X^2]$, therefore according to Proposition 3, the skew polynomial h is not divisible by f and it belongs to $\overline{\mathcal{H}}_{f^m}$.

The conclusion follows thanks to the following equivalence :

$$\begin{cases} h = H_1 \cdots H_m \\ \frac{1}{\mu_j} H_j^{\natural} \cdot \mu_j \cdot H_j = f \\ H_{j+1} \neq \frac{1}{\mu_j} H_j^{\natural} \cdot \mu_j \end{cases} \Leftrightarrow \begin{cases} h = \left(h_1 \cdot \frac{1}{\nu_1}\right) \cdots \left(h_m \cdot \frac{1}{\nu_m}\right) \cdot \prod_{j=1}^m \nu_j \\ h_j^{\natural} \cdot h_j = f \\ h_{j+1} \neq \nu_j h_j^{\natural} \cdot \frac{1}{\nu_j} \end{cases}$$

where $\mu_j = (H_1 \cdots H_{j-1})_0$ is the constant coefficient of $H_1 \cdots H_{j-1}$, ν_j is defined in \mathbb{F}_{p^2} by $\nu_j^2 = (H_j)_0 = (h_j)_0$ and $h_j = (\nu_0 \cdots \nu_j)H_j \cdot \frac{1}{\nu_0 \cdots \nu_j}$.

3. The number of elements of $\overline{\mathcal{H}}_{f^m}$ follows from the fact that $\overline{\mathcal{H}}_f$ has $1 + p^{\delta}$ elements (Proposition 6).

The construction of the set $\mathcal{H}_{f^{p^s}}$ for f in \mathcal{F} is deduced from Lemma 2 and Lemma 7. The whole construction is illustrated in Table 4.

Lemma 3 and Cauchy in- terpolation (Lemma 5)	Decomposition into the products of elements of $\overline{\mathcal{H}}_f$ (Proposition 3)	Partition (Lemma 2)
\downarrow	\downarrow	\downarrow
$\overline{\mathcal{H}}_f$ (Proposition 6) \rightarrow	$\overline{\mathcal{H}}_{f^m}$ (Lemma 7)	$\rightarrow \qquad \mathcal{H}_{f^{p^s}} $ (Proposition 9)

Table 4: Main steps of the construction of $\mathcal{H}_{f^{p^s}}$ for f in \mathcal{F}

Proposition 9 Consider p a prime number, θ the Frobenius automorphism over \mathbb{F}_{p^2} , $R = \mathbb{F}_{p^2}[X;\theta]$, s a nonnegative integer and $f = f(X^2)$ in \mathcal{F} with degree $d = 2\delta > 1$ in X^2 . The set $\mathcal{H}_{f^{p^s}}$ has $\frac{p^{\delta(p^s+1)}-1}{p^{\delta}-1}$ elements.

Proof. According to Lemma 2, $\mathcal{H}_{f^{p^s}} = \bigsqcup_{i=0}^{\lfloor \frac{p^s}{2} \rfloor} f^i \cdot \overline{\mathcal{H}}_{f^{p^s-2i}}$ and according to Lemma 7, $\overline{\mathcal{H}}_{f^m}$ has $(1+p^{\delta})(p^{\delta})^{m-1}$ if $m \neq 0$ and 1 element if m = 0. Therefore $\mathcal{H}_{f^{p^s}}$ has $\sum_{i=0}^{(p^s-1)/2} (1+p^{\delta})(p^{\delta})^{p^s-2i-1}$ elements if p is odd and $1 + \sum_{i=0}^{2^{s-1}-1} (1+2^{\delta})(2^{\delta})^{2^s-2i-1}$ elements otherwise. In both cases one gets $\#\mathcal{H}_{f^{p^s}} = \frac{p^{\delta(p^s+1)}-1}{p^{\delta}-1}$.

Example 8 Consider $\mathbb{F}_4 = \mathbb{F}_2(a)$, θ the Frobenius automorphism and $f(X^2) = X^4 + X^2 + 1$ in \mathcal{F} . According to Proposition 9, the set \mathcal{H}_{f^2} has $\frac{2^{1 \times (2^1+1)}-1}{2^1-1} = 7$ elements. More precisely, $\mathcal{H}_{f^2} = f^1 \cdot \overline{\mathcal{H}}_{f^0} \bigsqcup f^0 \cdot \overline{\mathcal{H}}_{f^2} = \{f\} \sqcup \overline{\mathcal{H}}_{f^2}$. Furthermore, according to Lemma 7, the elements of $\overline{\mathcal{H}}_{f^2}$ are constructed by using products of elements of $\overline{\mathcal{H}}_f = \{X^2 + a, X^2 + a^2, X^2 + X + 1\}$ (see Example 4 for the construction of $\overline{\mathcal{H}}_f$). Here are the 6 elements of $\overline{\mathcal{H}}_{f^2}$:

$$\left\{ \begin{array}{rrrr} (X^2+X+1)\cdot(1/1)(X^2+a)\cdot(1/a^2)a^2 &=& X^4+X^3+a^2X^2+a^2X+a\\ (X^2+X+1)\cdot(1/1)(X^2+a^2)\cdot(1/a)a &=& X^4+X^3+aX^2+aX+a^2\\ (X^2+a)\cdot(1/a^2)(X^2+a)\cdot(1/a^2)a &=& X^4+a^2\\ (X^2+a)\cdot(1/a^2)(X^2+X+1)\cdot a^2 &=& X^4+a^2X^3+a^2X^2+X+a\\ (X^2+a^2)\cdot(1/a)(X^2+a^2)\cdot(1/a)a^2 &=& X^4+a\\ (X^2+a^2)\cdot(1/a)(X^2+X+1)\cdot a &=& X^4+aX^3+aX^2+X+a^2. \end{array} \right.$$

In next subsection one constructs the set $\mathcal{H}_{f^{p^s}}$ when $f = f(X^2)$ belongs to \mathcal{G} .

5.2 Construction of $\mathcal{H}_{f^{p^s}}$ for f in \mathcal{G}

In this subsection, one computes $\mathcal{H}_{f^{p^s}}$ for f in \mathcal{G} (Proposition 11). The construction is summed up in Table 5.

Assume that $f = f(X^2) = g(X^2)g^{\natural}(X^2)$ with $g(X^2) \neq g^{\natural}(X^2)$ irreducible in $\mathbb{F}_p[X^2]$. Recall that the set $\overline{\mathcal{H}}_{f^m}$ is defined by $\overline{\mathcal{H}}_{f^m} = \{h \in \mathcal{H}_{f^m} \mid g(X^2) \text{ and } g^{\natural}(X^2) \text{ do not divide } h\}.$

One first starts with a partition of $\mathcal{H}_{f^{p^s}}$ which generalizes Lemma 2 :

Lemma 8 Consider p a prime number, θ the Frobenius automorphism over \mathbb{F}_{p^2} , $R = \mathbb{F}_{p^2}[X;\theta]$, $s \in \mathbb{N}$ and $f = f(X^2) = g(X^2)g^{\natural}(X^2)$ in \mathcal{G} with $g = g(X^2) \neq g^{\natural}(X^2)$ irreducible in $\mathbb{F}_p[X^2]$.

$$\mathcal{H}_{f^{p^{s}}} = \bigsqcup_{i=0}^{p^{s}} \bigsqcup_{j=0}^{p^{s}-i} g^{i} g^{\natural^{j}} \cdot \overline{\mathcal{H}}_{f^{p^{s}-(i+j)}}.$$
(17)

Proof. Consider h in $\mathcal{H}_{f^{p^s}}$ and $i \in \{0, \ldots, p^s\}, j \in \{0, \ldots, p^s - i\}$ such that $h = g(X^2)^i g^{\natural}(X^2)^j \cdot H$ where $g(X^2)$ and $g^{\natural}(X^2)$ do not divide H. One has $h^{\natural} = g^{\natural}(X^2)^i g(X^2)^j \cdot H^{\natural}$, therefore $H^{\natural} \cdot H = f^{p^s - (i+j)}$ and h belongs to the set $g(X^2)^i g^{\natural}(X^2)^j \cdot \overline{\mathcal{H}}_{f^{p^s - (i+j)}}$. Conversely, consider $i \in \{0, \ldots, p^s\}, j \in \{0, \ldots, p^s - i\}$ and H in $\overline{\mathcal{H}}_{f^{p^s - (i+j)}}$, then $g(X^2)^i g^{\natural}(X^2)^j \cdot H$ belongs to $\mathcal{H}_{f^{p^s}}$. Consider $(i, j) \neq (i', j')$ with $i > i' \in \{0, \ldots, p^s\}, j \in \{0, \ldots, p^s - i\}, j' \in \{0, \ldots, p^s - i\}, i' \in \{0, \ldots, p^s - i\}, i' \in \{0, \ldots, p^s - i\}, j' \in \{0, \ldots, p^s - i'\}, u \in \overline{\mathcal{H}}_{f^{p^s - (i+j)}}, u' \in \overline{\mathcal{H}}_{f^{p^s - (i'+j')}}$. Assume that $g(X^2)^i g^{\natural}(X^2)^j \cdot u = g(X^2)^{i'} g^{\natural}(X^2)^{j'} \cdot u'$. If $j \geq j'$ then $g(X^2)$ divides u' which is impossible, therefore j < j'. Necessarily, $g(X^2)$ divides $g^{\natural}(X^2)^{j'-j} \cdot u'$. As $g(X^2)$ and $g^{\natural}(X^2)$ are right coprime and belong to $\mathbb{F}_p[X^2]$ therefore their lclm coincides with their lcm i.e. $g(X^2)g^{\natural}(X^2)$. So one gets that $g(X^2)$ divides $g^{\natural}(X^2) \cdot u'$. As $g(X^2)$ and $g^{\natural}(X^2)u'$ their lclm $g(X^2)$ divides $g^{\natural}(X^2) \cdot u'$. As $g(X^2)$ and $g^{\natural}(X^2)u'$ their lclm $g(X^2)g^{\natural}(X^2)$ divides $g^{\natural}(X^2)u'$ therefore $g(X^2)$ divides u', contradiction.

In what follows one constructs the set $\overline{\mathcal{H}}_{f^m}$ for f in \mathcal{G} and m greater than 1 (Lemma 9). This construction requires a generalization of Proposition 3 :

Proposition 10 Consider p a prime number, θ the Frobenius automorphism over \mathbb{F}_{p^2} , $R = \mathbb{F}_{p^2}[X;\theta]$, $f(X^2) = g(X^2)g^{\natural}(X^2)$ in $\mathbb{F}_p[X^2]$ with degree d in X^2 where $g(X^2) \neq g^{\natural}(X^2)$ is irreducible in $\mathbb{F}_p[X^2]$. Assume that $h = h_1 \cdots h_m$ is a product of monic skew polynomials of degree d whose bound is $f(X^2)$. The following assertions are equivalent :

- (i) The above factorization of h is not unique.
- (ii) $g(X^2)$ or $g^{\natural}(X^2)$ divides h in R.

(iii) There exists i in $\{1, \ldots, m-1\}$ such that $g(X^2)$ or $g^{\natural}(X^2)$ divides $h_i \cdot h_{i+1}$ in R.

Proof. To simplify the presentation, one denotes $f = f(X^2)$, $g = g(X^2)$ and $g^{\natural} = g^{\natural}(X^2)$.

The implication $(iii) \Rightarrow (ii)$ comes from the fact that g and g^{\natural} are central. Let us prove that $(ii) \Rightarrow (i)$. If g divides h, then it divides h on the right so h has at least two distinct right factors u and v irreducible dividing g. As the bound of h_m is equal to $f = gg^{\natural}$, necessarily, hhas an irreducible right factor w dividing g^{\natural} . The skew polynomials lclm(u, w) and lclm(v, w)are two right factors of h with degree d dividing f. According to Theorem 13 of [16], as uand w are irreducible and do not have the same bound, the lclm-decomposition lclm(u, w)is unique. Similarly, the lclm decomposition lclm(v, w) is unique. Furthermore, u and v are distinct, so the skew polynomials lclm(u, w) and lclm(v, w) are distinct. Let us prove by induction on m that if $h_1 \cdots h_m = g_1 \cdots g_m$ are two distinct decompositions of h into the product of monic skew polynomials whose bound is f and whose degree is d, then there are two consecutive factors whose product is divisible by g or g^{\natural} .

Consider $h = h_1 \cdot h_2 = g_1 \cdot g_2$ where g_i, h_i are skew polynomials with degree d and with bound f. Assume that g and g^{\natural} do not divide h. Then gcrd(h,g) is an irreducible skew polynomial of degree δ dividing g which is also equal to $gcrd(h_2,g)$ and $gcrd(g_2,g)$. Similarly, $gcrd(h_2, g^{\natural}) = gcrd(g_2, g^{\natural})$. Furthermore, according to Theorem 4.1 of [6], $h_2 =$ $lclm(gcrd(h_2, g), gcrd(h_2, g^{\natural}))$ and $g_2 = lclm(gcrd(g_2, g), gcrd(g_2, g^{\natural}))$, therefore $g_2 = h_2$ and $(h_1, h_2) = (g_1, g_2)$.

Consider m > 2 and assume the property is true for m - 1. Consider two distinct decompositions of h into the product of monic skew polynomials with degree d ad bound f: $h = h_1 \cdots h_m = g_1 \cdots g_m$. Therefore, h_i and g_j are products of two irreducible monic skew polynomials of degree δ dividing g and g^{\natural} .

If $gcrd(h_m, g_m) = 1$ then $lclm(h_m, g_m) = \overline{h}_{m-1} \cdot h_m$ divides $h_1 \cdots h_m$ and \overline{h}_{m-1} is a monic skew polynomial of degree d dividing f which is the product of two irreducible monic skew polynomials of degree δ dividing g and g^{\natural} . Consider H in R such that $H\overline{h}_{m-1} =$ $h_1 \cdots h_{m-1}$. If $h_{m-1} = h_{m-1}$ then $\operatorname{lclm}(h_m, g_m) = h_{m-1} \cdot h_m$ has two factorizations into the product of two monic skew polynomials of degree d dividing f, therefore, g or g^{\natural} divides $h_{m-1} \cdot h_m$. Otherwise, as $h_1, \ldots, h_{m-1}, \overline{h}_{m-1}$ are the products of an irreducible polynomial dividing g and an irreducible polynomial dividing g^{\natural} , H is the product of m-2 irreducible polynomials dividing g and m-2 skew polynomials dividing g^{\natural} . In particular, H divides $g^{m-2}(g^{\natural})^{m-2}$. According to Theorem 4.1 of [6], $H = \operatorname{lclm}(G, \tilde{G})$ where $G = \operatorname{gcrd}(H, g^{m-2})$ and $\tilde{G} = \operatorname{gcrd}(H, (g^{\natural})^{m-2})$. As g (resp. g^{\natural}) is irreducible in $\mathbb{F}_p[X^2]$, the skew polynomial G (resp. G) is the product of N (resp. N) monic irreducible skew polynomials dividing g (resp. g^{\natural}). Without loss of generality, one can assume that $N \leq N$. Consider $G = G_1 \cdots G_N$ (resp. $G = G_1 \cdots G_{\tilde{N}}$ the factorization of G as the product of N (resp. N) monic irreducible factors dividing g (resp. g^{\natural}). As g (resp. g^{\natural}) does not divide G (resp \tilde{G}), according to Proposition 3, these factorizations are unique. Therefore, according to Theorem 14 of [16], $H = H_1 \cdots H_N$ where $H_i = \text{lclm}(\overline{G_i}, \overline{\tilde{G_i}})$ with $R/\overline{G_i}R$ and R/G_iR (resp. $R/\tilde{G_i}R$ and $R/\tilde{G_i}R$) isomorphic modules. As G_i divides g, according to Corollary of Theorem 10 of [9], $\overline{G_i}$ also divides g. As G_i and \tilde{G}_i are right coprime with same degree d/2, \overline{G}_i and \tilde{G}_i are also right coprime therefore H_i is a skew polynomial of degree d which divides f. Lastly, as H has degree (m-2)d one gets N = m-2. Therefore, H can be written as the product of m-2 monic skew polynomials of degree d dividing f and one can apply the induction hypothesis to $H \cdot h_{m-1}$.

Assume that $\operatorname{gcrd}(h_m, g_m) = u \neq 1$. Necessarily u is an irreducible monic skew polynomial of degree δ which divides g or g^{\natural} . Without loss of generality, one can assume that u divides g. Consider v such that $\operatorname{lclm}(g_m, h_m) = v \cdot h_m$ and H in R such that $h_1 \cdots h_m = H \cdot v \cdot h_m$ i.e $h_1 \cdots h_{m-1} = H \cdot v$. Necessarily v is an irreducible monic skew polynomial of degree δ dividing g^{\natural} and $H = \tilde{h_1} \cdots \tilde{h_{m-2}} \cdot w$ where w is an irreducible skew polynomial dividing g, \tilde{h}_i is a product of two irreducible monic skew polynomials of degree δ dividing g and g^{\natural} . If $h_{m-1} = w \cdot v$, then $w \cdot \operatorname{lclm}(g_m, h_m) = h_{m-1} \cdot h_m$ and one concludes that g or g^{\natural} divides $h_{m-1} \cdot h_m$. If $h_{m-1} \neq w \cdot v$ then $h_1 \cdots h_{m-1} = \tilde{h}_1 \cdots \tilde{h}_{m-2} \cdot (w \cdot v)$ where $w \cdot v$ is a monic skew polynomial of degree d dividing f, and one concludes using the induction hypothesis.

Remark 4 The unique factorization of h in Proposition 10 below is the unique representation of h as the product of maximal completely reducible factors as it is defined in [16] page 498.

If $f(X^2)$ is irreducible in $\mathbb{F}_p[X^2]$ (and $f(X^2) \in \mathbb{R}$ does not divide h), then this factorization coincides with the factorization of h into irreducible monic skew polynomials.

Lemma 9 generalizes Lemma 7 (where \mathcal{F} is replaced with \mathcal{G}). It uses the same type of arguments linked to the factorization of skew polynomials. The elements of $\overline{\mathcal{H}}_{f^m}$ are constructed by using products of elements of $\overline{\mathcal{H}}_f$.

Lemma 9 Consider p a prime number, θ the Frobenius automorphism over \mathbb{F}_{p^2} , $R = \mathbb{F}_{p^2}[X; \theta]$, m a nonnegative integer and $f = f(X^2)$ in \mathcal{G} with degree $d = 2\delta > 1$ in X^2 .

- 1. The constant coefficients of the elements of $\overline{\mathcal{H}}_f$ are squares in \mathbb{F}_{p^2} .
- 2. The set $\overline{\mathcal{H}}_{f^m}$ has $(1+p^{\delta})p^{\delta(m-1)}$ elements and is equal to

$$\left\{ \left(h_1 \cdot \frac{1}{\nu_1}\right) \cdots \left(h_m \cdot \frac{1}{\nu_m}\right) \cdot \left(\prod_{j=1}^m \nu_j\right) \mid h_j \in \overline{\mathcal{H}}_f, \nu_j^2 = (h_j)_0, h_j \neq \nu_{j-1} h_{j-1}^{\natural} \cdot \frac{1}{\nu_{j-1}} \right\}$$

Proof.

To simplify the presentation, the following notations will be used in this proof : h = h(X), $f = f(X^2) = g(X^2)g^{\natural}(X^2)$, $g = g(X^2)$ and $g^{\natural} = g^{\natural}(X^2)$.

- 1. If p = 2 the nonzero elements of \mathbb{F}_{p^2} are squares. Assume that p is an odd prime number. Consider h in $\overline{\mathcal{H}}_{f^m}$ with constant term h_0 , A(Z) and B(Z) defined in (9). Like in point 1. of Lemma 7, if B(Z) and f(Z) are coprime then $h_0^{p+1} = 1$ so h_0 is a square in \mathbb{F}_{p^2} . If B(Z) and f(Z) are not coprime, then according to Lemma 4, δ is necessarily odd and $A(Z) = \tilde{g}(Z)\Theta(\tilde{g}^{\natural})(Z)$ or $A(Z) = \tilde{g}^{\natural}(Z)\Theta(\tilde{g})(Z)$ where $g(Z) = \tilde{g}(Z)\Theta(\tilde{g})(Z)$ is the factorization of g(Z) in $\mathbb{F}_{p^2}[Z]$. Denote μ the constant coefficient of $\tilde{g}(Z)$, then the constant coefficient h_0 of A(Z) is such that $h_0 = \mu/\mu^p = 1/\mu^{p-1}$ if $A(Z) = \tilde{g}(Z)\Theta(\tilde{g}^{\natural})(Z)$ or such that $h_0 = \mu^p/\mu = \mu^{p-1}$ if $A(Z) = \tilde{g}^{\natural}(Z)\Theta(\tilde{g})(Z)$, therefore h_0 is a square in \mathbb{F}_{p^2} .
- 2. Like in Lemma 7, it suffices to prove that $\overline{\mathcal{H}}_{f^m} =$

$$\left\{H_1\cdots H_m \mid \frac{1}{\mu_i}H_i^{\natural} \cdot \mu_i \cdot H_i = f, \mu_i = (H_1\cdots H_{i-1})_0, H_{i+1} \neq \frac{1}{\mu_i}H_i^{\natural} \cdot \mu_i, g, g^{\natural}\right\}.$$

Consider h in $\overline{\mathcal{H}}_{f^m}$. Let us prove that h can be written as the product of m monic skew polynomials with degree d and with bound f. As h divides f^m , according to Theorem 4.1 of [6], $h = \operatorname{lclm}(G, \tilde{G})$ where $G = \operatorname{gcrd}(h, g^m)$ and $\tilde{G} = \operatorname{gcrd}(h, (g^{\natural})^m)$. As g (resp. g^{\natural}) is irreducible in $\mathbf{F}_p[X^2]$, the skew polynomial G (resp. \tilde{G}) is the product of N (resp. \tilde{N}) monic irreducible skew polynomials dividing g (resp. g^{\natural}). Without loss of generality, one can assume that $N \leq \tilde{N}$. Consider $G = G_1 \cdots G_N$ (resp. $\tilde{G} = \tilde{G}_1 \cdots \tilde{G}_{\tilde{N}}$) the factorization of G as the product of N (resp. \tilde{N}) monic irreducible factors dividing g (resp. g^{\natural}). According to Proposition 3, as g (resp. g^{\natural}) does not divide G (resp \tilde{G}), these factorizations are unique. Therefore, according to Theorem 14 of [16], $h = H_1 \cdots H_N$ where $H_i = \operatorname{lclm}(\overline{G_i}, \tilde{G_i})$ with $R/\overline{G_i}R$ and R/G_iR (resp. $R/\tilde{G_i}R$ and $R/\tilde{G_i}R$) isomorphic modules. As G_i divides g, according to Corollary of Theorem 10 of [9], $\overline{G_i}$ also divides g. As G_i and $\tilde{G_i}$ are right coprime with same degree d/2, $\overline{G_i}$ and $\tilde{G_i}$ are also coprime therefore H_i is a skew polynomial of degree d which divides f. Lastly, the degree of h is equal to $m \times d$ and one gets N = m. Therefore

$$h = \prod_{i=1}^{m} H_i, H_i \text{ monic, } \deg(H_i) = d, \ H_i \text{ divides } f, H_i \neq g, H_i \neq g^{\natural}$$

Consider $\mu_i = (H_1 \cdots H_{i-1})_0$ the constant coefficient of $H_1 \cdots H_{i-1}$. Using an induction argument, one has :

$$h^{\natural} = \prod_{i=0}^{m-1} \frac{1}{\mu_{m-i}} H_{m-i}^{\natural} \cdot \mu_{m-i}.$$

By hypothesis $h^{\natural} \cdot h = f^m$, therefore

$$\left(\frac{1}{\mu_m}H_m^{\natural}\cdot\mu_m\right)\cdots\left(\frac{1}{\mu_2}H_2^{\natural}\cdot\mu_2\right)\cdot\left(\frac{1}{\mu_1}H_1^{\natural}\cdot\mu_1\right)\cdot H_1\cdot H_2\cdots H_m=f^m$$

is the product of 2m monic factors with degree d and with bound f. As f^m is central, the above decomposition is not unique. Therefore, according to Proposition 10, there exists two consecutive factors in $h^{\natural} \cdot h$ whose product is divisible by g or g^{\natural} . Such a product can be of three types : $\left(\frac{1}{\mu_{i+1}}H_{i+1}^{\natural}\cdot\mu_{i+1}\right)\cdot\left(\frac{1}{\mu_i}H_i^{\natural}\cdot\mu_i\right)$, $H_i\cdot H_{i+1}$ or $\frac{1}{\mu_1}H_1^{\natural}\cdot\mu_1\cdot H_1$. However g and g^{\natural} do not divide $H_i \cdot H_{i+1}$, otherwise, they would divide h, and they do not divide $\frac{1}{\mu_{i+1}}H_{i+1}^{\natural}\cdot\mu_{i+1}\frac{1}{\mu_i}H_i^{\natural}\cdot\mu_i = \frac{1}{\mu_i}(H_i\cdot H_{i+1})^*\mu_i$ either. Therefore g or g^{\natural} divides $\frac{1}{\mu_1}H_1^{\natural}\cdot\mu_1\cdot H_1$. As g is central, one gets that g and g^{\natural} divide $\frac{1}{\mu_1}H_1^{\natural}\cdot\mu_1\cdot H_1$, therefore f divides $\frac{1}{\mu_1}H_1^{\natural}\cdot\mu_1\cdot H_1$ and as these two skew polynomials are monic with the same degree they are equal. By induction, one gets

$$\frac{1}{\mu_i}H_i^{\natural}\cdot \mu_i\cdot H_i=f, H_{i+1}\neq \frac{1}{\mu_i}H_i^{\natural}\cdot \mu_i, g, g^{\natural}.$$

Conversely, consider $h = \prod_{i=1}^{m} H_i$, with $\frac{1}{\mu_i} H_i^{\natural} \cdot \mu_i \cdot H_i = f$, $H_{i+1} \neq \frac{1}{\mu_i} H_i^{\natural} \cdot \mu_i, g, g^{\natural}$. One

has $h^{\natural} = \prod_{i=0}^{m-1} \frac{1}{\mu_{m-i}} H^{\natural}_{m-i} \cdot \mu_{m-i}$ therefore $h^{\natural} \cdot h = f^m$. It remains to prove that g and g^{\natural}

do not divide h. Assume that g divides h, all the skew factors H_i in the decomposition of h are monic, with degree d, divide f and are distinct of g, g^{\natural} , therefore, according to Proposition 10, there exists i such that g divides $H_i \cdot H_{i+1}$. Consider u in R such that $H_i \cdot H_{i+1} = g \cdot u$. As both H_i and H_{i+1} divide f without diving g or g^{\natural} , they are the products of two irreducible polynomials dividing respectively g and g^{\natural} , therefore the skew polynomial u is the product of two irreducible skew polynomials both dividing g^{\natural} and u divides $(g^{\natural})^2$. The relation $(H_i \cdot H_{i+1})^* = (g^{\natural} \cdot u)^*$ gives $H_{i+1}^{\natural} \cdot \lambda_i H_i^{\natural} = \lambda_i u^{\natural} \cdot g^{\natural}$ where λ_i is the constant coefficient of H_i . Multiplying the above equality on the left by $\mu_{i+1}H_{i+1} \cdot \frac{1}{\mu_{i+1}}$ and on the right by $\mu_i H_i \cdot \frac{1}{\mu_i}$ yields $f^2 = (\frac{1}{\lambda_i}\mu_{i+1}H_{i+1} \cdot \frac{1}{\mu_{i+1}}\lambda_i u^{\natural} \cdot g^{\natural} \cdot \mu_i$ $g^{\natural} \cdot \mu_i) \cdot (H_i \cdot \frac{1}{\mu_i})$. As f^2 is central, the two terms of the product commute and $f^2 =$ $H_i \cdot (\frac{1}{\mu_i} \frac{1}{\lambda_i} \mu_{i+1}) H_{i+1} \cdot (\frac{1}{\mu_{i+1}} \lambda_i) u^{\natural} \cdot g^{\natural} \cdot \mu_i = H_i \cdot H_{i+1} \cdot \frac{1}{\mu_i} \cdot u^{\natural} \cdot g^{\natural} \cdot \mu_i = g \cdot u \cdot \frac{1}{\mu_i} \cdot u^{\natural} \cdot g^{\natural} \cdot \mu_i$ therefore $(u \cdot \frac{1}{\mu_i} \cdot u^{\natural} \cdot \mu_i) \cdot g^{\natural} = g \cdot (g^{\natural})^2 = u \cdot v \cdot g$, where v in R is such that $u \cdot v = (g^{\natural})^2$. One gets the relation $\frac{1}{\mu_i} u^{\natural} \cdot \mu_i \cdot g^{\natural} = v \cdot g$. The skew polynomials g and g^{\natural} divide $v \cdot g$ and $\deg(v \cdot g) = \deg(f)$, therefore $f = v \cdot g, v = g^{\natural}$ and $u = g^{\natural}$ which is impossible because $H_i \cdot H_{i+1} \neq f$.

Lemma 3 and Cauchy in- terpolation (Lemma 6)	Decomposition into the products of elements of $\overline{\mathcal{H}}_f$ (Proposition 10)	Partition (Lemma 8)
\downarrow	\downarrow	\downarrow
$\overline{\mathcal{H}}_f$ (Proposition 7) \rightarrow	$\overline{\mathcal{H}}_{f^m}$ (Lemma 9)	$\rightarrow \qquad \mathcal{H}_{f^{p^s}} $ (Proposition 11)

Table 5: Main steps of the construction of $\mathcal{H}_{f^{p^s}}$ for f in \mathcal{G}

The number of elements of $\overline{\mathcal{H}}_{f^m}$ follows from the fact that $\overline{\mathcal{H}}_f$ has $1 + p^{\delta}$ elements (Proposition 7).

The construction of the set $\mathcal{H}_{f^{p^s}}$ for f in \mathcal{G} is deduced from Lemma 8 and Lemma 9. The whole construction is illustrated in Table 5.

Proposition 11 Consider p a prime number, θ the Frobenius automorphism over \mathbb{F}_{p^2} , $R = \mathbb{F}_{p^2}[X;\theta]$, s a nonnegative integer and $f = f(X^2)$ in \mathcal{G} with degree $d = 2\delta > 1$ in X^2 . The set $\mathcal{H}_{f^{p^s}}$ has $\frac{\left(p^{\delta(p^s+1)} - 2p^s - 3\right)\left(1 + p^\delta\right) + 4p^s + 4}{\left(p^\delta - 1\right)^2}$ elements.

Proof.

Consider $f = f(X^2) = g(X^2)g^{\natural}(X^2)$ in \mathcal{G} , then according to Lemma 8, one has the partition

 $\mathcal{H}_{f^{p^s}} = \bigsqcup_{i=0}^{p^s} \bigsqcup_{j=0}^{p^s-i} g(X^2)^j g^{\natural} (X^2)^{i-j} \cdot \overline{\mathcal{H}}_{f^{p^s-i-j}}$. Furthermore, according to Lemma 9,

the set $\overline{\mathcal{H}}_{f^m}$ has $(p^{\delta} + 1)p^{\delta(m-1)}$ if $m \geq 1$ and 1 element if m = 0. Therefore the number of elements of the set $\mathcal{H}_{f^{p^s}}$ is

$$\sum_{i=0}^{p^s} \left[\sum_{j=0}^{p^s-i-1} (1+p^{\delta})(p^{\delta})^{p^s-i-1-j} + 1 \right] = \frac{\left(p^{\delta(p^s+1)} - 2p^s - 3\right)\left(1+p^{\delta}\right) + 4p^s + 4}{\left(p^{\delta} - 1\right)^2}$$

Example 9 Consider $\mathbb{F}_4 = \mathbb{F}_2(a)$, $\theta : x \mapsto x^2$ and $f = f(X^2) = (X^6 + X^2 + 1)(X^6 + X^4 + 1) \in \mathcal{G}$ with degree d = 6 in X^2 . According to Proposition 11, the set \mathcal{H}_{f^2} has 93 elements. More precisely,

$$\mathcal{H}_{f^2} = \overline{\mathcal{H}}_{f^2} \sqcup (X^6 + X^2 + 1)\overline{\mathcal{H}}_f \sqcup (X^6 + X^4 + 1)\overline{\mathcal{H}}_f$$
$$\sqcup \{ (X^6 + X^2 + 1)^2, (X^6 + X^4 + 1)^2, (X^6 + X^2 + 1)(X^6 + X^4 + 1) \}.$$

There are 9 elements in $\overline{\mathcal{H}}_f = \mathcal{H}_f \setminus \{X^6 + X^2 + 1, X^6 + X^4 + 1\}$ (see example 6) and $72 = (1+2^3) \times 2^3$ skew polynomials in $\overline{\mathcal{H}}_{f^2}$. Here is one of these elements : $h = X^{12} + aX^{11} + a^2X^{10} + a^2X^7 + a^2X^6 + X^5 + a^2X^2 + aX + a = (h_1 \cdot \frac{1}{\nu_1}) \cdot (h_2 \cdot \frac{1}{\nu_2}) \cdot (\nu_1\nu_2)$ where $h_1 = X^6 + X^5 + aX^3 + a^2X + a, h_2 = X^6 + X^5 + X^4 + X^3 + X^2 + X + 1$ are two elements of $\overline{\mathcal{H}}_f$, $\nu_1 = a^2$ is the square root of the constant coefficient of h_1 and $\nu_2 = 1$.

5.3 Conclusion

The following theorem gives the number of self-dual θ -cyclic and θ -negacyclic codes of any dimension k over \mathbb{F}_{p^2} for p prime number and θ Frobenius automorphism. Tables 6 and 7 illustrate this theorem over \mathbb{F}_4 for $k = 2^s \times t$ and $t \in \{1, 3, 5, 7, 9\}$ and over \mathbb{F}_9 for $k = 3^s \times t$ and $t \in \{1, 2, 4, 5, 7\}$.

Theorem 1 Consider p prime number, θ the Frobenius automorphism over \mathbb{F}_{p^2} , k a positive integer, ε in $\{-1,1\}$, s,t two integers such that $k = p^s \times t$ and p does not divide t. The number of self-dual (θ, ε) -constacyclic codes of dimension k over \mathbb{F}_{p^2} is

$$N_{\varepsilon} \times \prod_{f \in \mathcal{F}_{k,\varepsilon}} \frac{p^{\delta(p^s+1)} - 1}{p^{\delta} - 1} \times \prod_{f \in \mathcal{G}_{k,\varepsilon}} \frac{\left(p^{\delta(p^s+1)} - 2p^s - 3\right)\left(1 + p^{\delta}\right) + 4p^s + 4p^s}{\left(p^{\delta} - 1\right)^2}$$

where

$$N_{1} = \begin{cases} 0 & \text{if } k \equiv 1 \pmod{2} \text{ and } p \equiv 1 \pmod{4} \\ & \text{or } k \equiv 0 \pmod{2} \text{ and } p \text{ odd} \\ 1 & \text{if } s = 0 \text{ and } p = 2 \\ 3 & \text{if } s > 0 \text{ and } p = 2 \\ 2\frac{p^{(p^{s}+1)/2} - 1}{p-1} & \text{if } k \equiv 1 \pmod{2} \text{ and } p \equiv 3 \pmod{4} \end{cases}$$

and

$$N_{-1} = \begin{cases} 0 & \text{if } k \equiv 1 \pmod{2} \text{ and } p \equiv 3 \pmod{4} \\ 1 & \text{if } k \equiv 0 \pmod{2} \text{ and } p \text{ odd} \\ 2\frac{p^{(p^s+1)/2} - 1}{p-1} & \text{if } k \equiv 1 \pmod{2} \text{ and } p \equiv 1 \pmod{4}. \end{cases}$$

Proof. According to Proposition 5, the number of self-dual (θ, ε) -constacyclic codes over \mathbb{F}_{p^2} with dimension k is

$$#\mathcal{H}_{X^{2k}-\varepsilon} = N_{\varepsilon} \times \prod_{f \in \mathcal{F}_{k,\varepsilon}} #\mathcal{H}_{f^{p^s}} \times \prod_{f \in \mathcal{G}_{k,\varepsilon}} #\mathcal{H}_{f^{p^s}}$$

where N_{ε} satisfies the above conditions. The final result follows from Proposition 9 and Proposition 11.

Remark 5 Proposition 4 is a particular case of Theorem 1 for t = 1 while Proposition 8 is a particular case for s = 0.

Dimension	cyclic	heta-cyclic
2^s	1	3 ([3])
3×2^s	$1 + 2^{s+1}$	$3 \times (2^{2^s+1}-1)$
5×2^s	1	$4^{2^s+1}-1$
7×2^s	$1 + 2^{s+1}$	$3 \times (9 \times 8^{2^{s+1}} - 7 \times 2^{s+1} - 23)/49$
9×2^s	$(1+2^{s+1})^2$	$3 \times (2^{2^s+1}-1) \times (8^{2^s+1}-1)/7$

Table 6: Number of self-dual cyclic codes (Theorem 3.6 of [11]) and self-dual θ -cyclic codes (Theorem 1) over \mathbb{F}_4 in dimension $t \times 2^s$ with $s \in \mathbb{N}^*$, $t \in \{1, 3, 5, 7, 9\}$ and $\theta : x \mapsto x^2$

Dimension	θ -cyclic	θ -negacyclic	
3^s	$3^{(3^s+1)/2} - 1$	0	
2×3^s	0	$(3^{3^{s+1}}-1)/2$	
4×3^s	0	$(5 \times 9^{3^s+1} - 8 \times 3^s - 13)/2^5$	
5×3^s	$(3^{(3^s+1)/2}-1) \times (9^{3^s+1}-1)/8$	0	
7×3^s	$(3^{(3^s+1)/2}-1) \times (27^{3^s+1}-1)/26$	0	

Table 7: Number of self-dual θ -cyclic and θ -negacyclic codes (Theorem 1) over \mathbb{F}_9 in dimensions $t \times 3^s$ with $s \in \mathbb{N}, t \in \{1, 2, 4, 5, 7\}$ and $\theta : x \mapsto x^3$.

6 Conclusion and perspectives

This text provides a construction and an enumeration of Euclidean self-dual θ -cyclic and θ negacyclic codes over \mathbb{F}_{p^2} where p is a prime number and θ is the Frobenius automorphism. The main ingredient of this study relies on the adaptation of Sloane and Thompson approach ([19]) to solve the self-dual skew equation over $\mathbb{F}_{p^2}[X;\theta]$. Some comparisons with the number of cyclic and negacyclic codes with the same dimensions are also provided.

This construction should be generalized to Hermitian self-dual θ -negacyclic codes over \mathbb{F}_{p^2} (work in progress). However, the question of the enumeration of self-dual skew codes over \mathbb{F}_{p^e} with e greater than 2 remains open. Namely, many properties in this text are specific to the ring $\mathbb{F}_{p^2}[X;\theta]$ and a new approach should be adopted to hope a generalization. Lastly, a lot of work still remains in the study of the minimal distances of the codes constructed in this text.

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