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Construction and number of self-dual skew codes over \mathbb{F}_{p^2}

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Abstract

The aim of this text is to construct and to count self-dual θ -cyclic and θ -negacyclic codes over \mathbb{F}_{p^2} where θ is the Frobenius automorphism.

1 Introduction

Self-dual codes have been extensively studied for their practical and theoretical importance. Cyclic, negacyclic and quasi-cyclic self-dual codes were investigated over finite fields in many ways ([1, 6, 8, 9, 11, 12, 14]). There is no cyclic self-dual code over \mathbb{F}_q when q is odd, and the number of self-dual cyclic codes over \mathbb{F}_{2^r} is given in Theorem 3.6 of [12] (see also [11]). For p prime number, negacyclic self-dual codes of dimension p^s over \mathbb{F}_q are constructed and counted in Corollary 3.3 of [6] when $q = p^r$ and in Theorems 3 and 4 of [1] when q is prime to p .

The goal of this text is to construct and to count self-dual θ -cyclic and θ -negacyclic codes over \mathbb{F}_{p^2} of any dimension where p is a prime number and θ is the Frobenius automorphism.

Over \mathbb{F}_{p^2} , θ -cyclic codes form, up to equivalence, a subclass of the class of 2-quasi-cyclic codes ([18]). The number of self-dual 2-quasi-cyclic codes of dimension prime to p over fields of characteristic p is given in Proposition 6.2 of [14]. A construction (without counting formula) of self-dual 2-quasi-cyclic codes of dimension not prime to p is given in Theorem 6.2 of [15].

Self-dual θ -cyclic and θ -negacyclic codes of dimension k are characterized by the equations $h^\natural h = X^{2k} - \epsilon$ in $\mathbb{F}_{p^2}[X; \theta]$ where h is the skew check polynomial of the code and h^\natural is the left skew reciprocal polynomial of h . The solutions of this equation are constructed in [Proposition 28 of [4]] as least common right multiples of intermediate skew polynomials which satisfy equations of the type $h^\natural h = f(X^2)^{p^s}$ where $f(Y) \in \mathbb{F}_p[Y]$ is a self-reciprocal polynomial either irreducible or product of two distinct irreducible polynomials. In the particular case when $p = 2$ and $f(X^2) = X^2 + 1$, these equations were solved for any s in [4] using factorization properties of h . This led to the characterization of self-dual skew codes of dimension 2^s over \mathbb{F}_4 . In the general case, up to now, these intermediate skew polynomials were determined by solving the polynomial systems satisfied by their coefficients for fixed values of s . In this text, one gives a new construction of these intermediate skew polynomials which enables to obtain a counting formula for self-dual skew codes. The main ingredients to achieve this goal are some factorization properties of skew polynomials in $\mathbb{F}_{p^2}[X; \theta]$ and Cauchy interpolations in $\mathbb{F}_{p^2}[Z]$.

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The text is organized as follows. In section 2 some facts about self-dual skew codes are recalled. In section 3 a strategy for constructing and counting self-dual θ -cyclic and θ -negacyclic codes over \mathbb{F}_{p^2} for p prime number and θ the Frobenius automorphism is set up. It is based on the resolution in $R = \mathbb{F}_{p^2}[X; \theta]$ of the equation $h^\natural h = X^{2k} - \epsilon$ where h^\natural is the left skew reciprocal polynomial of h . This equation can be reduced to intermediate equations $h^\natural h = f(X^2)^{p^s}$ where $f(Y) \in \mathbb{F}_p[Y]$ (Proposition 28 of [4]). The main idea is to split this equation into a finite number of equations of the type $h^\natural h = f(X^2)^m$ where h and $f(X^2)$ have no common non constant factor in $R \cap \mathbb{F}_p[X^2]$. In section 4, the equations $h^\natural h = (X^2 - \epsilon)^m$ are considered for p odd prime and this leads to a first counting formula, for self-dual θ -cyclic and θ -negacyclic codes over \mathbb{F}_{p^2} with dimension a power of p (Proposition 2). This construction generalizes the construction for $p = 2$ given in [4]. Some experimental results and a conjecture about the weight enumerators of these codes are also given. In section 5, the equations $h^\natural h = f(X^2)^m$ for f of degree > 1 are then considered. For $m = 1$, this equation is solved by Cauchy interpolation problems in $\mathbb{F}_{p^2}[Z]$ whereas for $m > 1$ one uses factorization properties of h . Lastly in section 6, the number of self-dual θ -cyclic and θ -negacyclic codes with any dimension is obtained (Theorem 1) and some perspectives are given.

2 Generalities on self-dual skew codes

For a finite field \mathbb{F}_q and θ an automorphism of \mathbb{F}_q we consider the ring $R = \mathbb{F}_q[X; \theta]$ where addition is defined to be the usual addition of polynomials and where multiplication is defined by the basic rule $X \cdot a = \theta(a)X$ ($a \in \mathbb{F}_q$) and extended to all elements of R by associativity and distributivity. The noncommutative ring R is called a skew polynomial ring or Ore ring (cf. [17]) and its elements are skew polynomials. It is a left and right Euclidean ring whose left and right ideals are principal. Left and right gcd and lcm exist in R and can be computed using the left and right Euclidean algorithm. The center of R is the commutative polynomial ring $Z(R) = \mathbb{F}_q^\theta[X^{|\theta|}]$ where \mathbb{F}_q^θ is the fixed field of θ and $|\theta|$ is the order of θ . The **bound** $B(h)$ of $h \in R$ with a nonzero constant term is the monic skew polynomial f with a non zero constant term belonging to $\mathbb{F}_q^\theta[X^{|\theta|}]$ of minimal degree such that h divides f on the right in R .

Definition 1 (definition 2 of [2] or definition 1 of [4]) A module θ -code (or module skew code) \mathcal{C} is a left R -submodule $Rg/Rf \subset R/Rf$ in the basis $1, X, \dots, X^{n-1}$ where $g \in R = \mathbb{F}_q[X; \theta]$ and f is a left multiple of g in R of degree n . If there exists an $a \in \mathbb{F}_q \setminus \{0\}$ such that g divides $X^n - a$ on the right, then the code \mathcal{C} is **(θ, \mathbf{a}) -constacyclic**. If $a = 1$, the code is **θ -cyclic** and if $a = -1$, it is **θ -negacyclic**. The skew polynomial g is called **skew generator polynomial** of \mathcal{C} .

If θ is the identity then a θ -cyclic (resp. θ -negacyclic) code is a cyclic code (resp. negacyclic) code.

The **(Euclidean) dual** of a linear code C of length n over \mathbb{F}_q is defined with the **Euclidean scalar product** $\langle x, y \rangle = \sum_{i=1}^n x_i y_i$ in \mathbb{F}_q^n as $C^\perp = \{x \in \mathbb{F}_q^n \mid \forall y \in C, \langle x, y \rangle = 0\}$. A linear code C over \mathbb{F}_q is **Euclidean self-dual** or **self-dual** if $C = C^\perp$. To characterize self-dual module θ -codes, the skew reciprocal polynomial of a skew polynomial (definition 3 of [4]) and also the left monic skew reciprocal polynomial will be useful.

Definition 2 ([4], **Definition 2**) *The skew reciprocal polynomial of $h = \sum_{i=0}^d h_i X^i \in R$ of degree d is $h^* = \sum_{i=0}^d X^{d-i} \cdot h_i = \sum_{i=0}^d \theta^i(h_{d-i}) X^i$. The left monic skew reciprocal polynomial of h is $h^\natural := \frac{1}{\theta^d(h_0)} \cdot h^*$. The polynomial h is **self-reciprocal** if $h = h^\natural$.*

According to Corollary 1 of [4], a module θ -code with skew generator polynomial $g \in R$ monic of degree k is self-dual if and only if there exists $h \in R$ monic (called **skew check polynomial** of the code) such that $g = h^\natural$ and

$$h^\natural h = X^{2k} - \varepsilon \text{ with } \varepsilon \in \{-1, 1\}. \quad (1)$$

In particular a self-dual θ -code must be either θ -cyclic or θ -negacyclic.

Some properties of the skew reciprocal polynomial and the left skew reciprocal polynomial will be useful (see also Lemma 24 of [4]).

Lemma 1 *Consider $R = \mathbb{F}_q[X; \theta]$ where q is a prime power, θ an automorphism of \mathbb{F}_q and Θ is the morphism defined on R by $\Theta(\sum a_i X^i) = \sum \theta(a_i) X^i$.*

1. *For $f, g \in R$, $(fg)^* = \Theta^k(g^*)f^*$, where $k = \deg(f)$.*
2. *For $f \in \mathbb{F}_q^\theta[X^{|\theta|}], g \in R$, $(fg)^* = (gf)^* = f^*g^*$ and $(fg)^\natural = f^\natural g^\natural$.*
3. *Consider $h = h_1 \cdots h_m \in R$ where for $i = 1, \dots, m$, h_i is a monic skew polynomial of R with degree d and constant coefficients λ_i . Then*

$$h^\natural = \prod_{i=m}^1 \Theta^{d(i-1)} \left(\frac{1}{\theta^d(\mu_i)} \cdot h_i^\natural \cdot \mu_i \right)$$

with $\mu_i = \lambda_1 \cdots \lambda_{i-1}$ and $\mu_1 = 1$.

Proof. Point 1 is in Lemma 4 of [4] and Point 2. is deduced from it. Point 3. is a generalization of Lemma 24 of [4]. For $m = 1$, the property is true. Consider m an integer ≥ 2 and assume that the property is true for $m - 1$. Consider $h = h_1 \cdots h_m$ and $H = h_1 \cdots h_{m-1}$.

$$\begin{aligned} h^\natural &= \frac{1}{\theta^{md}(\mu_{m+1})} (Hh_m)^* = \frac{1}{\theta^{md}(\mu_{m+1})} \Theta^{(m-1)d}(h_m^*) \cdot H^* \\ &= \frac{1}{\theta^{md}(\mu_{m+1})} \cdot \Theta^{(m-1)d} \left(\theta^d(\lambda_m) h_m^\natural \right) \cdot \theta^{(m-1)d}(\mu_m) \cdot H^\natural \\ &= \frac{1}{\theta^{md}(\mu_m)} \cdot \Theta^{(m-1)d} \left(h_m^\natural \right) \cdot \theta^{(m-1)d}(\mu_m) \cdot H^\natural \\ &= \Theta^{(m-1)d} \left(\frac{1}{\theta^d(\mu_m)} h_m^\natural \mu_m \right) \cdot \prod_{i=m-1}^1 \Theta^{d(i-1)} \left(\frac{1}{\theta^d(\mu_i)} h_i^\natural \mu_i \right) \\ &= \prod_{i=m}^1 \Theta^{d(i-1)} \left(\frac{1}{\theta^d(\mu_i)} \cdot h_i^\natural \cdot \mu_i \right). \end{aligned}$$

■

To solve the equation (1), a first approach consists in solving the polynomial system whose unknowns are the coefficients of h . If $q = p^2$ and θ is the Frobenius automorphism, another construction (Proposition 28 of [4]) is based on a lcrm computation of skew polynomials whose

coefficients are also solutions to auxiliary polynomial systems coming from equations of the type $H^\natural H = F(X^2)$ where $H \in R$ and $F(X^2) \in \mathbb{F}_p[X^2]$ is a self-reciprocal polynomial. The aim of this text is to give a new way to solve these equations which enables to give a counting formula.

3 Strategy

Consider, for $F(Y) = \sum a_i Y^i \in \mathbb{F}_p[Y]$ the skew polynomial $F(X^2) = \sum a_i X^{2i} \in R$ and define the set $\mathcal{H}_{F(X^2)}$ by

$$\mathcal{H}_{F(X^2)} := \{h \in R \mid h \text{ monic and } h^\natural h = F(X^2)\}. \quad (2)$$

Following (1), for $\epsilon \in \{-1, 1\}$, the set $\mathcal{H}_{X^{2k-\epsilon}}$ is the set of the skew check polynomials of all the self-dual (θ, ϵ) -constacyclic codes of dimension k over \mathbb{F}_{p^2} . The aim is therefore to provide a construction of the set $\mathcal{H}_{X^{2k-\epsilon}}$ which enables to count its number of elements. The starting point of this construction is given in Lemma 2 below and is based on Proposition 28 of [4] :

Lemma 2 *Consider p prime number, $\theta : x \mapsto x^p$ Frobenius automorphism over \mathbb{F}_{p^2} , $k \in \mathbb{N}^*$, $\epsilon \in \{-1, 1\}$, $s = \text{val}_p(k)$, $t = k/p^s$. The number of self-dual (θ, ϵ) -constacyclic codes of dimension k is*

$$\#\mathcal{H}_{X^{2k-\epsilon}} = N_\epsilon \times \prod_{f(Y) \in \mathcal{F}_{k,\epsilon}} \#\mathcal{H}_{f(X^2)^{p^s}} \times \prod_{f(Y) \in \mathcal{G}_{k,\epsilon}} \#\mathcal{H}_{f(X^2)^{p^s}}$$

where $\mathcal{F}_{k,\epsilon}$ is the set of all monic factors $f(Y) = f^\natural(Y)$ in $\mathbb{F}_p[Y]$ of $Y^t - \epsilon$ with degree > 1 such that $f(Y)$ is irreducible, $\mathcal{G}_{k,\epsilon}$ is the set of all monic factors $f(Y) = f^\natural(Y)$ in $\mathbb{F}_p[Y]$ of $Y^t - \epsilon$ with degree > 1 such that $f(Y) = g(Y)g^\natural(Y)$ with $g(Y) \neq g^\natural(Y)$ monic irreducible in $\mathbb{F}_p[Y]$ and N_ϵ is defined below :

$$\text{for } p = 2, N_\epsilon = \begin{cases} 1 & \text{if } s = 0 \\ 3 & \text{if } s > 0 \end{cases};$$

$$\text{for } p \text{ odd, } N_\epsilon = \begin{cases} \#\mathcal{H}_{(X^{2-\epsilon})^{p^s}} & \text{if } k \equiv 1 \pmod{2} \\ 1 & \text{if } k \equiv 0 \pmod{2}, \epsilon = -1 \\ \#\mathcal{H}_{(X^{2-1})^{p^s}} \times \#\mathcal{H}_{(X^{2+1})^{p^s}} & \text{if } k \equiv 0 \pmod{2}, \epsilon = 1. \end{cases}$$

Proof. Consider $t, s \in \mathbb{N}$ such that $k = p^s t$ with t prime to p and $Y^t - \epsilon = f_1(Y) \cdots f_r(Y)$ in $\mathbb{F}_p[Y]$ where $f_i(Y)$ are monic self-reciprocal polynomials which are either irreducible or products of two monic irreducible polynomials. Then according to Proposition 28 of [4], if $h \in \mathcal{H}_{X^{2k-\epsilon}}$, then $h = \text{lcrm}(h_1, \dots, h_r)$ where for all i in $\{1, \dots, r\}$, $h_i^\natural = \text{gcd}(f_i(X^2)^{p^s}, h^\natural)$ and h_i belongs to $\mathcal{H}_{f_i(X^2)^{p^s}}$. Consider therefore the application

$$\phi : \begin{cases} \mathcal{H}_{X^{2k-\epsilon}} & \rightarrow \mathcal{H}_{f_1(X^2)^{p^s}} \times \cdots \times \mathcal{H}_{f_r(X^2)^{p^s}} \\ h & \mapsto (h_1, \dots, h_r) \text{ where } h_i^\natural = \text{gcd}(f_i(X^2)^{p^s}, h^\natural). \end{cases}$$

Consider $(h_1, \dots, h_r) \in \mathcal{H}_{f_1(X^2)^{p^s}} \times \cdots \times \mathcal{H}_{f_r(X^2)^{p^s}}$ and $h = \text{lcrm}(h_1, \dots, h_r)$. According to Proposition 28 of [4], the skew polynomial h belongs to $\mathcal{H}_{X^{2k-\epsilon}}$. Let us prove that $h_i^\natural = \text{gcd}(f_i(X^2)^{p^s}, h^\natural)$. As h belongs to $\mathcal{H}_{X^{2k-\epsilon}}$, $h = \text{lcrm}(H_1, \dots, H_r)$, where $H_i^\natural = \text{gcd}(f_i(X^2)^{p^s}, h^\natural)$ and H_i belongs to $\mathcal{H}_{f_i(X^2)^{p^s}}$. As h_i^\natural divides $f_i(X^2)^{p^s}$ and h_i divides on

the left h , h_i^{\natural} divides on the right h and $f_i(X^2)^{p^s}$, therefore h_i divides on the right H_i , furthermore, h_i and H_i have the same degree because they both belong to $\mathcal{H}_{f_i(X^2)^{p^s}}$, so $h_i = H_i$ and $h_i^{\natural} = \text{gcd}(f_i(X^2)^{p^s}, h^{\natural})$. Therefore ϕ is bijective and one gets that

$$\#\mathcal{H}_{X^{2k-\epsilon}} = \prod_{i=1}^r \#\mathcal{H}_{f_i(X^2)^{p^s}} = N_{\epsilon} \times \prod_{f(Y) \in \mathcal{F}_{k,\epsilon}} \#\mathcal{H}_{f(X^2)^{p^s}} \times \prod_{f(Y) \in \mathcal{G}_{k,\epsilon}} \#\mathcal{H}_{f(X^2)^{p^s}}$$

where $N_{\epsilon} = \prod_{f(Y)=f^{\natural}(Y)|Y^t-\epsilon, \deg(f)=1} \#\mathcal{H}_{f(X^2)^{p^s}}$. The irreducible self-reciprocal polynomials of $\mathbb{F}_p[Y]$ of degree 1 are $Y+1$ and $Y-1$. For $p=2$, $N_{\epsilon} = \#\mathcal{H}_{(X^2+1)^{2^s}} = 1$ if $s=0$, 3 if $s>0$ according to Proposition 24 of [4]. If p is odd, then the set of self-reciprocal divisors of degree 1 of $Y^t - \epsilon$ is empty if $\epsilon = -1$ and $k \equiv 0 \pmod{2}$, $\{Y - \epsilon\}$ if $k \equiv 1 \pmod{2}$; $\{Y - 1, Y + 1\}$ if $\epsilon = 1$ and $k \equiv 0 \pmod{2}$. The expression of N_{ϵ} follows. ■

It remains now to determine the number of elements of each intermediate set $\mathcal{H}_{f(X^2)^{p^s}}$. These sets were computed in [4] for $p=2$ and $f(X^2) = X^2 - \epsilon$ by using factorization properties of the skew check polynomials.

The strategy here is based on a partition (Lemma 3) of the sets $\mathcal{H}_{f(X^2)^{p^s}}$ into subsets $\overline{\mathcal{H}}_{f(X^2)^m}$ who contains the elements of $\mathcal{H}_{f(X^2)^m}$ which are not divisible by the non constant factors of $f(X^2)$ in $\mathbb{F}_p[X^2]$.

Lemma 3 Consider $R = \mathbb{F}_{p^2}[X; \theta]$ with p prime number, $\theta : x \mapsto x^p$, $s \in \mathbb{N}$ and $f(Y) = f^{\natural}(Y)$ in $\mathbb{F}_p[Y]$. For $m \in \mathbb{N}$, consider $\overline{\mathcal{H}}_{f(X^2)^m}$ the set of elements of $\mathcal{H}_{f(X^2)^m}$ which are divisible by no non constant divisor in $\mathbb{F}_p[X^2]$ of $f(X^2)$.

1. If $f(Y) = f^{\natural}(Y)$ is irreducible in $\mathbb{F}_p[Y]$ then

$$\mathcal{H}_{f(X^2)^{p^s}} = \bigsqcup_{i=0}^{\lfloor \frac{p^s}{2} \rfloor} f(X^2)^i \cdot \overline{\mathcal{H}}_{f(X^2)^{p^s-2i}}. \quad (3)$$

2. If $f(Y) = g(Y)g^{\natural}(Y)$ with $g(Y) \neq g^{\natural}(Y)$ irreducible in $\mathbb{F}_p[Y]$ then

$$\mathcal{H}_{f(X^2)^{p^s}} = \bigsqcup_{i=0}^{p^s} \bigsqcup_{j=0}^{p^s-i} g(X^2)^i g^{\natural}(X^2)^j \cdot \overline{\mathcal{H}}_{f(X^2)^{p^s-(i+j)}}. \quad (4)$$

Proof.

1. Consider $M = \lfloor \frac{p^s}{2} \rfloor$, $h = h(X)$ in $\mathcal{H}_{f(X^2)^{p^s}}$ and i the biggest integer in $\{0, \dots, M\}$ such that $f(X^2)^i$ divides h . Consider $H = H(X)$ in R such that $h = f(X^2)^i H$ and $f(X^2)$ does not divide H . As $f(X^2)^i$ is central, according to point 2. of Lemma 1, $h^{\natural} = f(X^2)^i H^{\natural}$ and $H^{\natural} H = f(X^2)^{p^s-2i}$. Conversely, if $H^{\natural} H = f(X^2)^{p^s-2i}$ and $f(X^2)$ does not divide H , then $f(X^2)^i H$ belongs to $\mathcal{H}_{f(X^2)^{p^s}}$. Therefore $\mathcal{H}_{f(X^2)^{p^s}} = \bigcup_{i=0}^M f(X^2)^i \overline{\mathcal{H}}_{f(X^2)^{p^s-2i}}$. Furthermore consider $i > i'$, $H \in \mathcal{H}_{f^{p^s-2i}(X^2)}$ and $H' \in \mathcal{H}_{f^{p^s-2i'}(X^2)}$ such that $f(X^2)^i H = f(X^2)^{i'} H'$ then $f(X^2)^{i-i'}$ divides H' , which is impossible as $f(X^2)$ does not divide H' . Therefore, the sets $f(X^2)^i \cdot \overline{\mathcal{H}}_{f(X^2)^{p^s-2i}}$ and $f(X^2)^{i'} \cdot \overline{\mathcal{H}}_{f(X^2)^{p^s-2i'}}$ are disjoint.

2. Consider h in $\mathcal{H}_{f(X^2)^{p^s}}$ and i, j such that $h = g(X^2)^i g^{\natural}(X^2)^j H$ where $g(X^2)$ and $g^{\natural}(X^2)$ do not divide H . Then according to point 2. of Lemma 1, $h^{\natural} = g^{\natural}(X^2)^i g(X^2)^j H^{\natural}$, therefore $H^{\natural}H = f^{p^s-(i+j)}$ and $h \in g(X^2)^i g^{\natural}(X^2)^j \overline{\mathcal{H}}_{f(X^2)^{p^s-(i+j)}}$. Conversely, if $H^{\natural}H = f^{p^s-(i+j)}(X^2)$ and $f(X^2)$ does not divide H , then $g(X^2)^i g^{\natural}(X^2)^j H \in \mathcal{H}_{f(X^2)^m}$. It remains to prove that the sets are disjoint. Consider $(i, j) \neq (i', j')$ with $i > i'$ and $g(X^2)^i g^{\natural}(X^2)^j u = g(X^2)^{i'} g^{\natural}(X^2)^{j'} u'$ where $g(X^2)$ and $g^{\natural}(X^2)$ do not divide u and u' . If $j \geq j'$ then $g(X^2)$ divides u' which is impossible. Assume that $j < j'$ and denotes $\beta = j' - j$. Necessarily, $g(X^2)$ divides $g^{\natural}(X^2)^{\beta} u'$. As $g(X^2)$ and $g^{\natural}(X^2)$ both divide $g^{\natural}(X^2)^{\beta} u'$, there lclm is also a divisor of $g^{\natural}(X^2)^{\beta} u'$. But $g(X^2)$ and $g^{\natural}(X^2)$ are right coprime and belongs to $\mathbb{F}_p[X^2]$ therefore their lclm coincides with their lcm i.e. $g(X^2)g^{\natural}(X^2)$. So one gets that $g(X^2)$ divides $g^{\natural}(X^2)^{\beta-1} u'$. After repeating the same argument one gets that $g(X^2)$ divides $g^{\natural}(X^2)^{\beta-2} u', \dots, g^{\natural}(X^2) u'$, therefore $g(X^2)$ divides u' , contradiction.

■

The construction of the sets $\overline{\mathcal{H}}_{f(X^2)^m}$ for $m = 1$ will be reduced to Cauchy interpolation problems in the ring $\mathbb{F}_{p^2}[Z]$. For $m > 1$ this construction will require factorization properties given by Proposition 16 of [4] for $f(Y)$ irreducible. These factorization properties are recalled below in Proposition 1 which is a generalization of Proposition 16 of [4] to the case when $f(Y) = g(Y)g^{\natural}(Y)$ with $g(Y) \neq g^{\natural}(Y)$ irreducible in $\mathbb{F}_p[Y]$.

Remark 1 *The unique factorization of h in Proposition 1 below is the unique representation of h as the product of maximal completely reducible factors as it is defined in [17] page 498. If $f(Y)$ is irreducible in $\mathbb{F}_p[Y]$ (and $f(X^2) \in R$ does not divide h), then this factorization coincides with the factorization of h into irreducible monic skew polynomials.*

Proposition 1 *Consider $R = \mathbb{F}_{p^2}[X; \theta]$ where p is a prime number and $\theta : x \mapsto x^p$ the Frobenius automorphism. Consider $f(Y)$ in $\mathbb{F}_p[Y]$ self-reciprocal with degree d either irreducible in $\mathbb{F}_p[Y]$ or product of two distinct irreducible polynomials in $\mathbb{F}_p[Y]$. Assume that $h = h_1 \cdots h_m$ is a product of monic skew polynomials of degree d bounded by $f(X^2)$. The following assertions are equivalent :*

- (i) *the above factorization of h is unique;*
- (ii) *no non constant factor of $f(X^2)$ in $\mathbb{F}_p[X^2]$ divides h in R ;*
- (iii) *for all i in $\{1, \dots, m-1\}$, no non constant factor of $f(X^2)$ in $\mathbb{F}_p[X^2]$ divides $h_i h_{i+1}$ in R .*

Proof. If $f(Y)$ is irreducible, then the proof follows from Proposition 16 of [4] and Theorem 21, 24 of [10]. In the following, we assume that $f(Y) = g(Y)g^{\natural}(Y)$ with $g(Y) \neq g^{\natural}(Y)$ with same degree δ . To simplify the notations we denote $f = f(X^2)$, $g = g(X^2)$ and $g^{\natural} = g^{\natural}(X^2)$.

The implication (ii) \Rightarrow (iii) is immediate. Let us prove that (i) \Rightarrow (ii). If g divides h , then it divides h on the right and h has at least two distinct right factors u and v irreducible bounded by g . As the factors h_i of h are all bounded by $f = gg^{\natural}$, necessarily, h has an irreducible right factor w bounded by g^{\natural} . Then $\text{lclm}(u, w)$ and $\text{lclm}(v, w)$ are two right factors of h with degree d dividing f . According to Theorem 13 of [17], as u (resp. v) and

w are irreducible and do not have the same bound, the lclm-decompositions $\text{lclm}(u, w)$ and $\text{lclm}(v, w)$ are unique. Furthermore, u and v are distinct, so the skew polynomials $\text{lclm}(u, w)$ and $\text{lclm}(v, w)$ are distinct.

Let us prove by induction on m that if $h_1 \cdots h_m = g_1 \cdots g_m$ are two distinct decompositions of h into the product of monic skew polynomials of degree d dividing f , then there are two consecutive factors whose product is divisible by g or g^\natural .

Consider $h = h_1 h_2 = g_1 g_2$ where g_i, h_i are skew polynomials of degree d bounded by f . Assume that g and g^\natural do not divide h . Then $\text{gcd}(h, g)$ is an irreducible skew polynomial of degree δ dividing g which is also equal to $\text{gcd}(h_2, g)$ and $\text{gcd}(g_2, g)$. Similarly, $\text{gcd}(h_2, g^\natural) = \text{gcd}(g_2, g^\natural)$. Furthermore, according to Theorem 4.1 of [7], $h_2 = \text{lclm}(\text{gcd}(h_2, g), \text{gcd}(h_2, g^\natural))$ and $g_2 = \text{lclm}(\text{gcd}(g_2, g), \text{gcd}(g_2, g^\natural))$, therefore $g_2 = h_2$ and $(h_1, h_2) = (g_1, g_2)$.

Consider $m > 2$ and assume the property is true for $m - 1$. Consider two distinct decompositions of h into the product of monic skew polynomials of degree d bounded by f : $h = h_1 \cdots h_m = g_1 \cdots g_m$. Therefore, h_i and g_j are products of two irreducible monic skew polynomials of degree δ dividing g and g^\natural .

If $\text{gcd}(h_m, g_m) = 1$ then $\text{lclm}(h_m, g_m) = \bar{h}_{m-1} h_m$ divides $h_1 \cdots h_m$ and \bar{h}_{m-1} is a monic skew polynomial of degree d dividing f which is the product of two irreducible monic skew polynomials of degree δ dividing g and g^\natural . Consider $H \in R$ such that $H \bar{h}_{m-1} = h_1 \cdots h_{m-1}$. If $\bar{h}_{m-1} = h_{m-1}$ then $\text{lclm}(h_m, g_m) = h_{m-1} h_m$ has two factorizations into the product of two monic skew polynomials of degree d dividing f , therefore, g or g^\natural divides $h_{m-1} h_m$. Otherwise, as $h_1, \dots, h_{m-1}, \bar{h}_{m-1}$ are the products of an irreducible polynomial dividing g and an irreducible polynomial dividing g^\natural , H is the product of $m-2$ irreducible polynomials bounded by g and $m-2$ skew polynomials bounded by g^\natural . In particular, H divides $g^{m-2} (g^\natural)^{m-2}$. According to Theorem 4.1 of [7], $H = \text{lclm}(G, \tilde{G})$ where $G = \text{gcd}(H, g^{m-2})$ and $\tilde{G} = \text{gcd}(H, (g^\natural)^{m-2})$. As $g(Y)$ (resp. $g^\natural(Y)$) is irreducible in $\mathbb{F}_p[Y]$, the skew polynomial G (resp. \tilde{G}) is the product of N (resp. \tilde{N}) monic irreducible skew polynomials bounded by g (resp. g^\natural). Without loss of generality, one can assume that $N \leq \tilde{N}$. Consider $G = G_1 \cdots G_N$ (resp. $\tilde{G} = \tilde{G}_1 \cdots \tilde{G}_{\tilde{N}}$) the factorization of G as the product of N (resp. \tilde{N}) monic irreducible factors bounded by g (resp. g^\natural). As g (resp. g^\natural) does not divide G (resp. \tilde{G}), these factorizations are unique. Therefore, according to Theorem 14 of [17], $H = H_1 \cdots H_N$ where $H_i = \text{lclm}(\overline{G}_i, \tilde{\tilde{G}}_i)$ with $R/\overline{G}_i R$ and $R/\tilde{\tilde{G}}_i R$ (resp. $R/\overline{G}_i R$ and $R/\tilde{\tilde{G}}_i R$) isomorphic modules. As G_i is bounded by g , according to Corollary of Theorem 10 of [10], \overline{G}_i is also bounded by g . As G_i and $\tilde{\tilde{G}}_i$ are right coprime with same degree $d/2$, so are \overline{G}_i and $\tilde{\tilde{G}}_i$ therefore H_i is a skew polynomial of degree d which divides f . Lastly, as H has degree $(m-2)d$ one gets $N = m-2$. Therefore, H can be written as the product of $m-2$ monic skew polynomials of degree d dividing f and one can apply the induction hypothesis to $H \bar{h}_{m-1}$.

Assume that $\text{gcd}(h_m, g_m) = u \neq 1$. Necessarily u is an irreducible monic skew polynomial of degree δ which divides g or g^\natural let's say g . Consider v such that $\text{lclm}(g_m, h_m) = v h_m$ and $H \in R$ such that $h_1 \cdots h_m = H v h_m$ i.e $h_1 \cdots h_{m-1} = H v$. Necessarily v is an irreducible monic skew polynomial of degree δ dividing g^\natural and $H = \tilde{h}_1 \cdots \tilde{h}_{m-2} w$ where w is irreducible dividing g , \tilde{h}_i is a product of two irreducible monic skew polynomials of degree δ dividing g and g^\natural . If $h_{m-1} = wv$, then $\text{wlclm}(g_m, h_m) = h_{m-1} h_m$ and one concludes that g or g^\natural divides $h_{m-1} h_m$. If $h_{m-1} \neq wv$ then $h_1 \cdots h_{m-1} = \tilde{h}_1 \cdots \tilde{h}_{m-2} (wv)$ where wv is a monic skew polynomial of degree d dividing f , and one concludes using the induction hypothesis. ■

4 Construction of the sets $\mathcal{H}_{(X^2 \pm 1)^{p^s}}$

In [4], it is proven that there are three self-dual skew codes of dimension 2^s over \mathbb{F}_4 with $s \neq 0$ (and one self-dual skew code if $s = 0$). The aim of this section is to construct and count self-dual skew codes over \mathbb{F}_{p^2} with dimension a power of p when p is an odd prime number. Following Lemma 3, for p odd prime, the set $\mathcal{H}_{(X^2 - \epsilon)^{p^s}}$ is equal to the union of

pairwise disjoint sets $\bigsqcup_{i=0}^{\frac{p^s-1}{2}} (X^2 - \epsilon)^i \cdot \overline{\mathcal{H}}_{(X^2 - \epsilon)^{p^s-2i}}$ where $\overline{\mathcal{H}}_{(X^2 - \epsilon)^{p^s-2i}}$ is the set of elements of $\mathcal{H}_{(X^2 - \epsilon)^{p^s-2i}}$ which are not divisible by $X^2 - \epsilon$. Lemma 4 below gives for odd number m , a construction of the set $\overline{\mathcal{H}}_{(X^2 - \epsilon)^m}$ from which one easily deduces its number of elements. Lemma 3 and 4 enable therefore to derive a formula for the number of (θ, ϵ) -constacyclic codes of dimension a power of p over \mathbb{F}_{p^2} for p odd prime (Proposition 2).

Lemma 4 *Consider $\epsilon \in \{-1, 1\}$, $R = \mathbb{F}_{p^2}[X; \theta]$ with p odd prime number, θ the Frobenius automorphism, m odd number and $M = \frac{m-1}{2}$. If $\epsilon \neq (-1)^{\frac{p+1}{2}}$ then the set $\overline{\mathcal{H}}_{(X^2 - \epsilon)^m}$ is empty, otherwise it has $2p^M$ elements and it satisfies $\overline{\mathcal{H}}_{(X^2 - \epsilon)^m} =$*

$$\left\{ (X + \alpha_0) \prod_{i=1}^M (X + \alpha_i)(X + \theta(\alpha_i)) \mid \alpha_0^2 = -1, \alpha_i^{p+1} = \epsilon, \alpha_1 \neq \epsilon \alpha_0, \alpha_{i+1} \neq -\alpha_i \right\}.$$

Proof.

For $m = 1$, $\overline{\mathcal{H}}_{X^2 - \epsilon} = \mathcal{H}_{X^2 - \epsilon} = \{X + \alpha_0 \mid \theta(\alpha_0)^2 + 1 = 0, \theta(\alpha_0) = -\epsilon \alpha_0\} = \{X + \alpha_0 \mid \alpha_0^2 = -1, \alpha_0^{p+1} = \epsilon\}$.

- Consider $m \geq 3$. Assume that $h^\natural h = (X^2 - \epsilon)^m$ and $X^2 - \epsilon$ does not divide h . As h divides $(X^2 - \epsilon)^m$ and as $X^2 - \epsilon$ is irreducible in $\mathbb{F}_p[X^2]$, the bound of h is a power of $X^2 - \epsilon$, therefore the skew polynomial h is the product of irreducible monic skew polynomials bounded by $X^2 - \epsilon$. Furthermore all irreducible skew polynomials bounded by $X^2 - \epsilon$ are of degree 1 with a nonzero constant coefficient and the degree of h is equal to m , therefore h is the product of m irreducible monic skew polynomials of degree 1 :

$$h = (X + \lambda_1) \cdots (X + \lambda_m).$$

As $X^2 - \epsilon$ does not divide h , according to Proposition 1 (or Proposition 16 of [4]), this factorization is unique and

$$\forall i \in \{1, \dots, m-1\}, (X + \lambda_i)(X + \lambda_{i+1}) \neq X^2 - \epsilon. \quad (5)$$

Furthermore, according to Lemma 1,

$$h^\natural = (X + \tilde{\lambda}_m) \cdots (X + \tilde{\lambda}_2)(X + \tilde{\lambda}_1) \quad (6)$$

where $(\tilde{\lambda}_i)_{1 \leq i \leq m}$ is defined by

$$\tilde{\lambda}_{2i} = \frac{\theta(\mu_{2i})}{\mu_{2i+1}} \text{ and } \tilde{\lambda}_{2i+1} = \frac{\mu_{2i+1}}{\theta(\mu_{2i+2})} \quad (7)$$

and $(\mu_i)_{1 \leq i \leq m}$ is defined by

$$\mu_1 = 1, \forall i \in \{2, \dots, m\}, \mu_i = \lambda_1 \cdots \lambda_{i-1}. \quad (8)$$

This factorization of h^\natural into the product of monic irreducible skew polynomials is unique (otherwise the factorization of h would not be unique) therefore, according to Proposition 1 (or Proposition 16 of [4]),

$$\forall i \in \{1, \dots, m-1\}, (X + \tilde{\lambda}_{i+1})(X + \tilde{\lambda}_i) \neq X^2 - \epsilon. \quad (9)$$

As $h^\natural h = (X^2 - \epsilon)^m$ and $(X^2 - \epsilon)^m$ does not have a unique factorization into the product of monic linear polynomials, according to Proposition 1 (or Proposition 16 of [4]), there must exist two irreducible consecutive monic factors in the decomposition of $h^\natural h$ whose product is $X^2 - \epsilon$. According to the relations (5) and (9), the only possibility is $(X + \lambda_1)(X + \tilde{\lambda}_1) = X^2 - \epsilon$ and by induction one gets

$$\forall i \in \{1, \dots, m\}, (X + \lambda_i)(X + \tilde{\lambda}_i) = X^2 - \epsilon. \quad (10)$$

Developping (10) yields $\tilde{\lambda}_i = -\epsilon/\lambda_i = -\theta(\lambda_i)$ therefore $\lambda_i \theta(\lambda_i) = \lambda_i^{p+1} = \epsilon$ and $\tilde{\lambda}_{2i} \tilde{\lambda}_{2i+1} = \theta(\lambda_{2i}) \theta(\lambda_{2i+1})$ (note that one can consider this product for $i \geq 1$ because $m \geq 3$). The relation (7) gives $\tilde{\lambda}_{2i} \tilde{\lambda}_{2i+1} = \frac{\theta(\mu_{2i}) \mu_{2i+1}}{\mu_{2i+1} \theta(\mu_{2i+2})} = \frac{1}{\theta(\lambda_{2i}) \theta(\lambda_{2i+1})}$ therefore $(\lambda_{2i} \lambda_{2i+1})^2 = 1$. Furthermore, according to (5) and (10), $X + \lambda_{i+1} \neq X + \tilde{\lambda}_i$, therefore $\lambda_{2i+1} \neq \tilde{\lambda}_{2i} = -\frac{\epsilon}{\lambda_{2i}}$ and as p is *odd* the relation $(\lambda_{2i} \lambda_{2i+1})^2 = 1$ implies that $\lambda_{2i+1} = \frac{\epsilon}{\lambda_{2i}} = \theta(\lambda_{2i})$ (note that for $p = 2$, one gets an impossibility, see also Remark 2). Assume furthermore that m is odd and consider $M = \frac{m-1}{2}$. Considering $\alpha_0 = \lambda_1$ and for $i = 1 \dots M$, $\alpha_i = \lambda_{2i}$, one gets

$$h = (X + \alpha_0) \prod_{i=1}^M (X + \alpha_i)(X + \theta(\alpha_i))$$

with $\forall i \in \{0, \dots, M\}$, $\alpha_i \theta(\alpha_i) = \alpha_i^{p+1} = \epsilon$. Furthermore, $\tilde{\lambda}_1 = \frac{\mu_1}{\theta(\mu_2)}$, $\lambda_2 \neq \tilde{\lambda}_1$ and $\lambda_{2i+2} \neq \tilde{\lambda}_{2i+1}$ therefore $\alpha_0^2 = -1$, $\alpha_1 \neq \epsilon \alpha_0$ and $\alpha_{i+1} \neq -\alpha_i$.

- Conversely consider

$$h = (X + \alpha_0) \left[\prod_{i=1}^M (X + \alpha_i)(X + \theta(\alpha_i)) \right]$$

where $\alpha_0^2 = -1$, $\forall i \in \{0, \dots, M\}$, $\alpha_i \theta(\alpha_i) = \alpha_i^{p+1} = \epsilon$, $\alpha_1 \neq \epsilon \alpha_0$, $\forall i \in \{1, \dots, M\}$, $\alpha_{i+1} \neq -\alpha_i$. The expression of h^\natural is given by (6) with (8) and (7) where $\lambda_{2i} = \alpha_i$ and $\lambda_{2i+1} = \theta(\alpha_i)$. The expressions of μ_i and $\tilde{\lambda}_i$ can be simplified as follows: for $i \geq 1$,

$$\left\{ \begin{array}{l} \mu_{2i+1} = \mu_{2i} \alpha_i \\ \mu_{2i} = \alpha_0 \left[\prod_{j=1}^{i-1} \alpha_j \theta(\alpha_j) \right] \end{array} \right\} = \alpha_0 \prod_{j=1}^{i-1} \epsilon = \alpha_0 \epsilon^{i-1}$$

and

$$\begin{cases} \tilde{\lambda}_{2i} = \frac{\theta(\alpha_0)}{\alpha_0 \alpha_i} = \frac{\theta(\alpha_0) \alpha_0}{\alpha_0^2 \alpha_i} = -\frac{\epsilon}{\alpha_i} = -\theta(\alpha_i) \\ \tilde{\lambda}_{2i+1} = \frac{\mu_{2i+1}}{\theta(\mu_{2i+2})} = \frac{\alpha_0 \epsilon^{i-1} \alpha_i}{\theta(\alpha_0) \epsilon^i} = -\alpha_i. \end{cases}$$

Therefore the expression of h^\natural in (6) becomes

$$h^\natural = \left[\prod_{i=M}^1 (X - \alpha_i)(X - \theta(\alpha_i)) \right] (X - \theta(\alpha_0)).$$

Furthermore $(X - \theta(\alpha_0))(X + \alpha_0) = X^2 - \alpha_0 \theta(\alpha_0) = X^2 - \epsilon$ and for $i = 1, \dots, M$, $(X - \alpha_i)(X - \theta(\alpha_i))(X + \alpha_i)(X + \theta(\alpha_i)) = (X - \alpha_i)(X^2 - \epsilon)(X + \theta(\alpha_i)) = (X^2 - \epsilon)(X - \alpha_i)(X + \theta(\alpha_i)) = (X^2 - \epsilon)^2$. As $X^2 - \epsilon$ is central one gets $h^\natural h = (X^2 - \epsilon)^{1+2M}$.

Lastly, to prove that $X^2 - \epsilon$ does not divide h , according to Proposition 1, it suffices to prove that no product of two consecutive factors in the decomposition of $h = (X + \alpha_0) \prod_{i=1}^M (X + \alpha_i)(X + \theta(\alpha_i))$ is equal to $X^2 - \epsilon$ i.e. $(X + \alpha_0)(X + \alpha_1)$, $(X + \alpha_i)(X + \theta(\alpha_i))$ and $(X + \theta(\alpha_i))(X + \alpha_{i+1})$ are distinct from $X^2 - \epsilon$. The constant coefficients of these polynomials are all distinct from $-\epsilon$; namely $\alpha_0 \alpha_1 \neq \epsilon \alpha_0^2 = -\epsilon$, $\alpha_i \theta(\alpha_i) = \epsilon \neq -\epsilon$ and $\alpha_{i+1} \theta(\alpha_i) \neq -\alpha_i \theta(\alpha_i) = -\epsilon$, therefore no product of two consecutive factors in the decomposition of h is equal to $X^2 - \epsilon$.

- We now count the number of elements of the set $\overline{\mathcal{H}}_{(X^2 - \epsilon)^m}$: it is empty if and only if there is not $\alpha_0 \in \mathbb{F}_{p^2}$ such that $\alpha_0^2 = -1$ and $\alpha_0^{p+1} = \epsilon$. Over \mathbb{F}_{p^2} , -1 is always a square, therefore $\overline{\mathcal{H}}_{(X^2 - \epsilon)^m}$ is empty if and only if $(-1)^{(p+1)/2} \neq \epsilon$. (i.e. if and only if $p \equiv 1 \pmod{4}$ and $\epsilon = 1$ or $p \equiv 3 \pmod{4}$ and $\epsilon = -1$). If $\epsilon = (-1)^{(p+1)/2}$, then there are as many elements in $\overline{\mathcal{H}}_{(X^2 - \epsilon)^m}$ as the number of $(M+1)$ -uplets $(\alpha_0, \alpha_1, \dots, \alpha_M)$ such that $\alpha_0^2 = -1$, $\alpha_i^{p+1} = \epsilon$, $\alpha_1 \neq \epsilon \alpha_0$, $\alpha_{i+1} \neq -\alpha_i$. Therefore there are 2 choices for α_0 and p choices for $\alpha_1, \dots, \alpha_M$, so $\overline{\mathcal{H}}_{(X^2 - \epsilon)^m}$ has $2p^M$ elements.

■

Remark 2 From the first part of the proof of the previous lemma, one deduces that for $m \geq 3$ and $p = 2$, the sets $\overline{\mathcal{H}}_{(X^2 - \epsilon)^m}$ are empty. Therefore according to Lemma 3, for $s \in \mathbb{N}^*$,

$$\mathcal{H}_{(X^2+1)^{2s}} = (X^2 + 1)^{2^{s-1}} \overline{\mathcal{H}}_{(X^2+1)^0} \sqcup (X^2 + 1)^{2^{s-1}-1} \overline{\mathcal{H}}_{(X^2+1)^2}.$$

Furthermore $\overline{\mathcal{H}}_{(X^2+1)^2} = \{(X+1)(X+u) \mid u \in \{a, a^2\}\}$ where $\mathbb{F}_4 = \mathbb{F}_2(a)$. Therefore $\mathcal{H}_{(X^2+1)^{2s}} = \{(X+1)^{2^{s-1}}(X+u) \mid u \in \{1, a, a^2\}\}$ and one gets that for $s > 0$ there are only three self-dual skew codes of dimension 2^s over \mathbb{F}_4 (see Corollary 26 of [4]).

Proposition 2 below gives a formula for the number of self-dual θ -cyclic and θ -negacyclic codes whose dimension is a power of p for p odd prime. It is deduced from Lemma 3 and 4.

Proposition 2 Consider p an odd prime number, $s \in \mathbb{N}$, $\theta : x \mapsto x^p$ defined over \mathbb{F}_{p^2} and $\epsilon \in \{-1, 1\}$. The number of self-dual (θ, ϵ) -constacyclic codes of dimension p^s over \mathbb{F}_{p^2} is

$$\#\mathcal{H}_{(X^2 - \epsilon)^{p^s}} = \begin{cases} 0 & \text{if } \epsilon \neq (-1)^{\frac{p+1}{2}} \\ 2^{\frac{p^s + 1}{2} - 1} & \text{if } \epsilon = (-1)^{\frac{p+1}{2}}. \end{cases}$$

Proof.

Consider $s \in \mathbb{N}^*$ and $M = \frac{p^s-1}{2}$. According to Lemma 3, $\mathcal{H}_{X^{2p^s}-\epsilon} = \bigsqcup_{i=0}^M (X^2 - 1)^i \overline{\mathcal{H}}_{(X^2-\epsilon)^{p^s-2i}}$ and according to Lemma 4, each set $\overline{\mathcal{H}}_{(X^2-\epsilon)^{p^s-2i}}$ is empty if $\epsilon \neq (-1)^{\frac{p+1}{2}}$ and has $2p^{M-i}$ elements if $\epsilon = (-1)^{\frac{p+1}{2}}$. Therefore, if $\epsilon \neq (-1)^{\frac{p+1}{2}}$, $\mathcal{H}_{X^{2p^s}-\epsilon}$ is empty and otherwise it has $\sum_{i=0}^M 2p^{M-i} = 2 \frac{p^{M+1}-1}{p-1} = 2 \frac{p^{(p^s+1)/2}-1}{p-1}$ elements. ■

To conclude this section, some experimental results about the weight enumerators of self-dual skew codes with dimension p^s and their links to the sets $\overline{\mathcal{H}}_{(X^2-\epsilon)^m}$ are given below.

Over \mathbb{F}_4 the best minimum distance reached by self-dual θ -codes of dimension 2^s does not depend on s and is only 3 for $s = 2$ and 4 for $s \geq 3$ (Theorem 1 [3]). The following proposition gives a family of self-dual θ -codes of dimension p^s whose minimal distance depends only on p and does not grow with s either (which is quite bad !). The skew check polynomials of the corresponding codes all belong to $(X^2 - \epsilon)^{\frac{p^s-1}{2}} \overline{\mathcal{H}}_{X^2-\epsilon}$.

Proposition 3 *Consider p odd prime, $\alpha \in \mathbb{F}_{p^2}$ such that $\alpha^2 + 1 = 0$, $\epsilon = (-1)^{\frac{p+1}{2}}$ and $\theta : x \mapsto x^p$. For each nonnegative integer s , the (ϵ, θ) -constacyclic code over \mathbb{F}_{p^2} of dimension p^s generated by the skew polynomial $(X^2 - \epsilon)^{\frac{p^s-1}{2}}(X + \alpha)$ is a self-dual code with minimum distance $\leq \frac{p+3}{2}$.*

Proof. According to Lemma 4, the skew polynomial $h = (X^2 - \epsilon)^{\frac{p^s-1}{2}}(X - \alpha)$ belongs to $(X^2 - \epsilon)^{\frac{p^s-1}{2}} \overline{\mathcal{H}}_{X^2-\epsilon}$. Therefore, according to Lemma 3, it belongs to $\mathcal{H}_{X^{2p^s}-\epsilon}$ and $g = h^\natural = (X^2 - \epsilon)^{\frac{p^s-1}{2}}(X + \alpha)$ generates a self-dual θ -code of dimension p^s .

The skew polynomial $(X^2 - \epsilon)^{\frac{p^s-1}{2}(p+1)} = (X^{2p^{s-1}} - \epsilon)^{\frac{p+1}{2}}$ has $\frac{p+3}{2}$ terms and it is also a right multiple of degree $< 2p^s$ of the skew polynomial g as $(X^{2p^{s-1}} - \epsilon)^{\frac{p+1}{2}} = (X^2 - \epsilon)^{\frac{p^s-1}{2}-1}(X + \epsilon\alpha)g$. So the code has a word of weight $\frac{p+3}{2}$. ■

Table 1 shows that the weight enumerators of self-dual skew codes over \mathbb{F}_{p^2} (in the third column) with dimension p (first column) for $p = 3, 5, 7$ can be classified in function of the sets $\mathcal{H}^{(i)} := (X^2 - \epsilon)^i \overline{\mathcal{H}}_{(X^2-\epsilon)^{p-2i}}$ where $0 \leq i \leq \frac{p-1}{2}$ (second column). The computations were made using MAGMA. From this table, one can notice two facts about self-dual skew codes of dimension p over \mathbb{F}_{p^2} that we cannot explain at the moment. First for $p = 3, 5, 7$, two codes with skew check polynomials belonging to two distinct sets $\mathcal{H}^{(i)}$ and $\mathcal{H}^{(j)}$ cannot have the same weight enumerator and we have the following conjecture :

Conjecture 1 *Let us denote, for $0 \leq i \leq \frac{p^s-1}{2}$, $\mathcal{H}^{(i)} := (X^2 - \epsilon)^i \overline{\mathcal{H}}_{(X^2-\epsilon)^{p^s-2i}}$. Consider $i \neq j \in \{0, \dots, \frac{p^s-1}{2}\}$, $h \in \mathcal{H}^{(i)}$ and $h' \in \mathcal{H}^{(j)}$. The weight enumerators of the self-dual (θ, ϵ) -constacyclic codes of dimension p^s with skew check polynomials h and h' are distinct.*

Secondly, for each $p \in \{3, 5, 7\}$, the skew check polynomials of the self-dual θ -codes of dimension p who reach the best distances all belong to $\mathcal{H}^{(0)}$ (they are not divisible by $X^2 - \epsilon$ and have therefore a unique factorization into the product of monic skew polynomials of degree 1). Here are some examples :

- There are 4 $[6, 3, 4]_9$ self-dual θ -cyclic codes, one of them has check polynomial $h = (X + a^2)(X + 1)(X + 1) = (X + a^2)(X^2 + 2X + 1) = X^3 + aX^2 + a^5X + a^2$ and generator polynomial $g = X^3 + a^7X^2 + a^5X + a^2$ where $a^2 + 2a + 2 = 0$.
- There are 32 $[10, 5, 6]_{25}$ self-dual θ -negacyclic codes, one of them has check polynomial $h = (X + 2)[(X + a^{22})(X + a^{14})][(X + a^2)(X + a^{10})] = (X + 2)(X^2 + a^4X + 4)(X^2 + a^8X + 4) = X^5 + a^{17}X^4 + X^3 + 2X^2 + a^7X + 2$ and generator polynomial $g = X^5 + aX^4 + X^3 + 3X^2 + a^7X + 3$ where $a^2 + 4a + 2 = 0$.
- There are 72 $[14, 7, 8]_{49}$ self-dual θ -cyclic codes and one of them has check polynomial $h = (X + a^{36})[(X + a^6)(X + a^{42})][(X + a^6)(X + a^{42})][(X + a^{12})(X + a^{36})] = (X + a^{36})(X^2 + a^{22}X + 1)(X^2 + a^{22}X + 1)(X^2 + a^{28}X + 1) = X^7 + a^{47}X^6 + a^{38}X^5 + a^{36}X^4 + X^3 + a^{14}X^2 + a^{29}X + a^{36}$ and generator polynomial $g = X^7 + a^{17}X^6 + a^{38}X^5 + a^{36}X^4 + X^3 + a^{26}X^2 + a^{29}X + a^{36}$ where $a^2 + 6a + 3 = 0$.

Paradoxically, as proved below, the set $\mathcal{H}^{(0)}$ also provides skew codes with very bad distance (2) for any s : consider $\alpha \in \mathbb{F}_{p^2}$ such that $\alpha^2 = -1$, the skew polynomial $X^{p^s} + \alpha$ generates a self-dual θ -code of minimum distance 2. One can notice that it factorizes uniquely as the product of monic skew polynomials of degree 1 :

- if $p \equiv 1 \pmod{4}$, $X^{p^s} + \alpha = (X + \alpha)^{p^s}$
- if $p \equiv 3 \pmod{4}$, $X^{p^s} + \alpha = (X + \alpha)(X - \alpha)(X + \alpha) \cdots (X - \alpha)(X + \alpha)$.

Namely, as $X^{p^s} + \alpha$ divides on the right $(X^2 + 1)^{p^s}$, its factors are $X + u$ with $u^2 = -1$. As the bound of $X^{p^s} + \alpha$ is $(X^2 + 1)^{p^s}$, the factorization is unique and is necessarily the one above.

The self-dual θ -cyclic codes $[18, 9]_9$ are not optimal, as the best distance they reach is 8 which is less than the best known distance (9) for these codes. Conjecture 1 is however satisfied for these codes (i.e. for $p = 3$ and $s = 2$). Some of the $[54, 27]_9$ self-dual θ -cyclic codes reach the best known distance, 18. Their skew check polynomials belong to $\mathcal{H}^{(2)}$ (divisible exactly by $(X^2 - 1)^2$), $\mathcal{H}^{(1)}$ (divisible exactly by $(X^2 - 1)$) or $\mathcal{H}^{(0)}$ (not divisible by $X^2 - 1$), but none of them is divisible by $(X^2 - 1)^3$. As there are 4782968 self-dual θ -cyclic codes with dimension 27, it remains quite long to compute all the minimal distances. A good strategy would be to exploit the factorization of the skew check polynomial more deeply to keep the best codes.

5 Construction of the sets $\mathcal{H}_{f(X^2)^{p^s}}$ for f of degree > 1

For $f(Y) = f^{\natural}(Y)$ of degree $d > 1$ such that $f(Y)$ is irreducible in $\mathbb{F}_p[Y]$ (resp. such that $f(Y) = g(Y)g^{\natural}(Y)$ with $g(Y) \neq g^{\natural}(Y)$ irreducible in $\mathbb{F}_p[Y]$), the set $\mathcal{H}_{f(X^2)^{p^s}}$ will be constructed by using the partition (3) (resp. (4)) given in Lemma 3. It requires the computation of the sets $\overline{\mathcal{H}}_{f(X^2)^m}$ whose elements will be constructed (Lemma 8) using products of elements of $\overline{\mathcal{H}}_{f(X^2)}$. In the following subsection, the sets $\mathcal{H}_{f(X^2)}$ are constructed via Cauchy interpolations in $\mathbb{F}_{p^2}[Z]$.

p	i	Weight enumerators	nbe codes
3	1	$1 + 16Y^3 + 72Y^4 + 288Y^5 + 352Y^6$	2
	0	$1 + 24Y^2 + 192Y^4 + 512Y^6$	4
		$1 + 120Y^4 + 240Y^5 + 368Y^6$	
5	2	$1 + 240Y^4 + 1008Y^5 + \dots$	2
	1	$1 + 240Y^4 + 48Y^5 + \dots$	2
		$1 + 48Y^5 + \dots$	8
	0	$1 + 120Y^2 + \dots$	2
$1 + 120Y^4 + 240Y^5 + \dots$		8	
$1 + 120Y^4 + 480Y^5 + \dots$		8	
		$1 + 5040Y^6 + \dots$	32
7	3	$1 + 2016Y^5 + \dots$	2
	2	$1 + 672Y^6 + 5472Y^7 + \dots$	12
		$1 + 3024Y^6 + 8160Y^7 + \dots$	2
	1	$1 + 1008Y^4 + \dots$	2
		$1 + 336Y^6 + 2112Y^7 + \dots$	12
		$1 + 336Y^6 + 4800Y^7 + \dots$	12
		$1 + 336Y^6 + 4128Y^7 + \dots$	12
$1 + 672Y^6 + 4800Y^7 + \dots$		12	
$1 + 96Y^7 + 143472Y^8 + \dots$		12	
$1 + 2784Y^7 + 124656Y^8 + \dots$	12		
		$1 + 4128Y^7 + 115248Y^8 + \dots$	24

p	i	Weight enumerators	nbe codes
7	0	$1 + 336Y^2 + \dots$	2
		$1 + 336Y^4 + \dots$	36
		$1 + 336Y^6 + 2688Y^7 + \dots$	36
		$1 + 336Y^6 + 3360Y^7 + \dots$	12
		$1 + 336Y^6 + 1344Y^7 + \dots$	24
		$1 + 336Y^6 + 2016Y^7 + \dots$	60
		$1 + 336Y^6 + 4032Y^7 + \dots$	12
		$1 + 672Y^6 + 3360Y^7 + \dots$	12
		$1 + 672Y^6 + 4704Y^7 + \dots$	24
		$1 + 1008Y^6 + 4032Y^7 + \dots$	24
		$1 + 1008Y^6 + 4704Y^7 + \dots$	12
		$1 + 1680Y^6 + \dots$	12
		$1 + 672Y^7 + \dots$	84
		$1 + 1344Y^7 + \dots$	24
		$1 + 2016Y^7 + \dots$	72
		$1 + 2688Y^7 + \dots$	84
		$1 + 3360Y^7 + \dots$	36
$1 + 4032Y^7 + \dots$	4		
$1 + 4704Y^7 + \dots$	16		
$1 + 6048Y^7 + \dots$	28		
		$1 + 144144Y^8 + \dots$	72

Table 1: Weight enumerators of self-dual skew codes over \mathbb{F}_{p^2} with dimension p for $p = 3, 5, 7$ in function of $i = \frac{p-1}{2}, \dots, 1, 0$ where i is the biggest integer such that $(X^2 - \epsilon)^i$ divides the skew check polynomial of the codes

5.1 Construction of the sets $\mathcal{H}_{f(X^2)}$.

Consider $f(Y) \in \mathbb{F}_p[Y]$ with degree $d > 1$ such that $f(Y) = f^\natural(Y)$ and $f(Y)$ irreducible. According to [16], page 6 (or Lemma 1.4.11 of [5] with $e = 2$), the skew polynomial $f(X^2) \in R$ has $((p^2)^d - 1)/(p^d - 1) = p^d + 1$ irreducible monic right factors of degree d . The set $\mathcal{H}_{f(X^2)}$ is a subset of this set of factors. Its construction is mainly based on Cauchy interpolations (Lemma 6). The following Lemma (see also exercise 3.14 page 141 of [13]) will be useful next

Lemma 5 Consider $f(Y) \in \mathbb{F}_p[Y]$ with degree $d > 1$ such that $f(Y) = f^\natural(Y)$ and $f(Y)$ irreducible in $\mathbb{F}_p[Y]$. Consider $\alpha \in \mathbb{F}_{p^d}$ such that $f(\alpha) = 0$. Then d is necessarily even and $\alpha^{-1} = \alpha^{p^\delta}$ where $2\delta = d$.

Proof. Consider $f(Y) \in \mathbb{F}_p[Y]$ irreducible with degree $d > 1$ and such that $f(Y) = f^\natural(Y)$. Consider $\alpha \in \mathbb{F}_{p^d}$ such that $f(\alpha) = 0$ and j the order of α . As $f^\natural(Y) = f(Y)$, α^{-1} is a root of $f(Y)$, so there exists $k \in \{0, \dots, d-1\}$ such that $\alpha^{-1} = \alpha^{p^k}$. Therefore $p^k \equiv -1 \pmod{j}$ and $p^{2k} \equiv 1 \pmod{j}$. Furthermore, d is the order of p modulo j , so d divides $2k$. If d was odd then it would divide k , which is impossible as $p^k \equiv -1 \pmod{j}$. Consider δ such that $d = 2\delta$, it divides $k \in \{0, \dots, d-1\}$ therefore $k = \delta$ and $\alpha^{-1} = \alpha^{p^\delta}$. ■

The construction of $\mathcal{H}_{f(X^2)}$ below is given for $f(Y)$ self-reciprocal irreducible in $\mathbb{F}_p[Y]$ or product of two distinct irreducible polynomials.

Lemma 6 Consider $R = \mathbb{F}_{p^2}[X; \theta]$ with p prime number, $\theta : x \mapsto x^p$, $m \in \mathbb{N}^*$ and $f(Y)$ in $\mathbb{F}_p[Y]$ with degree $d = 2\delta > 1$ in Y such that $f(Y) = f^\natural(Y)$ with $f(Y)$ irreducible or product of two irreducible polynomials with degree δ . The set $\mathcal{H}_{f(X^2)}$ has $1 + p^\delta$ elements if f is irreducible and $3 + p^\delta$ elements if f is reducible.

Proof. Assume that $f(Y)$ is irreducible, as its degree is even (Lemma 5) $f(Z)$ factorizes in $\mathbb{F}_{p^2}[Z]$ as the product of two irreducible polynomials $\tilde{f}(Z)$ and $\Theta(\tilde{f})(z)$. If $h \in R \cap \mathbb{F}_{p^2}[X^2]$ then the equation $h^\natural h = f(X^2)$ in R is equivalent to the equation $A(X^2)A^\natural(X^2) = f(X^2)$

in $\mathbb{F}_{p^2}[X^2]$. If δ is even then $\tilde{f} = \tilde{f}^{\natural}$, so there is no solution; if δ is odd, then $\tilde{f}^{\natural} = \Theta(\tilde{f})$ therefore $\tilde{f}(X^2)$ and $\Theta(\tilde{f})(X^2)$ are solutions. Consider now $\alpha \in \mathbb{F}_{p^d}$ such that $f(\alpha) = 0$. If $h \in R \setminus \mathbb{F}_{p^2}[X^2]$, then writing $h(X) = A(X^2) + XB(X^2)$, with $B \neq 0$, one gets that $h^{\natural}h = f(X^2)$ is equivalent to

$$\begin{cases} A^*(X^2)A(X^2) + B^*(X^2)B(X^2)X^2 & = \lambda f(X^2) \in R \\ A^*(X^2)\Theta(B)(X^2) + B^*(X^2)\Theta(A)(X^2) & = 0 \in R. \end{cases}$$

The skew polynomials which appear in the equations belong to $\mathbb{F}_{p^2}[X^2; \theta] = \mathbb{F}_{p^2}[X^2]$ therefore, replacing X^2 with Z one gets the equivalent system in $\mathbb{F}_{p^2}[Z]$:

$$\begin{cases} A^*(Z)A(Z) + B^*(Z)B(Z)Z & = \lambda f(Z) \in \mathbb{F}_{p^2}[Z] \\ A^*(Z)\Theta(B)(Z) + B^*(Z)\Theta(A)(Z) & = 0 \in \mathbb{F}_{p^2}[Z] \end{cases} \quad (11)$$

where the expressions above belong now to the commutative ring $\mathbb{F}_{p^2}[Z]$ and the morphism of $\mathbb{F}_{p^2}[Z] : \sum a_i Z^i \mapsto \sum a_i^p Z^i$ is again denoted Θ . As $A(Z)$ and $B(Z)$ are polynomials of degree δ and $\leq \delta - 1$ whereas $f(Z)$ has degree 2δ , these two last equalities can be replaced with relations of divisibility :

$$\begin{cases} f(Z) & | & A^*(Z)A(Z) + B^*(Z)B(Z)Z \\ f(Z) & | & A^*(Z)\Theta(B)(Z) + B^*(Z)\Theta(A)(Z). \end{cases} \quad (12)$$

As the degree of $B(Z)$ is $\leq \delta - 1$ and as $f(Z)$ factors over \mathbb{F}_{p^2} into the product of two irreducible polynomials of degree δ , B and f are coprime i.e. $B(\alpha) \neq 0$ and $B(\alpha^p) \neq 0$. Evaluating right hand sides of (12) at α and α^p and replacing α^{-1} with α^{p^δ} (cf Lemma 5), one gets $h(X) = A(X^2) + XB(X^2) \in \mathcal{H}_{f(X^2)}$ with $B \neq 0$

$$\Leftrightarrow \begin{cases} A(\alpha^{p^\delta})A(\alpha) + B(\alpha^{p^\delta})B(\alpha) & = 0 \\ A(\alpha^{p^{\delta+1}})A(\alpha^p) + B(\alpha^{p^{\delta+1}})B(\alpha^p) & = 0 \\ \alpha A(\alpha^{p^\delta})\Theta(B)(\alpha) + B(\alpha^{p^\delta})\Theta(A)(\alpha) & = 0 \\ \alpha^p A(\alpha^{p^{\delta+1}})\Theta(B)(\alpha^p) + B(\alpha^{p^{\delta+1}})\Theta(A)(\alpha^p) & = 0. \end{cases}$$

If δ is even, then $A(\alpha^{p^\delta}) = A(\alpha)^{p^\delta}$ and $B(\alpha^{p^\delta}) = B(\alpha)^{p^\delta}$, therefore one gets the equivalent system

$$\begin{cases} u^{p^\delta+1} & = -1 \\ A(\alpha) & = u \times B(\alpha) \\ A(\theta(\alpha)) & = \theta(\frac{\alpha}{u}) \times B(\theta(\alpha)) \end{cases} \quad (13)$$

Consider for $u \in \mathbb{F}_{p^{2\delta}}$ satisfying $u^{p^\delta+1} = -1$ the unique polynomial P_u of $\mathbb{F}_{p^2}[Z]$ with degree $\leq 2\delta - 1$ defined by $P_u(\alpha) = u$ and $P_u(\theta(\alpha)) = \theta(\alpha/u)$. The relation (13) is equivalent to

$$\begin{cases} u^{p^\delta+1} & = -1 \\ A & \equiv P_u \times B \pmod{\tilde{f}} \\ A & \equiv P_u \times B \pmod{\Theta(\tilde{f})}. \end{cases}$$

Therefore $h(X) = A(X^2) + XB(X^2) \in \mathcal{H}_{f(X^2)}$ with $B \neq 0$ is equivalent to the following Cauchy interpolations problems

$$\begin{aligned} \mathcal{R}I_{P_u} : \quad & \gcd(B(Z), f(Z)) = 1 \\ & \frac{A(Z)}{B(Z)} \equiv P_u(Z) \pmod{f(Z)} \end{aligned}$$

for u such that $u^{p^\delta+1} = -1$.

As there is a unique solution (A, B) to $\mathcal{R}I_{P_u}$ with A monic (section 5.8 of [19]) for each u such that $u^{p^\delta+1} = -1$, and as two distinct u provide two distinct solutions, the set $\mathcal{H}_f(X^2) \setminus \mathbb{F}_{p^2}[X^2]$ has $p^\delta + 1$ elements.

If δ is odd, using the relations $A(\alpha^{p^\delta}) = A(\alpha^p)^{p^{\delta-1}}$ and $B(\alpha^{p^\delta}) = B(\alpha^p)^{p^{\delta-1}}$, one gets the equivalent system

$$\begin{cases} u^{p^\delta-1} & = -1/\alpha \\ A(\alpha) & = \frac{\alpha}{u} \times B(\alpha) \\ A(\theta(\alpha)) & = \theta(u) \times B(\theta(\alpha)) \end{cases}$$

which is equivalent to the interpolation problem $\mathcal{R}I_P$ where P is the unique polynomial of $\mathbb{F}_{p^2}[Z]$ with degree $\leq 2\delta - 1$ defined by $P(\alpha) = \frac{\alpha}{u}$ and $P(\theta(\alpha)) = \theta(u)$ for $u^{p^\delta-1} = -1/\alpha$. The number of elements of \mathcal{H}_f is therefore $p^\delta + 1 + 0$ if δ is even and $(p^\delta - 1) + 2 = p^\delta + 1$ if δ is odd.

When $f(Y)$ is reducible, the relation (12) still holds. If δ is even then $g(Z)$ is the product of two irreducible polynomials $\tilde{g}(Z)$ and $\Theta(\tilde{g})(Z)$ belonging to $\mathbb{F}_{p^2}[Z]$ and the set $\mathcal{H}_f \cap \mathbb{F}_{p^2}[X^2]$ is $\{g(X^2), g^\natural(X^2), \tilde{g}(X^2)\Theta(\tilde{g}^\natural)(X^2), \tilde{g}^\natural(X^2)\Theta(\tilde{g})(X^2)\}$. If δ is odd, then $g(Z)$ is irreducible in $\mathbb{F}_{p^2}[Z]$ and the set $\mathcal{H}_f \cap \mathbb{F}_{p^2}[X^2]$ is $\{g(X^2), g^\natural(X^2)\}$. Consider β in \mathbb{F}_{p^δ} such that $g(\beta) = 0$. Then, for δ even, if $h(X) = A(X^2) + XB(X^2)$, with $B \neq 0$ is such that $h^\natural h = f(X^2)$ then $f(Z)$ and $B(Z)$ are coprime i.e. $B(\beta), B(1/\beta), B(\beta^p), B(1/\beta^p) \neq 0$. Indeed if $B(\beta) = 0$ then $B(\beta^p), B(1/\beta), B(1/\beta^p) \neq 0$ otherwise the condition on the degree of B would give $B = 0$. Furthermore according to the second relation of (12), $B^*(\beta^p)\Theta(A)(\beta^p) = 0$, so $A(\beta) = 0$ and $A^*(1/\beta) = 0$. According to (12), $B^*(1/\beta)\Theta(A)(1/\beta) = 0$, so $\Theta(A)(1/\beta) = 0$ and $A(1/\beta^p) = 0$. Then first part of (12) gives $B^*(1/\beta^p)B(1/\beta^p) = 0$ which is impossible. In the same way, one gets that $B(1/\beta), B(\beta^p), B(1/\beta^p) \neq 0$. Therefore

$$h(X) = A(X^2) + XB(X^2) \in \mathcal{H}_f(X^2), B \neq 0 \Leftrightarrow$$

$$\begin{cases} A(\beta)A(1/\beta) + B(\beta)B(1/\beta) & = 0 \\ A(\beta^p)A(1/\beta^p) + B(\beta^p)B(1/\beta^p) & = 0 \\ \beta\Theta(B)(\beta)A(1/\beta) + \Theta(A)(\beta)B(1/\beta) & = 0 \\ 1/\beta\Theta(B)(1/\beta)A(\beta) + \Theta(A)(1/\beta)B(\beta) & = 0 \\ \beta^p\Theta(B)(\beta^p)A(1/\beta^p) + \Theta(A)(\beta^p)B(1/\beta^p) & = 0 \\ 1/\beta^p\Theta(B)(1/\beta^p)A(\beta^p) + \Theta(A)(1/\beta^p)B(\beta^p) & = 0 \end{cases} \Leftrightarrow \begin{cases} u^{p^\delta-1} & = 1 \\ A(\beta) & = u \times B(\beta) \\ A(\frac{1}{\beta}) & = -\frac{1}{u} \times B(\frac{1}{\beta}) \\ A(\theta(\beta)) & = \theta(\frac{\beta}{u}) \times B(\theta(\beta)) \\ A(\theta(\frac{1}{\beta})) & = -\theta(\frac{u}{\beta}) \times B(\theta(\frac{1}{\beta})) \end{cases}$$

which is equivalent to $\mathcal{R}I_P$ where P is the unique polynomial of $\mathbb{F}_{p^2}[Z]$ with degree $\leq 2\delta - 1$ defined by $P(\beta) = u, P(\frac{1}{\beta}) = -\frac{1}{u}, P(\theta(\beta)) = \theta(\frac{\beta}{u})$ and $P(\theta(\frac{1}{\beta})) = -\theta(\frac{u}{\beta})$ for $u^{p^\delta-1} = 1$. If δ is odd, then g is irreducible over \mathbb{F}_{p^2} and g^\natural is irreducible over \mathbb{F}_{p^2} with root $1/\beta$, therefore for $B \neq 0$ with degree $\deg(B) \leq \delta - 1$, $B(\beta), B(1/\beta) \neq 0$ and $h(X) = A(X^2) + XB(X^2) \in \mathcal{H}_f(X^2), B \neq 0 \Leftrightarrow$

$$\begin{cases} u\theta^\delta(u) & = \beta \\ A(\beta) & = u \times B(\beta) \\ A(\frac{1}{\beta}) & = -\frac{1}{u} \times B(\frac{1}{\beta}) \end{cases}$$

which is equivalent to $\mathcal{R}IP$ where $P(\beta) = u, P(\frac{1}{\beta}) = -\frac{1}{u}$ for $u^{p^\delta+1} = \beta$. The number of elements of \mathcal{H}_f is therefore $p^\delta - 1 + 4 = p^\delta + 3$ if δ is even and $(p^\delta + 1) + 2 = p^\delta + 3$ if δ is odd.

■

Example 1 Consider $\mathbb{F}_4 = \mathbb{F}_2(a)$, θ Frobenius automorphism, $R = \mathbb{F}_4[X, \theta]$ and $f(Y) = Y^2 + Y + 1 \in \mathbb{F}_2[Y]$ irreducible and self-reciprocal. The polynomial $f(Z)$ factorizes in $\mathbb{F}_4[Z]$ as $f(Z) = (Z+a)(Z+a^2)$ where $a^2 + a + 1 = 0$. As $\delta = \deg(f(Y))/2 = 1$ is odd, the equation $h^\natural h = f(X^2)$ has two solutions in $R \cap \mathbb{F}_4[X^2]$ namely $X^2 + a$ and $X^2 + a^2$. Consider $\alpha \in \mathbb{F}_4$ such that $f(\alpha) = 0$ and u such that $u^{2^\delta-1} = -1/\alpha$ i.e. $\alpha = a$ and $u = a^2$. The polynomial $P \in \mathbb{F}_4[Z]$ of degree ≤ 1 such that $P(\alpha) = \alpha/u$ and $P(\alpha^2) = u^2$ i.e. $P(a) = a^2$ and $P(a^2) = a$ is $P(Z) = Z+1$ and the unique solution (A, B) in $\mathbb{F}_4[Z] \times \mathbb{F}_4[Z]$ of $\frac{A}{B} \equiv Z+1 \pmod{Z^2+Z+1}$ is $(A, B) = (Z+1, 1)$. Therefore, the equation $h^\natural h = f(X^2)$ has one solution in $R \setminus \mathbb{F}_4[X^2]$ namely $(X^2 + 1) + X \times 1$. The set \mathcal{H}_f is $\{X^2 + a, X^2 + a^2, X^2 + X + 1\}$ and has $2^\delta + 1 = 3$ elements.

Example 2 Consider $\mathbb{F}_4 = \mathbb{F}_2(a)$, θ Frobenius automorphism, $R = \mathbb{F}_4[X, \theta]$ and $f(Y) = (Y^3 + Y + 1)(Y^3 + Y^2 + 1) \in \mathbb{F}_2[Y]$ self-reciprocal and product of two irreducible polynomials. As $\delta = \deg(f(Y))/2 = 1$ is odd, the equation $h^\natural h = f(X^2)$ has two solutions in $R \cap \mathbb{F}_4[X^2]$ namely $X^6 + X^2 + 1$ and $X^6 + X^4 + 1$. Consider $\beta \in \mathbb{F}_8$ such that $\beta^3 + \beta^2 + 1 = 0$ and u such that $u^{2^3+1} = \beta$ for example $u = v^3$ where $v^6 + v^4 + v^3 + v + 1 = 0$. The polynomial $P \in \mathbb{F}_4[Z]$ monic of degree ≤ 3 such that $P(\beta) = u$ and $P(1/\beta) = -1/u$ is $P(Z) = aZ^4 + Z^3 + Z^2 + 1$ and the unique solution (A, B) in $\mathbb{F}_4[Z] \times \mathbb{F}_4[Z]$ of $\frac{A}{B} \equiv P(Z) \pmod{f(Z)}$ with A monic is $(A, B) = (Z^3 + a, aZ^2 + a^2Z + 1)$. Therefore, $h(X) = X^6 + a + X(aX^4 + a^2X^2 + 1) = X^6 + a^2X^5 + aX^3 + X + a$ is solution to the equation $h^\natural h = f(X^2)$ in $R \setminus \mathbb{F}_4[X^2]$. The set \mathcal{H}_f is $\{X^6 + X^2 + 1, X^6 + X^4 + 1, X^6 + X^5 + aX^3 + a^2X + a, X^6 + a^2X^5 + X^4 + X^2 + aX + 1, X^6 + aX^5 + X^4 + X^2 + a^2X + 1, X^6 + X^5 + X^4 + X^3 + X^2 + X + 1, X^6 + aX^4 + aX^3 + X^2 + a, X^6 + a^2X^4 + a^2X^3 + X^2 + a^2, X^6 + aX^5 + a^2X^3 + X + a^2, X^6 + X^5 + a^2X^3 + aX + a^2, X^6 + a^2X^5 + aX^3 + X + a\}$. It has $2^\delta + 3 = 11$ elements.

Remark 3 If $f(Y)$ is irreducible then $\overline{\mathcal{H}}_{f(X^2)} = \mathcal{H}_{f(X^2)}$ and if $f(Y) = g(Y)g^\natural(Y)$ with $g(Y) \neq g^\natural(Y)$ irreducible, then $\overline{\mathcal{H}}_{f(X^2)} = \mathcal{H}_{f(X^2)} \setminus \{g(X^2), g^\natural(X^2)\}$. In both cases, $\overline{\mathcal{H}}_f$ has $p^\delta + 1$ elements.

The proposition below gives a formula for the number of self-dual θ -cyclic and θ -negacyclic codes whose dimension is prime to p . It is deduced from Lemma 6, Lemma 2 and Proposition 2.

Proposition 4 Consider p prime number, $\theta : \mathbb{F}_{p^2} \rightarrow \mathbb{F}_{p^2}, x \mapsto x^p$ the Frobenius automorphism, $k \in \mathbb{N}^*$ not divisible by p . For $\epsilon \in \{-1, 1\}$, consider $\mathcal{F}_{k, \epsilon}$ and $\mathcal{G}_{k, \epsilon}$ defined in Lemma 2. For $f(Y)$ in $\mathcal{F}_{k, \epsilon} \cup \mathcal{G}_{k, \epsilon}$, denote $\delta = \deg(f(Y))/2$ and for $p = 2$, $N_\epsilon = 1$; for p odd,

$$N_\epsilon = \begin{cases} 0 & \text{if } k \equiv 1 \pmod{2} \text{ and } \epsilon \neq (-1)^{\frac{p+1}{2}} \text{ or } k \equiv 0 \pmod{2} \text{ and } \epsilon = 1 \\ 1 & \text{if } k \equiv 0 \pmod{2} \text{ and } \epsilon = -1 \\ 2 & \text{if } k \equiv 1 \pmod{2} \text{ and } \epsilon = (-1)^{\frac{p+1}{2}}. \end{cases}$$

The number of self-dual (θ, ϵ) -constacyclic codes with dimension k defined over \mathbb{F}_{p^2} is

$$N_\epsilon \times \prod_{f(Y) \in \mathcal{F}_{k, \epsilon}} (p^\delta + 1) \times \prod_{f(Y) \in \mathcal{G}_{k, \epsilon}} (p^\delta + 3).$$

Proof. According to Lemma 2,

$$\#\mathcal{H}_{X^{2k-\epsilon}} = N_\epsilon \times \prod_{f(Y) \in \mathcal{F}_{k,\epsilon}} \#\mathcal{H}_{f(X^2)} \times \prod_{f(Y) \in \mathcal{G}_{k,\epsilon}} \#\mathcal{H}_{f(X^2)}$$

The expression of N_ϵ is deduced from Proposition 2 for p odd prime number and Remark 2 for $p = 2$. The expressions of the two products $\prod_{f(Y) \in \mathcal{F}_{k,\epsilon}} \#\mathcal{H}_{f(X^2)}$ and $\prod_{f(Y) \in \mathcal{G}_{k,\epsilon}} \#\mathcal{H}_{f(X^2)}$ come from Lemma 6. ■

Remark 4 According to Proposition 6.2 of [14], the number of self-dual 2-quasi-cyclic codes of dimension k over \mathbb{F}_{p^r} with k prime to p depends on the factorization of $Y^k - 1$ over $\mathbb{F}_{p^r}[Y]$:

$$N \prod_{f \in \mathcal{F}} (p^{r\delta} + 1) \prod_{f \in \mathcal{G}} (p^{r\delta} + 3)$$

where 2δ is the degree of f ; \mathcal{F} (resp. \mathcal{G}) is the set of all monic factors $f(Y) = f^\natural(Y)$ in $\mathbb{F}_{p^r}[Y]$ of $Y^k - 1$ with degree > 1 such that $f(Y)$ is irreducible (resp. $f(Y) = g(Y)g^\natural(Y)$ with $g(Y) \neq g^\natural(Y)$ monic irreducible in $\mathbb{F}_{p^r}[Y]$); $N = 1$ if $p = 2$; $N = 2$ if p is odd and $k \equiv 1 \pmod{2}$; $N = 4$ if p is odd and $k \equiv 0 \pmod{2}$. Therefore, for k odd number, $p = 2$ or $p \equiv 3 \pmod{4}$, the number of self-dual θ -cyclic codes over \mathbb{F}_{p^2} with dimension k is equal to the number of self-dual 2-quasi-cyclic codes over \mathbb{F}_p with dimension k .

In next subsection, the construction of the sets $\mathcal{H}_{f(X^2)^m}$ is considered.

5.2 Construction of the sets $\mathcal{H}_{f(X^2)^{p^s}}$ for $s > 0$

The following technical Lemma will be useful in the proof of Lemma 8. It can be deduced from the construction given in Lemma 6.

Lemma 7 Consider $R = \mathbb{F}_{p^2}[X; \theta]$ with p prime number, $\theta : x \mapsto x^p$, $m \in \mathbb{N}$ and $f(Y)$ in $\mathbb{F}_p[Y]$ with degree $d = 2\delta > 1$ such that $f(Y) = f^\natural(Y)$ with $f(Y)$ irreducible or product of two irreducible polynomials with degree δ . The constant coefficients of the elements of $\mathcal{H}_{f(X^2)}$ are squares in \mathbb{F}_{p^2} .

Proof.

Consider $h \in \mathcal{H}_{f(X^2)} \setminus \mathbb{F}_{p^2}[X^2]$ and $A(X^2), B(X^2) \in R$ with $B \neq 0$ such that $h(X) = A(X^2) + XB(X^2)$. According to (11), one gets $\Theta(A)(Z) \times (A^*(Z)A(Z) + B^*(Z)B(Z)Z) - (A^*(Z)\Theta(B)(Z) + B^*(Z)\Theta(A)(Z)) = \Theta(A)(Z)\lambda f(Z)$ where λ is the constant coefficient of h . As $B \neq 0$, $f(Z)$ and $A(Z)$ have no common factor, therefore, $f(Z)$ divides $A(Z)\Theta(A)(Z) - ZB(Z)\Theta(B)(Z)$. These two polynomials are monic with the same degree, therefore, they are equal and $A^* = \lambda\Theta(A)$. One deduces that $\lambda^{p+1} = 1$ and therefore, λ is a square in \mathbb{F}_{p^2} .

The elements of $\mathcal{H}_{f(X^2)} \cap \mathbb{F}_{p^2}[X^2]$ are given in proof of Lemma 6. Namely, if $f(Y)$ is irreducible,

$$\mathcal{H}_{f(X^2)} \cap \mathbb{F}_{p^2}[X^2] = \begin{cases} \emptyset & \text{if } \delta \equiv 1 \pmod{2} \\ \{\tilde{f}, \Theta(\tilde{f})\} & \text{if } \delta \equiv 0 \pmod{2}. \end{cases}$$

Denote λ the constant coefficient of \tilde{f} . As the constant coefficient of f is equal to 1 (because $f(Y) = f^\natural(Y)$ and f is monic), one has $1 = \lambda\lambda^p$, therefore λ is a square in \mathbb{F}_{p^2} .

If $f(Y) = g(Y)g^\natural(Y)$ then

$$\mathcal{H}_{f(X^2)} \cap \mathbb{F}_{p^2}[X^2] = \begin{cases} \{g, g^\natural, \tilde{g}\Theta(\tilde{g}^\natural), \tilde{g}^\natural\Theta(\tilde{g})\} & \text{if } \delta \equiv 1 \pmod{2} \\ \{g, g^\natural\} & \text{if } \delta \equiv 0 \pmod{2}. \end{cases}$$

As $g(Y)$ and $g^{\natural}(Y)$ belong to $\mathbb{F}_p[Y]$ their constant coefficients are squares in \mathbb{F}_{p^2} . Denotes μ the constant coefficient of \tilde{g} , then the constant coefficient of $\tilde{g}\Theta(\tilde{g}^{\natural})$ is $\mu\theta(\frac{1}{\mu}) = \frac{1}{\mu^{p-1}}$ which is a square in \mathbb{F}_{p^2} . ■

Lemma 8 generalizes Lemma 4 and uses the same type of arguments linked to the factorization of skew polynomials.

Lemma 8 Consider $R = \mathbb{F}_{p^2}[X; \theta]$ with p prime number, $\theta : x \mapsto x^p$, $m \in \mathbb{N}^*$ and $f(Y)$ in $\mathbb{F}_p[Y]$ with degree $d = 2\delta > 1$ in Y such that $f(Y) = f^{\natural}(Y)$ with $f(Y)$ irreducible or product of two irreducible polynomials with degree δ . The set $\overline{\mathcal{H}}_{f(X^2)^m}$ has $(1 + p^\delta)p^{\delta(m-1)}$ elements and is equal to

$$\left\{ \left(h_1 \frac{1}{\nu_1} \right) \cdots \left(h_m \frac{1}{\nu_m} \right) \left(\prod_{j=1}^m \nu_j \right) \mid h_j \in \overline{\mathcal{H}}_{f(X^2), \nu_j^2} = (h_j)_0, h_j \neq \nu_{j-1} h_{j-1}^{\natural} \frac{1}{\nu_{j-1}} \right\}.$$

Proof.

To simplify the presentation, the following notations will be used in this proof : $h = h(X)$, $f = f(X^2)$, $g = g(X^2)$ and $g^{\natural} = g^{\natural}(X^2)$.

1. Assume that $f(Y)$ is irreducible in $\mathbb{F}_p[Y]$.

Consider $h \in R$ such that $h^{\natural}h = f^m$ and $f \nmid h$. As f is irreducible in $\mathbb{F}_p[X^2]$ and central, all the irreducible factors of h are bounded by f with the same degree d (Lemma 13 (2) of [4] or [16] page 6) :

$$h = \prod_{i=1}^m H_i, H_i \text{ monic, } \deg(H_i) = d, B(H_i) = f.$$

Furthermore, f does not divide h , therefore according to Proposition 1 (or Proposition 16 of [4]), for all $j \in \{1 \dots m-1\}$, $H_j H_{j+1} \neq f$.

According to Lemma 5, d is even therefore the order of θ divides d and part 3. of Lemma 1 enables to conclude that

$$h^{\natural} = \prod_{i=m}^1 \frac{1}{\mu_i} H_i^{\natural} \mu_i$$

where $\mu_i = (H_1 \cdots H_{i-1})_0$. Furthermore, this factorization (into the product of irreducible monic polynomials of same degree d dividing f) is unique (because the factorization of h is unique).

As the factorization of f^m into the product of irreducible factors is not unique (because each factorization of f commutes), according to Proposition 1 (or Proposition 16 of [4]), $f^m = h^{\natural}h$ must have two consecutive irreducible monic factors whose product is f . As h and h^{\natural} do not possess two consecutive factors whose product is f , necessarily, $\frac{1}{\mu_1} H_1^{\natural} \mu_1 H_1 = f$ and proceeding by induction, one gets

$$\frac{1}{\mu_j} H_j^{\natural} \mu_j H_j = f \text{ and } H_{j+1} \neq \frac{1}{\mu_j} H_j^{\natural} \mu_j. \quad (14)$$

Conversely, if $h = H_1 \cdots H_m$ with $\frac{1}{\mu_j} H_j^{\natural} \mu_j H_j = f$, $H_{j+1} \neq \frac{1}{\mu_j} H_j^{\natural} \mu_j$, $\mu_j = (H_1 \cdots H_{j-1})_0$, then $h^{\natural}h = f^m$ and $H_j H_{j+1} \neq f$. Furthermore, the skew polynomials H_j are all

irreducible because they are nontrivial factors of f and $f(Y) \in \mathbb{F}_p[Y]$ is irreducible, therefore according to Proposition 1 (or Proposition 16 of [4]), the skew polynomial h is not divisible by f and it belongs to $\overline{\mathcal{H}}_{f(X^2)^m}$.

The set $\overline{\mathcal{H}}_{f^m}$ is therefore equal to the set

$$\left\{ H_1 \cdots H_m \mid \frac{1}{\mu_i} H_i^{\natural} \mu_i H_i = f, \mu_i = (H_1 \cdots H_{i-1})_0, H_{i+1} \neq \frac{1}{\mu_i} H_i^{\natural} \mu_i \right\}.$$

The conclusion follows by observing that $h = H_1 \cdots H_m$ with $\frac{1}{\mu_j} H_j^{\natural} \mu_j H_j = f$ and $H_{j+1} \neq \frac{1}{\mu_j} H_j^{\natural} \mu_j$ is equivalent to $h = \left(h_1 \frac{1}{\nu_1} \right) \cdots \left(h_m \frac{1}{\nu_m} \right) \prod_{j=1}^m \nu_j$ with $h_j^{\natural} h_j = f$ and $h_{j+1} \neq \nu_j h_j^{\natural} \frac{1}{\nu_j}$ where $h_j = (\nu_0 \cdots \nu_j) H_j \frac{1}{(\nu_0 \cdots \nu_j)}$ and ν_j is such that $\nu_j^2 = (H_j)_0 = (h_j)_0$ and ν_j belongs to \mathbb{F}_{p^2} (according to Lemma 7).

2. Assume that $f(Y) = g(Y)g^{\natural}(Y)$ with $g(Y) \neq g^{\natural}(Y)$ irreducible in $\mathbb{F}_p[Y]$. Like in the previous case it suffices to prove that $\overline{\mathcal{H}}_{f^m} = \left\{ H_1 \cdots H_m \mid \frac{1}{\mu_i} H_i^{\natural} \mu_i H_i = f, \mu_i = (H_1 \cdots H_{i-1})_0, H_{i+1} \neq \frac{1}{\mu_i} H_i^{\natural} \mu_i, g, g^{\natural} \right\}$.

Consider $h = h(X) \in R$ such that $h^{\natural} h = f^m$ and g, g^{\natural} do not divide h . Let us prove that h can be written as the product of m monic skew polynomials of degree d bounded by f . As h divides f^m , according to Theorem 4.1 of [7], $h = \text{lcm}(G, \tilde{G})$ where $G = \text{gcd}(h, g^m)$ and $\tilde{G} = \text{gcd}(h, (g^{\natural})^m)$. As $g(Y)$ (resp. $g^{\natural}(Y)$) is irreducible in $\mathbb{F}_p[Y]$, the skew polynomial G (resp. \tilde{G}) is the product of N (resp. \tilde{N}) monic irreducible skew polynomials bounded by g (resp. g^{\natural}). Without loss of generality, one can assume that $N \leq \tilde{N}$. Consider $G = G_1 \cdots G_N$ (resp. $\tilde{G} = \tilde{G}_1 \cdots \tilde{G}_{\tilde{N}}$) the factorization of G as the product of N (resp. \tilde{N}) monic irreducible factors bounded by g (resp. g^{\natural}). According to Proposition 1, as g (resp. g^{\natural}) does not divide G (resp. \tilde{G}), these factorizations are unique. Therefore, according to Theorem 14 of [17], $h = H_1 \cdots H_N$ where $H_i = \text{lcm}(\overline{G_i}, \tilde{G}_i)$ with $R/\overline{G_i}R$ and R/G_iR (resp. R/\tilde{G}_iR and R/\tilde{G}_iR) isomorphic modules. As G_i is bounded by g , according to Corollary of Theorem 10 of [10], $\overline{G_i}$ is also bounded by g . As G_i and \tilde{G}_i are right coprime with same degree $d/2$, so are $\overline{G_i}$ and \tilde{G}_i therefore H_i is a skew polynomial of degree d which divides f . Lastly, as h has degree md one gets $N = m$. Therefore

$$h = \prod_{i=1}^m H_i, H_i \text{ monic, } \deg(H_i) = d, B(H_i) = f.$$

By hypothesis $h^{\natural} h = f^m = \prod_{i=1}^m \frac{1}{\mu_i} H_i^{\natural} \mu_i \prod_{i=1}^m H_i$ is the product of $2m$ monic factors of degree d bounded by f and as the decomposition of f^m (as the product of monic factors of degree d dividing f) is not unique, according to Proposition 1, there exists two consecutive factors in $h^{\natural} h$ whose product is divisible by g or g^{\natural} . Such a product can be of three types : $\frac{1}{\mu_{i+1}} H_{i+1}^{\natural} \mu_{i+1} \frac{1}{\mu_i} H_i^{\natural} \mu_i$, $H_i H_{i+1}$ or $\frac{1}{\mu_1} H_1^{\natural} \mu_1 H_1$. However g and g^{\natural} do not divide $H_i H_{i+1}$, otherwise, they would divide h , and they do not divide $\frac{1}{\mu_{i+1}} H_{i+1}^{\natural} \mu_{i+1} \frac{1}{\mu_i} H_i^{\natural} \mu_i = \frac{1}{\mu_i} (H_i H_{i+1})^* \mu_i$ either. Therefore g or g^{\natural} divides $\frac{1}{\mu_1} H_1^{\natural} \mu_1 H_1$. Using Lemma 1, one gets that g and g^{\natural} divide $\frac{1}{\mu_1} H_1^{\natural} \mu_1 H_1$, therefore f divides $\frac{1}{\mu_1} H_1^{\natural} \mu_1 H_1$

and as these two skew polynomials are monic with the same degree they are equal. By induction, one gets

$$\frac{1}{\mu_i} H_i^{\natural} \mu_i H_i = f, H_{i+1} \neq \frac{1}{\mu_i} H_i^{\natural} \mu_i, g, g^{\natural}.$$

Conversely, consider $h = \prod_{i=1}^m H_i$, with $\frac{1}{\mu_i} H_i^{\natural} \mu_i H_i = f$, $H_{i+1} \neq \frac{1}{\mu_i} H_i^{\natural} \mu_i, g, g^{\natural}$. Accord-

ing to Lemma 1, $h^{\natural} = \prod_{i=m}^1 \frac{1}{\mu_i} H_i^{\natural} \mu_i$, therefore $h^{\natural} h = f^m$. It remains to prove that

g and g^{\natural} do not divide h . Assume that g divides h , all the factors H_i in the decomposition of h are monic, with degree d , divide f and are distinct of g, g^{\natural} , therefore, according to Proposition 1, there exists i such that g divides $H_i H_{i+1}$. Consider $u \in R$ such that $H_i H_{i+1} = gu$. As both H_i and H_{i+1} are bounded by f , they are the products of two irreducible polynomials bounded by g and g^{\natural} , therefore the skew polynomial u is the product of two irreducible skew polynomials bounded by g^{\natural} and u divides $(g^{\natural})^2$. The relation $(H_i H_{i+1})^* = (g^{\natural} u)^*$ gives $H_{i+1}^{\natural} \lambda_i H_i^{\natural} = \lambda_i u^{\natural} g^{\natural}$ where $\lambda_i := (H_i)_0$. Multiplying on the left by $\mu_{i+1} H_{i+1} \frac{1}{\mu_{i+1}}$ and on the right by $\mu_i H_i \frac{1}{\mu_i}$ yields $f^2 = (\frac{1}{\lambda_i} \mu_{i+1} H_{i+1} \frac{1}{\mu_{i+1}} \lambda_i u^{\natural} g^{\natural} \mu_i) (H_i \frac{1}{\mu_i})$. As f^2 is central, the two terms of the product commute and $f^2 = H_i (\frac{1}{\mu_i} \frac{1}{\lambda_i} \mu_{i+1}) H_{i+1} (\frac{1}{\mu_{i+1}} \lambda_i) u^{\natural} g^{\natural} \mu_i = H_i H_{i+1} \frac{1}{\mu_i} u^{\natural} g^{\natural} \mu_i = gu \frac{1}{\mu_i} u^{\natural} g^{\natural} \mu_i$ therefore $(u \frac{1}{\mu_i} u^{\natural} \mu_i) g^{\natural} = g (g^{\natural})^2 = uvg$, where $v \in R$ is such that $uv = (g^{\natural})^2$. One gets the relation $\frac{1}{\mu_i} u^{\natural} \mu_i g^{\natural} = vg$. The skew polynomials g and g^{\natural} divide vg and $\deg(vg) = \deg(f)$, therefore $f = vg$, $v = g^{\natural}$ and $u = g^{\natural}$ which is impossible because $H_i H_{i+1} \neq f$.

3. The number of elements of $\overline{\mathcal{H}}_{f(X^2)^m}$ follows from the fact that $\overline{\mathcal{H}}_{f(X^2)}$ has $1+p^{\delta}$ elements.

■

Example 3 . Consider $\mathbb{F}_4 = \mathbb{F}_2(a)$, $\theta : x \mapsto x^2$ and $f(Y) = Y^2 + Y + 1 \in \mathbb{F}_2[Y]$. According to Example 1, the set $\mathcal{H}_{f(X^2)}$ is $\{X^2 + X + 1, X^2 + a, X^2 + a^2\}$. The $6 = (1 + 2^1) \times 2^1$ skew polynomials of $\overline{\mathcal{H}}_{f(X^2)^2}$ are :

$$\begin{aligned} X^4 + X^3 + a^2 X^2 + a^2 X + a &= (X^2 + X + 1)(1/1)(X^2 + a)(1/a^2)a^2, \\ X^4 + X^3 + aX^2 + aX + a^2 &= (X^2 + X + 1)(1/1)(X^2 + a^2)(1/a)a, \\ X^4 + a^2 &= (X^2 + a)(X^2 + a) = (X^2 + a)(1/a^2)(X^2 + a)(1/a^2)a, \\ X^4 + a^2 X^3 + a^2 X^2 + X + a &= (X^2 + a)(X^2 + a^2 X + 1) = (X^2 + a)(1/a^2)(X^2 + X + 1)(1/1)a^2, \\ X^4 + a &= (X^2 + a^2)(X^2 + a^2) = (X^2 + a^2)(1/a)(X^2 + a^2)(1/a)a^2, \\ X^4 + aX^3 + aX^2 + X + a^2 &= (X^2 + a^2)(X^2 + aX + 1) = (X^2 + a^2)(1/a)(X^2 + X + 1)(1/1)a. \end{aligned}$$

Example 4 . Consider $\mathbb{F}_4 = \mathbb{F}_2(a)$, $\theta : x \mapsto x^2$ and $f(Y) = (Y^3 + Y + 1)(Y^3 + Y^2 + 1) \in \mathbb{F}_2[Y]$. There are $72 = (1+2^3) \times 2^3$ skew polynomials in $\overline{\mathcal{H}}_{f(X^2)^2}$. Here is one of these elements : $h = X^{12} + aX^{11} + a^2 X^{10} + a^2 X^7 + a^2 X^6 + X^5 + a^2 X^2 + aX + a = (h_1 \frac{1}{\nu_1}) \times (h_2 \frac{1}{\nu_2}) \times (\nu_1 \nu_2)$ where $h_1 = X^6 + X^5 + aX^3 + a^2 X + a$, $h_2 = X^6 + X^5 + X^4 + X^3 + X^2 + X + 1$ are two elements of $\overline{\mathcal{H}}_f$ (constructed in example 2), $\nu_1 = a^2$ is the square root of the constant coefficient of h_1 and $\nu_2 = 1$.

Proposition 5 Consider $R = \mathbb{F}_{p^2}[X; \theta]$ with p prime number, $\theta : x \mapsto x^p$, $s \in \mathbb{N}$ and $f(Y)$ in $\mathbb{F}_p[Y]$ with degree $d = 2\delta > 1$ in Y such that $f(Y) = f^{\natural}(Y)$.

- If $f(Y)$ is irreducible, the set $\mathcal{H}_{f(X^2)^{p^s}}$ has $\frac{p^{\delta(p^s+1)} - 1}{p^\delta - 1}$ elements.
- If $f(Y) = g(Y)g^{\natural}(Y)$ with $g(Y) \neq g^{\natural}(Y)$ irreducible in $\mathbb{F}_p[Y]$, the set $\mathcal{H}_{f(X^2)^{p^s}}$ has $\frac{(p^{\delta(p^s+1)} - 2p^s - 3)(1 + p^\delta) + 4p^s + 4}{(p^\delta - 1)^2}$ elements.

Proof.

Assume that $f(Y)$ is irreducible in $\mathbb{F}_p[Y]$. According to Lemma 3, $\mathcal{H}_{f(X^2)^{p^s}} = \bigsqcup_{i=0}^{\lfloor \frac{p^s}{2} \rfloor} f^i \overline{\mathcal{H}}_{f(X^2)^{p^s-2i}}$ and according to Lemma 8, $\overline{\mathcal{H}}_{f(X^2)^m}$ has $(1 + p^\delta)(p^\delta)^{m-1}$ if $m \neq 0$ and 1 element if $m = 0$. Therefore $\mathcal{H}_{f(X^2)^{p^s}}$ has $\sum_{i=0}^{(p^s-1)/2} (1 + p^\delta)(p^\delta)^{p^s-2i-1}$ elements if p is odd and $1 + \sum_{i=0}^{2^{s-1}-1} (1 + 2^\delta)(2^\delta)^{2^s-2i-1}$ elements otherwise. In both cases one gets $\#\mathcal{H}_{f(X^2)^{p^s}} = \frac{p^{\delta(p^s+1)} - 1}{p^\delta - 1}$. Assume that $f(Y) = g(Y)g^{\natural}(Y)$ with $g(Y) \neq g^{\natural}(Y)$ irreducible in $\mathbb{F}_p[Y]$, then according to Lemma 3,

$$\mathcal{H}_{f(X^2)^{p^s}} = \bigsqcup_{i=0}^{p^s} \bigsqcup_{j=0}^{p^s-i} g(X^2)^j g^{\natural}(X^2)^{i-j} \overline{\mathcal{H}}_{f(X^2)^{p^s-i-j}}.$$

Furthermore, the set $\overline{\mathcal{H}}_{f(X^2)^m}$ has $(p^\delta + 1)p^{\delta(m-1)}$ if $m \geq 1$ and 1 element if $m = 0$. Therefore the number of elements of the set $\mathcal{H}_{f(X^2)^{p^s}}$ is $\sum_{i=0}^{p^s} \left[\sum_{j=0}^{p^s-i-1} (1 + p^\delta)(p^\delta)^{p^s-i-1-j} + 1 \right] = \frac{(p^{\delta(p^s+1)} - 2p^s - 3)(1 + p^\delta) + 4p^s + 4}{(p^\delta - 1)^2}$.

■

Example 5 Consider $f(Y) = Y^2 + Y + 1$ defined in Example 3. The skew polynomials h satisfying $h^{\natural}h = f^2$ are $f = X^4 + X^2 + 1 \in f^1 \times \overline{\mathcal{H}}_{f^0}$ (whose factorization is not unique : $f = (X^2 + X + 1)(X^2 + X + 1) = (X^2 + a)(X^2 + a^2) = (X^2 + a^2)(X^2 + a)$) and the six skew polynomials, who have a unique factorization into the product of irreducible monic polynomials and who describe $f^0 \times \overline{\mathcal{H}}_{f(X^2)^2}$ given in Example 3.

Example 6 Consider $f(Y) = (Y^3 + Y^2 + 1)(Y^3 + Y + 1)$ with degree $d = 6$ defined in Example 4. There are 93 skew polynomials h satisfying $h^{\natural}h = (X^6 + X^4 + 1)^2(X^6 + X^2 + 1)^2$. The only one which is divisible by $X^6 + X^4 + 1$ and $X^6 + X^2 + 1$ is $(X^6 + X^4 + 1)(X^6 + X^2 + 1)$. There are 10 polynomials which are divisible by $X^6 + X^4 + 1$ without being divisible by $X^6 + X^2 + 1$; 10 other skew polynomials which are divisible by $X^6 + X^2 + 1$ without being divisible by $X^6 + X^4 + 1$ and 72 skew polynomials which are not divisible by $X^6 + X^2 + 1$ or $X^6 + X^4 + 1$.

6 Conclusion and perspectives

The following theorem gives the number of self-dual θ -cyclic and θ -negacyclic codes of any dimension over \mathbb{F}_{p^2} for θ Frobenius automorphism.

Theorem 1 Consider p prime number, $\theta : x \mapsto x^p$ Frobenius automorphism, $k \in \mathbb{N}^*$ and $s, t \in \mathbb{N}$ such that $k = p^s t$ with t not divisible by p . For $\epsilon \in \{-1, 1\}$, consider $\mathcal{F}_{k, \epsilon}$ and $\mathcal{G}_{k, \epsilon}$ defined in Lemma 2. For $f(Y)$ in $\mathcal{F}_{k, \epsilon} \cup \mathcal{G}_{k, \epsilon}$, denote $\delta = \deg(f(Y))/2$ and for $p = 2$,

$$N_\epsilon = \begin{cases} 1 & \text{if } s = 0 \\ 3 & \text{if } s > 0 \end{cases}; \text{ for } p \text{ odd,}$$

$$N_\epsilon = \begin{cases} 0 & \text{if } k \equiv 1 \pmod{2} \text{ and } \epsilon \neq (-1)^{\frac{p+1}{2}} \\ & \text{or } k \equiv 0 \pmod{2} \text{ and } \epsilon = 1 \\ 1 & \text{if } k \equiv 0 \pmod{2} \text{ and } \epsilon = -1 \\ 2 \frac{p^{(p^s+1)/2} - 1}{p-1} & \text{if } k \equiv 1 \pmod{2} \text{ and } \epsilon = (-1)^{\frac{p+1}{2}}. \end{cases}$$

The number of self-dual (θ, ϵ) -constacyclic codes of dimension k over \mathbb{F}_{p^2} is

$$N_\epsilon \times \prod_{f(Y) \in \mathcal{F}_{k,\epsilon}} \frac{p^{\delta(p^s+1)} - 1}{p^\delta - 1} \times \prod_{f(Y) \in \mathcal{G}_{k,\epsilon}} \frac{(p^{\delta(p^s+1)} - 2p^s - 3)(1 + p^\delta) + 4p^s + 4}{(p^\delta - 1)^2}.$$

Proof. According to Lemma 2,

$$\#\mathcal{H}_{X^{2k-\epsilon}} = N_\epsilon \times \prod_{f(Y) \in \mathcal{F}_{k,\epsilon}} \#\mathcal{H}_{f(X^2)^{p^s}} \times \prod_{f(Y) \in \mathcal{G}_{k,\epsilon}} \#\mathcal{H}_{f(X^2)^{p^s}}.$$

The expression of N_ϵ is deduced from Proposition 2 for p odd prime number and Remark 2 for $p = 2$. The expressions of the two products $\prod_{f(Y) \in \mathcal{F}_{k,\epsilon}} \#\mathcal{H}_{f(X^2)^{p^s}}$ and $\prod_{f(Y) \in \mathcal{G}_{k,\epsilon}} \#\mathcal{H}_{f(X^2)^{p^s}}$ come from Proposition 5. ■

Remark 5 Proposition 2 is a particular case of Theorem 1 for $t = 1$ while Proposition 4 is a particular case for $s = 0$.

To conclude, this formula should be generalized for self-dual (θ, ϵ) -codes defined over \mathbb{F}_q with θ automorphism of order 2. A future work would consist in studying the case when θ is an automorphism of order > 2 .

Lastly, one can hope that these constructions help for the study of the minimal distances of self-dual skew codes. Some experimental results and observations in dimension a power of p were made and need to be examined in more details.

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