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ACZÉLIAN n -ARY SEMIGROUPS

MIGUEL COUCEIRO AND JEAN-LUC MARICHAL

ABSTRACT. We show that the real continuous, symmetric, and cancellative n -ary semigroups are topologically order-isomorphic to additive real n -ary semigroups. The binary case ($n = 2$) was originally proved by Aczél [1]; there symmetry was redundant.

1. INTRODUCTION

Let I be a nontrivial real interval (i.e., nonempty and not a singleton) and let $n \geq 2$ be an integer. Recall that an n -ary function $f: I^n \rightarrow I$ is said to be *associative* if it solves the following system of $n - 1$ functional equations:

$$\begin{aligned} f(x_1, \dots, f(x_i, \dots, x_{i+n-1}), x_{i+n}, \dots, x_{2n-1}) \\ = f(x_1, \dots, x_i, f(x_{i+1}, \dots, x_{i+n}), \dots, x_{2n-1}), \quad i = 1, \dots, n-1. \end{aligned}$$

The pair (I, f) is then called an n -ary semigroup (see Dörnte [5] and Post [9]).

A function $f: I^n \rightarrow I$ is said to be *cancellative* if it is one-to-one in each variable; that is, for every $k \in [n] = \{1, \dots, n\}$ and every $\mathbf{x} = (x_1, \dots, x_n) \in I^n$ and $\mathbf{x}' = (x'_1, \dots, x'_n) \in I^n$,

$$(x_i = x'_i \text{ for all } i \in [n] \setminus \{k\} \text{ and } f(\mathbf{x}) = f(\mathbf{x}')) \Rightarrow x_k = x'_k.$$

Also, a function $f: I^n \rightarrow I$ is said to be *symmetric* if, for every permutation σ on $[n]$, we have $f(x_1, \dots, x_n) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)})$.

In this paper we present a complete description of those associative functions $f: I^n \rightarrow I$ which are continuous, symmetric, and cancellative. Our main result can be stated as follows.

Main Theorem. *A function $f: I^n \rightarrow I$ is continuous, symmetric, cancellative, and associative if and only if there exists a continuous and strictly monotonic function $\varphi: I \rightarrow J$ such that*

$$(1) \quad f(\mathbf{x}) = \varphi^{-1} \left(\sum_{i=1}^n \varphi(x_i) \right),$$

where J is a real interval of one of the forms $]-\infty, b[$, $]-\infty, b]$, $]a, \infty[$, $]a, \infty[$ or $\mathbb{R} =]-\infty, \infty[$ ($b \leq 0 \leq a$). For such a function f , I is necessarily open at least on one end. Moreover, φ can be chosen to be strictly increasing. In other words, the n -ary semigroup (I, f) is topologically order-isomorphic to the n -ary semigroup $(J, +)$.

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The binary case ($n = 2$) of the Main Theorem, for which symmetry is not needed, was first stated and proved by J. Aczél [1]. A shorter, more technical proof of Aczél's result was then provided by Craigen and Páles [4] (see also [2] for a recent survey). The corresponding binary semigroups are called *Aczélian* (see Ling [7, Section 3.2]).

We say that an n -ary semigroup is *Aczélian* if it satisfies the assumptions of the Main Theorem. Thus the Main Theorem provides an explicit description of the class of Aczélian n -ary semigroups. Although this result is not a trivial derivation of the binary case, we prove it by following more or less the same steps as in [4].

The following example shows that the symmetry assumption is necessary for every odd integer $n \geq 3$.

Example 1.1. Let $n \geq 3$ be an odd integer. The function $f: \mathbb{R}^n \rightarrow \mathbb{R}$, defined by

$$f(\mathbf{x}) = \sum_{i=1}^n (-1)^{i-1} x_i,$$

is continuous, cancellative, and associative. However, it cannot be of the form (1) with a continuous and strictly monotonic function φ . Indeed, if the latter would be the case, then by identifying the variables, we would have $f(x^n) = x$ and hence $\varphi(x) = \varphi(f(x^n)) = n\varphi(x)$, a contradiction.

This paper is organized as follows. In Section 2 we show how n -ary associative functions can be extended to associative functions of certain higher arities. In Section 3 we provide the proof of the Main Theorem.

To avoid cumbersome notation, we henceforth regard tuples \mathbf{x} in I^n as n -strings over I and we write $|\mathbf{x}| = n$. The 0-string or *empty* string is denoted by ε so that $I^0 = \{\varepsilon\}$. We denote by I^* the set of all strings over I , that is, $I^* = \bigcup_{n \in \mathbb{N}} I^n$, where $\mathbb{N} = \{0, 1, 2, \dots\}$. Moreover, we consider I^* endowed with concatenation for which we adopt the juxtaposition notation. For instance, if $\mathbf{x} \in I^n$, $y \in I$, and $\mathbf{z} \in I^m$, then $\mathbf{x}y\mathbf{z} \in I^{n+1+m}$.

Remark 1. Using this notation, we immediately see that a function $f: I^n \rightarrow I$ is associative if and only if we have $f(\mathbf{x}f(\mathbf{y})\mathbf{z}) = f(\mathbf{x}'f(\mathbf{y}')\mathbf{z}')$ for every $\mathbf{x}y\mathbf{z}, \mathbf{x}'y'\mathbf{z}' \in I^{2n-1}$ such that $\mathbf{y}, \mathbf{y}' \in I^n$ and $\mathbf{x}y\mathbf{z} = \mathbf{x}'y'\mathbf{z}'$. Similarly, f is cancellative if and only if, for every $\mathbf{x}z \in I^{n-1}$ and every $y, y' \in I$, the equality $f(\mathbf{x}y\mathbf{z}) = f(\mathbf{x}y'\mathbf{z})$ implies $y = y'$.

For $x \in I$, we also use the short-hand notation $x^m = x \cdots x \in I^m$. Given a function $g: I^* \rightarrow I$, we denote by g_m the restriction of g to I^m , i.e. $g_m := g|_{I^m}$. We convey that g_0 is defined by $g_0(\varepsilon) = \varepsilon$.

2. ASSOCIATIVE EXTENSIONS

Recall that a binary function $f: I^2 \rightarrow I$ is said to be *associative* if

$$f(f(xy)z) = f(xf(yz)) \quad \text{for all } x, y, z \in I.$$

Using an infix notation, we can also write this property as

$$(x \diamond y) \diamond z = x \diamond (y \diamond z) \quad \text{for all } x, y, z \in I.$$

Since associativity expresses that the order in which variables are bracketed is not relevant, it can be easily extended to functions $g: I^* \rightarrow I$ by defining

$$g_m(x_1 \cdots x_m) = x_1 \diamond \cdots \diamond x_m$$

for every integer $m \geq 2$. The latter definition can be reformulated in prefix notation as $g_2 = f$ and

$$(2) \quad g_m(x_1 \cdots x_m) = g_2(g_2(\cdots g_2(g_2(g_2(x_1 x_2) x_3) x_4) \cdots) x_m)$$

for every $m > 2$. Equivalently, we may write $g_2 = f$ and

$$g_m(x_1 \cdots x_m) = g_2(g_{m-1}(x_1 \cdots x_{m-1}) x_m)$$

for every $m > 2$.

Note that the unary function g_1 is not involved in this construction and so it could be chosen arbitrarily. However, as we will see in Proposition 2.2, it is convenient to ask g_1 to satisfy the following condition:

$$(3) \quad g_1 \circ g = g \quad \text{and} \quad g(\mathbf{x} g_1(\mathbf{y}) \mathbf{z}) = g(\mathbf{x} \mathbf{y} \mathbf{z}) \quad \text{for all } \mathbf{x} \mathbf{y} \mathbf{z} \in I^*.$$

Definition 2.1. A function $g: I^* \rightarrow I$ is said to be *associative* if

- (i) g_2 is associative,
- (ii) condition (2) holds for every $m > 2$ and every $x_1, \dots, x_m \in I$, and
- (iii) condition (3) holds.

By definition, an associative function $g: I^* \rightarrow I$ can always be constructed from a binary associative function $f: I^2 \rightarrow I$ by defining $g_2 = f$, using (2), and choosing a unary function g_1 satisfying (3) (e.g., the identity function).¹ Such a function g , which is completely determined by g_1 and $g_2 = f$, will be called an *associative extension* of f .

The following proposition provides concise reformulations of associativity of functions $g: I^* \rightarrow I$ and justifies condition (3). We will prove a more general statement in Proposition 2.5. The equivalence of assertions (ii)–(iv) was proved in [3].

Proposition 2.2. *Let $g: I^* \rightarrow I$ be a function. The following assertions are equivalent.*

- (i) g is associative.
- (ii) $g(\mathbf{x} g(\mathbf{y}) \mathbf{z}) = g(\mathbf{x}' g(\mathbf{y}') \mathbf{z}')$ for every $\mathbf{x} \mathbf{y} \mathbf{z}, \mathbf{x}' \mathbf{y}' \mathbf{z}' \in I^*$ such that $\mathbf{x} \mathbf{y} \mathbf{z} = \mathbf{x}' \mathbf{y}' \mathbf{z}'$.
- (iii) $g(\mathbf{x} g(\mathbf{y}) \mathbf{z}) = g(\mathbf{x} \mathbf{y} \mathbf{z})$ for every $\mathbf{x} \mathbf{y} \mathbf{z} \in I^*$.
- (iv) $g(g(\mathbf{x}) g(\mathbf{y})) = g(\mathbf{x} \mathbf{y})$ for every $\mathbf{x} \mathbf{y} \in I^*$.

For any integer $n \geq 2$, define the sets

$$A_n = \{m \in \mathbb{N} : m \equiv 1 \pmod{n-1}\} \quad \text{and} \quad I^{(n)} = \bigcup_{m \in A_n} I^m = I \times (I^{n-1})^*.$$

Just as associativity for binary functions can be extended to functions $g: I^* \rightarrow I$, one can also extend the associativity of n -ary functions to functions $g: I^{(n)} \rightarrow I$ as follows.² Given an associative function $f: I^n \rightarrow I$, we define $g: I^{(n)} \rightarrow I$ as $g_n = f$ and

$$(4) \quad g_m(x_1 \cdots x_m) = g_n(g_n(\cdots g_n(g_n(x_1 \cdots x_n) x_{n+1} \cdots x_{2n-1}) \cdots) x_{m-n+2} \cdots x_m)$$

for every $m \in A_n$ and $m > n$. Equivalently, we may write $g_n = f$ and

$$g_m(x_1 \cdots x_m) = g_n(g_{m-n+1}(x_1 \cdots x_{m-n+1}) x_{m-n} \cdots x_m)$$

for every $m \in A_n$ and $m > n$.

¹Note that g_1 necessarily solves the idempotency equation $g_1 \circ g_1 = g_1$.

²This construction is inspired from Dörnte [5] and Post [9].

Once again, the unary function g_1 can be chosen arbitrarily. However, we ask g_1 to satisfy the following condition:

$$(5) \quad g_1 \circ g = g \quad \text{and} \quad g(\mathbf{x}g_1(\mathbf{y})\mathbf{z}) = g(\mathbf{x}\mathbf{y}\mathbf{z}) \quad \text{for all } \mathbf{x}\mathbf{y}\mathbf{z} \in I^{(n)}.$$

Definition 2.3. A function $g: I^{(n)} \rightarrow I$ is said to be *n-associative* if

- (i) g_n is associative,
- (ii) condition (4) holds for every $m \in A_n$, $m > n$, and every $x_1, \dots, x_m \in I$, and
- (iii) condition (5) holds.

By definition, an *n-associative* function $g: I^{(n)} \rightarrow I$ can always be constructed from an *n-ary* associative function $f: I^n \rightarrow I$ by defining $g_n = f$, using (4), and choosing a unary function g_1 satisfying (5) (e.g., the identity function). Such a function g , which is completely determined by g_1 and $g_n = f$, will be called an *n-associative extension* of f .

Example 2.4. From the ternary associative function $f: \mathbb{R}^3 \rightarrow \mathbb{R}$, defined by $f(x_1x_2x_3) = x_1 - x_2 + x_3$, we can construct the 3-associative extension $g: \mathbb{R}^{(3)} \rightarrow \mathbb{R}$ as

$$g_m(x_1 \cdots x_m) = \sum_{i=1}^m (-1)^{i-1} x_i \quad (m \geq 3, \text{ odd}),$$

for which (5) provides the unique solution $g_1 = \text{id}$.

The following proposition generalizes Proposition 2.2 and provides concise reformulations of *n-associativity* of functions $g: I^{(n)} \rightarrow I$ and justifies condition (5).

Proposition 2.5. *Let $g: I^{(n)} \rightarrow I$ be a function. The following assertions are equivalent.*

- (i) g is *n-associative*.
- (ii) $g_1 \circ g = g$ and $g(\mathbf{x}g(\mathbf{y})\mathbf{z}) = g(\mathbf{x}'g(\mathbf{y}')\mathbf{z}')$ for every $\mathbf{x}\mathbf{y}\mathbf{z}, \mathbf{x}'\mathbf{y}'\mathbf{z}' \in I^{(n)}$ such that $\mathbf{y}, \mathbf{y}' \in I^{(n)}$ and $\mathbf{x}\mathbf{y}\mathbf{z} = \mathbf{x}'\mathbf{y}'\mathbf{z}'$.
- (iii) $g(\mathbf{x}g(\mathbf{y})\mathbf{z}) = g(\mathbf{x}\mathbf{y}\mathbf{z})$ for every $\mathbf{x}\mathbf{y}\mathbf{z} \in I^{(n)}$ such that $\mathbf{y} \in I^{(n)}$.
- (iv) $g_1 \circ g = g$ and $g(g(\mathbf{x}_1) \cdots g(\mathbf{x}_n)) = g(\mathbf{x}_1 \cdots \mathbf{x}_n)$ for every $\mathbf{x}_1, \dots, \mathbf{x}_n \in I^{(n)}$.

Proof. Implications (iii) \Rightarrow (i), (iii) \Rightarrow (ii), and (iii) \Rightarrow (iv) are easy to verify.

To prove (ii) \Rightarrow (iii) simply take $\mathbf{y}' = \mathbf{x}\mathbf{y}\mathbf{z}$ (i.e., $\mathbf{x}'\mathbf{z}' = \varepsilon$).

Let us now prove that (iv) \Rightarrow (iii). Let $\mathbf{x}\mathbf{y}\mathbf{z} \in I^{(n)}$ such that $\mathbf{y} \in I^{(n)}$. We write $\mathbf{x}g(\mathbf{y})\mathbf{z} = t_1 \cdots t_m$, with $t_k = g(\mathbf{y})$. By (iv) we have

$$g(\mathbf{x}g(\mathbf{y})\mathbf{z}) = g(t_1 \cdots t_m) = g(g(t_1) \cdots g(t_{n-1})g(t_n \cdots t_m)).$$

If $k \leq n-1$, then

$$\begin{aligned} g(\mathbf{x}g(\mathbf{y})\mathbf{z}) &= g(g(t_1) \cdots g(t_k) \cdots g(t_{n-1})g(t_n \cdots t_m)) \\ &= g(g(t_1) \cdots g(\mathbf{y}) \cdots g(t_{n-1})g(t_n \cdots t_m)) = g(\mathbf{x}\mathbf{y}\mathbf{z}). \end{aligned}$$

If $k \geq n$, we proceed similarly with $g(t_n \cdots t_m)$, unless $n = m$ in which case the result follows immediately.

Let us establish that (i) \Rightarrow (iii). We only need to prove that $g(\mathbf{x}g(\mathbf{y})\mathbf{z}) = g(\mathbf{x}\mathbf{y}\mathbf{z})$ for every $\mathbf{x}\mathbf{y}\mathbf{z} \in I^{(n)}$ such that $|\mathbf{y}| \geq 2$ and $|\mathbf{x}\mathbf{z}| \geq 1$. Using (4) twice and the associativity of g_n , we can rewrite the function $\mathbf{x}\mathbf{y}\mathbf{z} \mapsto g(\mathbf{x}g(\mathbf{y})\mathbf{z})$ in terms of nested g_n 's only. Then, using the associativity of g_n again, we can move all the g_n 's to the left to obtain the right-hand side of (4), which reduces to $g(\mathbf{x}\mathbf{y}\mathbf{z})$.

To illustrate, consider the following example with $n = 3$:

$$\begin{aligned} g(x_1x_2x_3g(x_4x_5x_6x_7x_8)x_9) &= g(x_1g(x_2x_3g(x_4g(x_5x_6x_7)x_8))x_9) \\ &= g(g(g(x_1x_2x_3)x_4x_5)x_6x_7)x_8x_9) \\ &= g(x_1x_2x_3x_4x_5x_6x_7x_8x_9). \quad \square \end{aligned}$$

Remark 2. Proposition 2.2 follows from Proposition 2.5. Note that the condition $g_1 \circ g = g$ is not needed in assertions (ii) and (iv) of Proposition 2.2 since I^* is used instead of $I^{(n)}$, thus allowing the use of the empty string ε .

3. PROOF OF THE MAIN THEOREM

It is easy to show that the condition in the Main Theorem is sufficient. To show that the condition is necessary, let I be a nontrivial real interval, let $f: I^n \rightarrow I$ be a continuous, symmetric, cancellative, and associative function, and let $g: I^{(n)} \rightarrow I$ be the unique n -associative extension of f such that $g_1 = \text{id}$ (see the observation following Definition 2.3).

Claim 1. f is strictly increasing in each variable.

Proof. Since f is continuous and cancellative, it must be strictly monotonic in each variable. Suppose it is strictly decreasing in the first variable. Then, by associativity, for every $\mathbf{y} \in I^{n-1}$, $u \in I$, and $\mathbf{v} \in I^{n-2}$, the unary function $x \mapsto f(f(xy)u\mathbf{v}) = f(xf(\mathbf{y}u)\mathbf{v})$ is both strictly increasing and strictly decreasing, which leads to a contradiction. Thus f must be strictly increasing in the first variable and hence in every variable by symmetry. \square

An element $e \in I$ is said to be an *idempotent* for f if $f(e^n) = e$. For instance, any real number is an idempotent for the function defined in Example 1.1.

Claim 2. There cannot be two distinct idempotents for f .

Proof. Otherwise, if d and e were distinct idempotents, we would have

$$f(de^{n-1}) = f(f(d^n)e^{n-1}) = f(df(d^{n-1}e)e^{n-2})$$

and hence (by cancellation), $e = f(d^{n-1}e) = f(e d^{n-1})$. Similarly, $d = f(e^{n-1}d) = f(de^{n-1})$. Now, if $e < d$, then $d = f(de^{n-1}) < f(d^{n-1}e) = e$ (by Claim 1), a contradiction. We arrive at a similar contradiction if $d < e$. \square

Because of Claim 2, there is a $c \in I$ such that either $c < f(c^n)$ or $c > f(c^n)$. We assume w.l.o.g. that the former holds and fix such a c . The latter case can be dealt with similarly.

Claim 3. For all fixed $x \in I$, we have $x < f(xc^{n-1})$. Thus the sequence $x_m = f(x_{m-1}c^{n-1})$ strictly increases, and $\lim x_m \notin I$ (hence $\lim x_m = \sup I$ and I is open from above).

Proof. Since $c < f(c^n)$, we have $f(cx^{n-1}) < f(f(c^n)x^{n-1}) = f(cf(c^{n-1}x)x^{n-2})$ and hence (by strict monotonicity) $x < f(c^{n-1}x) = f(xc^{n-1})$. Thus $x_m = f(x_{m-1}c^{n-1}) > x_{m-1}$. If $\lim x_m = x'$ and $x' \in I$, continuity gives the following:

$$x' = \lim x_m = \lim f(x_{m-1}c^{n-1}) = f(\lim x_{m-1}c^{n-1}) = f(x'c^{n-1}),$$

a contradiction. Thus $x' \notin I$, so $\lim x_m = \sup I$. \square

Hereinafter we work on the extended real line so that suprema of arbitrary sets exist and all monotone sequences converge.

Claim 4. Let $x \in I$ and let $j, k, p, q \in \mathbb{N}$ such that $j+1, k, p, q+1 \in A_n$. Then we have

$$g(c^p) > g(x c^q) \iff g(c^{kp}) > g(x^k c^{kq}) \iff g(c^{p+j}) > g(x c^{q+j}).$$

The same equivalence holds if “<” or “=” replaces “>”.

Proof. Assume that $g(c^p) > g(x c^q)$. Then, by Proposition 2.5(iv), Claim 1, and symmetry, we have $g(c^{kp}) = g(g(c^p)^k) > g(g(x c^q)^k) = g(x^k c^{kq})$, which proves the first equivalence (since the same conclusion clearly holds if “<” or “=” replaces “>”). For the second equivalence, assume again that $g(c^p) > g(x c^q)$. Then, as before, we have $g(c^{p+j}) = g(g(c^p) c^j) > g(g(x c^q) c^j) = g(x c^{q+j})$. \square

Let x be any fixed element of I . Define S_x to be the subset of all rational numbers r for which there exist $k, p, q \in \mathbb{N}$ such that $k, p, q+1 \in A_n$, $g(c^p) > g(x^k c^q)$, and $r = (p-q)/k$. Now, if $r = (p-q)/k = (p'-q')/k'$, then we have $pk' + q'k = p'k + qk'$ and it follows from Claim 4 that

$$\begin{aligned} g(c^p) > g(x^k c^q) &\iff g(c^{pk'}) > g(x^{kk'} c^{qk'}) \\ &\iff g(c^{pk'+q'k}) > g(x^{kk'} c^{qk'+q'k}) \\ &\iff g(c^{p'k+qk'}) > g(x^{kk'} c^{q'k+qk'}) \\ &\iff g(c^{p'k}) > g(x^{kk'} c^{q'k}) \\ &\iff g(c^{p'}) > g(x^{k'} c^{q'}). \end{aligned}$$

Hence S_x is in fact the subset of rational numbers r for which every representation $r = (p-q)/k$ with $k, p, q+1 \in A_n$ satisfies $g(c^p) > g(x^k c^q)$.

Claim 5. The set $S = \{\frac{p-q}{k} : k, p, q+1 \in A_n\}$ is dense in \mathbb{R} .

Proof. For every $a, b \in \mathbb{N}$, the sequence

$$x_m = \frac{1 \pm a m (n-1)}{1 + b m (n-1)}$$

of S converges to $\pm a/b$. Thus S is dense in \mathbb{Q} and hence (by transitivity) in \mathbb{R} . \square

Claim 6. Any two numbers $r, r' \in S$ may be written $r = (p-q)/k$, $r' = (p'-q)/k$ for suitable $k, p, p', q+1 \in A_n$.

Proof. Let $r = (p-q)/k$ and $r' = (p'-q')/k'$, with $k, k', p, p', q+1, q'+1 \in A_n$. Assume w.l.o.g. that $r' > r$. Setting $\tilde{k} = k k'$, $\tilde{q} = |\tilde{k} r - 1|$, $\tilde{p} = \tilde{k} r + \tilde{q}$, and $\tilde{p}' = \tilde{k} r' + \tilde{q}$, we have $r = (\tilde{p} - \tilde{q})/\tilde{k}$, $r' = (\tilde{p}' - \tilde{q})/\tilde{k}$ with $\tilde{k}, \tilde{p}, \tilde{p}', \tilde{q} + 1 \in A_n$. \square

Claim 7. S_x is a nonempty, proper, and upper subset of S (“upper” means that if $r \in S_x$ and $r' \in S$, $r' > r$, then $r' \in S_x$).

Proof. To show that S_x is an upper subset, let $r = (p-q)/k \in S_x$ and $r' = (p'-q)/k > r$ (cf. Claim 6). Then $p' > p$ and, since $p, p' \in A_n$, we have $p' = p + j(n-1)$ for some integer $j \geq 1$. Using the definition of S_x and the first part of Claim 3, we obtain

$$\begin{aligned} g(x^k c^q) < g(c^p) &< g(g(c^p) c^{n-1}) = g(c^p c^{n-1}) \\ &< g(g(c^p c^{n-1}) c^{n-1}) = g(c^p c^{2(n-1)}) \\ &< \dots \\ &< g(c^p c^{j(n-1)}) = g(c^{p'}). \end{aligned}$$

Hence $r' \in S_x$. Now, by Claim 3, $\lim f(c^{m(n-1)+1}) = \sup I > g(xc^{n-1})$, and hence there is some $p \in A_n$ with $g(c^p) > g(xc^{n-1})$. Hence $r = (p - (n - 1))/1 \in S_x$, and so S_x is nonempty. Similarly, since $\lim g(xc^{m(n-1)}) = \sup I$, there must a q such that $q + 1 \in A_n$ and $g(c) < g(xc^q)$, and so $(1 - q)/1 \notin S_x$. \square

Now, by Claim 7, S_x is precisely the set of elements in S which are greater than (and possibly equal to) $\inf S_x$. Using this fact, let $\varphi: I \rightarrow \mathbb{R}$ be the function given by

$$\varphi(x) := \inf S_x.$$

Claim 8. If $g(c^p) = g(x^k c^q)$, then $\varphi(x) = (p - q)/k$. In particular, $\varphi(c) = 1$.

Proof. Note that $g(c^p) = g(x^k c^q)$ implies $r = (p - q)/k \notin S_x$. Moreover, by Claim 7 it follows that if $r' = (p' - q)/k > r$ (resp. $r' < r$), then $g(c^{p'}) > g(c^p) = g(x^k c^q)$ (resp. $g(c^{p'}) < g(c^p) = g(x^k c^q)$), and hence $r' \in S_x$ (resp. $r' \notin S_x$). Thus $\inf S_x = (p - q)/k$ by Claim 5. For the last claim just note that $g(c^{q+1}) = g(c c^q)$. \square

Claim 9. We have $\varphi(g(x_1 \cdots x_n)) = \sum_{i=1}^n \varphi(x_i)$ for every $x_1, \dots, x_n \in I$.

Proof. Let $r_i = (p_i - q)/k > \varphi(x_i)$ for all $i \in [n]$. Then $g(c^{p_i}) > g(x_i^k c^q)$, and by Proposition 2.5(iv), Claim 1, and symmetry, we have

$$g(c^{\sum_{i=1}^n p_i}) = g(g(c^{p_1}) \cdots g(c^{p_n})) > g(g(x_1^k c^q) \cdots g(x_n^k c^q)) = g(g(x_1 \cdots x_n)^k c^{nq}).$$

By Claim 8, $(\sum_{i=1}^n p_i - nq)/k \in S_{g(x_1 \cdots x_n)}$. Thus $\sum_{i=1}^n r_i > \varphi(g(x_1 \cdots x_n))$. Similarly, if $r_i \leq \varphi(x_i)$ for all $i \in [n]$, then $\sum_{i=1}^n r_i \leq \varphi(g(x_1 \cdots x_n))$. The result then follows from Claim 5. \square

Claim 10. φ is nondecreasing.

Proof. Suppose $y > x$ and $(p - q)/k \in S_y$. Then $g(c^p) > g(y^k c^q) > g(x^k c^q)$ and hence $S_y \subseteq S_x$ and so $\varphi(y) = \inf S_y \geq \inf S_x = \varphi(x)$. \square

Claim 11. φ is continuous.

Proof. Since φ is nondecreasing, the only possible sort of discontinuity is a gap discontinuity. Hence, if φ is discontinuous, there must exist $x, y \in I$, say $x < y$, and an interval, and thus a rational $r \notin \varphi(I)$, such that $\varphi(x) < r < \varphi(y)$. Now if $r = (p - q)/k$, then $g(x^k c^q) < g(c^p) \leq g(y^k c^q)$. By continuity of g_{k+q} , there is $t \in]x, y]$ such that $g(c^p) = g(t^k c^q)$. By Claim 8 it then follows that $\varphi(t) = r$, which yields the desired contradiction. \square

Claim 12. φ is strictly increasing.

Proof. For the sake of contradiction, suppose that there are $x, y \in I$ such that $x < y$ and $\varphi(x) = \varphi(y) = a$. Since φ is nondecreasing, there is an interval I' containing x and y , and such that $\varphi(z) = a$, for all $z \in I'$. Let I' be the largest interval having this property, and set $t = \sup I'$. If $t \notin I$, then for every $z > x$, $\varphi(z) = a$. Now $g(xc^{n-1}) > x$ (by Claim 3) and hence $a = \varphi(g(xc^{n-1})) = a + (n - 1) > a$ (by Claim 9), a contradiction. Thus $t \in I$, and $\varphi(t) = a$ by Claim 11. We have $g(xt^{n-1}) < g(t^n)$ and, by Claim 3, there exists q such that $q + 1 \in A_n$ and $g(t^n) < g(xc^{q(n-1)}) = g(xg(c^q)^{n-1})$ and $g(c^q) > t$. By continuity of g_n , there is $z \in I$ such that $t < z < g(c^q)$ (and so $z \notin I'$) and $g(xz^{n-1}) = g(t^n)$. Thus

$$a + (n - 1) \varphi(z) = \varphi(x) + (n - 1) \varphi(z) = \varphi(g(xz^{n-1})) = \varphi(g(t^n)) = n \varphi(t) = n a,$$

and we obtain $\varphi(z) = a$, so $z \in I'$, a contradiction. \square

Thus φ is a continuous strictly increasing n -ary semigroup homomorphism and, by Claim 9, its range J is a connected real additive n -ary semigroup. Hence the only possibilities for J are $]-\infty, b[$, $]-\infty, b]$, $]a, \infty[$, $]a, \infty[$ or $]-\infty, \infty[$ ($b \leq 0 \leq a$); see final comments in [4]. This completes the proof of the Main Theorem. \square

Remark 3. The function φ is determined up to a multiplicative constant, that is, with φ all functions $\psi = r\varphi$ ($r \neq 0$) belong to the same function f , and only these; see the “Uniqueness” section in [2].

Remark 4. An n -ary semigroup (I, f) is said to be *reducible to* (or *derived from*) a binary semigroup (I, \diamond) if there is an associative extension $g: I^* \rightarrow I$ of \diamond such that $g_n = f$; that is, $f(x_1 \cdots x_n) = x_1 \diamond \cdots \diamond x_n$ (see [5, 9]). Dudek and Mukhin [6] showed that an n -ary semigroup is reducible if and only if we can adjoint an n -ary neutral element to it. This shows that the n -ary semigroup given in Example 1.1 is not reducible since we cannot adjoint any n -ary neutral element (for an alternative proof, see [8]). However, the Main Theorem shows that every Aczélian n -ary semigroup is reducible and hence we can always adjoint an n -ary neutral element to it (if $0 \in J$, then the neutral element is $e = \varphi^{-1}(0)$; otherwise fix $e \notin I$ and extend φ to $\varphi': I \cup \{e\} \rightarrow J \cup \{0\}$ by the rule $\varphi'(x) = \varphi(x)$ if $x \in I$ and $\varphi'(e) = 0$).

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