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# ACZÉLIAN $n$ -ARY SEMIGROUPS

MIGUEL COUCEIRO AND JEAN-LUC MARICHAL

ABSTRACT. We show that the real continuous, symmetric, and cancellative  $n$ -ary semigroups are topologically order-isomorphic to additive real  $n$ -ary semigroups. The binary case ( $n = 2$ ) was originally proved by Aczél [1]; there symmetry was redundant.

## 1. INTRODUCTION

Let  $I$  be a nontrivial real interval (i.e., nonempty and not a singleton) and let  $n \geq 2$  be an integer. Recall that an  $n$ -ary function  $f: I^n \rightarrow I$  is said to be *associative* if it solves the following system of  $n - 1$  functional equations:

$$\begin{aligned} f(x_1, \dots, f(x_i, \dots, x_{i+n-1}), x_{i+n}, \dots, x_{2n-1}) \\ = f(x_1, \dots, x_i, f(x_{i+1}, \dots, x_{i+n}), \dots, x_{2n-1}), \quad i = 1, \dots, n-1. \end{aligned}$$

The pair  $(I, f)$  is then called an  $n$ -ary semigroup (see Dörnte [5] and Post [9]).

A function  $f: I^n \rightarrow I$  is said to be *cancellative* if it is one-to-one in each variable; that is, for every  $k \in [n] = \{1, \dots, n\}$  and every  $\mathbf{x} = (x_1, \dots, x_n) \in I^n$  and  $\mathbf{x}' = (x'_1, \dots, x'_n) \in I^n$ ,

$$(x_i = x'_i \text{ for all } i \in [n] \setminus \{k\} \text{ and } f(\mathbf{x}) = f(\mathbf{x}')) \Rightarrow x_k = x'_k.$$

Also, a function  $f: I^n \rightarrow I$  is said to be *symmetric* if, for every permutation  $\sigma$  on  $[n]$ , we have  $f(x_1, \dots, x_n) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)})$ .

In this paper we present a complete description of those associative functions  $f: I^n \rightarrow I$  which are continuous, symmetric, and cancellative. Our main result can be stated as follows.

**Main Theorem.** *A function  $f: I^n \rightarrow I$  is continuous, symmetric, cancellative, and associative if and only if there exists a continuous and strictly monotonic function  $\varphi: I \rightarrow J$  such that*

$$(1) \quad f(\mathbf{x}) = \varphi^{-1} \left( \sum_{i=1}^n \varphi(x_i) \right),$$

where  $J$  is a real interval of one of the forms  $]-\infty, b[$ ,  $]-\infty, b]$ ,  $]a, \infty[$ ,  $]a, \infty[$  or  $\mathbb{R} = ]-\infty, \infty[$  ( $b \leq 0 \leq a$ ). For such a function  $f$ ,  $I$  is necessarily open at least on one end. Moreover,  $\varphi$  can be chosen to be strictly increasing. In other words, the  $n$ -ary semigroup  $(I, f)$  is topologically order-isomorphic to the  $n$ -ary semigroup  $(J, +)$ .

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The binary case ( $n = 2$ ) of the Main Theorem, for which symmetry is not needed, was first stated and proved by J. Aczél [1]. A shorter, more technical proof of Aczél's result was then provided by Craigen and Páles [4] (see also [2] for a recent survey). The corresponding binary semigroups are called *Aczélian* (see Ling [7, Section 3.2]).

We say that an  $n$ -ary semigroup is *Aczélian* if it satisfies the assumptions of the Main Theorem. Thus the Main Theorem provides an explicit description of the class of Aczélian  $n$ -ary semigroups. Although this result is not a trivial derivation of the binary case, we prove it by following more or less the same steps as in [4].

The following example shows that the symmetry assumption is necessary for every odd integer  $n \geq 3$ .

**Example 1.1.** Let  $n \geq 3$  be an odd integer. The function  $f: \mathbb{R}^n \rightarrow \mathbb{R}$ , defined by

$$f(\mathbf{x}) = \sum_{i=1}^n (-1)^{i-1} x_i,$$

is continuous, cancellative, and associative. However, it cannot be of the form (1) with a continuous and strictly monotonic function  $\varphi$ . Indeed, if the latter would be the case, then by identifying the variables, we would have  $f(x^n) = x$  and hence  $\varphi(x) = \varphi(f(x^n)) = n\varphi(x)$ , a contradiction.

This paper is organized as follows. In Section 2 we show how  $n$ -ary associative functions can be extended to associative functions of certain higher arities. In Section 3 we provide the proof of the Main Theorem.

To avoid cumbersome notation, we henceforth regard tuples  $\mathbf{x}$  in  $I^n$  as  $n$ -strings over  $I$  and we write  $|\mathbf{x}| = n$ . The 0-string or *empty* string is denoted by  $\varepsilon$  so that  $I^0 = \{\varepsilon\}$ . We denote by  $I^*$  the set of all strings over  $I$ , that is,  $I^* = \bigcup_{n \in \mathbb{N}} I^n$ , where  $\mathbb{N} = \{0, 1, 2, \dots\}$ . Moreover, we consider  $I^*$  endowed with concatenation for which we adopt the juxtaposition notation. For instance, if  $\mathbf{x} \in I^n$ ,  $y \in I$ , and  $\mathbf{z} \in I^m$ , then  $\mathbf{x}y\mathbf{z} \in I^{n+1+m}$ .

*Remark 1.* Using this notation, we immediately see that a function  $f: I^n \rightarrow I$  is associative if and only if we have  $f(\mathbf{x}f(\mathbf{y})\mathbf{z}) = f(\mathbf{x}'f(\mathbf{y}')\mathbf{z}')$  for every  $\mathbf{x}y\mathbf{z}, \mathbf{x}'y'\mathbf{z}' \in I^{2n-1}$  such that  $\mathbf{y}, \mathbf{y}' \in I^n$  and  $\mathbf{x}y\mathbf{z} = \mathbf{x}'y'\mathbf{z}'$ . Similarly,  $f$  is cancellative if and only if, for every  $\mathbf{x}z \in I^{n-1}$  and every  $y, y' \in I$ , the equality  $f(\mathbf{x}y\mathbf{z}) = f(\mathbf{x}y'\mathbf{z})$  implies  $y = y'$ .

For  $x \in I$ , we also use the short-hand notation  $x^m = x \cdots x \in I^m$ . Given a function  $g: I^* \rightarrow I$ , we denote by  $g_m$  the restriction of  $g$  to  $I^m$ , i.e.  $g_m := g|_{I^m}$ . We convey that  $g_0$  is defined by  $g_0(\varepsilon) = \varepsilon$ .

## 2. ASSOCIATIVE EXTENSIONS

Recall that a binary function  $f: I^2 \rightarrow I$  is said to be *associative* if

$$f(f(xy)z) = f(xf(yz)) \quad \text{for all } x, y, z \in I.$$

Using an infix notation, we can also write this property as

$$(x \diamond y) \diamond z = x \diamond (y \diamond z) \quad \text{for all } x, y, z \in I.$$

Since associativity expresses that the order in which variables are bracketed is not relevant, it can be easily extended to functions  $g: I^* \rightarrow I$  by defining

$$g_m(x_1 \cdots x_m) = x_1 \diamond \cdots \diamond x_m$$

for every integer  $m \geq 2$ . The latter definition can be reformulated in prefix notation as  $g_2 = f$  and

$$(2) \quad g_m(x_1 \cdots x_m) = g_2(g_2(\cdots g_2(g_2(g_2(x_1 x_2) x_3) x_4) \cdots) x_m)$$

for every  $m > 2$ . Equivalently, we may write  $g_2 = f$  and

$$g_m(x_1 \cdots x_m) = g_2(g_{m-1}(x_1 \cdots x_{m-1}) x_m)$$

for every  $m > 2$ .

Note that the unary function  $g_1$  is not involved in this construction and so it could be chosen arbitrarily. However, as we will see in Proposition 2.2, it is convenient to ask  $g_1$  to satisfy the following condition:

$$(3) \quad g_1 \circ g = g \quad \text{and} \quad g(\mathbf{x} g_1(\mathbf{y}) \mathbf{z}) = g(\mathbf{x} \mathbf{y} \mathbf{z}) \quad \text{for all } \mathbf{x} \mathbf{y} \mathbf{z} \in I^*.$$

**Definition 2.1.** A function  $g: I^* \rightarrow I$  is said to be *associative* if

- (i)  $g_2$  is associative,
- (ii) condition (2) holds for every  $m > 2$  and every  $x_1, \dots, x_m \in I$ , and
- (iii) condition (3) holds.

By definition, an associative function  $g: I^* \rightarrow I$  can always be constructed from a binary associative function  $f: I^2 \rightarrow I$  by defining  $g_2 = f$ , using (2), and choosing a unary function  $g_1$  satisfying (3) (e.g., the identity function).<sup>1</sup> Such a function  $g$ , which is completely determined by  $g_1$  and  $g_2 = f$ , will be called an *associative extension* of  $f$ .

The following proposition provides concise reformulations of associativity of functions  $g: I^* \rightarrow I$  and justifies condition (3). We will prove a more general statement in Proposition 2.5. The equivalence of assertions (ii)–(iv) was proved in [3].

**Proposition 2.2.** *Let  $g: I^* \rightarrow I$  be a function. The following assertions are equivalent.*

- (i)  $g$  is associative.
- (ii)  $g(\mathbf{x} g(\mathbf{y}) \mathbf{z}) = g(\mathbf{x}' g(\mathbf{y}') \mathbf{z}')$  for every  $\mathbf{x} \mathbf{y} \mathbf{z}, \mathbf{x}' \mathbf{y}' \mathbf{z}' \in I^*$  such that  $\mathbf{x} \mathbf{y} \mathbf{z} = \mathbf{x}' \mathbf{y}' \mathbf{z}'$ .
- (iii)  $g(\mathbf{x} g(\mathbf{y}) \mathbf{z}) = g(\mathbf{x} \mathbf{y} \mathbf{z})$  for every  $\mathbf{x} \mathbf{y} \mathbf{z} \in I^*$ .
- (iv)  $g(g(\mathbf{x}) g(\mathbf{y})) = g(\mathbf{x} \mathbf{y})$  for every  $\mathbf{x} \mathbf{y} \in I^*$ .

For any integer  $n \geq 2$ , define the sets

$$A_n = \{m \in \mathbb{N} : m \equiv 1 \pmod{n-1}\} \quad \text{and} \quad I^{(n)} = \bigcup_{m \in A_n} I^m = I \times (I^{n-1})^*.$$

Just as associativity for binary functions can be extended to functions  $g: I^* \rightarrow I$ , one can also extend the associativity of  $n$ -ary functions to functions  $g: I^{(n)} \rightarrow I$  as follows.<sup>2</sup> Given an associative function  $f: I^n \rightarrow I$ , we define  $g: I^{(n)} \rightarrow I$  as  $g_n = f$  and

$$(4) \quad g_m(x_1 \cdots x_m) = g_n(g_n(\cdots g_n(g_n(x_1 \cdots x_n) x_{n+1} \cdots x_{2n-1}) \cdots) x_{m-n+2} \cdots x_m)$$

for every  $m \in A_n$  and  $m > n$ . Equivalently, we may write  $g_n = f$  and

$$g_m(x_1 \cdots x_m) = g_n(g_{m-n+1}(x_1 \cdots x_{m-n+1}) x_{m-n} \cdots x_m)$$

for every  $m \in A_n$  and  $m > n$ .

<sup>1</sup>Note that  $g_1$  necessarily solves the idempotency equation  $g_1 \circ g_1 = g_1$ .

<sup>2</sup>This construction is inspired from Dörnte [5] and Post [9].

Once again, the unary function  $g_1$  can be chosen arbitrarily. However, we ask  $g_1$  to satisfy the following condition:

$$(5) \quad g_1 \circ g = g \quad \text{and} \quad g(\mathbf{x}g_1(\mathbf{y})\mathbf{z}) = g(\mathbf{x}\mathbf{y}\mathbf{z}) \quad \text{for all } \mathbf{x}\mathbf{y}\mathbf{z} \in I^{(n)}.$$

**Definition 2.3.** A function  $g: I^{(n)} \rightarrow I$  is said to be *n-associative* if

- (i)  $g_n$  is associative,
- (ii) condition (4) holds for every  $m \in A_n$ ,  $m > n$ , and every  $x_1, \dots, x_m \in I$ , and
- (iii) condition (5) holds.

By definition, an *n-associative* function  $g: I^{(n)} \rightarrow I$  can always be constructed from an *n-ary* associative function  $f: I^n \rightarrow I$  by defining  $g_n = f$ , using (4), and choosing a unary function  $g_1$  satisfying (5) (e.g., the identity function). Such a function  $g$ , which is completely determined by  $g_1$  and  $g_n = f$ , will be called an *n-associative extension* of  $f$ .

**Example 2.4.** From the ternary associative function  $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ , defined by  $f(x_1x_2x_3) = x_1 - x_2 + x_3$ , we can construct the 3-associative extension  $g: \mathbb{R}^{(3)} \rightarrow \mathbb{R}$  as

$$g_m(x_1 \cdots x_m) = \sum_{i=1}^m (-1)^{i-1} x_i \quad (m \geq 3, \text{ odd}),$$

for which (5) provides the unique solution  $g_1 = \text{id}$ .

The following proposition generalizes Proposition 2.2 and provides concise reformulations of *n-associativity* of functions  $g: I^{(n)} \rightarrow I$  and justifies condition (5).

**Proposition 2.5.** *Let  $g: I^{(n)} \rightarrow I$  be a function. The following assertions are equivalent.*

- (i)  $g$  is *n-associative*.
- (ii)  $g_1 \circ g = g$  and  $g(\mathbf{x}g(\mathbf{y})\mathbf{z}) = g(\mathbf{x}'g(\mathbf{y}')\mathbf{z}')$  for every  $\mathbf{x}\mathbf{y}\mathbf{z}, \mathbf{x}'\mathbf{y}'\mathbf{z}' \in I^{(n)}$  such that  $\mathbf{y}, \mathbf{y}' \in I^{(n)}$  and  $\mathbf{x}\mathbf{y}\mathbf{z} = \mathbf{x}'\mathbf{y}'\mathbf{z}'$ .
- (iii)  $g(\mathbf{x}g(\mathbf{y})\mathbf{z}) = g(\mathbf{x}\mathbf{y}\mathbf{z})$  for every  $\mathbf{x}\mathbf{y}\mathbf{z} \in I^{(n)}$  such that  $\mathbf{y} \in I^{(n)}$ .
- (iv)  $g_1 \circ g = g$  and  $g(g(\mathbf{x}_1) \cdots g(\mathbf{x}_n)) = g(\mathbf{x}_1 \cdots \mathbf{x}_n)$  for every  $\mathbf{x}_1, \dots, \mathbf{x}_n \in I^{(n)}$ .

*Proof.* Implications (iii)  $\Rightarrow$  (i), (iii)  $\Rightarrow$  (ii), and (iii)  $\Rightarrow$  (iv) are easy to verify.

To prove (ii)  $\Rightarrow$  (iii) simply take  $\mathbf{y}' = \mathbf{x}\mathbf{y}\mathbf{z}$  (i.e.,  $\mathbf{x}'\mathbf{z}' = \varepsilon$ ).

Let us now prove that (iv)  $\Rightarrow$  (iii). Let  $\mathbf{x}\mathbf{y}\mathbf{z} \in I^{(n)}$  such that  $\mathbf{y} \in I^{(n)}$ . We write  $\mathbf{x}g(\mathbf{y})\mathbf{z} = t_1 \cdots t_m$ , with  $t_k = g(\mathbf{y})$ . By (iv) we have

$$g(\mathbf{x}g(\mathbf{y})\mathbf{z}) = g(t_1 \cdots t_m) = g(g(t_1) \cdots g(t_{n-1})g(t_n \cdots t_m)).$$

If  $k \leq n-1$ , then

$$\begin{aligned} g(\mathbf{x}g(\mathbf{y})\mathbf{z}) &= g(g(t_1) \cdots g(t_k) \cdots g(t_{n-1})g(t_n \cdots t_m)) \\ &= g(g(t_1) \cdots g(\mathbf{y}) \cdots g(t_{n-1})g(t_n \cdots t_m)) = g(\mathbf{x}\mathbf{y}\mathbf{z}). \end{aligned}$$

If  $k \geq n$ , we proceed similarly with  $g(t_n \cdots t_m)$ , unless  $n = m$  in which case the result follows immediately.

Let us establish that (i)  $\Rightarrow$  (iii). We only need to prove that  $g(\mathbf{x}g(\mathbf{y})\mathbf{z}) = g(\mathbf{x}\mathbf{y}\mathbf{z})$  for every  $\mathbf{x}\mathbf{y}\mathbf{z} \in I^{(n)}$  such that  $|\mathbf{y}| \geq 2$  and  $|\mathbf{x}\mathbf{z}| \geq 1$ . Using (4) twice and the associativity of  $g_n$ , we can rewrite the function  $\mathbf{x}\mathbf{y}\mathbf{z} \mapsto g(\mathbf{x}g(\mathbf{y})\mathbf{z})$  in terms of nested  $g_n$ 's only. Then, using the associativity of  $g_n$  again, we can move all the  $g_n$ 's to the left to obtain the right-hand side of (4), which reduces to  $g(\mathbf{x}\mathbf{y}\mathbf{z})$ .

To illustrate, consider the following example with  $n = 3$ :

$$\begin{aligned} g(x_1x_2x_3g(x_4x_5x_6x_7x_8)x_9) &= g(x_1g(x_2x_3g(x_4g(x_5x_6x_7)x_8))x_9) \\ &= g(g(g(x_1x_2x_3)x_4x_5)x_6x_7)x_8x_9) \\ &= g(x_1x_2x_3x_4x_5x_6x_7x_8x_9). \quad \square \end{aligned}$$

*Remark 2.* Proposition 2.2 follows from Proposition 2.5. Note that the condition  $g_1 \circ g = g$  is not needed in assertions (ii) and (iv) of Proposition 2.2 since  $I^*$  is used instead of  $I^{(n)}$ , thus allowing the use of the empty string  $\varepsilon$ .

### 3. PROOF OF THE MAIN THEOREM

It is easy to show that the condition in the Main Theorem is sufficient. To show that the condition is necessary, let  $I$  be a nontrivial real interval, let  $f: I^n \rightarrow I$  be a continuous, symmetric, cancellative, and associative function, and let  $g: I^{(n)} \rightarrow I$  be the unique  $n$ -associative extension of  $f$  such that  $g_1 = \text{id}$  (see the observation following Definition 2.3).

*Claim 1.*  $f$  is strictly increasing in each variable.

*Proof.* Since  $f$  is continuous and cancellative, it must be strictly monotonic in each variable. Suppose it is strictly decreasing in the first variable. Then, by associativity, for every  $\mathbf{y} \in I^{n-1}$ ,  $u \in I$ , and  $\mathbf{v} \in I^{n-2}$ , the unary function  $x \mapsto f(f(xy)u\mathbf{v}) = f(xf(\mathbf{y}u)\mathbf{v})$  is both strictly increasing and strictly decreasing, which leads to a contradiction. Thus  $f$  must be strictly increasing in the first variable and hence in every variable by symmetry.  $\square$

An element  $e \in I$  is said to be an *idempotent* for  $f$  if  $f(e^n) = e$ . For instance, any real number is an idempotent for the function defined in Example 1.1.

*Claim 2.* There cannot be two distinct idempotents for  $f$ .

*Proof.* Otherwise, if  $d$  and  $e$  were distinct idempotents, we would have

$$f(de^{n-1}) = f(f(d^n)e^{n-1}) = f(df(d^{n-1}e)e^{n-2})$$

and hence (by cancellation),  $e = f(d^{n-1}e) = f(e d^{n-1})$ . Similarly,  $d = f(e^{n-1}d) = f(de^{n-1})$ . Now, if  $e < d$ , then  $d = f(de^{n-1}) < f(d^{n-1}e) = e$  (by Claim 1), a contradiction. We arrive at a similar contradiction if  $d < e$ .  $\square$

Because of Claim 2, there is a  $c \in I$  such that either  $c < f(c^n)$  or  $c > f(c^n)$ . We assume w.l.o.g. that the former holds and fix such a  $c$ . The latter case can be dealt with similarly.

*Claim 3.* For all fixed  $x \in I$ , we have  $x < f(xc^{n-1})$ . Thus the sequence  $x_m = f(x_{m-1}c^{n-1})$  strictly increases, and  $\lim x_m \notin I$  (hence  $\lim x_m = \sup I$  and  $I$  is open from above).

*Proof.* Since  $c < f(c^n)$ , we have  $f(cx^{n-1}) < f(f(c^n)x^{n-1}) = f(cf(c^{n-1}x)x^{n-2})$  and hence (by strict monotonicity)  $x < f(c^{n-1}x) = f(xc^{n-1})$ . Thus  $x_m = f(x_{m-1}c^{n-1}) > x_{m-1}$ . If  $\lim x_m = x'$  and  $x' \in I$ , continuity gives the following:

$$x' = \lim x_m = \lim f(x_{m-1}c^{n-1}) = f(\lim x_{m-1}c^{n-1}) = f(x'c^{n-1}),$$

a contradiction. Thus  $x' \notin I$ , so  $\lim x_m = \sup I$ .  $\square$

Hereinafter we work on the extended real line so that suprema of arbitrary sets exist and all monotone sequences converge.

*Claim 4.* Let  $x \in I$  and let  $j, k, p, q \in \mathbb{N}$  such that  $j+1, k, p, q+1 \in A_n$ . Then we have

$$g(c^p) > g(x c^q) \iff g(c^{kp}) > g(x^k c^{kq}) \iff g(c^{p+j}) > g(x c^{q+j}).$$

The same equivalence holds if “<” or “=” replaces “>”.

*Proof.* Assume that  $g(c^p) > g(x c^q)$ . Then, by Proposition 2.5(iv), Claim 1, and symmetry, we have  $g(c^{kp}) = g(g(c^p)^k) > g(g(x c^q)^k) = g(x^k c^{kq})$ , which proves the first equivalence (since the same conclusion clearly holds if “<” or “=” replaces “>”). For the second equivalence, assume again that  $g(c^p) > g(x c^q)$ . Then, as before, we have  $g(c^{p+j}) = g(g(c^p) c^j) > g(g(x c^q) c^j) = g(x c^{q+j})$ .  $\square$

Let  $x$  be any fixed element of  $I$ . Define  $S_x$  to be the subset of all rational numbers  $r$  for which there exist  $k, p, q \in \mathbb{N}$  such that  $k, p, q+1 \in A_n$ ,  $g(c^p) > g(x^k c^q)$ , and  $r = (p-q)/k$ . Now, if  $r = (p-q)/k = (p'-q')/k'$ , then we have  $pk' + q'k = p'k + qk'$  and it follows from Claim 4 that

$$\begin{aligned} g(c^p) > g(x^k c^q) &\iff g(c^{pk'}) > g(x^{kk'} c^{qk'}) \\ &\iff g(c^{pk'+q'k}) > g(x^{kk'} c^{qk'+q'k}) \\ &\iff g(c^{p'k+qk'}) > g(x^{kk'} c^{q'k+qk'}) \\ &\iff g(c^{p'k}) > g(x^{kk'} c^{q'k}) \\ &\iff g(c^{p'}) > g(x^{k'} c^{q'}). \end{aligned}$$

Hence  $S_x$  is in fact the subset of rational numbers  $r$  for which every representation  $r = (p-q)/k$  with  $k, p, q+1 \in A_n$  satisfies  $g(c^p) > g(x^k c^q)$ .

*Claim 5.* The set  $S = \{\frac{p-q}{k} : k, p, q+1 \in A_n\}$  is dense in  $\mathbb{R}$ .

*Proof.* For every  $a, b \in \mathbb{N}$ , the sequence

$$x_m = \frac{1 \pm a m (n-1)}{1 + b m (n-1)}$$

of  $S$  converges to  $\pm a/b$ . Thus  $S$  is dense in  $\mathbb{Q}$  and hence (by transitivity) in  $\mathbb{R}$ .  $\square$

*Claim 6.* Any two numbers  $r, r' \in S$  may be written  $r = (p-q)/k$ ,  $r' = (p'-q)/k$  for suitable  $k, p, p', q+1 \in A_n$ .

*Proof.* Let  $r = (p-q)/k$  and  $r' = (p'-q')/k'$ , with  $k, k', p, p', q+1, q'+1 \in A_n$ . Assume w.l.o.g. that  $r' > r$ . Setting  $\tilde{k} = k k'$ ,  $\tilde{q} = |\tilde{k} r - 1|$ ,  $\tilde{p} = \tilde{k} r + \tilde{q}$ , and  $\tilde{p}' = \tilde{k} r' + \tilde{q}$ , we have  $r = (\tilde{p} - \tilde{q})/\tilde{k}$ ,  $r' = (\tilde{p}' - \tilde{q})/\tilde{k}$  with  $\tilde{k}, \tilde{p}, \tilde{p}', \tilde{q} + 1 \in A_n$ .  $\square$

*Claim 7.*  $S_x$  is a nonempty, proper, and upper subset of  $S$  (“upper” means that if  $r \in S_x$  and  $r' \in S$ ,  $r' > r$ , then  $r' \in S_x$ ).

*Proof.* To show that  $S_x$  is an upper subset, let  $r = (p-q)/k \in S_x$  and  $r' = (p'-q)/k > r$  (cf. Claim 6). Then  $p' > p$  and, since  $p, p' \in A_n$ , we have  $p' = p + j(n-1)$  for some integer  $j \geq 1$ . Using the definition of  $S_x$  and the first part of Claim 3, we obtain

$$\begin{aligned} g(x^k c^q) < g(c^p) &< g(g(c^p) c^{n-1}) = g(c^p c^{n-1}) \\ &< g(g(c^p c^{n-1}) c^{n-1}) = g(c^p c^{2(n-1)}) \\ &< \dots \\ &< g(c^p c^{j(n-1)}) = g(c^{p'}). \end{aligned}$$

Hence  $r' \in S_x$ . Now, by Claim 3,  $\lim f(c^{m(n-1)+1}) = \sup I > g(xc^{n-1})$ , and hence there is some  $p \in A_n$  with  $g(c^p) > g(xc^{n-1})$ . Hence  $r = (p - (n - 1))/1 \in S_x$ , and so  $S_x$  is nonempty. Similarly, since  $\lim g(xc^{m(n-1)}) = \sup I$ , there must a  $q$  such that  $q + 1 \in A_n$  and  $g(c) < g(xc^q)$ , and so  $(1 - q)/1 \notin S_x$ .  $\square$

Now, by Claim 7,  $S_x$  is precisely the set of elements in  $S$  which are greater than (and possibly equal to)  $\inf S_x$ . Using this fact, let  $\varphi: I \rightarrow \mathbb{R}$  be the function given by

$$\varphi(x) := \inf S_x.$$

*Claim 8.* If  $g(c^p) = g(x^k c^q)$ , then  $\varphi(x) = (p - q)/k$ . In particular,  $\varphi(c) = 1$ .

*Proof.* Note that  $g(c^p) = g(x^k c^q)$  implies  $r = (p - q)/k \notin S_x$ . Moreover, by Claim 7 it follows that if  $r' = (p' - q)/k > r$  (resp.  $r' < r$ ), then  $g(c^{p'}) > g(c^p) = g(x^k c^q)$  (resp.  $g(c^{p'}) < g(c^p) = g(x^k c^q)$ ), and hence  $r' \in S_x$  (resp.  $r' \notin S_x$ ). Thus  $\inf S_x = (p - q)/k$  by Claim 5. For the last claim just note that  $g(c^{q+1}) = g(c c^q)$ .  $\square$

*Claim 9.* We have  $\varphi(g(x_1 \cdots x_n)) = \sum_{i=1}^n \varphi(x_i)$  for every  $x_1, \dots, x_n \in I$ .

*Proof.* Let  $r_i = (p_i - q)/k > \varphi(x_i)$  for all  $i \in [n]$ . Then  $g(c^{p_i}) > g(x_i^k c^q)$ , and by Proposition 2.5(iv), Claim 1, and symmetry, we have

$$g(c^{\sum_{i=1}^n p_i}) = g(g(c^{p_1}) \cdots g(c^{p_n})) > g(g(x_1^k c^q) \cdots g(x_n^k c^q)) = g(g(x_1 \cdots x_n)^k c^{nq}).$$

By Claim 8,  $(\sum_{i=1}^n p_i - nq)/k \in S_{g(x_1 \cdots x_n)}$ . Thus  $\sum_{i=1}^n r_i > \varphi(g(x_1 \cdots x_n))$ . Similarly, if  $r_i \leq \varphi(x_i)$  for all  $i \in [n]$ , then  $\sum_{i=1}^n r_i \leq \varphi(g(x_1 \cdots x_n))$ . The result then follows from Claim 5.  $\square$

*Claim 10.*  $\varphi$  is nondecreasing.

*Proof.* Suppose  $y > x$  and  $(p - q)/k \in S_y$ . Then  $g(c^p) > g(y^k c^q) > g(x^k c^q)$  and hence  $S_y \subseteq S_x$  and so  $\varphi(y) = \inf S_y \geq \inf S_x = \varphi(x)$ .  $\square$

*Claim 11.*  $\varphi$  is continuous.

*Proof.* Since  $\varphi$  is nondecreasing, the only possible sort of discontinuity is a gap discontinuity. Hence, if  $\varphi$  is discontinuous, there must exist  $x, y \in I$ , say  $x < y$ , and an interval, and thus a rational  $r \notin \varphi(I)$ , such that  $\varphi(x) < r < \varphi(y)$ . Now if  $r = (p - q)/k$ , then  $g(x^k c^q) < g(c^p) \leq g(y^k c^q)$ . By continuity of  $g_{k+q}$ , there is  $t \in ]x, y]$  such that  $g(c^p) = g(t^k c^q)$ . By Claim 8 it then follows that  $\varphi(t) = r$ , which yields the desired contradiction.  $\square$

*Claim 12.*  $\varphi$  is strictly increasing.

*Proof.* For the sake of contradiction, suppose that there are  $x, y \in I$  such that  $x < y$  and  $\varphi(x) = \varphi(y) = a$ . Since  $\varphi$  is nondecreasing, there is an interval  $I'$  containing  $x$  and  $y$ , and such that  $\varphi(z) = a$ , for all  $z \in I'$ . Let  $I'$  be the largest interval having this property, and set  $t = \sup I'$ . If  $t \notin I$ , then for every  $z > x$ ,  $\varphi(z) = a$ . Now  $g(xc^{n-1}) > x$  (by Claim 3) and hence  $a = \varphi(g(xc^{n-1})) = a + (n - 1) > a$  (by Claim 9), a contradiction. Thus  $t \in I$ , and  $\varphi(t) = a$  by Claim 11. We have  $g(xt^{n-1}) < g(t^n)$  and, by Claim 3, there exists  $q$  such that  $q + 1 \in A_n$  and  $g(t^n) < g(xc^{q(n-1)}) = g(xg(c^q)^{n-1})$  and  $g(c^q) > t$ . By continuity of  $g_n$ , there is  $z \in I$  such that  $t < z < g(c^q)$  (and so  $z \notin I'$ ) and  $g(xz^{n-1}) = g(t^n)$ . Thus

$$a + (n - 1) \varphi(z) = \varphi(x) + (n - 1) \varphi(z) = \varphi(g(xz^{n-1})) = \varphi(g(t^n)) = n \varphi(t) = n a,$$

and we obtain  $\varphi(z) = a$ , so  $z \in I'$ , a contradiction.  $\square$



Thus  $\varphi$  is a continuous strictly increasing  $n$ -ary semigroup homomorphism and, by Claim 9, its range  $J$  is a connected real additive  $n$ -ary semigroup. Hence the only possibilities for  $J$  are  $]-\infty, b[$ ,  $]-\infty, b]$ ,  $]a, \infty[$ ,  $]a, \infty[$  or  $]-\infty, \infty[$  ( $b \leq 0 \leq a$ ); see final comments in [4]. This completes the proof of the Main Theorem.  $\square$

*Remark 3.* The function  $\varphi$  is determined up to a multiplicative constant, that is, with  $\varphi$  all functions  $\psi = r\varphi$  ( $r \neq 0$ ) belong to the same function  $f$ , and only these; see the “Uniqueness” section in [2].

*Remark 4.* An  $n$ -ary semigroup  $(I, f)$  is said to be *reducible to* (or *derived from*) a binary semigroup  $(I, \diamond)$  if there is an associative extension  $g: I^* \rightarrow I$  of  $\diamond$  such that  $g_n = f$ ; that is,  $f(x_1 \cdots x_n) = x_1 \diamond \cdots \diamond x_n$  (see [5, 9]). Dudek and Mukhin [6] showed that an  $n$ -ary semigroup is reducible if and only if we can adjoint an  $n$ -ary neutral element to it. This shows that the  $n$ -ary semigroup given in Example 1.1 is not reducible since we cannot adjoint any  $n$ -ary neutral element (for an alternative proof, see [8]). However, the Main Theorem shows that every Aczélian  $n$ -ary semigroup is reducible and hence we can always adjoint an  $n$ -ary neutral element to it (if  $0 \in J$ , then the neutral element is  $e = \varphi^{-1}(0)$ ; otherwise fix  $e \notin I$  and extend  $\varphi$  to  $\varphi': I \cup \{e\} \rightarrow J \cup \{0\}$  by the rule  $\varphi'(x) = \varphi(x)$  if  $x \in I$  and  $\varphi'(e) = 0$ ).

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