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On the decay in time of solutions of some generalized regularized long waves equations

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Abstract. We consider the generalized Benjamin-Ono equation, regularized in the same manner that the Benjamin-Bona-Mahony equation is found from the Korteweg-de Vries equation [3], namely the equation $u_t + u_x + u^\rho u_x + H(u_{xt}) = 0$, where H is the Hilbert transform. In a second time, we consider the generalized Kadomtsev-Petviashvili-II equation, also regularized, namely the equation $u_t + u_x + u^\rho u_x - u_{xxt} + \partial_x^{-1} u_{yy} = 0$. We are interested in dispersive properties of these equations for small initial data. We will show that, if the power ρ of the nonlinearity is higher than 3, the respective solution of these equations tends to zero when time rises with a decay rate of order close to $\frac{1}{2}$.

Keywords. BO, BBM, KP equations, decay in time

MS Codes. 35B05, 35B40, 35Q53, 76B03, 76B55

1 Introduction

The small amplitude long waves moving inside a nonhomogeneous fluid are modelled in dimension 1 by the Benjamin-Ono equation [2, 4, 11]

$$u_t + u_x + uu_x - H(u_{xx}) = 0,$$

where H indicates the Hilbert transform in the direction x . Since this equation is obtained correcting, at the second order, the transport equation $u_t + u_x = 0$, the BO-BBM equation is found [3]

$$u_t + u_x + uu_x + H(u_{xt}) = 0.$$

We consider here a generalization of this equation, namely the gBO-BBM equation

$$u_t + u_x + u^\rho u_x + H(u_{xt}) = 0, \tag{1.1}$$

where ρ is a nonnegative integer. We are interested here in the decay in time for small amplitude solution of the equation (1.1).

For $1 < p < 2$ and $(m_0, m_1) \in \mathbb{R}^2$, we denote $X^{m_0, m_1, p}(\mathbb{R}) = H^{m_0}(\mathbb{R}) \cap W^{m_1, p}(\mathbb{R})$ the space of functions f such that the norm

$$\|f\|_{X^{m_0, m_1, p}} := \|f\|_{H^{m_0}} + \|f\|_{W^{m_1, p}}$$

is finite. Our result reads as follows.

Theorem 1.1

Let $\rho \geq 3$. For $0 < \delta < \frac{1}{3} - \frac{8}{9\rho}$, we set $m = \frac{1}{2\delta} - 1$. We choose $1 < p < 2$ and $q > 2$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and $0 < \frac{1}{q} < \frac{1}{2} - \frac{4}{3\rho(1-3\delta)}$.

Then there exists $\varepsilon > 0$ sufficiently small so that for all $f \in X^{m+1, 4-\frac{2}{p}, p}(\mathbb{R})$ such that $\|f\|_{X^{m+1, 4-\frac{2}{p}, p}} \leq \varepsilon$, there exists a unique global in time solution $u \in C(\mathbb{R}; X^{m+1, 4-\frac{2}{p}, p}(\mathbb{R}))$ of the gBO-BBM equation (1.1) with initial datum f .

Moreover this solution verifies that there exists a constant $C > 0$, depending only on m , p and ε such that for all time $t \in \mathbb{R}$, we have

$$\|u(t)\|_{L^q} + \|u_x(t)\|_{L^q} \leq C(1 + |t|)^{(-\frac{1}{2} + \frac{3\delta}{2})(1 - \frac{2}{q})}.$$

We remark that if δ approaches zero then the decay rate of the norm L^q of $u(t)$, for $q \in [2, +\infty[$, approaches $t^{-\frac{1}{2}(1 - \frac{2}{q})}$, but we need to impose more regularity on the initial datum. We notice that $t^{-\frac{1}{2}(1 - \frac{2}{q})}$ is the decay rate of the norm L^q of the linear evolution.

We are inspired by the method described by Albert [1] for generalized Benjamin-Bona-Mahony equation

$$u_t + u_x + u^\rho u_x - u_{xxt} = 0.$$

In this paper, Albert shows that, for a power ρ strictly higher than 4, the solution of the preceding equation decreases in time with a rate of order $\frac{1}{3}$, what is the same order as the rate of the decay in time of the solution of the generalized Korteweg-de Vries equation [10, 14]

$$u_t + u_x + u^\rho u_x + u_{xxx} = 0.$$

In our work, we prove that the decay rate in time of the solution of the gBO-BBM equation is of order close to $\frac{1}{2}$, what is equal to the decay rate in time of the solution of generalized Benjamin-Ono equation [6].

In a second time, we study the small amplitude long waves in shallow water moving mainly in the direction x , which are modelled in dimension 2 by the Kadomtsev-Petviashvili equations [8]

$$u_t + u_x + uu_x + \sigma u_{xxx} + \partial_x^{-1} u_{yy} = 0,$$

called KP-I if $\sigma = -1$ and KP-II if $\sigma = 1$, according to whether the surface tension is or isn't neglected. Since this equation is also obtained correcting, at the second order, the transport equation $u_t + u_x = 0$, the KP-BBM equations are found [3]

$$u_t + u_x + uu_x - \sigma u_{xxt} + \partial_x^{-1} u_{yy} = 0,$$

called KP-BBM-I if $\sigma = -1$ and KP-BBM-II if $\sigma = 1$. The KP-BBM-I is not well posed in $L^2(\mathbb{R}^2)$, and so we study here only the KP-BBM-II equation.

We consider a generalization of this equation, namely the gKP-BBM-II equation

$$u_t + u_x + u^\rho u_x - u_{xxt} + \partial_x^{-1} u_{yy} = 0, \tag{1.2}$$

where ρ is a nonnegative integer. We prove a similar result as the theorem 1.1 for the equation (1.2).

Theorem 1.2

Let $\rho \geq 3$. For $0 < \delta < \frac{1}{5} - \frac{8}{15\rho}$, we set $m = \frac{1}{2\delta} - 2$. We choose $1 < p < 2$ and $q > 2$ such that

$$\frac{1}{p} + \frac{1}{q} = 1 \text{ and } 0 < \frac{1}{q} < \frac{1}{2} - \frac{4}{3\rho(1-5\delta)}.$$

Then there exists $\varepsilon > 0$ sufficiently small so that for all $f \in X^{2m+1, 7-\frac{4}{p}, p}(\mathbb{R}^2)$ such that $\|f\|_{X^{2m+1, 7-\frac{4}{p}, p}} \leq \varepsilon$,

there exists an unique global in time solution $u \in C\left(\mathbb{R}; X^{2m+1, 7-\frac{4}{p}, p}(\mathbb{R}^2)\right)$ of the gKP-BBM-II equation with initial datum f .

Moreover this solution verifies that there exists a constant $C > 0$, depending only on m, p and ε such that for all time $t \in \mathbb{R}$, we have

$$\|u(t)\|_{L^q} + \|u_x(t)\|_{L^q} \leq C(1 + |t|)^{\left(-\frac{1}{2} + \frac{5\delta}{2}\right)\left(1 - \frac{2}{q}\right)}.$$

We recall that, for the generalized KP equations, Hayashi, Naumkin and Saut [7] proved that for $\rho \geq 3$, the decay rate of the solution of these equations is of order 1.

Whether it is for the gBO-BBM or the gKP-BBM-II, the Strauss method [14] will be used to prove the decay in time. For $T > 0$, it consists of choosing a norm N_T , which depends on time *a priori*, such that for a solution of the gBO-BBM equation, all the derivatives act like a linear term and are included in a Sobolev norm of sufficiently high order and such that the power ρ are linked with the decay in time. More precisely, the Duhamel formula is written for the gBO-BBM equation (1.1) and for $t \geq 0$

$$\Phi u(t) = S_t f - \frac{1}{\rho + 1} \int_0^t S_{t-\tau} \left(\frac{D_x}{1 + |D_x|} u^{\rho+1} \right) (\tau) d\tau,$$

where $(S_t)_{t \geq 0}$ is the semi-group of the gBO-BBM evolution and $\frac{D_x}{1 + |D_x|}$ denotes the operator defined by

the Fourier multiplier $\sigma(k) := \frac{ik}{1 + |k|}$. The fractional Leibniz rule implies for $m \geq 0$ and $1 < p < +\infty$, there exists a constant $C > 0$ such that for $t \in [-T, T]$,

$$\|D^m (u^{\rho+1})\|_{L^p}(t) \leq C \|D^m u\|_{L^p}(t) \|u\|_{L^\infty}^\rho(t).$$

The first term of this inequality is a linear term and a Sobolev norm, and for the second one, we choose $\theta > 0$ such that

$$\|u\|_{L^\infty}(t) \leq C N_T(u) (1 + t)^{-\theta}.$$

Then we show that for sufficiently small initial datum, this norm N_T is bounded independently of time.

The principal difficulty here is that the operator $\frac{D_x}{1 + |D_x|}$ is not bounded in $L^1(\mathbb{R})$ but only in $L^p(\mathbb{R})$ for

$1 < p < +\infty$. In dimension 2, we have to be careful that the operator $\frac{D_x}{1 + D_x^2}$ is not bounded in $L^1(\mathbb{R}^2)$ but only in $L^p(\mathbb{R}^2)$ for $1 < p < +\infty$.

We will use the following notations: for $n = 1, 2$ and $1 \leq p < \infty$, we denote $L^p(\mathbb{R}^n)$ the space of p -power integrable functions equipped with the norm

$$\|f\|_{L^p} := \left(\int_{\mathbb{R}^n} |f(x)|^p dx \right)^{1/p},$$

we denote $L^\infty(\mathbb{R}^n)$ the functions space equipped with the norm

$$\|f\|_\infty = \text{sup ess } (f) := \inf \{c; |f(x)| \leq c \text{ almost everywhere in } \mathbb{R}^n\}.$$

Let T an operator from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ with $\frac{1}{p} + \frac{1}{q} = 1$, we define the operator norm by

$$\|T\|_{L^q} := \sup_{\|f\|_{L^p}=1} \|Tf\|_{L^q}.$$

The Schwartz space is denoted by $\mathcal{S}(\mathbb{R}^n)$ and for $1 \leq p \leq \infty$, $W^{m,p}(\mathbb{R}^n)$ is the Sobolev space equipped with the norm

$$\|f\|_{W^{m,p}} := \left\| (1 - \Delta)^{m/2} f \right\|_{L^p}.$$

In particular, we will note $H^m(\mathbb{R}^n)$ the Sobolev space $W^{m,2}(\mathbb{R}^n)$.

We organise the paper as follows. In the first section, we will give estimates for the linear Cauchy problem associated with the gBO-BBM equation (1.1), and in the second one, the existence and uniqueness of global solution with the decay in time will be done. Finally, the decay in time for the gKP-BBM-II equation (1.2) will be studied.

2 Estimates for the linear gBO-BBM equation

We consider the linear Cauchy problem

$$u_t + u_x + H(u_{xt}) = 0; \quad u(x, 0) = f(x).$$

Let us suppose that the initial datum f belongs to the Schwartz space $\mathcal{S}(\mathbb{R})$. Then the Fourier transform in space implies

$$\hat{u}_t(1 + |k|) + ik\hat{u} = 0; \quad \hat{u}(k, 0) = \hat{f}(k).$$

The solution u of this ordinary differential equation is given by, for all $x \in \mathbb{R}$ and $t \in \mathbb{R}^*$,

$$u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ith_\alpha(k)} \hat{f}(k) dk, \quad (2.1)$$

with $h_\alpha(k) = \frac{k}{1 + |k|} - \alpha k$ and $\alpha = \frac{x}{t}$.

We recall the Van der Corput lemma [13].

Lemma 2.1

For all $a \leq b$, $\lambda > 0$, $t \neq 0$ and for all function $h \in C^\infty([a, b])$ real valued satisfying for all $k \in [a, b]$, $|h''(k)| \geq \lambda$, we have

$$\left| \int_a^b e^{-ith(k)} dk \right| \leq \frac{10}{(\lambda|t|)^{1/2}}. \quad (2.2)$$

Proof. In [13], the proof is done for $\lambda = 1$. Here it is enough to set $g(k) = \frac{h(k)}{\lambda}$. We have then

$$\int_a^b e^{-ith(k)} dk = \int_a^b e^{-i(\lambda t)g(k)} dk,$$

with $k \in [a, b]$, $|g''(k)| \geq 1$. □

Now, the Van der Corput lemma is applied with the function h_α .

Lemma 2.2

Let $0 \leq \delta < \frac{1}{3}$. For all $\alpha \in \mathbb{R}$ and all time $|t| \geq 1$, we have

$$\left| \int_{-|t|^\delta+1}^{|t|^\delta-1} e^{-ith_\alpha(k)} dk \right| \leq 10\sqrt{2}|t|^{-\frac{1}{2} + \frac{3\delta}{2}}.$$

Proof. We only consider the contribution for $k \geq 0$, the contribution for $k \leq 0$ being dealt with similarly.

For $k \geq 0$, we have

$$h_\alpha(k) = \frac{k}{1+k} - \alpha k, \quad h'_\alpha(k) = \frac{1}{(1+k)^2} - \alpha, \quad h''_\alpha(k) = \frac{-2}{(1+k)^3}.$$

For $k \in [0, |t|^\delta - 1]$, we have

$$|h''_\alpha(k)| = \left| \frac{-2}{(1+k)^3} \right| = \frac{2}{(1+k)^3} \geq 2|t|^{-3\delta}.$$

The lemma 2.1 is then applied to find

$$\left| \int_0^{|t|^\delta - 1} e^{-ith_\alpha(k)} dk \right| \leq \frac{10}{(2|t|^{-3\delta}|t|)^{1/2}} \leq 5\sqrt{2}|t|^{-\frac{1}{2} + \frac{3\delta}{2}}.$$

□

The oscillating integral (2.1) can be majorized.

Lemma 2.3

Let $0 < \delta < \frac{1}{3}$ and $m = \frac{1}{2\delta} - 1$. For all $f \in H^m(\mathbb{R})$, $\alpha \in \mathbb{R}$ and all time $|t| \geq 1$, we have

$$\left| \int_{|t|^\delta - 1}^{+\infty} e^{-ith_\alpha(k)} \hat{f}(k) dk \right| \leq \sqrt{\frac{2^m}{2m-1}} \|f\|_{H^m} |t|^{-\frac{1}{2} + \frac{3\delta}{2}},$$

and

$$\left| \int_{-\infty}^{-|t|^\delta + 1} e^{-ith_\alpha(k)} \hat{f}(k) dk \right| \leq \sqrt{\frac{2^m}{2m-1}} \|f\|_{H^m} |t|^{-\frac{1}{2} + \frac{3\delta}{2}}.$$

Proof. We have

$$\left| \int_{|t|^\delta - 1}^{+\infty} e^{-ith_\alpha(k)} \hat{f}(k) dk \right| \leq \int_{|t|^\delta - 1}^{+\infty} |\hat{f}(k)| dk = \int_{|t|^\delta - 1}^{+\infty} \frac{(1+k)^m}{(1+k)^m} |\hat{f}(k)| dk,$$

and the Cauchy-Schwarz inequality gives

$$\left| \int_{|t|^\delta - 1}^{+\infty} e^{-ith_\alpha(k)} \hat{f}(k) dk \right| \leq \left(\int_{|t|^\delta - 1}^{+\infty} (1+k)^{2m} |\hat{f}(k)|^2 dk \right)^{1/2} \left(\int_{|t|^\delta - 1}^{+\infty} \frac{dk}{(1+k)^{2m}} \right)^{1/2}.$$

In one hand, since for $0 < \delta < \frac{1}{3}$ and $m = \frac{1}{2\delta} - 1$, the order m of the Sobolev space is strictly higher than $\frac{1}{2}$ and we have

$$\begin{aligned} \left(\int_{|t|^\delta - 1}^{+\infty} \frac{dk}{(1+k)^{2m}} \right)^{1/2} &= \left(\left[\frac{-1}{(2m-1)(1+k)^{2m-1}} \right]_{|t|^\delta - 1}^{+\infty} \right)^{1/2} \\ &= \frac{1}{\sqrt{2m-1}} |t|^{\frac{\delta}{2}(1-2m)} = \frac{1}{\sqrt{2m-1}} |t|^{-\frac{1}{2} + \frac{3\delta}{2}}. \end{aligned}$$

In the other hand, since for $k \in \mathbb{R}$, we have $(1+k)^2 \leq 2(1+k^2)$, the first integral becomes

$$\begin{aligned} \left(\int_{|t|^{\delta-1}}^{+\infty} (1+k)^{2m} |\hat{f}(k)|^2 dk \right)^{1/2} &\leq \sqrt{2^m} \left(\int_{|t|^{\delta-1}}^{+\infty} (1+k^2)^m |\hat{f}(k)|^2 dk \right)^{1/2} \\ &\leq \sqrt{2^m} \|f\|_{H^m}. \end{aligned}$$

Finally, we find

$$\left| \int_{|t|^{\delta-1}}^{+\infty} e^{-ith_\alpha(k)} \hat{f}(k) dk \right| \leq \sqrt{\frac{2^m}{2m-1}} \|f\|_{H^m} |t|^{-\frac{1}{2} + \frac{3\delta}{2}}.$$

□

By abuse, when there is no ambiguity, we will write C and C_m the different constants appearing in the following results.

Proposition 1

Let $0 < \delta < \frac{1}{3}$ and $m = \frac{1}{2\delta} - 1$. There exists a constant $C_m > 0$, depending only on m , such that for all function $f \in L^1(\mathbb{R}) \cap H^m(\mathbb{R})$, $\alpha \in \mathbb{R}$ and all time $t \in \mathbb{R}$, we have

$$\left| \int_{-\infty}^{+\infty} e^{-ith_\alpha(k)} \hat{f}(k) dk \right| \leq C_m (\|f\|_{L^1} + \|f\|_{H^m}) (1 + |t|)^{-\frac{1}{2} + \frac{3\delta}{2}}. \quad (2.3)$$

Proof. Let $0 < \delta < \frac{1}{3}$ and $|t| \geq 1$, we write

$$\int_0^{+\infty} e^{-ith_\alpha(k)} \hat{f}(k) dk = \int_0^{|t|^{\delta-1}} e^{-ith_\alpha(k)} \hat{f}(k) dk + \int_{|t|^{\delta-1}}^{+\infty} e^{-ith_\alpha(k)} \hat{f}(k) dk.$$

For the first integral, the Fubini theorem implies

$$\begin{aligned} \int_0^{|t|^{\delta-1}} e^{-ith_\alpha(k)} \hat{f}(k) dk &= \int_0^{|t|^{\delta-1}} e^{-ith_\alpha(k)} \left(\int_{-\infty}^{+\infty} e^{-ikx'} f(x') dx' \right) dk \\ &= \int_{-\infty}^{+\infty} \left(\int_0^{|t|^{\delta-1}} e^{ik(x-x')} e^{-it\frac{k}{1+k}} dk \right) f(x') dx'. \end{aligned}$$

We deduce from it that

$$\left| \int_0^{|t|^{\delta-1}} e^{-ith_\alpha(k)} \hat{f}(k) dk \right| \leq \left\| \int_0^{|t|^{\delta-1}} e^{-ith_\alpha(k)} dk \right\|_{\infty} \|f\|_{L^1}.$$

The lemmas 2.2 and 2.3 are applied to give

$$\left| \int_0^{+\infty} e^{-ith_\alpha(k)} \hat{f}(k) dk \right| \leq \max \left(10\sqrt{2}; \sqrt{\frac{2^m}{2m-1}} \right) (\|f\|_{L^1} + \|f\|_{H^m}) |t|^{-\frac{1}{2} + \frac{3\delta}{2}}.$$

Since $|t| \geq 1$, we have $|t| \geq \frac{1+|t|}{2}$, thus we obtain finally

$$\left| \int_0^{+\infty} e^{-ith_\alpha(k)} \hat{f}(k) dk \right| \leq C_m (\|f\|_{L^1} + \|f\|_{H^m}) (1 + |t|)^{-\frac{1}{2} + \frac{3\delta}{2}}. \quad (2.4)$$

The contribution for $k \leq 0$ is dealt with similarly.
Let $|t| \leq 1$. We have directly

$$\left| \int_{-\infty}^{+\infty} e^{-ith_\alpha(k)} \hat{f}(k) dk \right| \leq \int_{-\infty}^{+\infty} |\hat{f}(k)| dk = \int_{-\infty}^{+\infty} \frac{(1+k^2)^{m/2}}{(1+k^2)^{m/2}} |\hat{f}(k)| dk,$$

and the Cauchy-Schwarz inequality gives

$$\left| \int_{-\infty}^{+\infty} e^{-ith_\alpha(k)} \hat{f}(k) dk \right| \leq C_m \|f\|_{H^m}.$$

Since $|t| \leq 1$, we have $1 \geq \frac{1+|t|}{2}$, thus we obtain finally

$$\left| \int_{-\infty}^{+\infty} e^{-ith_\alpha(k)} \hat{f}(k) dk \right| \leq 2^{\frac{1}{2}-\frac{3\delta}{2}} C_m \|f\|_{H^m} (1+|t|)^{-\frac{1}{2}+\frac{3\delta}{2}}. \quad (2.5)$$

The inequalities (2.4) and (2.5) give the result. \square

We can deduce the following corollary.

Corollary 1

Let $0 < \delta < \frac{1}{3}$ and $m = \frac{1}{2\delta} - 1$. There exists a constant $C_m > 0$, depending only on m , such that for all function $f \in L^p(\mathbb{R}) \cap H^m(\mathbb{R})$, with $1 \leq p \leq 2$, for all $\alpha \in \mathbb{R}$ and all time $t \in \mathbb{R}$, we have

$$\left\| \int_{-\infty}^{+\infty} e^{-ith_\alpha(k)} \hat{f}(k) dk \right\|_{L^q} \leq C_m (\|f\|_{L^p} + \|f\|_{H^m}) (1+|t|)^{\left(-\frac{1}{2}+\frac{3\delta}{2}\right)\left(1-\frac{2}{q}\right)}, \quad (2.6)$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Let $0 < \delta < \frac{1}{3}$ and $|t| \geq 1$, we define the operator $I_t^{(1)}$ from $L^1(\mathbb{R})$ to $L^\infty(\mathbb{R})$ by, for $f \in L^1(\mathbb{R})$ and $x \in \mathbb{R}$,

$$I_t^{(1)} f(x) := \int_0^{|t|^\delta-1} e^{-ith_\alpha(k)} \hat{f}(k) dk.$$

The lemma 2.2 gives that there exists a constant $C > 0$ such that

$$\|I_t^{(1)} f\|_\infty \leq C |t|^{-\frac{1}{2}+\frac{3\delta}{2}} \|f\|_{L^1},$$

thus

$$\| \|I_t^{(1)} \| \|_\infty \leq C |t|^{-\frac{1}{2}+\frac{3\delta}{2}}. \quad (2.7)$$

The operator $I_t^{(1)}$ is rewritten

$$I_t^{(1)} f(x) := \int_{-\infty}^{\infty} e^{ikx} e^{-it\frac{k}{1+|k|}} \hat{f}(k) \mathbf{1}_{[0,|t|^\delta-1]} dk,$$

thus

$$\|I_t^{(1)} f\|_{L^2} \leq \|f\|_{L^2},$$

and

$$\| \|I_t^{(1)} \| \|_{L^2} \leq 1. \quad (2.8)$$

The Riesz-Thorin interpolation theorem then implies that, for $1 \leq p \leq 2$, and $\frac{1}{p} + \frac{1}{q} = 1$, the operator $I_t^{(1)}$ is continuous from $L^p(\mathbb{R})$ to $L^q(\mathbb{R})$, and we have according to the inequalities (2.7) and (2.8)

$$\|I_t^{(1)}\|_{L^q} \leq C|t|^{(-\frac{1}{2} + \frac{3\delta}{2})(1 - \frac{2}{q})}.$$

We conclude that for $f \in L^p(\mathbb{R})$,

$$\|I_t^{(1)}f\|_{L^q} \leq C|t|^{(-\frac{1}{2} + \frac{3\delta}{2})(1 - \frac{2}{q})}\|f\|_{L^p} \leq C(1 + |t|)^{(-\frac{1}{2} + \frac{3\delta}{2})(1 - \frac{2}{q})}\|f\|_{L^p}.$$

Let $m = \frac{1}{2\delta} - 1$, we define now the operator $I_t^{(2)}$ from $H^m(\mathbb{R})$ to $L^\infty(\mathbb{R})$ by, for $f \in H^m(\mathbb{R})$ and $x \in \mathbb{R}$,

$$I_t^{(2)}f(x) := \int_{|t|^\delta - 1}^{+\infty} e^{-ith_\alpha(k)} \hat{f}(k) dk.$$

The lemma 2.3 implies that there exists a constant $C_m > 0$, depending only on m , such that

$$\|I_t^{(2)}f\|_\infty \leq C_m|t|^{-\frac{1}{2} + \frac{3\delta}{2}}\|f\|_{H^m}. \quad (2.9)$$

Since the operator $I_t^{(2)}$ can be rewritten

$$I_t^{(2)}f(x) := \int_{-\infty}^{+\infty} e^{ikx} e^{-it \frac{k}{1+|k|}} \hat{f}(k) \mathbf{1}_{[|t|^\delta - 1, +\infty]} dk,$$

we have

$$\|I_t^{(2)}f\|_{L^2} \leq \|f\|_{L^2},$$

and since $m > \frac{1}{2}$,

$$\|I_t^{(2)}f\|_{L^2} \leq \|f\|_{H^m}. \quad (2.10)$$

The Hölder inequality then implies that, for $2 \leq q \leq \infty$, according to the inequalities (2.9) and (2.10), we have

$$\|I_t^{(2)}f\|_{L^q} \leq \|I_t^{(2)}f\|_\infty^{1 - \frac{2}{q}} \|I_t^{(2)}f\|_{L^2}^{\frac{2}{q}} \leq C_m(1 + |t|)^{(-\frac{1}{2} + \frac{3\delta}{2})(1 - \frac{2}{q})}\|f\|_{H^m}.$$

The contribution for $k \leq 0$ is dealt with similarly.

For $|t| \leq 1$, we define the operator $I_t^{(3)}$ from $H^m(\mathbb{R})$ to $L^\infty(\mathbb{R})$ by, for $f \in H^m(\mathbb{R})$ and $x \in \mathbb{R}$,

$$I_t^{(3)}f(x) := \int_{-\infty}^{+\infty} e^{-ith_\alpha(k)} \hat{f}(k) dk.$$

As for the proposition 1, we have directly that there exists a constant $C_m > 0$, depending only on m , such that

$$\|I_t^{(3)}f\|_\infty \leq C_m\|f\|_{H^m}(1 + |t|)^{(-\frac{1}{2} + \frac{3\delta}{2})}. \quad (2.11)$$

Since the operator $I_t^{(3)}$ can be rewritten

$$I_t^{(3)}f(x) := \int_{-\infty}^{+\infty} e^{ikx} e^{-it \frac{k}{1+|k|}} \hat{f}(k) \mathbf{1}_{[0, +\infty[} dk,$$

we have, because $m > \frac{1}{2}$,

$$\|I_t^{(3)}f\|_{L^2} \leq \|f\|_{H^m}. \quad (2.12)$$

The Hölder inequality then implies that, for $2 \leq q \leq \infty$, according to the inequalities (2.11) and (2.12), we have

$$\|I_t^{(3)}f\|_{L^q} \leq \|I_t^{(3)}f\|_\infty^{1 - \frac{2}{q}} \|I_t^{(3)}f\|_{L^2}^{\frac{2}{q}} \leq C_m\|f\|_{H^m}(1 + |t|)^{(-\frac{1}{2} + \frac{3\delta}{2})(1 - \frac{2}{q})}.$$

□

3 Preliminary results

We now quote some useful results for the proof of our principal result. We will need the Mikhlín-Hörmander theorem proved in [12, chapter 6 - proposition 4.4].

Theorem 3.1

Let $\sigma : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{C}$ satisfying for all multi-index of length $0 \leq |\alpha| < l$, where l is the smallest integer with $l > \frac{n}{2}$, and for all $k \in \mathbb{R}^n \setminus \{0\}$,

$$|\partial_k^\alpha \sigma(k)| \leq C_\alpha |k|^{-|\alpha|}.$$

Then, for all $1 < p < +\infty$, there exists a constant $C_p > 0$, depending only on p , such that for all $u \in L^p(\mathbb{R}^n)$, we have

$$\|\mathcal{F}^{-1}(\sigma \hat{u})\|_{L^p} \leq C_p \|u\|_{L^p}.$$

The fractional Leibniz rule will be also used [9, lemma X.4].

Theorem 3.2

For all $m \geq 0$ and $1 < p < +\infty$, there exists a constant $C_{m,p} > 0$, depending on m and p such that for all u and v in $\mathcal{S}(\mathbb{R}^n)$, we have

$$\|uv\|_{W^{m,p}} \leq C_{m,p} (\|u\|_{W^{m,p}} \|v\|_\infty + \|u\|_\infty \|v\|_{W^{m,p}}).$$

We need the Sobolev inequality [5, chapter 1 - theorem 9.3].

Theorem 3.3

Let $1 \leq p, q \leq +\infty$ and $0 \leq j < m$, there exists a constant $C > 0$, depending on p, q, j and m , such that for all $u \in \mathcal{S}(\mathbb{R}^n)$, we have

$$\|(-\Delta)^{\frac{j}{2}} u\|_{L^r} \leq C \|(-\Delta)^{\frac{m}{2}} u\|_{L^p}^a \|u\|_{L^q}^{1-a},$$

where $\frac{1}{r} = \frac{j}{n} + a \left(\frac{1}{p} - \frac{m}{n} \right) + \frac{1-a}{q}$ and $\frac{j}{m} \leq a < 1$.

Finally, we prove an integration's lemma.

Lemma 3.4

Let $\alpha > 1$ and $\beta > 0$. Then there exists $0 < \beta_1 < 1$ and $C > 0$ such that for all time $t \geq 0$

$$\int_0^t \frac{d\tau}{(1+\tau)^\alpha (1+t-\tau)^\beta} \leq \frac{C}{(1+t)^{\beta_1}},$$

with $\beta_1 = \beta$ if $0 < \beta < 1$.

Proof. If $t = 0$, the result is obvious. Let $t > 0$, we separate the time integral in two parts

$$\int_0^t \frac{d\tau}{(1+\tau)^\alpha (1+t-\tau)^\beta} = \int_0^{t/2} \frac{d\tau}{(1+\tau)^\alpha (1+t-\tau)^\beta} + \int_{t/2}^t \frac{d\tau}{(1+\tau)^\alpha (1+t-\tau)^\beta}.$$

For the first integral, we have

$$\begin{aligned} \int_0^{t/2} \frac{d\tau}{(1+\tau)^\alpha (1+t-\tau)^\beta} &\leq \left(1 + \frac{t}{2}\right)^{-\beta} \int_0^{t/2} \frac{d\tau}{(1+\tau)^\alpha} \\ &\leq \left(1 + \frac{t}{2}\right)^{-\beta} \left[\frac{(1+\tau)^{-\alpha+1}}{-\alpha+1} \right]_0^{t/2}, \end{aligned}$$

since $-\alpha + 1 < 0$, we find

$$\int_0^{t/2} \frac{d\tau}{(1+\tau)^\alpha(1+t-\tau)^\beta} \leq C \left(1 + \frac{t}{2}\right)^{-\beta} \left[1 - \left(1 + \frac{t}{2}\right)^{-\alpha+1}\right] \leq C \left(1 + \frac{t}{2}\right)^{-\beta},$$

and it is enough to remark $1 + \frac{t}{2} \geq \frac{1}{2}(1+t)$ to obtain

$$\int_0^{t/2} \frac{d\tau}{(1+\tau)^\alpha(1+t-\tau)^\beta} \leq \frac{C}{(1+t)^\beta}.$$

Let us suppose first of all that $0 < \beta < 1$. For the second integral, we have

$$\begin{aligned} \int_{t/2}^t \frac{d\tau}{(1+\tau)^\alpha(1+t-\tau)^\beta} &\leq \left(1 + \frac{t}{2}\right)^{-\alpha} \int_{t/2}^t \frac{d\tau}{(1+t-\tau)^\beta} \\ &= \left(1 + \frac{t}{2}\right)^{-\alpha} \left[-\frac{(1+t-\tau)^{-\beta+1}}{-\beta+1} \right]_{t/2}^t. \end{aligned}$$

Since $-\beta + 1 > 0$ and $-\alpha + 1 < 0$, we have

$$\int_{t/2}^t \frac{d\tau}{(1+\tau)^\alpha(1+t-\tau)^\beta} \leq C \left(1 + \frac{t}{2}\right)^{-\alpha-\beta+1} \leq C (1+t)^{-\beta}.$$

Let us suppose now $\beta \geq 1$. Then for $0 < \beta_1 < 1$, we have for all $t \geq 0$

$$\int_0^t \frac{d\tau}{(1+\tau)^\alpha(1+t-\tau)^\beta} \leq \int_0^t \frac{d\tau}{(1+\tau)^\alpha(1+t-\tau)^{\beta_1}},$$

and the first part of the proof gives the result. \square

4 Existence and Uniqueness of global solution of the gBO-BBM equation

Let us return to the nonlinear problem. We consider the Cauchy problem

$$u_t + u_x + H(u_{xt}) + u^\rho u_x = 0 \quad (4.1)$$

$$u(x, 0) = f(x). \quad (4.2)$$

We show from now the principal result of this paper.

Proof. [Proof of the theorem 1.1.] The proof is carried out for positive time. By abuse, we will write C_m and $C_{m,p}$ the different constants depending respectively on m , and on m and p .

To simplify the writings, we denote $\frac{D_x}{1+|D_x|}$ the operator defined by the Fourier multiplier $\sigma(k) := \frac{ik}{1+|k|}$,

and we denote also $0 < \theta = \left(\frac{1}{2} - \frac{3\delta}{2}\right) \left(1 - \frac{2}{q}\right) < \frac{1}{2}$.

We remark that

$$\frac{3\theta\rho}{4} > 1. \quad (4.3)$$

Indeed we have

$$\frac{3\theta\rho}{4} > 1 \iff \frac{3}{4} \left(\frac{1}{2} - \frac{3\delta}{2} \right) \left(1 - \frac{2}{q} \right) \rho > 1 \iff (1 - 3\delta) \left(1 - \frac{2}{q} \right) > \frac{8}{3\rho},$$

since $1 - 3\delta > 0$, we find

$$\frac{3\theta\rho}{4} > 1 \iff \frac{1}{q} < \frac{1}{2} - \frac{4}{3\rho(1 - 3\delta)},$$

which inequality is the assumption on q .

The Duhamel formula implies that u is the solution of the gBO-BBM equation (4.1)-(4.2) if and only if u is the solution of the following equation, for $t \geq 0$

$$u(t) = \Phi u(t) := S_t f - \frac{1}{\rho + 1} \int_0^t S_{t-\tau} \left(\frac{D_x}{1 + |D_x|} u^{\rho+1} \right) (\tau) d\tau, \quad (4.4)$$

where $S_t f := \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ikx - it \frac{k}{1+|k|}} \hat{f}(k) dk$.

We remark first of all that in one hand, we have for $u \in \mathcal{S}(\mathbb{R})$

$$\begin{aligned} \left\| \frac{D_x}{1 + |D_x|} u^{\rho+1} \right\|_{H^m} &= \left(\int_{-\infty}^{+\infty} (1 + k^2)^m \left(\frac{ik}{1 + |k|} \right)^2 |\widehat{u^{\rho+1}}|^2 dk \right)^{1/2} \\ &\leq \left(\int_{-\infty}^{+\infty} (1 + k^2)^m |\widehat{u^{\rho+1}}|^2 dk \right)^{1/2} = \|u^{\rho+1}\|_{H^m}. \end{aligned} \quad (4.5)$$

In the other hand, we have

$$\text{for } k \geq 0, \quad |\sigma'(k)| = \frac{1}{(1+k)^2} \quad \text{and for } k \leq 0, \quad |\sigma'(k)| = \frac{1}{(1-k)^2}.$$

The Mihlin-Hörmander theorem 3.1 is applied to give, for $1 < p < 2$, there exists a constant $C_p > 0$ depending only on p such that

$$\left\| \frac{D_x}{1 + |D_x|} u^{\rho+1} \right\|_{L^p} \leq C_p \|u^{\rho+1}\|_{L^p}. \quad (4.6)$$

Let $T > 0$, we define the norm N_T by, for $u \in X^{m+1, 4 - \frac{2}{p}, p}(\mathbb{R})$,

$$N_T(u) := \sup_{0 \leq \tau \leq T} \left[(\|u\|_{L^q}(\tau) + \|u_x\|_{L^q}(\tau))(1 + \tau)^\theta + \|u\|_{X^{m+1, 4 - \frac{2}{p}, p}}(\tau) \right]. \quad (4.7)$$

We will proved separately some technical lemmas.

Lemma 4.1

There exists a constant $C_{m,p} > 0$ such that for all u and v in $X^{m+1, 4 - \frac{2}{p}, p}(\mathbb{R})$, we have

$$\|\Phi u - \Phi v\|_{L^q}(t) \leq C_{m,p} \left(\sum_{i=0}^{\rho} N_T(u)^{\rho-i} N_T(v)^i \right) (1+t)^{-\theta} N_T(u-v), \quad (4.8)$$

and

$$\|(\Phi u - \Phi v)_x\|_{L^q}(t) \leq C_{m,p} \left(\sum_{i=0}^{\rho} N_T(u)^{\rho-i} N_T(v)^i \right) (1+t)^{-\theta} N_T(u-v). \quad (4.9)$$

Proof. Let u and v two elements of $X^{m+1,4-\frac{2}{p},p}(\mathbb{R})$, the Duhamel formula gives

$$\|\Phi u - \Phi v\|_{L^q} \leq \frac{1}{\rho+1} \int_0^t \left\| S_{t-\tau} \left(\frac{D_x}{1+|D_x|} (u^{\rho+1} - v^{\rho+1}) \right) \right\|_{L^q}(\tau) d\tau.$$

The corollary 1 implies that there exists a constant $C_m > 0$, depending only on m , such that

$$\|\Phi u - \Phi v\|_{L^q} \leq C_m \int_0^t \frac{\left(\left\| \frac{D_x}{1+|D_x|} (u^{\rho+1} - v^{\rho+1}) \right\|_{L^p} + \left\| \frac{D_x}{1+|D_x|} (u^{\rho+1} - v^{\rho+1}) \right\|_{H^m} \right) (\tau)}{(1+t-\tau)^\theta} d\tau,$$

and the inequalities (4.5) and (4.6) imply

$$\|\Phi u - \Phi v\|_{L^q} \leq C_{m,p} \int_0^t \frac{(\|u^{\rho+1} - v^{\rho+1}\|_{L^p} + \|u^{\rho+1} - v^{\rho+1}\|_{H^m}) (\tau)}{(1+t-\tau)^\theta} d\tau.$$

From now, when there is no ambiguity, we will use the following notation: for any positive A and B , the notation $A \lesssim B$ means that there exists a constant $C > 0$ such that $A \leq C B$.

Since $u^{\rho+1} - v^{\rho+1} = (u - v) \sum_{i=0}^{\rho} u^{\rho-i} v^i$, we have according to the fractional Leibniz theorem 3.2 with $p = 2$ and the Minkowski inequality

$$\begin{aligned} \|u^{\rho+1} - v^{\rho+1}\|_{H^m}(\tau) &\lesssim \|u - v\|_{H^m} \sum_{i=0}^{\rho} \|u^{\rho-i} v^i\|_{\infty}(\tau) + \|u - v\|_{\infty} \sum_{i=0}^{\rho} \|u^{\rho-i} v^i\|_{H^m}(\tau) \\ &= \text{I}(\tau) + \text{II}(\tau). \end{aligned}$$

On the other hand, the Sobolev inequality 3.3 with $\frac{1}{p} + \frac{1}{q} = 1$, $r = \infty$, $j = 0$, $a = \frac{1}{4}$ and $m = 3 - \frac{2}{p}$ gives

$$\|u\|_{\infty} \leq C_p \|u\|_{W^{3-\frac{2}{p},p}}^{1/4} \|u\|_{L^q}^{3/4}. \quad (4.10)$$

According to the definition (4.7) of the norm N_T , we can see that the decay in time will be linked to the norm L^q and the power $3/4$. We deduce from the inequality (4.10) that for $\text{I}(\tau)$, we have

$$\begin{aligned} \text{I}(\tau) &\lesssim \|u - v\|_{H^m} \sum_{i=0}^{\rho} \left(\|u\|_{L^q}^{3/4} \|u\|_{W^{3-\frac{2}{p},p}}^{1/4} \right)^{\rho-i} \left(\|v\|_{L^q}^{3/4} \|v\|_{W^{3-\frac{2}{p},p}}^{1/4} \right)^i (\tau) \\ &\lesssim \|u - v\|_{H^m} \sum_{i=0}^{\rho} \left(\|u\|_{L^q} \frac{(1+\tau)^\theta}{(1+\tau)^\theta} \right)^{\frac{3(\rho-i)}{4}} \|u\|_{W^{3-\frac{2}{p},p}}^{\frac{\rho-i}{4}} \left(\|v\|_{L^q} \frac{(1+\tau)^\theta}{(1+\tau)^\theta} \right)^{\frac{3i}{4}} \|v\|_{W^{3-\frac{2}{p},p}}^{\frac{i}{4}} \\ &\lesssim \left(\sum_{i=0}^{\rho} N_T(u)^{\rho-i} N_T(v)^i \right) (1+\tau)^{-\frac{3\theta\rho}{4}} N_T(u-v). \end{aligned}$$

For $\text{II}(\tau)$, the theorem 3.2 is again applied and we have

$$\begin{aligned} \text{II}(\tau) &\lesssim \|u - v\|_{\infty} \left(\|v\|_{\infty}^{\rho-1} \|v\|_{H^m} + \sum_{i=0}^{\rho-1} \|u\|_{\infty}^{\rho-i-1} \|u\|_{H^m} \|v\|_{\infty}^i \right. \\ &\quad \left. + \|u\|_{\infty}^{\rho-1} \|u\|_{H^m} + \sum_{i=1}^{\rho} \|u\|_{\infty}^{\rho-i} \|v\|_{\infty}^{i-1} \|v\|_{H^m} \right) (\tau), \end{aligned}$$

and the inequality (4.10) gives

$$\begin{aligned}
\Pi(\tau) &\lesssim \|u - v\|_{L^q}^{3/4} \|u - v\|_{W^{3-\frac{2}{p},p}}^{1/4} \times \left[\left(\|v\|_{L^q}^{3/4} \|v\|_{W^{3-\frac{2}{p},p}}^{1/4} \right)^{\rho-1} \|v\|_{H^m} + \left(\|u\|_{L^q}^{3/4} \|u\|_{W^{3-\frac{2}{p},p}}^{1/4} \right)^{\rho-1} \|u\|_{H^m} \right. \\
&\quad + \sum_{i=0}^{\rho-1} \left(\|u\|_{L^q}^{3/4} \|u\|_{W^{3-\frac{2}{p},p}}^{1/4} \right)^{\rho-i-1} \|u\|_{H^m} \left(\|v\|_{L^q}^{3/4} \|v\|_{W^{3-\frac{2}{p},p}}^{1/4} \right)^i \\
&\quad \left. + \sum_{i=1}^{\rho} \left(\|u\|_{L^q}^{3/4} \|u\|_{W^{3-\frac{2}{p},p}}^{1/4} \right)^{\rho-i} \left(\|v\|_{L^q}^{3/4} \|v\|_{W^{3-\frac{2}{p},p}}^{1/4} \right)^{i-1} \|v\|_{H^m} \right] (\tau) \\
&\lesssim \left(\sum_{i=0}^{\rho} N_T(u)^{\rho-i} N_T(v)^i \right) (1 + \tau)^{-\frac{3\theta\rho}{4}} N_T(u - v).
\end{aligned}$$

In the same manner, we have

$$\begin{aligned}
\|u^{\rho+1} - v^{\rho+1}\|_{L^p}(\tau) &\leq \|u - v\|_{L^p} \sum_{i=0}^{\rho} \|u\|_{\infty}^{\rho-i} \|v\|_{\infty}^i(\tau) \\
&\lesssim \left(\sum_{i=0}^{\rho} N_T(u)^{\rho-i} N_T(v)^i \right) (1 + \tau)^{-\frac{3\theta\rho}{4}} N_T(u - v).
\end{aligned}$$

Finally, we find

$$\|\Phi u - \Phi v\|_{L^q}(t) \leq C_{m,p} \left(\sum_{i=0}^{\rho} N_T(u)^{\rho-i} N_T(v)^i \right) N_T(u - v) \int_0^t \frac{d\tau}{(1 + \tau)^{\frac{3\theta\rho}{4}} (1 + t - \tau)^\theta}.$$

Since $\frac{3\theta\rho}{4} > 1$ by the inequality (4.3) and $0 < \theta < \frac{1}{2}$, the lemma 3.4 can be applied to give

$$\|\Phi u - \Phi v\|_{L^q}(t) \leq C_{m,p} \left(\sum_{i=0}^{\rho} N_T(u)^{\rho-i} N_T(v)^i \right) (1 + t)^{-\theta} N_T(u - v).$$

We have, for the space derivate,

$$(\Phi u - \Phi v)_x(t) = \int_0^t S_{t-\tau} \left(\frac{D_x}{1 + |D_x|} (u^\rho u_x - v^\rho v_x) \right) (\tau) d\tau,$$

the corollary 1 and the inequalities (4.5) and (4.6) imply

$$\|(\Phi u - \Phi v)_x\|_{L^q}(t) \leq C_{m,p} \int_0^t \frac{(\|u^\rho u_x - v^\rho v_x\|_{L^p} + \|u^\rho u_x - v^\rho v_x\|_{H^m})(\tau)}{(1 + t - \tau)^\theta} d\tau.$$

Since $u^{\rho+1} - v^{\rho+1} = (u - v) \sum_{i=0}^{\rho} u^{\rho-i} v^i$, the fractional Leibniz rule gives

$$\begin{aligned}
\|u^\rho u_x - v^\rho v_x\|_{H^m}(\tau) &= \|(u - v) \sum_{i=0}^{\rho} u^{\rho-i} v^i\|_{H^{m+1}}(\tau) \\
&\lesssim \|u - v\|_{H^{m+1}} \sum_{i=0}^{\rho} \|u^{\rho-i} v^i\|_{\infty}(\tau) + \|u - v\|_{\infty} \sum_{i=0}^{\rho} \|u^{\rho-i} v^i\|_{H^{m+1}}(\tau) = \text{I}(\tau) + \text{II}(\tau).
\end{aligned}$$

We deduce from the inequality (4.10) that for $I(\tau)$, we have

$$\begin{aligned} I(\tau) &\lesssim \|u - v\|_{H^{m+1}} \sum_{i=0}^{\rho} \left(\|u\|_{L^q}^{3/4} \|u\|_{W^{3-\frac{2}{p},p}}^{1/4} \right)^{\rho-i} \left(\|v\|_{L^q}^{3/4} \|v\|_{W^{3-\frac{2}{p},p}}^{1/4} \right)^i (\tau) \\ &\lesssim \left(\sum_{i=0}^{\rho} N_T(u)^{\rho-i} N_T(v)^i \right) (1 + \tau)^{-\frac{3\theta\rho}{4}} N_T(u - v). \end{aligned}$$

For the second norm, the theorem 3.2 is again applied and we have

$$\begin{aligned} II(\tau) &\lesssim \|u - v\|_{\infty} \left(\|v\|_{\infty}^{\rho-1} \|v\|_{H^{m+1}} + \sum_{i=0}^{\rho-1} \|u\|_{\infty}^{\rho-i-1} \|u\|_{H^{m+1}} \|v\|_{\infty}^i \right. \\ &\quad \left. + \|u\|_{\infty}^{\rho-1} \|u\|_{H^{m+1}} + \sum_{i=1}^{\rho} \|u\|_{\infty}^{\rho-i} \|v\|_{\infty}^{i-1} \|v\|_{H^{m+1}} \right) (\tau), \end{aligned}$$

and the inequality (4.10) gives

$$\begin{aligned} II(\tau) &\lesssim \|u - v\|_{L^q}^{3/4} \|u - v\|_{W^{3-\frac{2}{p},p}}^{1/4} \left[\left(\|v\|_{L^q}^{3/4} \|v\|_{W^{3-\frac{2}{p},p}}^{1/4} \right)^{\rho-1} \|v\|_{H^{m+1}} + \left(\|u\|_{L^q}^{3/4} \|u\|_{W^{3-\frac{2}{p},p}}^{1/4} \right)^{\rho-1} \|u\|_{H^{m+1}} \right. \\ &\quad + \sum_{i=0}^{\rho-1} \left(\|u\|_{L^q}^{3/4} \|u\|_{W^{3-\frac{2}{p},p}}^{1/4} \right)^{\rho-i-1} \|u\|_{H^{m+1}} \left(\|v\|_{L^q}^{3/4} \|v\|_{W^{3-\frac{2}{p},p}}^{1/4} \right)^i \\ &\quad \left. + \sum_{i=1}^{\rho} \left(\|u\|_{L^q}^{3/4} \|u\|_{W^{3-\frac{2}{p},p}}^{1/4} \right)^{\rho-i} \left(\|v\|_{L^q}^{3/4} \|v\|_{W^{3-\frac{2}{p},p}}^{1/4} \right)^{i-1} \|v\|_{H^{m+1}} \right] (\tau) \\ &\lesssim \left(\sum_{i=0}^{\rho} N_T(u)^{\rho-i} N_T(v)^i \right) (1 + \tau)^{-\frac{3\theta\rho}{4}} N_T(u - v). \end{aligned}$$

In the same manner,

$$\begin{aligned} \|u^\rho u_x - v^\rho v_x\|_{L^p}(\tau) &\lesssim \|(u - v)_x\|_{L^p} \sum_{i=0}^{\rho} \|u\|_{\infty}^{\rho-i} \|v\|_{\infty}^i (\tau) \\ &\quad + \|u - v\|_{L^p} \left(\sum_{i=0}^{\rho-1} \|u\|_{\infty}^{\rho-i-1} \|u_x\|_{\infty} \|v\|_{\infty}^i + \sum_{i=1}^{\rho} \|u\|_{\infty}^{\rho-i} \|v\|_{\infty}^{i-1} \|v_x\|_{\infty} \right) (\tau). \end{aligned}$$

According to the inequality (4.10), $\|u_x\|_{\infty} \leq \|u_x\|_{L^q}^{3/4} \|u_x\|_{W^{4-\frac{2}{p},p}}^{1/4} = \|u_x\|_{L^q}^{3/4} \|u\|_{W^{4-\frac{2}{p},p}}^{1/4}$, we deduce

$$\begin{aligned} \|u^\rho u_x - v^\rho v_x\|_{L^p}(\tau) &\lesssim \|u - v\|_{W^{1,p}} \sum_{i=0}^{\rho} \left(\|u\|_{L^q}^{3/4} \|u\|_{W^{3-\frac{2}{p},p}}^{1/4} \right)^{\rho-i} \left(\|v\|_{L^q}^{3/4} \|v\|_{W^{3-\frac{2}{p},p}}^{1/4} \right)^i (\tau) \\ &\quad + \|u - v\|_{L^p} \left[\sum_{i=0}^{\rho-1} \left(\|u\|_{L^q}^{3/4} \|u\|_{W^{3-\frac{2}{p},p}}^{1/4} \right)^{\rho-i-1} \left(\|u_x\|_{L^q}^{3/4} \|u\|_{W^{4-\frac{2}{p},p}}^{1/4} \right) \left(\|v\|_{L^q}^{3/4} \|v\|_{W^{3-\frac{2}{p},p}}^{1/4} \right)^i \right. \\ &\quad \left. + \sum_{i=1}^{\rho} \left(\|u\|_{L^q}^{3/4} \|u\|_{W^{3-\frac{2}{p},p}}^{1/4} \right)^{\rho-i} \left(\|v\|_{L^q}^{3/4} \|v\|_{W^{3-\frac{2}{p},p}}^{1/4} \right)^{i-1} \left(\|v_x\|_{L^q}^{3/4} \|v\|_{W^{4-\frac{2}{p},p}}^{1/4} \right) \right] (\tau), \end{aligned}$$

and since $4 - \frac{2}{p} > 1$, we have

$$\|u^\rho u_x - v^\rho v_x\|_{L^p}(\tau) \lesssim \left(\sum_{i=0}^{\rho} N_T(u)^{\rho-i} N_T(v)^i \right) (1+\tau)^{-\frac{3\theta\rho}{4}} N_T(u-v).$$

Thus we find

$$\|(\Phi u - \Phi v)_x\|_{L^q}(t) \leq C_{m,p} \left(\sum_{i=0}^{\rho} N_T(u)^{\rho-i} N_T(v)^i \right) (1+t)^{-\theta} N_T(u-v).$$

□

Lemma 4.2

There exists a constant $C_{m,p} > 0$ such that for all u and v in $X^{m+1,4-\frac{2}{p},p}(\mathbb{R})$, we have

$$\|\Phi u - \Phi v\|_{X^{m+1,4-\frac{2}{p},p}}(t) \leq C_{m,p} \left(\sum_{i=0}^{\rho} N_T(u)^{\rho-i} N_T(v)^i \right) N_T(u-v). \quad (4.11)$$

Proof. We now majorize the norm $\|(\Phi u - \Phi v)_x\|_{X^{m+1,4-\frac{2}{p},p}}(t)$,

$$\|\Phi u - \Phi v\|_{X^{m+1,4-\frac{2}{p},p}}(t) \leq \frac{1}{\rho+1} \int_0^t \left\| S_{t-\tau} \left(\frac{D_x}{1+|D_x|} (u^{\rho+1} - v^{\rho+1}) \right) \right\|_{X^{m+1,4-\frac{2}{p},p}}(\tau) d\tau,$$

the inequalities (4.5) and (4.6) imply

$$\|\Phi u - \Phi v\|_{X^{m+1,4-\frac{2}{p},p}}(t) \leq \frac{1}{\rho+1} \int_0^t \|u^{\rho+1} - v^{\rho+1}\|_{X^{m+1,4-\frac{2}{p},p}}(\tau) d\tau.$$

Since $X^{m+1,4-\frac{2}{p},p}(\mathbb{R}) = H^{m+1}(\mathbb{R}) \cap W^{4-\frac{2}{p},p}(\mathbb{R})$ and $u^{\rho+1} - v^{\rho+1} = (u-v) \sum_{i=0}^{\rho} u^{\rho-i} v^i$, we have

$$\|u^{\rho+1} - v^{\rho+1}\|_{X^{m+1,4-\frac{2}{p},p}}(\tau) = \|u^{\rho+1} - v^{\rho+1}\|_{H^{m+1}}(\tau) + \|u^{\rho+1} - v^{\rho+1}\|_{W^{4-\frac{2}{p},p}}(\tau),$$

and as previously,

$$\|u^{\rho+1} - v^{\rho+1}\|_{H^{m+1}}(\tau) \lesssim \left(\sum_{i=0}^{\rho} N_T(u)^{\rho-i} N_T(v)^i \right) N_T(u-v) (1+\tau)^{-\frac{3\theta\rho}{4}}.$$

Since $4 - \frac{2}{p} \geq 0$, the fractional Leibniz theorem 3.2 gives

$$\begin{aligned} \|u^{\rho+1} - v^{\rho+1}\|_{W^{4-\frac{2}{p},p}}(\tau) &\lesssim \|u-v\|_{W^{4-\frac{2}{p},p}} \sum_{i=0}^{\rho} \|u^{\rho-i} v^i\|_{\infty}(\tau) + \|u-v\|_{\infty} \sum_{i=0}^{\rho} \|u^{\rho-i} v^i\|_{W^{4-\frac{2}{p},p}}(\tau) \\ &= \text{I}(\tau) + \text{II}(\tau). \end{aligned}$$

We deduce from the inequality (4.10)

$$\begin{aligned} \text{I}(\tau) &\lesssim \|u-v\|_{W^{4-\frac{2}{p},p}} \sum_{i=0}^{\rho} \left(\|u\|_{L^q}^{3/4} \|u\|_{W^{3-\frac{2}{p},p}}^{1/4} \right)^{\rho-i} \left(\|v\|_{L^q}^{3/4} \|v\|_{W^{3-\frac{2}{p},p}}^{1/4} \right)^i(\tau) \\ &\lesssim \left(\sum_{i=0}^{\rho} N_T(u)^{\rho-i} N_T(v)^i \right) (1+\tau)^{-\frac{3\theta\rho}{4}} N_T(u-v). \end{aligned}$$

For $\Pi(\tau)$, the theorem 3.2 is again applied and we have

$$\begin{aligned} \Pi(\tau) \lesssim & \|u - v\|_\infty \left(\|v\|_\infty^{\rho-1} \|v\|_{W^{4-\frac{2}{p},p}} + \sum_{i=0}^{\rho-1} \|u\|_\infty^{\rho-i-1} \|u\|_{W^{4-\frac{2}{p},p}} \|v\|_\infty^i \right. \\ & \left. + \|u\|_\infty^{\rho-1} \|u\|_{W^{4-\frac{2}{p},p}} + \sum_{i=1}^{\rho} \|u\|_\infty^{\rho-i} \|v\|_\infty^{i-1} \|v\|_{W^{4-\frac{2}{p},p}} \right) (\tau), \end{aligned}$$

and the inequality (4.10) gives

$$\Pi(\tau) \lesssim \left(\sum_{i=0}^{\rho} N_T(u)^{\rho-i} N_T(v)^i \right) (1 + \tau)^{-\frac{3\theta\rho}{4}} N_T(u - v).$$

Finally we find

$$\|u^{\rho+1} - v^{\rho+1}\|_{X^{m+1,4-\frac{2}{p},p}}(\tau) \lesssim \left(\sum_{i=0}^{\rho} N_T(u)^{\rho-i} N_T(v)^i \right) N_T(u - v) (1 + \tau)^{-\frac{3\theta\rho}{4}},$$

thus

$$\|\Phi u - \Phi v\|_{X^{m+1,4-\frac{2}{p},p}}(t) \lesssim \left(\sum_{i=0}^{\rho} N_T(u)^{\rho-i} N_T(v)^i \right) N_T(u - v) \int_0^t (1 + \tau)^{-\frac{3\theta\rho}{4}} d\tau,$$

and thanks to the inequality (4.3),

$$\|\Phi u - \Phi v\|_{X^{m+1,4-\frac{2}{p},p}}(t) \leq C_{m,p} \left(\sum_{i=0}^{\rho} N_T(u)^{\rho-i} N_T(v)^i \right) N_T(u - v).$$

□

Lemma 4.3

There exists a constant $C_{m,p} > 0$ such that for all u and v in $X^{m+1,4-\frac{2}{p},p}(\mathbb{R})$, we have

$$N_T(\Phi u - \Phi v) \leq C_{m,p} \left(\sum_{i=0}^{\rho} N_T(u)^{\rho-i} N_T(v)^i \right) N_T(u - v), \quad (4.12)$$

and

$$N_T(\Phi u) \leq C_{m,p} \left(\|f\|_{X^{m+1,4-\frac{2}{p},p}} + N_T(u)^{\rho+1} \right). \quad (4.13)$$

Proof. The inequalities (4.8), (4.9) and (4.11) give the inequality (4.12). For the second inequality, it is enough to take $v = 0$. □

Let $M > 0$, we consider the closed ball

$$\overline{B}_{T,M} := \left\{ u \in \mathcal{C}([-T, T]; X^{m+1,4-\frac{2}{p},p}(\mathbb{R})); N_T(u) \leq M \right\}.$$

We would like to show that there exists a unique solution u of the equation (4.4) in this ball by using the fixed point theorem.

First, there exists $\varepsilon > 0$ sufficiently small such that if $\|f\|_{X^{m+1,4-\frac{2}{p},p}} \leq \varepsilon$, even if we take $C_{m,p}M$ instead of M , it is enough to take $M > 0$ satisfying $\varepsilon + M^{\rho+1} \leq M$ so that the inequality (4.13) implies that

the image of the closed ball $\overline{B}_{T,M}$ by the map Φ is include in itself. Here, the crucial point is that ε is independent of T . Secondly, we prove that the map Φ is a contraction on this ball for M sufficiently small. Let u and v two elements of the closed ball $\overline{B}_{T,M}$. The inequality (4.12) gives

$$N_T(\Phi u - \Phi v) \leq C_{m,p} M^\rho N_T(u - v),$$

and it is enough to take $M > 0$ sufficiently small so that the quantity $C_{m,p} M^\rho < 1$. Then, the fixed point theorem is applied and there exists an unique solution of the equation (4.4) in the closed ball $\overline{B}_{T,M}$. It remains to prove that this unique solution can be prolonged in time with all $[0, +\infty[$. By uniqueness of the solution, the inequality (4.13) is written

$$N_T(u) \leq C_{m,p} \left(\|f\|_{X^{m+1,4-\frac{2}{p},p}} + N_T(u)^{\rho+1} \right). \quad (4.14)$$

Since there exists $\varepsilon > 0$ sufficiently small such that $\|f\|_{X^{m+1,4-\frac{2}{p},p}} \leq \varepsilon$, we can find $M > 0$ such that

$$\begin{aligned} N_0(u) &< M \\ C_{m,p} (\varepsilon + M^{\rho+1}) &\leq M. \end{aligned}$$

Then for all $T > 0$, we have $N_T(u) < M$. Indeed, if not by continuity, there exists a time $T > 0$ such that

$$\begin{aligned} N_T(u) &= M \\ &> C_{m,p} (\varepsilon + M^{\rho+1}) \\ &> C_{m,p} (\varepsilon + N_T(u)^{\rho+1}), \end{aligned}$$

what contradicts the inequality (4.14). Finally, there exists a constant $M > 0$ such that for all $T > 0$, $N_T(u) < M$. In particular, we have for all time $t \geq 0$

$$\begin{aligned} \|u(t)\|_{L^q} + \|u_x(t)\|_{L^q} &\leq C_{m,p} \left(\|f\|_{X^{m+1,4-\frac{2}{p},p}} + N_T(u)^{\rho+1} \right) (1+t)^{-\theta} \\ &\leq C_{m,p} (\varepsilon + M^{\rho+1}) (1+t)^{-\theta}. \end{aligned} \quad (4.15)$$

We reason in a similar manner for negative times. \square

5 Decay in time of the gKP-BBM-II equation

We consider the linear Cauchy problem

$$u_t + u_x - u_{xxt} + \partial_x^{-1} u_{yy} = 0; \quad u(x, y, 0) = f(x, y).$$

Let us suppose that the initial datum f belongs to the Schwartz space $\mathcal{S}(\mathbb{R}^2)$. Then the Fourier transform in space implies

$$\hat{u}_t(1+k^2) + i \left(k + \frac{l^2}{k} \right) \hat{u} = 0; \quad \hat{u}(k, l, 0) = \hat{f}(k, l).$$

The solution u of this ordinary differential equation is given by, for all $(x, y) \in \mathbb{R}^2$ and $t \in \mathbb{R}^*$,

$$u(x, y, t) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{-ith_{\alpha,\beta}(k,l)} \hat{f}(k, l) dk dl, \quad (5.1)$$

with $h_{\alpha,\beta}(k,l) = \frac{k + \frac{l^2}{k}}{1 + k^2} - \alpha k - \beta l$, $\alpha = \frac{x}{t}$ and $\beta = \frac{y}{t}$. We have to majorize this oscillating integral in function of time.

Lemma 5.1

Let $0 \leq \delta < \frac{1}{5}$. For all $(\alpha, \beta) \in \mathbb{R}^2$, and all time $|t| \geq 1$, we have

$$\left| \int_{-|t|^\delta}^{|t|^\delta} \int_{-\infty}^{+\infty} e^{-ith_{\alpha,\beta}(k,l)} dl dk \right| \leq 3\sqrt{\pi} |t|^{-\frac{1}{2} + \frac{5\delta}{2}}.$$

Proof. We adapt the Haysahi-Naumkin-Saut method [7] to the generalized KP-BBM-II equation. It consists of transforming the oscillating double integral (5.1) in a simple one. For $t \in \mathbb{R}^*$, we do the change

of variables $l = l' \sqrt{\frac{|k|(1+k^2)}{|t|}}$ to obtain

$$\begin{aligned} \int_{-|t|^\delta}^{|t|^\delta} \int_{-\infty}^{+\infty} e^{-ith_{\alpha,\beta}(k,l)} dl dk &= \int_{-|t|^\delta}^{|t|^\delta} \int_{-\infty}^{+\infty} e^{ikx + iy l' \sqrt{\frac{|k|(1+k^2)}{|t|}} - it \frac{k}{1+k^2} - il'^2 \operatorname{sgn}(\frac{k}{t})} \sqrt{\frac{|k|(1+k^2)}{|t|}} dl' dk \\ &= \frac{1}{|t|^{1/2}} \int_{-|t|^\delta}^{|t|^\delta} \sqrt{|k|(1+k^2)} e^{ikx - it \frac{k}{1+k^2}} \left(\int_{-\infty}^{+\infty} e^{iy l' \sqrt{\frac{|k|(1+k^2)}{|t|}} - il'^2 \operatorname{sgn}(\frac{k}{t})} dl' \right) dk. \end{aligned}$$

However,

$$\begin{aligned} iy l' \sqrt{\frac{|k|(1+k^2)}{|t|}} - il'^2 \operatorname{sgn}(\frac{k}{t}) &= -i \operatorname{sgn}(\frac{k}{t}) \left(l'^2 - l' y \sqrt{\frac{|k|(1+k^2)}{|t|}} \right) \\ &= -i \operatorname{sgn}(\frac{k}{t}) \left(l'^2 - \frac{y}{2} \sqrt{\frac{|k|(1+k^2)}{|t|}} \right)^2 + i \frac{y^2 k (1+k^2)}{4t}. \end{aligned}$$

A last change of variable gives

$$\begin{aligned} \int_{-\infty}^{+\infty} e^{iy l' \sqrt{\frac{|k|(1+k^2)}{|t|}} - il'^2 \operatorname{sgn}(\frac{k}{t})} dl' &= e^{\frac{y^2 k (1+k^2)}{4t}} \int_{-\infty}^{+\infty} e^{\pm i l^2} dl \\ &= \sqrt{\pi} e^{\pm i \pi / 4} e^{\frac{y^2 k (1+k^2)}{4t}}. \end{aligned}$$

Finally, we find

$$\int_{-|t|^\delta}^{|t|^\delta} \int_{-\infty}^{+\infty} e^{-ith_{\alpha,\beta}(k,l)} dl dk = \frac{\sqrt{\pi} e^{\pm i \pi / 4}}{|t|^{1/2}} \int_{-|t|^\delta}^{|t|^\delta} \sqrt{|k|(1+k^2)} e^{ikx - it \frac{k}{1+k^2} + \frac{y^2}{4t^2} (k+k^3)} dk. \quad (5.2)$$

We deduce then

$$\begin{aligned} \left| \int_{-|t|^\delta}^{|t|^\delta} \int_{-\infty}^{+\infty} e^{-ith_{\alpha,\beta}(k,l)} dl dk \right| &= \sqrt{\pi} |t|^{-1/2} \int_{-|t|^\delta}^{|t|^\delta} \sqrt{|k|(1+k^2)} dk \\ &\leq 3\sqrt{\pi} |t|^{-\frac{1}{2} + \frac{5\delta}{2}}. \end{aligned}$$

□

The oscillating integral (5.1) can be majorized.

Lemma 5.2

Let $0 < \delta < \frac{1}{5}$ and $m = \frac{1}{2\delta} - 2$. There exists a constant $C_m > 0$, depending only on m , such that for all $f \in H^{2m}(\mathbb{R}^2)$, $(\alpha, \beta) \in \mathbb{R}^2$ and all time $|t| \geq 1$, we have

$$\left| \int_{|t|^\delta}^{+\infty} \int_{-\infty}^{+\infty} e^{-ith_{\alpha,\beta}(k,l)} \hat{f}(k,l) dl dk \right| \leq C_m \|f\|_{H^{2m}} |t|^{-\frac{1}{2} + \frac{5\delta}{2}},$$

and

$$\left| \int_{-\infty}^{-|t|^\delta} \int_{-\infty}^{+\infty} e^{-ith_{\alpha,\beta}(k,l)} \hat{f}(k,l) dl dk \right| \leq C_m \|f\|_{H^{2m}} |t|^{-\frac{1}{2} + \frac{5\delta}{2}}.$$

Proof. We have

$$\left| \int_{|t|^\delta}^{+\infty} \int_{-\infty}^{+\infty} e^{-ith_{\alpha,\beta}(k,l)} \hat{f}(k,l) dl dk \right| \leq \int_{|t|^\delta}^{+\infty} \int_{-\infty}^{+\infty} |\hat{f}(k,l)| dl dk,$$

the Fubini theorem implies

$$\left| \int_{|t|^\delta}^{+\infty} \int_{-\infty}^{+\infty} e^{-ith_{\alpha,\beta}(k,l)} \hat{f}(k,l) dl dk \right| \leq \int_{-\infty}^{+\infty} \frac{(1+l^2)^{m/2}}{(1+l^2)^{m/2}} \int_{|t|^\delta}^{+\infty} \frac{(1+k)^m}{(1+k)^m} |\hat{f}(k,l)| dk dl,$$

and the Cauchy-Schwarz inequality, first in k and secondly in l , gives the result. \square

Proposition 2

Let $0 < \delta < \frac{1}{5}$ and $m = \frac{1}{2\delta} - 2$. There exists a constant $C_m > 0$, depending only on m , such that for all function $f \in L^1(\mathbb{R}^2) \cap H^{2m}(\mathbb{R}^2)$, $(\alpha, \beta) \in \mathbb{R}^2$ and all time $t \in \mathbb{R}$, we have

$$\left| \int_{\mathbb{R}^2} e^{-ith_{\alpha,\beta}(k,l)} \hat{f}(k,l) dl dk \right| \leq C_m (\|f\|_{L^1} + \|f\|_{H^{2m}}) (1 + |t|)^{-\frac{1}{2} + \frac{5\delta}{2}}. \quad (5.3)$$

Proof. It is enough to separate only the integral in direction k and to take again the proof of the proposition 1. \square

We can deduce in the same way as the corollary 1 the following result.

Corollary 2

Let $0 < \delta < \frac{1}{5}$ and $m = \frac{1}{2\delta} - 2$. There exists a constant $C_m > 0$, depending only on m , such that for all function $f \in L^p(\mathbb{R}^2) \cap H^{2m}(\mathbb{R}^2)$, with $1 \leq p \leq 2$, for all $(\alpha, \beta) \in \mathbb{R}^2$ and all time $t \in \mathbb{R}$, we have

$$\left\| \int_{\mathbb{R}^2} e^{-ith_{\alpha,\beta}(k,l)} \hat{f}(k,l) dl dk \right\|_{L^q} \leq C_m (\|f\|_{L^p} + \|f\|_{H^{2m}}) (1 + |t|)^{\left(-\frac{1}{2} + \frac{5\delta}{2}\right) \left(1 - \frac{2}{q}\right)}, \quad (5.4)$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Let us return to the nonlinear problem. We consider the Cauchy problem

$$u_t + u_x + u^\rho u_x - u_{xxt} + \partial_x^{-1} u_{yy} = 0 \quad (5.5)$$

$$u(x, y, 0) = f(x, y). \quad (5.6)$$

The existence and uniqueness of global solution, given by the theorem 1.2, is proved.

Proof. [Proof of the theorem 1.2] To simplify the writings, we denote $\frac{D_x}{1+D_x^2}$ the operator defined by

the Fourier multiplier $\sigma(k, l) := \frac{ik}{1+k^2}$, and we denote also

$$0 < \theta = \left(\frac{1}{2} - \frac{5\delta}{2}\right) \left(1 - \frac{2}{q}\right) < \frac{1}{2}.$$

We remark that

$$\frac{3\theta\rho}{4} > 1. \quad (5.7)$$

The Duhamel formula implies that u is the solution of the gKP-BBM-II equation (5.5)-(5.6) if and only if u is the solution of the following equation, for $t \geq 0$

$$u(t) = \Phi u(t) := S_t f - \frac{1}{\rho+1} \int_0^t S_{t-\tau} \left(\frac{D_x}{1+D_x^2} u^{\rho+1} \right) (\tau) d\tau, \quad (5.8)$$

where $S_t f := \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{ikx+ily-it\frac{k+l^2/k}{1+k^2}} \hat{f}(k, l) dl dk$.

We remark first of all that in one hand, we have for $u \in \mathcal{S}(\mathbb{R}^2)$

$$\begin{aligned} \left\| \frac{D_x}{1+D_x^2} u^{\rho+1} \right\|_{H^m} &= \left(\int_{\mathbb{R}^2} (1+k^2+l^2)^m \left(\frac{ik}{1+k^2} \right)^2 |\widehat{u^{\rho+1}}|^2 dk dl \right)^{1/2} \\ &\leq \left(\int_{\mathbb{R}^2} (1+k^2+l^2)^m |\widehat{u^{\rho+1}}|^2 dk dl \right)^{1/2} = \|u^{\rho+1}\|_{H^m}. \end{aligned} \quad (5.9)$$

In the other hand, the Mihlin-Hörmander theorem 3.1 is applied with σ , defined above, to give, for $1 < p < 2$, there exists a constant $C_p > 0$ depending only on p such that

$$\left\| \frac{D_x}{1+D_x^2} u^{\rho+1} \right\|_{L^p} \leq C_p \|u^{\rho+1}\|_{L^p}. \quad (5.10)$$

Let $T > 0$, we define the norm N_T by, for $u \in X^{2m+1, 7-\frac{4}{p}, p}(\mathbb{R}^2)$,

$$N_T(u) := \sup_{0 \leq \tau \leq T} \left[(\|u\|_{L^q}(\tau) + \|u_x\|_{L^q}(\tau))(1+\tau)^\theta + \|u\|_{X^{2m+1, 7-\frac{4}{p}, p}}(\tau) \right]. \quad (5.11)$$

From now, the end of the proof is similar to that of the theorem 1.1 by remarking that the Sobolev inequality 3.3 in dimension 2 with $\frac{1}{p} + \frac{1}{q} = 1$, $r = \infty$, $j = 0$, $a = \frac{1}{4}$ and $m = 6 - \frac{4}{p}$ gives

$$\|u\|_\infty \leq C_p \|u\|_{W^{6-\frac{4}{p}, p}}^{1/4} \|u\|_{L^q}^{3/4}. \quad (5.12)$$

According to the definition (5.11) of the norm N_T , we can see that the decay in time will be linked to the norm L^q and the power $3/4$. \square

Remark 1 *If we compare the generalized Korteweg-de Vries equation and his BBM version, we remark that the decay rate in time is of the same order. In dimension 2, Haysahi, Naumkin and Saut [7] found, for the generalized KP equation, a decay rate in time of order 1. However, nothing says that our result is optimal and we wonder if we can find a decay rate in time better than 1/2, and approach 1 for the BBM version. We plan to study this issue in a future work.*

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