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# Unique continuation property for the KP-BBM-II equation

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**Abstract.** We prove, by using a method introduced by Constantin [6], that if the solution of the Cauchy problem associated with the KP-BBM-II equation, has a compact support for all times, then this solution vanishes identically. The only restriction is that the support in the  $y$ -direction has to be small.

**Keywords.** KP equation, unique continuation property, compact support

**MS Codes.** 35B60, 35Q53

## Introduction

The small amplitude long waves in shallow water moving mainly in the  $x$ -direction are modelled in dimension 2 by the Kadomtsev-Petviashvili equations [7]

$$u_t + u_x + \alpha uu_x + s\beta u_{xxx} + \gamma \partial_x^{-1} u_{yy} = 0,$$

called KP-I if  $s = -1$  and KP-II if  $s = 1$ , according to whether the surface tension is or is not neglected, and where  $\alpha$  denotes the quotient between the waves amplitude and the depth of the water,  $\beta$  the square of the quotient between the depth and the wavelength,  $\gamma$  the square of the quotient between the wavelength in the two directions of the plan and  $\partial_x^{-1}$  denotes the anti-derivative, defined by the Fourier symbol  $\widehat{\partial_x^{-1} u}(k) := \frac{\hat{u}(k)}{ik}$ . We recall that  $0 \leq \alpha, \beta, \gamma \leq 1$ . Since this equation is obtained correcting, at the second order, the transport equation  $u_t + u_x = 0$ , we find the KP-BBM equations [1]

$$u_t + u_x + \alpha uu_x - s\beta u_{xxt} + \gamma \partial_x^{-1} u_{yy} = 0,$$

called KP-BBM-I if  $s = -1$  and KP-BBM-II if  $s = 1$ . The KP-BBM-I equation is not well posed in  $L^2(\mathbb{R}^2)$  and so we study here only the KP-BBM-II equation. Indeed, the KP-BBM-I equation can be rewritten

$$(1 + \beta \partial_x^2) u_t = -u_x - \alpha uu_x - \gamma \partial_x^{-1} u_{yy},$$

and the operator  $(1 + \beta \partial_x^2)$  is not invertible in  $L^2(\mathbb{R}^2)$ .

The goal of this work is to show that if the solution of the Cauchy problem associated with the KP-BBM-II equation has a compact support on a time interval, then this solution vanishes identically.

In dimension 1, Saut and Scheurer [11] proved the unique continuation property for a general class of dispersive equation, in particular for the Korteweg-de Vries equation, by using the Carleman's estimates

[5]. In 1997, Bourgain introduced another method [4] based on entire function estimates. Thanks to the Paley-Wiener theorem, the entire function here is given by an analytic continuation of the Fourier transform in space of the Duhamel formula. In this way, he proved the unique continuation property for the KdV equation and Panthee [9, 10] expanded this property to the KP-II equation. But it is not yet clear if the proof can be expanded to the KP-BBM-II equation. Indeed, to use Bourgain's method, we need to prove that for all  $R > 0$ , there exists  $k \in \mathbb{R}$ , with  $|k| > R$ , such that

$$|L'(k)| \geq |f(k)|, \quad \text{with} \quad \lim_{|k| \rightarrow \infty} |f(k)| = +\infty. \quad (0.1)$$

where  $L$  is the symbol of the linear evolution. In the context of the KdV equation, we choose  $f(k) = k^2$  [4]. We notice for the BBM equation

$$u_t + u_x + \alpha u u_x - \beta u_{xxt} = 0,$$

the symbol of the linear evolution is given by  $L(k) = \frac{k}{1 + \beta k^2}$ , thus  $L'(k) = \frac{1 - \beta k^2}{(1 + \beta k^2)^2}$  tends to zero at infinity and the condition (0.1) is not satisfied.

In dimension 2, this condition becomes for all  $R > 0$ , there exists  $(k, l) \in \mathbb{R}^2$ , with  $|k| + |l| > R$ , such that

$$|\partial_k L(k, l)| \geq |f(k, l)|, \quad \text{with} \quad \lim_{|k| + |l| \rightarrow \infty} |f(k, l)| = +\infty. \quad (0.2)$$

Let us see how behave the KP equations

$$u_t + u_x + \alpha u u_x + s \beta u_{xxx} + \gamma \partial_x^{-1} u_{yy} = 0.$$

We have  $L(k, l) = k - s \beta k^3 + \gamma l^2/k$ , and

$$\partial_k L(k, l) = 1 - 3s \beta k^2 - \gamma \frac{l^2}{k^2}.$$

If  $s = 1$ , *i.e.* for the KP-II equation, we choose  $f(k, l) = |k| + |l|$  and the condition (0.2) is satisfied [10]. If  $s = -1$ , *i.e.* for the KP-I equation, there exists  $(k, l)$  large ( $l \sim k^2$ ) such that  $\partial_k L(k, l)$  vanishes, and the method introduced by Bourgain no longer applies.

In a similar way, the symbol of the linear evolution is given for the KP-BBM-II equation by

$$L(k, l) = \frac{k + \gamma l^2/k}{1 + \beta k^2}, \quad \text{thus}$$

$$\partial_k L(k, l) = \frac{(1 - \beta k^2)k^2 - \gamma l^2(1 + 3\beta k^2)}{k^2(1 + \beta k^2)^2}.$$

For  $k$  and  $l$  large,  $\partial_k L(k, l)$  can be very small if  $|l| \ll |k|$ .

In order to study the unique continuation property for the KP-BBM-II equation, we have to envisage another way. We are then inspired by a simpler method introduced by Constantin [6] based on integration by parts. If we denote  $X^s(\mathbb{R}^2)$  the space of functions  $f$  in  $H^s(\mathbb{R}^2)$  such that  $\partial_x^{-1} f$  belongs to  $H^s(\mathbb{R}^2)$ , provided with the norm

$$|f|_s = \left( \|f\|_s^2 + \|\partial_x^{-1} f\|_s^2 \right)^{1/2},$$

our result reads as follows.

**Theorem 0.1**

Let  $s > 2$  and  $u \in \mathcal{C}([-T, T]; X^s(\mathbb{R}^2))$  the solution of the Cauchy problem associated with the KP-BBM-II equation. Let us suppose that there exists  $0 < B < +\infty$  and  $C > D$  in  $\mathbb{R}$ , with  $\frac{|D - C|}{\sqrt{\beta\gamma}} < \pi$ , such that for all  $t \in [-T, T]$

$$\text{supp } u(t) \subseteq [-B, B] \times [D, C].$$

Then  $u$  vanishes identically.

We recall that, for all positive real number  $s$ ,  $H^s(\mathbb{R}^2)$  is the Sobolev space of square integrable functions, which the  $s$  first derivatives are also square integrable, equipped with the norm

$$\|f\|_s^2 = \int_{\mathbb{R}^2} (1 + k^2 + l^2)^s |\hat{f}(k, l)|^2 dk dl,$$

where the Fourier transform in space is defined by

$$\hat{f}(k, l) = \int_{\mathbb{R}^2} e^{-i(kx+ly)} f(x, y) dx dy.$$

## 1 Summary of existence theory

We consider the Cauchy problem

$$u_t + u_x + \alpha u u_x - \beta u_{xxt} + \gamma \partial_x^{-1} u_{yy} = 0 \quad (1.1)$$

$$u(x, y, 0) = f(x, y). \quad (1.2)$$

This Cauchy problem is dealt in the articles of Bona, Liu and Tom [2], or Saut and Tzvetkov [12]. They proved that for all initial datum in the subspace of  $L^2(\mathbb{R}^2)$  provided with the norm  $(\|u\|^2 + \|u_x\|^2)^{1/2}$ , there exists a unique global in time solution.

## 2 Unique continuation property

We start proving that the anti-derivative  $\partial_x^{-1}$  can be written as an integral.

### Lemma 2.1

Let  $s > 2$  and  $u \in \mathcal{C}([-T, T]; X^s(\mathbb{R}^2))$  solution of the Cauchy problem (1.1)-(1.2). If for all  $t \in [-T, T]$ ,  $u(t)$  has a compact support, then for all  $(x, y) \in \mathbb{R}^2$  and  $t \in [-T, T]$ ,

$$\partial_x^{-1} u(x, y, t) = \int_{-\infty}^x u(x', y, t) dx'.$$

Moreover  $u$  is integrable with mean zero.

**Proof.** Let  $u \in \mathcal{C}([-T, T]; X^s(\mathbb{R}^2))$  solution of the Cauchy problem (1.1)-(1.2). We denote

$$v(x, y, t) := \partial_x^{-1} u(x, y, t) \quad \text{and} \quad w(x, y, t) := \int_{-\infty}^x u(x', y, t) dx'.$$

We deduce that, for all  $t \in [-T, T]$ ,  $\hat{v}(t) = \hat{w}(t)$  in  $\mathcal{S}'(\mathbb{R}^2)$ , where  $\mathcal{S}'(\mathbb{R}^2)$  is the space of tempered distributions. Since the Fourier transform is bijective on the tempered distribution, we conclude by definition of the Sobolev spaces with respect to  $\mathcal{S}'(\mathbb{R}^2)$ .

Let us prove now that  $u$  is integrable with mean zero. Since  $u$  has compact support, and  $u \in \mathcal{C}([-T, T]; X^s(\mathbb{R}^2))$ , with  $s > 2$ , the Cauchy-Schwarz inequality implies that  $u$  is integrable over the  $x$  and  $y$ -directions. Moreover,  $\partial_x^{-1} u \in \mathcal{C}([-T, T]; H^s(\mathbb{R}^2))$ , thus, by property of Sobolev space,

$$\lim_{x \rightarrow +\infty} \partial_x^{-1} u(x, y, t) = 0 = \int_{-\infty}^{+\infty} u(x, y, t) dx.$$

□

**Lemma 2.2**

Let  $s > 2$  and  $u \in \mathcal{C}([-T, T]; X^s(\mathbb{R}^2))$  solution of the Cauchy problem (1.1)-(1.2). If for all  $t \in [-T, T]$ ,  $u(t)$  has a compact support, then for all  $t \in [-T, T]$  and  $g \in \mathcal{C}^\infty(\mathbb{R})$ ,

$$\int_{\mathbb{R}^2} e^{\frac{x}{\sqrt{\beta}}} g(y) (u - \beta u_{xx})(x, y, t) dx dy = 0.$$

**Proof.** Let  $G \in \mathcal{C}^\infty(\mathbb{R}^2; \mathbb{R})$  and  $t \in [-T, T]$ . Since  $u(t)$  has a compact support, an integration by parts gives

$$\int_{\mathbb{R}^2} G(x, y) (u - \beta u_{xx})(x, y, t) dx dy = 0,$$

if

$$G(x, y) - \beta G_{xx}(x, y) = 0.$$

We set then  $G(x, y) = e^{\frac{x}{\sqrt{\beta}}} g(y)$ . □

We can prove now the main result of this paper.

**Proof of the theorem 0.1.** We have, thanks to the differentiation under the integral sign theorem of Lebesgue,

$$\begin{aligned} 0 &= \frac{d}{dt} \int_{\mathbb{R}^2} e^{\frac{x}{\sqrt{\beta}}} g(y) (u - \beta u_{xx})(x, y, t) dx dy \\ &= \int_{\mathbb{R}^2} e^{\frac{x}{\sqrt{\beta}}} g(y) (u_t - \beta u_{xxt})(x, y, t) dx dy, \end{aligned}$$

and since  $u$  is the solution of the equation (1.1), we find

$$0 = \int_{\mathbb{R}^2} e^{\frac{x}{\sqrt{\beta}}} g(y) (u_x + \alpha u u_x + \gamma \partial_x^{-1} u_{yy})(x, y, t) dx dy.$$

According to the lemma 2.1, we apply an integration by parts in the  $x$ -direction to find

$$0 = \int_{\mathbb{R}^2} e^{\frac{x}{\sqrt{\beta}}} \frac{1}{\sqrt{\beta}} g(y) \left( u + \frac{\alpha}{2} u^2 \right) + e^{\frac{x}{\sqrt{\beta}}} \sqrt{\beta} \gamma g(y) u_{yy}(x, y, t) dx dy.$$

Two new integrations by parts for the third term in the  $y$ -direction give

$$0 = \int_{\mathbb{R}^2} e^{\frac{x}{\sqrt{\beta}}} \left( \frac{1}{\sqrt{\beta}} g(y) + \sqrt{\beta} \gamma g''(y) \right) u(x, y, t) + \frac{\alpha}{2\sqrt{\beta}} e^{\frac{x}{\sqrt{\beta}}} g(y) u^2(x, y, t) dx dy.$$

Our aim is to solve the ordinary differential equation

$$g(y) + \beta \gamma g''(y) = 0 \quad \text{with } g(y) > 0. \quad (2.1)$$

One solution is given by  $g(y) = \cos\left(\frac{y + y_0}{\sqrt{\beta\gamma}}\right)$ , with  $y_0$  to be chosen. Finally, we find

$$\int_{\mathbb{R}^2} e^{\frac{x}{\sqrt{\beta}}} \cos\left(\frac{y + y_0}{\sqrt{\beta\gamma}}\right) u^2(x, y, t) dx dy = 0.$$

We choose then  $y_0$  such that  $\frac{y + y_0}{\sqrt{\beta\gamma}} \in \left] -\frac{\pi}{2}, \frac{\pi}{2} \right]$ . However, for all  $t \in [-T, T]$ , the support of  $u(t)$  verifies that  $\text{supp } u(t) \subseteq [-B, B] \times [D, C]$ , with  $\frac{|D - C|}{\sqrt{\beta\gamma}} < \pi$ , and this implies that  $e^{\frac{x}{\sqrt{\beta}}} \cos\left(\frac{y + y_0}{\sqrt{\beta\gamma}}\right) > 0$  on the support of  $u(t)$ . In particular  $u$  vanishes identically. □

**Remark 2.3**

If we consider the KP equations written under the form

$$u_t + u_x + \alpha uu_x + \beta u_{xxx} + s\gamma \partial_x^{-1} u_{yy} = 0,$$

and since from the modelling point of view it is permissible to change  $u_x$  by  $-u_t$  in the fourth term [1], we find another KP-BBM type equations

$$u_t + u_x + \alpha uu_x - \beta u_{xxt} + s\gamma \partial_x^{-1} u_{yy} = 0. \quad (2.2)$$

In particular, the equation (2.2) with  $s = -1$  is here well-posed [12], and we can expand the result of unique continuation property as follows.

**Theorem 2.4**

Let  $s > 2$  and  $u \in C([-T, T]; X^s(\mathbb{R}^2))$  solution of the Cauchy problem associated with the equation (2.2) for  $s = -1$ . Let us suppose that there exists  $0 < B < +\infty$  such that for all  $t \in [-T, T]$

$$\text{supp } u(t) \subseteq [-B, B] \times [-B, B].$$

Then  $u$  vanishes identically.

**Proof.** The proof is similar to the KP-BBM-II one by taking the function  $G(x, y) = e^{\frac{x}{\sqrt{\beta}} + \frac{y}{\sqrt{\beta}\gamma}}$ . □

Let us notice that there is no restriction on the support in the  $y$ -direction in this case.

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