On the decay in time of solutions of the generalized regularized Boussinesq system

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November 2009

Abstract. We are interested in dispersive properties of the Boussinesq system for small initial data. We prove that, for a nonlinearity of sufficiently high order, the solution of the Boussinesq system exists for all time and tends to zero when the time tends to infinity.

Keywords. Boussinesq system, decay in time

MS Codes. 35B40, 35Q53, 76B03, 76B15

1 Introduction

The surface η and the velocity v of the two-way propagation of small amplitude, long wavelength, gravity waves in shallow water was first derived by Boussinesq [5, 11] from the Euler equation as a system of two equations of the form

$$\begin{aligned} \eta_t + v_x + (v\eta)_x &= 0 \\ v_t + \eta_x + vv_x - v_{xxt} &= 0. \end{aligned}$$

The goal of this work is to study the decay in time for small amplitude solutions of a more general family of Boussinesq system (gBS) below [3, 4]

$$\eta_t + v_x + (v^\rho \eta)_x + (v\eta^\rho)_x + av_{xxx} - b\eta_{xxt} = 0 \tag{1.1}$$

$$v_t + \eta_x + (v^{\rho+1})_x + a\eta_{xxx} - bv_{xxt} = 0, \tag{1.2}$$

with $a \in \mathbb{R}$, $b > 0$ and ρ a positive integer.
For $s \in \mathbb{R}$, we denote $X^s(\mathbb{R}) = H^s(\mathbb{R}) \cap L^1(\mathbb{R}) \times H^s(\mathbb{R}) \cap L^1(\mathbb{R})$ the space of functions (η, v) such that the norm

$$||(\eta, v)||_{X^s} := ||\eta||_{H^s} + ||v||_{H^s} + ||\eta||_{L^1} + ||v||_{L^1}$$

is finite. Our result reads as follows.

Theorem 1.1
Let $\rho \geq 4$. For $1/9 \leq \delta < 1/3 - 2/(3\rho)$, we set $s = 1/(2\delta) - 1$.

Then there exists $\varepsilon > 0$ such that for all $(\eta_0, v_0) \in X^s(\mathbb{R})$ with $||(\eta_0, v_0)||_{X^s} \leq \varepsilon$, there exists a unique global in time solution $(\eta, v) \in \mathcal{C}(\mathbb{R}; X^s(\mathbb{R}))$ of the gBS system (1.1)-(1.2) with (η_0, v_0) as initial datum. Moreover this solution verifies that there exists a constant $C > 0$, depending only on s and ε, such that for all time $t \in \mathbb{R}$, for all $x \in \mathbb{R}$, we have

$$|\eta(x,t)| + |v(x,t)| \leq C(1+|t|)^{-\frac{1}{2}+\frac{3\delta}{2}}.$$

We notice that if δ tends to $1/9$ then the decay rate of the $L^\infty-$norm of $(\eta, v)(t)$ tends to $1/3$, but we need to impose more regularity on the initial datum. On the other hand, the more δ is close to $1/3$, the more the regularity is small, but the decay in time becomes slow.
We are inspired by the method described by Albert [1] for the generalized Benjamin-Bona-Mahony (gBBM) equation [2]

$$\eta_t + \eta_x + \eta^\rho \eta_x - \eta_{xxt} = 0. \tag{1.3}$$

In this paper, Albert shows that, for a power $\rho > 4$, the solution of (1.3), with a small initial datum, decreases in time with a rate of order $1/3$. A same result is found for the solution of the generalized Korteweg-de Vries equation [7, 10]

$$\eta_t + \eta_x + \eta^\rho \eta_x + \eta_{xxx} = 0. \tag{1.4}$$

In our work, we prove that the decay rate in time of the solution of the gBS system is also of order to $1/3$, but the result remains true for $\rho = 4$. However, the gBBM equation can be considered as a special case of the gBS system. It is enough to take $v = \eta$, $a = 0$ in the theorem 1.1 to slightly improve the result of Albert [1].

Corollary 1.2
Let $\rho \geq 4$. For $1/9 \leq \delta < 1/3 - 2/(3\rho)$, we set $s = 1/(2\delta) - 1$.
Then there exists $\varepsilon > 0$ such that for all $\eta_0 \in H^s(\mathbb{R}) \cap L^1(\mathbb{R})$ with $||\eta_0||_{H^s \cap L^1} \leq \varepsilon$, there exists a unique global in time solution $\eta \in \mathcal{C}\left(\mathbb{R}; H^s(\mathbb{R}) \cap L^1(\mathbb{R})\right)$ of the gBBM equation (1.3) with η_0 as initial datum. Moreover this solution verifies that there exists $C > 0$, depending only on s and ε, such that for all time $t \in \mathbb{R}$, for all $x \in \mathbb{R}$, we have

$$|\eta(x,t)| \leq C(1+|t|)^{-\frac{1}{2}+\frac{3\delta}{2}}.$$

The Strauss method [10] will be used to prove these results. It consists of proving that, for a small initial datum, all the derivatives of the solution of the gBS system behave like a linear term, and so equivalent to a Sobolev norm of sufficiently high order, and the power ρ relates to the dispersion. The decay in time is given by the free gBS system evolution.

We will use the following notations: for $1 \leq p < \infty$, we denote $L^p(\mathbb{R})$ the space of p-power integrable functions equipped with the norm

$$||f||_{L^p} := \left(\int_{-\infty}^{+\infty} |f(x)|^p \, dx \right)^{1/p},$$

we denote $L^\infty(\mathbb{R})$ the functions space equipped with the norm

$$||f||_\infty = \sup \operatorname{ess}(f) := \inf \{c \, ; |f(x)| \leq c \text{ almost everywhere in } \mathbb{R}\}.$$

The Schwartz space is denoted by $\mathcal{S}(\mathbb{R})$ and for $1 \leq p \leq \infty$, $W^{s,p}(\mathbb{R})$ is the Sobolev space equipped with the norm

$$||f||_{W^{s,p}} := \left|\left|(1-\partial_x^2)^{s/2} f\right|\right|_{L^p}.$$

In particular, $H^s(\mathbb{R}) := W^{s,2}(\mathbb{R})$.

We organise the paper as follows. In the first section, we give estimates concerning the linear Cauchy problem. In the second one, the existence and the uniqueness of global solution with the decay in time are proved.

2 Estimates for the linear gBS system

We consider the linear Cauchy problem

$$\begin{aligned} \eta_t + v_x + av_{xxx} - b\eta_{xxt} &= 0 \\ v_t + \eta_x + a\eta_{xxx} - bv_{xxt} &= 0 \\ \eta(x,0) \ = \ \eta_0(x), v(x,0) &= v_0(x), \end{aligned}$$

with $a \in \mathbb{R}$, $b > 0$.
Let us suppose that the initial datum (η_0, v_0) is in the Schwartz space $\mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R})$. The solution (η, v) of the ordinary differential equation, deduced from the Fourier transform in space, is given by, for all $x \in \mathbb{R}$ and $t \in \mathbb{R}^*$,

$$\begin{aligned} \eta(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{ikx}\cos\left(tk\frac{1-ak^2}{1+bk^2}\right)\hat{\eta}_0(k) - i\mathrm{e}^{ikx}\sin\left(tk\frac{1-ak^2}{1+bk^2}\right)\hat{v}_0(k)\,dk \\ &= \frac{1}{4\pi}\int_{-\infty}^{+\infty}\left(\mathrm{e}^{ith_\alpha(k)} + \mathrm{e}^{itg_\alpha(k)}\right)\hat{\eta}_0(k) - \left(\mathrm{e}^{ith_\alpha(k)} - \mathrm{e}^{itg_\alpha(k)}\right)\hat{v}_0(k)\,dk \qquad (2.1) \\ v(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{ikx}\cos\left(tk\frac{1-ak^2}{1+bk^2}\right)\hat{v}_0(k) - i\mathrm{e}^{ikx}\sin\left(tk\frac{1-ak^2}{1+bk^2}\right)\hat{\eta}_0(k)\,dk \\ &= \frac{1}{4\pi}\int_{-\infty}^{+\infty}\left(\mathrm{e}^{ith_\alpha(k)} + \mathrm{e}^{itg_\alpha(k)}\right)\hat{v}_0(k) - \left(\mathrm{e}^{ith_\alpha(k)} - \mathrm{e}^{itg_\alpha(k)}\right)\hat{\eta}_0(k)\,dk, \qquad (2.2) \end{aligned}$$

with $h_\alpha(k) = k(1-ak^2)/(1+bk^2) + \alpha k, g_\alpha(k) = -k(1-ak^2)/(1+bk^2) + \alpha k$, and $\alpha = x/t$.
We first recall the Van der Corput lemma [9].

Lemma 2.1
Let $n \geq 2$. Then there exists $C > 0$ such that for all $\alpha \leq \beta$, $\lambda > 0$, $t \in \mathbb{R}^*$ and for all function $h \in \mathcal{C}^\infty([\alpha,\beta])$ real valued satisfying for all $k \in [\alpha,\beta]$, $|h^{(n)}(k)| \geq \lambda$, we have

$$\left|\int_a^b e^{ith(k)}\,dk\right| \leq \frac{C}{(\lambda|t|)^{1/n}}.$$

The Van der Corput lemma is now applied with the functions h_α and g_α.

Lemma 2.2
There exists $C > 0$ such that for all $1/9 \leq \delta < 1/3$, $\alpha \in \mathbb{R}$ and all time $|t| > \left((3+2\sqrt{2})/b\right)^{1/2\delta}$, we have

$$\left|\int_{-|t|^\delta}^{|t|^\delta} \mathrm{e}^{ith_\alpha(k)}\,dk\right| \leq C|t|^{-\frac{1}{2}+\frac{3\delta}{2}}, \ \left|\int_{-|t|^\delta}^{|t|^\delta} \mathrm{e}^{itg_\alpha(k)}\,dk\right| \leq C|t|^{-\frac{1}{2}+\frac{3\delta}{2}}.$$

Proof. We only consider the cases h_α and $k \geq 0$, the other cases being dealt with similarly.
We have for $k \in [0, |t|^\delta]$

$$|h''_\alpha(k)| = \left|\frac{2(a+b)k(bk^2-3)}{(1+bk^2)^3}\right|, \ |h'''_\alpha(k)| = \left|\frac{6(a+b)(b^2k^4-6bk^2+1)}{(1+bk^2)^4}\right|.$$

We notice that $|h''_\alpha(k)|$ vanishes if $k = 0, \sqrt{3/b}$ and $|h'''_\alpha(k)|$ vanishes if $k = \sqrt{(3-2\sqrt{2})/b}, \sqrt{(3+2\sqrt{2})/b}$. The interval $[0, |t|^\delta]$ is divided in four pieces $[0, A] \cup [A, B] \cup [B, C] \cup [C, |t|^\delta]$, with

$$0 < A < \sqrt{\frac{3-2\sqrt{2}}{b}} < B < \sqrt{\frac{3}{b}} < C < \sqrt{\frac{3+2\sqrt{2}}{b}} < |t|^\delta.$$

In particular, for $k \in [0, A] \cup [B, C]$, we have $|h'''_\alpha(k)| \geq C > 0$, for $k \in [A, B]$, $|h''_\alpha(k)| \geq C > 0$ and for $k \in [C, |t|^\delta]$, $|h''_\alpha(k)| \geq |h''_\alpha(|t|^\delta)| \geq C|t|^{-3\delta}$, where C is independent on t. Finally, the lemma 2.1 is applied on each interval to give

$$\left| \int_0^{|t|^\delta} \mathrm{e}^{ith_\alpha(k)}\, dk \right| \ \leq \ C|t|^{-\frac{1}{3}} + C|t|^{-\frac{1}{2}+\frac{3\delta}{2}},$$

and since $\delta \geq 1/9$, we have $-1/2 + 3\delta/2 \geq -1/3$. □

The oscillating integrals (2.1) and (2.2) can be majorized. When there is no ambiguity, we will write C and C_s the different constants appearing in the following results.

Lemma 2.3
Let $1/9 \leq \delta < 1/3$ and $s = 1/(2\delta) - 1$. For all $f \in H^s(\mathbb{R})$, $\alpha \in \mathbb{R}$ and all time $|t| > \left((3+2\sqrt{2})/b\right)^{1/2\delta}$, we have

$$\left| \int_{|k|>|t|^\delta} \mathrm{e}^{ith_\alpha(k)} \hat{f}(k)\, dk \right| \leq C_s ||f||_{H^s} |t|^{-\frac{1}{2}+\frac{3\delta}{2}}, \ \left| \int_{|k|>|t|^\delta} \mathrm{e}^{itg_\alpha(k)} \hat{f}(k)\, dk \right| \leq C_s ||f||_{H^s} |t|^{-\frac{1}{2}+\frac{3\delta}{2}}.$$

Proof. We have

$$\left| \int_{|t|^\delta}^{+\infty} \mathrm{e}^{ith_\alpha(k)} \hat{f}(k)\, dk \right| \ \leq \ \int_{|t|^\delta}^{+\infty} \left| \hat{f}(k) \right| \, dk \ = \ \int_{|t|^\delta}^{+\infty} \frac{(1+k)^s}{(1+k)^s} \left| \hat{f}(k) \right| \, dk,$$

and the Cauchy-Schwarz inequality gives

$$\left| \int_{|t|^\delta}^{+\infty} \mathrm{e}^{ith_\alpha(k)} \hat{f}(k)\, dk \right| \ \leq \ \left(\int_{|t|^\delta}^{+\infty} (1+k)^{2s} |\hat{f}(k)|^2\, dk \right)^{1/2} \left(\int_{|t|^\delta}^{+\infty} \frac{dk}{(1+k)^{2s}} \right)^{1/2}.$$

On one hand, since $s > 1/2$, it gets

$$\begin{aligned} \left(\int_{|t|^\delta}^{+\infty} \frac{dk}{(1+k)^{2s}} \right)^{1/2} &= \left(\left[\frac{-1}{(2s-1)(1+k)^{2s-1}} \right]_{|t|^\delta}^{+\infty} \right)^{1/2} \\ &\leq C_s |t|^{\frac{\delta}{2}(1-2s)} \ = \ C_s |t|^{-\frac{1}{2}+\frac{3\delta}{2}}. \end{aligned}$$

On the other hand, since for $k \in \mathbb{R}$, we have $(1+k)^2 \leq 2(1+k^2)$, the first integral becomes

$$\begin{aligned} \left(\int_{|t|^\delta}^{+\infty} (1+k)^{2s} |\hat{f}(k)|^2\, dk \right)^{1/2} &\leq \sqrt{2^s} \left(\int_{|t|^\delta}^{+\infty} (1+k^2)^s |\hat{f}(k)|^2\, dk \right)^{1/2} \\ &\leq C_s ||f||_{H^s}. \end{aligned}$$

Finally, we find

$$\left| \int_{|t|^\delta}^{+\infty} \mathrm{e}^{ith_\alpha(k)} \hat{f}(k)\, dk \right| \leq C_s ||f||_{H^s} |t|^{-\frac{1}{2}+\frac{3\delta}{2}}.$$

□

Proposition 2.4
Let $1/9 \leq \delta < 1/3$ and $s = 1/(2\delta) - 1$. There exists a constant $C_s > 0$, depending only on s, such that for all function $f \in L^1(\mathbb{R}) \cap H^s(\mathbb{R})$, $\alpha \in \mathbb{R}$ and all time $t \in \mathbb{R}$, we have

$$\begin{aligned}\left|\int_{-\infty}^{+\infty} \mathrm{e}^{ith_\alpha(k)} \hat{f}(k)\, dk\right| &\leq C_s(||f||_{L^1} + ||f||_{H^s})(1+|t|)^{-\frac{1}{2}+\frac{3\delta}{2}} \\ \left|\int_{-\infty}^{+\infty} \mathrm{e}^{itg_\alpha(k)} \hat{f}(k)\, dk\right| &\leq C_s(||f||_{L^1} + ||f||_{H^s})(1+|t|)^{-\frac{1}{2}+\frac{3\delta}{2}}.\end{aligned}$$

Proof. Let $1/9 \leq \delta < 1/3$ and $|t| > \left((3+2\sqrt{2})/b\right)^{1/2\delta} =: k_1$, we write

$$\int_0^{+\infty} \mathrm{e}^{ith_\alpha(k)} \hat{f}(k)\, dk = \int_0^{|t|^\delta} \mathrm{e}^{ith_\alpha(k)} \hat{f}(k)\, dk + \int_{|t|^\delta}^{+\infty} \mathrm{e}^{ith_\alpha(k)} \hat{f}(k)\, dk.$$

For the first integral, the Fubini theorem implies

$$\begin{aligned}\int_0^{|t|^\delta} \mathrm{e}^{ith_\alpha(k)} \hat{f}(k)\, dk &= \int_0^{|t|^\delta} \mathrm{e}^{ith_\alpha(k)} \left(\int_{-\infty}^{+\infty} \mathrm{e}^{-ikx'} f(x')\, dx'\right) dk \\ &= \int_{-\infty}^{+\infty} \left(\int_0^{|t|^\delta} \mathrm{e}^{ik(x-x')} \mathrm{e}^{-it\frac{k(1-ak^2)}{1+bk^2}}\, dk\right) f(x')\, dx'.\end{aligned}$$

We deduce that

$$\left|\int_0^{|t|^\delta} \mathrm{e}^{ith_\alpha(k)} \hat{f}(k)\, dk\right| \leq \left|\left|\int_0^{|t|^\delta} \mathrm{e}^{ith_\alpha(k)}\, dk\right|\right|_\infty ||f||_{L^1}.$$

The lemmas 2.2 and 2.3 provide

$$\left|\int_0^{+\infty} \mathrm{e}^{ith_\alpha(k)} \hat{f}(k)\, dk\right| \leq C_s\left(||f||_{L^1} + ||f||_{H^s}\right)|t|^{-\frac{1}{2}+\frac{3\delta}{2}}.$$

Since $|t| > k_1$, we have $1 + |t| < (1/k_1 + 1)|t|$, thus we obtain

$$\left|\int_0^{+\infty} \mathrm{e}^{ith_\alpha(k)} \hat{f}(k)\, dk\right| \leq C_s\left(||f||_{L^1} + ||f||_{H^s}\right)(1+|t|)^{-\frac{1}{2}+\frac{3\delta}{2}}. \tag{2.3}$$

The contribution for $k \leq 0$ is dealt with similarly.
Let $|t| \leq k_1$. We immediately have

$$\left|\int_{-\infty}^{+\infty} \mathrm{e}^{ith_\alpha(k)} \hat{f}(k)\, dk\right| \leq \int_{-\infty}^{+\infty} \left|\hat{f}(k)\right| dk = \int_{-\infty}^{+\infty} \frac{(1+k^2)^{s/2}}{(1+k^2)^{s/2}} \left|\hat{f}(k)\right| dk,$$

and the Cauchy-Schwarz inequality gives, with $s > 1/2$,

$$\left|\int_{-\infty}^{+\infty} \mathrm{e}^{ith_\alpha(k)} \hat{f}(k)\, dk\right| \leq C_s||f||_{H^s}.$$

Since $|t| \leq k_1$, we have $(1+|t|)/(1+k_1) \leq 1$, thus

$$\left|\int_{-\infty}^{+\infty} \mathrm{e}^{ith_\alpha(k)} \hat{f}(k)\, dk\right| \leq C_s||f||_{H^s}(1+|t|)^{-\frac{1}{2}+\frac{3\delta}{2}}. \tag{2.4}$$

The result is obtained from the inequalities (2.3) and (2.4). □

We can generalize the preceding result.

Corollary 2.5
Let $1/9 \leq \delta < 1/3$ and $s = 1/(2\delta) - 1$. There exists a constant $C_s > 0$, depending only on s, such that for all function $f \in L^p(\mathbb{R}) \cap H^s(\mathbb{R})$, with $1 \leq p \leq 2$, for all $\alpha \in \mathbb{R}$ and all time $t \in \mathbb{R}$, we have

$$\begin{aligned}\left|\left|\int_{-\infty}^{+\infty} \mathrm{e}^{itg_\alpha(k)} \hat{f}(k)\, dk\right|\right|_{L^q} &\leq C_s(||f||_{L^p} + ||f||_{H^s})(1+|t|)^{\left(-\frac{1}{2}+\frac{3\delta}{2}\right)\left(1-\frac{2}{q}\right)} \\ \left|\left|\int_{-\infty}^{+\infty} \mathrm{e}^{itg_\alpha(k)} \hat{f}(k)\, dk\right|\right|_{L^q} &\leq C_s(||f||_{L^p} + ||f||_{H^s})(1+|t|)^{\left(-\frac{1}{2}+\frac{3\delta}{2}\right)\left(1-\frac{2}{q}\right)},\end{aligned}$$

where $1/p + 1/q = 1$.

Proof. See [8] for a complete proof by interpolation. □

We apply the proposition 2.4 to the linear Cauchy problem.

Corollary 2.6
Let $1/9 \leq \delta < 1/3$ and $s = 1/(2\delta) - 1$. There exists $C_s > 0$, depending only on s, such that for all function $(\eta_0, v_0) \in (L^1(\mathbb{R}) \cap H^s(\mathbb{R}))^2$, $\alpha \in \mathbb{R}$ and all time $t \in \mathbb{R}$, we have

$$\left|\frac{1}{4\pi}\int_{-\infty}^{+\infty} \left(\mathrm{e}^{ith_\alpha(k)} + \mathrm{e}^{itg_\alpha(k)}\right)\hat{\eta}_0(k) - \left(\mathrm{e}^{ith_\alpha(k)} - \mathrm{e}^{itg_\alpha(k)}\right)\hat{v}_0(k)\, dk\right|$$

$$\leq C_s(||\eta_0||_{L^1} + ||v_0||_{L^1} + ||\eta_0||_{H^s} + ||v_0||_{H^s})(1+|t|)^{-\frac{1}{2}+\frac{3\delta}{2}}$$

$$\left|\frac{1}{4\pi}\int_{-\infty}^{+\infty} \left(\mathrm{e}^{ith_\alpha(k)} + \mathrm{e}^{itg_\alpha(k)}\right)\hat{v}_0(k) - \left(\mathrm{e}^{ith_\alpha(k)} - \mathrm{e}^{itg_\alpha(k)}\right)\hat{\eta}_0(k)\, dk\right|$$

$$\leq C_s(||\eta_0||_{L^1} + ||v_0||_{L^1} + ||\eta_0||_{H^s} + ||v_0||_{H^s})(1+|t|)^{-\frac{1}{2}+\frac{3\delta}{2}}.$$

3 Preliminary results

We now quote some useful results. The fractional Leibniz rule will be used [6, lemma X.4].

Theorem 3.1
For all $s \geq 0$ and $1 < p < +\infty$, there exists $C_{s,p} > 0$, depending on s and p such that for all u and v in $\mathcal{S}(\mathbb{R})$, we have

$$||u\, v||_{W^{s,p}} \leq C_{s,p}\left(||u||_{W^{s,p}}||v||_\infty + ||u||_\infty||v||_{W^{s,p}}\right).$$

We will need an integration's lemma proved in [8].

Lemma 3.2
Let $\alpha > 1$ and $0 < \beta < 1$. Then there exists $C > 0$ such that for all time $t \geq 0$

$$\int_0^t \frac{d\tau}{(1+\tau)^\alpha(1+t-\tau)^\beta} \leq \frac{C}{(1+t)^\beta}.$$

4 Existence and Uniqueness of global solution of the gBS system

We consider the nonlinear Cauchy problem

$$\begin{aligned} \eta_t + v_x + (v^\rho \eta)_x + (v\eta^\rho)_x + a v_{xxx} - b\eta_{xxt} &= 0 \quad (4.1)\\ v_t + \eta_x + (v^{\rho+1})_x + a\eta_{xxx} - b v_{xxt} &= 0 \quad (4.2)\\ \eta(x,0) \;=\; \eta_0(x),\; v(x,0) &= v_0(x), \quad (4.3) \end{aligned}$$

with $a \in \mathbb{R}$, $b > 0$ and ρ a positive integer. The main result of this paper is now proved.

Proof of the theorem 1.1. The proof is done for positive times.
To simplify the writings, we define the operator $\partial_x/(1-b\partial_x^2)$ by the Fourier symbol $\sigma(k) := ik/(1+bk^2)$, and we also denote $0 < \theta := (1/2 - 3\delta/2) \leq 1/3$.
Our assumptions on ρ and δ imply that

$$\theta\rho > 1. \quad (4.4)$$

According to the Duhamel formula, (η, v) is the solution of the gBS system (4.1)-(4.2)-(4.3) if and only if (η, v) is the solution of the following equation, for $t \geq 0$

$$\Phi(\eta, v)(t) = S_t(\eta_0, v_0) - \int_0^t S_{t-\tau}\left(\frac{\partial_x}{1-b\partial_x^2}\left(v^\rho\eta + v\eta^\rho, v^{\rho+1}\right)\right)(\tau)\, d\tau, \quad (4.5)$$

where $S_t(\eta, v)$ is defined by (2.1)-(2.2).
First of all, we notice that, for $f \in \mathcal{S}(\mathbb{R})$

$$\left|\left|\frac{\partial_x}{1-b\partial_x^2} f\right|\right|_{H^s} = \left(\int_{-\infty}^{+\infty} (1+k^2)^s \left(\frac{ik}{1+bk^2}\right)^2 |\widehat{f}(k)|^2\, dk\right)^{1/2} \leq ||f||_{H^{s-1}} \leq ||f||_{H^s}. \quad (4.6)$$

On the other hand, we have

$$\left|\left|\frac{\partial_x}{1-b\partial_x^2} f\right|\right|_{L^1} = \left|\left|\left(-\frac{1}{2b^{3/2}}\mathrm{sgn}(x)\mathrm{e}^{-|x|/\sqrt{b}}\right) * f\right|\right|_{L^1} \leq C||f||_{L^1}. \quad (4.7)$$

Let $T > 0$, we define the norm N_T by, for $(\eta, v) \in X^s(\mathbb{R})$,

$$N_T(\eta, v) := \sup_{0\leq\tau\leq T} \left[(||\eta||_{L^\infty}(\tau) + ||v||_{L^\infty}(\tau))(1+\tau)^\theta + ||(\eta, v)||_{X^s}(\tau)\right]. \quad (4.8)$$

We separately prove some technical lemmas.

Lemma 4.1
There exists a constant $C_s > 0$ such that for all (η_1, v_1) and (η_2, v_2) in $X^s(\mathbb{R})$, we have

$$|\Phi(\eta_1, v_1) - \Phi(\eta_2, v_2)|(t) \leq C_s \left(N_T(\eta_1, v_1)^\rho + \sum_{i=0}^{\rho} N_T(\eta_1, v_1)^{\rho-i} N_T(\eta_2, v_2)^i\right) \times (1+t)^{-\theta} N_T(\eta_1 - \eta_2, v_1 - v_2). \quad (4.9)$$

Proof. Let (η_1, v_1) and (η_2, v_2) be in $X^s(\mathbb{R})$, the Duhamel formula gives

$$|\Phi(\eta_1, v_1) - \Phi(\eta_2, v_2)|(t) \leq \int_0^t \left|S_{t-\tau}\left(\frac{\partial_x}{1-b\partial_x^2}(v_1^\rho\eta_1 - v_2^\rho\eta_2 + v_1\eta_1^\rho - v_2\eta_2^\rho, v_1^{\rho+1} - v_2^{\rho+1})\right)\right|(\tau)\, d\tau.$$

The corollary 2.6 implies that there exists $C_s > 0$, depending only on s, such that

$$|\Phi(\eta_1, v_1) - \Phi(\eta_2, v_2)|(t) \leq C_s \int_0^t \frac{\left(\left|\left|\left(\frac{\partial_x}{1-b\partial_x^2}(v_1^\rho \eta_1 - v_2^\rho \eta_2 + v_1\eta_1^\rho - v_2\eta_2^\rho), \frac{\partial_x}{1-b\partial_x^2}(v_1^{\rho+1} - v_2^{\rho+1})\right)\right|\right|_{X^s}\right)(\tau)}{(1+t-\tau)^\theta} d\tau,$$

and the inequalities (4.6) and (4.7) provide

$$\begin{aligned}|\Phi(\eta_1, v_1) - \Phi(\eta_2, v_2)|(t) &\leq C_s \int_0^t \frac{\left(||v_1^\rho \eta_1 - v_2^\rho \eta_2 + v_1\eta_1^\rho - v_2\eta_2^\rho||_{L^1} + ||v_1^\rho \eta_1 - v_2^\rho \eta_2 + v_1\eta_1^\rho - v_2\eta_2^\rho||_{H^s}\right)(\tau)}{(1+t-\tau)^\theta} d\tau \\ &\quad + C_s \int_0^t \frac{\left(\left|\left|v_1^{\rho+1} - v_2^{\rho+1}\right|\right|_{L^1} + \left|\left|v_1^{\rho+1} - v_2^{\rho+1}\right|\right|_{H^s}\right)(\tau)}{(1+t-\tau)^\theta} d\tau,\end{aligned}$$

From now, when there is no ambiguity, we use the following notation: for any positive A and B, the notation $A \lesssim B$ means that there exists a constant $C > 0$ such that $A \leq C\, B$.

Since $v_1^{\rho+1} - v_2^{\rho+1} = (v_1 - v_2)\sum_{i=0}^{\rho} v_1^{\rho-i} v_2^i$, we have according to the fractional Leibniz theorem 3.1 and the Minkowski inequality

$$\begin{aligned}||v_1^{\rho+1} - v_2^{\rho+1}||_{H^s}(\tau) &\lesssim ||v_1 - v_2||_{H^s} \sum_{i=0}^{\rho} ||v_1^{\rho-i} v_2^i||_{L^\infty}(\tau) + ||v_1 - v_2||_{L^\infty} \sum_{i=0}^{\rho} ||v_1^{\rho-i} v_2^i||_{H^s}(\tau) \\ &= \mathrm{I}(\tau) + \mathrm{II}(\tau).\end{aligned}$$

According to the definition (4.8) of the norm N_T, we can note that the decay in time relates to the $L^\infty-$norm. We deduce

$$\begin{aligned}\mathrm{I}(\tau) &\lesssim ||v_1 - v_2||_{H^s} \sum_{i=0}^{\rho} ||v_1||_{L^\infty}^{\rho-i} ||v_2||_{L^\infty}^i(\tau) \\ &\lesssim ||v_1 - v_2||_{H^s} \sum_{i=0}^{\rho} \left(||v_1||_{L^\infty} \frac{(1+\tau)^\theta}{(1+\tau)^\theta}\right)^{\rho-i} \left(||v_2||_{L^\infty} \frac{(1+\tau)^\theta}{(1+\tau)^\theta}\right)^i \\ &\lesssim \left(\sum_{i=0}^{\rho} N_T(\eta_1, v_1)^{\rho-i} N_T(\eta_2, v_2)^i\right)(1+\tau)^{-\theta\rho} N_T(\eta_1 - \eta_2, v_1 - v_2).\end{aligned}$$

For $\mathrm{II}(\tau)$, the theorem 3.1 is applied again to find

$$\begin{aligned}\mathrm{II}(\tau) \lesssim ||v_1 - v_2||_\infty \Bigg(&||v_2||_\infty^{\rho-1} ||v_2||_{H^s} + \sum_{i=0}^{\rho-1} ||v_1||_\infty^{\rho-i-1} ||v_1||_{H^s} ||v_2||_\infty^i \\ &+ ||v_1||_\infty^{\rho-1} ||v_1||_{H^s} + \sum_{i=1}^{\rho} ||v_1||_\infty^{\rho-i} ||v_2||_\infty^{i-1} ||v_2||_{H^s}\Bigg)(\tau),\end{aligned}$$

and as for $\mathrm{I}(\tau)$, it gets

$$\mathrm{II}(\tau) \lesssim \left(\sum_{i=0}^{\rho} N_T(\eta_1, v_1)^{\rho-i} N_T(\eta_2, v_2)^i\right)(1+\tau)^{-\theta\rho} N_T(\eta_1 - \eta_2, v_1 - v_2).$$

We also have

$$||v_1^{\rho+1} - v_2^{\rho+1}||_{L^1}(\tau) \leq ||v_1 - v_2||_{L^1} \sum_{i=0}^{\rho} ||v_1||_\infty^{\rho-i} ||v_2||_\infty^i(\tau)$$

$$\lesssim \quad \left(\sum_{i=0}^{\rho} N_T(\eta_1,v_1)^{\rho-i} N_T(\eta_2,v_2)^i\right)(1+\tau)^{-\theta\rho} N_T(\eta_1-\eta_2,v_1-v_2).$$

By noticing that $v_1^\rho\eta_1 - v_2^\rho\eta_2 = v_1^\rho(\eta_1-\eta_2) + \eta_2(v_1-v_2)\sum_{i=0}^{\rho-1} v_1^{\rho-1-i}v_2^i$, we obtain in the same manner

$$\begin{aligned}
||v_1^\rho\eta_1 - v_2^\rho\eta_2||_{H^s}(\tau) &\lesssim \left(N_T(\eta_1,v_1)^\rho + \sum_{i=1}^{\rho} N_T(\eta_1,v_1)^{\rho-i}N_T(\eta_2,v_2)^i\right)\times(1+\tau)^{-\theta\rho}N_T(\eta_1-\eta_2,v_1-v_2)\\
||v_1^\rho\eta_1 - v_2^\rho\eta_2||_{L^1}(\tau) &\lesssim \left(N_T(\eta_1,v_1)^\rho + \sum_{i=1}^{\rho} N_T(\eta_1,v_1)^{\rho-i}N_T(\eta_2,v_2)^i\right)\times(1+\tau)^{-\theta\rho}N_T(\eta_1-\eta_2,v_1-v_2),
\end{aligned}$$

and idem for $v_1\eta_1^\rho - v_2\eta_2^\rho$. Finally, it remains

$$|\Phi(\eta_1,v_1)-\Phi(\eta_1,v_1)|(t)\lesssim\left(N_T(\eta_1,v_1)^\rho+\sum_{i=0}^{\rho} N_T(\eta_1,v_1)^{\rho-i}N_T(\eta_2,v_2)^i\right)\int_0^t \frac{d\tau}{(1+\tau)^{\theta\rho}(1+t-\tau)^\theta} N_T(\eta_1-\eta_2,v_1-v_2).$$

Since $\theta\rho>1$ and $0<\theta\leq 1/3<1$, we conclude thanks to the lemma 3.2.

□

Lemma 4.2
There exists $C_s>0$ such that for all (η_1,v_1) and (η_2,v_2) in $X^s(\mathbb{R})$, we have

$$||\Phi(\eta_1,v_1)-\Phi(\eta_2,v_2)||_{X^s}(t)\leq C_s\left((N_T(\eta_1,v_1)^\rho+\sum_{i=0}^{\rho} N_T(\eta_1,v_1)^{\rho-i}N_T(\eta_2,v_2)^i\right)N_T(\eta_1-\eta_2,v_1-v_2). \tag{4.10}$$

Proof. We now majorize the norm $||\Phi(\eta_1,v_1)-\Phi(\eta_2,v_2)||_{X^s}(t)$.

$$||\Phi(\eta_1,v_1)-\Phi(\eta_2,v_2)||_{X^s}(t)\leq\int_0^t\left|\left|S_{t-\tau}\left(\frac{\partial_x}{1-b\partial_x^2}(v_1^\rho\eta_1-v_2^\rho\eta_2+v_1\eta_1^\rho-v_2\eta_2^\rho,v_1^{\rho+1}-v_2^{\rho+1})\right)\right|\right|_{X^s}(\tau)\,d\tau,$$

the inequalities (4.6) and (4.7) imply

$$||\Phi(\eta_1,v_1)-\Phi(\eta_2,v_2)||_{X^s}(t)\ \leq\ \int_0^t\left|\left|(v_1^\rho\eta_1-v_2^\rho\eta_2+v_1\eta_1^\rho-v_2\eta_2^\rho,v_1^{\rho+1}-v_2^{\rho+1})\right|\right|_{X^s}(\tau)\,d\tau.$$

Since $X^s(\mathbb{R})=\left(H^s(\mathbb{R})\cap L^1(\mathbb{R})\right)^2$, as for the preceding lemma, we obtain

$$||\Phi(\eta_1,v_1)-\Phi(\eta_2,v_2)||_{X^s}(t)\lesssim\left(N_T(\eta_1,v_1)^\rho+\sum_{i=0}^{\rho} N_T(\eta_1,v_1)^{\rho-i}N_T(\eta_2,v_2)^i\right)\int_0^t\frac{d\tau}{(1+\tau)^{\theta\rho}}$$
$$\times N_T(\eta_1-\eta_2,v_1-v_2).$$

However, according the inequalities (4.4),

$$\int_0^t\frac{d\tau}{(1+\tau)^{\theta\rho}}\lesssim 1.$$

□

Lemma 4.3
There exists a constant $C_s > 0$ such that for all (η_1, v_1) and (η_2, v_2) in $X^s(\mathbb{R})$, we have

$$N_T(\Phi(\eta_1, v_1) - \Phi(\eta_2, v_2)) \leq C_s \left(N_T(\eta_1, v_1)^\rho + \sum_{i=0}^{\rho} N_T(\eta_1, v_1)^{\rho-i} N_T(\eta_2, v_2)^i \right) N_T(\eta_1 - \eta_2, v_1 - v_2). \quad (4.11)$$

and

$$N_T(\Phi(\eta_1, v_1)) \leq C_s \left(||(\eta_0, v_0)||_{X^s} + N_T(\eta_1, v_1)^{\rho+1} \right). \quad (4.12)$$

Proof. The inequalities (4.9) and (4.10) provide (4.11). For the second inequality, it is enough to take $(\eta_2, v_2) = (0, 0)$.

□

Let $M > 0$, we consider the closed ball

$$\overline{B}_{T,M} := \{(\eta, v) \in \mathcal{C}([-T, T]; X^s(\mathbb{R})); \ N_T(\eta, v) \leq M\}.$$

We show that there exists a unique solution (η, v) of the system (4.1)-(4.2)-(4.3) in this ball by using the fixed point theorem.
First, there exists $\varepsilon > 0$ sufficiently small such that if $||(\eta_0, v_0)||_{X^s} \leq \varepsilon$, even if we take $C_s M$ instead of M, it is enough to take $M > 0$ satisfying $\varepsilon + M^{\rho+1} \leq M$ so that the inequality (4.12) implies $\Phi(\overline{B}_{T,M}) \subseteq \overline{B}_{T,M}$. Here, the crucial point is that ε is independent of T. Secondly, we prove that the map Φ is a contraction on this ball for M sufficiently small. Let (η_1, v_1) and (η_2, v_2) be in the closed ball $\overline{B}_{T,M}$. The inequality (4.11) gives

$$N_T(\Phi(\eta_1, v_1) - \Phi(\eta_2, v_2)) \leq C_s \, M^\rho N_T(\eta_1 - \eta_2, v_1 - v_2),$$

and, if we need to take $M > 0$ smaller again, it gets $C_s \, M^\rho < 1$. Then, the fixed point theorem is applied and there exists a unique solution of the system (4.1)-(4.2)-(4.3) in the closed ball $\overline{B}_{T,M}$.
It remains to prove that this unique solution can go on in time with all $[0, +\infty[$. By uniqueness of the solution, the inequality (4.12) is written

$$N_T(\eta, v) \leq C_s \left(||(\eta_0, v_0)||_{X^s} + N_T(\eta, v)^{\rho+1} \right). \quad (4.13)$$

Since there exists $\varepsilon > 0$ sufficiently small such that, if $||(\eta_0, v_0)||_{X^s} \leq \varepsilon$, we can find $M > 0$ such that

$$\begin{aligned} N_0(\eta, v) &< M \\ C_s \left(\varepsilon + M^{\rho+1} \right) &\leq M. \end{aligned}$$

Then for all $T > 0$, we have $N_T(\eta, v) < M$. Indeed, if not by continuity, there exists a time $T > 0$ such that

$$\begin{aligned} N_T(\eta, v) &= M \\ &> C_s \left(\varepsilon + M^{\rho+1} \right) \\ &> C_s \left(\varepsilon + N_T(\eta, v)^{\rho+1} \right), \end{aligned}$$

what contradicts the inequality (4.13). To summarize, there exists a constant $M > 0$ such that for all $T > 0$, $N_T(\eta, v) < M$. In particular, we have for all time $t \geq 0$ and for all $x \in \mathbb{R}$

$$\begin{aligned} |\eta(x,t)| + |v(x,t)| &\leq C_s \left(||(\eta_0, v_0)||_{X^s} + N_T(\eta_1, v_1)^{\rho+1} \right) (1+t)^{-\theta} \\ &\leq C_s \left(\varepsilon + M^{\rho+1} \right) (1+t)^{-\theta}. \end{aligned} \quad (4.14)$$

Negative times are dealt with similarly. □

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