



HAL
open science

Continuation of Time Bounds for a Regularized Boussinesq System

Youcef Mammeri

► **To cite this version:**

Youcef Mammeri. Continuation of Time Bounds for a Regularized Boussinesq System. *Acta Applicandae Mathematicae*, 2012, 117, pp.1 - 13. 10.1007/s10440-011-9647-1 . hal-01090364

HAL Id: hal-01090364

<https://hal.science/hal-01090364>

Submitted on 3 Dec 2014

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Continuation of time bounds for a regularized Boussinesq system

Y. Mammeri

Laboratoire Amiénois de Mathématique Fondamentale et Appliquée d'Amiens,
CNRS UMR 6140, Université de Picardie Jules Verne,
80069 Amiens, France.

Email: youcef.mammeri@u-picardie.fr

October 2011

Abstract. We study the periodic solution of a perturbed regularized Boussinesq system [2, 5], namely the system $\eta_t + u_x + \beta(-\eta_{xxt} + u_{xxx}) + \alpha((\eta u)_x + \eta\eta_x + uu_x) = 0$, $u_t + \eta_x + \beta(\eta_{xxx} - u_{xxt}) + \alpha((\eta u)_x + \eta\eta_x + uu_x) = 0$, with $0 < \alpha, \beta \leq 1$. We prove that the solution, starting from an initial datum of size ε , remains smaller than ε for a time scale of order $(\varepsilon^{-1}\alpha^{-1}\beta)^2$, whereas the natural time is of order $\varepsilon^{-1}\alpha^{-1}\beta$.

Keywords. Boussinesq system, Sobolev bounds, normal form

MS Codes. 35B60, 35Q53, 76B03, 76B15

Introduction

The two-way propagation of small amplitude, long wavelength, gravity waves in shallow water, described by its surface η and its velocity u , was first derived by Boussinesq [4, 2, 3, 8] as a system of the form

$$\begin{aligned}\eta_t + u_x + \alpha(\eta u)_x &= 0 \\ u_t + \eta_x + \alpha uu_x - \beta u_{xxt} &= 0,\end{aligned}$$

where α denotes the quotient between the characteristic waves amplitude and the depth of the water, β is the square of the quotient between this depth and the wavelength.

In this paper, we consider the following regularized Boussinesq system, proposed in [5] as a model of interactions between internal solitary waves,

$$\eta_t + u_x + \beta(-\eta_{xxt} + u_{xxx}) + \alpha((\eta u)_x + \eta\eta_x + uu_x) = 0 \quad (0.1)$$

$$u_t + \eta_x + \beta(\eta_{xxx} - u_{xxt}) + \alpha((\eta u)_x + \eta\eta_x + uu_x) = 0. \quad (0.2)$$

As a consequence of the local in time well-posedness of the Boussinesq system (0.1)-(0.2) with (η_0, u_0) as initial datum, for $0 < \alpha, \beta \leq 1$ and $s > 1/2$, there exist two constants $C_1 > 0$ and $C_2 > 0$ such that if $\|\eta_0\|_s + \|u_0\|_s \leq \varepsilon$, then for $|t| \leq C_2 \varepsilon^{-1}\alpha^{-1}\beta$, the solution (η, u) satisfies

$$\|\eta(t)\|_s + \|u(t)\|_s \leq C_1 \varepsilon. \quad (0.3)$$

Our goal is to prove (0.3) on a longer time scale for ε sufficiently small. We can notice that the H^1 -norm is preserved by the flow, i.e. if the solution exists, we have for all $t \in \mathbb{R}$

$$\|\eta(t)\|_1 + \|u(t)\|_1 \leq (\|\eta_0\|_1 + \|u_0\|_1)/\beta,$$

(with equality if $\beta = 1$) and the bound (0.3) with $s = 1$ is true for all time. If $s > 3/2$, our result reads as follows.

Theorem 0.1

Let $0 < \alpha, \beta \leq 1$ and $s > 3/2$. There exist $0 < \varepsilon_0 < \alpha^{-1}\beta^2$, $C_1 > 0$ and $C_2 > 0$ such that if $(\eta_0, u_0) \in H_0^s(\mathbb{T}) \times H_0^s(\mathbb{T})$ with $\|\eta_0\|_s + \|u_0\|_s \leq \varepsilon$, for $\varepsilon \in]0, \varepsilon_0[$, then the unique solution (η, u) of the Boussinesq system (0.1)-(0.2) with (η_0, u_0) as initial datum, satisfies for $|t| \leq C_2(\varepsilon^{-1}\alpha^{-1}\beta)^2$,

$$\|\eta(t)\|_s + \|u(t)\|_s \leq C_1\varepsilon. \tag{0.4}$$

The principle of the proof bases on the Poincaré’s theory of normal forms [1]. It consists in finding a map Λ , such that $(\mu, v) = \Lambda(\eta, u)$ satisfies

$$(\mu, v)_t + L(\mu, v) = F(\mu, v)$$

where L is the linear operator given by the free evolution of the initial system and F a multilinear operator of order strictly higher than 2 to improve the quadratic nature of the non-linearity of the Boussinesq system [6, 7].

Remark 0.2

- Let us notice that, when $\alpha^{-1}\beta^2 \geq 1$, the constraint of smallness on ε_0 does not depend on α and β . On the other hand, ε_0 chosen in the such way $\varepsilon < \alpha^{-1}\beta^2$, with $0 < \varepsilon < \varepsilon_0$, implies that the time $(\varepsilon^{-1}\alpha^{-1}\beta)^2$ is higher than the one found from the local existence, namely $\varepsilon^{-1}\alpha^{-1}\beta$.
- Fixing $\alpha = 1$ and $\eta = u$, when β tends to zero, the persistence time of the bound (0.4) given by the theorem 0.1 also tends to zero. That is not surprising, the formal limit, when β tends to zero, of the Boussinesq system being the Burgers equation

$$u_t + u_x + uu_x = 0.$$

However the existence time of the solution of the Cauchy problem with an initial datum of size ε , is of order ε^{-1} and this time is maximum.

- The case $1/2 < s \leq 3/2$ remains an open problem.

We use the following notations : $\mathbb{T} = \mathbb{R} / (2\pi\mathbb{Z})$ is the one-dimensional torus, \mathbb{Z}^* the set of nonzero integers. For $s \in \mathbb{R}$, we define $H_0^s(\mathbb{T})$ the space of zero x -mean value functions equipped with the norm

$$\|u(t)\|_s = \left(\sum_{k \in \mathbb{Z}^*} |k|^{2s} |\hat{u}(k)|^2 \right)^{1/2},$$

where \hat{u} denotes the Fourier transform defined by

$$\hat{u}(k) = \int_{\mathbb{T}} e^{-ikx} u(x) d\mu(x).$$

The measure $d\mu(x)$ is chosen proportional to the Lebesgue one on \mathbb{T} and normalized such that

$$u(x) = \sum_{k \in \mathbb{Z}^*} e^{ikx} \hat{u}(k).$$

The paper is organised as follows. In Section 1, the local in time well-posedness is established. The Section 2 deals with the proof of the main theorem.

1 Summary of existence theory

We consider the Cauchy problem, for $t \in \mathbb{R}$, and $x \in \mathbb{T}$,

$$\eta_t + u_x + \beta(-\eta_{xxt} + u_{xxx}) + \alpha((\eta u)_x + \eta\eta_x + uu_x) = 0 \quad (1.1)$$

$$u_t + \eta_x + \beta(\eta_{xxx} - u_{xxt}) + \alpha((\eta u)_x + \eta\eta_x + uu_x) = 0 \quad (1.2)$$

$$u(x, 0) = u_0(x), \eta(x, 0) = \eta_0(x). \quad (1.3)$$

The existence and the uniqueness of local in time solution are proved.

Theorem 1.1

Let $0 < \alpha, \beta \leq 1$, $s > 1/2$ and $(\eta_0, u_0) \in H_0^s(\mathbb{T}) \times H_0^s(\mathbb{T})$. There exists a constant $C_0 > 0$, depending only on s , such that for

$$T = \frac{C_0}{\|\eta_0\|_s + \|u_0\|_s} \frac{\beta}{\alpha},$$

there exists a unique solution $(\eta, u) \in \mathcal{C}([-T, T]; H_0^s(\mathbb{T})) \times \mathcal{C}([-T, T]; H_0^s(\mathbb{T}))$ of the Cauchy problem (1.1)-(1.2)-(1.3).

Moreover, for all $M > 0$ with $\|\eta_0\|_s + \|u_0\|_s \leq M$ and $\|\mu_0\|_s + \|v_0\|_s \leq M$, there exists $C_1 > 0$ such that solutions (η, u) and (μ, v) , of initial data (η_0, u_0) and (μ_0, v_0) respectively, satisfy for $t \in [-T, T]$, with $T = C_0\alpha^{-1}\beta/M$,

$$\|\eta(t) - \mu(t)\|_s + \|u(t) - v(t)\|_s \leq C_1(\|\eta_0 - \mu_0\|_s + \|u_0 - v_0\|_s).$$

Proof. Let $T > 0$. The Duhamel's formula implies that (η, u) is the solution of the Cauchy problem (1.1)-(1.2)-(1.3) if and only if (η, u) is the solution of the following equation, for $t \in [0, T]$,

$$(\eta, u)(t) = \Phi(\eta, u)(t) := S_t(\eta_0, u_0) - \frac{\alpha}{2} \int_0^t S_{t-\tau} \left(\frac{\partial_x}{1 - \beta\partial_x^2} ((\eta + u)^2, (\eta + u)^2) \right) (\tau) d\tau, \quad (1.4)$$

with

$$S_t(\eta, u) := \sum_{k \in \mathbb{Z}^*} e^{ikx} \left(\cos(tk \frac{1 - \beta k^2}{1 + \beta k^2}) \hat{\eta}(k) - i \sin(tk \frac{1 - \beta k^2}{1 + \beta k^2}) \hat{u}(k), \right. \\ \left. \cos(tk \frac{1 - \beta k^2}{1 + \beta k^2}) \hat{u}(k) - i \sin(tk \frac{1 - \beta k^2}{1 + \beta k^2}) \hat{\eta}(k) \right). \quad (1.5)$$

We aim at applying the fixed point theorem. We deduce from the Duhamel's formula, for $t \in [0, T]$,

$$\|\Phi(\eta, u)(t)\|_s \leq C(\|\eta_0\|_s + \|u_0\|_s) + C\alpha \int_0^t \left\| \frac{\partial_x}{1 - \beta\partial_x^2} (u + \eta)^2 \right\|_s (\tau) d\tau.$$

For $0 < \beta \leq 1$, the definition of the Sobolev norm provides

$$\left\| \frac{\partial_x}{1 - \beta\partial_x^2} (u + \eta)^2 \right\|_s = \left(\sum_{k \in \mathbb{Z}^*} |k|^{2s} \left| \frac{ik}{1 + \beta k^2} \widehat{(u + \eta)^2}(k) \right|^2 \right)^{1/2} \leq \frac{1}{\beta} \|(u + \eta)^2\|_s \\ \leq \frac{C_s}{\beta} \|u + \eta\|_\infty \|u + \eta\|_s.$$

Since $s > 1/2$, the Sobolev embedding implies that there exists a constant $C_s > 0$, depending only on s , such that

$$\|\Phi(\eta, u)(t)\|_s \leq C(\|\eta_0\|_s + \|u_0\|_s) + \frac{\alpha}{\beta} C_s T \left(\sup_{t \in [0, T]} (\|\eta(t)\|_s + \|u(t)\|_s) \right)^2. \quad (1.6)$$

Then there exists $C_0 > 0$ such that for $T = C_0\alpha^{-1}\beta/(\|\eta_0\|_s + \|u_0\|_s)$, the closed ball

$$\overline{B}_T := \left\{ (\eta, u) \in \mathcal{C}([0, T]; H_0^s(\mathbb{T})) \times \mathcal{C}([0, T]; H_0^s(\mathbb{T})) ; \sup_{t \in [0, T]} (\|\eta(t)\|_s + \|u(t)\|_s) \leq 2C(\|\eta_0\|_s + \|u_0\|_s) \right\}$$

satisfies $\Phi(\overline{B}_T) \subseteq \overline{B}_T$. Indeed, let $(\eta, u) \in \overline{B}_T$, the inequality (1.6) becomes

$$\|\Phi(\eta, u)(t)\|_s \leq C(\|\eta_0\|_s + \|u_0\|_s)(1 + 4CC_sC_0),$$

and $C(\|\eta_0\|_s + \|u_0\|_s)(1 + 4CC_sC_0) \leq 2C(\|\eta_0\|_s + \|u_0\|_s)$ if $C_0 \leq 1/(4CC_s)$.

Let (η, u) and (μ, v) be in \overline{B}_T . The Duhamel's formula (1.4) provides, for $t \in [0, T]$,

$$\|\Phi(\eta, u)(t) - \Phi(\mu, v)(t)\|_s \leq C\alpha \int_0^t \left\| \frac{\partial_x}{1 - \beta\partial_x^2}(\eta u - \mu v) \right\|_s + \left\| \frac{\partial_x}{1 - \beta\partial_x^2}(u^2 - v^2 + \eta^2 - \mu^2) \right\|_s(\tau) d\tau.$$

Noticing that $\eta u - \mu v = 1/2((\eta + \mu)(u - v) + (\eta - \mu)(u + v))$, $u^2 - v^2 = (u + v)(u - v)$ and applying the Sobolev embedding, one gets

$$\begin{aligned} \|\Phi(\eta, u)(t) - \Phi(\mu, v)(t)\|_s &\leq \frac{12C^2\alpha}{\beta} C_s T (\|\eta_0\|_s + \|u_0\|_s) \left(\sup_{t \in [-T, T]} (\|\eta - \mu\|_s(t) + \|u - v\|_s(t)) \right) \\ &= 12C^2C_sC_0 \left(\sup_{t \in [-T, T]} (\|\eta - \mu\|_s(t) + \|u - v\|_s(t)) \right). \end{aligned}$$

For $C_0 < 1/(12C^2C_s)$, the map Φ is a contraction on \overline{B}_T . Finally, according to the fixed point theorem, there exists a unique solution (η, u) of $\Phi(\eta, u)(t) = (\eta, u)(t)$ in \overline{B}_T .

It remains to prove the continuity with the initial datum. Let (η, u) and (μ, v) be solutions of the Cauchy problem (1.1)-(1.2)-(1.3) with initial datum (η_0, u_0) and (μ_0, v_0) respectively, such that $\|\eta_0\|_s + \|u_0\|_s \leq M$ and $\|\mu_0\|_s + \|v_0\|_s \leq M$. The Duhamel's formula (1.4) gives for $t \in [0, T]$, with $T = C_0\alpha^{-1}\beta/M$, with $C_0 \ll 1$,

$$\begin{aligned} \|(\eta, u)(t) - (\mu, v)(t)\|_s &\leq C(\|\eta_0 - \mu_0\|_s + \|u_0 - v_0\|_s) \\ &\quad + C\alpha \int_0^t \left\| \frac{\partial_x}{1 - \beta\partial_x^2}(\eta u - \mu v) \right\|_s + \left\| \frac{\partial_x}{1 - \beta\partial_x^2}(u^2 - v^2 + \eta^2 - \mu^2) \right\|_s(\tau) d\tau \\ &\leq C(\|\eta_0 - \mu_0\|_s + \|u_0 - v_0\|_s) + \frac{1}{2} \left(\sup_{t \in [0, T]} \|\eta - \mu\|_s(t) + \sup_{t \in [0, T]} \|u - v\|_s(t) \right), \end{aligned}$$

thus

$$\sup_{t \in [0, T]} (\|\eta - \mu\|_s + \|u - v\|_s) \leq 2C(\|\eta_0 - \mu_0\|_s + \|u_0 - v_0\|_s).$$

□

Remark 1.2

The time given in the preceding theorem could be higher, especially in the case $0 < \beta \leq \alpha \leq 1$, but we have to impose more regular initial data. More precisely, let us suppose $s > 3/2$ and $(\eta_0, u_0) \in H_0^s(\mathbb{T}) \times H_0^s(\mathbb{T})$. Then, there exists a constant $C_0 > 0$, depending only on s , such that for

$$T = \frac{C_0}{\|\eta_0\|_s + \|u_0\|_s} \frac{1}{\alpha},$$

there exists a unique solution $(\eta, u) \in \mathcal{C}([-T, T]; H_0^s(\mathbb{T})) \times \mathcal{C}([-T, T]; H_0^s(\mathbb{T}))$ of the Cauchy problem (1.1)-(1.2)-(1.3). On the other hand, the choice of ε_0 in the theorem 0.1 also implies that $(\varepsilon^{-1}\alpha^{-1}\beta)^2 \geq \varepsilon^{-1}\alpha^{-1}$.

2 Long time bounds

A consequence of the local well-posedness is that for $0 < \alpha, \beta \leq 1$, $s > 1/2$, $\varepsilon > 0$ and $(\eta_0, u_0) \in H_0^s(\mathbb{T}) \times H_0^s(\mathbb{T})$ with $\|\eta_0\|_s + \|u_0\|_s \leq \varepsilon$, there exist $C_0 > 0$ and $C_1 > 0$ such that the solution (η, u) of the Cauchy problem (1.1)-(1.2)-(1.3) satisfies for $|t| \leq C_0 \varepsilon^{-1} \alpha^{-1} \beta$,

$$\|\eta(t)\|_s + \|u(t)\|_s \leq C_1 \varepsilon.$$

We wonder if the solution exists and remains small longer, the normal form is used [6, 7]. To simplify the writings, we define L by the Fourier symbol $\sigma(k) := ik(1 - \beta k^2)/(1 + \beta k^2)$. Let $D := \{(k, k_1) \in \mathbb{Z}^2; k \neq 0, k_1 \neq 0, k \neq k_1\}$, we define the operator Λ by,

$$\Lambda(\eta, u) := (\eta + B(\eta + u, \eta + u), u + B(\eta + u, \eta + u)),$$

with the bilinear operator

$$B(u, v) := -\frac{\alpha}{2} \sum_D e^{ikx} \frac{ik}{1 + \beta k^2} \frac{\hat{u}(k_1) \hat{v}(k - k_1)}{\sigma(k_1) + \sigma(k - k_1) - \sigma(k)},$$

and for $\delta > 0$, $\mathcal{V}_\delta := \{(\eta, u) \in H_0^s(\mathbb{T}) \times H_0^s(\mathbb{T}); \|\eta\|_s + \|u\|_s < \delta\}$.

Introducing Λ is used to define $(\mu, v) = \Lambda(\eta, u)$ so that (μ, v) is solution of the equation

$$\mu_t + L(v) = F(\eta, u), \quad v_t + L(\mu) = F(\eta, u),$$

with F trilinear whereas (η, u) is solution of a quadratic Boussinesq system. Thus the well-posedness of (μ, v) and the definition of B are used to estimate (η, u) with respect to (μ, v) and to extend its well-posedness.

Proposition 2.1

Let $s > 3/2$. Then there exist $0 < \delta' < \alpha^{-1} \beta^2$, $\delta > 0$, and $C > 0$ such that for all $(\mu, v) \in \mathcal{V}_\delta$, there exists a unique $(\eta, u) \in \mathcal{V}_{\delta'}$ such that $\Lambda(\eta, u) = (\mu, v)$. Moreover

$$\|\eta\|_s + \|u\|_s \leq C(\|\mu\|_s + \|v\|_s).$$

Proof. We first prove some useful lemmas.

Lemma 2.2

The Fourier symbol σ satisfies for all k and k_1 in \mathbb{Z}

$$|\sigma(k_1) + \sigma(k - k_1) - \sigma(k)| = \frac{|kk_1(k - k_1)| |6\beta + 2\beta^2(k^2 - kk_1 + k_1^2)|}{(1 + \beta k^2)(1 + \beta k_1^2)(1 + \beta(k - k_1)^2)}.$$

Lemma 2.3

Let $s > 3/2$. There exists a constant $C > 0$ such that for all u and v in $H_0^s(\mathbb{T})$

$$\|B(u, v)\|_s \leq C \frac{\alpha}{\beta^2} \|u\|_s \|v\|_s.$$

Proof. By duality, to prove the lemma is equivalent to prove for all $w \in \mathcal{C}^\infty(\mathbb{T})$

$$\left| \sum_D \widehat{B(u, v)}(k) \hat{w}(k) \right| \leq C \frac{\alpha}{\beta^2} (\|u\|_s \|v\|_s) \|w\|_{-s}. \quad (2.1)$$

Indeed, we have

$$\|B(u, v)\|_s^2 = \sum_{k \in \mathbb{Z}^*} |k|^{2s} |\widehat{B(u, v)}(k)|^2 = \sum_{k \in \mathbb{Z}^*} \widehat{B(u, v)}(k) \left(|k|^{2s} \widehat{B(u, v)}(k) \right).$$

We set $\hat{w}(k) = |k|^{2s} \widehat{B(u, v)}(k)$ and we write

$$\|B(u, v)\|_s^2 = \sum_{k \in \mathbb{Z}^*} \widehat{B(u, v)}(k) \hat{w}(k),$$

and according to the inequality (2.1)

$$\|B(u, v)\|_s^2 \leq C \frac{\alpha}{\beta^2} (\|u\|_s \|v\|_s) \|w\|_{-s}.$$

However

$$\begin{aligned} \|w\|_{-s} &= \left(\sum_{k \in \mathbb{Z}^*} |k|^{-2s} |k|^{4s} |\widehat{B(u, v)}(k)|^2 \right)^{1/2} \\ &= \left(\sum_{k \in \mathbb{Z}^*} |k|^{2s} |\widehat{B(u, v)}(k)|^2 \right)^{1/2} = \|B(u, v)\|_s. \end{aligned}$$

We define

$$\hat{u}_1(k) = |k|^s \hat{u}(k), \quad \hat{v}_1(k) = |k|^s \hat{v}(k) \quad \text{and} \quad \hat{w}_1(k) = |k|^{-s} \hat{w}(k).$$

In particular, it implies

$$\|u_1\|_{L^2} = \|u\|_s, \quad \|v_1\|_{L^2} = \|v\|_s \quad \text{and} \quad \|w_1\|_{L^2} = \|w\|_{-s}.$$

We then find

$$\begin{aligned} \widehat{B(u, v)}(k) &= -\frac{\alpha}{2} \frac{ik}{1 + \beta k^2} \sum_{\substack{k_1 \in \mathbb{Z}^* \\ k_1 \neq k}} \frac{\hat{u}(k_1) \hat{v}(k - k_1)}{\sigma(k_1) + \sigma(k - k_1) - \sigma(k)} \\ &= -\frac{\alpha}{2} \frac{ik}{1 + \beta k^2} \sum_{\substack{k_1 \in \mathbb{Z}^* \\ k_1 \neq k}} \frac{|k|^s \hat{u}_1(k_1) \hat{v}_1(k - k_1)}{|k_1|^s |k - k_1|^s (\sigma(k_1) + \sigma(k - k_1) - \sigma(k))}. \end{aligned}$$

Finally, it is enough to prove

$$\left| \frac{\alpha}{2} \sum_D \frac{ik}{1 + \beta k^2} \frac{|k|^s \hat{u}_1(k_1) \hat{v}_1(k - k_1) \hat{w}_1(k)}{|k_1|^s |k - k_1|^s (\sigma(k_1) + \sigma(k - k_1) - \sigma(k))} \right| \leq C \frac{\alpha}{\beta^2} (\|u_1\|_{L^2} \|v_1\|_{L^2}) \|w\|_{L^2}.$$

Lemma 2.4

We have for k and k_1 in D

$$\left| \frac{ik}{1 + \beta k^2} \frac{1}{\sigma(k_1) + \sigma(k - k_1) - \sigma(k)} \right| \leq \frac{2}{\beta^2} \left| \frac{k - k_1}{k} \right|.$$

Proof. According to the lemma 2.2, we have for k and k_1 in D ,

$$\left| \frac{ik}{1 + \beta k^2} \frac{1}{\sigma(k_1) + \sigma(k - k_1) - \sigma(k)} \right| \leq \frac{(1 + \beta k_1^2)(1 + \beta(k - k_1)^2)}{2\beta^2 |k_1(k - k_1)(k^2 - kk_1 + k_1^2)|}.$$

Since $0 < \beta \leq 1$, $1 + \beta k_1^2 \leq 2k_1^2$ and $|k^2 - kk_1 + k_1^2| \geq |kk_1|$, we have

$$\left| \frac{ik}{1 + \beta k^2} \frac{1}{\sigma(k_1) + \sigma(k - k_1) - \sigma(k)} \right| \leq \frac{2}{\beta^2} \left| \frac{k - k_1}{k} \right|.$$

□

For $s \geq 1$, the triangle inequality implies

$$\frac{|k|^{s-1}}{|k_1|^s |k - k_1|^{s-1}} \leq C \left(\frac{1}{|k_1|^s} + \frac{1}{|k_1| |k - k_1|^{s-1}} \right) \leq C \left(\frac{1}{|k_1|^s} + \frac{1}{|k - k_1|^{s-1}} \right).$$

According to the preceding lemma, it remains to bound

$$\frac{\alpha}{\beta^2} \sum_D \frac{|\hat{u}_1(k_1)| |\hat{v}_1(k - k_1)| |\hat{w}_1(k)|}{|k_1|^s} + \frac{\alpha}{\beta^2} \sum_D \frac{|\hat{u}_1(k_1)| |\hat{v}_1(k - k_1)| |\hat{w}_1(k)|}{|k - k_1|^{s-1}} =: \frac{\alpha}{\beta^2} (\text{I} + \text{II}).$$

The Cauchy-Schwarz inequality in k gives for the first term of the preceding sum

$$\text{I} \leq \left(\sum_{k \in \mathbb{Z}^*} \left(\sum_{\substack{k_1 \in \mathbb{Z}^* \\ k_1 \neq k}} \frac{|\hat{u}_1(k_1)| |\hat{v}_1(k - k_1)|}{|k_1|^s} \right)^2 \right)^{1/2} \left(\sum_{k \in \mathbb{Z}^*} |\hat{w}_1(k)|^2 \right)^{1/2},$$

then the Cauchy-Schwarz inequality in k_1 is applied again,

$$\text{I} \leq \left(\sum_{\substack{k_1 \in \mathbb{Z}^* \\ k_1 \neq k}} \frac{1}{|k_1|^{2s}} \right)^{1/2} \left(\sum_{k \in \mathbb{Z}^*} \sum_{\substack{k_1 \in \mathbb{Z}^* \\ k_1 \neq k}} |\hat{u}_1(k_1)|^2 |\hat{v}_1(k - k_1)|^2 \right)^{1/2} \left(\sum_{k \in \mathbb{Z}^*} |\hat{w}_1(k)|^2 \right)^{1/2}.$$

Since $s > 1/2$, there exists a constant $C > 0$ such that

$$\text{I} \leq C (\|u_1\|_{L^2} \|v_1\|_{L^2}) \|w_1\|_{L^2}.$$

By symmetry, a similar inequality for II is verified if $2(s - 1) > 1$, i.e. $s > 3/2$. □

The differential of this operator is given by, for all $(\varphi, \psi) \in \mathcal{C}^\infty(\mathbb{T}) \times \mathcal{C}^\infty(\mathbb{T})$

$$\langle d\Lambda(\eta, u), (\varphi, \psi) \rangle = \begin{pmatrix} \varphi + 2B(\eta + u, \varphi) & 2B(\eta + u, \psi) \\ 2B(\eta + u, \varphi) & \psi + 2B(\eta + u, \psi) \end{pmatrix},$$

the preceding lemma implies that $d\Lambda$ is continuous on $H_0^s(\mathbb{T}) \times H_0^s(\mathbb{T})$. Since $d\Lambda(0, 0)$ is the identity, the inverse function theorem is applied to give the following lemma. □

We aim at setting which equation is satisfied by Λ .

Proposition 2.5

Let $s > 3/2$. There exists a trilinear operator

$$F : H_0^s(\mathbb{T}) \times H_0^s(\mathbb{T}) \times H_0^s(\mathbb{T}) \longrightarrow H_0^s(\mathbb{T})$$

satisfying that there exists a constant $C > 0$ such that for all $(u_1, u_2, u_3) \in H_0^s(\mathbb{T}) \times H_0^s(\mathbb{T}) \times H_0^s(\mathbb{T})$

$$\|F(u_1, u_2, u_3)\|_s \leq C \frac{\alpha^2}{\beta^2} \|u_1\|_s \|u_2\|_s \|u_3\|_s,$$

and, if $(\eta, u) \in \mathcal{C}([-T, T]; H_0^s(\mathbb{T})) \times \mathcal{C}([-T, T]; H_0^s(\mathbb{T}))$ is the solution of the system (1.1)-(1.2), then (μ, v) defined by, for $t \in [-T, T]$

$$\begin{aligned} \mu(t) &:= \eta(t) + B(\eta(t) + u(t), \eta(t) + u(t)) \\ v(t) &:= u(t) + B(\eta(t) + u(t), \eta(t) + u(t)), \end{aligned}$$

is solution of

$$\begin{aligned} \mu_t + L(v) &= F(\eta + u, \eta + u, \eta + u) \\ v_t + L(\mu) &= F(\eta + u, \eta + u, \eta + u). \end{aligned}$$

Proof. We notice firstly that $(\mu, v) \in \mathcal{C}([-T, T]; H_0^s(\mathbb{T})) \times \mathcal{C}([-T, T]; H_0^s(\mathbb{T}))$ according to the lemma 2.3. We write

$$\begin{aligned} \mu_t + L(v) &= \eta_t + \partial_t B(\eta + u, \eta + u) + L(u) + L(B(\eta + u, \eta + u)) \\ &= -\frac{\alpha}{2} \frac{\partial_x}{1 - \beta \partial_x^2} (\eta + u)^2 + \partial_t B(\eta + u, \eta + u) + L(B(\eta + u, \eta + u)). \end{aligned}$$

On one hand, we have

$$\begin{aligned} \partial_t B(\eta + u, \eta + u) &= B(\eta_t + u_t, \eta + u) + B(\eta + u, \eta_t + u_t) = 2B(\eta_t + u_t, \eta + u) \\ &= -\alpha \sum_D e^{ikx} \frac{ik}{1 + \beta k^2} \frac{(\hat{\eta}_t(k_1) + \hat{u}_t(k_1))(\hat{\eta}(k - k_1) + \hat{u}(k - k_1))}{\sigma(k_1) + \sigma(k - k_1) - \sigma(k)}. \end{aligned}$$

Since (η, u) is solution of the system (1.1)-(1.3), we obtain by symmetry

$$\begin{aligned} \partial_t B(\eta + u, \eta + u) &= -\alpha^2 \sum_D e^{ikx} \frac{kk_1}{(1 + \beta k^2)(1 + \beta k_1^2)} \frac{\widehat{(\eta + u)^2}(k_1)(\hat{\eta}(k - k_1) + \hat{u}(k - k_1))}{\sigma(k_1) + \sigma(k - k_1) - \sigma(k)} \\ &\quad + \frac{\alpha}{2} \sum_D e^{ikx} \frac{ik}{1 + \beta k^2} \frac{(\sigma(k_1) + \sigma(k - k_1))(\hat{\eta}(k_1) + \hat{u}(k_1))(\hat{\eta}(k - k_1) + \hat{u}(k - k_1))}{\sigma(k_1) + \sigma(k - k_1) - \sigma(k)}. \end{aligned}$$

On the other hand, we have

$$L(B(\eta + u, \eta + u)) = -\frac{\alpha}{2} \sum_D e^{ikx} \frac{ik}{1 + \beta k^2} \frac{\sigma(k)(\hat{\eta}(k_1) + \hat{u}(k_1))(\hat{\eta}(k - k_1) + \hat{u}(k - k_1))}{\sigma(k_1) + \sigma(k - k_1) - \sigma(k)}.$$

The last term gives

$$-\frac{\alpha}{2} \frac{\partial_x}{1 - \beta \partial_x^2} (\eta + u)^2 = -\frac{\alpha}{2} \sum_D e^{ikx} \frac{ik}{1 + \beta k^2} (\hat{\eta}(k_1) + \hat{u}(k_1))(\hat{\eta}(k - k_1) + \hat{u}(k - k_1)).$$

Finally, it follows

$$\mu_t + L(v) = -\alpha^2 \sum_D e^{ikx} \frac{kk_1}{(1 + \beta k^2)(1 + \beta k_1^2)} \frac{\widehat{(\eta + u)^2}(k_1)(\hat{\eta}(k - k_1) + \hat{u}(k - k_1))}{\sigma(k_1) + \sigma(k - k_1) - \sigma(k)}. \quad (2.2)$$

We denote $D_1 := \{(k, k_1, k_2) \in \mathbb{Z}^3; k \neq 0, k_1 \neq 0, k_2 \neq 0, k \neq k_1, k_1 \neq k_2\}$ and we deduce from (2.2) that F is defined by

$$F(u_1, u_2, u_3) := -\alpha^2 \sum_{D_1} e^{ikx} \frac{kk_1}{(1 + \beta k^2)(1 + \beta k_1^2)} \frac{\hat{u}_1(k_2)\hat{u}_2(k_1 - k_2)\hat{u}_3(k - k_1)}{\sigma(k_1) + \sigma(k - k_1) - \sigma(k)}.$$

Lemma 2.6

We have for k, k_1 and k_2 in D_1

$$\left| \frac{kk_1}{(1 + \beta k^2)(1 + \beta k_1^2)} \frac{1}{\sigma(k_1) + \sigma(k - k_1) - \sigma(k)} \right| \leq \frac{1}{\beta^2} \left| \frac{k - k_1}{k} \right|.$$

Proof. According to the lemma 2.2, we have for k, k_1 and k_2 in D_1 ,

$$\left| \frac{kk_1}{(1 + \beta k^2)(1 + \beta k_1^2)} \frac{1}{\sigma(k_1) + \sigma(k - k_1) - \sigma(k)} \right| \leq \frac{1 + \beta(k - k_1)^2}{2\beta^2 |kk_1(k - k_1)|},$$

and since $0 < \beta \leq 1$ and $|k_1| \geq 1$, we have

$$\left| \frac{kk_1}{(1 + \beta k^2)(1 + \beta k_1^2)} \frac{1}{\sigma(k_1) + \sigma(k - k_1) - \sigma(k)} \right| \leq \frac{1}{\beta^2} \left| \frac{k - k_1}{k} \right|.$$

□

In the same way as lemma 2.3, by duality, it is enough to bound

$$I := \sum_{D_1} \frac{|k|^{s-1} |\hat{u}_1(k_2)| |\hat{u}_2(k_1 - k_2)| |\hat{u}_3(k - k_1)| |\hat{u}_4(k)|}{|k_2|^s |k_1 - k_2|^s |k - k_1|^{s-1}}.$$

The Cauchy-Schwarz inequality is first applied in k to give

$$I \leq \left(\sum_{k \in \mathbb{Z}^*} \left(\sum_{\substack{(k_1, k_2) \in (\mathbb{Z}^*)^2 \\ k_1 \neq k, k_2 \neq k_1}} \frac{|k|^{s-1} |\hat{u}_1(k_2)| |\hat{u}_2(k_1 - k_2)| |\hat{u}_3(k - k_1)|}{|k_2|^s |k_1 - k_2|^s |k - k_1|^{s-1}} \right)^2 \right)^{1/2} \|u_4\|_{L^2},$$

and then in (k_2, k_1) ,

$$I \leq \left(\sum_{k \in \mathbb{Z}^*} \sum_{\substack{(k_1, k_2) \in (\mathbb{Z}^*)^2 \\ k_1 \neq k, k_2 \neq k_1}} |\hat{u}_1(k_2)|^2 |\hat{u}_2(k_1 - k_2)|^2 |\hat{u}_3(k - k_1)|^2 \times \sum_{\substack{(k_1, k_2) \in (\mathbb{Z}^*)^2 \\ k_1 \neq k, k_2 \neq k_1}} \frac{|k|^{2(s-1)}}{|k_2|^{2s} |k_1 - k_2|^{2s} |k - k_1|^{2(s-1)}} \right)^{1/2} \|u_4\|_{L^2},$$

However, since $s > 3/2 \geq 1$, the triangle inequality implies

$$\frac{|k|^{2(s-1)}}{|k_2|^{2s} |k_1 - k_2|^{2s} |k - k_1|^{2(s-1)}} \leq C \left(\frac{1}{|k_2|^{2s} |k_1 - k_2|^{2s}} + \frac{1}{|k_2|^2 |k_2 - k_1|^{2s} |k - k_1|^{2(s-1)}} + \frac{1}{|k_2|^{2s} |k_1 - k_2|^2 |k - k_1|^{2(s-1)}} \right),$$

thus

$$\sup_{k \in \mathbb{Z}^*} \sum_{\substack{(k_1, k_2) \in (\mathbb{Z}^*)^2 \\ k_1 \neq k, k_2 \neq k_1}} \frac{|k|^{2(s-1)}}{|k_2|^{2s} |k_1 - k_2|^{2s} |k - k_1|^{2(s-1)}} < +\infty.$$

□

The main result of this paper is now proved.

Proof of the theorem 0.1. We suppose $t > 0$, the proof is similar for negative time. Let δ and δ' be the positive constants involved in the lemma 2.1.

Thanks to the local well-posedness theorem 1.1, there exists $\varepsilon_0 > 0$ such that if $(\eta_0, u_0) \in \mathcal{V}_\varepsilon$, for $\varepsilon \in]0, \varepsilon_0[$, then for $t \leq C_2 \varepsilon^{-1} \alpha^{-1} \beta$,

$$(\eta(t) + B(\eta + u, \eta + u)(t), u(t) + B(\eta + u, \eta + u)(t)) \in \mathcal{V}_\delta, \text{ and } (\eta(t), u(t)) \in \mathcal{V}_{\delta'}.$$

Thus $(\eta, u)(t) = \Lambda^{-1}(\mu, v)(t)$, for $t \leq C_2 \varepsilon^{-1} \alpha^{-1} \beta =: T$.

The Duhamel's formula gives for $t \in [0, T]$

$$\begin{aligned} (\mu, v)(t) &= S_t(\eta_0 + B(\eta_0 + u_0, \eta_0 + u_0), u_0 + B(\eta_0 + u_0, \eta_0 + u_0)) \\ &\quad + \int_0^t S_{t-\tau}(F(\eta + u, \eta + u, \eta + u), F(\eta + u, \eta + u, \eta + u))(\tau) d\tau, \end{aligned} \quad (2.3)$$

where S_t is defined in (1.5). According to the lemma 2.3, there exists a constant $C_1 > 0$ such that we have

$$\begin{aligned} \|S_t(\eta_0 + B(\eta_0 + u_0, \eta_0 + u_0), u_0 + B(\eta_0 + u_0, \eta_0 + u_0))\|_s &\leq \|\eta_0\|_s + \|u_0\|_s + C \frac{\alpha}{\beta^2} (\|\eta_0\|_s + \|u_0\|_s)^2 \\ &\leq C_1 \varepsilon \left(2 + 4\varepsilon \frac{\alpha}{\beta^2} \right). \end{aligned}$$

Even if we take $\varepsilon_0 > 0$ smaller, we have

$$\|S_t(\eta_0 + B(\eta_0 + u_0, \eta_0 + u_0), u_0 + B(\eta_0 + u_0, \eta_0 + u_0))\|_s \leq 3C_1 \varepsilon. \quad (2.4)$$

Lemmas 2.1 and 2.5 imply that there exists a constant $C_3 > 0$ such that, if $(\mu(t), v(t)) \in \mathcal{V}_\delta$ for $t \in [0, T]$, then

$$\left\| \int_0^t S_{t-\tau}(F(\eta + u, \eta + u, \eta + u), F(\eta + u, \eta + u, \eta + u))(\tau) d\tau \right\|_{L^\infty([0, T]; H_\delta^s(\mathbb{T}))} \leq \quad (2.5)$$

$$C_3 \frac{\alpha^2}{\beta^2} T (\|\mu\|_{L^\infty([0, T]; H_\delta^s(\mathbb{T}))} + \|v\|_{L^\infty([0, T]; H_\delta^s(\mathbb{T}))})^3.$$

If we need to take $\varepsilon_0 > 0$ smaller again, we impose $4C_1 \varepsilon_0 < \delta$. We set $C_0 = 1/(128C_1^2 C_3)$ and $T_0 = C_0 (\varepsilon^{-1} \alpha^{-1} \beta)^2$. It then follows

$$\|\mu\|_{L^\infty([0, T_0]; H_\delta^s(\mathbb{T}))} + \|v\|_{L^\infty([0, T_0]; H_\delta^s(\mathbb{T}))} \leq 4C_1 \varepsilon. \quad (2.6)$$

Let us suppose that the inequality (2.6) fails. Since

$$\|\mu(0)\|_s + \|v(0)\|_s = \|\eta_0 + B(\eta_0 + u_0, \eta_0 + u_0)\|_s + \|u_0 + B(\eta_0 + u_0, \eta_0 + u_0)\|_s \leq C_1 \varepsilon \left(2 + 4\varepsilon \frac{\alpha}{\beta^2} \right) < 4C_1 \varepsilon,$$

by continuity with time, there exists $\tau \in [0, T_0]$ such that for $t \in [0, \tau]$

$$\|\mu(t)\|_s + \|v(t)\|_s \leq 4C_1\varepsilon \quad \text{and} \quad \|\mu(\tau)\|_s + \|v(\tau)\|_s = 4C_1\varepsilon.$$

Let C be the positive constant involved in the lemma 2.1, we also impose $4C_1C\varepsilon_0 < \delta'$. We know that $(\eta(t), u(t)) \in \mathcal{V}_{\delta'}$ for $|t| \leq T$, and with this choice of ε , it follows that $(\eta(t), u(t)) \in \mathcal{V}_{\delta'}$ for $t \in [0, \tau]$. Indeed, if there exists $\tau_0 \in [0, \tau]$ such that for $t \in [0, \tau_0]$

$$\|\eta(t)\|_s + \|u(t)\|_s < \delta' \quad \text{and} \quad \|\eta(\tau_0)\|_s + \|u(\tau_0)\|_s = \delta',$$

then by continuity with time and according to the lemma 2.1, we have

$$\begin{aligned} \delta' = \|\eta(\tau_0)\|_s + \|u(\tau_0)\|_s &= \lim_{t \rightarrow \tau_0} \|\eta(t)\|_s + \|u(t)\|_s \leq \sup_{0 \leq t < \tau_0} \|\eta(t)\|_s + \|u(t)\|_s \\ &\leq C \sup_{0 \leq t < \tau_0} \|\mu(t)\|_s + \|v(t)\|_s \leq 4C_1C\varepsilon < \delta', \end{aligned}$$

which is impossible.

Finally, we find from the Duhamel's formula (2.3) and from inequalities (2.4) and (2.5)

$$\|\mu\|_{L^\infty([0, \tau]; H_0^s(\mathbb{T}))} + \|v\|_{L^\infty([0, \tau]; H_0^s(\mathbb{T}))} \leq 3C_1\varepsilon + C_3 \frac{\alpha^2}{\beta^2} \tau (\|\mu\|_{L^\infty([0, \tau]; H_0^s(\mathbb{T}))} + \|v\|_{L^\infty([0, \tau]; H_0^s(\mathbb{T}))})^3,$$

or equivalently

$$4C_1\varepsilon \leq 3C_1\varepsilon + C_3 \frac{\alpha^2}{\beta^2} \tau (4C_1\varepsilon)^3,$$

which gives

$$\tau \geq 2C_0 \left(\varepsilon^{-1} \frac{\beta}{\alpha} \right)^2 = 2T_0.$$

And this is a contradiction with $\tau \in [0, T_0]$. Then the inequality (2.6) is true and using the lemma 2.1 we have for $t \in [0, T_0]$

$$\|\eta(t)\|_s + \|u(t)\|_s \leq C(\|\mu(t)\|_s + \|v(t)\|_s) \leq 4C_1C\varepsilon.$$

□

References

- [1] V. I. Arnold: Geometric Methods in the Theory of Ordinary Differential Equations. Springer-Verlag, New York, (1983).
- [2] J.L. Bona, M. Chen, J.-C. Saut: Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media. I. Derivation and linear theory. *J. Nonlinear Sci.* **12** (2002), no. 4, 283–318.
- [3] J.L. Bona, M. Chen, J.-C. Saut: Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media. II. The nonlinear theory. *Nonlinearity* **17** (2004), no. 3, 925–952.
- [4] J. V. Boussinesq: Théorie générale des mouvements qui sont propagés dans un canal rectangulaire horizontal. *C.R. Acad. Sci. Paris* **73** (1871) 256–260.
- [5] J. A. Gear, R. Grimshaw: Weak and strong interactions between internal solitary waves. *Stud. Appl. Math.* **70** (3) (1984), 235–258.
- [6] J. Shatah: Normal forms and quadratic nonlinear Klein-Gordon equations. *Comm. Pure Appl. Math.* **38** (1985), 685–696.
- [7] N. Tzvetkov: Long time bounds for the periodic KP-II equation. *Int. Math. Res. Not.* **46** (2004), 2485–2496.
- [8] G. B. Whitham: Linear and Nonlinear Waves. Wiley, New York (1999).