Continuation of Time Bounds for a Regularized Boussinesq System

Youcef Mammeri

Continuation of time bounds for a regularized Boussinesq system

Y. Mammeri
Laboratoire Amiénois de Mathématique Fondamentale et Appliquée d'Amiens,
CNRS UMR 6140, Université de Picardie Jules Verne,
80069 Amiens, France.
Email: youcef.mammeri@u-picardie.fr

October 2011

Abstract. We study the periodic solution of a perturbed regularized Boussinesq system [2, 5], namely the system $\eta_t + u_x + \beta(-\eta_{xxt} + u_{xxx}) + \alpha((\eta u)_x + \eta\eta_x + uu_x) = 0, u_t + \eta_x + \beta(\eta_{xxx} - u_{xxt}) + \alpha((\eta u)_x + \eta\eta_x + uu_x) = 0$, with $0 < \alpha, \beta \leq 1$. We prove that the solution, starting from an initial datum of size ε, remains smaller than ε for a time scale of order $(\varepsilon^{-1}\alpha^{-1}\beta)^2$, whereas the natural time is of order $\varepsilon^{-1}\alpha^{-1}\beta$.

Keywords. Boussinesq system, Sobolev bounds, normal form

MS Codes. 35B60, 35Q53, 76B03, 76B15

Introduction

The two-way propagation of small amplitude, long wavelength, gravity waves in shallow water, described by its surface η and its velocity u, was first derived by Boussinesq [4, 2, 3, 8] as a system of the form

$$\begin{aligned} \eta_t + u_x + \alpha(\eta u)_x &= 0 \\ u_t + \eta_x + \alpha u u_x - \beta u_{xxt} &= 0, \end{aligned}$$

where α denotes the quotient between the characteristic waves amplitude and the depth of the water, β is the square of the quotient between this depth and the wavelength.
In this paper, we consider the following regularized Boussinesq system, proposed in [5] as a model of interactions between internal solitary waves,

$$\eta_t + u_x + \beta(-\eta_{xxt} + u_{xxx}) + \alpha((\eta u)_x + \eta\eta_x + uu_x) = 0 \quad (0.1)$$
$$u_t + \eta_x + \beta(\eta_{xxx} - u_{xxt}) + \alpha((\eta u)_x + \eta\eta_x + uu_x) = 0. \quad (0.2)$$

As a consequence of the local in time well-posedness of the Boussinesq system (0.1)-(0.2) with (η_0, u_0) as initial datum, for $0 < \alpha, \beta \leq 1$ and $s > 1/2$, there exist two constants $C_1 > 0$ and $C_2 > 0$ such that if $||\eta_0||_s + ||u_0||_s \leq \varepsilon$, then for $|t| \leq C_2\,\varepsilon^{-1}\alpha^{-1}\beta$, the solution (η, u) satisfies

$$||\eta(t)||_s + ||u(t)||_s \leq C_1\varepsilon. \quad (0.3)$$

Our goal is to prove (0.3) on a longer time scale for ε sufficiently small. We can notice that the H^1-norm is preserved by the flow, i.e. if the solution exists, we have for all $t \in \mathbb{R}$

$$||\eta(t)||_1 + ||u(t)||_1 \leq (||\eta_0||_1 + ||u_0||_1)/\beta,$$

(with equality if $\beta = 1$) and the bound (0.3) with $s = 1$ is true for all time. If $s > 3/2$, our result reads as follows.

Theorem 0.1
Let $0 < \alpha, \beta \leq 1$ and $s > 3/2$. There exist $0 < \varepsilon_0 < \alpha^{-1}\beta^2$, $C_1 > 0$ and $C_2 > 0$ such that if $(\eta_0, u_0) \in H_0^s(\mathbb{T}) \times H_0^s(\mathbb{T})$ with $||\eta_0||_s + ||u_0||_s \leq \varepsilon$, for $\varepsilon \in]0, \varepsilon_0[$, then the unique solution (η, u) of the Boussinesq system (0.1)-(0.2) with (η_0, u_0) as initial datum, satisfies for $|t| \leq C_2(\varepsilon^{-1}\alpha^{-1}\beta)^2$,

$$||\eta(t)||_s + ||u(t)||_s \leq C_1\varepsilon. \tag{0.4}$$

The principle of the proof bases on the Poincaré's theory of normal forms [1]. It consists in finding a map Λ, such that $(\mu, v) = \Lambda(\eta, u)$ satisfies

$$(\mu, v)_t + L(\mu, v) = F(\mu, v)$$

where L is the linear operator given by the free evolution of the initial system and F a multilinear operator of order strictly higher than 2 to improve the quadratic nature of the non-linearity of the Boussinesq system [6, 7].

Remark 0.2

- *Let us notice that, when $\alpha^{-1}\beta^2 \geq 1$, the constraint of smallness on ε_0 does not depend on α and β. On the other hand, ε_0 chosen in the such way $\varepsilon < \alpha^{-1}\beta^2$, with $0 < \varepsilon < \varepsilon_0$, implies that the time $(\varepsilon^{-1}\alpha^{-1}\beta)^2$ is higher than the one found from the local existence, namely $\varepsilon^{-1}\alpha^{-1}\beta$.*
- *Fixing $\alpha = 1$ and $\eta = u$, when β tends to zero, the persistence time of the bound (0.4) given by the theorem 0.1 also tends to zero. That is not surprising, the formal limit, when β tends to zero, of the Boussinesq system being the Burgers equation*

$$u_t + u_x + uu_x = 0.$$

  However the existence time of the solution of the Cauchy problem with an initial datum of size ε, is of order ε^{-1} and this time is maximum.
- *The case $1/2 < s \leq 3/2$ remains an open problem.*

We use the following notations : $\mathbb{T} = \mathbb{R} / (2\pi\mathbb{Z})$ is the one-dimensional torus, $\mathbb{Z}^*$ the set of nonzero integers. For $s \in \mathbb{R}$, we define $H_0^s(\mathbb{T})$ the space of zero x-mean value functions equipped with the norm

$$||u(t)||_s = \left(\sum_{k \in \mathbb{Z}^*} |k|^{2s} |\hat{u}(k)|^2 \right)^{1/2},$$

where $\hat{u}$ denotes the Fourier transform defined by

$$\hat{u}(k) = \int_{\mathbb{T}} \mathrm{e}^{-ikx} u(x) d\mu(x).$$

The measure $d\mu(x)$ is chosen proportional to the Lebesgue one on $\mathbb{T}$ and normalized such that

$$u(x) = \sum_{k \in \mathbb{Z}^*} \mathrm{e}^{ikx} \hat{u}(k).$$

The paper is organised as follows. In Section 1, the local in time well-posedness is established. The Section 2 deals with the proof of the main theorem.

1 Summary of existence theory

We consider the Cauchy problem, for $t \in \mathbb{R}$, and $x \in \mathbb{T}$,

$$\eta_t + u_x + \beta(-\eta_{xxt} + u_{xxx}) + \alpha((\eta u)_x + \eta\eta_x + uu_x) = 0 \tag{1.1}$$

$$u_t + \eta_x + \beta(\eta_{xxx} - u_{xxt}) + \alpha((\eta u)_x + \eta\eta_x + uu_x) = 0 \tag{1.2}$$

$$u(x,0) = u_0(x), \eta(x,0) = \eta_0(x). \tag{1.3}$$

The existence and the uniqueness of local in time solution are proved.

Theorem 1.1
Let $0 < \alpha, \beta \leq 1$, $s > 1/2$ and $(\eta_0, u_0) \in H_0^s(\mathbb{T}) \times H_0^s(\mathbb{T})$. There exists a constant $C_0 > 0$, depending only on s, such that for

$$T = \frac{C_0}{||\eta_0||_s + ||u_0||_s} \frac{\beta}{\alpha},$$

there exists a unique solution $(\eta, u) \in \mathcal{C}([-T,T]; H_0^s(\mathbb{T})) \times \mathcal{C}([-T,T]; H_0^s(\mathbb{T}))$ of the Cauchy problem (1.1)-(1.2)-(1.3).
Moreover, for all $M > 0$ with $||\eta_0||_s + ||u_0||_s \leq M$ and $||\mu_0||_s + ||v_0||_s \leq M$, there exists $C_1 > 0$ such that solutions (η, u) and (μ, v), of initial data (η_0, u_0) and (μ_0, v_0) respectively, satisfy for $t \in [-T, T]$, with $T = C_0\alpha^{-1}\beta/M$,

$$||\eta(t) - \mu(t)||_s + ||u(t) - v(t)||_s \leq C_1(||\eta_0 - \mu_0||_s + ||u_0 - v_0||_s).$$

Proof. Let $T > 0$. The Duhamel's formula implies that (η, u) is the solution of the Cauchy problem (1.1)-(1.2)-(1.3) if and only if (η, u) is the solution of the following equation, for $t \in [0, T]$,

$$(\eta, u)(t) = \Phi(\eta, u)(t) := S_t(\eta_0, u_0) - \frac{\alpha}{2}\int_0^t S_{t-\tau}\left(\frac{\partial_x}{1-\beta\partial_x^2}\left((\eta+u)^2, (\eta+u)^2\right)\right)(\tau)\, d\tau, \tag{1.4}$$

with

$$\begin{aligned} S_t(\eta, u) \quad := \quad \sum_{k\in\mathbb{Z}^*} \mathrm{e}^{ikx} \Bigg(& \cos(tk\frac{1-\beta k^2}{1+\beta k^2})\hat{\eta}(k) - i\sin(tk\frac{1-\beta k^2}{1+\beta k^2})\hat{u}(k), \\ & \cos(tk\frac{1-\beta k^2}{1+\beta k^2})\hat{u}(k) - i\sin(tk\frac{1-\beta k^2}{1+\beta k^2})\hat{\eta}(k)\Bigg). \end{aligned} \tag{1.5}$$

We aim at applying the fixed point theorem. We deduce from the Duhamel's formula, for $t \in [0, T]$,

$$||\Phi(\eta, u)(t)||_s \leq C(||\eta_0||_s + ||u_0||_s) + C\alpha\int_0^t \left|\left|\frac{\partial_x}{1-\beta\partial_x^2}(u+\eta)^2\right|\right|_s(\tau)\, d\tau.$$

For $0 < \beta \leq 1$, the definition of the Sobolev norm provides

$$\begin{aligned} \left|\left|\frac{\partial_x}{1-\beta\partial_x^2}(u+\eta)^2\right|\right|_s &= \left(\sum_{k\in\mathbb{Z}^*}|k|^{2s}\left|\frac{ik}{1+\beta k^2}\widehat{(u+\eta)}^2(k)\right|^2\right)^{1/2} \leq \frac{1}{\beta}||(u+\eta)^2||_s \\ &\leq \frac{C_s}{\beta}||u+\eta||_\infty||u+\eta||_s. \end{aligned}$$

Since $s > 1/2$, the Sobolev embedding implies that there exists a constant $C_s > 0$, depending only on s, such that

$$||\Phi(\eta, u)(t)||_s \leq C(||\eta_0||_s + ||u_0||_s) + \frac{\alpha}{\beta}C_s T\left(\sup_{t\in[0,T]}(||\eta(t)||_s + ||u(t)||_s)\right)^2. \tag{1.6}$$

Then there exists $C_0 > 0$ such that for $T = C_0\alpha^{-1}\beta/(||\eta_0||_s + ||u_0||_s)$, the closed ball

$$\overline{B}_T := \left\{(\eta,u) \in \mathcal{C}([0,T];H_0^s(\mathbb{T})) \times \mathcal{C}([0,T];H_0^s(\mathbb{T}))\ ;\ \sup_{t\in[0,T]}(||\eta(t)||_s + ||u(t)||_s) \leq 2C(||\eta_0||_s + ||u_0||_s)\right\}$$

satisfies $\Phi(\overline{B}_T) \subseteq \overline{B}_T$. Indeed, let $(\eta,u) \in \overline{B}_T$, the inequality (1.6) becomes

$$||\Phi(\eta,u)(t)||_s \leq C(||\eta_0||_s + ||u_0||_s)(1 + 4CC_sC_0),$$

and $C(||\eta_0||_s + ||u_0||_s)(1 + 4CC_sC_0) \leq 2C(||\eta_0||_s + ||u_0||_s)$ if $C_0 \leq 1/(4CC_s)$.
Let (η,u) and (μ,v) be in $\overline{B}_T$. The Duhamel's formula (1.4) provides, for $t \in [0,T]$,

$$||\Phi(\eta,u)(t) - \Phi(\mu,v)(t)||_s \quad\leq\quad C\alpha\int_0^t \left|\left|\frac{\partial_x}{1-\beta\partial_x^2}(\eta u - \mu v)\right|\right|_s + \left|\left|\frac{\partial_x}{1-\beta\partial_x^2}(u^2 - v^2 + \eta^2 - \mu^2)\right|\right|_s(\tau)\,d\tau.$$

Noticing that $\eta u - \mu v = 1/2((\eta+\mu)(u-v) + (\eta-\mu)(u+v))$, $u^2 - v^2 = (u+v)(u-v)$ and applying the Sobolev embedding, one gets

$$\begin{aligned}||\Phi(\eta,u)(t) - \Phi(\mu,v)(t)||_s &\leq \frac{12C^2\alpha}{\beta}\,C_sT(||\eta_0||_s + ||u_0||_s)\left(\sup_{t\in[-T,T]}(||\eta-\mu||_s(t) + ||u-v||_s(t))\right)\\ &= 12C^2C_sC_0\left(\sup_{t\in[-T,T]}(||\eta-\mu||_s(t) + ||u-v||_s(t))\right).\end{aligned}$$

For $C_0 < 1/(12C^2C_s)$, the map Φ is a contraction on $\overline{B}_T$. Finally, according to the fixed point theorem, there exists a unique solution (η,u) of $\Phi(\eta,u)(t) = (\eta,u)(t)$ in $\overline{B}_T$.
It remains to prove the continuity with the initial datum. Let (η,u) and (μ,v) be solutions of the Cauchy problem (1.1)-(1.2)-(1.3) with initial datum (η_0,u_0) and (μ_0,v_0) respectively, such that $||\eta_0||_s+||u_0||_s \leq M$ and $||\mu_0||_s + ||v_0||_s \leq M$. The Duhamel's formula (1.4) gives for $t \in [0,T]$, with $T = C_0\alpha^{-1}\beta/M$, with $C_0 \ll 1$,

$$\begin{aligned}||(\eta,u)(t) - (\mu,v)(t)||_s &\leq C(||\eta_0-\mu_0||_s + ||u_0-v_0||_s)\\ &\quad + C\alpha\int_0^t \left|\left|\frac{\partial_x}{1-\beta\partial_x^2}(\eta u-\mu v)\right|\right|_s + \left|\left|\frac{\partial_x}{1-\beta\partial_x^2}(u^2-v^2+\eta^2-\mu^2)\right|\right|_s(\tau)\,d\tau\\ &\leq C(||\eta_0-\mu_0||_s + ||u_0-v_0||_s) + \frac{1}{2}(\sup_{t\in[0,T]}||\eta-\mu||_s(t) + \sup_{t\in[0,T]}||u-v||_s(t)),\end{aligned}$$

thus

$$\sup_{t\in[0,T]}(||\eta-\mu||_s + ||u-v||_s) \leq 2C(||\eta_0-\mu_0||_s + ||u_0-v_0||_s).$$

□

Remark 1.2
The time given in the preceding theorem could be higher, especially in the case $0 < \beta \leq \alpha \leq 1$, but we have to impose more regular initial data. More precisely, let us suppose $s > 3/2$ and $(\eta_0,u_0) \in H_0^s(\mathbb{T}) \times H_0^s(\mathbb{T})$. Then, there exists a constant $C_0 > 0$, depending only on s, such that for

$$T = \frac{C_0}{||\eta_0||_s + ||u_0||_s}\frac{1}{\alpha},$$

there exists a unique solution $(\eta,u) \in \mathcal{C}([-T,T];H_0^s(\mathbb{T}))\times\mathcal{C}([-T,T];H_0^s(\mathbb{T}))$ of the Cauchy problem (1.1)-(1.2)-(1.3). On the other hand, the choice of ε_0 in the theorem 0.1 also implies that $(\varepsilon^{-1}\alpha^{-1}\beta)^2 \geq \varepsilon^{-1}\alpha^{-1}$.

2 Long time bounds

A consequence of the local well-posedness is that for $0 < \alpha, \beta \leq 1$, $s > 1/2$, $\varepsilon > 0$ and $(\eta_0, u_0) \in H_0^s(\mathbb{T}) \times H_0^s(\mathbb{T})$ with $||\eta_0||_s + ||u_0||_s \leq \varepsilon$, there exist $C_0 > 0$ and $C_1 > 0$ such that the solution (η, u) of the Cauchy problem (1.1)-(1.2)-(1.3) satisfies for $|t| \leq C_0\, \varepsilon^{-1}\alpha^{-1}\beta$,

$$||\eta(t)||_s + ||u(t)||_s \leq C_1 \varepsilon.$$

We wonder if the solution exists and remains small longer, the normal form is used [6, 7].
To simplify the writings, we define L by the Fourier symbol $\sigma(k) := ik(1-\beta k^2)/(1+\beta k^2)$.
Let $D := \{(k, k_1) \in \mathbb{Z}^2\ ;\ k \neq 0, k_1 \neq 0, k \neq k_1\}$, we define the operator Λ by,

$$\Lambda(\eta, u) := (\eta + B(\eta + u, \eta + u), u + B(\eta + u, \eta + u)),$$

with the bilinear operator

$$B(u, v) := -\frac{\alpha}{2} \sum_D \mathrm{e}^{ikx} \frac{ik}{1 + \beta k^2} \frac{\hat{u}(k_1)\, \hat{v}(k - k_1)}{\sigma(k_1) + \sigma(k - k_1) - \sigma(k)},$$

and for $\delta > 0$, $\mathcal{V}_\delta := \{(\eta, u) \in H_0^s(\mathbb{T}) \times H_0^s(\mathbb{T}); ||\eta||_s + ||u||_s < \delta\}$.
Introducing Λ is used to define $(\mu, v) = \Lambda(\eta, u)$ so that (μ, v) is solution of the equation

$$\mu_t + L(v) = F(\eta, u),\ v_t + L(\mu) = F(\eta, u),$$

with F trilinear whereas (η, u) is solution of a quadratic Boussinesq system. Thus the well-posedness of (μ, v) and the definition of B are used to estimate (η, u) with respect to (μ, v) and to extend its well-posedness.

Proposition 2.1
Let $s > 3/2$. Then there exist $0 < \delta' < \alpha^{-1}\beta^2$, $\delta > 0$, and $C > 0$ such that for all $(\mu, v) \in \mathcal{V}_\delta$, there exists a unique $(\eta, u) \in \mathcal{V}_{\delta'}$ such that $\Lambda(\eta, u) = (\mu, v)$. Moreover

$$||\eta||_s + ||u||_s \leq C(||\mu||_s + ||v||_s).$$

Proof. We first prove some useful lemmas.

Lemma 2.2
The Fourier symbol σ satisfies for all k and k_1 in $\mathbb{Z}$

$$|\sigma(k_1) + \sigma(k - k_1) - \sigma(k)| = \frac{|kk_1(k - k_1)|\, |6\beta + 2\beta^2(k^2 - kk_1 + k_1^2)|}{(1 + \beta k^2)(1 + \beta k_1^2)(1 + \beta(k - k_1)^2)}.$$

Lemma 2.3
Let $s > 3/2$. There exists a constant $C > 0$ such that for all u and v in $H_0^s(\mathbb{T})$

$$||B(u, v)||_s \quad \leq \quad C\frac{\alpha}{\beta^2} ||u||_s ||v||_s.$$

Proof. By duality, to prove the lemma is equivalent to prove for all $w \in \mathcal{C}^\infty(\mathbb{T})$

$$\left| \sum_D \widehat{B(u, v)}(k) \hat{w}(k) \right| \leq C \frac{\alpha}{\beta^2} \left(||u||_s ||v||_s\right) ||w||_{-s}. \tag{2.1}$$

Indeed, we have

$$||B(u,v)||_s^2 \;=\; \sum_{k\in\mathbb{Z}^*} |k|^{2s}|\widehat{B(u,v)}(k)|^2 \;=\; \sum_{k\in\mathbb{Z}^*} \widehat{B(u,v)}(k)\left(|k|^{2s}\widehat{B(u,v)}(k)\right).$$

We set $\hat{w}(k) = |k|^{2s}\widehat{B(u,v)}(k)$ and we write

$$||B(u,v)||_s^2 = \sum_{k\in\mathbb{Z}^*} \widehat{B(u,v)}(k)\,\hat{w}(k),$$

and according to the inequality (2.1)

$$||B(u,v)||_s^2 \le C\frac{\alpha}{\beta^2}\left(||u||_s||v||_s\right)||w||_{-s}.$$

However

$$\begin{aligned} ||w||_{-s} &= \left(\sum_{k\in\mathbb{Z}^*} |k|^{-2s}|k|^{4s}|\widehat{B(u,v)}(k)|^2\right)^{1/2} \\ &= \left(\sum_{k\in\mathbb{Z}^*} |k|^{2s}|\widehat{B(u,v)}(k)|^2\right)^{1/2} = ||B(u,v)||_s. \end{aligned}$$

We define

$$\hat{u}_1(k) = |k|^s\hat{u}(k)\ ,\ \hat{v}_1(k) = |k|^s\hat{v}(k)\ \text{ and }\ \hat{w}_1(k) = |k|^{-s}\hat{w}(k).$$

In particular, it implies

$$||u_1||_{L^2} = ||u||_s\ ,\ ||v_1||_{L^2} = ||v||_s\ \text{ and }\ ||w_1||_{L^2} = ||w||_{-s}.$$

We then find

$$\begin{aligned} \widehat{B(u,v)}(k) &= -\frac{\alpha}{2}\frac{ik}{1+\beta k^2}\sum_{\substack{k_1\in\mathbb{Z}^*\\ k_1\neq k}} \frac{\hat{u}(k_1)\hat{v}(k-k_1)}{\sigma(k_1)+\sigma(k-k_1)-\sigma(k)} \\ &= -\frac{\alpha}{2}\frac{ik}{1+\beta k^2}\sum_{\substack{k_1\in\mathbb{Z}^*\\ k_1\neq k}} \frac{|k|^s\hat{u}_1(k_1)\hat{v}_1(k-k_1)}{|k_1|^s|k-k_1|^s(\sigma(k_1)+\sigma(k-k_1)-\sigma(k))}. \end{aligned}$$

Finally, it is enough to prove

$$\left|\frac{\alpha}{2}\sum_{D}\frac{ik}{1+\beta k^2}\frac{|k|^s\hat{u}_1(k_1)\hat{v}_1(k-k_1)\hat{w}_1(k)}{|k_1|^s|k-k_1|^s(\sigma(k_1)+\sigma(k-k_1)-\sigma(k))}\right| \quad\le\quad C\frac{\alpha}{\beta^2}\left(||u_1||_{L^2}||v_1||_{L^2}\right)||w||_{L^2}.$$

Lemma 2.4
We have for k and k_1 in D

$$\left|\frac{ik}{1+\beta k^2}\frac{1}{\sigma(k_1)+\sigma(k-k_1)-\sigma(k)}\right| \le \frac{2}{\beta^2}\left|\frac{k-k_1}{k}\right|.$$

Proof. According to the lemma 2.2, we have for k and k_1 in D,

$$\left|\frac{ik}{1+\beta k^2}\frac{1}{\sigma(k_1)+\sigma(k-k_1)-\sigma(k)}\right| \le \frac{(1+\beta k_1^2)(1+\beta(k-k_1)^2)}{2\beta^2|k_1(k-k_1)(k^2-kk_1+k_1^2)|}.$$

Since $0 < \beta \leq 1$, $1 + \beta k_1^2 \leq 2k_1^2$ and $|k^2 - kk_1 + k_1^2| \geq |kk_1|$, we have

$$\left| \frac{ik}{1+\beta k^2} \frac{1}{\sigma(k_1) + \sigma(k-k_1) - \sigma(k)} \right| \leq \frac{2}{\beta^2} \left| \frac{k-k_1}{k} \right| .$$

$\square$

For $s \geq 1$, the triangle inequality implies

$$\frac{|k|^{s-1}}{|k_1|^s |k-k_1|^{s-1}} \leq C \left(\frac{1}{|k_1|^s} + \frac{1}{|k_1||k-k_1|^{s-1}} \right) \leq C \left(\frac{1}{|k_1|^s} + \frac{1}{|k-k_1|^{s-1}} \right) .$$

According to the preceding lemma, it remains to bound

$$\frac{\alpha}{\beta^2} \sum_D \frac{|\hat{u}_1(k_1)|\,|\hat{v}_1(k-k_1)||\hat{w}_1(k)|}{|k_1|^s} \; + \; \frac{\alpha}{\beta^2} \sum_D \frac{|\hat{u}_1(k_1)|\,|\hat{v}_1(k-k_1)||\hat{w}_1(k)|}{|k-k_1|^{s-1}} =: \frac{\alpha}{\beta^2} (\mathrm{I} + \mathrm{II}).$$

The Cauchy-Schwarz inequality in k gives for the first term of the preceding sum

$$\mathrm{I} \;\leq\; \left(\sum_{k \in \mathbb{Z}^*} \left(\sum_{\substack{k_1 \,\in\, \mathbb{Z}^* \\ k_1 \neq k}} \frac{|\hat{u}_1(k_1)|\,|\hat{v}_1(k-k_1)|}{|k_1|^s} \right)^2 \right)^{1/2} \left(\sum_{k \in \mathbb{Z}^*} |\hat{w}_1(k)|^2 \right)^{1/2} ,$$

then the Cauchy-Schwarz inequality in k_1 is applied again,

$$\mathrm{I} \leq \left(\sum_{\substack{k_1 \,\in\, \mathbb{Z}^* \\ k_1 \neq k}} \frac{1}{|k_1|^{2s}} \right)^{1/2} \left(\sum_{k \in \mathbb{Z}^*} \sum_{\substack{k_1 \,\in\, \mathbb{Z}^* \\ k_1 \neq k}} |\hat{u}_1(k_1)|^2 |\hat{v}_1(k-k_1)|^2 \right)^{1/2} \left(\sum_{k \in \mathbb{Z}^*} |\hat{w}_1(k)|^2 \right)^{1/2} .$$

Since $s > 1/2$, there exists a constant $C > 0$ such that

$$\mathrm{I} \;\leq\; C \left(||u_1||_{L^2} ||v_1||_{L^2} \right) ||w_1||_{L^2} .$$

By symmetry, a similar inequality for II is verified if $2(s-1) > 1$, i.e. $s > 3/2$. $\square$

The differential of this operator is given by, for all $(\varphi, \psi) \in \mathcal{C}^\infty(\mathbb{T}) \times \mathcal{C}^\infty(\mathbb{T})$

$$\langle\, d\Lambda(\eta, u)\,,\, (\varphi, \psi) \rangle = \begin{pmatrix} \varphi + 2B(\eta + u, \varphi) & 2B(\eta + u, \psi) \\ 2B(\eta + u, \varphi) & \psi + 2B(\eta + u, \psi) \end{pmatrix} ,$$

the preceding lemma implies that $d\Lambda$ is continuous on $H_0^s(\mathbb{T}) \times H_0^s(\mathbb{T})$. Since $d\Lambda(0,0)$ is the identity, the inverse function theorem is applied to give the following lemma. $\square$

We aim at setting which equation is satisfied by Λ.

Proposition 2.5
Let $s > 3/2$. There exists a trilinear operator

$$F : H_0^s(\mathbb{T}) \times H_0^s(\mathbb{T}) \times H_0^s(\mathbb{T}) \longrightarrow H_0^s(\mathbb{T})$$

satisfying that there exists a constant $C>0$ such that for all $(u_1,u_2,u_3)\in H_0^s(\mathbb{T})\times H_0^s(\mathbb{T})\times H_0^s(\mathbb{T})$

$$||F(u_1,u_2,u_3)||_s \le C\frac{\alpha^2}{\beta^2}||u_1||_s||u_2||_s||u_3||_s,$$

and, if $(\eta,u)\in\mathcal{C}([-T,T];H_0^s(\mathbb{T}))\times\mathcal{C}([-T,T];H_0^s(\mathbb{T}))$ is the solution of the system (1.1)-(1.2), then (μ,v) defined by, for $t\in[-T,T]$

$$\begin{aligned}\mu(t) &:= \eta(t)+B(\eta(t)+u(t),\eta(t)+u(t))\\ v(t) &:= u(t)+B(\eta(t)+u(t),\eta(t)+u(t)),\end{aligned}$$

is solution of

$$\begin{aligned}\mu_t+L(v) &= F(\eta+u,\eta+u,\eta+u)\\ v_t+L(\mu) &= F(\eta+u,\eta+u,\eta+u).\end{aligned}$$

Proof. We notice firstly that $(\mu,v)\in\mathcal{C}([-T,T];H_0^s(\mathbb{T}))\times\mathcal{C}([-T,T];H_0^s(\mathbb{T}))$ according to the lemma 2.3. We write

$$\begin{aligned}\mu_t+L(v) &= \eta_t+\partial_t B(\eta+u,\eta+u)+L(u)+L(B(\eta+u,\eta+u))\\ &= -\frac{\alpha}{2}\frac{\partial_x}{1-\beta\partial_x^2}(\eta+u)^2+\partial_t B(\eta+u,\eta+u)+L(B(\eta+u,\eta+u)).\end{aligned}$$

On one hand, we have

$$\begin{aligned}\partial_t B(\eta+u,\eta+u) &= B(\eta_t+u_t,\eta+u)+B(\eta+u,\eta_t+u_t)\ =\ 2B(\eta_t+u_t,\eta+u)\\ &= -\alpha\sum_D \mathrm{e}^{ikx}\frac{ik}{1+\beta k^2}\frac{(\hat\eta_t(k_1)+\hat u_t(k_1))(\hat\eta(k-k_1)+\hat u(k-k_1))}{\sigma(k_1)+\sigma(k-k_1)-\sigma(k)}.\end{aligned}$$

Since (η,u) is solution of the system (1.1)-(1.3), we obtain by symmetry

$$\begin{aligned}\partial_t B(\eta+u,\eta+u) &= -\alpha^2\sum_D \mathrm{e}^{ikx}\frac{kk_1}{(1+\beta k^2)(1+\beta k_1^2)}\frac{\widehat{(\eta+u)^2}(k_1)(\hat\eta(k-k_1)+\hat u(k-k_1))}{\sigma(k_1)+\sigma(k-k_1)-\sigma(k)}\\ &\quad+\frac{\alpha}{2}\sum_D \mathrm{e}^{ikx}\frac{ik}{1+\beta k^2}\frac{(\sigma(k_1)+\sigma(k-k_1))(\hat\eta(k_1)+\hat u(k_1))(\hat\eta(k-k_1)+\hat u(k-k_1))}{\sigma(k_1)+\sigma(k-k_1)-\sigma(k)}.\end{aligned}$$

On the other hand, we have

$$L(B(\eta+u,\eta+u)) = -\frac{\alpha}{2}\sum_D \mathrm{e}^{ikx}\frac{ik}{1+\beta k^2}\frac{\sigma(k)(\hat\eta(k_1)+\hat u(k_1))(\hat\eta(k-k_1)+\hat u(k-k_1))}{\sigma(k_1)+\sigma(k-k_1)-\sigma(k)}.$$

The last term gives

$$-\frac{\alpha}{2}\frac{\partial_x}{1-\beta\partial_x^2}(\eta+u)^2=-\frac{\alpha}{2}\sum_D \mathrm{e}^{ikx}\frac{ik}{1+\beta k^2}(\hat\eta(k_1)+\hat u(k_1))(\hat\eta(k-k_1)+\hat u(k-k_1)).$$

Finally, it follows

$$\mu_t+L(v) = -\alpha^2\sum_D \mathrm{e}^{ikx}\frac{kk_1}{(1+\beta k^2)(1+\beta k_1^2)}\frac{\widehat{(\eta+u)^2}(k_1)(\hat\eta(k-k_1)+\hat u(k-k_1))}{\sigma(k_1)+\sigma(k-k_1)-\sigma(k)}. \tag{2.2}$$

We denote $D_1 := \{(k,k_1,k_2)\in\mathbb{Z}^3; k\neq 0, k_1\neq 0, k_2\neq 0, k\neq k_1, k_1\neq k_2\}$ and we deduce from (2.2) that F is defined by

$$F(u_1,u_2,u_3) := -\alpha^2 \sum_{D_1} \mathrm{e}^{ikx} \frac{kk_1}{(1+\beta k^2)(1+\beta k_1^2)} \frac{\hat{u}_1(k_2)\hat{u}_2(k_1-k_2)\hat{u}_3(k-k_1)}{\sigma(k_1)+\sigma(k-k_1)-\sigma(k)}.$$

Lemma 2.6
We have for k, k_1 and k_2 in D_1

$$\left|\frac{kk_1}{(1+\beta k^2)(1+\beta k_1^2)} \frac{1}{\sigma(k_1)+\sigma(k-k_1)-\sigma(k)}\right| \leq \frac{1}{\beta^2}\left|\frac{k-k_1}{k}\right|.$$

Proof. According to the lemma 2.2, we have for k, k_1 and k_2 in D_1,

$$\left|\frac{kk_1}{(1+\beta k^2)(1+\beta k_1^2)} \frac{1}{\sigma(k_1)+\sigma(k-k_1)-\sigma(k)}\right| \leq \frac{1+\beta(k-k_1)^2}{2\beta^2|kk_1(k-k_1)|},$$

and since $0<\beta\leq 1$ and $|k_1|\geq 1$, we have

$$\left|\frac{kk_1}{(1+\beta k^2)(1+\beta k_1^2)} \frac{1}{\sigma(k_1)+\sigma(k-k_1)-\sigma(k)}\right| \leq \frac{1}{\beta^2}\left|\frac{k-k_1}{k}\right|.$$

$\square$

In the same way as lemma 2.3, by duality, it is enough to bound

$$\mathrm{I} := \sum_{D_1} \frac{|k|^{s-1}|\hat{u}_1(k_2)|\,|\hat{u}_2(k_1-k_2)|\,|\hat{u}_3(k-k_1)|\,|\hat{u}_4(k)|}{|k_2|^s|k_1-k_2|^s|k-k_1|^{s-1}}.$$

The Cauchy-Schwarz inequality is first applied in k to give

$$\mathrm{I} \leq \left(\sum_{k\in\mathbb{Z}^*}\left(\sum_{\substack{(k_1,k_2)\in(\mathbb{Z}^*)^2\\ k_1\neq k,\, k_2\neq k_1}} \frac{|k|^{s-1}|\hat{u}_1(k_2)|\,|\hat{u}_2(k_1-k_2)|\,|\hat{u}_3(k-k_1)|}{|k_2|^s|k_1-k_2|^s|k-k_1|^{s-1}}\right)^2\right)^{1/2} ||u_4||_{L^2},$$

and then in (k_2,k_1),

$$\mathrm{I} \leq \left(\sum_{k\in\mathbb{Z}^*}\sum_{\substack{(k_1,k_2)\in(\mathbb{Z}^*)^2\\ k_1\neq k,\, k_2\neq k_1}} |\hat{u}_1(k_2)|^2\,|\hat{u}_2(k_1-k_2)|^2\,|\hat{u}_3(k-k_1)|^2 \times \sum_{\substack{(k_1,k_2)\in(\mathbb{Z}^*)^2\\ k_1\neq k,\, k_2\neq k_1}} \frac{|k|^{2(s-1)}}{|k_2|^{2s}|k_1-k_2|^{2s}|k-k_1|^{2(s-1)}}\right)^{1/2} ||u_4||_{L^2},$$

However, since $s>3/2\geq 1$, the triangle inequality implies

$$\frac{|k|^{2(s-1)}}{|k_2|^{2s}|k_1-k_2|^{2s}|k-k_1|^{2(s-1)}} \leq C\left(\frac{1}{|k_2|^{2s}|k_1-k_2|^{2s}} + \frac{1}{|k_2|^2|k_2-k_1|^{2s}|k-k_1|^{2(s-1)}} + \frac{1}{|k_2|^{2s}|k_1-k_2|^2|k-k_1|^{2(s-1)}}\right),$$

thus

$$\sup_{k\in\mathbb{Z}^*} \sum_{\substack{(k_1,k_2)\in(\mathbb{Z}^*)^2\\ k_1\neq k,\, k_2\neq k_1}} \frac{|k|^{2(s-1)}}{|k_2|^{2s}|k_1-k_2|^{2s}|k-k_1|^{2(s-1)}} < +\infty.$$

$\square$

The main result of this paper is now proved.

Proof of the theorem 0.1. We suppose $t>0$, the proof is similar for negative time. Let δ and δ' be the positive constants involved in the lemma 2.1.

Thanks to the local well-posedness theorem 1.1, there exists $\varepsilon_0>0$ such that if $(\eta_0,u_0)\in\mathcal{V}_\varepsilon$, for $\varepsilon\in]0,\varepsilon_0[$, then for $t\leq C_2\,\varepsilon^{-1}\alpha^{-1}\beta$,

$$(\eta(t)+B(\eta+u,\eta+u)(t),u(t)+B(\eta+u,\eta+u)(t))\in\mathcal{V}_\delta,\ \text{and}\ (\eta(t),u(t))\in\mathcal{V}_{\delta'}.$$

Thus $(\eta,u)(t)=\Lambda^{-1}(\mu,v)(t)$, for $t\leq C_2\,\varepsilon^{-1}\alpha^{-1}\beta=:T$.

The Duhamel's formula gives for $t\in[0,T]$

$$\begin{aligned}(\mu,v)(t) \quad &= \quad S_t\left(\eta_0+B(\eta_0+u_0,\eta_0+u_0),u_0+B(\eta_0+u_0,\eta_0+u_0)\right)\\ &\quad +\int_0^t S_{t-\tau}\left(F(\eta+u,\eta+u,\eta+u),F(\eta+u,\eta+u,\eta+u)\right)(\tau)\,d\tau,\end{aligned} \tag{2.3}$$

where S_t is defined in (1.5). According to the lemma 2.3, there exists a constant $C_1>0$ such that we have

$$\begin{aligned}||S_t\left(\eta_0+B(\eta_0+u_0,\eta_0+u_0),u_0+B(\eta_0+u_0,\eta_0+u_0)\right)||_s \quad &\leq \quad ||\eta_0||_s+||u_0||_s+C\frac{\alpha}{\beta^2}(||\eta_0||_s+||u_0||_s)^2\\ &\leq \quad C_1\varepsilon\left(2+4\varepsilon\frac{\alpha}{\beta^2}\right).\end{aligned}$$

Even if we take $\varepsilon_0>0$ smaller, we have

$$||S_t\left(\eta_0+B(\eta_0+u_0,\eta_0+u_0),u_0+B(\eta_0+u_0,\eta_0+u_0)\right)||_s\leq 3\,C_1\varepsilon. \tag{2.4}$$

Lemmas 2.1 and 2.5 imply that there exists a constant $C_3>0$ such that, if $(\mu(t),v(t))\in\mathcal{V}_\delta$ for $t\in[0,T]$, then

$$\begin{aligned}&\left|\left|\int_0^t S_{t-\tau}\left(F(\eta+u,\eta+u,\eta+u),F(\eta+u,\eta+u,\eta+u)\right)(\tau)\,d\tau\right|\right|_{L^\infty([0,T];H^s_0(\mathbb{T}))} \quad \leq\\ &\qquad C_3\frac{\alpha^2}{\beta^2}\,T\,(||\mu||_{L^\infty([0,T];H^s_0(\mathbb{T}))}+||v||_{L^\infty([0,T];H^s_0(\mathbb{T}))})^3.\end{aligned} \tag{2.5}$$

If we need to take $\varepsilon_0>0$ smaller again, we impose $4C_1\varepsilon_0<\delta$. We set $C_0=1/(128C_1^2C_3)$ and $T_0=C_0\left(\varepsilon^{-1}\alpha^{-1}\beta\right)^2$. It then follows

$$||\mu||_{L^\infty([0,T_0];H^s_0(\mathbb{T}))}+||v||_{L^\infty([0,T_0];H^s_0(\mathbb{T}))}\leq 4\,C_1\varepsilon. \tag{2.6}$$

Let us suppose that the inequality (2.6) fails. Since

$$||\mu(0)||_s+||v(0)||_s=||\eta_0+B(\eta_0+u_0,\eta_0+u_0)||_s+||u_0+B(\eta_0+u_0,\eta_0+u_0)||_s\leq C_1\varepsilon\left(2+4\varepsilon\frac{\alpha}{\beta^2}\right)<4\,C_1\varepsilon,$$

by continuity with time, there exists $\tau \in [0, T_0]$ such that for $t \in [0, \tau]$

$$||\mu(t)||_s + ||v(t)||_s \leq 4\, C_1 \varepsilon \;\text{ and }\; ||\mu(\tau)||_s + ||v(\tau)||_s = 4\, C_1 \varepsilon.$$

Let C be the positive constant involved in the lemma 2.1, we also impose $4\, C_1 C \varepsilon_0 < \delta'$. We know that $(\eta(t), u(t)) \in \mathcal{V}_{\delta'}$ for $|t| \leq T$, and with this choice of ε, it follows that $(\eta(t), u(t)) \in \mathcal{V}_{\delta'}$ for $t \in [0, \tau]$. Indeed, if there exists $\tau_0 \in [0, \tau]$ such that for $t \in [0, \tau_0]$

$$||\eta(t)||_s + ||u(t)||_s < \delta' \;\text{ and }\; ||\eta(\tau_0)||_s + ||u(\tau_0)||_s = \delta',$$

then by continuity with time and according to the lemma 2.1, we have

$$\begin{aligned} \delta' = ||\eta(\tau_0)||_s + ||u(\tau_0)||_s &= \lim_{t \to \tau_0} ||\eta(t)||_s + ||u(t)||_s \leq \sup_{0 \leq t < \tau_0} ||\eta(t)||_s + ||u(t)||_s \\ &\leq C \sup_{0 \leq t < \tau_0} ||\mu(t)||_s + ||v(t)||_s \leq 4 C_1 C \varepsilon < \delta', \end{aligned}$$

which is impossible.
Finally, we find from the Duhamel's formula (2.3) and from inequalities (2.4) and (2.5)

$$||\mu||_{L^\infty([0,\tau];H_0^s(\mathbb{T}))} + ||v||_{L^\infty([0,\tau];H_0^s(\mathbb{T}))} \leq 3\, C_1 \varepsilon + C_3 \frac{\alpha^2}{\beta^2} \tau (||\mu||_{L^\infty([0,\tau];H_0^s(\mathbb{T}))} + ||v||_{L^\infty([0,\tau];H_0^s(\mathbb{T}))})^3,$$

or equivalently

$$4\, C_1 \varepsilon \leq 3\, C_1 \varepsilon + C_3 \frac{\alpha^2}{\beta^2} \tau \left(4\, C_1 \varepsilon\right)^3,$$

which gives

$$\tau \geq 2\, C_0 \left(\varepsilon^{-1} \frac{\beta}{\alpha}\right)^2 = 2T_0.$$

And this is a contradiction with $\tau \in [0, T_0]$. Then the inequality (2.6) is true and using the lemma 2.1 we have for $t \in [0, T_0]$

$$||\eta(t)||_s + ||u(t)||_s \leq C(||\mu(t)||_s + ||v(t)||_s) \leq 4\, C_1 C \varepsilon.$$

□

References

[1] V. I. Arnold: Geometric Methods in the Theory of Ordinary Differential Equations. Springer-Verlag, New York, (1983).

[2] J.L. Bona, M. Chen, J.-C. Saut: Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media. I. Derivation and linear theory. J. Nonlinear Sci. **12** (2002), no. 4, 283–318.

[3] J.L. Bona, M. Chen, J.-C. Saut: Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media. II. The nonlinear theory. Nonlinearity **17** (2004), no. 3, 925–952.

[4] J. V. Boussinesq: Théorie générale des mouvements qui sont propagés dans un canal rectangulaire horizontal. C.R. Acad. Sci. Paris **73** (1871) 256–260.

[5] J. A. Gear, R. Grimshaw: Weak and strong interactions between internal solitary waves. Stud. Appl. Math. **70** (3) (1984), 235–258.

[6] J. Shatah: Normal forms and quadratic nonlinear Klein-Gordon equations. Comm. Pure Appl. Math. **38** (1985), 685–696.

[7] N. Tzvetkov: Long time bounds for the periodic KP-II equation. Int. Math. Res. Not. **46** (2004), 2485–2496.

[8] G. B. Whitham: Linear and Nonlinear Waves. Wiley, New York (1999).