



**HAL**  
open science

# Asymptotic behavior of small solution of the Benjamin-Ono equations with time dependent coefficients

Youcef Mammeri

► **To cite this version:**

Youcef Mammeri. Asymptotic behavior of small solution of the Benjamin-Ono equations with time dependent coefficients. *Journal of Applied Analysis*, 2015, 10.1515/jaa-2015-0002 . hal-01090309

**HAL Id: hal-01090309**

**<https://hal.science/hal-01090309>**

Submitted on 3 Dec 2014

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Asymptotic behavior of small solution of the Benjamin-Ono equations with time dependent coefficients

Y. Mammeri

Laboratoire Amiénois de Mathématique Fondamentale et Appliquée,  
CNRS UMR 7352, Université de Picardie Jules Verne,  
80069 Amiens, France.

Email: youcef.mammeri@u-picardie.fr

November 2013

**Abstract.** We study the behavior of small solutions depending on time of the generalized and regularized Benjamin-Ono equation in both continuous and periodic context. In particular, we prove that these solutions remain small for a time scale improving the natural time given by the local well-posedness. In the continuous case, the result becomes global in time.

**Keywords.** Benjamin-Ono, dispersion, normal form.

**MS Codes.** 35B40, 35Q53, 76B03, 76B15

## Introduction

We study the behavior of small solutions depending on time of the generalized and regularized Benjamin-Ono equation (grBO)

$$(1 + b(t)H\partial_x)u_t + u_x + a(t)u^\rho u_x = 0,$$

where  $H$  denotes the Hilbert transform,  $\rho \geq 1$  an integer and  $a(t), b(t)$  real valued functions. This model is deduced from the Benjamin-Ono equation using the BBM trick [2, 10, 3]

$$u_t + u_x - bHu_{xx} + au^\rho u_x = 0.$$

Here coefficients  $a$  and  $b$  represent respectively the ratio between amplitude and depth, and between depth and wavelength. This equation models the long waves at the interface of two fluids with the top layer of infinite depth and the bottom of finite depth.

Dispersive inequalities such as

$$\|u(t)\|_{L^q} \leq C\varepsilon(1 + |t|)^{-\gamma(q)},$$

with  $\gamma(q) > 0$ , can be proved for a small initial datum using a method from Strauss [13]. This decay time is obtained by studying the linearized equation around zero and Van der Corput's lemma. Concerning the model with time-dependent coefficients, our result reads as follows.

**Theorem 0.1** *Let  $\rho \geq 3$  and  $\forall t \in \mathbb{R}, a(t) > 0, b(t) > 0$  be real valued bounded functions. Let*

$$\begin{aligned} 0 < \gamma &< 1/3 - 8/(9\rho), \\ s &= 1/(2\gamma) - 1, \\ 1/p + 1/q &= 1 \text{ with } 0 < 1/q < 1/2 - 4/(3\rho(1 - 3\gamma)), \\ 1/m + 1/n &= 1 \text{ with } \frac{8}{3\rho(1 - 3\gamma)(1 - 2/q)} < n < \frac{2}{(1 - 3\gamma)(1 - 2/q)}. \end{aligned}$$

*Assume that  $a/\tilde{B} \in L^m(\tilde{B})$ , where  $\tilde{B}(t) := \int_0^{|t|} b(\tau)/(1 + b(\tau))^3 d\tau$ , more precisely*

$$\left( \int_0^{+\infty} \left( \frac{a(\tau)}{\tilde{B}(\tau)} \right)^m \tilde{B}(\tau) d\tau \right)^{1/m} < +\infty.$$

*Then there exists  $\varepsilon > 0$  such that for  $u_0 \in H^s \cap W^{3-2/p,p}(\mathbb{R})$ , of norm smaller than  $\varepsilon$ , there exists a unique global in time solution  $u \in C(\mathbb{R}, H^s \cap W^{3-2/p,p}(\mathbb{R}))$  of the grBO equation.*

*Moreover  $\forall t \in \mathbb{R}$ ,*

$$\|u(t)\|_{L^q} \leq C(1 + \tilde{B}(t))^{-\frac{1+3\gamma}{2}(1-\frac{2}{q})}.$$

This result follows the ones of Bisognin and Menzala [5, 6]. They proved such a result for the time-dependent Korteweg-de Vries and the Benjamin-Bona-Mahony equations in the continuous context.

As regards the periodic case, the decay of the solution thanks to the oscillatory integral is no longer available. Based on a method of normal form [1, 14], one still manages to show that, starting from a small initial datum, the solution remains small (but not necessarily decreasing to zero) for a finite time interval. The method of normal form consists in finding a bijective map which combines the solution  $u$  of the Benjamin-Ono equation with  $v$  solution of nonlinear partial differential equation with high-order nonlinearity. The well-posedness of  $v$  then extends the one of  $u$ .

In this paper, we prove the following result.

**Theorem 0.2** *Let  $\rho = 1$  and  $\forall t \in \mathbb{R}, a(t) > 0, b(t) > 0$  be real valued bounded functions. Then there exists  $\varepsilon > 0$  such that for  $u_0 \in H^s(\mathbb{T})$ ,  $u_0$  zero mean valued, with  $\|u_0\|_s \leq \varepsilon$  and the time-derivative  $\|a'\|_\infty \leq \varepsilon\|a/b\|_\infty$ , there exists a unique solution  $u \in C([-T, T], H^s(\mathbb{T}))$  of the grBO equation. Here*

$$T := \frac{1}{C(\|a/b\|_\infty \|u_0\|_s)^2}.$$

*Moreover  $\forall t \in [-T, T]$ ,*

$$\|u(t)\|_s \leq C\varepsilon.$$

We notice that the time given by this result is the square of the one given by the standard result of local well-posedness. Such a result can be found for the time-dependent Korteweg-de Vries and the Benjamin-Bona-Mahony equations with periodic conditions.

We use the following notations: for  $\Omega = \mathbb{R}$  or  $\mathbb{T}$ , and  $1 \leq p < \infty$ , we denote  $L^p(\Omega)$  the space of  $p$ -power integrable functions equipped with the norm

$$\|u\|_{L^p} := \left( \int_\Omega |u(x)|^p d\mu(x) \right)^{1/p},$$

where either the measure  $d\mu(x)$  is the Lebesgue one on  $\mathbb{R}$  such that

$$u(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ikx} \hat{u}(k) dk, \hat{u}(k) = \int_{\mathbb{R}} e^{-ikx} u(x) dx,$$

or  $d\mu(x)$  is chosen proportional to the Lebesgue one on  $\mathbb{T}$  and normalized such that

$$u(x) = \sum_{k \in \mathbb{Z}^*} e^{ikx} \hat{u}(k), \quad \hat{u}(k) = \int_{\mathbb{T}} e^{-ikx} u(x) d\mu(x).$$

We denote  $L^\infty(\Omega)$  the functions space equipped with the norm

$$\|u\|_\infty = \text{supess}(u) := \inf \{c; |u(x)| \leq c \text{ almost everywhere in } \Omega\},$$

and for  $1 \leq p \leq \infty$ ,  $W^{s,p}(\mathbb{R})$  is the Sobolev space, and  $H^s(\mathbb{R}) := W^{s,2}(\mathbb{R})$ . We also denote  $H_0^s(\mathbb{T})$  the space of zero  $x$ -mean value functions equipped with the norm

$$\|u\|_s = \left( \sum_{k \in \mathbb{Z}^*} |k|^{2s} |\hat{u}(k)|^2 \right)^{1/2},$$

When there is no ambiguity, for any positive  $A$  and  $B$ , the notation  $A \lesssim B$  means that there exists a constant  $C > 0$  such that  $A \leq CB$ .

The paper is organised the paper as follows. Section 1 is devoted to the continuous context, we establish the dispersion properties of the solution. The periodic context is studied in Section 2 using a method of normal form.

## 1 The continuous context

We consider the following initial value problem

$$\begin{aligned} (1 + b(t)H\partial_x)u_t + u_x + a(t)u^\rho u_x &= 0, \\ u(x, 0) &= u_0(x) \end{aligned}$$

### 1.1 Local well-posedness

**Theorem 1.1** *Let  $\forall t \in \mathbb{R}, a(t) > 0, b(t) > 0$  be real valued bounded functions and  $s > 1/2$ . Let  $u_0 \in H^s(\mathbb{R})$ . There exists a constant  $C > 0$ , depending only on  $s$ , such that for*

$$T = \frac{1}{C \|a/b\|_\infty \|u_0\|_s^\rho},$$

*there exists a unique solution  $u \in \mathcal{C}([-T, T]; H^s(\mathbb{R}))$  of the Cauchy problem.*

*Moreover, for all  $M > 0$  with  $\|u_0\|_s \leq M$  and  $\|v_0\|_s \leq M$ , there exists  $C_1 > 0$  such that solutions  $u$  and  $v$ , with  $u_0$  and  $v_0$  as initial data respectively, satisfy for  $t \in [-T, T]$ , with  $T = 1/(CM^\rho \|a/b\|_\infty)$ ,*

$$\|u(t) - v(t)\|_s \leq C_1 \|u_0 - v_0\|_s.$$

**Proof.** The proof is done for positive time.

Let  $T > 0$ . According to Duhamel's formula,  $u$  is the solution of the Cauchy problem if and only if  $u$  is the solution of the following equation, for  $t \in [0, T]$ ,

$$\begin{aligned} u(t) = \Phi u(t) &:= S_t u_0 - \frac{1}{\rho+1} \int_0^t S_{t-\tau} (a(\tau)(1 + b(\tau)H\partial_x)^{-1} \partial_x u^{\rho+1})(\tau) d\tau \\ S_t v &:= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ikx - iB(k,t)} \hat{v}(k) dk, \quad B(k, t) := k \int_0^t \frac{d\tau}{1 + b(\tau)|k|}. \end{aligned}$$

The Sobolev embedding and lemma X.4 from [8] provide for  $s > 1/2$

$$\begin{aligned} \|\Phi u(t)\|_s &\leq \|u_0\|_s + \frac{1}{\rho+1} \int_0^t \|a(\tau)(1+b(\tau)H\partial_x)^{-1}\partial_x u^{\rho+1}(\tau)\|_s d\tau \\ &\leq \|u_0\|_s + C_s \sup_{t \in \mathbb{R}} \left( \frac{a(t)}{b(t)} \right) T \left( \sup_{t \in [0, T]} \|u(t)\|_s \right)^{\rho+1}, \end{aligned}$$

because

$$\|(1+b(\tau)H\partial_x)^{-1}\partial_x u\|_s \leq \left( \int_{-\infty}^{+\infty} (1+k^2)^s \left| \frac{ik}{1+b(\tau)|k|} \right|^2 |\hat{u}(k)|^2 dk \right)^{1/2} \leq \frac{1}{b(\tau)} \|u\|_s.$$

Then for  $T = (C\|a/b\|_\infty \|u_0\|_s^\rho)^{-1}$ , the closed ball

$$\bar{B}_T := \left\{ u \in \mathcal{C}([0, T]; H_0^s(\mathbb{T})) ; \sup_{t \in [0, T]} \|u(t)\|_s \leq 2\|u_0\|_s \right\}$$

verifies  $\Phi(\bar{B}_T) \subset \bar{B}_T$  if  $C \geq 2^{\rho+1} C_s$ .

Let  $u$  and  $v$  in  $\bar{B}_T$ . Duhamel's formula gives, for  $t \in [0, T]$ , with  $u^{\rho+1} - v^{\rho+1} = (u-v) \sum_{i=0}^{\rho} u^{\rho-i} v^i$ ,

$$\begin{aligned} \|\Phi u(t) - \Phi v(t)\|_s &\leq \frac{1}{\rho+1} \int_0^t \|a(\tau)(1+b(\tau)H\partial_x)^{-1}\partial_x(u^{\rho+1} - v^{\rho+1})\|_s d\tau, \\ &\leq \frac{2^\rho \rho}{\rho+1} C_s \|u_0\|_s^\rho \sup_{t \in \mathbb{R}} \left( \frac{a(t)}{b(t)} \right) T \sup_{t \in [-T, T]} \|u-v\|_s(t) \\ &\leq \frac{2^\rho C_s}{C} \sup_{t \in [0, T]} \|u-v\|_s(t). \end{aligned}$$

For  $C > 2^\rho C_s$ , the map  $\Phi$  is a contraction on  $\bar{B}_T$ . The fixed point theorem implies that the existence and uniqueness of the solution  $u$  of  $\Phi u(t) = u(t)$  in  $\bar{B}_T$ .

It remains to prove the continuity with the initial datum. Let  $u$  and  $v$  solutions of the Cauchy problem with initial data  $u_0$  and  $v_0$  respectively, such that  $\|u_0\|_s \leq M$  and  $\|v_0\|_s \leq M$ . Duhamel's formula gives for  $t \in [0, T]$ , with  $T = 1/(CM^\rho \sup_{t \in \mathbb{R}} \left( \frac{a(t)}{b(t)} \right))$

$$\begin{aligned} \|u(t) - v(t)\|_s &\leq \|u_0 - v_0\|_s + \frac{1}{\rho+1} \int_0^t \|a(\tau)(1+b(\tau)H\partial_x)^{-1}\partial_x(u^{\rho+1} - v^{\rho+1})\|_s d\tau \\ &\leq \|u_0 - v_0\|_s + \frac{2^\rho C_s}{CM} \sum_{i=0}^{\rho} \|u_0\|_s^{\rho-i} \|v_0\|_s^i \sup_{t \in [0, T]} \|u-v\|_s(t). \end{aligned}$$

Since  $C_0 > 2^\rho C_s$ , we find

$$\|u(t) - v(t)\|_s \leq \|u_0 - v_0\|_s + \frac{1}{2} \sup_{t \in [0, T]} \|u-v\|_s(t),$$

thus

$$\sup_{t \in [0, T]} \|u-v\|_s \leq 2\|u_0 - v_0\|_s.$$

□

## 1.2 Estimates for the linear equation

We consider the linear initial-value problem, for  $(x, t) \in \mathbb{R} \times \mathbb{R}$ ,

$$\begin{aligned} (1 + b(t)H\partial_x)u_t + u_x &= 0 \\ u(x, 0) &= u_0(x). \end{aligned}$$

Let  $u_0 \in \mathcal{S}(\mathbb{R})$ . The Fourier transform in space provides the solution  $u$  of the ordinary differential equation, for all  $x \in \mathbb{R}$  and  $t \in \mathbb{R}^*$ ,

$$u(x, t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ikx - iB(k, t)} \hat{u}_0(k) dk, \text{ with } B(k, t) := \int_0^t \frac{k d\tau}{1 + b(\tau)|k|}. \quad (1.1)$$

Let us recall Van der Corput's lemma [12].

**Lemma 1.2** *Let  $n \geq 2$ . There exists  $C > 0$  such that for all  $a \leq b$ ,  $\lambda > 0$ , and  $h \in C^\infty([a, b])$  real valued satisfying for all  $k \in [a, b]$ ,  $|h^{(n)}(k)| \geq \lambda$ , we have*

$$\left| \int_a^b e^{-ih(k)} dk \right| \leq \frac{C}{\lambda^{1/n}}. \quad (1.2)$$

The following dispersive inequality is obtained.

**Proposition 1.3** *Let  $0 < \gamma < 1/3$  and  $s = 1/(2\gamma) - 1$ . There exists a constant  $C_s > 0$ , depending only on  $s$ , such that  $\forall u_0 \in L^1(\mathbb{R}) \cap H^s(\mathbb{R})$ ,  $\forall t \in \mathbb{R}$ , we have*

$$\begin{aligned} \left| \int_{-\infty}^{+\infty} e^{ikx - iB(k, t)} \hat{u}_0(k) dk \right| &\leq C_s (\|u_0\|_{L^1} + \|u_0\|_{H^s}) \left(1 + \tilde{B}(t)\right)^{-1/2 + 3\gamma/2}, \\ \tilde{B}(t) &:= \int_0^{|t|} \frac{b(\tau) d\tau}{(1 + b(\tau))^3}. \end{aligned} \quad (1.3)$$

**Proof.** Let  $r \geq 1$ ,  $0 \leq k \leq r$  and  $\tilde{B}(t) \geq 1$ , we define

$$h(k) := \int_0^t \frac{k d\tau}{1 + b(\tau)k} - kx, \quad h'(k) = \int_0^t \frac{d\tau}{(1 + b(\tau)k)^2} - x, \quad h''(k) = \int_0^t \frac{-2b(\tau) d\tau}{(1 + b(\tau)k)^3}.$$

We deduce

$$|h''(k)| \geq 2 \int_0^t \frac{b(\tau) d\tau}{(1 + b(\tau)r)^3} \geq Cr^{-3} \int_0^t \frac{b(\tau) d\tau}{(1 + b(\tau))^3}.$$

Van der Corput's lemma is then applied to find

$$\left| \int_0^r e^{ikx - iB(k, t)} dk \right| = \left| \int_0^r e^{ikx - i \int_0^t \frac{k d\tau}{1 + b(\tau)k}} dk \right| \leq C \left( r^{-3} \int_0^t \frac{b(\tau) d\tau}{(1 + b(\tau))^3} \right)^{-1/2} = Cr^{3/2} \tilde{B}(t)^{-1/2}.$$

The Fubini theorem provides

$$\left| \int_0^r e^{ikx - iB(k, t)} \hat{u}_0(k) dk \right| \leq C \|u_0\|_{L^1} r^{3/2} \tilde{B}(t)^{-1/2}.$$

On the other hand, the Cauchy-Schwarz inequality gives, for  $s > 1/2$ ,

$$\begin{aligned} \left| \int_r^{+\infty} e^{ikx - iB(k, t)} \hat{u}_0(k) dk \right| &\leq \int_r^{+\infty} |\hat{u}_0(k)| dk = \int_r^{+\infty} \frac{(1+k)^s}{(1+k)^s} |\hat{u}_0(k)| dk, \\ &\leq \left( \int_r^{+\infty} (1+k)^{2s} |\hat{u}_0(k)|^2 dk \right)^{1/2} \left( \int_r^{+\infty} \frac{dk}{(1+k)^{2s}} \right)^{1/2} \\ &\leq C_s \|u_0\|_{H^s} r^{-(2s-1)/2} \end{aligned}$$

It follows, choosing  $r = \tilde{B}(t)^\gamma$  and  $s = 1/(2\gamma) - 1 > 1$ , that

$$\left| \int_{-\infty}^{+\infty} e^{ikx - iB(k,t)} \hat{u}_0(k) dk \right| \leq C_s (\|u_0\|_{L^1} + \|u_0\|_{H^s}) \tilde{B}(t)^{-1/2+3\gamma/2}.$$

Since  $\tilde{B}(t) \geq 1$ , we have  $\tilde{B}(t) \geq (1 + \tilde{B}(t))/2$ , and the result is found.

If  $\tilde{B}(t) \leq 1$ , the Cauchy-Schwarz inequality provides

$$\left| \int_0^{+\infty} e^{ikx - iB(k,t)} \hat{u}_0(k) dk \right| \leq \int_0^{+\infty} |\hat{u}_0(k)| dk = \int_0^{+\infty} \frac{(1+k^2)^{s/2}}{(1+k^2)^{s/2}} |\hat{u}_0(k)| dk \leq C_s \|u_0\|_{H^s}.$$

We conclude using  $\tilde{B}(t) \leq 1 \implies 1 \geq (1 + \tilde{B}(t))/2$ .

The contribution for  $k \leq 0$  is dealt with similar.  $\square$

**Corollary 1.4** *Let  $0 < \gamma < 1/3$  and  $s = 1/(2\gamma) - 1$ . There exists a constant  $C_s > 0$  such that for  $u_0 \in L^p(\mathbb{R}) \cap H^s(\mathbb{R})$ , with  $1 \leq p \leq 2$ , for all time  $t \in \mathbb{R}$ , we have*

$$\left\| \int_{-\infty}^{+\infty} e^{ikx - iB(k,t)} \hat{u}_0(k) dk \right\|_{L^q} \leq C_s (\|u_0\|_{L^p} + \|u_0\|_{H^s}) (1 + \tilde{B}(t))^{(-1/2+3\gamma/2)(1-2/q)}, \quad (1.4)$$

where  $\frac{1}{p} + \frac{1}{q} = 1$ .

**Proof.** The proof is done via the Riesz-Thorin interpolation theorem.  $\square$

### 1.3 Existence and Uniqueness of global solution

We consider the following nonlinear initial value problem

$$(1 + b(t)Hu_x)u_t + u_x + a(t)u^\rho u_x = 0 \quad (1.5)$$

$$u(x, 0) = u_0(x). \quad (1.6)$$

We now prove the theorem 0.1. The proof is carried out for positive time. Let us denote  $\theta := (1/2 - 3\gamma/2)(1 - 2/q) > 0$ .

Thanks to Duhamel's formula,  $u$  is the solution of (1.5)-(1.6) if and only if  $u$  is the solution of the following equation, for  $t \geq 0$

$$u(t) = \Phi u(t) := S_t u_0 - \frac{1}{\rho + 1} \int_0^t S_{t-\tau} (a(\tau)(1 + b(\tau)H\partial_x)^{-1} \partial_x u^{\rho+1})(\tau) d\tau \quad (1.7)$$

$$S_t u := \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ikx - iB(k,t)} \hat{u}(k) dk.$$

On the other hand, the Mihlin-Hörmander theorem [12] provides, for  $1 < p < 2$ , a constant  $C_p > 0$  depending only on  $p$  such that

$$\|(1 + b(t)H\partial_x)^{-1} \partial_x u\|_{L^p} \leq C_p \|u\|_{L^p}. \quad (1.8)$$

Let  $T > 0$ , we define the norm  $N_T$  by

$$N_T(u) := \sup_{0 \leq \tau \leq T} \left[ \|u(\tau)\|_{L^q} (1 + \tilde{B}(\tau))^\theta + \|u(\tau)\|_{H^s} + \|u(\tau)\|_{W^{3-2/p,p}} \right]. \quad (1.9)$$

We prove a few technical lemmata first.

**Lemma 1.5** *There exists a constant  $C_{s,p} > 0$  such that for all  $u$  and  $v$  in  $H^s(\mathbb{R}) \cap W^{3-2/p,p}(\mathbb{R})$*

$$\|\Phi u - \Phi v\|_{L^q}(t) \leq C_{s,p} \|a/\tilde{B}\|_{L^r(\tilde{B})} \left( \sum_{i=0}^{\rho} N_T(u)^{\rho-i} N_T(v)^i \right) (1 + \tilde{B}(t))^{-\theta} N_T(u-v), \quad (1.10)$$

**Proof.** Let  $u$  and  $v$  two elements of  $H^s(\mathbb{R}) \cap W^{3-2/p,p}(\mathbb{R})$ , the Duhamel's formula and Corollary 1.4 give

$$\begin{aligned} \|\Phi u - \Phi v\|_{L^q} &\leq \frac{1}{\rho+1} \int_0^t a(\tau) \|S_{t-\tau}((1+b(t)H\partial_x)^{-1}\partial_x(u^{\rho+1}-v^{\rho+1}))\|_{L^q}(\tau) d\tau \\ &\leq C_{s,p} \int_0^t \frac{a(\tau) (\|u^{\rho+1}-v^{\rho+1}\|_{L^p} + \|u^{\rho+1}-v^{\rho+1}\|_{H^s})(\tau)}{(1+\tilde{B}(t-\tau))^\theta} d\tau. \end{aligned}$$

Since  $u^{\rho+1}-v^{\rho+1} = (u-v) \sum_{i=0}^{\rho} u^{\rho-i}v^i$ , thanks to the fractional Leibniz rule [8] and the Minkowski inequality we obtain

$$\begin{aligned} \|u^{\rho+1}-v^{\rho+1}\|_{H^s}(\tau) &\lesssim \|u-v\|_{H^s} \sum_{i=0}^{\rho} \|u^{\rho-i}v^i\|_{\infty}(\tau) + \|u-v\|_{\infty} \sum_{i=0}^{\rho} \|u^{\rho-i}v^i\|_{H^s}(\tau) \\ &= \text{I}(\tau) + \text{II}(\tau). \end{aligned}$$

**Lemma 1.6** [7, Chapter 1 - Theorem 9.3] *Let  $1 \leq p, q \leq +\infty$  and  $0 \leq j < s$ , there exists a constant  $C > 0$ , depending on  $p, q, j$  and  $s$ , such that for all  $u \in \mathcal{S}(\mathbb{R})$ , we have*

$$\|(-\partial_x)^{j/2} u\|_{L^r} \leq C \|(-\partial_x)^{s/2} u\|_{L^p}^a \|u\|_{L^q}^{1-a},$$

where  $1/r = j + a(1/p - s) + (1-a)/q$  and  $j/s \leq a \leq 1$ , with the following exception: if  $s - j - 1/p$  is a non-negative integer, then the above inequality holds for  $j/s \leq a < 1$ .

The Sobolev inequality with  $1/p + 1/q = 1$ ,  $r = \infty$ ,  $j = 0$ ,  $a = 1/4$  and  $s = 3 - 2/p$  provides

$$\|u\|_{\infty} \leq C_p \|u\|_{W^{3-\frac{2}{p},p}}^{1/4} \|u\|_{L^q}^{3/4}. \quad (1.11)$$

We deduce

$$\begin{aligned} \text{I}(\tau) &\lesssim \|u-v\|_{H^s} \sum_{i=0}^{\rho} \left( \|u\|_{L^q}^{3/4} \|u\|_{W^{3-\frac{2}{p},p}}^{1/4} \right)^{\rho-i} \left( \|v\|_{L^q}^{3/4} \|v\|_{W^{3-\frac{2}{p},p}}^{1/4} \right)^i(\tau) \\ &\lesssim \|u-v\|_{H^s} \sum_{i=0}^{\rho} \left( \|u\|_{L^q} \frac{(1+\tilde{B}(\tau))^\theta}{(1+\tilde{B}(\tau))^\theta} \right)^{\frac{3(\rho-i)}{4}} \|u\|_{W^{3-\frac{2}{p},p}}^{\frac{\rho-i}{4}} \left( \|v\|_{L^q} \frac{(1+\tilde{B}(\tau))^\theta}{(1+\tilde{B}(\tau))^\theta} \right)^{\frac{3i}{4}} \|v\|_{W^{3-\frac{2}{p},p}}^{\frac{i}{4}} \\ &\lesssim \left( \sum_{i=0}^{\rho} N_T(u)^{\rho-i} N_T(v)^i \right) (1+\tilde{B}(\tau))^{-\frac{3\theta\rho}{4}} N_T(u-v). \end{aligned}$$

For  $\text{II}(\tau)$ , the Leibniz rule is applied again

$$\begin{aligned} \text{II}(\tau) &\lesssim \|u-v\|_{\infty} \left( \|v\|_{\infty}^{\rho-1} \|v\|_{H^s} + \sum_{i=0}^{\rho-1} \|u\|_{\infty}^{\rho-i-1} \|u\|_{H^s} \|v\|_{\infty}^i \right. \\ &\quad \left. + \|u\|_{\infty}^{\rho-1} \|u\|_{H^s} + \sum_{i=1}^{\rho} \|u\|_{\infty}^{\rho-i} \|v\|_{\infty}^{i-1} \|v\|_{H^s} \right)(\tau), \end{aligned}$$



and the inequality (1.11) gives

$$\begin{aligned}
\Pi(\tau) &\lesssim \|u - v\|_{L^q}^{3/4} \|u - v\|_{W^{3-\frac{2}{p},p}}^{1/4} \times \\
&\quad \left[ \left( \|v\|_{L^q}^{3/4} \|v\|_{W^{3-\frac{2}{p},p}}^{1/4} \right)^{\rho-1} \|v\|_{H^s} + \left( \|u\|_{L^q}^{3/4} \|u\|_{W^{3-\frac{2}{p},p}}^{1/4} \right)^{\rho-1} \|u\|_{H^s} \right. \\
&\quad + \sum_{i=0}^{\rho-1} \left( \|u\|_{L^q}^{3/4} \|u\|_{W^{3-\frac{2}{p},p}}^{1/4} \right)^{\rho-i-1} \|u\|_{H^s} \left( \|v\|_{L^q}^{3/4} \|v\|_{W^{3-\frac{2}{p},p}}^{1/4} \right)^i \\
&\quad \left. + \sum_{i=1}^{\rho} \left( \|u\|_{L^q}^{3/4} \|u\|_{W^{3-\frac{2}{p},p}}^{1/4} \right)^{\rho-i} \left( \|v\|_{L^q}^{3/4} \|v\|_{W^{3-\frac{2}{p},p}}^{1/4} \right)^{i-1} \|v\|_{H^s} \right] (\tau) \\
&\lesssim \left( \sum_{i=0}^{\rho} N_T(u)^{\rho-i} N_T(v)^i \right) (1 + \tilde{B}(\tau))^{-\frac{3\theta\rho}{4}} N_T(u - v).
\end{aligned}$$

Similar computations yield

$$\begin{aligned}
\|u^{\rho+1} - v^{\rho+1}\|_{L^p}(\tau) &\leq \|u - v\|_{L^p} \sum_{i=0}^{\rho} \|u\|_{\infty}^{\rho-i} \|v\|_{\infty}^i(\tau) \\
&\lesssim \left( \sum_{i=0}^{\rho} N_T(u)^{\rho-i} N_T(v)^i \right) (1 + \tilde{B}(\tau))^{-\frac{3\theta\rho}{4}} N_T(u - v).
\end{aligned}$$

To sum up, we find

$$\|\Phi u - \Phi v\|_{L^q}(t) \leq C_{s,p} \left( \sum_{i=0}^{\rho} N_T(u)^{\rho-i} N_T(v)^i \right) N_T(u - v) \int_0^t \frac{a(\tau) d\tau}{(1 + \tilde{B}(\tau))^{\frac{3\theta\rho}{4}} (1 + \tilde{B}(t - \tau))^\theta}.$$

However, the Hölder inequality implies for  $1/m + 1/n = 1$

$$\int_0^t \frac{a(\tau) d\tau}{(1 + \tilde{B}(\tau))^{\frac{3\theta\rho}{4}} (1 + \tilde{B}(t - \tau))^\theta} \leq \left( \int_0^t \left( \frac{a(\tau)}{\tilde{B}(\tau)} \right)^m \tilde{B}(\tau) d\tau \right)^{1/m} \left( \int_0^{\tilde{B}(t)} \frac{d\tau}{(1 + \tau)^{\frac{3n\theta\rho}{4}} (1 + \tilde{B}(t) - \tau)^{\theta n}} \right)^{1/n}.$$

**Lemma 1.7** [11] *Let  $a > 0, b > 1$  and  $F(t)$  be continuous real-valued function with, for all  $t \geq 0$ ,  $F(t) \geq 0$ . Then there exists  $0 < c < 1$ , with  $c = a$  if  $0 < a < 1$ , such that  $\forall t \geq 0$*

$$\int_0^{F(t)} \frac{d\tau}{(1 + F(t) - \tau)^a (1 + \tau)^b} \lesssim \frac{1}{(1 + F(t))^c}.$$

The lemma 1.7 is applied with  $b = 3n\theta\rho/4 > 1$  and  $0 < a = c = \theta n < 1$  to give

$$\|\Phi u - \Phi v\|_{L^q}(t) \leq C_{s,p} \|a/\tilde{B}\|_{L^m(\tilde{B})} \left( \sum_{i=0}^{\rho} N_T(u)^{\rho-i} N_T(v)^i \right) (1 + \tilde{B}(t))^{-\theta} N_T(u - v).$$

Here  $b > 1$  and  $0 < a < 1$  translated in

$$\begin{aligned}
\frac{8}{3\rho(1 - 3\gamma)(1 - 2/q)} &< n < \frac{2}{(1 - 3\gamma)(1 - 2/q)}, \\
0 < 1/q &< 1/2 - 4/(3\rho(1 - 3\gamma)), \\
0 < \gamma &< 1/3 - 8/(9\rho), \\
\rho &\geq 3.
\end{aligned}$$

□

**Lemma 1.8** *There exists a constant  $C_{s,p} > 0$  such that for all  $u$  and  $v$  in  $H^s(\mathbb{R}) \cap W^{3-2/p,p}(\mathbb{R})$ , we have*

$$\|\Phi u - \Phi v\|_{H^s}(t) + \|\Phi u - \Phi v\|_{W^{3-2/p,p}}(t) \leq C_{s,p} \|a/\tilde{B}\|_{L^m(\tilde{B})} \left( \sum_{i=0}^{\rho} N_T(u)^{\rho-i} N_T(v)^i \right) N_T(u-v). \quad (1.12)$$

**Proof.** We have

$$\begin{aligned} \|\Phi u - \Phi v\|_{W^{3-2/p,p}}(t) &\lesssim \int_0^t a(\tau) \|S_{t-\tau}((1+b(t)H\partial_x)^{-1}\partial_x(u^{\rho+1} - v^{\rho+1}))\|_{W^{3-2/p,p}}(\tau) d\tau \\ &\lesssim \int_0^t a(\tau) \|u^{\rho+1} - v^{\rho+1}\|_{W^{3-2/p,p}}(\tau) d\tau \end{aligned}$$

As previously, the fractional Leibniz rule gives

$$\|u^{\rho+1} - v^{\rho+1}\|_{W^{4-\frac{2}{p},p}}(\tau) \lesssim \left( \sum_{i=0}^{\rho} N_T(u)^{\rho-i} N_T(v)^i \right) N_T(u-v) (1 + \tilde{B}(\tau))^{-\frac{3\theta\rho}{4}}$$

thus, thanks to the Hölder inequality implies

$$\begin{aligned} \|\Phi u - \Phi v\|_{W^{3-2/p,p}}(t) &\lesssim \left( \sum_{i=0}^{\rho} N_T(u)^{\rho-i} N_T(v)^i \right) N_T(u-v) \int_0^t a(\tau) (1 + \tilde{B}(\tau))^{-\frac{3\theta\rho}{4}} d\tau \\ &\lesssim \left( \sum_{i=0}^{\rho} N_T(u)^{\rho-i} N_T(v)^i \right) N_T(u-v) \\ &\quad \times \left( \int_0^t \left( \frac{a(\tau)}{\tilde{B}(\tau)} \right)^m \tilde{B}(\tau) d\tau \right)^{1/m} \left( \int_0^{\tilde{B}(t)} \frac{d\tau}{(1+\tau)^{\frac{3m\theta\rho}{4}}} \right)^{1/n} \\ &\leq C_{s,p} \|a/\tilde{B}\|_{L^m(\tilde{B})} \left( \sum_{i=0}^{\rho} N_T(u)^{\rho-i} N_T(v)^i \right) N_T(u-v). \end{aligned}$$

Same computations for  $\|\Phi u - \Phi v\|_{H^s}$  allow to conclude.  $\square$

**Lemma 1.9** *There exists a constant  $C_{s,p} > 0$  such that for all  $u$  and  $v$  in  $H^s(\mathbb{R}) \cap W^{3-2/p,p}(\mathbb{R})$ , we have*

$$N_T(\Phi u - \Phi v) \leq C_{s,p} \|a/\tilde{B}\|_{L^m(\tilde{B})} \left( \sum_{i=0}^{\rho} N_T(u)^{\rho-i} N_T(v)^i \right) N_T(u-v), \quad (1.13)$$

and

$$N_T(\Phi u) \leq C_{s,p} \|a/\tilde{B}\|_{L^m(\tilde{B})} (\|u_0\|_{H^s} + \|u_0\|_{W^{3-2/p,p}} + N_T(u)^{\rho+1}). \quad (1.14)$$

**Proof.** Inequalities (1.10) and (1.12) give (1.13). Taking  $v = 0$  provides the second inequality.  $\square$

**Proof of theorem 0.1.** Let  $M > 0$ , we consider the closed ball

$$\bar{B}_{T,M} := \left\{ u \in \mathcal{C}([-T, T]; H^s(\mathbb{R}) \cap W^{3-2/p,p}(\mathbb{R})); N_T(u) \leq M \right\}.$$

We aim at showing that there exists a unique solution  $u$  of the equation (1.7) in this ball by using Banach fixed point theorem.

First, there exists  $\varepsilon > 0$  sufficiently small such that if  $\|u_0\|_{H^s} + \|u_0\|_{W^{3-2/p,p}} \leq \varepsilon$ , even if we take

$C_{s,p}M$  instead of  $M$ , it is enough to take  $M > 0$  satisfying  $\varepsilon + M^{\rho+1} \leq M$  so that the inequality (1.14) implies that the image of the closed ball  $\bar{B}_{T,M}$  by the map  $\Phi$  is included in itself. Here, the crucial point is that  $\varepsilon$  is independent of  $T$ . Secondly, we prove that the map  $\Phi$  is a contraction on this ball for  $M$  sufficiently small. Let  $u$  and  $v$  two elements of the closed ball  $\bar{B}_{T,M}$ . The inequality (1.13) gives

$$N_T(\Phi u - \Phi v) \leq C_{s,p} \|a/\tilde{B}\|_{L^m(\tilde{B})} M^\rho N_T(u - v),$$

and it is enough to take  $M > 0$  sufficiently small so that the quantity  $C_{s,p} M^\rho < 1$ . Then, Banach fixed point theorem is applied and there exists a unique solution of the equation (1.7) in the closed ball  $\bar{B}_{T,M}$ .

It remains to prove that this unique solution can be prolonged in time with all  $[0, +\infty[$ . By uniqueness of the solution, the inequality (1.14) is written

$$N_T(u) \leq C_{s,p} \|a/\tilde{B}\|_{L^m(\tilde{B})} (\|u_0\|_{H^s} + \|u_0\|_{W^{3-2/p,p}} + N_T(u)^{\rho+1}). \quad (1.15)$$

Since there exists  $\varepsilon > 0$  sufficiently small such that  $\|u_0\|_{H^s} + \|u_0\|_{W^{3-2/p,p}} \leq \varepsilon$ , we can find  $M > 0$  such that

$$\begin{aligned} N_0(u) &< M \\ C_{s,p} \|a/\tilde{B}\|_{L^m(\tilde{B})} (\varepsilon + M^{\rho+1}) &\leq M. \end{aligned}$$

Then for all  $T > 0$ , we have  $N_T(u) < M$ . Indeed, if not, by continuity, there exists a time  $T > 0$  such that

$$\begin{aligned} N_T(u) &= M \\ &> C_{s,p} \|a/\tilde{B}\|_{L^m(\tilde{B})} (\varepsilon + M^{\rho+1}) \\ &> C_{s,p} \|a/\tilde{B}\|_{L^m(\tilde{B})} (\varepsilon + N_T(u)^{\rho+1}), \end{aligned}$$

which contradicts the inequality (1.15). Finally, there exists a constant  $M > 0$  such that for all  $T > 0$ ,  $N_T(u) < M$ . In particular, we have for all time  $t \geq 0$

$$\begin{aligned} \|u(t)\|_{L^q} &\leq C_{s,p} \|a/\tilde{B}\|_{L^m(\tilde{B})} (\|u_0\|_{H^s} + \|u_0\|_{W^{3-2/p,p}} + N_T(u)^{\rho+1}) (1 + \tilde{B}(t))^{-\theta} \\ &\leq C_{s,p} \|a/\tilde{B}\|_{L^m(\tilde{B})} (\varepsilon + M^{\rho+1}) (1 + \tilde{B}(t))^{-\theta}. \end{aligned} \quad (1.16)$$

□

**Remark 1.10** *Similar result including the Korteweg-de Vries and the Benjamin-Bona-Mahony equations with time-dependent coefficients can be found in [5, 6].*

**Remark 1.11** *Concerning the Benjamin-Ono equation*

$$u_t + u_x - b(t)Hu_{xx} + a(t)u^\rho u_x = 0,$$

*we obtain in a similar way the global well-posedness with the following decay rate*

$$\|u(t)\|_{L^\infty} \leq C \|u(t)\|_{L^1} (1 + B(t))^{-1/2}, \text{ with } B(t) := \int_0^{|t|} b(\tau) d\tau.$$

## 2 The periodic context

In this section, we only consider the case  $\rho = 1$ . The local in time well-posedness can be obtained in the same manner as for the continuous case.

We define, for  $k \in \mathbb{Z}$ ,  $\sigma(k) := ik/(1 + b(t)|k|)$ . The Fourier symbol  $\sigma$  satisfies for all  $k$  and  $k_1$  in  $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$

$$|\sigma(k_1) + \sigma(k - k_1) - \sigma(k)| \geq b(t)^2 \frac{|kk_1(k - k_1)|}{(1 + b(t)|k|)(1 + b(t)|k_1|)(1 + b(t)|k - k_1|)}.$$

**Proof.** Indeed, six cases have to be studied according to the sign of  $k, k_1$  and  $k - k_1$ .

i. If  $k > 0, k_1 > 0$  and  $k - k_1 > 0$ , we have

$$|\sigma(k_1) + \sigma(k - k_1) - \sigma(k)| = \frac{|b(t)(2 + b(t)k)k_1(k - k_1)|}{(1 + b(t)|k|)(1 + b(t)|k_1|)(1 + b(t)|k - k_1|)}.$$

ii. If  $k > 0, k_1 > 0$  and  $k - k_1 < 0$ ,

$$|\sigma(k_1) + \sigma(k - k_1) - \sigma(k)| = \frac{|b(t)k(2 + b(t)k_1)(k - k_1)|}{(1 + b(t)|k|)(1 + b(t)|k_1|)(1 + b(t)|k - k_1|)}.$$

iii. If  $k > 0$  and  $k_1 < 0$ ,

$$|\sigma(k_1) + \sigma(k - k_1) - \sigma(k)| = \frac{|b(t)kk_1(2 + b(t)(k - k_1))|}{(1 + b(t)|k|)(1 + b(t)|k_1|)(1 + b(t)|k - k_1|)}.$$

iv. If  $k < 0$  and  $k_1 > 0$ ,

$$\begin{aligned} |\sigma(k_1) + \sigma(k - k_1) - \sigma(k)| &= \frac{|-b(t)kk_1(2 - b(t)(k - k_1))|}{(1 + b(t)|k|)(1 + b(t)|k_1|)(1 + b(t)|k - k_1|)} \\ &= \frac{b(t)|k|k_1(2 + b(t)|k - k_1|)}{(1 + b(t)|k|)(1 + b(t)|k_1|)(1 + b(t)|k - k_1|)}. \end{aligned}$$

v. If  $k < 0, k_1 < 0$  and  $k - k_1 > 0$ ,

$$\begin{aligned} |\sigma(k_1) + \sigma(k - k_1) - \sigma(k)| &= \frac{|-b(t)k(2 - b(t)k_1)(k - k_1)|}{(1 + b(t)|k|)(1 + b(t)|k_1|)(1 + b(t)|k - k_1|)} \\ &= \frac{b(t)|k|(2 + b(t)|k_1|)(k - k_1)}{(1 + b(t)|k|)(1 + b(t)|k_1|)(1 + b(t)|k - k_1|)}. \end{aligned}$$

vi. If  $k < 0, k_1 < 0$  and  $k - k_1 < 0$ ,

$$\begin{aligned} |\sigma(k_1) + \sigma(k - k_1) - \sigma(k)| &= \frac{|(2 - b(t)k)(-b(t)k_1)(k - k_1)|}{(1 + b(t)|k|)(1 + b(t)|k_1|)(1 + b(t)|k - k_1|)} \\ &= \frac{(2 + b(t)|k|)b(t)|k_1|(k - k_1)}{(1 + b(t)|k|)(1 + b(t)|k_1|)(1 + b(t)|k - k_1|)}. \end{aligned}$$

□

For  $\delta > 0$ , we set  $U_\delta := \{u \in H_0^s(\mathbb{T}); \|u\|_s < \delta\}$ . Let  $D := \{(k, k_1) \in \mathbb{Z}^2; k \neq 0, k_1 \neq 0, k \neq k_1\}$ , we define the operator  $\Lambda$  by  $\Lambda u := u + K(u, u)$  for  $u \in H_0^s(\mathbb{T})$ , with

$$K(u, v) := -\frac{1}{2} \sum_D e^{ikx} \frac{ik}{1 + b(t)|k|} \frac{\hat{u}(k_1) \hat{v}(k - k_1)}{\sigma(k_1) + \sigma(k - k_1) - \sigma(k)}.$$

Introducing  $\Lambda$  is used to define  $v = \Lambda u$  so that  $v$  is solution of the equation

$$v_t + L(v) = F(u),$$

with  $F$  trilinear whereas  $u$  is solution of a quadratic Boussinesq system. Thus the well-posedness of  $v$  and the definition of  $K$  are used to estimate  $u$  with respect to  $v$  and to extend its well-posedness.

**Lemma 2.1** *Let  $s > 1/2$ . There exists a constant  $C > 0$  such that for all  $u$  and  $v$  in  $H_0^s(\mathbb{T})$*

$$\|K(u, v)\|_s \leq \frac{C}{b(t)^2} \|u\|_s \|v\|_s.$$

**Proof.** By duality, to prove the lemma is equivalent to proving for all  $w \in \mathcal{C}^\infty(\mathbb{T})$

$$\left| \sum_D \widehat{K(u, v)}(k) \widehat{w}(k) \right| \leq \frac{C}{b(t)^2} (\|u\|_s \|v\|_s) \|w\|_{-s}. \quad (2.1)$$

Indeed, we have

$$\|K(u, v)\|_s^2 = \sum_{k \in \mathbb{Z}^*} |k|^{2s} |\widehat{K(u, v)}(k)|^2 = \sum_{k \in \mathbb{Z}^*} \widehat{K(u, v)}(k) \left( |k|^{2s} \widehat{K(u, v)}(k) \right).$$

We set  $\widehat{w}(k) = |k|^{2s} \widehat{K(u, v)}(k)$  and we write

$$\|K(u, v)\|_s^2 = \sum_{k \in \mathbb{Z}^*} \widehat{K(u, v)}(k) \widehat{w}(k),$$

and according to the inequality (2.1)

$$\|K(u, v)\|_s^2 \leq \frac{C}{b(t)^2} (\|u\|_s \|v\|_s) \|w\|_{-s}.$$

However

$$\begin{aligned} \|w\|_{-s} &= \left( \sum_{k \in \mathbb{Z}^*} |k|^{-2s} |k|^{4s} |\widehat{K(u, v)}(k)|^2 \right)^{1/2} \\ &= \left( \sum_{k \in \mathbb{Z}^*} |k|^{2s} |\widehat{K(u, v)}(k)|^2 \right)^{1/2} = \|K(u, v)\|_s. \end{aligned}$$

If we define

$$\widehat{u}_1(k) = |k|^s \widehat{u}(k), \quad \widehat{v}_1(k) = |k|^s \widehat{v}(k) \quad \text{and} \quad \widehat{w}_1(k) = |k|^{-s} \widehat{w}(k),$$

it is enough to prove

$$\left| \frac{1}{2} \sum_D \frac{ik}{1 + b(t)|k|} \frac{|k|^s \widehat{u}_1(k_1) \widehat{v}_1(k - k_1) \widehat{w}_1(k)}{|k_1|^s |k - k_1|^s (\sigma(k_1) + \sigma(k - k_1) - \sigma(k))} \right| \leq \frac{C}{b(t)^2} (\|u_1\|_{L^2} \|v_1\|_{L^2}) \|w\|_{L^2}.$$

On one hand, we have for  $k$  and  $k_1$  in  $D$

$$\left| \frac{ik}{1 + b(t)|k|} \frac{1}{\sigma(k_1) + \sigma(k - k_1) - \sigma(k)} \right| \lesssim \frac{1}{b(t)^2}.$$

Indeed, since  $b(t)$  is a non-negative real valued bounded function, and  $k, k_1 \in \mathbb{Z}^*$ , it gets

$$\left| \frac{ik}{1 + b(t)|k|} \frac{1}{\sigma(k_1) + \sigma(k - k_1) - \sigma(k)} \right| \leq \frac{(1 + b(t)|k_1|)(1 + b(t)|k - k_1|)}{b(t)^2 |k_1(k - k_1)|}.$$

On the other hand, for  $s \geq 0$ , the triangle inequality implies

$$\frac{|k|^s}{|k_1|^s |k - k_1|^s} \lesssim \left( \frac{1}{|k_1|^s} + \frac{1}{|k - k_1|^s} \right),$$

and it remains to bound

$$\frac{1}{b(t)^2} \sum_D \frac{|\hat{u}_1(k_1)| |\hat{v}_1(k-k_1)| |\hat{w}_1(k)|}{|k_1|^s} + \frac{1}{b(t)^2} \sum_D \frac{|\hat{u}_1(k_1)| |\hat{v}_1(k-k_1)| |\hat{w}_1(k)|}{|k-k_1|^s} =: \frac{1}{b(t)^2} (\text{I} + \text{II}).$$

The Cauchy-Schwarz inequality in  $k$ , then in  $k_1$ , provides

$$\begin{aligned} \text{I} &\leq \left( \sum_{k \in \mathbb{Z}^*} \left( \sum_{\substack{k_1 \in \mathbb{Z}^* \\ k_1 \neq k}} \frac{|\hat{u}_1(k_1)| |\hat{v}_1(k-k_1)|}{|k_1|^s} \right)^2 \right)^{1/2} \left( \sum_{k \in \mathbb{Z}^*} |\hat{w}_1(k)|^2 \right)^{1/2} \\ &\leq \left( \sum_{\substack{k_1 \in \mathbb{Z}^* \\ k_1 \neq k}} \frac{1}{|k_1|^{2s}} \right)^{1/2} \left( \sum_{k \in \mathbb{Z}^*} \sum_{\substack{k_1 \in \mathbb{Z}^* \\ k_1 \neq k}} |\hat{u}_1(k_1)|^2 |\hat{v}_1(k-k_1)|^2 \right)^{1/2} \left( \sum_{k \in \mathbb{Z}^*} |\hat{w}_1(k)|^2 \right)^{1/2}. \end{aligned}$$

Since  $s > 1/2$ , there exists  $C > 0$  such that

$$\text{I} \leq C (\|u_1\|_{L^2} \|v_1\|_{L^2}) \|w_1\|_{L^2}.$$

A similar inequality for II is found by symmetry.  $\square$

**Proposition 2.2** *Let  $s > 1/2$ . Then there exist  $0 < \delta' < \min\{b(t)^2/a(t), t \in \mathbb{R}\}$ ,  $\delta > 0$ , and  $C > 0$  such that for all  $v \in U_\delta$ , there exists a unique  $u \in U_{\delta'}$  such that  $\Lambda u = v$ . Moreover*

$$\|u\|_s \leq C \|v\|_s,$$

and the map  $\Lambda^{-1}$  is of class  $\mathcal{C}^1$ .

**Proof.** For  $u \in H_0^s(\mathbb{T})$ , The differential of this operator is given by, for all  $\varphi \in \mathcal{C}^\infty(\mathbb{T})$

$$\langle d\Lambda(u), \varphi \rangle = \varphi + 2K(u, \varphi),$$

the preceding lemma implies that  $d\Lambda$  is continuous from  $H_0^s(\mathbb{T})$  to itself. Since  $d\Lambda(0)$  is the identity, the inverse function theorem is applied to give the following lemma.  $\square$

**Proposition 2.3** *Let  $s > 1/2$ . There exists a trilinear operator*

$$K_1 : H_0^s(\mathbb{T}) \times H_0^s(\mathbb{T}) \times H_0^s(\mathbb{T}) \longrightarrow H_0^s(\mathbb{T})$$

such that, if  $u \in \mathcal{C}([-T, T]; H_0^s(\mathbb{T}))$  is solution of

$$(1 + b(t)Hu_x)u_t + u_x + a(t)u^p u_x = 0,$$

then  $v$  defined by

$$v(t) := u(t) + a(t)K(u(t), u(t)), \text{ for } t \in [-T, T],$$

is solution of

$$v_t + (1 + b(t)H\partial_x)^{-1}v_x = a'(t)K(u, u) + a(t)^2 K_1(u, u, u).$$

Moreover, there exists a constant  $C > 0$  such that for all  $(u_1, u_2, u_3) \in H_0^s(\mathbb{T}) \times H_0^s(\mathbb{T}) \times H_0^s(\mathbb{T})$

$$\|K_1(u_1, u_2, u_3)\|_s \leq C \frac{1}{b(t)^2} \|u_1\|_s \|u_2\|_s \|u_3\|_s,$$

**Proof.** We write

$$\begin{aligned}
v_t + (1 + b(t)H\partial_x)^{-1}v_x &= u_t + a'(t)K(u, u) + a(t)\partial_t K(u, u) + (1 + b(t)H\partial_x)^{-1}u_x \\
&\quad + a(t)(1 + b(t)H\partial_x)^{-1}\partial_x K(u, u) \\
&= -\frac{a(t)}{2}(1 + b(t)H\partial_x)^{-1}\partial_x u^2 + a'(t)K(u, u) + a(t)\partial_t K(u, u) \\
&\quad + a(t)(1 + b(t)H\partial_x)^{-1}\partial_x K(u, u).
\end{aligned}$$

On one hand, we have by symmetry and for  $u$  solution of (1.5),

$$\begin{aligned}
\partial_t K(u, u) &= K(u_t, u) + K(u, u_t) = 2K(u_t, u) \\
&= -\sum_D e^{ikx} \frac{ik}{1 + b(t)|k|} \frac{\hat{u}_t(k_1)\hat{u}(k - k_1)}{\sigma(k_1) + \sigma(k - k_1) - \sigma(k)} \\
&= -\frac{a(t)}{2} \sum_D e^{ikx} \frac{kk_1}{(1 + b(t)|k|)(1 + b(t)|k_1|)} \frac{\widehat{u^2}(k_1)\hat{u}(k - k_1)}{\sigma(k_1) + \sigma(k - k_1) - \sigma(k)} \\
&\quad + \frac{1}{2} \sum_D e^{ikx} \frac{ik}{1 + b(t)|k|} \frac{(\sigma(k_1) + \sigma(k - k_1))\hat{u}(k_1)\hat{u}(k - k_1)}{\sigma(k_1) + \sigma(k - k_1) - \sigma(k)}.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
(1 + b(t)H\partial_x)^{-1}\partial_x K(u, u) &= -\frac{1}{2} \sum_D e^{ikx} \frac{ik}{1 + b(t)|k|} \frac{\sigma(k)\hat{u}(k_1)\hat{u}(k - k_1)}{\sigma(k_1) + \sigma(k - k_1) - \sigma(k)} \\
(1 + b(t)H\partial_x)^{-1}\partial_x u^2 &= \sum_D e^{ikx} \frac{ik}{1 + b(t)|k|} \hat{u}(k_1)\hat{u}(k - k_1).
\end{aligned}$$

Finally, we find

$$v_t + (1 + b(t)H\partial_x)^{-1}v_x = a'(t)K(u, u) + a(t)^2 K_1(u, u, u).$$

where  $K_1$  is defined, for  $D_1 := \{(k, k_1, k_2) \in \mathbb{Z}^3; k \neq 0, k_1 \neq 0, k_2 \neq 0, k \neq k_1, k_1 \neq k_2\}$

$$K_1(u_1, u_2, u_3) := -\frac{1}{2} \sum_{D_1} e^{ikx} \frac{kk_1}{(1 + b(t)|k|)(1 + b(t)|k_1|)} \frac{\hat{u}_1(k_2)\hat{u}_2(k_1 - k_2)\hat{u}_3(k - k_1)}{\sigma(k_1) + \sigma(k - k_1) - \sigma(k)}.$$

However, we have for  $k, k_1$  and  $k_2$  in  $D_1$

$$\left| \frac{kk_1}{(1 + b(t)|k|)(1 + b(t)|k_1|)} \frac{1}{\sigma(k_1) + \sigma(k - k_1) - \sigma(k)} \right| \lesssim \frac{1}{b(t)^2}.$$

In the same manner as lemma 2.1, it is enough to bound

$$\text{I} := \frac{1}{2b(t)^2} \sum_{D_1} \frac{|k|^s |\hat{u}_1(k_2)| |\hat{u}_2(k_1 - k_2)| |\hat{u}_3(k - k_1)| |\hat{u}_4(k)|}{|k_2|^s |k_1 - k_2|^s |k - k_1|^s}.$$

The Cauchy-Schwarz inequality is applied first in  $k$  to give

$$\text{I} \leq \frac{1}{b(t)^2} \left( \sum_{k \in \mathbb{Z}^*} \left( \sum_{\substack{(k_1, k_2) \in (\mathbb{Z}^*)^2 \\ k_1 \neq k, k_2 \neq k_1}} \frac{|k|^s |\hat{u}_1(k_2)| |\hat{u}_2(k_1 - k_2)| |\hat{u}_3(k - k_1)|}{|k_2|^s |k_1 - k_2|^s |k - k_1|^s} \right) \right)^{1/2} \|u_4\|_{L^2},$$

and then in  $(k_2, k_1)$ ,

$$I \leq \frac{1}{b(t)^2} \left( \sum_{k \in \mathbb{Z}^*} \sum_{\substack{(k_1, k_2) \in (\mathbb{Z}^*)^2 \\ k_1 \neq k, k_2 \neq k_1}} |\hat{u}_1(k_2)|^2 |\hat{u}_2(k_1 - k_2)|^2 |\hat{u}_3(k - k_1)|^2 \right)^{1/2} \\ \times \sum_{\substack{(k_1, k_2) \in (\mathbb{Z}^*)^2 \\ k_1 \neq k, k_2 \neq k_1}} \frac{|k|^{2s}}{|k_2|^{2s} |k_1 - k_2|^{2s} |k - k_1|^{2s}} \|u_4\|_{L^2},$$

The triangle inequality implies

$$\frac{|k|^{2s}}{|k_2|^{2s} |k_1 - k_2|^{2s} |k - k_1|^{2s}} \leq C \left( \frac{1}{|k_2|^{2s} |k_1 - k_2|^{2s}} + \frac{1}{|k_2|^{2s} |k - k_1|^{2s}} + \frac{1}{|k_1 - k_2|^{2s} |k - k_1|^{2s}} \right),$$

thus

$$\sup_{k \in \mathbb{Z}^*} \sum_{\substack{(k_1, k_2) \in (\mathbb{Z}^*)^2 \\ k_1 \neq k, k_2 \neq k_1}} \frac{|k|^{2s}}{|k_2|^{2s} |k_1 - k_2|^{2s} |k - k_1|^{2s}} < +\infty.$$

□

**Proof of the theorem 0.2.** We suppose  $t > 0$ , the proof being similar for negative time. Let us denote  $X := L^\infty([0, T]; H_0^s(\mathbb{T}))$ . Let  $\delta$  and  $\delta'$  the positive constants involved in Lemma 2.2. According to the local well-posedness theorem, there exists  $\varepsilon_0 > 0$  such that if  $u_0 \in U_\varepsilon$ , for  $\varepsilon \in ]0, \varepsilon_0[$ , then for  $t \leq C_1 (\varepsilon \sup_{t \in \mathbb{R}} a(t)/b(t))^{-1} =: T$ ,

$$u(t) + a(t)K(u, u)(t) \in U_\delta, u(t) \in U_{\delta'}, \text{ and } u(t) = \Lambda^{-1}v(t).$$

Duhamel's formula gives for  $t \in [0, T]$

$$v(t) = S_t(u_0 + a(t)K(u_0, u_0)) + \int_0^t S_{t-\tau} (a'(\tau)K(u, u) + a(\tau)^2 K_1(u, u, u))(\tau) d\tau \quad (2.2)$$

$$S_t v := \sum_{k \in \mathbb{Z}^*} e^{ikx - iB(k,t)} \hat{v}(k). \quad (2.3)$$

Thanks to Lemma 2.1, and for  $\varepsilon_0 > 0$  sufficiently small, we have

$$\begin{aligned} \|S_t(u_0 + a(t)K(u_0, u_0))\|_s &\leq \|u_0\|_s + \|K(u_0, u_0)\|_s \\ &\leq \|u_0\|_s + C \frac{a(t)}{b(t)^2} \|u_0\|_s^2 \leq C_1 \varepsilon \left( 1 + \varepsilon \frac{a(t)}{b(t)^2} \right) \\ &\leq 2C_2 \varepsilon. \end{aligned} \quad (2.4)$$

The preceding lemmas yield for  $t \in [0, T]$

$$\left\| \int_0^t S_{t-\tau} (a(\tau)^2 K_1(\Lambda^{-1}v, \Lambda^{-1}v, \Lambda^{-1}v))(\tau) d\tau \right\|_X \leq C_3 \sup_{t \in [0, T]} \left( \frac{a(t)}{b(t)} \right)^2 T \|v\|_X^3 \quad (2.5)$$

$$\left\| \int_0^t S_{t-\tau} (a'(\tau)K(\Lambda^{-1}v, \Lambda^{-1}v))(\tau) d\tau \right\|_X \leq C_3 \sup_{t \in [0, T]} (a'(t)) T \|v\|_X^2. \quad (2.6)$$

Even if we take  $\varepsilon_0 > 0$  smaller, we assume  $2C_1\varepsilon_0 < \delta$ . We set  $C_0 = (18C_2C_3(1 + 3C_2))^{-1}$  and  $T_0 = C_0(\varepsilon \sup_{t \in \mathbb{R}} a(t)/b(t))^{-2}$ . It follows then

$$\|v\|_{L^\infty([0, T_0]; H_0^s(\mathbb{T}))} \leq 3C_2\varepsilon. \quad (2.7)$$



Let us suppose that the inequality (2.7) fails. Since

$$\|v(0)\|_s = \|u_0 + a(t)K(u_0, u_0)\|_s \leq C_2\varepsilon \left(1 + \varepsilon \frac{a(t)}{b(t)^2}\right) < 3C_2\varepsilon,$$

by continuity with time, there exists  $\tau \in [0, T_0]$  such that for  $t \in [0, \tau]$

$$\|v(t)\|_s \leq 3C_2\varepsilon \quad \text{and} \quad \|v(\tau)\|_s = 3C_2\varepsilon.$$

Let  $C$  the positive constant involved in Lemma 2.2, we also impose  $3C_1C\varepsilon_0 < \delta'$ . We know that  $u(t) \in U_{\delta'}$  for  $|t| \leq T$ , and with this choice of  $\varepsilon$ , it follows that  $u(t) \in U_{\delta'}$  for  $t \in [0, \tau]$ . Indeed, if there exists  $\tau_1 \in [0, \tau]$  such that for  $t \in [0, \tau_1]$

$$\|u(t)\|_s < \delta' \quad \text{and} \quad \|u(\tau_1)\|_s = \delta',$$

then by continuity with time and according to Lemma 2.2, we have

$$\begin{aligned} \delta' = \|u(\tau_1)\|_s &= \lim_{t \rightarrow \tau_1} \|u(t)\|_s \leq \sup_{0 \leq t < \tau_1} \|u(t)\|_s \\ &\leq \sup_{0 \leq t < \tau_1} C\|v(t)\|_s \leq 3C_2C\varepsilon < \delta', \end{aligned}$$

which is a contradiction.

Finally, we deduce from Duhamel's formula and inequalities (2.4), (2.5) and (2.6)

$$\|v\|_X \leq 2C_2\varepsilon + C_3 \sup_{t \in [0, \tau]} \left(\frac{a(t)}{b(t)}\right)^2 \tau \|v\|_X^3 + C_3 \sup_{t \in [0, \tau]} (a'(t)) \tau \|v\|_X^2.$$

Since  $\sup_{t \in [0, \tau]} (a'(t)) \leq \sup_{t \in [0, \tau]} (a(t)/b(t))^2 \varepsilon$ , we find

$$\tau \geq 2C_0 \left( \varepsilon \sup_{t \in [0, \tau]} \left(\frac{a(t)}{b(t)}\right) \right)^{-2} = 2T_0.$$

This is in contradiction with the fact  $\tau \in [0, T_0]$ . Then the inequality (2.7) is true and using Lemma 2.2, it get for  $t \in [0, T_0]$

$$\|u(t)\|_s \leq C\|v(t)\|_s \leq 3C_2C\varepsilon.$$

□

**Remark 2.4** *This method does not apply to the Benjamin-Ono equation*

$$u_t + u_x - b(t)Hu_{xx} + a(t)u^\rho u_x = 0.$$

Indeed, here  $\sigma(k) = |k|k$  and for  $k, k_1$  in  $D$

$$\left| \frac{kk_1}{\sigma(k_1) + \sigma(k - k_1) - \sigma(k)} \right|$$

is not bounded from above.

### 3 Numerical simulations

We propose to present in this section some numerical illustrations showing the dispersive properties for some (not exhaustive) choices of functions  $a(t), b(t)$ . The aim of this section is not to perform a complete numerical study, but rather to present some relevant simulations. We use a relaxation

scheme proposed in [4]. This method allows us to replace the solution of a nonlinear problem and thus provides a time saving. We rewrite the grBO equation

$$\begin{aligned} (1 + b(t)H\partial_x)u_t + u_x + \frac{a(t)}{\rho + 1}\Phi_x &= 0 \\ \Phi &= u^{\rho+1} \end{aligned}$$

We consider a bounded domain  $[-L, L], L > 0$  (large fixed value). We denote  $N_x > 0$  the number of Fourier modes,  $\Delta t > 0$  the time step, and for  $n \in \mathbb{N}$ ,  $\hat{u}_n$ , resp.  $\hat{\Phi}^{n+1/2}$ , is the approximation of  $\hat{u}(n\Delta t)$ , resp.  $\hat{\Phi}((n + 1/2)\Delta t)$ .

**Algorithm:**

- Set  $u_0$  and  $\hat{\Phi}_{-1/2} = \widehat{u_0^{\rho+1}}$ .
- For  $n = 0, 1, \dots$ , compute:

$$\begin{aligned} \frac{\hat{\Phi}_{n+1/2}(\xi) + \hat{\Phi}_{n-1/2}(\xi)}{2} &= \widehat{u_n^{\rho+1}}(\xi) \\ (1 + b_n|\xi|) \left( \frac{\hat{u}_{n+1}(\xi) - \hat{u}_n(\xi)}{\Delta t} \right) + i\xi \left( \frac{\hat{u}_{n+1}(\xi) + \hat{u}_n(\xi)}{2} \right) + i\xi a_n \hat{\Phi}_{n+1/2}(\xi) &= 0 \\ \xi = k \frac{\pi}{L}, -\frac{N_x}{2} \leq k \leq \frac{N_x}{2} - 1. \end{aligned}$$

Simulations are performed with  $L = 3000, N_x = 2^{14}, \Delta t = 0.02$  and start from the initial datum:

$$u_0(x) = \frac{4d}{1 + (\frac{d}{d+1})^2 x^2}.$$

This initial datum provides, in the case  $\rho = 1, a(t) = b(t) = 1$ , a solitary wave solution of the regularized Benjamin-Ono equation, given by  $u(x, t) = u_0(x - ct)$ . We can notice, for  $\rho = 1, a(t) = b(t) = 1$ , the  $H^{1/2}$ -norm is conserved, i.e.  $\forall t \in \mathbb{R}, \|u(t)\|_{H^{1/2}} = \|u_0\|_{H^{1/2}}$ . Figure 1 shows the evolution with time of the solution, the  $H^{1/2}$  and  $L^\infty$ -norms, and the error between the approximate solution and the solitary wave.

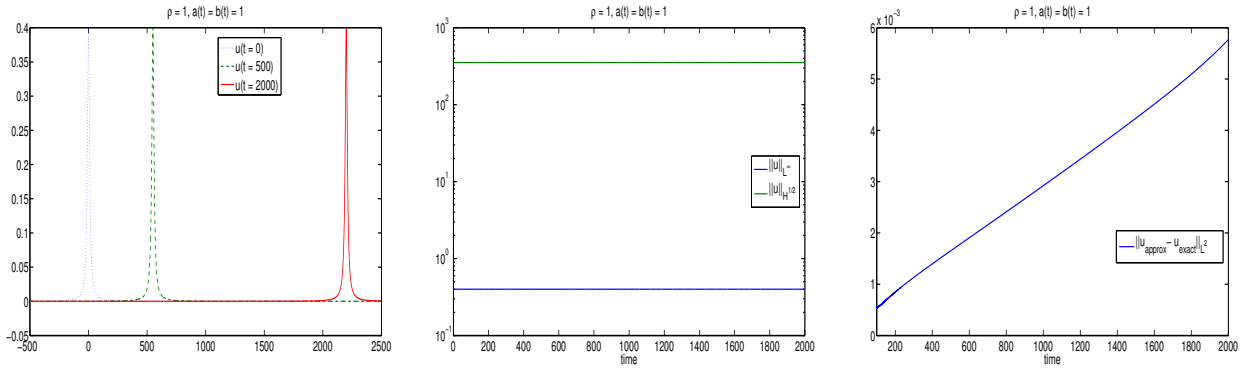


Figure 1: Result for  $\rho = 1, a(t) = b(t) = 1$ . On the left, the solution at time  $t = 0, 500, 2000$ . In the center, the  $H^{1/2}$  and  $L^\infty$ -norms. On the right, the error between the approximate solution and the solitary wave.

We notice that the approximate solution remains close to the soliton. The norms are well preserved and the error remains small. The numerical scheme appears to be relevant for the simulations. We represent in Figure 2 the evolution of the solution starting from a solitary wave with  $\rho = 3, a(t) = b(t) = 1$ .

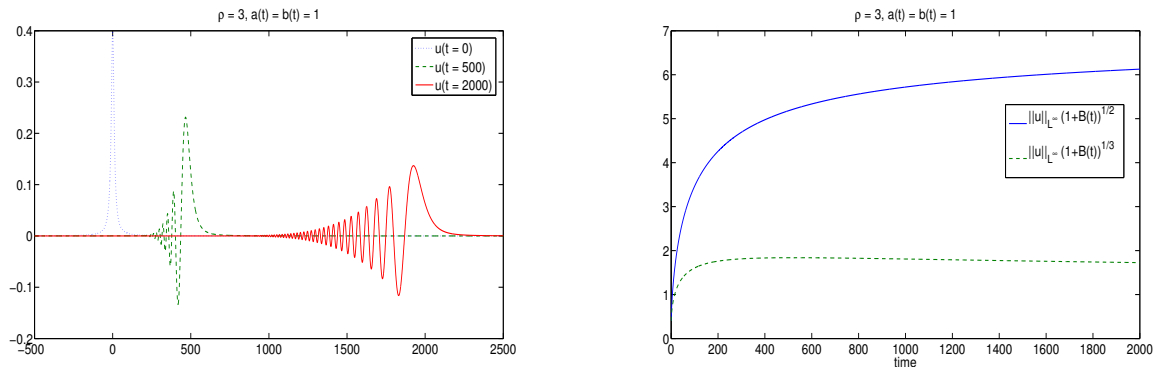


Figure 2: Result for  $\rho = 3, a(t) = b(t) = 1$ . On the left, the solution at time  $t = 0, 500, 2000$ . On the right, the  $L^\infty$ -norm.

We observe that the solution decreases and the decay rate is close to  $(1+t)^{-1/2}$ . We also note that the Fourier modes of the wave split.

The test with  $\rho = 3, a(t) = b(t) = 1/(1 + \log(1+t))$  is illustrated in Figure 3.

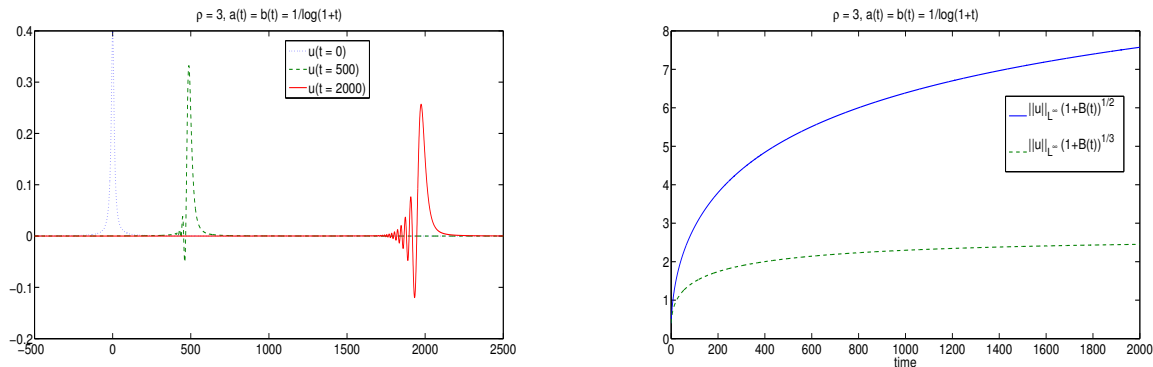


Figure 3: Result for  $\rho = 3, a(t) = b(t) = 1/(1 + \log(1+t))$ . On the left, the solution at time  $t = 0, 500, 2000$ . On the right, the  $L^\infty$ -norm.

The solution decreases slowly and the Fourier modes also disperse more slowly.

In contrast, when  $\rho = 3, a(t) = b(t) = \log(1+t)$ , the solution given by Figure 4, decreases rapidly as well as its Fourier modes.

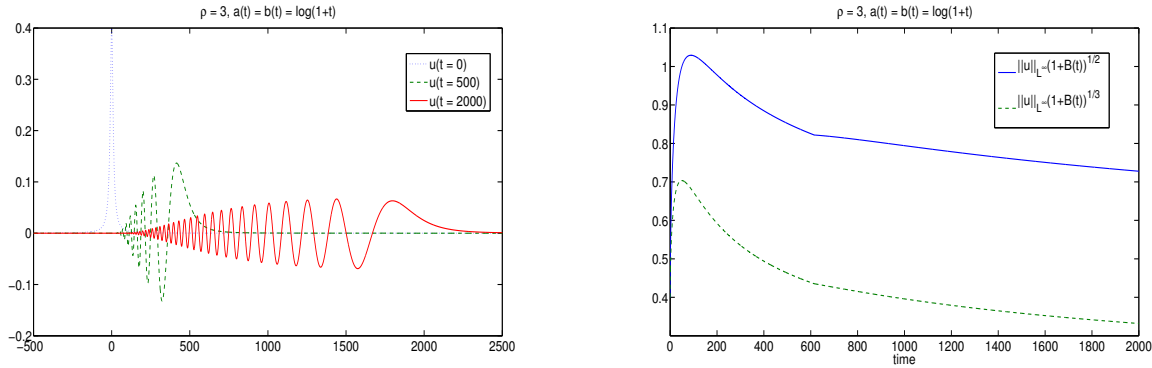


Figure 4: Result for  $\rho = 3, a(t) = b(t) = \log(1+t)$ . On the left, the solution at time  $t = 0, 500, 2000$ . On the right, the  $L^\infty$ -norm.

## References

- [1] V.I. Arnold, *Geometric Methods in the Theory of Ordinary Differential Equations*, Springer-Verlag, New York, 1983.
- [2] T.B. Benjamin, Internal waves of permanent form in fluids of great depth, *J. Fluid. Mech.* **29** (1967), 559–592.
- [3] T. B. Benjamin, J. L. Bona, and J. J. Mahony, Model equations for long waves in nonlinear dispersive systems, *Philos. Trans. Roy. Soc. London Ser. A* **272** (1972), no. 1220, 47-78.
- [4] C. Besse, A relaxation scheme for the nonlinear Schrödinger equation, *SIAM J. Numer. Anal.* **42** (2004), no. 3, 934-952.
- [5] V. Bisognin and G.P. Menzala, Asymptotic behaviour of nonlinear dispersive models with variable coefficient, *Ann. Mat. Pura Appl.*, **CLXVIII** (1995), 219-235.
- [6] V. Bisognin and G.P. Menzala, Asymptotic behaviour in time of KdV type equations with time dependent coefficients, *Appl. Math. Lett.* **7**, (1994), no. 6, 85-89.
- [7] A. Friedman, *Partial Differential Equations*, Holt, Rinehart and Winston, New York, 1969.
- [8] T. Kato and G. Ponce, Commutator estimates and the Euler and Navier-Stokes equations, *Commun. Pure Appl. Math.* **41** (1988), no. 7, 891-907.
- [9] C. E. Kenig, G. Ponce, L. Vega, On the (generalized) Korteweg-de Vries equation, *Duke Math. J.* **59** (1989), no. 3, 585–610
- [10] H. Ono, Algebraic solitary waves in stratified fluids, *J. Phys. Soc. Japan* **39** (4) (1975), 1082–1091.
- [11] R. Racke, *Lectures on nonlinear evolution equations, Initial value problems, Aspects of Mathematics*, Vieweg, 1991.
- [12] E. Stein, *Harmonic Analysis : real-variable methods, orthogonality, and oscillatory integrals*, Princeton University Press, 1993.
- [13] W. A. Strauss, Dispersion of low-energy waves for two conservative equations, *Arch. Rational Mech. Anal.* **55** (1974), 86-92.
- [14] N. Tzvetkov, Long time bounds for the periodic KP-II equation, *Int. Math. Res. Not.* (46) (2004) 2485–2496.