Well-posedness of the generalized Korteweg-de Vries-Burgers equation with nonlinear dispersion and nonlinear dissipation

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Abstract. We prove the well-posedness of the generalized Korteweg-de Vries-Burgers equation with nonlinear dispersion and nonlinear dissipation

$$u_t + f(u)_x - \delta g(u_{xx})_x - \varepsilon h(u_x)_x = 0.$$

Contrary to the linear case, the dispersion properties of the free evolution are useless and a vanishing parabolic regularization is then used.

Keywords. KdV equation, nonlinear dispersion, nonlinear dissipation, Cauchy problem.

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1 I ntroduction

Fully nonlinear equations

$$u_t + f(u)_x - \delta g(u_{xx})_x + \varepsilon u_{xxxx} = 0,$$

proposed by Brenier and Levy [5], can be viewed as a generalization of the Korteweg-de Vries-Burgers equation. When $f(u) = u^2/2, g(u) = u, \varepsilon = 0$, the equation turns to the classical KdV equation [11], which describes the propagation of the one-dimensional gravity waves in shallow water. Such nonlinear dispersion, $g(u_{xx})_x$, significantly affects the dispersive behavior of the solutions what differs completely from the linear case. In particular, Brenier and Levy obtain dissipative behavior as soon as g is a nonlinear even concave function. The nonlinear dispersion has a tendency to stabilize the solutions. It is then conjectured [5] that, for f strictly convex and g concave functions, the solution converges, when δ and $\varepsilon\delta^{-1}$ go to zero, to the unique entropy solution of the hyperbolic conservation law

$$u_t + f(u)_x = 0.$$

Contrary to the linear case, when the considered flux $f(u) = u^2/2$ and $g(u_{xx}) = u_{xx}$, the solution converges under the condition $\delta = \mathcal{O}(\varepsilon)$ [7, 14, 13, 16, 19, 20]. The study of nonlinear dispersion is also of physical interest. Rosenau and Hyman highlight notable dispersive effects and obtain a new class of compactly supported solitary waves [17, 18]. Although the literature proposed many results related with the well-posedness [2, 3, 4, 9, 12]and the vanishing limit with nonlinear viscosity and linear dispersion [6, 15], this paper is one of the first theoretical proof dealing with nonlinear dispersion [1].

In this paper, we study the initial value problem for a more general class of dissipative-dispersive hyperbolic conservation law defined by

$$u_t + f(u)_x - \delta g(u_{xx})_x - \varepsilon h(u_x)_x = 0,$$

for $x, t \in \mathbb{R}$ where h represents the dissipation satisfying

$$\int_{-\infty}^{+\infty} u_x h(u_x) \geq 0.$$

In this case, we can not take the advantage of the dispersive properties of the free evolution to obtain Strichartz type estimates [3, 12]. A fourth order regularization is applied to avoid the third space derivative of the nonlinearity g. Nevertheless, to obtain the well-posedness, a condition which links the dispersion to the dissipation is needed. For all u_0 sufficiently smooth initial data, the condition can be written as follows,

$$\int_{-\infty}^{+\infty} u_{0,5x}^2 \left(-\varepsilon h'(u_{0,x}) + C_1\delta ||u_0||_4^{\alpha_g}\right) dx \leq 0, \tag{1.1}$$

for a dispersion $|g'(u)| \leq C_g|u|^{\alpha_g}$. To keep a nonlinear dispersion, a superlinear condition $h'(u) \geq C_0 > 0$ is imposed. Then, as soon as $||u_0||_4^{\alpha_g} \lesssim \varepsilon/\delta$, the condition (1.1) is satisfied. Notice that, when g is linear, it allows to consider a large range of dissipation, the inequality (1.1) being reduced to

$$-\int_{-\infty}^{+\infty} u_{0,5x}^2 \varepsilon h'(u_{0,x}) dx \leq 0.$$

With such a dissipation, the result remains true for nonlinear dispersion of type $g(u_{xxx})$ and $g(u)_{xxx}$, including the $K(m, n)$ equations [17]. To improve this constraint on the dissipation, we consider in a second part nonlinear dispersion of type $g(u_x)_{xx}$. The inequality (1.1) becomes

$$\int_{-\infty}^{+\infty} u_{0,5x}^2 \left(-\varepsilon C_0 + C_1\delta ||u_0||_4^{\alpha_g - \alpha_h}\right) dx \leq 0.$$

and it allows us to consider dissipation in such a way $c_h|u|^{\alpha_h} \leq h'(u) \leq C_h|u|^{\alpha_h}$.

The paper is organised as follows. Section 2 deals with the well-posedness of the Cauchy problem with nonlinear dispersion of type $g(u_{xx})_x$. The fourth order regularization is introduced, then the regularization limit is obtained. In Section 3, we present the result concerning the nonlinearity $g(u_x)_{xx}$. Finally, we state the well-posedness regarding nonlinearities $g(u_{xxx})$ and $g(u)_{xxx}$.

2 Nonlinearity of type $g(u_{xx})_x$

2.1 Regularization

Let first consider the parabolic equation

$$u_t + \mu(-1)^q \partial_x^{2q} u = 0$$

with $\mu > 0$ and $q \in \mathbb{N}^*$. The semi-group is given by

$$S_t u(x) := \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{e}^{i\xi x - \mu\xi^{2q}t} \hat{u}(\xi) d\xi,$$

and satisfies the following regularization property.

Lemma 2.1 *Let $r, s \geq 0$ and $u \in H^s(\mathbb{R})$. Then for all $t \in \mathbb{R}$,*

$$||S_t u||_{r+s} \leq C_r \left(1 + \left(\frac{1}{2\mu|t|}\right)^{r/q}\right)^{1/2} ||u||_s.$$

Proof. We have

$$\begin{aligned} ||S_t u||_{r+s}^2 &\leq \int_{-\infty}^{+\infty} (1+\xi^2)^{r+s} \mathrm{e}^{-2\mu\xi^{2q}t} |\hat{u}(\xi)|^2 d\xi \\ &\leq \sup_{\xi \in \mathbb{R}} \left((1+\xi^2)^r \mathrm{e}^{-2\mu\xi^{2q}t}\right) ||u||_s^2. \end{aligned}$$

However,

$$\xi^{2r} \mathrm{e}^{-2\mu\xi^{2q}t} = \left(\frac{\xi^{2q}}{\mathrm{e}^{2\mu\xi^{2q}t/(r/q)}}\right)^{r/q} \leq \left(\frac{r}{q}\right)^{r/q} \left(\frac{1}{2\mu t}\right)^{r/q}.$$

□

To compute the well-posedness of the initial value problem, the following parabolic regularization is used

$$u_t + f(u)_x - \delta g(u_{xx})_x - \varepsilon h(u_x)_x + \mu u_{xxxx} = 0 \quad (2.1)$$

$$u(x,0) = u_0(x). \quad (2.2)$$

Lemma 2.2 *Assume that there exists $s \geq 3$ such that the functions f, g, h are locally Lipschitzian in the Sobolev space $H^s(\mathbb{R}), H^{s-2}(\mathbb{R})$ and $H^{s-1}(\mathbb{R})$ respectively, with $f(0) = g(0) = h(0) = 0$. Then there exists $T_\mu > 0$, depending on μ, such that*

$$\phi(u)(t) := S_t u_0 - \int_0^t S_{t-\tau}(f(u)_x - \delta g(u_{xx})_x - \varepsilon h(u_x)_x)(\tau) d\tau, \quad (2.3)$$

is a contraction mapping on the closed ball

$$\overline{B}(T_\mu) = \{u \in \mathcal{C}([0, T_\mu]; H^s(\mathbb{R})); ||u(t) - u_0||_s \leq 3||u_0||_s\}.$$

Moreover, there exists $C > 0$ such that the solutions u and v, with u_0 and v_0 as initial datum respectively, satisfy for $t \leq T_\mu$,

$$||u(t) - v(t)||_s \leq C||u_0 - v_0||_s.$$

Proof. Let us denote $C_f, C_g, C_h > 0$ the Lipschitz constants of the functions f, g, h respectively. Let $u, v \in \overline{B}(T_\mu)$. We have

$$\phi(u)(t) - \phi(v)(t) = -\int_0^t S_{t-\tau}((f(u)_x - f(v)_x) - \delta(g(u_{xx})_x - g(v_{xx})_x) - \varepsilon(h(u_x)_x - h(v_x)_x))(\tau) d\tau.$$

On one hand, thanks to Lemma 2.1 with $q = 2, r = 1$, we write

$$\begin{aligned} ||S_{t-\tau}(f(u)_x - f(v)_x)(\tau)||_s &= ||S_{t-\tau}(f(u)_x - f(v)_x)(\tau)||_{(s-1)+1} \\ &\leq C_1 \left(1 + \left(\frac{1}{2\mu(t-\tau)}\right)^{1/2}\right)^{1/2} ||f(u)_x - f(v)_x||_{s-1}, \end{aligned}$$

and

$$||f(u)_x - f(v)_x||_{s-1} = ||f(u) - f(v)||_s \leq C_f ||u - v||_s.$$

On the other hand, it gets with $q = 2, r = 3$

$$\begin{aligned}
||S_{t-\tau}(g(u_{xx})_x - g(v_{xx})_x)(\tau)||_s &= ||S_{t-\tau}(g(u_{xx})_x - g(v_{xx})_x)(\tau)||_{(s-3)+3} \\
&\leq C_3 \left(1 + \left(\frac{1}{2\mu(t-\tau)}\right)^{3/2}\right)^{1/2} ||g(u_{xx})_x - g(v_{xx})_x||_{s-3} \\
&\leq C_3 C_g \left(1 + \left(\frac{1}{2\mu(t-\tau)}\right)^{3/2}\right)^{1/2} ||u - v||_s,
\end{aligned}$$

and with $q = 2, r = 2$,

$$\begin{aligned}
||S_{t-\tau}(h(u_x)_x - h(v_x)_x)(\tau)||_s &= ||S_{t-\tau}(h(u_x)_x - h(v_x)_x)(\tau)||_{(s-2)+2} \\
&\leq C_2 \left(1 + \frac{1}{2\mu(t-\tau)}\right)^{1/2} ||h(u_x)_x - h(v_x)_x||_{s-2} \\
&\leq C_2 C_h \left(1 + \frac{1}{2\mu(t-\tau)}\right)^{1/2} ||u - v||_s.
\end{aligned}$$

We deduce

$$\begin{aligned}
\sup_{t\in[0,T]} ||\phi(u)(t) - \phi(v)(t)||_s &\leq C_1 C_f \int_0^t \left(1 + \left(\frac{1}{2\mu(t-\tau)}\right)^{1/2}\right)^{1/2} d\tau \sup_{t\in[0,T]} ||u - v||_s \\
&\quad + \delta C_3 C_g \int_0^t \left(1 + \left(\frac{1}{2\mu(t-\tau)}\right)^{3/2}\right)^{1/2} d\tau \sup_{t\in[0,T]} ||u - v||_s \\
&\quad + \varepsilon C_2 C_h \int_0^t \left(1 + \frac{1}{2\mu(t-\tau)}\right)^{1/2} d\tau \sup_{t\in[0,T]} ||u - v||_s \\
&\leq C(\mu, T) \sup_{t\in[0,T]} ||u - v||_s,
\end{aligned}$$

and we choose $T > 0$, proportional to μ, such that $C(\mu, T)$ is small enough to ensure the contraction mapping of ϕ in $\overline{B}(T_\mu)$.

In the same manner, it comes for $u \in \overline{B}(T_\mu)$

$$\begin{aligned}
\sup_{t\in[0,T]} ||\phi(u)(t) - u_0||_s &= \sup_{t\in[0,T]} ||(\phi(u)(t) - S_t u_0) + (S_t u_0 - u_0)||_s \\
&\leq 2||u_0||_s + C_f \widetilde{C_1} ||u_0||_s \int_0^t \left(1 + \left(\frac{1}{2\mu(t-\tau)}\right)^{1/2}\right)^{1/2} d\tau \\
&\quad + \delta C_g \widetilde{C_3} ||u_0||_s \int_0^t \left(1 + \left(\frac{1}{2\mu(t-\tau)}\right)^{3/2}\right)^{1/2} d\tau \\
&\quad + \varepsilon C_h \widetilde{C_2} ||u_0||_s \int_0^t \left(1 + \frac{1}{2\mu(t-\tau)}\right)^{1/2} d\tau \\
&\leq \widetilde{C}(\mu, T) ||u_0||_s,
\end{aligned}$$

we choose $T > 0$ such that $\widetilde{C}(\mu, T) \leq 3$ to obtain $\phi(u) \in \overline{B}(T_\mu)$.

It remains to prove the continuity with respect to the initial data. Let u and v in $\overline{B}(T_\mu)$, with u_0 and v_0 as initial datum respectively. From (2.3), it comes

$$u(t) - v(t) = S_t(u_0 - v_0) - \int_0^t S_{t-\tau}((f(u)_x - f(v)_x) - \delta(g(u_{xx})_x - g(v_{xx})_x) - \varepsilon(h(u_x)_x - h(v_x)_x))(\tau) d\tau,$$

thus

$$\sup_{t\in[0,T]} ||u(t)-v(t)||_s \leq ||u_0-v_0||_s + C(\mu,T) \sup_{t\in[0,T]} ||u(t)-v(t)||_s.$$

And as soon as $c = 1/(1-C(\mu,T)) > 0$, we have

$$\sup_{t\in[0,T]} ||u(t)-v(t)||_s \leq c||u_0-v_0||_s. \qquad \square$$

Remark 2.3 *If f, g, h are polynomial functions, the Sobolev embedding, with $s > 1/2, 3/2, 5/2$ respectively, implies that f, h, g are locally Lipschitz in $H^s(\mathbb{R})$ respectively. Indeed, suppose $f(u) = u^{\alpha_f+1}$, we have [10]*

$$\begin{aligned}
||f(u)-f(v)||_s &= \left|\left|(u-v)\sum_{i=0}^{\alpha_f} u^i v^{\alpha-i}\right|\right|_s \\
&\leq C_s\left(||u-v||_s \sum_{i=0}^{\alpha_f} ||u^i v^{\alpha-i}||_\infty + ||u-v||_\infty \sum_{i=0}^{\alpha_f} ||u^i v^{\alpha-i}||_s\right),
\end{aligned}$$

and the Sobolev embedding with $s > 1/2$ gives

$$||f(u)-f(v)||_s \leq C_s\left(\sum_{i=0}^{\alpha_f} ||u||_s^i ||v||_s^{\alpha-i}\right) ||u-v||_s.$$

2.2 Regularization limit

We wish to determine if the limit as μ goes to 0 exists. We first show that the time T_μ can be fixed independent of μ.

Proposition 2.4 *Assume that there exists $C_0, C_f, C_g, C_h > 0$ such that*

$$\begin{aligned}
|f^{(i)}(u)| &\leq C_f|u|^{\alpha_f+1-i} \text{ for } 0 \leq i \leq 2 \\
|g^{(j)}(u)| &\leq C_g|u|^{\alpha_g+1-j} \text{ for } 0 \leq j \leq 7 \\
|h^{(k)}(u)| &\leq C_h|u|^{\alpha_h+1-k} \text{ for } 0 \leq k \leq 6,
\end{aligned}$$

with

$$h'(u) \geq C_0 > 0.$$

Define

$$\alpha := \begin{cases} \min(\alpha_f, \alpha_g, \alpha_h) \text{ if } ||u||_4 < 1 \\ \max(\alpha_f, \alpha_g, \alpha_h) \text{ if } ||u||_4 \geq 1 \end{cases}$$

with $\alpha_g \geq 1$. Then there exists a constant $K > 0$ such that for $u_0 \in H^4(\mathbb{R})$ satisfying

$$||u_0||_4^{\alpha_g} \leq K\frac{\varepsilon}{\delta}, \tag{2.4}$$

the time T of well-posedness in the preceding lemma can be chosen independent of μ. Moreover, for all $t \in [-T, T]$, we have

$$||u(t)||_4 \leq 2^{1/\alpha}||u_0||_4. \tag{2.5}$$

Proof. Multiplying the equation (2.1) by $\sum_{i=0}^4(-1)^i\partial_x^{2i}u$ and integrating over space give

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}||u(t)||_4^2 + \mu\sum_{i=2}^{6}\int_{-\infty}^{+\infty}(\partial_x^i u)^2 dx &= \sum_{i=0}^{4}\int_{-\infty}^{+\infty}(-1)^{i+1}(\partial_x^{2i}u)f(u)_x dx \\
&\quad + \sum_{i=0}^{4}\int_{-\infty}^{+\infty}\delta(-1)^i(\partial_x^{2i}u)g(u_{xx})_x + \varepsilon(-1)^i(\partial_x^{2i}u)h(u_x)_x dx \\
&=: \mathrm{I} + \mathrm{II}.
\end{aligned}$$

Lemma 2.5 *There exist $C_1, C_2 > 0$ such that*

$$\mathrm{II} \quad \leq \quad C_2 \left(||u||_4^{\alpha_g+2} + ||u||_4^{\alpha_h+2} \right) + \int_{-\infty}^{+\infty} u_{5x}^2 \left(-\varepsilon h'(u_x) + C_1 \delta ||u||_4^{\alpha_g} \right) dx.$$

Proof. On the one hand,

$$\begin{aligned}
\int_{-\infty}^{+\infty} u h(u_x)_x dx &= -\int_{-\infty}^{+\infty} u_x h(u_x) dx \leq 0 \\
\int_{-\infty}^{+\infty} u_{xx} h(u_x)_x dx &= \int_{-\infty}^{+\infty} u_{xx}^2 h'(u_x) dx \\
\int_{-\infty}^{+\infty} u_{4x} h(u_x)_x dx &= \int_{-\infty}^{+\infty} -u_{xxx}^2 h'(u_x) + \frac{1}{3} u_{xx}^4 h'''(u_x) dx \\
\int_{-\infty}^{+\infty} u_{6x} h(u_x)_x dx &= \int_{-\infty}^{+\infty} u_{4x}^2 h'(u_x) - \frac{3}{2} u_{xxx}^3 h''(u_x) - \frac{9}{2} u_{xxx}^2 u_{xx}^2 h'''(u_x) + \frac{1}{5} u_{xx}^6 h^{(5)}(u_x) dx \\
\int_{-\infty}^{+\infty} u_{8x} h(u_x)_x dx &= \int_{-\infty}^{+\infty} -u_{5x}^2 h'(u_x) + 8 u_{4x}^2 u_{xxx} h''(u_x) + 8 u_{4x}^2 u_{xx}^2 h'''(u_x) - 5 u_{xxx}^4 h'''(u_x) \\
&\quad - 20 u_{xxx}^3 u_{xx}^2 h^{(4)}(u_x) - 10 u_{xxx}^2 u_{xx}^4 h^{(5)}(u_x) + \frac{1}{7} u_{xx}^8 h^{(7)}(u_x) dx.
\end{aligned}$$

And on the other hand,

$$\begin{aligned}
\int_{-\infty}^{+\infty} u g(u_{xx})_x dx &= -\int_{-\infty}^{+\infty} u_x g(u_{xx}) dx \\
\int_{-\infty}^{+\infty} u_{xx} g(u_{xx})_x dx &= -\int_{-\infty}^{+\infty} u_{xxx} g(u_{xx}) dx = [G(u_{xx}]_{-\infty}^{+\infty} = 0 \\
\int_{-\infty}^{+\infty} u_{4x} g(u_{xx})_x dx &= -\int_{-\infty}^{+\infty} \frac{1}{2} u_{xxx}^3 g''(u_{xx}) dx \\
\int_{-\infty}^{+\infty} u_{6x} g(u_{xx})_x dx &= \int_{-\infty}^{+\infty} \frac{5}{2} u_{4x}^2 u_{xxx} g''(u_{xx}) - \frac{1}{4} u_{xxx}^5 g^{(4)}(u_{xx}) dx \\
\int_{-\infty}^{+\infty} u_{8x} g(u_{xx})_x dx &= \int_{-\infty}^{+\infty} -\frac{7}{2} u_{5x}^2 u_{xxx} g''(u_{xx}) + u_{4x}^3 u_{xxx} g'''(u_{xx}) + 10 u_{4x}^2 u_{xxx}^3 g^{(4)}(u_{xx}) \\
&\qquad + \frac{1}{6} u_{xxx}^7 g^{(6)}(u_{xx}) dx.
\end{aligned}$$

Let us remind the Gagliardo-Nirenberg inequality [8]. Let $1 \leq p, q \leq \infty, 0 \leq j < m$, then there exists $C = C(p, q, j, m) > 0$ such that

$$||\partial_x^j v||_{L^r} \leq C ||\partial_x^m v||_{L^p}^a ||v||_{L^q}^{1-a},$$

where

$$\frac{1}{r} = j + a \left(\frac{1}{p} - m \right) + \frac{1-a}{q} \text{ and } \frac{j}{m} \leq a < 1.$$

In particular, we have with $p = 2, q = \infty, r = 3, j = 2, m = 3, a = 2/3$

$$||\partial_x^2 v||_{L^3}^3 \leq C ||\partial_x^3 v||_{L^2}^2 ||v||_\infty. \tag{2.6}$$

We deduce

$$\begin{aligned}\int_{-\infty}^{+\infty} u_{5x}^2 u_{xxx} g''(u_{xx})dx &\leq ||u_{xxx}g''(u_{xx})||_\infty \int_{-\infty}^{+\infty} u_{5x}^2 dx \leq C||u||_4^{\alpha_g} \int_{-\infty}^{+\infty} u_{5x}^2 dx \\ \int_{-\infty}^{+\infty} u_{4x}^3 u_{xxx} g'''(u_{xx})dx &\leq ||u_{xxx}g'''(u_{xx})||_\infty ||\partial_x^2 u_{xx}||_{L^3}^3 \leq C(||u_{xxx}||_\infty ||u_{xx}||_\infty^{\alpha_g -1})||\partial_x^3 u_{xx}||_{L^2}^2 \\ &\leq C||u||_4^{\alpha_g} \int_{-\infty}^{+\infty} u_{5x}^2 dx.\end{aligned}$$

Other terms can be bounded by the Sobolev norm $H^4(\mathbb{R})$ thanks to the Sobolev embedding. $\square$

Lemma 2.6 *Let $s > 3/2$. There exists $C_3 > 0$, depending only on s, such that*

$$< u, f(u)_x >_s \leq C_3 ||u||_s^{\alpha_f + 2}, \tag{2.7}$$

where the scalar product is defined as

$$< u, v >_s = \int_{-\infty}^{+\infty} (1+\xi^2)^s \hat{u}(\xi)\overline{\hat{v}(\xi)} d\xi.$$

Proof. We define

$$J^s(v) := \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{e}^{ix\xi}(1+\xi^2)^{s/2}\hat{v}(\xi)d\xi.$$

We remind that Kato and Ponce [10] show that there exists $C_s > 0$, depending only on s, such that

$$||[J^s, u](v)||_{L^2} \leq C_s \left(||u||_\infty ||J^{s-1}(v)||_{L^2} + ||J^s(u)||_{L^2}||v||_\infty \right), \tag{2.8}$$

where $[J^s, u](v) = J^s(uv) - uJ^s(v)$.
We deduce, since u is real valued,

$$\overline{J^s(u)} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{-ix\xi}(1+\xi^2)^{s/2}\overline{\hat{u}(\xi)}d\xi = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{-ix\xi}(1+\xi^2)^{s/2}\left(\int_{-\infty}^{+\infty} \mathrm{e}^{i\widetilde{x}\xi}u(\widetilde{x})d\widetilde{x}\right),$$

and the change of variables $\xi \to -\xi$ implies $\overline{J^s(u)} = J^s(u)$. Thus

$$< u, f(u)_x >_s = \int_{-\infty}^{+\infty} J^s(u)\overline{J^s(f(u)_x)}\, dx = \int_{-\infty}^{+\infty} J^s(u)J^s(f(u)_x)\, dx.$$

However, $f(u)_x = u_x f'(u)$ and $J^s(f(u)_x) = J^s(u_x f'(u)) = f'(u)J^s(u_x) + [J^s, f'(u)](u_x)$, thus

$$< u, f(u)_x >_s = \int_{-\infty}^{+\infty} f'(u)J^s(u_x)J^s(u)\, dx + \int_{-\infty}^{+\infty} [J^s, f'(u)](u_x)J^s(u)\, dx =: \widetilde{\mathrm{I}} + \widetilde{\mathrm{II}}.$$

On one hand, we obtain from the Sobolev embedding and using $J(u_x) = J(u)_x$

$$\begin{aligned}|\widetilde{\mathrm{I}}| &= \left|\int_{-\infty}^{+\infty} f'(u)\left(\frac{J^s(u)^2}{2}\right)_x dx\right| = \frac{1}{2}\left|\int_{-\infty}^{+\infty} u_x f''(u)J^s(u)^2\, dx\right| \\ &\leq \frac{1}{2}||u_x f''(u)||_\infty ||J^s u||_{L^2}^2 \leq C_s ||u||_s^{\alpha_f+2}.\end{aligned}$$

On the other hand, the Cauchy-Schwarz inequality provides

$$|\widetilde{\mathrm{II}}| \leq \left|\int_{-\infty}^{+\infty} [J^s, f'(u)](u_x)J^s(u)\, dx\right| \leq ||[J^s, f'(u)](u_x)||_{L^2}\, ||J^s(u)||_{L^2},$$

and the Kato-Ponce inequality (2.8) and the Sobolev embedding yield

$$|\widetilde{\mathrm{II}}| \leq C_s \left(||f'(u)||_\infty ||J^{s-1}(u_x)||_{L^2} + ||J^s(f'(u))||_{L^2}||u_x||_\infty\right) ||J^s(u)||_{L^2} \leq C_s||u||_s^{\alpha_f+2}. \qquad \square$$

We deduce from the preceding lemmata that

$$\begin{aligned} \frac{d}{dt}||u(t)||_4^2 + \mu \sum_{i=2}^{6} ||\partial_x^i u||_{L^2}^2 \leq \quad & C_3||u||_4^{\alpha_f+2} + C_2(||u||_4^{\alpha_g+2} + ||u||_4^{\alpha_h+2}) \\ & + \int_{-\infty}^{+\infty} u_{5x}^2 \left(-\varepsilon h'(u_x) + C_1\delta||u||_4^{\alpha_g}\right) dx \\ \leq \quad & C||u||_4^{\alpha+2} + \int_{-\infty}^{+\infty} u_{5x}^2 \left(-\varepsilon h'(u_x) + C_1\delta||u||_4^{\alpha_g}\right) dx, \end{aligned} \tag{2.9}$$

where

$$C = 3\max(C_2, C_3) \text{ and } \alpha = \begin{cases} \min(\alpha_f, \alpha_g, \alpha_h) \text{ if } ||u||_4 < 1 \\ \max(\alpha_f, \alpha_g, \alpha_h) \text{ if } ||u||_4 \geq 1 \end{cases}.$$

We notice that, since $h'(u) \geq C_0 > 0$,

$$\begin{aligned} &\int_{-\infty}^{+\infty} u_{0,5x}^2 \left(-\varepsilon h'(u_{0,x}) + C_1\delta||u_0||_4^{\alpha_g}\right) dx \\ &\qquad \leq \int_{-\infty}^{+\infty} u_{0,5x}^2 \varepsilon h'(u_{0,x}) \left(-1 + \frac{C_1\delta}{C_0\varepsilon}||u_0||_4^{\alpha_g}\right) dx. \end{aligned}$$

From (2.4), it gets for $K \leq C_0/C_1$,

$$\frac{C_1\delta}{C_0\varepsilon}||u_0||_4^{\alpha_g} \leq \frac{C_1K}{C_0} \leq 1,$$

thus

$$\int_{-\infty}^{+\infty} u_{0,5x}^2 \left(-\varepsilon h'(u_{0,x}) + C_1\delta||u_0||_4^{\alpha_g}\right) dx \leq 0.$$

Then we can choose $K > 0$ and $T > 0$ such for all $t \leq T$

$$\int_{-\infty}^{+\infty} u_{5x}^2 \left(-\varepsilon h'(u_x) + C_1\delta||u||_4^{\alpha_g}\right) dx \leq 0.$$

Indeed, from (2.9), we deduce that $||u(t)||_4 \leq m(t)^{1/2}$ where m is solution of

$$\begin{aligned} m'(t) &= 2Cm(t)^{(\alpha+2)/2} \\ m(0) &= ||u_0||_4^2. \end{aligned}$$

The solution of this ordinary differential equation is explicitly given by

$$m(t)^{\alpha/2} = \frac{||u_0||_4^\alpha}{1 - \alpha C||u_0||_4^\alpha t}$$

and

$$||u(t)||_4 \leq m(t)^{1/2} \leq 2^{1/\alpha}||u_0||_4 \text{ if } t \leq T = \frac{1}{2\alpha C||u_0||_4^\alpha}.$$

Then, it is enough to choose $K \leq C_0/(2^{\alpha_g/\alpha}C_1)$ to obtain

$$\frac{C_1\delta}{C_0\varepsilon}||u||_4^{\alpha_g} \leq \frac{C_1\delta}{C_0\varepsilon}2^{\alpha_g/\alpha}||u_0||_4^{\alpha_g} \leq \frac{2^{\alpha_g/\alpha}C_1K}{C_0} \leq 1. \qquad \square$$

Theorem 2.7 *Let $u_0 \in H^4(\mathbb{R})$ with*

$$||u_0||_4^{\alpha_g} \leq K\frac{\varepsilon}{\delta}.$$

There exists $T > 0$, inversely proportional to $||u_0||_4$, such that there exists a unique solution $u \in \mathcal{C}([-T,T], H^4(\mathbb{R}))$ of the initial value problem

$$\begin{array}{rcl} u_t + f(u)_x - \delta g(u_{xx})_x - \varepsilon h(u_x)_x & = & 0 \\ u(x,0) & = & u_0(x). \end{array}$$

Proof. We show that the solution $(u^\mu(t))_\mu$ is a Cauchy sequence for $t \in [0,T]$. Let $\mu, \nu \geq 0$, and u^μ, v^ν be the respective solution of (2.1)-(2.2). We have, for $t \in [0,T]$,

$$\begin{array}{rcl} \partial_t ||u^\mu - v^\nu||^2 & = & 2 < u-v, u_t - v_t > \\ & = & -2 < u-v, f(u)_x - f(v)_x > +2\delta < u-v, g(u_{xx})_x - g(v_{xx})_x > \\ & & + 2\varepsilon < u-v, h(u_x)_x - h(v_x)_x > - < u-v, \mu u_{xxxx} - \nu v_{xxxx} > . \end{array}$$

We notice that

$$\begin{array}{rcl} < u-v, \mu u_{xxxx} - \nu v_{xxxx} > & = & \mu < u-v, u_{xxxx} - v_{xxxx} > +(\mu-\nu) < u-v, v_{xxxx} > \\ < u-v, u_{xxxx} - v_{xxxx} > & = & < (u-v)_{xx}, (u-v)_{xx} > \geq 0, \end{array}$$

and

$$\begin{aligned} < u-v, f(u)_x - f(v)_x > &= \int_{-\infty}^{+\infty} -(u-v)_x (f(u)-f(v)) dx = \int_{-\infty}^{+\infty} -(u-v)_x (u-v) \left(\int_0^1 f'(z_\lambda) d\lambda \right) dx \\ &= \int_{-\infty}^{+\infty} - \left(\frac{(u-v)^2}{2} \right)_x \left(\int_0^1 f'(z_\lambda) d\lambda \right) dx = \int_{-\infty}^{+\infty} \frac{(u-v)^2}{2} \left(\int_0^1 f'(z_\lambda) d\lambda \right)_x dx, \end{aligned}$$

where $z_\lambda := (1-\lambda)u + \lambda v$. In the same way, we find

$$\begin{aligned} &\varepsilon < u-v, h(u_x)_x - h(v_x)_x > +\delta < u-v, g(u_{xx})_x - g(v_{xx})_x > \\ &\qquad = \int ((u-v)_x)^2 \left(\int_0^1 -\varepsilon h'(z_{\lambda,x}) + \frac{\delta}{2} z_{\lambda,xxx} g''(z_{\lambda,xx}) d\lambda \right) dx \leq 0, \end{aligned}$$

because $h'(z_{\lambda,x}) \geq C_0 > 0$ implies

$$\left(\int_0^1 -\varepsilon h'(z_{\lambda,x}) + \frac{\delta}{2} z_{\lambda,xxx} g''(z_{\lambda,xx}) d\lambda \right) \leq \left(\int_0^1 \varepsilon h'(z_{\lambda,x}) (-1 + \frac{\delta}{2C_0\varepsilon} ||z_{\lambda,xxx} g''(z_{\lambda,xx})||_\infty d\lambda \right)$$

and, from (2.5), as soon as $K \leq C_0/(2^{\alpha_g + \alpha_g/\alpha} C_1)$,

$$\frac{\delta}{2C_0\varepsilon} ||z_{\lambda,xxx} g''(z_{\lambda,xx})||_\infty \leq \frac{C_1\delta}{C_0\varepsilon} ||z_\lambda||_4^{\alpha_g} \leq \frac{C_1\delta}{C_0\varepsilon} (||u||_4 + ||v||_4)^{\alpha_g} \leq \frac{2^{\alpha_g + \alpha_g/\alpha} C_1 K}{C_0} \leq 1.$$

Finally, it comes

$$\begin{array}{rcl} \partial_t ||u^\mu - v^\nu||^2 & = & -2 \displaystyle\int_{-\infty}^{+\infty} \frac{(u-v)^2}{2} \left(\int_0^1 z_{\lambda,x} f''(z_\lambda) d\lambda \right) dx \\ & & -2 \displaystyle\int_{-\infty}^{+\infty} ((u-v)_x)^2 \left(\int_0^1 -\varepsilon h'(z_{\lambda,x}) + \frac{\delta}{2} z_{\lambda,xxx} g''(z_{\lambda,xx}) d\lambda \right) dx \\ & & -\mu \displaystyle\int_{-\infty}^{+\infty} ((u-v)_{xx})^2 dx + (\mu-\nu) \int_{-\infty}^{+\infty} (u-v)_{xx} v_{xx} dx \\ & \leq & \left| \displaystyle\int_{-\infty}^{+\infty} (u-v)^2 \left(\int_0^1 z_{\lambda,x} f''(z_\lambda) d\lambda \right) dx \right| + |\mu-\nu| \left| \int_{-\infty}^{+\infty} (u-v)_{xx} v_{xx} dx \right| . \end{array}$$

Denoting $M = \sup_{t\in[0,T]} m(t)$, we have from the preceding proposition $||u^\mu(t)||_4 \leq M^{1/2}$ and $||v^\nu(t)||_4 \leq M^{1/2}$. We deduce that there exists a constant $C_M > 0$, depending only on M, such that

$$\partial_t ||u^\mu - v^\nu||^2 \quad \leq \quad C_M ||u^\mu - v^\nu||^2 + C_M|\mu - \nu|.$$

The Gronwall lemma implies that $(u^\mu(t))_\mu$ is a Cauchy sequence in the complete space $L^2(\mathbb{R})$ and then it converges to a limit $u(t)$. Moreover, since $u^\mu(t)$ is continuous with respect to time and uniformly bounded by $M^{1/2}$, the sequence $(u^\mu(t))_\mu$ is also weakly convergent in $H^4(\mathbb{R})$ to the limit $u(t)$. $\square$

Remark 2.8 *We can easily improve the assumptions by setting only*

$$\begin{aligned} |f^{(2)}(u) - f^{(2)}(0)| &\leq C_f|u|^{\alpha_f - 1} \\ |g^{(7)}(u) - g^{(7)}(0)| &\leq C_g|u|^{\alpha_g - 6} \\ |h^{(6)}(u) - h^{(6)}(0)| &\leq C_h|u|^{\alpha_h - 5}. \end{aligned}$$

The inequality (2.9) becoming

$$\frac{d}{dt}||u(t)||_4^2 + \mu \sum_{i=2}^{6} ||\partial_x^i u||_{L^2}^2 \leq C(1 + ||u||_4^{\alpha+2}) + \int_{-\infty}^{+\infty} u_{5x}^2 \left(-\varepsilon h'(u_x) + C_1||u||_4^{\alpha_g}\right) dx,$$

the rest of the proof is dealt with similarly.

Remark 2.9 *The time T, proportional to $1/||u_0||_4$, is also the time well-posedness of the purely hyperbolic initial value problem.*

Remark 2.10 *Concerning the fully nonlinear dispersive equation (i.e. $\varepsilon = \mu = 0$), one can not control the sign of $z_{xxx}g''(z_{xx})$. Nevertheless, the regularized problem*

$$u_t + f(u)_x - \delta g(u_{xx})_x - \mu u_{xxx} = 0$$

remains well-posed on a time-scale inversely proportional to $||u_0||_4$ if the initial datum satisfies

$$||u_0||_4 \leq K\mu/\delta.$$

Remark 2.11 *Concerning the case of linear dispersion, same ideas provide the well-posedness of the initial value problem in $H^2(\mathbb{R})$ for a large range of dissipation.*

3 Nonlinearity of type $g(u_x)_{xx}$

To improve the assumptions concerning the dissipation, we now focus on nonlinear dispersion of type $g(u_x)_{xx}$. It allows us to regard more generalized dissipation.

As for the preceding section, we consider the regularized Cauchy problem

$$\begin{aligned} u_t + f(u)_x - \delta g(u_x)_{xx} - \varepsilon h(u_x)_x + \mu u_{xxxx} &= 0 && (3.1) \\ u(x,0) &= u_0(x). && (3.2) \end{aligned}$$

Proposition 3.1 *Assume that*

$$\begin{aligned} |f^{(i)}(u)| &\leq C_f|u|^{\alpha_f+1-i} \text{ for } 0 \leq i \leq 2 \\ |g^{(j)}(u)| &\leq C_g|u|^{\alpha_g+1-j} \text{ for } 0 \leq j \leq 8 \\ |h^{(k)}(u)| &\leq C_h|u|^{\alpha_h+1-k} \text{ for } 0 \leq k \leq 7, \end{aligned}$$

and

$$h'(u) \geq c_h |u|^{\alpha_h}.$$

Suppose that $\alpha_g \geq \alpha_h + 1$. Then there exists $K > 0$ such that for $u_0 \in H^4(\mathbb{R})$ with

$$||u_0||_4^{\alpha_g - \alpha_h} \leq K \frac{\varepsilon}{\delta},$$

there exists $T > 0$, depending only on $||u_0||_4$ and independent on μ, such that there exists a unique solution $u \in \mathcal{C}([-T, T], H^4(\mathbb{R}))$ of the initial value problem (3.1)-(3.2).
Moreover, there exists $C > 0$ such that the solutions u and v, with u_0 and v_0 as initial datum respectively, satisfy for $|t| \leq T$,

$$||u(t) - v(t)||_4 \leq C ||u_0 - v_0||_4.$$

Proof. In the exact same way that we prove Lemma (2.2), we first show there exists a unique solution $u \in \mathcal{C}([-T_\mu, T_\mu], H^4(\mathbb{R}))$ of the initial value problem (3.1)-(3.2) where T_μ depends on μ using Duhamel's formula. It remains to prove that this time can be chosen independently on μ. The equation (3.1) is multiplied by $\sum_{i=0}^4 (-1)^i \partial_x^{2i} u$ and the result is integrated over space to supply

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}||u(t)||_4^2 + \mu \sum_{i=2}^{6} \int_{-\infty}^{+\infty} (\partial_x^i u)^2 dx \quad &= \quad \sum_{i=0}^{4} \int_{-\infty}^{+\infty} (-1)^{i+1} (\partial_x^{2i} u) f(u)_x dx \\
&\quad + \sum_{i=0}^{4} \int_{-\infty}^{+\infty} \delta (-1)^i (\partial_x^{2i} u) g(u_x)_{xx} dx + \varepsilon (-1)^i (\partial_x^{2i} u) h(u_x)_x dx \\
&=: \quad \mathrm{I} + \mathrm{II}.
\end{aligned}$$

Lemma 3.2 *There exist $C_1, C_2 > 0$ such that*

$$\mathrm{II} \quad \leq \quad \int_{-\infty}^{+\infty} \varepsilon h'(u_x) u_{5x}^2 \left(-1 + \frac{C_1 \delta}{\varepsilon} ||u||_4^{\alpha_g - \alpha_h} \right) dx + C_2 (||u||_4^{\alpha_g + 2} + ||u||_4^{\alpha_h + 2}).$$

Proof. We note, on the one hand,

$$\begin{aligned}
\int_{-\infty}^{+\infty} u h(u_x)_x dx \quad &= \quad -\int_{-\infty}^{+\infty} u_x h(u_x) dx \leq 0 \\
\int_{-\infty}^{+\infty} u_{xx} h(u_x)_x dx \quad &= \quad \int_{-\infty}^{+\infty} u_{xx}^2 h'(u_x) dx \\
\int_{-\infty}^{+\infty} u_{4x} h(u_x)_x dx \quad &= \quad \int_{-\infty}^{+\infty} -u_{xxx}^2 h'(u_x) + \frac{1}{3} u_{xx}^4 h'''(u_x) dx \\
\int_{-\infty}^{+\infty} u_{6x} h(u_x)_x dx \quad &= \quad \int_{-\infty}^{+\infty} u_{4x}^2 h'(u_x) - \frac{3}{2} u_{xxx}^3 h''(u_x) - \frac{9}{2} u_{xxx}^2 u_{xx}^2 h'''(u_x) + \frac{1}{5} u_{xx}^6 h^{(5)}(u_x) dx \\
\int_{-\infty}^{+\infty} u_{8x} h(u_x)_x dx \quad &= \quad \int_{-\infty}^{+\infty} -u_{5x}^2 h'(u_x) + 8 u_{4x}^2 u_{xxx} h''(u_x) + 8 u_{4x}^2 u_{xx}^2 h'''(u_x) - 5 u_{xxx}^4 h'''(u_x) \\
&\quad - 20 u_{xxx}^3 u_{xx}^2 h^{(4)}(u_x) - 10 u_{xxx}^2 u_{xx}^4 h^{(5)}(u_x) + \frac{1}{7} u_{xx}^8 h^{(7)}(u_x) dx,
\end{aligned}$$

and on the other hand

$$\begin{aligned}
\int_{-\infty}^{+\infty} ug(u_x)_{xx}dx &= \int_{-\infty}^{+\infty} u_{xx}g(u_x)dx = [G(u_x]_{-\infty}^{+\infty} = 0\\
\int_{-\infty}^{+\infty} u_{xx}g(u_x)_{xx}dx &= \int_{-\infty}^{+\infty} \frac{1}{2}u_{xx}^3 g''(u_x)dx\\
\int_{-\infty}^{+\infty} u_{4x}g(u_x)_{xx}dx &= \int_{-\infty}^{+\infty} -\frac{5}{2}u_{xxx}^2 u_{xx}g''(u_x) + \frac{1}{4}u_{xx}^5 g^{(4)}(u_x)dx\\
\int_{-\infty}^{+\infty} u_{6x}g(u_x)_{xx}dx &= \int_{-\infty}^{+\infty} \frac{7}{2}u_{4x}^2 u_{xx}g''(u_x) - 7u_{xxx}^3 u_{xx}g'''(u_x) - 7u_{xxx}^2 u_{xx}^3 g^{(4)}(u_x) + \frac{1}{6}u_{xx}^7 g^{(6)}(u_x)dx\\
\int_{-\infty}^{+\infty} u_{8x}g(u_x)_{xx}dx &= \int_{-\infty}^{+\infty} -\frac{9}{2}u_{5x}^2 u_{xx}g''(u_x) + u_{5x}u_{4x}u_{xxx}g''(u_x) + 40u_{4x}^2 u_{xxx}u_{xx}g'''(u_x) + 15u_{4x}^2 u_{xx}^3 g^{(4)}(u_x)dx\\
&\quad - \frac{145}{4}u_{xxx}^4 u_{xx}g^{(4)}(u_x) - 45u_{xxx}^3 u_{xx}^3 g^{(5)}(u_x) - \frac{15}{2}u_{xxx}^2 u_{xx}^5 g^{(6)}(u_x) - 6u_{xxx}^2 u_{xx}^5 g^{(6)}(u_x) + \frac{1}{8}u_{xx}^9 g^{(8)}(u_x).
\end{aligned}$$

Finally, according to the Sobolev embedding,

$$\mathrm{II} \le \int_{-\infty}^{+\infty} -\varepsilon h'(u_x)u_{5x}^2 - \frac{9\delta}{2}u_{5x}^2 u_{xx}g''(u_x) + \delta u_{5x}u_{4x}u_{xxx}g''(u_x)dx + C_2(||u||_4^{\alpha_g+2} + ||u||_4^{\alpha_h+2}).$$

We have, since $\alpha_g \ge \alpha_h + 1$,

$$\int_{-\infty}^{+\infty} u_{5x}^2 u_{xx}g''(u_x)dx \le \left|\left|\frac{u_{xx}g''(u_x)}{h'(u_x)}\right|\right|_\infty \int_{-\infty}^{+\infty} h'(u_x)u_{5x}^2 dx \le C||u||_4^{\alpha_g-\alpha_h}\int_{-\infty}^{+\infty} h'(u_x)u_{5x}^2 dx$$

and from the Young inequality $2ab \le a^2 + b^2$

$$\begin{aligned}
\int_{-\infty}^{+\infty} u_{5x}u_{4x}u_{xxx}g''(u_x)dx &\le \int_{-\infty}^{+\infty} u_{5x}^2|u_{xxx}g''(u_x)|dx + \int_{-\infty}^{+\infty} u_{4x}^2|u_{xxx}g''(u_x)|dx\\
&\le \left|\left|\frac{u_{xxx}g''(u_x)}{h'(u_x)}\right|\right|_\infty \int_{-\infty}^{+\infty} h'(u_x)u_{5x}^2 dx + \int_{-\infty}^{+\infty} u_{4x}^2|u_{xxx}g''(u_x)|dx\\
&\le C||u||_4^{\alpha_g-\alpha_h}\int_{-\infty}^{+\infty} h'(u_x)u_{5x}^2 dx + C||u||_4^{\alpha_g+2}.
\end{aligned}$$

□

Lemma 2.6 provides the following inequality.

Lemma 3.3 *There exists a constant $C_4 > 0$ such that*

$$|\mathrm{I}| \le C_4||u||_4^{\alpha_f+2}.$$

We deduce from Lemmata 3.2 and 3.3

$$\frac{d}{dt}||u(t)||_4^2 + \mu\sum_{i=2}^{6}||\partial_x^i u||_{L^2}^2 \le C||u||_4^{\alpha+2} + \int_{-\infty}^{+\infty}\varepsilon h'(u_x)u_{5x}^2\left(-1 + \frac{C_1\delta}{\varepsilon}||u||_4^{\alpha_g-\alpha_h}\right)dx.$$

Then we can choose $T = 1/(2\alpha C||u_0||_4^\alpha) > 0$ such that for all $t \le T$

$$\int_{-\infty}^{+\infty}\varepsilon h'(u_x)u_{4x}^2\left(-1 + \frac{C_1\delta}{\varepsilon}||u||_4^{\alpha_g-\alpha_h}\right)dx \le 0$$

as soon as

$$\frac{C_1\delta}{\varepsilon}||u(t)||_4^{\alpha_g-\alpha_h} \le \frac{C_1\delta}{\varepsilon}2^{(\alpha_g-\alpha_h)/\alpha}||u_0||_4^{\alpha_g-\alpha_h} \le \frac{C_1 2^{(\alpha_g-\alpha_h)/\alpha}K}{\varepsilon} \le 1.$$

□

Theorem 3.4 *Let $u_0 \in H^4(\mathbb{R})$ with*

$$||u_0||_4^{\alpha_g-\alpha_h} \leq K\frac{\varepsilon}{\delta}.$$

There exists $T > 0$, inversely proportional to $||u_0||_4$, such that there exists a unique solution $u \in \mathcal{C}([-T,T], H^4(\mathbb{R}))$ of the initial value problem

$$\begin{aligned} u_t + f(u)_x - \delta g(u_x)_{xx} - \varepsilon h(u_x)_x &= 0 \\ u(x,0) &= u_0(x). \end{aligned}$$

Proof. Once again, we show that the solution $(u^\mu(t))_\mu$ is a Cauchy sequence for $t \in [0,T]$. Let $\mu, \nu \geq 0$, and u^μ, v^ν be the respective solution of (2.1)-(2.2). We have, for $t \in [0,T]$,

$$\begin{aligned} \partial_t ||u^\mu - v^\nu||^2 &= 2 < u-v, u_t - v_t > \\ &= -2 < u-v, f(u)_x - f(v)_x > + 2\delta < u-v, g(u_x)_{xx} - g(v_x)_{xx} > \\ &\quad + 2\varepsilon < u-v, h(u_x)_x - h(v_x)_x > - < u-v, \mu u_{xxxx} - \nu v_{xxxx} > . \end{aligned}$$

We see that

$$\begin{aligned} < u-v, \mu u_{xxxx} - \nu v_{xxxx} > &= \mu < u-v, u_{xxxx} - v_{xxxx} > + (\mu-\nu) < u-v, v_{xxxx} > \\ < u-v, u_{xxxx} - v_{xxxx} > &= < (u-v)_{xx}, (u-v)_{xx} > \geq 0, \end{aligned}$$

and, setting $z_\lambda = (1-\lambda)u + \lambda v$

$$< u-v, f(u)_x - f(v)_x > = -\int_{-\infty}^{+\infty} (u-v)_x (f(u) - f(v)) dx = \int_{-\infty}^{+\infty} \frac{(u-v)^2}{2} \left(\int_0^1 z_{\lambda,x} f''(z_\lambda) d\lambda \right) dx.$$

In the same way, we find

$$\begin{aligned} &\varepsilon < u-v, h(u_x)_x - h(v_x)_x > + \delta < u-v, g(u_x)_{xx} - g(v_x)_{xx} > \\ &\qquad = -\int ((u-v)_x)^2 \left(\int_0^1 \varepsilon h'(z_{\lambda,x}) + \frac{\delta}{2} z_{\lambda,xx} g''(z_{\lambda,x}) d\lambda \right) \leq 0, \end{aligned}$$

because

$$\frac{\delta}{2\varepsilon} \left|\left| \frac{z_{\lambda,xx} g''(z_{\lambda,x})}{h'(z_{\lambda,x})} \right|\right|_\infty \leq \frac{C_1 \delta}{\varepsilon} ||z_\lambda||_3^{\alpha_g - \alpha_h} \leq \frac{2^{(\alpha_g-\alpha_h)+(\alpha_g-\alpha_h)/\alpha} C_1 K}{\varepsilon} \leq 1.$$

Finally, we obtain

$$\begin{aligned} \partial_t ||u^\mu - v^\nu||^2 &= -2 \int_{-\infty}^{+\infty} \frac{(u-v)^2}{2} \left(\int_0^1 z_{\lambda,x} f''(z_\lambda) d\lambda \right) dx \\ &\quad -2 \int_{-\infty}^{+\infty} ((u-v)_x)^2 \left(\int_0^1 \varepsilon h'(z_{\lambda,x}) + \frac{\delta}{2} z_{\lambda,xx} g''(z_{\lambda,x}) d\lambda \right) dx \\ &\quad -\mu \int_{-\infty}^{+\infty} ((u-v)_{xx})^2 dx + (\mu - \nu) \int_{-\infty}^{+\infty} (u-v)_{xx} v_{xx} dx \\ &\leq \left| \int_{-\infty}^{+\infty} (u-v)^2 \left(\int_0^1 z_{\lambda,x} f''(z_\lambda) d\lambda \right) dx \right| + |\mu - \nu| \left| \int_{-\infty}^{+\infty} (u-v)_{xx} v_{xx} dx \right| . \end{aligned}$$

We have $||u^\mu(t)||_4 \leq M^{1/2}$ and $||v^\nu(t)||_4 \leq M^{1/2}$. Then we deduce that there exists a constant $C_M > 0$, depending only on M, such that

$$\partial_t ||u^\mu - v^\nu||^2 \leq C_M ||u^\mu - v^\nu||^2 + C_M |\mu - \nu|.$$

The Gronwall lemma implies that $(u^\mu(t))_\mu$ is a Cauchy sequence in the complete space $L^2(\mathbb{R})$ and then it converges to a limit $u(t)$. Moreover, since $u^\mu(t)$ is continuous with respect to time and uniformly bounded by $M^{1/2}$, the sequence $(u^\mu(t))_\mu$ is also weakly convergent in $H^4(\mathbb{R})$ to the limit $u(t)$. $\square$

Remark 3.5 *When $h'(u) \geq C_0 > 0$, the H^3-regularity is enough to obtain the well-posedness with the nonlinear dispersion $g(u_x)_{xx}$.*

4 Nonlinearities of type $g(u_{xxx})$ and $g(u)_{xxx}$

In a similar manner, we can study nonlinear dispersions $g(u_{xxx})$ and $g(u)_{xxx}$. The proofs are sketched.

Theorem 4.1 *Assume that there exists $C_0, C_f, C_g, C_h > 0$ such that*

$$\begin{aligned} |f^{(i)}(u)| &\leq C_f|u|^{\alpha_f+1-i} \text{ for } 0 \leq i \leq 2 \\ |g^{(j)}(u)| &\leq C_g|u|^{\alpha_g+1-j} \text{ for } 0 \leq j \leq 6 \\ |h^{(k)}(u)| &\leq C_h|u|^{\alpha_h+1-j} \text{ for } 0 \leq k \leq 6, \end{aligned}$$

with

$$h'(u) \geq C_0 > 0,$$

and $\alpha_g \geq 1$. Then there exists $K > 0$ such that for $u_0 \in H^7(\mathbb{R})$ satisfying

$$||u_0||_7^{\alpha_g} \leq K\frac{\varepsilon}{\delta},$$

there exists $T > 0$, inversely proportional to $||u_0||_7$, such that there exists a unique solution $u \in \mathcal{C}([-T,T], H^7(\mathbb{R}))$ of the initial value problem

$$\begin{aligned} u_t + f(u)_x - \delta g(u_{xxx}) - \varepsilon h(u_x)_x &= 0 \\ u(x,0) &= u_0(x). \end{aligned}$$

Moreover, for all $t \in [-T,T]$, we have

$$||u(t)||_7 \leq C||u_0||_7.$$

Proof. Let $n \in \mathbb{N}^*$. Multiplying the equation

$$u_t + f(u)_x - \delta g(u_{xxx}) - \varepsilon h(u_x)_x + \mu u_{xxxx} = 0$$

by $\sum_{i=0}^n (-1)^i \partial_x^{2i} u$ and integrating over space give

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}||u(t)||_n^2 + \mu\sum_{i=2}^{n+2}\int_{-\infty}^{+\infty}(\partial_x^i u)^2 dx &= \sum_{i=0}^{n}\int_{-\infty}^{+\infty}(-1)^{i+1}(\partial_x^{2i}u)f(u)_x dx \\ &+ \sum_{i=0}^{n}\int_{-\infty}^{+\infty}\delta(-1)^i(\partial_x^{2i}u)g(u_{xxx}) + \varepsilon(-1)^i(\partial_x^{2i}u)h(u_x)_x dx. \end{aligned}$$

The Leibniz rule points to

$$\begin{aligned} \int_{-\infty}^{+\infty}(-1)^n(\partial_x^{2n}u)g(u_{xxx})dx &= -\int_{-\infty}^{+\infty}(\partial_x^{n+1}u)\partial_x^{n-1}g(u_{xxx})dx = -\int_{-\infty}^{+\infty}(\partial_x^{n+1}u)\partial_x^{n-2}(u_{4x}g'(u_{xxx}))dx \\ &= -\int_{-\infty}^{+\infty}(\partial_x^{n+1}u)\sum_{j=0}^{n-2}\begin{pmatrix} n-2 \\ j \end{pmatrix}(\partial_x^{n-2-j}u_{4x})(\partial_x^j g'(u_{xxx}))dx \\ &= -\int_{-\infty}^{+\infty}(\partial_x^{n+1}u)\sum_{j=0}^{n-2}\begin{pmatrix} n-2 \\ j \end{pmatrix}(\partial_x^{n+2-j}u)(\partial_x^j g'(u_{xxx}))dx. \end{aligned}$$

We focus on the high order space derivatives. The other derivatives can be included in the Sobolev norm $H^n(\mathbb{R})$ using the Gagliardo-Nirenberg inequality and the Sobolev embedding. Then, we obtain from $j = 0, 1, 2$

$$\begin{aligned}
\int_{-\infty}^{+\infty} (\partial_x^{n+1}u)(\partial_x^{n+2}u)g'(u_{xxx})dx &= \int_{-\infty}^{+\infty} -\frac{(\partial_x^{n+1}u)^2}{2}u_{4x}g''(u_{xxx})dx \\
\int_{-\infty}^{+\infty} (\partial_x^{n+1}u)(\partial_x^{n+1}u)\partial_x(g'(u_{xxx}))dx &= \int_{-\infty}^{+\infty} (\partial_x^{n+1}u)^2 u_{4x}g''(u_{xxx})dx \\
\int_{-\infty}^{+\infty} (\partial_x^{n+1}u)(\partial_x^{n}u)\partial_x^2(g'(u_{xxx}))dx &= \int_{-\infty}^{+\infty} -\frac{(\partial_x^{n}u)^2}{2}\partial_x^3(g'(u_{xxx}))dx \\
&= \int_{-\infty}^{+\infty} -\frac{(\partial_x^{n}u)^2}{2}\left(u_{6x}g''(u_{xxx}) + 3u_{5x}u_{4x}g'''(u_{xxx}) + u_{4x}^3 g^{(4)}(u_{xxx})\right)dx.
\end{aligned}$$

We notice that, if $n \geq 7$,

$$\begin{aligned}
\left|\int_{-\infty}^{+\infty} (\partial_x^{n+1}u)^2 u_{4x}g''(u_{xxx})dx\right| = ||u_{4x}g''(u_{xxx})||_\infty \int_{-\infty}^{+\infty} (\partial_x^{n+1}u)^2 dx &\leq C||u||_n^{\alpha_g}\int_{-\infty}^{+\infty} (\partial_x^{n+1}u)^2 dx \\
\left|\int_{-\infty}^{+\infty} \frac{(\partial_x^{n}u)^2}{2}\left(u_{6x}g''(u_{xxx}) + 3u_{5x}u_{4x}g'''(u_{xxx}) + u_{4x}^3 g^{(4)}(u_{xxx})\right)dx\right| &\leq C||u||_n^{\alpha_g+2}.
\end{aligned}$$

Same equalities hold for $j = n-1, n-2, n-3$ and for h. Finally, the inequality (2.9) is now written as

$$\frac{d}{dt}||u(t)||_7^2 \leq C||u||_7^{\alpha+2} + \int_{-\infty}^{+\infty} u_{8x}^2\left(-\varepsilon h'(u_x) + C_7||u||_7^{\alpha_g}\right)dx,$$

and the rest of the proof is dealt with similarly as the previous theorems. □

Theorem 4.2 *Assume that there exists $C_0, C_f, C_g, C_h > 0$ such that*

$$\begin{aligned}
|f^{(i)}(u)| &\leq C_f|u|^{\alpha_f+1-i} \text{ for } 0 \leq i \leq 2 \\
|g^{(j)}(u)| &\leq C_g|u|^{\alpha_g+1-j} \text{ for } 0 \leq j \leq 3 \\
|h^{(k)}(u)| &\leq C_h|u|^{\alpha_h+1-j} \text{ for } 0 \leq k \leq 6,
\end{aligned}$$

with

$$h'(u) \geq C_0 > 0,$$

and $\alpha_g \geq 1$. Then there exists $K > 0$ such that for $u_0 \in H^4(\mathbb{R})$ with

$$||u_0||_4^{\alpha_g} \leq K\frac{\varepsilon}{\delta},$$

there exists $T > 0$, inversely proportional to $||u_0||_4$, such that there exists a unique solution $u \in \mathcal{C}([-T,T], H^4(\mathbb{R}))$ of the initial value problem

$$\begin{aligned}
u_t + f(u)_x - \delta g(u)_{xxx} - \varepsilon h(u_x)_x &= 0 \\
u(x,0) &= u_0(x).
\end{aligned}$$

Moreover, for all $t \in [-T,T]$, we have

$$||u(t)||_4 \leq C||u_0||_4.$$

Proof. Let $n \in \mathbb{N}^*$. The equation

$$u_t + f(u)_x - \delta g(u)_{xxx} - \varepsilon h(u_x)_x + \mu u_{xxxx} = 0$$

is multiplied by $\sum_{i=0}^{n}(-1)^i \partial_x^{2i} u$ and by integrating over space, we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}||u(t)||_n^2 + \mu \sum_{i=2}^{n+2} \int_{-\infty}^{+\infty} (\partial_x^i u)^2 dx &= \sum_{i=0}^{n} \int_{-\infty}^{+\infty} (-1)^{i+1} (\partial_x^{2i} u) f(u)_x dx \\
&+ \sum_{i=0}^{n} \int_{-\infty}^{+\infty} \delta (-1)^i (\partial_x^{2i} u) g(u)_{xxx} + \varepsilon (-1)^i (\partial_x^{2i} u) h(u_x)_x dx.
\end{aligned}$$

The Leibniz rule implies

$$\begin{aligned}
\int_{-\infty}^{+\infty} (-1)^n (\partial_x^{2n} u) g(u)_{xxx} &= -\int_{-\infty}^{+\infty} (\partial_x^{n+1} u) \partial_x^{n-1} (g(u)_{xxx}) \\
= -\int_{-\infty}^{+\infty} (\partial_x^{n+1} u) & \sum_{j=0}^{n-1} \left(\begin{array}{c} n-1 \\ j \end{array} \right) \partial_x^{n-1-j} (u_{xxx}) (\partial_x^j g'(u)) \\
& + 3\partial_x^{n-1-j} (u_{xx} u_x) (\partial_x^j g''(u)) + \partial_x^{n-1-j} (u_x^3) (\partial_x^j g'''(u)).
\end{aligned}$$

As before, to control the norm of the derivatives with the Sobolev norm $H^n(\mathbb{R})$, we need $n \geq 4$, the greatest orders being such that

$$\begin{aligned}
\left| \int_{-\infty}^{+\infty} (\partial_x^{n+1} u)^2 u_{3x} g''(u) dx \right| \leq ||u_{3x} g''(u)||_\infty \int_{-\infty}^{+\infty} (\partial_x^{n+1} u)^2 dx &\leq C||u||_n^{\alpha_g} \int_{-\infty}^{+\infty} (\partial_x^{n+1} u)^2 dx \\
\left| \int_{-\infty}^{+\infty} (\partial_x^n u)^2 u_{3x} g''(u) dx \right| &\leq C||u||_n^{\alpha_g + 2}.
\end{aligned}$$

Finally, it comes

$$\frac{d}{dt}||u(t)||_4^2 \leq C||u||_4^{\alpha+2} + \int_{-\infty}^{+\infty} u_{5x}^2 \left(-\varepsilon h'(u_x) + C_4 ||u||_4^{\alpha_g} \right) dx. \qquad \square$$

Remark 4.3 *Regarding the nonlinear dispersions $g(u_{xxx})$ and $g(u)_{xxx}$, it could be possible to reduce the regularity of the initial datum by writing more precisely the derivatives appearing the integrations by parts. For example, to apply the Leibniz rule with the nonlinear dispersion $g(u_{xx})_x$ gives $n = 6$ whereas $n = 4$ is enough.*

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