Carleman estimates and unique continuation property for the Kadomtsev–Petviashvili equations

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Abstract. We study the unique continuation property for the generalized Kadomtsev-Petviashvili equations and its regularized version. We use Carleman estimates to prove that if the solution of the Kadomtsev-Petviashvili equations vanishes in an open subset, then this solution is identically equal to zero in the horizontal component of the open subset.

Keywords. KP equations, BBM trick, Carleman estimates, UCP, Treves' inequality.

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Introduction

The small amplitude long waves in shallow water moving mainly in the $x-$direction are described by the Kadomtsev-Petviashvili equations [10]

$$(u_t + u_x + \gamma u_{xxx} + uu_x)_x + u_{yy} = 0,$$

called KP-I if $\gamma = -1$ and KP-II if $\gamma = 1$, depending on whether the surface tension is negligible or not. These equations are obtained by correcting the transport equation $u_t + u_x = O(\varepsilon)$ to second order. Thus, by replacing the space-derivative of order 3, we obtain the following KP-BBM equations [3, 11]

$$(u_t + u_x - \gamma u_{xxt} + uu_x)_x + u_{yy} = 0.$$

It is noted that the equation is ill-posed in $L^2(\mathbb{R}^2)$ if $\gamma = -1$ because $(1 + \partial_x^2)$ is not invertible in $L^2(\mathbb{R}^2)$. The purpose of this work is to prove a unique continuation property. More precisely, we show that if $u = u(x, y, t)$ is solution of the equations and u vanishes on an open subset Ω of $\mathbb{R}^2 \times \mathbb{R}$, then u vanishes identically on the horizontal component $\Omega_h := \{(x, y, t) \in \mathbb{R}^2 \times \mathbb{R}; \exists (x_1, y_1) \text{ with } (x_1, y_1, t) \in \Omega\}$ of Ω. Carleman estimates can be used. These estimates are based on estimates with exponential weight for the solution of the equation. More precisely, if u is solution of $\mathcal{L}u = Vu$, with $\mathcal{L}$ a linear operator, V a well-defined potential, the Carleman estimates is written, for Ψ a convex function and $\tau > 0$ to choose,

$$||\mathrm{e}^{\tau\Psi(x)}u|| \leq C||\mathrm{e}^{\tau\Psi(x)}\mathcal{L}u||.$$

Such a result was shown by Saut and Scheurer [15] for a general class of dispersive equations, including the Korteweg-de Vries one.
An alternative approach was suggested by Bourgain [1]. The method here is based on an analytic continuation of the Fourier transform via the theorem of Paley-Wiener. Therefore Panthee proved a unique continuation property of the solution of KP-II [14]. The method of Bourgain is not suited to KP-I and KP-BBM. Indeed, it depends on the relation dispersion σ which has to satisfy $\forall\, R>0, \exists \max(|k|,|l|)>R$ such that

$$\frac{\partial\sigma}{\partial k}(k,l) \geq f(k,l) \text{ with } \lim_{|k|+|l|\to\infty} f(k,l) = +\infty.$$

In the case of the Kadomtsev-Petviashvili equations, we obtain for KP and KP-BBM-II respectively

$$\begin{aligned} \sigma(k,l) &= k-\gamma k^3 + l^2/k,\ \frac{\partial\sigma}{\partial k}(k,l) = 1-3\gamma k^2 - l^2/k^2, \\ \sigma(k,l) &= \frac{k+l^2/k}{1+k^2},\ \frac{\partial\sigma}{\partial k}(k,l) = \frac{(1-k^2)k^2 - l^2(1+3k^2)}{k^2(1+k^2)}. \end{aligned}$$

Surprisingly and unexpectedly, one can prove a unique continuation property using only integrations by parts. The simplicity of the proof is paid by a restrictive condition on the support of the solution [5, 12]. We propose here to prove a unique continuation property using a Carleman estimate. We are inspired by a work of Davila and Menzala [6] who proved a unique continuation property of the one-dimensional Benjamin-Bona-Mahony equation. The first section is devoted to the solution of KP-BBM-II equation. We establish a Carleman estimate and then the unique continuation property. The second section deals with the Carleman estimate corresponding to the solution of the KP equation.

1 Unique continuation property for the regularized KP equations

We consider the initial value problem, for $(x,y)\in\mathbb{R}^2, t\in\mathbb{R}, \rho\geq 1$ an integer,

$$\begin{aligned} (u_t + u_x - u_{xxt} + u^\rho u_x)_x + u_{yy} &= 0 \\ u(x,y,0) &= u_0(x,y). \end{aligned}$$

We denote $X^s(\mathbb{R}^2) := \{u \in H^s(\mathbb{R}^2); \partial_x^{-1}u \in H^s(\mathbb{R}^2)\}$ equipped with the norm $|u|_s = ||u||_s + ||\partial_x^{-1}u||_s$, where ∂_x^{-1} is defined by the Fourier symbol $\widehat{\partial_x^{-1}u}(\xi) = \hat{u}(\xi)/(i\xi)$. The existence and uniqueness of local in time solution reads as follows [11].

Theorem 1.1
Let $u_0 \in X^s(\mathbb{R}^2), s>2$. There exist $T = T(|u_0|_s^{-1}) > 0$ and a unique solution $u \in \mathcal{C}^1([-T,T]; X^s(\mathbb{R}^2))$ of the KP-BBM equation.

The existence and uniqueness of global in time solution for $\rho = 1$ is due to Saut and Tzvetkov [16].

1.1 Carleman estimates

Let us start by recalling the Treves' inequality [17].

Theorem 1.2
Let $P = P(D)$ be a differential operator of order m with constant coefficients. Then for all $\alpha\in\mathbb{N}^n, \xi\in\mathbb{R}^n$ and $\Phi\in\mathcal{C}_0^\infty(\mathbb{R}^n)$,

$$\frac{2^{|\alpha|}}{\alpha!}\xi^{2\alpha}\int_{\mathbb{R}^n} |P^{(\alpha)}(D)\Phi|^2 \mathrm{e}^{\Psi(X,\xi)}dX \leq C(m,\alpha)\int_{\mathbb{R}^n}|P(D)\Phi|^2\mathrm{e}^{\Psi(X,\xi)}dX,$$

with

$$\Psi(X,\xi)=\sum_{j=1}^{n}X_j^2\xi_j^2, |\alpha|=\sum_{j=1}^{n}\alpha_j, \alpha!=\alpha_1!\dots\alpha_n! \text{ and } C(m,\alpha)=\left\{\begin{array}{ll}\sup\limits_{|r+\alpha|\le m}\begin{pmatrix} r+\alpha\\ \alpha\end{pmatrix} & \text{if } |\alpha|\le m\\ 0 & \text{if } |\alpha|>m\end{array}\right. .$$

This inequality can be rewritten in two dimension.

Corollary 1.3
Let $P=P(\partial_x,\partial_y,\partial_t)$ be a differential operator of order m with constant coefficients. Then for all $\alpha=(\alpha_1,\alpha_2,\alpha_3)\in\mathbb{N}^3,\delta>0,\tau>0,\Phi\in\mathcal{C}_0^\infty(\mathbb{R}^3)$ and $\Psi(x,y,t)=(x-\delta)^2/2+(y-\delta)^2/2+\delta^2t^2$,

$$\frac{2^{|\alpha|+\alpha_3}\tau^{|\alpha|}\delta^{2\alpha_3}}{\alpha!}\int_{\mathbb{R}^3}|P^{(\alpha)}(D)\Phi|^2\mathrm{e}^{2\tau\Psi}dxdydt\le C(m,\alpha)\int_{\mathbb{R}^3}|P(D)\Phi|^2\mathrm{e}^{2\tau\Psi}dxdydt.$$

Proof. The Treves' inequality is applied with $X=(x-\delta,y-\delta,t),\xi=(\sqrt{\tau},\sqrt{\tau},\sqrt{2\tau}\delta)$ and $\Psi(X,\xi)=2\tau((x-\delta)^2/2+(y-\delta)^2/2+\delta^2t^2)$. □

Now we prove a Carleman estimate for the solution of KP-BBM.

Proposition 1.4
We define

$$\mathcal{L}:=\partial_{xt}-\partial_{xxxt}+c_1\partial_{xx}+c_2\partial_{xxxx}+c_3\partial_{xy}+c_4\partial_{xxxy}+\partial_{yy}+f_1(x,y,t)\partial_x+f_2(x,y,t)\partial_{xx},$$

where c_1,c_2,c_3,c_4 are constant in $\mathbb{R}$, $f_1,f_2\in L^\infty(\mathbb{R}^3)$. Let $\delta>0$ and $B_\delta:=\{(x,y,t)\in\mathbb{R}^3;x^2+y^2+t^2<\delta^2\}$. Then, there exists $C>0$ such that for all $\Phi\in\mathcal{C}_0^\infty(B_\delta)$, $\Psi(x,y,t)=(x-\delta)^2/2+(y-\delta)^2/2+\delta^2t^2$ and $\tau>0$ with

$$\frac{||f_1||_\infty^2}{\tau^3\delta^2}+\frac{||f_2||_\infty^2}{\tau^2\delta^2}\left(1+\frac{1}{\tau^2}\right)\le\frac{1}{4},$$

we have

$$\begin{aligned}\tau^4\delta^2\int_{B_\delta}|\Phi|^2\mathrm{e}^{2\tau\Psi}dxdydt+\tau^3\delta^2\int_{B_\delta}|\Phi_x|^2\mathrm{e}^{2\tau\Psi}dxdydt&\\ +C\tau^2\delta^2\int_{B_\delta}|\Phi_{xx}|^2\mathrm{e}^{2\tau\Psi}dxdydt &\le C\int_{B_\delta}|\mathcal{L}\Phi|^2\mathrm{e}^{2\tau\Psi}dxdydt.\end{aligned}\tag{1.1}$$

Proof. We define the differential operator

$$P:=-(\partial_{xt}-\partial_{xxxt}+c_1\partial_{xx}+c_2\partial_{xxxx}+c_3\partial_{xy}+c_4\partial_{xxxy}+\partial_{yy}).$$

The Fourier transform gives for $(\xi_1,\xi_2,\tau)\in\mathbb{R}^3$

$$\widehat{P}(\xi_1,\xi_2,\tau)=\xi_1\tau+\xi_1^3\tau+c_1\xi_1^2-c_2\xi_1^4+c_3\xi_1\xi_2-c_4\xi_1^3\xi_2+\xi_2^2.$$

Lemma 1.5
For all $\Phi\in\mathcal{C}_0^\infty(B_\delta)$, $\Psi(x,y,t)=(x-\delta)^2/2+(y-\delta)^2/2+\delta^2t^2$ and $\tau>0$, we have

$$\begin{aligned}(\tau^4-\tau^2)\delta^2\int_{B_\delta}|\Phi|^2\mathrm{e}^{2\tau\Psi}dxdydt+\tau^3\delta^2\int_{B_\delta}|\Phi_x|^2\mathrm{e}^{2\tau\Psi}dxdydt&\\ +\tau^2\delta^2\int_{B_\delta}|\Phi_{xx}|^2\mathrm{e}^{2\tau\Psi}dxdydt &\le 3\int_{B_\delta}|P\Phi|^2\mathrm{e}^{2\tau\Psi}dxdydt.\end{aligned}$$

Proof. With the same notations of Treves' inequality, we have for $\alpha = (3, 0, 1)$

$$\frac{\widehat{P}^{|\alpha|}(\xi_1, \xi_2, \tau)}{\partial \xi_1^3 \partial \tau} = 6, P^{|\alpha|}\Phi = 6\Phi, C(4, (3, 0, 1)) = \sup_{|r+\alpha|\leq 4} \left(\begin{array}{c} r+\alpha \\ \alpha \end{array} \right) = 1.$$

Applying the Treves' inequality with P and α gives

$$\frac{2^5\tau^4\delta^2}{6} \int_{\mathbb{R}^3} |6\Phi|^2 \mathrm{e}^{2\tau\Psi} dxdydt \leq \int_{\mathbb{R}^3} |P\Phi|^2 \mathrm{e}^{2\tau\Psi} dxdydt,$$

what implies

$$\tau^4\delta^2 \int_{\mathbb{R}^3} |\Phi|^2 \mathrm{e}^{2\tau\Psi} dxdydt \leq \int_{\mathbb{R}^3} |P\Phi|^2 \mathrm{e}^{2\tau\Psi} dxdydt.$$

In the same way, we obtain for $\alpha = (2, 0, 1)$

$$\frac{\widehat{P}^{|\alpha|}(\xi_1, \xi_2, \tau)}{\partial \xi_1^2 \partial \tau} = 6\xi_1, P^{|\alpha|}\Phi = -6i\Phi_x, C(3, (2, 0, 1)) = 3,$$

and the Treves' inequality provides

$$\tau^3\delta^2 \int_{\mathbb{R}^3} |\Phi_x|^2 \mathrm{e}^{2\tau\Psi} dxdydt \leq \int_{\mathbb{R}^3} |P\Phi|^2 \mathrm{e}^{2\tau\Psi} dxdydt.$$

Finally, with $\alpha = (1, 0, 1)$, one gets

$$\frac{\widehat{P}^{|\alpha|}(\xi_1, \xi_2, \tau)}{\partial \xi_1 \partial \tau} = 1 + 3\xi_1^2, P^{|\alpha|}\Phi = \Phi - 3\Phi_{xx}, C(4, (1, 0, 1)) = 2,$$

$$\frac{2^3\tau^2\delta^2}{1} \int_{\mathbb{R}^3} |\Phi - 3\Phi_{xx}|^2 \mathrm{e}^{2\tau\Psi} dxdydt \leq 2 \int_{\mathbb{R}^3} |P\Phi|^2 \mathrm{e}^{2\tau\Psi} dxdydt.$$

On the other hand, we have $|\Phi - 3\Phi_{xx}|^2 \geq -2\Phi^2 + 6\Phi_{xx}^2$. We deduce

$$\tau^2\delta^2 \int_{B_\delta} (|\Phi_{xx}|^2 - |\Phi|^2) \mathrm{e}^{2\tau\Psi} dxdydt \leq \int_{B_\delta} |P\Phi|^2 \mathrm{e}^{2\tau\Psi} dxdydt,$$

thus

$$\tau^2\delta^2 \int_{B_\delta} |\Phi_{xx}|^2 \mathrm{e}^{2\tau\Psi} dxdydt \leq \left(1 + \frac{1}{\tau^2}\right) \int_{B_\delta} |P\Phi|^2 \mathrm{e}^{2\tau\Psi} dxdydt.$$

$\square$

Lemma 1.6
We have

$$\int_{B_\delta} |f_1\, \Phi_x|^2 \mathrm{e}^{2\tau\Psi} dxdydt + \int_{B_\delta} |f_2\, \Phi_{xx}|^2 \mathrm{e}^{2\tau\Psi} dxdydt \leq \left(\frac{||f_1||_\infty^2}{\tau^3\delta^2} + \frac{||f_2||_\infty^2}{\tau^2\delta^2} \left(1 + \frac{1}{\tau^2}\right) \right)$$

$$\times \left(\int_{B_\delta} |\mathcal{L}\Phi|^2 \mathrm{e}^{2\tau\Psi} dxdydt + \int_{B_\delta} |f_1\, \Phi_x|^2 \mathrm{e}^{2\tau\Psi} dxdydt + \int_{B_\delta} |f_2\, \Phi_{xx}|^2 \mathrm{e}^{2\tau\Psi} dxdydt \right).$$

Proof. We have thanks to the lemma 1.5

$$\begin{aligned}
\int_{B_\delta} |f_1(x,y,t)\Phi_x|^2 \mathrm{e}^{2\tau\Psi} dxdydt &\leq ||f_1||_\infty^2 \int_{B_\delta} |\Phi_x|^2 \mathrm{e}^{2\tau\Psi} dxdydt \\
&\leq \frac{||f_1||_\infty^2}{\tau^3\delta^2} \int_{B_\delta} |P\Phi|^2 \mathrm{e}^{2\tau\Psi} dxdydt \\
&\leq \frac{2||f_1||_\infty^2}{\tau^3\delta^2} \int_{B_\delta} \left(|\mathcal{L}\Phi|^2 + |f_1\,\Phi_x|^2 + |f_2\Phi_{xx}|^2\right) \mathrm{e}^{2\tau\Psi} dxdydt.
\end{aligned}$$

In the same manner, it gets

$$\begin{aligned}
\int_{B_\delta} |f_2(x,y,t)\Phi_{xx}|^2 \mathrm{e}^{2\tau\Psi} dxdydt &\leq ||f_2||_\infty^2 \int_{B_\delta} |\Phi_{xx}|^2 \mathrm{e}^{2\tau\Psi} dxdydt \\
&\leq \frac{||f_2||_\infty^2}{\tau^2\delta^2} \left(1 + \frac{1}{\tau^2}\right) \int_{B_\delta} |P\Phi|^2 \mathrm{e}^{2\tau\Psi} dxdydt \\
&\leq \frac{2||f_2||_\infty^2}{\tau^2\delta^2} \left(1 + \frac{1}{\tau^2}\right) \int_{B_\delta} (|\mathcal{L}\Phi|^2 + |f_1\,\Phi_x|^2 + |f_2\,\Phi_{xx}|^2) \mathrm{e}^{2\tau\Psi} dxdydt.
\end{aligned}$$

□

To conclude, it is enough to choose $\tau > 0$ large enough with

$$\frac{2||f_1||_\infty^2}{\tau^3\delta^2} + \frac{2||f_2||_\infty^2}{\tau^2\delta^2}\left(1 + \frac{1}{\tau^2}\right) \leq \frac{1}{2}.$$

□

Corollary 1.7
Let $T > 0$. If $\Phi \in \mathcal{C}([-T,T]; H^4(\mathbb{R}^2)), \Phi_{xt} \in \mathcal{C}([-T,T]; H^2(\mathbb{R}^2))$ and $\operatorname{supp}\Phi \subseteq B_\delta$, then the inequality (1.1) remains true.

Proof. The proof is done by regularization. □

1.2 Unique continuation property

The unique continuation property is now proven. The proof is similar to the one-dimensional case of the paper of Davila and Menzala [6].

Lemma 1.8
Let $T > 0, s \geq 4, f_1, f_2 \in L^\infty(\mathbb{R}^2 \times [-T,T]), c_1, c_2, c_3, c_4 \in \mathbb{R}$. Let $u \in \mathcal{C}^1([-T,T]; H^s(\mathbb{R}^2))$ be the solution of $\mathcal{L}u = 0$. Assume that $u \equiv 0$ when $x < |t|$ and $y < |t|$ in a neighborhood of $(0,0,0)$. Then there exists a neighborhood of $(0,0,0)$ in which $u \equiv 0$.

Remark 1.9 *If $u \in \mathcal{C}^1([-T,T]; H^s(\mathbb{R}^2))$ is solution of $\mathcal{L}u = 0$, since*

$$(1 - \partial_x^2)u_{xt} = -(u_{xx} + (u^\rho u_x)_x + u_{yy}),$$

then $u_{xt} \in \mathcal{C}([-T,T]; H^{s-2}(\mathbb{R}^2))$. Therefore the Carleman estimate (1.1) holds if $s \geq 4$.

Proof. Let $0 < \delta < 1$, choose $\chi \in \mathcal{C}_0^\infty(B_\delta)$ such that $\chi = 1$ in a neighborhood $\mathcal{O}_1$ of $(0,0,0)$ and define $\Phi := \chi u$. It follows that $\Phi \in \mathcal{C}([-T,T]; H^s(\mathbb{R}^2)), \Phi_{xt} \in \mathcal{C}([-T,T]; H^{s-2}(\mathbb{R}^2))$ and $\operatorname{supp}\Phi \subseteq B_\delta$. We deduce thanks to the preceding corollary that there exists $C > 0$ such that, for $\tau > 0$ large enough,

$$\tau^4\delta^2 \int_{B_\delta} |\Phi|^2 \mathrm{e}^{2\tau\Psi} dxdydt \quad \leq \quad C \int_{B_\delta} |\mathcal{L}\Phi|^2 \mathrm{e}^{2\tau\Psi} dxdydt. \tag{1.2}$$

The right hand side integral holds on $B_\delta \backslash \mathcal{O}_1$, since $\mathcal{L}\Phi = 0$ in $\mathcal{O}_1$.
For $(x,y,t) \neq 0$ in $\operatorname{supp}\Phi$, we have $0 \leq |t| \leq x < \delta < 1, 0 \leq |t| \leq y < \delta < 1$ and

$$\Psi(x,y,t) \quad = \quad (x-\delta)^2/2 + (y-\delta)^2/2 + \delta^2 t^2 < (|t|-\delta)^2 + \delta^2 t^2 < \delta^2 \text{ and } \Psi(0,0,0) = \delta^2.$$

Then for $(x,y,t) \in \operatorname{supp}\mathcal{L}\Phi \subseteq B_\delta$, there exists $0 < \varepsilon < \delta^2$ such that $\Psi(x,y,t) \leq \delta^2 - \varepsilon$. On the other hand, we can choose $\mathcal{O}_2$ a neighborhood of $(0,0,0)$ with $\Psi(x,y,t) > \delta^2 - \varepsilon$. in $\mathcal{O}_2$. The inequality (1.2) is then written for all $\tau > 0$ large enough

$$\tau^4\delta^2 \mathrm{e}^{2\tau(\delta^2-\varepsilon)} \int_{\mathcal{O}_2} |\Phi|^2 dxdydt \leq C\mathrm{e}^{2\tau(\delta^2-\varepsilon)} \int_{B_\delta\backslash\mathcal{O}_1} |\mathcal{L}\Phi|^2 dxdydt.$$

Tending τ to infinity implies Φ vanishes in $\mathcal{O}_2$. However $u = \Phi$ in $\mathcal{O}_2 \subseteq \mathcal{O}_1$ and $u \equiv 0$ in $\mathcal{O}_2$. □

Corollary 1.10
Let $T > 0, s \geq 4, A, B \in L^\infty(\mathbb{R}^2 \times [-T,T])$. Let $u \in \mathcal{C}^1([-T,T]; H^s(\mathbb{R}^2))$ be solution of

$$u_{xt} - u_{xxxt} + A(x,y,t)u_{xx} + B(x,y,t)u_x + u_{yy} = 0.$$

We consider the surface $(x,y) = \mu(t) := (\mu_1(t), \mu_2(t)), \mu(0) = (0,0), \mu$ a continuously differential function in a neighborhood of $(0,0,0)$. Assume that $u \equiv 0$ when $x < \mu_1(t)$ and $y < \mu_2(t)$ in a neighborhood of $(0,0,0)$. Then there exists a neighborhood of $(0,0,0)$ in which $u \equiv 0$.

Proof. We consider the change of variables $(x,y,t) \longrightarrow (X,Y,T)$ with

$$\begin{aligned} X &= x - \mu_1(t) + |t| \\ Y &= y - \mu_2(t) + |t| \\ T &= t. \end{aligned}$$

This change of variables provides $U = U(X,Y,T)$ satisfying $U \equiv 0$ when $X < |T|$ and $Y < |T|$ in a neighborhood of $(0,0,0)$ and $\mathcal{L}U = 0$ with

$$\begin{aligned} \mathcal{L} \; := \; & \partial_{XT} + (-\mu_1'(T) + \operatorname{sgn}(T))\partial_{XX} + (-\mu_2'(T) + \operatorname{sgn}(T))\partial_{XY} - \partial_{XXXT} - (-\mu_1'(T) + \operatorname{sgn}(T))\partial_{XXXX} \\ & + (-\mu_2'(T) + \operatorname{sgn}(T))\partial_{XXXY} + A\partial_{XX} + B\partial_X + \partial_{YY}. \end{aligned}$$

□

Theorem 1.11
Let $T > 0, s \geq 4$ and $u \in \mathcal{C}^1([-T,T]; H^s(\mathbb{R}^2))$ be solution of the KP-BBM equation. If $u \equiv 0$ in an open subset $\Omega \subseteq \mathbb{R}^2 \times [-T,T]$, then $u \equiv 0$ in the horizontal component of Ω.

Proof. The proof follows [13] or [6] applying the preceding corollary with $A = (1 + u^\rho)$ and $B = u_x$. Thanks to the Sobolev embedding with $s > 2$, A and B belong to $L^\infty(\mathbb{R}^2 \times [-T,T])$. □

2 Unique continuation property for the KP equations

We consider the initial value problem, for $(x,y)\in\mathbb{R}^2, t\in\mathbb{R}, \gamma=\pm1, \rho\geq 1$ an integer,

$$\begin{aligned}(u_t+u_x+\gamma u_{xxx}+u^\rho u_x)_x+u_{yy} &= 0\\ u(x,y,0) &= u_0(x,y).\end{aligned}$$

Theorem 2.1 *[7]*
Let $u_0\in X^s(\mathbb{R}^2), s>2$. There exist $T=T(|u_0|_s^{-1})$ and a unique solution $u\in\mathcal{C}^1([-T,T];X^s(\mathbb{R}^2))$ of the KP equation.

The existence and uniqueness of global in time solution for $\rho=1$ and $\gamma=1$ is due to Bourgain [2].

2.1 Carleman estimates

Proposition 2.2
We define

$$\mathcal{L}:=\partial_{xt}+\gamma\partial_{xxxx}+c_1\partial_{xx}+c_2\partial_{xy}+\partial_{yy}+f_1(x,y,t)\partial_x+f_2(x,y,t)\partial_{xx},$$

where c_1,c_2 are constant in $\mathbb{R}$, $f_1,f_2\in L^\infty(\mathbb{R}^3)$. Let $\delta>0$ and $B_\delta:=\{(x,y,t)\in\mathbb{R}^3; x^2+y^2+t^2<\delta^2\}$. Then, there exists $C>0$ such that for all $\Phi\in\mathcal{C}_0^\infty(B_\delta)$, $\Psi(x,y,t)=(x-\delta)^2/2+(y-\delta)^2/2+\delta^2t^2$ and $\tau>0$ with

$$\frac{||f_1||_\infty^2}{\tau\delta^2}+\frac{||f_2||_\infty^2(1+c_1^2/\delta^2)}{\tau^2}\leq\frac{1}{4},$$

we have

$$\begin{aligned}\tau^2\delta^2\int_{B_\delta}|\Phi|^2\mathrm{e}^{2\tau\Psi}dxdydt+\tau\delta^2\int_{B_\delta}|\Phi_x|^2\mathrm{e}^{2\tau\Psi}dxdydt&\\ +\tau^2\int_{B_\delta}|\Phi_{xx}|^2\mathrm{e}^{2\tau\Psi}dxdydt &\leq C\int_{B_\delta}|\mathcal{L}\Phi|^2\mathrm{e}^{2\tau\Psi}dxdydt. \qquad (2.1)\end{aligned}$$

Proof. We define the differential operator

$$P:=-(\partial_{xt}+c_1\partial_{xx}+c_2\partial_{xy}+\gamma\partial_{xxxx}+\partial_{yy}).$$

The Fourier transform gives for $(\xi_1,\xi_2,\tau)\in\mathbb{R}^3$

$$\widehat{P}(\xi_1,\xi_2,\tau)=\xi_1\tau+c_1\xi_1^2+c_2\xi_1\xi_2-\gamma\xi_1^4+\xi_2^2.$$

Lemma 2.3
For all $\Phi\in\mathcal{C}_0^\infty(B_\delta)$, $\Psi(x,y,t)=(x-\delta)^2/2+(y-\delta)^2/2+\delta^2t^2$ and $\tau>0$, we have

$$\begin{aligned}(\tau^2\delta^2-c_1\tau^2)\int_{B_\delta}|\Phi|^2\mathrm{e}^{2\tau\Psi}dxdydt+\tau\delta^2\int_{B_\delta}|\Phi_x|^2\mathrm{e}^{2\tau\Psi}dxdydt&\\ +\tau^2\int_{B_\delta}|\Phi_{xx}|^2\mathrm{e}^{2\tau\Psi}dxdydt &\leq 3\int_{B_\delta}|P\Phi|^2\mathrm{e}^{2\tau\Psi}dxdydt.\end{aligned}$$

Proof. With the same notations of Treves' inequality, we have for $\alpha=(1,0,1)$

$$\frac{\widehat{P}^{|\alpha|}(\xi_1,\xi_2,\tau)}{\partial\xi_1\partial\tau}=1, P^{|\alpha|}\Phi=\Phi, C(4,(1,0,1))=\sup_{|r+\alpha|\leq 4}\begin{pmatrix} r+\alpha\\ \alpha\end{pmatrix}=2.$$

Applying the Treves' inequality with P and α gives

$$\frac{2^3\tau^2\delta^2}{1}\int_{\mathbb{R}^3}|\Phi|^2 \mathrm{e}^{2\tau\Psi}dxdydt \leq 2\int_{\mathbb{R}^3}|P\Phi|^2\mathrm{e}^{2\tau\Psi}dxdydt,$$

what implies

$$\tau^2\delta^2\int_{\mathbb{R}^3}|\Phi|^2 \mathrm{e}^{2\tau\Psi}dxdydt \leq \int_{\mathbb{R}^3}|P\Phi|^2\mathrm{e}^{2\tau\Psi}dxdydt.$$

In the same way, we obtain for $\alpha=(0,0,1)$

$$\frac{\widehat{P}^{|\alpha|}(\xi_1,\xi_2,\tau)}{\partial\tau}=\xi_1, P^{|\alpha|}\Phi=-i\Phi_x, C(4,(0,0,1))=4,$$

$$\tau\delta^2\int_{\mathbb{R}^3}|\Phi_x|^2 \mathrm{e}^{2\tau\Psi}dxdydt \leq \int_{\mathbb{R}^3}|P\Phi|^2\mathrm{e}^{2\tau\Psi}dxdydt.$$

Finally, with $\alpha=(2,0,0)$, it gets

$$\frac{\widehat{P}^{|\alpha|}(\xi_1,\xi_2,\tau)}{\partial\xi_1^2}=2c_1-12\gamma\xi_1^2, P^{|\alpha|}\Phi=2c_1\Phi-12\gamma\Phi_{xx}, C(4,(2,0,0))=3,$$

$$\frac{2^4\tau^2}{2}\int_{\mathbb{R}^3}|2c_1\Phi-12\gamma\Phi_{xx}|^2 \mathrm{e}^{2\tau\Psi}dxdydt \leq 3\int_{\mathbb{R}^3}|P\Phi|^2\mathrm{e}^{2\tau\Psi}dxdydt.$$

On the other hand, we have $|2c_1\Phi-12\gamma\Phi_{xx}|^2\geq -140c_1^2\Phi^2+143\Phi_{xx}^2$. We deduce

$$\tau^2\int_{B_\delta}(|\Phi_{xx}|^2-c_1^2|\Phi|^2)\mathrm{e}^{2\tau\Psi}dxdydt \leq \int_{B_\delta}|P\Phi|^2\mathrm{e}^{2\tau\Psi}dxdydt,$$

thus

$$\tau^2\int_{B_\delta}|\Phi_{xx}|^2\mathrm{e}^{2\tau\Psi}dxdydt \leq \left(1+\frac{c_1^2}{\delta^2}\right)\int_{B_\delta}|P\Phi|^2\mathrm{e}^{2\tau\Psi}dxdydt.$$

□

Lemma 2.4
We have

$$\int_{B_\delta}|f_1\,\Phi_x|^2\mathrm{e}^{2\tau\Psi}dxdydt+\int_{B_\delta}|f_2\,\Phi_{xx}|^2\mathrm{e}^{2\tau\Psi}dxdydt \leq \left(\frac{2||f_1||_\infty^2}{\tau\delta^2}+\frac{2||f_2||_\infty^2(1+c_1^2/\delta^2)}{\tau^2}\right)$$
$$\times\left(\int_{B_\delta}|\mathcal{L}\Phi|^2\mathrm{e}^{2\tau\Psi}dxdydt+\int_{B_\delta}|f_1\,\Phi_x|^2\mathrm{e}^{2\tau\Psi}dxdydt+\int_{B_\delta}|f_2\,\Phi_{xx}|^2\mathrm{e}^{2\tau\Psi}dxdydt\right).$$

Proof. We have thanks to the lemma 2.3

$$\begin{aligned}
\int_{B_\delta}|f_1(x,y,t)\Phi_x|^2\mathrm{e}^{2\tau\Psi}dxdydt &\leq ||f_1||_\infty^2\int_{B_\delta}|\Phi_x|^2\mathrm{e}^{2\tau\Psi}dxdydt\\
&\leq \frac{||f_1||_\infty^2}{\tau\delta^2}\int_{B_\delta}|P\Phi|^2\mathrm{e}^{2\tau\Psi}dxdydt\\
&\leq \frac{2||f_1||_\infty^2}{\tau\delta^2}\int_{B_\delta}\left(|\mathcal{L}\Phi|^2+|f_1\,\Phi_x|^2+|f_2\Phi_{xx}|^2\right)\mathrm{e}^{2\tau\Psi}dxdydt.
\end{aligned}$$

In the same manner, it gets

$$\begin{aligned}\int_{B_\delta} |f_2(x,y,t)\Phi_{xx}|^2 \mathrm{e}^{2\tau\Psi} dxdydt &\le ||f_2||_\infty^2 \int_{B_\delta} |\Phi_{xx}|^2 \mathrm{e}^{2\tau\Psi} dxdydt \\ &\le \frac{||f_2||_\infty^2 (1 + c_1^2/\delta^2)}{\tau^2} \int_{B_\delta} |P\Phi|^2 \mathrm{e}^{2\tau\Psi} dxdydt \\ &\le \frac{2||f_2||_\infty^2 (1 + c_1^2/\delta^2)}{\tau^2} \int_{B_\delta} (|\mathcal{L}\Phi|^2 + |f_1\,\Phi_x|^2 + |f_2\,\Phi_{xx}|^2) \mathrm{e}^{2\tau\Psi} dxdydt.\end{aligned}$$

□

To conclude, it is enough to choose $\tau > 0$ large enough with

$$\frac{2||f_1||_\infty^2}{\tau\delta^2} + \frac{2||f_2||_\infty^2 (1 + c_1^2/\delta^2)}{\tau^2} \le \frac{1}{2}.$$

□

Corollary 2.5
Let $T > 0$. If $\Phi \in \mathcal{C}([-T,T]; H^4(\mathbb{R}^2)), \Phi_{xt} \in \mathcal{C}([-T,T]; L^2(\mathbb{R}^2))$ and $\operatorname{supp}\Phi \subseteq B_\delta$, then the inequality (2.1) remains true.

2.2 Unique continuation property

Theorem 2.6
Let $T > 0, s \ge 4$ and $u \in \mathcal{C}([-T,T]; H^s(\mathbb{R}^2))$ solution of the KP equation. If $u \equiv 0$ in an open subset $\Omega \subseteq \mathbb{R}^2 \times [-T,T]$, then $u \equiv 0$ in the horizontal component of Ω.

Remark 2.7 *If $u \in \mathcal{C}^1([-T,T]; H^s(\mathbb{R}^2))$ is solution of the KP equation, since*

$$u_{xt} = -(u_{xx} + \gamma u_{xxxx} + (uu_x)_x + u_{yy}),$$

then $u_{xt} \in \mathcal{C}([-T,T]; H^{s-4}(\mathbb{R}^2))$. Therefore the Carleman estimate (1.1) holds if $s \ge 4$.

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