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ON UNIQUENESS AND NON-DEGENERACY OF ANISOTROPIC POLARONS

JULIEN RICAUD

ABSTRACT. We study the anisotropic Choquard–Pekar equation which describes a polaron in an anisotropic medium. We prove the uniqueness and non-degeneracy of minimizers in a weakly anisotropic medium. In addition, for a wide range of anisotropic media, we derive the symmetry properties of minimizers and prove that the kernel of the associated linearized operator is reduced, apart from three functions coming from the translation invariance, to the kernel on the subspace of functions that are even in each of the three principal directions of the medium.

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1. INTRODUCTION

A *polaron* describes a quantum electron in a polar crystal. The atoms of the crystal are displaced due to the electrostatic force induced by the charge of the electron, and the resulting deformation is then felt by the electron itself. This coupled system (the electron and its polarization cloud) is a quasi-particle, called a polaron.

When the polaron extends over a domain much larger than the characteristic length of the underlying lattice, the crystal can be approximated by a continuous polarizable medium, leading to the so-called Pekar nonlinear model [13, 14]. In this theory, the energy functional is

$$\mathcal{E}^V(\psi) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \psi(x)|^2 dx - \frac{1}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |\psi(y)|^2 |\psi(x)|^2 V(x-y) dy dx, \quad (1.1)$$

where ψ is the wave function of the electron, in units such that the vacuum permittivity, the mass, and the charge of the electron are all normalized to one:

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$4\pi\epsilon_0 = m_{e^-} = e = 1$. While, on the other hand,

$$-V \star |\psi|^2(x) = - \int_{\mathbb{R}^3} |\psi(y)|^2 V(x-y) dy$$

is the mean-field self-trapping potential felt by the electron.

For an isotropic and homogeneous medium, characterized by its *relative permittivity* (or *relative dielectric constant*) $\varepsilon_M \geq 1$, the effective interaction potential is

$$V(x) = \frac{1 - \varepsilon_M^{-1}}{|x|}. \quad (1.2)$$

For $\varepsilon_M > 1$, the so-called *Choquard–Pekar* or *Schrödinger–Newton* equation

$$\left(-\frac{\Delta}{2} - V \star |\psi|^2 \right) \psi = -\mu \psi \quad (1.3)$$

is obtained by minimizing the energy $\mathcal{E}^V$ in (1.1) under the constraint $\int_{\mathbb{R}^3} |\psi|^2 = 1$, with associated Lagrange multiplier $\mu > 0$. Lieb proved in [9] the uniqueness of minimizers, up to space translations and multiplication by a phase factor. This ground state Q is positive, smooth, radial decreasing, and has an exponential decay at infinity. That Q is also the unique positive solution to (1.3) was proved in [12].

In 2009, Lenzmann proved in [7] that Q is nondegenerate. Namely, the linearization of (1.3),

$$\mathfrak{L}_Q \xi = -\frac{1}{2} \Delta \xi + \mu \xi - (V \star |Q|^2) \xi - 2Q (V \star (Q\xi)), \quad (1.4)$$

has the trivial kernel

$$\ker_{L^2(\mathbb{R}^3)} \mathfrak{L}_Q = \text{span} \{ \partial_{x_1} Q, \partial_{x_2} Q, \partial_{x_3} Q \} \quad (1.5)$$

which stems from the translation invariance. This nondegeneracy result is an important property which is useful in implicit function type arguments. Uniqueness and nondegeneracy were originally used in [7] to study a pseudo-relativistic model, and then in [5, 11, 18, 20, 4, 19] for other models.

The purpose of this paper is to study the case of *anisotropic* media, for which the corresponding potential is

$$V(x) = \frac{1}{|x|} - \frac{1}{\sqrt{\det(M^{-1})} |M^{1/2} x|}, \quad 0 < M \leq 1, \quad (1.6)$$

where $M^{-1} \geq 1$ is the (real and symmetric) static dielectric matrix of the medium. The mathematical expression is simpler in the Fourier domain:

$$\hat{V}(k) = 4\pi \left(\frac{1}{|k|^2} - \frac{1}{k^T M^{-1} k} \right).$$

The form of the potential V in the anisotropic case is well-known in the physics literature and it has recently been derived by Lewin and Rougerie from a microscopic model of quantum crystals in [8].

From a technical point of view, the fact that V in (1.6) is a difference of two Coulomb type potentials complicates the analysis. For this reason, we will also consider a simplified anisotropic model where V is replaced by

$$V(x) = \frac{1}{|(1-S)^{-1}x|}, \quad 0 \leq S < 1, \quad (1.7)$$

and S is also a real and symmetric matrix. This simplified potential can be seen as an approximation of the potential (1.6) in the weakly anisotropic regime, that is, when M is close to an homothecy.

In this paper, we derive several properties of minimizers of $\mathcal{E}^V$ and of positive solutions to the nonlinear equation (1.3), when V is given by formulas (1.6) and (1.7).

After some preparations in Section 2, we quickly discuss the existence of minimizers and the compactness of minimizing sequences in Section 3. Then, based on the fundamental non degeneracy result by Lenzmann [7], we prove in Section 4 the uniqueness and non-degeneracy of minimizers in a weakly anisotropic material. In Section 5, considering back general anisotropic materials, we investigate the symmetry properties of minimizers using rearrangement inequalities. Finally we discuss the linearized operator in Section 6. By using Perron–Frobenius type arguments, we are able to prove that for ψ a positive solution of the so-called *Choquard–Pekar* equation (1.3) sharing the symmetry properties of V , we have

$$\ker \mathfrak{L}_\psi = \text{span} \{ \partial_x \psi, \partial_y \psi, \partial_z \psi \} \bigoplus \ker (\mathfrak{L}_\psi)|_{L^2_{\text{sym}}(\mathbb{R}^3)}. \quad (1.8)$$

Where $L^2_{\text{sym}}(\mathbb{R}^3)$ is the subspace of function in $L^2(\mathbb{R}^3)$ sharing the symmetry properties of V . For instance, in the general case where the three eigenvalues of M (or S) are distinct from each other and V is decreasing in the corresponding directions, $L^2_{\text{sym}}(\mathbb{R}^3)$ is the subspace of functions that are even in these directions. On the other hand, if exactly two eigenvalues are equal, it is the subspace of cylindrical functions that are also even in the directions of the principal axis.

The main difficulty in proving (1.8) is that the operator $\mathfrak{L}_\psi$ is non-local and therefore the ordering of its eigenvalues is not obvious. The next step would be to prove that $\ker \mathfrak{L}_\psi|_{L^2_{\text{sym}}(\mathbb{R}^3)} = \{0\}$ which we only know for now in the weakly anisotropic regime (Theorem 7 below) and in the radial case (see [7]). We hope to come back to this problem in the future.

2. ELEMENTARY PROPERTIES

We define the energy $\mathcal{E}^V$ as in (1.1) and consider, for all $\lambda > 0$, the minimization problem

$$I^V(\lambda) := \min_{\substack{\psi \in H^1(\mathbb{R}^3) \\ \|\psi\|_2^2 = \lambda}} \mathcal{E}^V(\psi). \quad (2.1)$$

Let (e_1, e_2, e_3) be the principal axis of the medium, that is, such that each $e_i \in \mathbb{R}^3$ is a normalized eigenvector associated with the eigenvalue m_i of the real symmetric matrix M , where $0 < m_1 \leq m_2 \leq m_3 \leq 1$ (we demand, in addition, that at least $m_2 < 1$), or associated with the eigenvalue s_i of the real symmetric matrix S where $0 \leq s_3 \leq s_2 \leq s_1 < 1$ in the simplified model.

We define the map $M \mapsto V$ as

$$\{0 < M \leq 1 \mid M \text{ symmetric real}\} \rightarrow L^2(\mathbb{R}^3) + L^4(\mathbb{R}^3) \\ M \mapsto V(x) = \frac{1}{|x|} - \frac{1}{\sqrt{\det(M^{-1})}|M^{1/2}x|} \quad (2.2)$$

with, in particular, $M \equiv \text{Id} \mapsto V \equiv \bar{0}$ which corresponds to the vacuum. And, in the simplified model, $S \mapsto V$ is defined as

$$\{0 \leq M < 1 \mid M \text{ symmetric real}\} \rightarrow L^2(\mathbb{R}^3) + L^4(\mathbb{R}^3) \\ S \mapsto V(x) = |(1 - S)^{-1}x|^{-1} \quad (2.3)$$

with, in particular, $S \equiv 0 \mapsto V \equiv V_0$. We denote the isotropic potentials by $V_c(x) = (1 - c)|x|^{-1}$, for $0 \leq c \leq 1$, and I_c^V the associated minimization problem.

Both maps are well-defined. Indeed, let V as in (2.2) or (2.3) then one can easily show that there exist $a > b \geq 0$ such that

$$\forall x \in \mathbb{R}_*^3, \quad 0 \leq b|x|^{-1} \leq V(x) \leq a|x|^{-1} \leq |x|^{-1}. \quad (2.4)$$

Consequently, $V \in L^2(\mathbb{R}^3) + L^4(\mathbb{R}^3)$. Moreover, if we restrict ourselves to $0 < M < 1$ then there exist $a > b > 0$ such that

$$\forall x \in \mathbb{R}_*^3, \quad 0 < b|x|^{-1} \leq V(x) \leq a|x|^{-1} \leq |x|^{-1}. \quad (2.5)$$

Lemma 1. *Let $M \mapsto V$ be defined as in (2.2), $S \mapsto V$ as in (2.3) and let f, g be two functions in $H^1(\mathbb{R}^3)$. Then $V \star (fg) \in W^{1,\infty}$ and, for any $0 < \alpha < 1$, we have*

(1) *locally Lipschitzity of*

$$\begin{aligned} \{\alpha < M \leq 1 \mid M \text{ symmetric real}\} \times H^1 \times H^1 &\rightarrow W^{1,\infty} \\ (M, f, g) &\mapsto V \star (fg), \end{aligned}$$

(2) *uniform Lipschitzity of*

$$\begin{aligned} \{0 \leq S < \alpha \mid S \text{ symmetric real}\} \times H^1 \times H^1 &\rightarrow W^{1,\infty} \\ (S, f, g) &\mapsto V \star (fg). \end{aligned}$$

Proof of Lemma 1. First, for any $f \in L^2(\mathbb{R}^3)$ and $g \in H^1(\mathbb{R}^3)$, by (2.4) together with Hardy inequality, $|V \star (fg)(x)| \leq (|\cdot|^{-1} \star |fg|)(x) \leq 2\|f\|_2 \|\nabla g\|_2$ holds. Consequently, for any $f, g \in H^1(\mathbb{R}^3)$, we have

$$\begin{aligned} \|V \star (fg)\|_{W^{1,\infty}} &\leq \|V \star (fg)\|_\infty + \|V \star (g\nabla f)\|_\infty + \|V \star (f\nabla g)\|_\infty \\ &\leq 2\|f\|_2 \|\nabla g\|_2 + 4\|\nabla f\|_2 \|\nabla g\|_2 \\ &\leq 6\|f\|_{H^1} \|g\|_{H^1}. \end{aligned}$$

Thus $V \star (fg)$ is in $W^{1,\infty}$. For the rest of the proof, we denote by $\|M\|$ the spectral norm of M and fix an α such that $0 < \alpha < 1$.

For $(S, T) \in \{0 \leq M < \alpha \mid M \text{ symmetric real}\}^2$, $f \in L^2(\mathbb{R}^3)$, $g \in H^1(\mathbb{R}^3)$ and $x \in \mathbb{R}^3$, we have

$$\begin{aligned} |(V_S - V_T) \star (fg)(x)| &\leq \left| \frac{|(1-T)^{-1} \cdot| - |(1-S)^{-1} \cdot|}{|(1-S)^{-1} \cdot| |(1-T)^{-1} \cdot|} \star |fg|(x) \right| \\ &\leq \frac{|[(1-T)^{-1} - (1-S)^{-1}] \cdot|}{|\cdot|^2} \star |fg|(x) \\ &\leq \frac{|(1-S)^{-1}(T-S)(1-T)^{-1} \cdot|}{|\cdot|^2} \star |fg|(x) \\ &\leq \|(1-S)^{-1}\| \|T-S\| \|(1-T)^{-1}\| \frac{1}{|\cdot|} \star |fg|(x) \\ &\leq 2(1-\alpha)^{-2} \|f\|_2 \|\nabla g\|_2 \|S-T\|. \end{aligned}$$

Thus, for any $f, g \in H^1(\mathbb{R}^3)$, we have

$$\|(V_S - V_T) \star (fg)\|_{W^{1,\infty}} \leq 6(1-\alpha)^{-2} \|f\|_{H^1} \|g\|_{H^1} \|S-T\|,$$

which concludes the proof of (2).

For $(M, N) \in \{\alpha < M \leq 1 \mid M \text{ symmetric real}\}^2$, we have

$$\begin{aligned} M^{1/2} - N^{1/2} &= \pi^{-1} \int_0^\infty \left(\frac{M}{s+M} - \frac{N}{s+N} \right) \frac{ds}{\sqrt{s}} \\ &= \pi^{-1} \int_0^\infty \frac{1}{s+M} (M-N) \frac{1}{s+N} \sqrt{s} ds, \end{aligned}$$

which leads to

$$\left\| M^{\frac{1}{2}} - N^{\frac{1}{2}} \right\| \leq \frac{\|M-N\|}{\pi} \int_0^\infty \frac{\sqrt{s}}{(s+\alpha)^2} ds = \frac{\|M-N\|}{\pi\sqrt{\alpha}} \int_0^\infty \frac{\sqrt{s}}{(s+1)^2} ds = \frac{\|M-N\|}{2\sqrt{\alpha}}.$$

Moreover, with a similar computation and since $\det M, \det N > \alpha^3$, we obtain

$$\left| \sqrt{\det M} - \sqrt{\det N} \right| \leq \frac{|\det M - \det N|}{2\alpha^{3/2}}.$$

Thus, for $f \in L^2(\mathbb{R}^3)$, $g \in H^1(\mathbb{R}^3)$ and $x \in \mathbb{R}^3$, we have

$$\begin{aligned} |(V_M - V_N) \star (fg)(x)| &\leq \frac{1}{\sqrt{\det N^{-1}}} |fg| \star \frac{\left| |M^{1/2} \cdot| - |N^{1/2} \cdot| \right|}{|M^{1/2} \cdot| |N^{1/2} \cdot|} (x) \\ &\quad + \left| \frac{1}{\sqrt{\det N^{-1}}} - \frac{1}{\sqrt{\det M^{-1}}} \right| |fg| \star |M^{1/2} \cdot|^{-1} (x) \\ &\leq 2\sqrt{\det N} \|M^{-1}\|^{1/2} \|N^{-1}\|^{1/2} \|f\|_2 \|\nabla g\|_2 \|M^{1/2} - N^{1/2}\| \\ &\quad + 2\|M^{-1}\|^{1/2} \|f\|_2 \|\nabla g\|_2 \left| \sqrt{\det N} - \sqrt{\det M} \right| \\ &\leq \left(\|M - N\| + \alpha^{-1/2} |\det N - \det M| \right) \alpha^{-3/2} \|f\|_2 \|\nabla g\|_2. \end{aligned}$$

Finally, the determinant being locally Lipschitz, we obtain that $M \mapsto V \star (fg)$ is locally Lipschitz. $\square$

Since M^{-1} is real and symmetric, there exists $R \in O(3)$ such that

$$R^T M R = \text{diag}(m_3, m_2, m_1)$$

and so, for any $x \in \mathbb{R}^3$, after a simple computation, we have

$$V(Rx) = |x|^{-1} - \left| \text{diag} \left((m_1 m_2)^{-1/2}, (m_1 m_3)^{-1/2}, (m_2 m_3)^{-1/2} \right) x \right|^{-1},$$

where $0 < \sqrt{m_1 m_2} \leq \sqrt{m_1 m_3} \leq \sqrt{m_2 m_3} < 1$ since $m_2 < 1$. Thus, we can consider, without any loss of generality, that

$$\begin{cases} M = \text{diag}(m_1, m_2, m_3) & \text{with } 0 < m_3 \leq m_2 \leq m_1 < 1, \\ M \mapsto V(x) = \frac{1}{|x|} - \frac{1}{|M^{-1}x|} \end{cases} \quad (2.6)$$

Similarly, for the simplified model, we can also assume that

$$V(x) = |\text{diag}(1 - s_1, 1 - s_2, 1 - s_3)^{-1} x|^{-1}, \quad 0 \leq s_3 \leq s_2 \leq s_1 < 1. \quad (2.7)$$

For clarity, from now on we denote by $\mathcal{E}_M$ (resp. $\mathcal{E}_S$) the energy and by $I_M(\lambda)$ (resp. $I_S(\lambda)$) the minimization problem since V depends only on the matrix M (resp. on the matrix S). However, for shortness, we will omit the subscripts when no confusion is possible.

Lemma 2. *Let $\psi \in H^1(\mathbb{R}^3)$ be a solution of the equation (1.3), for V defined as in (2.6) or in (2.7), then $(x, y, z) \mapsto \psi(\pm x, \pm y, \pm z)$ are $H^1(\mathbb{R}^3)$ -solutions to (1.3).*

Proof of Lemma 2. This follows from the symmetry properties of V . $\square$

3. EXISTENCE OF MINIMIZERS

We state in this section the existence of minimizers for the minimization problems (see Appendix in [17] for the proof). We first give some properties of these variational problems.

Lemma 3. *Let V be defined as in (2.6) or (2.7) and I be defined as in (2.1). Then*

$$I(\lambda) = \lambda^3 I(1) < 0, \text{ if } \lambda > 0. \quad (3.1)$$

Consequently,

- (1) $\lambda \mapsto I(\lambda)$ is C^∞ on $\mathbb{R}^+$,
- (2) $I(\lambda) < I(\lambda - \lambda') + I(\lambda')$, for any λ et λ' such that $0 < \lambda' < \lambda$,

and, in particular:

(3) $I(\lambda) < I(\lambda')$, for any $0 \leq \lambda' < \lambda$.

Proof of Lemma 3. Let $\psi \in H^1(\mathbb{R}^3)$ with $\|\psi\|_{L^2}^2 = 1$, then $\psi_\lambda := \lambda^2 \psi(\lambda \cdot) \in H^1(\mathbb{R}^3)$ and $\|\psi_\lambda\|_{L^2}^2 = \lambda$ and, by a direct computation, $\mathcal{E}(\psi_\lambda) = \lambda^3 \mathcal{E}(\psi)$ which leads to $I(\lambda) = \lambda^3 I(1)$. If we now define $\psi_t = t^{3/2} \psi(t \cdot)$ and use (2.5), we find that

$$\begin{aligned} \mathcal{E}(\psi_t) &\leq \frac{1}{2} \|\nabla \psi_t\|_{L^2}^2 - \frac{b}{2} \| |\psi_t|^2 \star |\cdot|^{-1} \|_{L^2}^2 \\ &\leq \frac{t^2}{2} \|\nabla \psi\|_{L^2}^2 - \frac{bt}{2} \| |\psi|^2 \star |\cdot|^{-1} \|_{L^2}^2, \end{aligned}$$

and taking t small enough leads to the claimed strict negativity. The rest follows immediately. $\square$

Lemma 4. *Let V be defined as in (2.6) or (2.7). Let I be as in (2.1) and let $\lambda > 0$. Then $I(t\lambda) > tI(\lambda)$, for all $t \in (0, 1)$.*

Proof of Lemma 4. Let $t \in (0, 1)$. By Lemma 3, $0 > I(t\lambda) = t^3 I(\lambda) > tI(\lambda)$. $\square$

These two lemmas imply the existence of minimizers and the compactness of minimizing sequences, as stated in the following theorem which gives also some properties of these minimizers.

Theorem 5 (Existence of a minimizer). *Let V be as in (2.6) or (2.7) and $\lambda > 0$. Then $I(\lambda)$ has a minimizer and any minimizing sequence strongly converges in $H^1(\mathbb{R}^3)$ to a minimizer, up to extraction of a subsequence and after an appropriate space translation.*

Moreover for any minimizer ψ , we have

- (1) ψ is a $H^2(\mathbb{R}^3)$ -solution of the Choquard–Pekar equation (1.3) with $-\mu = \frac{d}{d\lambda} I(\lambda) < 0$ being the smallest eigenvalue of the self-adjoint operator $H_\psi := -\Delta/2 - |\psi|^2 \star V$ and being simple;
- (2) $\mu\lambda = -\lambda \frac{d}{d\lambda} I(\lambda) = -3\lambda^3 I(1) = \frac{3}{2} \|\nabla \psi\|_2^2 = \frac{3}{4} \langle V \star |\psi|^2, |\psi|^2 \rangle$; (3.2)
- (3) $|\psi|$ is a minimizer and $|\psi| > 0$;
- (4) $\psi = z|\psi|$ for a given $|z| = 1$.

See Appendix in [17] for the proof. Note that for the isotropic potentials V_c , Lieb proved several of these results in 1977 [9] using only the fact that $|x|^{-1}$ is radially decreasing.

4. UNIQUENESS IN A WEAKLY ANISOTROPIC MATERIAL

We recall that the uniqueness of the minimizer, up to phases and space translations, in the isotropic case, was proven by Lieb in [9]. In this section, we extend this result to the case of *weakly anisotropic materials*, meaning that we consider static dielectric matrices close to an homothecy.

We first prove the continuity of $I_M(\lambda)$, with respect to (M, λ) , which we will need in the proof of uniqueness.

Lemma 6 (Minimums' convergence). *Let V be defined as in (2.6) or (2.7), I be defined as in (2.1) and $(\lambda, \lambda') \in (\mathbb{R}_+^*)^2$. Then*

$$I_{M'}(\lambda') \xrightarrow[\substack{\|M' - M\| \rightarrow 0 \\ |\lambda' - \lambda| \rightarrow 0}]{} I_M(\lambda).$$

Thus, the continuity of the corresponding Euler-Lagrange multiplier, $-\mu_{M',\lambda'}$, holds as well:

$$\mu_{M',\lambda'} \xrightarrow[\substack{\|M'-M\| \rightarrow 0 \\ |\lambda'-\lambda| \rightarrow 0}]{} \mu_{M,\lambda}.$$

Proof of Lemma 6. Let ψ (resp. ψ') be a minimizer of $I_M(\lambda)$ (resp. $I_{M'}(\lambda)$) for a given $\lambda > 0$.

First, for any $\varphi \in H^1(\mathbb{R}^3)$, we have

$$|\mathcal{E}_M(\varphi) - \mathcal{E}_{M'}(\varphi)| = \frac{1}{2} |\langle |\varphi|^2, |\varphi|^2 \star (V - V') \rangle| \leq \frac{1}{2} \| |\varphi|^2 \star (V - V') \|_\infty \|\varphi\|_2^2.$$

Thus, by Lemma 1, $M \mapsto \mathcal{E}_M(\varphi)$ is Lipschitz for any $\varphi \in H^1(\mathbb{R}^3)$. Moreover

$$\mathcal{E}_M(\psi) - \mathcal{E}_{M'}(\psi) \leq I_M(\lambda) - I_{M'}(\lambda) \leq \mathcal{E}_M(\psi') - \mathcal{E}_{M'}(\psi'),$$

which implies that $M \mapsto I_M(\lambda)$ is Lipschitz for any $\lambda > 0$.

Thanks to Lemma 3, we conclude the proof of the convergence of I since

$$|I_M(\lambda) - I_{M'}(\lambda')| \lesssim |I_M(1)| |\lambda^3 - (\lambda')^3| + \|M - M'\|.$$

Then, the equality $-\mu_{M,\lambda} = 3\lambda^2 I_M(1)$ gives the convergence of the $\mu_{M',\lambda'}$'s. $\square$

We now give our theorem of uniqueness in the weakly anisotropic case.

Theorem 7 (Uniqueness and non-degeneracy in the weakly anisotropic case).

Let $\lambda > 0$.

- i. Let $0 < s < 1$. There exists $\varepsilon > 0$ such that, for every real symmetric 3×3 matrix $0 < M < 1$ with $\|M - s \cdot \text{Id}\| < \varepsilon$, the minimizer ψ of the minimization problem $I_M(\lambda)$, for $V(x) = |x|^{-1} - |M^{-1}x|^{-1}$ as in (2.6), is unique up to phase and space translations.
- ii. Let $0 \leq s < 1$. There exists $\varepsilon > 0$ such that, for every real symmetric 3×3 matrix $0 \leq S < 1$ with $\|S - s \cdot \text{Id}\| < \varepsilon$, the minimizer ψ of the minimization problem $I_S(\lambda)$, for $V(x) = |(1-S)^{-1}x|^{-1}$ as in (2.7), is unique up to phase and space translations.

Moreover, in both cases, the minimizer is even along each eigenvectors of M and $\ker \mathfrak{L}_\psi = \text{span}\{\partial_x \psi, \partial_y \psi, \partial_z \psi\}$, where $\mathfrak{L}_\psi$ is the linearized operator defined in (1.4).

The proof of this theorem is based on a perturbative argument around the isotropic case, using the implicit functions theorem. The fundamental nondegeneracy result in the isotropic case, proved by Lenzmann in [7], is a key ingredient of our proof.

Proof of Theorem 7. The proof of ii being similar to the one of i, we will only give the latter. Let us fix $0 < s < 1$, define $\mathcal{D} := \{0 < M < 1 \mid M \text{ symmetric real}\}$ and denote by Q the unique positive minimizer of the isotropic minimization problem $I(\lambda) := I_{s \cdot \text{Id}}(\lambda)$ for $V(x) = V_{s \cdot \text{Id}}(x) = (1-s)|x|^{-1}$, which is radial and solves (1.3):

$$-\frac{1}{2} \Delta Q + \mu Q - (|Q|^2 \star V)Q = 0,$$

with $\|Q\|_2^2 = \lambda$. There λ is fixed hence is $\mu := \mu_{s \cdot \text{Id}, \lambda} > 0$ by Lemma 3.

Step 1: Implicit function theorem and local uniqueness. By Proposition 5 in [7], we know that the linearized operator $\mathfrak{L}_Q$ given by

$$\mathfrak{L}_Q \xi = -\frac{1}{2} \Delta \xi + \mu \xi - (V \star |Q|^2) \xi - 2Q(V \star (Q\xi)), \quad (4.1)$$

acting on $L^2(\mathbb{R}^3)$ with domain $H^2(\mathbb{R}^3)$, has the kernel

$$\ker \mathfrak{L}_Q = \text{span}\{\partial_{x_1} Q, \partial_{x_2} Q, \partial_{x_3} Q\}. \quad (4.2)$$

Let us define u as

$$\begin{aligned} H^1(\mathbb{R}^3, \mathbb{R}) \times \mathcal{D} &\xrightarrow{u} L^2(\mathbb{R}^3, \mathbb{R}) \\ (\psi, M) &\mapsto -(|\psi|^2 \star V) \psi \end{aligned}$$

and G as

$$\begin{aligned} (\ker \mathfrak{L}_Q)^\perp \times \mathcal{D} &\xrightarrow{G} H^1(\mathbb{R}^3, \mathbb{R}) \\ (\psi, M) &\mapsto \psi + (-\Delta/2 + \mu_M)^{-1} u(\psi, M), \end{aligned}$$

where $(\ker \mathfrak{L}_Q)^\perp$ is the orthogonal of $\ker \mathfrak{L}_Q$ for the scalar product of $L^2(\mathbb{R}^3)$, which we endow with the norm of $H^1(\mathbb{R}^3)$, and $\mu_M := \mu_{M,\lambda} = 3\lambda^2 I_M(1)$. We emphasize here that we consider real valued functions, meaning that we are constructing a branch of real valued solutions. Moreover, $G(\psi, M) = 0$ is equivalent to $-\frac{1}{2}\Delta\psi + \mu\psi - (|\psi|^2 \star V)\psi = 0$. Differentiating with respect to x_i , for $i = 1, 2, 3$, we get $\mathfrak{L}_\psi \partial_{x_i} \psi = 0$, for $i = 1, 2, 3$, and thus $\text{span}\{\partial_x \psi, \partial_y \psi, \partial_z \psi\} \subset \ker \mathfrak{L}_\psi$.

By the Hardy-Littlewood-Sobolev and Sobolev inequalities, u is well defined. Moreover, splitting $u(\psi, M) - u(\psi', M')$ into three pieces and using (2.5) together with the Hardy inequality, one obtains

$$\begin{aligned} \|u(\psi, M) - u(\psi', M')\|_{L^2} &\leq \|V \star |\psi|^2\|_{L^\infty} \|\psi - \psi'\|_{L^2} + \|(V - V') \star |\psi|^2\|_{L^\infty} \|\psi'\|_{L^2} \\ &\quad + \|V' \star (|\psi|^2 - |\psi'|^2)\|_{L^\infty} \|\psi - \psi'\|_{L^2} \\ &\leq \|V \star |\psi|^2\|_{L^\infty} \|\psi - \psi'\|_{L^2} + \|(V - V') \star |\psi|^2\|_{L^\infty} \|\psi'\|_{L^2} \\ &\quad + 2(\|\psi\|_{H^1} + \|\psi'\|_{H^1}) \|\psi - \psi'\|_{L^2}^2. \end{aligned}$$

Therefore, using Lemma 1, u is locally Lipschitz on $H^1(\mathbb{R}^3, \mathbb{R}) \times \mathcal{D}$. Then, since $(-\Delta/2 + \mu_M)^{-1}$ maps $L^2(\mathbb{R}^3)$ onto $H^2(\mathbb{R}^3) \subset H^1(\mathbb{R}^3)$, G is also well defined. Moreover, since $\|(-\Delta + \nu)^{-1}\|_{L^2 \rightarrow H^2} \leq \max\{1, \nu^{-1}\}$ (for $\nu > 0$) and

$$(-\Delta/2 + a)^{-1} - (-\Delta/2 + b)^{-1} = (b - a)(-\Delta/2 + a)^{-1}(-\Delta/2 + b)^{-1},$$

for all $a, b > 0$, we have

$$\begin{aligned} \|G(\psi, M) - G(\psi', M')\|_{H^1} &\leq \|\psi - \psi'\|_{H^1} + \|(-\Delta/2 + \mu_M)^{-1} (u(\psi, M) - u(\psi', M'))\|_{H^1} \\ &\quad + |\mu_{M'} - \mu_M| \|(-\Delta/2 + \mu_M)^{-1} (-\Delta/2 + \mu_{M'})^{-1} u(\psi', M')\|_{H^1} \\ &\lesssim \|\psi - \psi'\|_{H^1} + \max\{2, (\mu_M)^{-1}\} \|u(\psi, M) - u(\psi', M')\|_{L^2} \\ &\quad + \max\{2, (\mu_M)^{-1}\} \max\{2, (\mu_{M'})^{-1}\} \|u(\psi', M')\|_{L^2} \|M' - M\|, \end{aligned}$$

which proves that G is also locally Lipschitz.

A simple computation shows that

$$\partial_\psi u(\psi, M) \xi = -(|\psi|^2 \star V) \xi - 2\psi ((\psi \xi) \star V), \quad (4.3)$$

acting on $\xi \in (\ker \mathfrak{L}_Q)^\perp$, and that

$$\partial_\psi G(\psi, M) = 1 + (-\Delta/2 + \mu_M)^{-1} \partial_\psi u(\psi, M). \quad (4.4)$$

We claim $\partial_\psi G(\varphi, M)$, defined from $(\ker \mathfrak{L}_Q)^\perp \times \mathcal{D}$ into $\mathcal{L}((\ker \mathfrak{L}_Q)^\perp, L^2(\mathbb{R}^3, \mathbb{R}))$, to be continuous. Indeed,

$$\|\partial_\psi u(\psi, M) \xi\|_{L^2} \leq \|\xi\|_{L^2} \|V \star |\psi|^2\|_{L^\infty} + 2\|\psi\|_{L^2} \|(\psi \xi) \star V\|_{L^\infty} \leq 3\|\psi\|_{H^1} \|\psi\|_{L^2} \|\xi\|_{L^2},$$

thus $\partial_\psi u(\psi, M) \xi \in L^2(\mathbb{R}^3, \mathbb{R})$ for any $(\varphi, M, \xi) \in (\ker \mathfrak{L}_Q)^\perp \times \mathcal{D} \times (\ker \mathfrak{L}_Q)^\perp$.

Splitting again the term into pieces and using (2.5), for $\xi \in L^2(\mathbb{R}^3, \mathbb{R})$, one obtains

$$\begin{aligned} & \|\partial_\psi u(\psi, M)\xi - \partial_1 u(\psi', M')\xi\|_{L^2} \\ & \leq \|V \star (|\psi|^2 - |\psi'|^2)\|_{L^\infty} \|\xi\|_{L^2} + \|(V - V') \star |\psi'|^2\|_{L^\infty} \|\xi\|_{L^2} \\ & \quad + 2\|V \star (\psi\xi)\|_{L^\infty} \|\psi - \psi'\|_{L^2} + 2\|V \star ((\psi - \psi')\xi)\|_{L^\infty} \|\psi'\|_{L^2} \\ & \quad + 2\|(V - V') \star (\psi'\xi)\|_{L^\infty} \|\psi'\|_{L^2} \\ & = O\left(\|(\psi, M) - (\psi', M')\|_{H^1 \times \mathcal{D}}\right) \|\xi\|_{L^2}. \end{aligned}$$

Then, since

$$\begin{aligned} & \|\partial_\psi G(\psi, M)\xi - \partial_\psi G(\psi', M')\xi\|_{H^1} \\ & \lesssim \max\{2, (\mu_M)^{-1}\} \|\partial_\psi u(\psi, M) - \partial_\psi u(\psi', M')\|_{L^2} \\ & \quad + \max\{2, (\mu_M)^{-1}\} \max\{2, (\mu_{M'})^{-1}\} \|\partial_\psi u(\psi', M')\|_{L^2} \|M' - M\|, \end{aligned}$$

we have

$$\|\partial_\psi G(\psi, M) - \partial_\psi G(\psi', M')\| \rightarrow 0, \text{ if } \|(\psi, M) - (\psi', M')\|_{H^1 \times \mathcal{D}} \rightarrow 0.$$

This concludes the proof of the continuity of $\partial_\psi G(\varphi, M)$ from $(\ker \mathfrak{L}_Q)^\perp \times \mathcal{D}$ into $\mathcal{L}((\ker \mathfrak{L}_Q)^\perp, H^1(\mathbb{R}^3, \mathbb{R}))$.

We now apply the implicit function theorem to G . Indeed, by the definition of $(\ker \mathfrak{L}_Q)^\perp$, the restriction of $\mathfrak{L}_Q$ to $(\ker \mathfrak{L}_Q)^\perp$ has a trivial kernel. On the other hand, the operator $(-\Delta/2 + \mu_M)^{-1} \partial_\psi u(Q, s \cdot \text{Id})$ is compact on $L^2(\mathbb{R}^3)$ (see Appendix in [17]), therefore -1 does not belong to its spectrum. We deduce from this the existence of the inverse operator

$$(\partial_\psi G(Q, s \cdot \text{Id}))^{-1} : \text{Ran}(G) \subset H^1(\mathbb{R}^3, \mathbb{R}) \rightarrow (\ker \mathfrak{L}_Q)^\perp. \quad (4.5)$$

Then, by the continuity of G and $\partial_\psi G$, the existence of $(\partial_\psi G(Q_s, s \cdot \text{Id}))^{-1}$ and since $G(Q, s \cdot \text{Id}) = 0$, the inverse function theorem 1.2.1 of [3] implies that there exist $\delta, \varepsilon > 0$ such that there exists a unique $\psi(M) \in (\ker \mathfrak{L}_Q)^\perp$ satisfying:

$$G(\psi(M), M) = 0 \quad \text{for } \|M - s \cdot \text{Id}\| \leq \varepsilon \text{ and } \|\psi(M) - Q\|_{H^1} \leq \delta. \quad (4.6)$$

Moreover, the map $M \mapsto \psi(M)$ is continuous.

Additionally, $\ker \partial_\psi G(\psi(M), M) = \{0\}$, i.e. $\ker|_{(\ker \mathfrak{L}_Q)^\perp} \mathfrak{L}_\psi = \{0\}$ which leads to $\dim \ker(\mathfrak{L}_\psi) \leq 3$ since $\dim \ker(\mathfrak{L}_Q) = 3$ by (4.2).

We now claim that $\psi(M)$ is symmetric with respect to the three eigenvectors of M , $\{e_i\}_{i=1,2,3}$, and consequently that, for $i = 1, 2, 3$, $\partial_{x_i} \psi(M)$ is odd along e_i and even along e_j for $j \neq i$. Indeed V being symmetric, the eight functions $(x, y, z) \mapsto \psi(M)(\pm x, \pm y, \pm z)$, which are in $(\ker \mathfrak{L}_Q)^\perp$, are zeros of $G(\cdot, M)$. If $\psi(M)$ were not symmetric with respect to each e_i , then at least two of the functions $(x, y, z) \mapsto \psi(M)(\pm x, \pm y, \pm z)$ would be distinct functions but both verifying (4.6), since Q is symmetric with respect to each e_i , which is impossible by local uniqueness.

Thus the $\partial_{x_i} \psi(M)$'s are orthogonal and we have $\dim \text{span}\{\partial_x \psi, \partial_y \psi, \partial_z \psi\} = 3$. Since $\text{span}\{\partial_x \psi, \partial_y \psi, \partial_z \psi\} \subset \ker \mathfrak{L}_\psi$, this leads to $\dim \ker(\mathfrak{L}_\psi) \geq 3$. Which proves that $\ker \mathfrak{L}_\psi = \text{span}\{\partial_x \psi, \partial_y \psi, \partial_z \psi\}$.

Let us emphasize that, at this point, we do not know the masses $\|\psi(M)\|_2^2$ of those $\psi(M)$. Note also that we could prove here that $|\psi| > 0$, since $-\mu_M$ stays the first eigenvalue by continuity and with a Perron–Frobenius type argument, but we do not give the details here since this fact will be a consequence of Step 2.

Step 2: Global uniqueness. Let $(M_n)_n$ be a sequence of matrices in $\mathcal{D}$ such that $M_n \xrightarrow{n \rightarrow \infty} s \cdot \text{Id}$ and let $(\psi_{M_n})_n$ be a sequence of minimizers of $(I_{M_n}(\lambda))_n$ which we can suppose, up to phase, strictly positive by Theorem 5 and, up to a space translation (for each M_n), in $(\ker \mathfrak{L}_Q)^\perp$. Indeed, for any $\psi \in H^1(\mathbb{R}^3)$, let us define the continuous function $f(\tau) := \int \nabla Q(\cdot) \psi(\cdot - \tau)$ which is bounded, by the Cauchy-Schwarz inequality. Then $\int f(\tau) d\tau = \int \psi(x) \int \nabla Q(x - \tau) d\tau dx = 0$ since $\int \nabla Q = 0$. Thus, f being continuous, there exists τ such that $f(\tau) = \int \psi(x - \tau) \nabla Q(x) dx = 0$, i.e. $\psi(\cdot - \tau) \in (\ker \mathfrak{L}_Q)^\perp$ since $\ker \mathfrak{L}_Q = \text{span}\{\partial_{x_1} Q, \partial_{x_2} Q, \partial_{x_3} Q\}$.

By continuity of $(I_{M_n}(\lambda))_n$, given by Lemma 6, $(\psi_{M_n})_n$ is a minimizing sequence of $I_{s \cdot \text{Id}}(\lambda)$. So, by Theorem 5, $(\psi_{M_n})_n$ strongly converges in $H^1(\mathbb{R}^3)$ to a minimizer of $I_{s \cdot \text{Id}}(\lambda)$, up to extraction of a subsequence. But, since the ψ_{M_n} are positive and in $(\ker \mathfrak{L}_Q)^\perp$, they converge to a positive minimizer of $I_{s \cdot \text{Id}}(\lambda)$ in $(\ker \mathfrak{L}_Q)^\perp$ which is Q .

So, there exists $\varepsilon' \leq \varepsilon$ such that if $\|M - s \cdot \text{Id}\|_\infty \leq \varepsilon'$, then each ψ_{M_n} verifies $G(\psi_{M_n}, M_n) = 0$, by definition of $(\psi_{M_n})_n$, and $\|\psi_{M_n} - Q\|_{H^1} \leq \delta$ i.e. verifies (4.6). So the ψ_{M_n} are unique (up to phases and spaces translation). Which concludes the proof of Theorem 7.

Moreover, we now know that, in fact, the masses $\|\psi(M_n)\|_2^2$ of the unique $\psi(M_n)$ found in the local result were in fact all equal to λ . We also proved incidentally that our choice of translation to obtain $(\psi_{M_n})_n \subset (\ker \mathfrak{L}_Q)^\perp$ was, in fact, unique. $\square$

5. REARRANGEMENTS AND SYMMETRIES

The goal of this section is to prove that minimizers are symmetric and strictly decreasing in the directions along which V is decreasing, without assuming that V is close to the isotropic case as we did in the previous section. More precisely, we will consider here the general anisotropic case $m_3 \leq m_2 \leq m_1$ (resp. $s_3 \leq s_2 \leq s_1$) and, in particular, the two cylindrical cases $m_3 = m_2 < m_1$ (resp. $s_3 = s_2 < s_1$) and $m_3 < m_2 = m_1$ (resp. $s_3 < s_2 = s_1$). Our main result in this section is Theorem 9 below. As a preparation, we first give conditions for V to be its own Steiner symmetrization.

As in [2], for f defined on $\mathbb{R}^n = \text{span}\{e_1, \dots, e_n\}$, we denote:

- by f^* its *Schwarz symmetrization*, for $n \geq 1$;
- by $\text{St}_{i_1, \dots, i_k}(f)$ its *Steiner symmetrization (in codimension k) with respect to the subspace spanned by $e_{i_1}, \dots, e_{i_k}$* , for $n \geq 2$ and $1 \leq k < n$.

Let us remark that the Steiner symmetrization $\text{St}_{i_1, \dots, i_k}(f)$ of f is the Schwarz symmetrization of the function $(x_{i_1}, \dots, x_{i_k}) \mapsto f(x_1, \dots, x_n)$.

Proposition 8 (Criterion for V to be its own Steiner symmetrization).

- (1) Let V be given by (2.6), with $0 < m_3 \leq m_2 \leq m_1 < 1$. Then $V = \text{St}_1(V)$ (thus V is e_1 -symmetric strictly decreasing). Moreover, for $k \in \{2, 3\}$, $V = \text{St}_k(V)$ (thus V is e_k -symmetric strictly decreasing) if and only if

$$m_1^3 \leq m_k^2. \quad (5.1_k)$$

Moreover,

- if $m_3 < m_2 = m_1$, then $V = \text{St}_{1,2}(V)$. Thus V is (e_1, e_2) -radial strictly decreasing.
- if $m_3 = m_2 < m_1$, then $V = \text{St}_{2,3}(V)$ — thus V is (e_2, e_3) -radial strictly decreasing — if and only if

$$m_1^3 \leq m_2^2 = m_3^2; \quad (5.2)$$

- (2) Let V be given by (2.7), with $0 \leq s_3 \leq s_2 \leq s_1 < 1$. Then $V = \text{St}_k(V)$ (thus V is e_k -symmetric strictly decreasing) for $k = 1, 2, 3$. Moreover,

- i. if $s_3 < s_2 = s_1$, then $V = \text{St}_{1,2}(V)$. Thus V is (e_1, e_2) -radial strictly decreasing;
- ii. if $s_3 = s_2 < s_1$, then $V = \text{St}_{2,3}(V)$. Thus V is (e_2, e_3) -radial strictly decreasing.

Proof of Proposition 8. Suppose V is given by (2.6), then it obviously has the claimed properties of symmetry and, moreover, the cylindrical ones in cases i. and ii.. So the proof that V is equal to its symmetrization is reduced to the proof of decreasing properties.

For any $x \neq 0$ and $k = 1, 2, 3$, we have

$$\partial_{|x_k|} V(x_1, x_2, x_3) = \frac{m_k^{-2} |x_k|}{(m_1^{-2} x_1^2 + m_2^{-2} x_2^2 + m_3^{-2} x_3^2)^{3/2}} - \frac{|x_k|}{(x_1^2 + x_2^2 + x_3^2)^{3/2}}. \quad (5.3)$$

Thus, $V = \text{St}_k(V)$ and V is radially decreasing with respect to x_k if and only if

$$0 \leq (m_1^{-2} - m_k^{-4/3})x_1^2 + (m_2^{-2} - m_k^{-4/3})x_2^2 + (m_3^{-2} - m_k^{-4/3})x_3^2 \quad \text{a.e. on } \mathbb{R}^3$$

which is equivalent to $m_1 \leq m_k^{2/3}$. Consequently, $V = \text{St}_1(V)$ always holds.

If $m_3 = m_2 < m_1$, denoting $u = |(x_2, x_3)|$, and computing $\partial_u V$, we obtain that $V = \text{St}_{2,3}(V)$ if and only if $m_1 \leq m_2^{2/3} = m_3^{2/3}$, in which case V is (e_2, e_3) -radial decreasing.

If $m_3 < m_2 = m_1$, denoting $u = |(x_1, x_2)|$, and computing $\partial_u V$, we obtain that $V = \text{St}_{1,2}(V)$ if and only if $m_3 \leq m_2^{2/3} = m_1^{2/3}$, which always holds thus V is (e_1, e_2) -radial decreasing.

We now need to prove the strict monotonicity. Thanks to (5.3), $\nabla V = 0$ holds only on measure-zero sets (note that we use the computation but do not use any condition on m_1, m_2 and m_3 except that they are strictly less than 1). Thus $|\{V = t\}| = 0$ for any $t \in \mathbb{R}_+$ and then $|\{V^* = t\}| = 0$ for any $t \in \mathbb{R}_+$. Hence V^* is radially strictly decreasing. Same results of strict decreasing hold for Steiner symmetrizations since, as noted before, a Steiner symmetrization is a Schwarz symmetrization on a subspace.

The proof for V given by (2.7) is very similar and easier. $\square$

We now state our main result about the symmetries of minimizers.

Theorem 9 (Symmetries of minimizers). *Let $\lambda > 0$.*

- (1) *Let V be given by (2.6) and $\psi_M \geq 0$ be a minimizer of $I_M(\lambda)$. Then, up to a space translation, ψ_M is e_1 -symmetric strictly decreasing. If $m_1^3 \leq m_2^2$ as in (5.12), then ψ_M is also e_2 -symmetric strictly decreasing. Finally, if $m_1^3 \leq m_3^2$ as in (5.13), then ψ_M is additionally e_3 -symmetric strictly decreasing. Moreover,*
 - i. *if $m_3 < m_2 = m_1$, then ψ_M is cylindrical strictly decreasing with axis e_3 . Meaning that ψ_M is (e_1, e_2) -radial strictly decreasing. If additionally (5.13) holds, then ψ_M is cylindrical-even strictly decreasing with axis e_3 . This means that ψ_M is cylindrical strictly decreasing with axis e_3 and e_3 -symmetric strictly decreasing;*
 - ii. *if $m_3 = m_2 < m_1$ and $m_1^3 \leq m_2^2 = m_3^2$ as in (5.2), then ψ_M is cylindrical-even strictly decreasing with axis e_1 .*
- (2) *Let V be given by (2.7) and $\psi_S \geq 0$ be a minimizer of $I_S(\lambda)$. Then, up to a space translation, ψ_S is e_k -symmetric strictly decreasing for $k = 1, 2, 3$. Moreover,*
 - i. *if $s_3 < s_2 = s_1$, then ψ_S is cylindrical-even strictly decreasing with axis e_3 ;*
 - ii. *If $s_3 = s_2 < s_1$, then ψ_S is cylindrical-even strictly decreasing with axis e_1 .*

To prove the symmetry properties of the minimizers, we need symmetrizations of a minimizer to be minimizers, which is proved in the following lemma.

Lemma 10. *Suppose that V , given by (2.6) or by (2.7), verifies one of the symmetric strictly decreasing property (resp. radial strictly decreasing property) described in Proposition 8, and define ψ^{St} the symmetrization of ψ corresponding to this symmetric strictly decreasing property of V .*

If ψ is a minimizer then ψ^{St} too. Moreover the following equalities hold

- i. $\|\nabla\psi\|_2^2 = \|\nabla\psi^{\text{St}}\|_2^2$,
- ii. $\langle |\psi|^2, |\psi|^2 \star V \rangle_2 = \langle |\psi^{\text{St}}|^2, |\psi^{\text{St}}|^2 \star V \rangle_2$.

Proof of Lemma 10. On one hand, since the symmetrization conserves the L^2 norm and ψ is a minimizer, we have $\mathcal{E}(\psi) \leq \mathcal{E}(\psi^{\text{St}})$. On the other hand, given the Riesz inequality (see [1]), the fact that the kinetic energy is decreasing under symmetrizations (see Theorem 2.1 in [2]) and since $V = V^{\text{St}}$ by Proposition 8, we have $\mathcal{E}(\psi) \geq \mathcal{E}(\psi^{\text{St}})$. So finally $I(\lambda) = \mathcal{E}(\psi) = \mathcal{E}(\psi^{\text{St}})$. Consequently, given (3.2) in Theorem 5 and that minimizers ψ and ψ^{St} have the same Lagrange multiplier $\mu = -3\lambda^2 I(1)$, we immediately obtain both equalities. $\square$

Using the analyticity of minimizers (Lemma 12) we can now prove the strict monotonicity of Steiner symmetrizations of minimizers.

Lemma 11. *Let $\lambda > 0$ and ψ be a real minimizer of $I(\lambda)$ for V given by (2.6) or by (2.7), then ψ^* is radially strictly decreasing. Moreover, for any permutation $\{i, j, k\}$ of $\{1, 2, 3\}$, we have*

- i. *for any $x \in \text{span}\{e_j, e_k\}$, $\text{St}_i(\psi)(x, \cdot)$ is radially strictly decreasing,*
- ii. *for any $x \in \text{span}\{e_i\}$, $\text{St}_{j,k}(\psi)(x, \cdot)$ is radially strictly decreasing.*

Proof of Lemma 11. By Theorem 5, ψ is a solution of (1.3) in $H^2(\mathbb{R}^3, \mathbb{R})$ with a real Lagrange multiplier μ . Then, by the following lemma (see the Appendix of [17] for the proof), ψ is real analytic.

Lemma 12. *Any $\psi \in H^2(\mathbb{R}^3, \mathbb{R})$ solution of (1.3) for $\mu \in \mathbb{R}$ is analytic.*

Thus $|\{\psi = t\}| = 0$ for any $t \in \mathbb{R}_+$ and this is equivalent to $|\{\psi^* = t\}| = 0$ for any $t \in \mathbb{R}_+$. Hence ψ^* is radially strictly decreasing.

Given that for any $1 \leq k < 3$ and any $x \in \mathbb{R}^{3-k}$, $\psi(x, \cdot)$ is analytic and since a Steiner symmetrization is a Schwarz symmetrization, we obtain ii. and iii. by the same reasoning to $\psi(x, \cdot)$. $\square$

Finally, to prove our Theorem 9 on the symmetries of minimizers, we need a result on the case of equality in Riesz' inequality for Steiner's symmetrizations. We emphasize that different Steiner symmetrizations do not commute in general. However, if the Steiner symmetrizations are made with respect to the vectors of an orthogonal basis then the radial strictly decreasing properties are preserved.

For shortness, we write $u^{\text{St}_k} := \text{St}_k(u)$ and, in cylindrical cases, $u^{\text{St}_{1,2}} := \text{St}_{1,2}(u)$ and $u^{\text{St}_{2,3}} := \text{St}_{2,3}(u)$.

Proposition 13 (Steiner symmetrization: case of equality for g strictly decreasing). *Let f, g, h be three measurable functions on $\mathbb{R}^3$ such that $g > 0$ and $f, h \geq 0$ where $0 \neq f \in L^p(\mathbb{R}^3)$, with $1 \leq p \leq +\infty$, and $0 \neq h \in L^q(\mathbb{R}^3)$, with $1 \leq q \leq +\infty$. Define*

$$J(f, g, h) = \frac{1}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f(x) g(x-y) h(y) dx dy \leq \infty.$$

- (1) Let (i, j, k) be a permutation of $(1, 2, 3)$ and $J(f^{\text{St}_i}, g, h^{\text{St}_i}) < \infty$. If for any $(x_j, x_k) \in \mathbb{R}^2$ the functions g , f^{St_i} and h^{St_i} are all strictly decreasing with respect to $|x_i|$, then

$$J(f, g, h) = J(f^{\text{St}_i}, g, h^{\text{St}_i}) \Leftrightarrow \exists a \in \mathbb{R}^3, \begin{cases} f = f^{\text{St}_i}(\cdot - a), \\ h = h^{\text{St}_i}(\cdot - a), \end{cases} \quad \text{a.e. on } \mathbb{R}^3.$$

- (2) Let (i, j, k) be a permutation of $(1, 2, 3)$ and $J(f^{\text{St}_{j,k}}, g, h^{\text{St}_{j,k}}) < \infty$. If for any $x_i \in \mathbb{R}$ the functions g , $f^{\text{St}_{j,k}}$ and $h^{\text{St}_{j,k}}$ are all radially strictly decreasing with respect to (x_j, x_k) , then

$$J(f, g, h) = J(f^{\text{St}_{j,k}}, g, h^{\text{St}_{j,k}}) \Leftrightarrow \exists a \in \mathbb{R}^3, \begin{cases} f = f^{\text{St}_{j,k}}(\cdot - a), \\ h = h^{\text{St}_{j,k}}(\cdot - a). \end{cases} \quad \text{a.e. on } \mathbb{R}^3.$$

- (3) Let St and St' be two Steiner symmetrizations, acting on two orthogonal directions, $T = \text{St}' \circ \text{St}$ and $J(f^T, g, h^T) < \infty$. If the functions g , f^{St} , h^{St} are all radially strictly decreasing in the direction (or the plane) of St , and g , $f^{\text{St}'}$ and $h^{\text{St}'}$ are all radially strictly decreasing in the direction (or the plane) of St' , then

$$J(f, g, h) = J(f^T, g, h^T) \Leftrightarrow \exists a \in \mathbb{R}^3, \begin{cases} f = f^T(\cdot - a), \\ h = h^T(\cdot - a). \end{cases} \quad \text{a.e. on } \mathbb{R}^3.$$

Proof of Proposition 13. The implications $\Leftarrow$ all follow from a simple changes of variable. We show the implications $\Rightarrow$ and start with (1). Define, for any permutation (i, j, k) of $(1, 2, 3)$ and any $(x_j, x'_j, x_k, x'_k) \in \mathbb{R}^4$, the functions

$$J_i(f, g, h)(x_j, x'_j, x_k, x'_k) = \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} f(X) g(X - X') h(X') dx_i dx'_i,$$

where $X = (x_1, x_2, x_3)$ and $X' = (x'_1, x'_2, x'_3)$. We claim that for almost all $(x_j, x'_j, x_k, x'_k) \in \mathbb{R}^4$, we have

$$J_i(f, g, h)(x_j, x'_j, x_k, x'_k) = J_i(f^{\text{St}_i}, g, h^{\text{St}_i})(x_j, x'_j, x_k, x'_k).$$

Indeed, assume that there exists a non-zero measure set $E \subset \mathbb{R}^2 \times \mathbb{R}^2$ such that $J_i(f, g, h)(y, y') \neq J_i(f^{\text{St}_i}, g, h^{\text{St}_i})(y, y')$ for any $(y, y') \in E$. Thus, by Riesz inequality on $\mathbb{R}$, $J_i(f, g, h) < J_i(f^{\text{St}_i}, g, h^{\text{St}_i})$ necessarily holds on E , since $g = g^{\text{St}_i}$, and consequently $J(f, g, h) < J(f^{\text{St}_i}, g, h^{\text{St}_i})$, reaching a contradiction.

We now use the following result of Lieb [9]:

Lemma 14 ([9, Lemma 3]: Case of equality in Riesz' inequality for g strictly decreasing). *Suppose g is a positive spherically symmetric strictly decreasing function on $\mathbb{R}^n$, $f \in L^p(\mathbb{R}^n)$ and $h \in L^q(\mathbb{R}^n)$ are two nonnegative functions, with $p, q \in [1; +\infty]$, such that $J(f^*, g, h^*) < \infty$. Then*

$$J(f, g, h) = J(f^*, g, h^*) \Rightarrow \exists a \in \mathbb{R}^n, f = f^*(\cdot - a) \text{ and } h = h^*(\cdot - a) \quad \text{a.e..}$$

Thus, for almost all $(y, y') \in \mathbb{R}^2 \times \mathbb{R}^2$, there exists $a_i(y, y') \in \mathbb{R}$ such that $f(y, x_i) = f^{\text{St}_i}(y, x_i - a_i(y, y'))$ and $h(y', x_i) = h^{\text{St}_i}(y', x_i - a_i(y, y'))$, for almost all $x_i \in \mathbb{R}$. Using now the assumed strict monotonicity of $f^{\text{St}_i}(y, \cdot)$ and $h^{\text{St}_i}(y', \cdot)$, it follows that a_i does not depend on (y, y') , and (1) is proved.

The case (2) is very similar, defining this time

$$J_{j,k}(f, g, h)(x_i, x'_i) = \frac{1}{2} \langle f(\cdot, x_i), g(\cdot, x_i - x'_i) \star h(\cdot, x'_i) \rangle_{L^2(\mathbb{R}^2)}, \quad \forall (x_i, x'_i) \in \mathbb{R}^2.$$

We now prove (3). Let St be one of the Steiner's symmetrization described (1) and (2) and the same for St' . We claim that

$$J_{\text{St}}(f, g, h) = J_{\text{St}}(f^{\text{St}}, g, h^{\text{St}}) \text{ and } J_{\text{St}'}(f, g, h) = J_{\text{St}'}(f^{\text{St}'}, g, h^{\text{St}'}), \text{ a.e..}$$

Indeed, Riesz inequality gives $J(f, g, h) \leq J(f^{\text{St}}, g, h^{\text{St}}) \leq J(f^T, g, h^T)$. Since first and third terms are equal, the three of them are. From the first equality, there exists $a \in \mathbb{R}^\ell$ ($\ell = 1, 2$) such that $f = f^{\text{St}}(\cdot - a, \cdot)$ and $h = h^{\text{St}}(\cdot - a, \cdot)$. Then, since St and St' act on orthogonal directions, we have

$$J(f^T, g, h^T) = J\left(f^{\text{St}'}(\cdot + a, \cdot), g, h^{\text{St}'}(\cdot + a, \cdot)\right) = J(f^{\text{St}'}, g, h^{\text{St}'})$$

and so the second claim holds true too. Then, for almost every $y := (x, z) \in \mathbb{R}^3$, we have

$$\begin{cases} f^T(y - (a', a)) = (f^{\text{St}}(x - a, \cdot))^{\text{St}'}(z - a') = f^{\text{St}'}(x, z - a') = f(x, z) = f(y), \\ h^T(y - (a', a)) = (h^{\text{St}}(x - a, \cdot))^{\text{St}'}(z - a') = h^{\text{St}'}(x, z - a') = h(x, z) = h(y). \end{cases}$$

□

We now have all the ingredients to prove Theorem 9.

Proof of Theorem 9. Let ψ be a minimizer and ψ^{St} one (or a composition) of its Steiner symmetrizations with a direction (or a plane) for which $V = V^{\text{St}}$.

We take $f = h = |\psi|^2 \in$ and $g = V$. So we have $f = h > 0$ (thanks to Theorem 5), $g > 0$ (thanks to (2.5)) and $J(f^{\text{St}}, V, f^{\text{St}})$ finite. Indeed by the Hardy-Littlewood-Sobolev inequality and (2.5), $J(f^{\text{St}}, V, f^{\text{St}}) \lesssim \|f^{\text{St}}\|_{6/5}^2 = \|f\|_{6/5}^2 < +\infty$ since $f \in H^1(\mathbb{R}^3)$. Moreover, the assumption on the m_k 's gives that $g = g^{\text{St}}$ is radially strictly decreasing by Proposition 8, and the strict monotonicity of $f^{\text{St}} = h^{\text{St}}$ is obtained by Lemma 11.

Finally, by Lemma 10, ψ^{St} is a minimizer and

$$J(|\psi|^2, V, |\psi|^2) = J((|\psi|^2)^{\text{St}}, (V)^{\text{St}}, (|\psi|^2)^{\text{St}}) = J((|\psi|^2)^{\text{St}}, V, (|\psi|^2)^{\text{St}}).$$

By Proposition 13, there exists a such that $|\psi|^2 = (|\psi|^2)^{\text{St}}(\cdot - a) = (|\psi|^{\text{St}})^2(\cdot - a)$ holds a.e. thus $\psi = \psi^{\text{St}}(\cdot - a)$ since $\psi \geq 0$. This concludes the proof of Theorem 9. □

6. STUDY OF THE LINEARIZED OPERATOR

In this section we study the linearized operator $\mathfrak{L}_Q$, on $L^2(\mathbb{R}^3)$ with domain $H^2(\mathbb{R}^3)$, associated with the Euler-Lagrange equation $-\Delta Q + Q - (|Q|^2 \star V)Q = 0$ (1.3), which is given by

$$\boxed{\mathfrak{L}_Q \xi = -\Delta \xi + \xi - (V \star |Q|^2) \xi - 2Q(V \star (Q\xi))}, \quad (6.1)$$

and we give partial characterization of its kernel. We first consider the true model (2.6) for which, following the scheme in [7], we will use a Perron-Frobenius argument on subspaces adapted to the symmetries of the problem. The main difficulty will stand in dealing with the non-local operator $Q(V \star (Q\xi))$ and, in particular, with proving that this operator is positivity improving. The fundamental use of Newton's theorem in the proof of this property in the isotropic case does not work here, therefore we need a new argument. Our proof will rely on the conditions (5.1_k)'s for which V is e_k -symmetric strictly decreasing for each k (see Proposition 8). Then we discuss in a similar way the cylindrical case for the simplified model (2.7), which will need another argument.

6.1. The linearized operator in the symmetric decreasing case. We consider the general case for V , given by (2.6), verifying the three conditions (5.1_k), for $k = 1, 2, 3$, and define the subspaces of $L^2(\mathbb{R}^3)$

$$L^2_{\tau_x, \tau_y, \tau_z} := \left\{ f \in L^2(\mathbb{R}^3) \left| \begin{array}{l} f(-x, y, z) = \tau_x f(x, y, z), \\ f(x, -y, z) = \tau_y f(x, y, z), \\ f(x, y, -z) = \tau_z f(x, y, z) \end{array} \right. \right\}, \quad (6.2)$$

obtained by choosing $\tau_x, \tau_y, \tau_z \in \{\pm 1\}$. We prove the following theorem which is basically saying that the kernel of the linearized operator around solutions is reduced to the kernel on functions that are even in all three directions.

Theorem 15. *Let V , be given by (2.6), verifying (5.1_k), for all k , and let Q be a positive and symmetric strictly decreasing (with respect to each e_k separately) solution of (1.3). Then*

$$\ker \mathfrak{L}_Q = \text{span} \{ \partial_x Q, \partial_y Q, \partial_z Q \} \bigoplus \ker (\mathfrak{L}_Q)|_{L^2_{+,+,+}}. \quad (6.3)$$

For instance, Q could be a minimizer for $I_M(\lambda)$.

The proof of this result is inspired by Lenzmann's proof in [7] of the fundamental similar result for the linearized operator in the radial case which corresponds to $m_1 = m_2 = m_3$. In that case, Lenzmann proved that $\ker (\mathfrak{L}_Q)|_{L^2_{+,+,+}} = \{0\}$. Note that by the result of Section 4, we know that this is still true in the weakly anisotropic case. Moreover, a theorem similar to Theorem 15 holds true for the simplified model (2.7) (with no conditions on the matrix S) but we do not state it here for shortness.

The rest of this Section 6.1 being dedicated to the proof of the theorem, let V and Q verify the assumptions of Theorem 15 for the entire Section 6.1.

6.1.1. Direct sum decomposition. First, one can easily verify that $\mathfrak{L}_Q$ stabilizes the spaces $L^2_{\tau_x, \tau_y, \tau_z}$. Let us then introduce the direct sum decomposition

$$L^2(\mathbb{R}^3) = L^2_{x-} \oplus L^2_{x+} = L^2_{y-} \oplus L^2_{y+} = L^2_{z-} \oplus L^2_{z+}$$

where

$$\begin{cases} L^2_{x-} := \bigoplus_{\tau_y, \tau_z = \pm} L^2_{-, \tau_y, \tau_z}, & L^2_{x+} := \bigoplus_{\tau_y, \tau_z = \pm} L^2_{+, \tau_y, \tau_z} \\ L^2_{y-} := \bigoplus_{\tau_x, \tau_z = \pm} L^2_{\tau_x, -, \tau_z}, & L^2_{y+} := \bigoplus_{\tau_x, \tau_z = \pm} L^2_{\tau_x, +, \tau_z} \\ L^2_{z-} := \bigoplus_{\tau_x, \tau_y = \pm} L^2_{\tau_x, \tau_y, -}, & L^2_{z+} := \bigoplus_{\tau_x, \tau_y = \pm} L^2_{\tau_x, \tau_y, +}. \end{cases}$$

We claim that those spaces — with corresponding projectors $P^{x-}, P^{x+}, P^{y-}, P^{y+}, P^{z-}$ and P^{z+} — reduce the linearized operator $\mathfrak{L}_Q$ (see [21] for a definition of reduction), where

$$P^{x\pm} \psi(r, \varphi, z) = \frac{\psi(x, y, z) \pm \psi(-x, y, z)}{2}$$

and similarly for the other projections. The reduction property is straightforward for $-\Delta + 1 - (V \star |Q|^2)$. Moreover, since Q is even in x , we have

$$\begin{aligned} V \star (Q P^{x\pm} \psi) &= \frac{V \star (Q \varphi) \pm V \star (Q \varphi(-\cdot, \cdot, \cdot))}{2} \\ &= \frac{V \star (Q \varphi) \pm [V \star (Q \varphi)](-\cdot, \cdot, \cdot)}{2} = P^{x\pm} [V \star (Q \psi)] \end{aligned}$$

and, Q being also even in y and in z , we obtain the result for the other projections. Thus we can apply [21, Lemma 2.24] which gives us that

$$\mathfrak{L}_Q = \mathfrak{L}_Q^{x-} \oplus \mathfrak{L}_Q^{x+} = \mathfrak{L}_Q^{y-} \oplus \mathfrak{L}_Q^{y+} = \mathfrak{L}_Q^{z-} \oplus \mathfrak{L}_Q^{z+},$$

with the six operators being self-adjoint operators on the corresponding $L^2(\mathbb{R}^3)$ spaces with domains $P^w H^2(\mathbb{R}^3)$, and $w \in \{x-, x+, y-, y+, z-, z+\}$. Note that $P^{x-} H^2(\mathbb{R}^3) = H_{x-}^2(\mathbb{R}^3) := H^2(\mathbb{R}^3) \cap L_{x-}^2(\mathbb{R}^3)$ and similarly for P^{y-} and P^{z-} .

Let us then redefine for now on the operator $\mathfrak{L}_Q^{x-}$ (resp. $\mathfrak{L}_Q^{y-}$ and $\mathfrak{L}_Q^{z-}$) by restricting it to x -odd (resp. y -odd and z -odd) functions through the isomorphic identifications $L_{x-}^2(\mathbb{R}_+^* \times \mathbb{R}^2) \approx L_{x-}^2(\mathbb{R}^3)$ and $H_{x-}^2(\mathbb{R}_+^* \times \mathbb{R}^2) \approx H_{x-}^2(\mathbb{R}^3)$. Thus, $\mathfrak{L}_Q^{x-}$, as an operator on $L_{x-}^2(\mathbb{R}_+^* \times \mathbb{R}^2)$ with domain $H_{x-}^2(\mathbb{R}_+^* \times \mathbb{R}^2)$, can be written

$$\mathfrak{L}_Q^{x-} = -\Delta + 1 + \Phi_{(-)} + W_{(-)}$$

where the strictly negative multiplication local operator, on $\mathbb{R}_+^* \times \mathbb{R}^2$, is

$$\begin{aligned} \Phi_{(-)}(x, Y) &= -(V \star |Q|^2)(x, Y) \\ &= -\int_{\mathbb{R}_+^* \times \mathbb{R}^2} Q^2(x', Y') [V(x - x', Y - Y') + V(x + x', Y - Y')] dY' dx' \end{aligned}$$

and the non-local term $W_{(-)}$, on $\mathbb{R}_+^* \times \mathbb{R}^2$, is

$$\begin{aligned} (W_{(-)}f)(x, Y) &= -2Q(x, Y) \times \\ &\times \int_{\mathbb{R}_+^* \times \mathbb{R}^2} Q(x', Y') f(x', Y') [V(x - x', Y - Y') - V(x + x', Y - Y')] dY' dx'. \end{aligned}$$

The same properties hold for $\mathfrak{L}_Q^{y-}$ and $\mathfrak{L}_Q^{z-}$.

The key fact to deal with the non-local operator, in order to adapt Lenzmann's proof to anisotropic case, is the positivity improving property of $-W_{(-)}$.

Lemma 16. *The operator $-W_{(-)}$ is positivity improving on $L_{x-}^2(\mathbb{R}_+^* \times \mathbb{R}^2)$.*

Proof of Lemma 16. Since $X \mapsto V(X, Y)$ is $|X|$ -strictly decreasing, due to conditions (5.1_k), and $x + x' > |x - x'|$ on $(\mathbb{R}_+^*)^2$, we obtain, for $x, x' > 0$ and $(Y, Y') \in (\mathbb{R}^2)^2$, that $V(x - x', Y - Y') - V(x + x', Y - Y') > 0$. Moreover $Q > 0$. Thus, $-W_{(-)}$ is positivity improving on $L_{x-}^2(\mathbb{R}_+^* \times \mathbb{R}^2)$. $\square$

6.1.2. Perron–Frobenius property. We can now prove that the three operators $\mathfrak{L}_Q^{x-}$, $\mathfrak{L}_Q^{y-}$ and $\mathfrak{L}_Q^{z-}$ verify a Perron–Frobenius property.

Proposition 17 (Perron–Frobenius properties). *The operators $\mathfrak{L}_Q^{x-}$, $\mathfrak{L}_Q^{y-}$ and $\mathfrak{L}_Q^{z-}$ are self-adjoint on $L_{x-}^2(\mathbb{R}_+^* \times \mathbb{R}^2)$ with domain $H_{x-}^2(\mathbb{R}_+^* \times \mathbb{R}^2)$ and bounded below.*

Moreover they have the Perron–Frobenius property: if λ_0^{x-} (resp. λ_0^{y-} and λ_0^{z-}) denotes the lowest eigenvalue of $\mathfrak{L}_Q^{x-}$ (resp. $\mathfrak{L}_Q^{y-}$ and $\mathfrak{L}_Q^{z-}$), then λ_0^{x-} (resp. λ_0^{y-} and λ_0^{z-}) is simple and the corresponding eigenfunction ψ_0^{x-} (resp. ψ_0^{y-} and ψ_0^{z-}) is strictly positive.

Proof of Proposition 17. We follow the structure of the proof of [7, Lemma 8]. Moreover, we only write the proof for $\mathfrak{L}_Q^{x-}$ which we denote $\mathfrak{L}_Q^-$ for simplicity. The argument is the same for the other directions.

Self-adjointness. We have $Q \in H^2(\mathbb{R}^3) \subset C^0(\mathbb{R}^3) \cap L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ and, by (2.5), $V \star |Q|^2$ is in $L^4(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ since $V = V_2 + V_4 \in L^2(\mathbb{R}^3) + L^4(\mathbb{R}^3)$. Defining, for any $f \in L_{x-}^2(\mathbb{R}_+^* \times \mathbb{R}^2)$, $\tilde{f} \in L_{x-}^2(\mathbb{R}^3)$ by $f(\cdot, z) = \tilde{f}(\cdot, z)$ for $z \geq 0$, we

have $2\langle f, g \rangle_{L^2_-(\mathbb{R}^*_+ \times \mathbb{R}^2)} = \langle \tilde{f}, \tilde{g} \rangle_{L^2_-(\mathbb{R}^3)}$ and so $\Phi_{(-)} + 1$ is bounded on $L^2_-(\mathbb{R}^*_+ \times \mathbb{R}^2)$. Moreover, by Young inequalities, for any $\xi \in L^2_-(\mathbb{R}^*_+ \times \mathbb{R}^2)$,

$$\|V \star (Q\tilde{\xi})\|_{L^\infty} \leq (\|V_4\|_{L^4}\|Q\|_{L^4} + \|V_2\|_{L^2}\|Q\|_{L^\infty}) \|\tilde{\xi}\|_{L^2}$$

holds. Thus, for $p \in [2, \infty]$, we have

$$\|W_{(-)}\xi\|_{L^p(\mathbb{R}^*_+ \times \mathbb{R}^2)} \leq 2\|Q\|_{L^p(\mathbb{R}^*_+ \times \mathbb{R}^2)}\|V \star (Q\tilde{\xi})\|_{L^\infty(\mathbb{R}^*_+ \times \mathbb{R}^2)} \leq \|Q\|_{L^p}\|V \star (Q\tilde{\xi})\|_{L^\infty}$$

and $W_{(-)}\xi \in L^2_-(\mathbb{R}^*_+ \times \mathbb{R}^2) \cap L^\infty(\mathbb{R}^*_+ \times \mathbb{R}^2)$. Finally, $1 + \Phi_{(-)} + W_{(-)}$ and, thus, $\mathfrak{L}_Q^-$ are bounded below on $L^2_-(\mathbb{R}^*_+ \times \mathbb{R}^2)$.

Finally, we deduce the self-adjointness of the operator $\mathfrak{L}_Q^-$ on $L^2_-(\mathbb{R}^*_+ \times \mathbb{R}^2)$ with domain $H^2_-(\mathbb{R}^*_+ \times \mathbb{R}^2)$ from the self-adjointness of the operator $\mathfrak{L}_Q^-$ on $L^2_-(\mathbb{R}^3)$ with domain $H^2_-(\mathbb{R}^3)$.

Positivity improving. We know (see [10] for example) that

$$(-\Delta + \mu)^{-1}\xi(X) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{e^{-\sqrt{\mu}|X-Y|}}{|X-Y|} \xi(Y) dY, \quad \forall \mu > 0, \forall \xi \in L^2(\mathbb{R}^3).$$

Consequently, for $\xi \in L^2_-(\mathbb{R}^*_+ \times \mathbb{R}^2)$ and $(x, \tilde{x}) \in \mathbb{R}^*_+ \times \mathbb{R}^2$, we have

$$(-\Delta + \mu)^{-1}\xi(x, \tilde{x}) = \frac{1}{4\pi} \int_{\mathbb{R}^*_+ \times \mathbb{R}^2} \left[\frac{e^{-\sqrt{\mu}|(x-y, \tilde{x}-\tilde{y})|}}{|(x-y, \tilde{x}-\tilde{y})|} - \frac{e^{-\sqrt{\mu}|(x+y, \tilde{x}-\tilde{y})|}}{|(x+y, \tilde{x}-\tilde{y})|} \right] \xi(y, \tilde{y}) dy d\tilde{y}.$$

Thus, with the same arguments as in the proof of Lemma 16, $(-\Delta + \mu)^{-1}$ is positivity improving on $L^2_-(\mathbb{R}^*_+ \times \mathbb{R}^2)$ for all $\mu > 0$. Moreover, $-(\Phi_{(-)} + W_{(-)})$ is positivity improving on $L^2_-(\mathbb{R}^*_+ \times \mathbb{R}^2)$ since $-\Phi_{(-)}$ is a positive multiplication operator and $-W_{(-)}$ is positivity improving by Lemma 16. Then similarly to the proof of [7, Lemma 8], for $\mu \gg 1$, we have

$$(\mathfrak{L}_Q^- + \mu)^{-1} = (-\Delta + \mu + 1)^{-1} \cdot (1 + (\Phi_{(-)} + W_{(-)})(-\Delta + \mu + 1)^{-1})^{-1}.$$

Since $(\Phi_{(-)} + W_{(-)})$ is bounded, we have

$$\|(\Phi_{(-)} + W_{(-)})(-\Delta + \mu)^{-1}\|_{L^2_-(\mathbb{R}^*_+ \times \mathbb{R}^2)} < 1,$$

for μ large enough. This implies, for $\mu \gg 1$, by Neumann's expansion that

$$(\mathfrak{L}_Q^- + \mu)^{-1} = (-\Delta + \mu + 1)^{-1} \sum_{p=0}^{\infty} [-(\Phi_{(-)} + W_{(-)})(-\Delta + \mu + 1)^{-1}]^p,$$

which is consequently positivity improving on $L^2_-(\mathbb{R}^*_+ \times \mathbb{R}^2)$ for $\mu \gg 1$.

Conclusion. We choose $\mu \gg 1$ such that $(\mathfrak{L}_Q^- + \mu)^{-1}$ is positivity improving and bounded. Then, by [15, Thm XIII.43], the largest eigenvalue $\sup \sigma((\mathfrak{L}_Q^- + \mu)^{-1})$ is simple and the associated eigenfunction $\psi_0^- \in L^2_-(\mathbb{R}^*_+ \times \mathbb{R}^2)$ is strictly positive. Since, for any $\psi \in L^2_-(\mathbb{R}^*_+ \times \mathbb{R}^2)$, having ψ being an eigenfunction of $\mathfrak{L}_Q^-$ for the eigenvalue λ is equivalent to having ψ being an eigenfunction of $(\mathfrak{L}_Q^- + \mu)^{-1}$ for the eigenvalue $(\lambda + \mu)^{-1}$, we have proved Proposition 17. $\square$

6.1.3. Proof of Theorem 15. Differentiating, with respect to x the Euler-Lagrange equation $-\Delta Q + Q - (|Q|^2 \star V)Q = 0$ (1.3), we obtain $\mathfrak{L}_Q \partial_x Q \equiv 0$. Moreover, Q is positive *symmetric strictly decreasing*, thus $\partial_x Q \in L^2_{x-}(\mathbb{R}^3)$, and this shows that $\mathfrak{L}_Q^{x-} \partial_x Q \equiv 0$. Then, $Q > 0$ being *symmetric strictly decreasing*, $\partial_x Q < 0$ on $\mathbb{R}^*_+ \times \mathbb{R}^2$ and, by the Perron-Frobenius property, it is (up to sign) the unique

eigenvector associated with the lowest eigenvalue of $\mathfrak{L}_Q^{x-}$, namely $\lambda_0^{x-} = 0$. Since $\mathfrak{L}_Q^{x-}$ acts on $L_{x-}^2 := \bigoplus_{\tau_y, \tau_z = \pm} L_{-, \tau_y, \tau_z}^2$, we obtain

$$\begin{cases} \ker(\mathfrak{L}_Q)|_{L_{-, +, +}^2(\mathbb{R}^3)} = \text{span}\{\partial_x Q\}; \\ \ker(\mathfrak{L}_Q)|_{L_{-, -, +}^2(\mathbb{R}^3)} = \ker(\mathfrak{L}_Q)|_{L_{-, +, -}^2(\mathbb{R}^3)} = \ker(\mathfrak{L}_Q)|_{L_{-, -, -}^2(\mathbb{R}^3)} = \{0\}. \end{cases}$$

This the exact same arguments for the two other directions we finally obtain that

$$\ker \mathfrak{L}_Q = \text{span}\{\partial_x Q, \partial_y Q, \partial_z Q\} \bigoplus \ker(\mathfrak{L}_Q)|_{L_{+, +, +}^2(\mathbb{R}^3)}.$$

Which concludes the proof of Theorem 15. $\square$

6.2. The linearized operator in the cylindrical case. We now consider the case where the static dielectric matrix has exactly two identical eigenvalues. Obviously, Theorem 15 holds and it tells us that the kernel is reduced to the kernel on functions that are even in the z -direction and even in any direction of the plane orthogonal to z . However, this does not tell us that it is reduced to the kernel on cylindrical functions, which is what we are interested in. Indeed, instead of the kernel of $\mathfrak{L}_Q$ on $L_{+, +, +}^2(\mathbb{R}^3)$, we want the remaining term in the direct sum to be the kernel on $L_{\text{rad}, +}^2(\mathbb{R}^3)$, namely the subset of cylindrical functions that are also even in the direction of their principal axis.

Unfortunately, our method fails to prove it for V given by (2.6) since we are not able to prove a positivity improving property for the non local operator. Therefore, in this section, we will only consider the simplified model where V is given by (2.7).

We use the cylindrical coordinates (r, z) where e_z is the vector orthogonal to the plane of symmetry. Namely, $e_z = e_3$ if $s_3 < s_2 = s_1$ and $e_z = e_1$ if $s_3 = s_2 < s_1$. We then define the following subspaces

$$\begin{aligned} L_\tau^2(\mathbb{R}^3) &:= \{f \in L^2(\mathbb{R}^3) \mid f(x, y, -z) = \tau f(x, y, z)\}, & \text{for } \tau = \pm \\ L_+^2(\mathbb{R}_+^* \times \mathbb{R}) &:= \{f \in L^2(\mathbb{R}_+^* \times \mathbb{R}, r \, dr \, dz) \mid f(r, -z) = f(r, z)\} \\ L_{\text{rad}, +}^2(\mathbb{R}^3) &:= L_+^2(\mathbb{R}_+^* \times \mathbb{R}) \otimes \mathbb{1}. \end{aligned} \quad (6.4)$$

Theorem 18. *Let V be given by (2.7) with S having one eigenvalue of multiplicity 2 and let Q be a cylindrical-even decreasing and positive solution of (1.3). Then*

$$\ker \mathfrak{L}_Q = \text{span}\{\partial_x Q, \partial_y Q, \partial_z Q\} \bigoplus \ker(\mathfrak{L}_Q)|_{L_{\text{rad}, +}^2(\mathbb{R}^3)}. \quad (6.5)$$

For instance, Q could be a minimizer for $I_S(\lambda)$.

Several parts of the proof of this theorem being identical to the ones in the proof of the Theorem 15, we will only give the details for the parts that differ.

6.2.1. Cylindrical decomposition. Since V is *cylindrical-even strictly decreasing* by Proposition 8 and since minimizers are *cylindrical-even strictly decreasing* by Proposition 9, $\mathfrak{L}_Q$ commutes with rotation in the plane of symmetry. Let us then introduce the direct sum decomposition

$$L^2(\mathbb{R}^3) = L_-^2(\mathbb{R}^3) \bigoplus \bigoplus_{n \geq 0, \sigma = \pm} L_+^2(\mathbb{R}_+^* \times \mathbb{R}) \otimes Y_n^\sigma, \quad (6.6)$$

with

$$\begin{cases} Y_0^+ \equiv (2\pi)^{-\frac{1}{2}}; Y_0^- \equiv 0; \\ Y_n^+ \equiv \pi^{-\frac{1}{2}} \cos(n\cdot); Y_n^- \equiv \pi^{-\frac{1}{2}} \sin(n\cdot), \quad \text{for } n \geq 1. \end{cases} \quad (6.7)$$

The operator $\mathfrak{L}_Q$ stabilizes $L_-^2(\mathbb{R}^3)$ and the spaces $L_+^2(\mathbb{R}_+^* \times \mathbb{R}) \otimes Y_n^\sigma$.

Let us immediately decompose the potential V in order to give the fundamental property in the cylindrical case (Proposition 19 below), which is what allows us

to adapt the original work of Lenzmann, namely the strict positivity of each z -odd terms of the cylindrical decomposition of V . For any $\mathbf{r} = (r, \varphi, z) \in \mathbb{R}^3$ and $\mathbf{r}' = (r', \varphi', z') \in \mathbb{R}^3$, defining $\rho := (r - r', 0, z - z')$ and $\theta := \varphi - \varphi'$, we have, as soon as $(r', z') \neq (r, z)$:

$$\theta \mapsto V(\mathbf{r} - \mathbf{r}') = \frac{1}{\sqrt{|(1-S)^{-1}\rho|^2 + 2(1-s_2)^{-2}rr'(1-\cos\theta)}} > 0, \quad (6.8)$$

which is in $C^\infty(\mathbb{R})$, 2π -periodic and even. Thus, for any $\mathbf{r} \neq \mathbf{r}'$,

$$V(\mathbf{r} - \mathbf{r}') = \sum_{n=0}^{\infty} v_n(r, r', z - z') Y_n^+(\varphi - \varphi'),$$

with

$$v_n(r, r', z - z') = \int_{-\pi}^{\pi} V(\mathbf{r} - \mathbf{r}') Y_n^+(\theta) d\theta = 2 \int_0^{\pi} V(\mathbf{r} - \mathbf{r}') Y_n^+(\theta) d\theta. \quad (6.9)$$

Proposition 19. *Let V be given by (2.7), the Y_n^+ 's by (6.7) and the v_n 's by (6.9) for any $(n, r, r', z, z') \in \mathbb{N} \times \mathbb{R}_+^* \times \mathbb{R}_+^* \times \mathbb{R} \times \mathbb{R}$. Then*

$$v_n(r, r', z - z') > 0, \quad \forall (n, r, r', z, z') \in \mathbb{N} \times \mathbb{R}_+^* \times \mathbb{R}_+^* \times \mathbb{R} \times \mathbb{R}.$$

Proof of Proposition 19. Defining for $r, r' > 0$,

$$\begin{aligned} m^\pm &:= \sqrt{\left(\frac{r+r'}{1-s_2}\right)^2 + \left(\frac{Z}{1-s_z}\right)^2} \pm \sqrt{\left(\frac{r-r'}{1-s_2}\right)^2 + \left(\frac{Z}{1-s_z}\right)^2} \\ &= \max_{\varphi-\varphi'} |(1-S)^{-1}(\mathbf{r} - \mathbf{r}')| \pm \min_{\varphi-\varphi'} |(1-S)^{-1}(\mathbf{r} - \mathbf{r}')| > 0, \end{aligned}$$

we note that $m^+ > m^-$ and obtain

$$V(\mathbf{r} - \mathbf{r}') = \frac{2}{m^+} \frac{1}{\sqrt{1 - 2\frac{m^-}{m^+} \cos \theta + \left(\frac{m^-}{m^+}\right)^2}}.$$

We now give the explicit expansion of $(1 - 2t \cos \theta + t^2)^{-1/2}$ in the following lemma.

Lemma 20. *For $(0, 1) \neq (\theta, t) \in \mathbb{R} \times [0, 1]$, we have*

$$\begin{aligned} \frac{1}{\sqrt{1 - 2t \cos \theta + t^2}} &= \sum_{k=0}^{\infty} \beta_{0,2k} t^{2k} + \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} \beta_{2n,2k} t^{2k} Y_{2n}^+(\theta) \\ &\quad + \sum_{n=0}^{\infty} \sum_{k=n}^{\infty} \beta_{2n+1,2k+1} t^{2k+1} Y_{2n+1}^+(\theta). \end{aligned} \quad (6.10)$$

with

$$\left\{ \begin{array}{ll} \beta_{0,2k} = \sqrt{2\pi} \frac{\binom{2k}{k}^2}{2^{4k}} > 0, & 0 \leq k; \\ \beta_{2n,2k} = 2\sqrt{\pi} \frac{\binom{2(k+n)}{k+n} \binom{2(k-n)}{k-n}}{2^{4k}} > 0, & 0 < n \leq k; \\ \beta_{2n+1,2k+1} = 2\sqrt{\pi} \frac{\binom{2(k+n+1)}{k+n+1} \binom{2(k-n)}{k-n}}{2^{4k+2}} > 0, & 0 \leq n \leq k. \end{array} \right.$$

Proof of Lemma 20. The proof of this lemma is entirely inspired by the original computation of Legendre¹ in his famous *mémoire* [6] where he introduced the polynomials that are nowadays called after him. Let us first rewrite the fraction, for

¹or Legendre and Laplace, according to a famous paternity controversy.

$(0, 1) \neq (\theta, t) \in \mathbb{R} \times [0, 1]$:

$$\frac{1}{\sqrt{1 - 2t \cos \theta + t^2}} = (1 - e^{i\theta}t)^{-1/2}(1 - e^{-i\theta}t)^{-1/2}.$$

Then, since $\Gamma(1/2 - p) = \frac{(-4)^p p!}{(2p)!} \Gamma(1/2)$ and using the following expansion

$$(1 - x)^{-1/2} = \sum_{p=0}^{\infty} \frac{\Gamma(1/2)}{\Gamma(1/2 - p)\Gamma(p + 1)} (-x)^p = \sum_{p=0}^{\infty} \frac{\binom{2p}{p}}{2^{2p}} x^p,$$

we obtain:

$$\begin{aligned} \frac{1}{\sqrt{1 - 2t \cos \theta + t^2}} &= \sum_{(p,q) \in \mathbb{N}^2} \frac{\binom{2p}{p} \binom{2q}{q}}{2^{2(p+q)}} e^{i(p-q)\theta} t^{p+q} \\ &= \sum_{\substack{k=0 \\ k \text{ even}}}^{\infty} \sum_{\substack{n=-k \\ n \text{ even}}}^k \frac{\binom{k+n}{(k+n)/2} \binom{k-n}{(k-n)/2}}{2^{2k}} e^{in\theta} t^k \\ &\quad + \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \sum_{\substack{n=-k \\ n \text{ odd}}}^k \frac{\binom{k+n}{(k+n)/2} \binom{k-n}{(k-n)/2}}{2^{2k}} e^{in\theta} t^k \\ &= \sum_{k=0}^{\infty} \frac{\binom{2k}{k}^2}{2^{4k}} t^{2k} + \sum_{k=0}^{\infty} \sum_{n=1}^k \frac{\binom{2(k+n)}{k+n} \binom{2(k-n)}{k-n}}{2^{4k}} 2 \cos(2n\theta) t^{2k} \\ &\quad + \sum_{k=0}^{\infty} \sum_{n=0}^k \frac{\binom{2(k+n+1)}{k+n+1} \binom{2(k-n)}{k-n}}{2^{4k+2}} 2 \cos((2n+1)\theta) t^{2k+1} \\ &= \sum_{n=0}^{\infty} \frac{\binom{2k}{k}^2}{2^{4k}} t^{2k} + \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} \frac{\binom{2(k+n)}{k+n} \binom{2(k-n)}{k-n}}{2^{4k}} 2 \cos(2n\theta) t^{2k} \\ &\quad + \sum_{n=0}^{\infty} \sum_{k=n}^{\infty} \frac{\binom{2(k+n+1)}{k+n+1} \binom{2(k-n)}{k-n}}{2^{4k+2}} 2 \cos((2n+1)\theta) t^{2k+1}. \end{aligned}$$

With the definition of the Y_n^+ 's, this concludes the proof of Lemma 20. $\square$

Defining all the others $\beta_{p,q}$'s to be zero, this proves Proposition 19:

$$v_n(r, r', z - z') = \frac{2}{m^+} \sum_{k=n}^{\infty} \beta_{n,k} \left(\frac{m^-}{m^+} \right)^k > 0,$$

for $n \geq 0$, $r, r' > 0$ and $z, z' \in \mathbb{R}$. Moreover, for $\mathbf{r} \neq \mathbf{r}'$, we have

$$V(\mathbf{r} - \mathbf{r}') = \sum_{n=0}^{\infty} \frac{2}{m^+} \left(\sum_{k=n}^{\infty} \beta_{n,k} \left(\frac{m^-}{m^+} \right)^k \right) Y_n^+(\theta).$$

$\square$

Remark 21. (The anisotropic potential (2.6)) If we define v_n in a similar fashion for the true model based on (2.6), even with the conditions (5.1_k) and (5.2), the v_n 's have no sign for $n \geq 2$, since we have

$$v_n(r, r', z - z') = \sum_{k=n}^{\infty} 2\beta_{n,k} \left(\frac{1}{m_{Id}^+} \left(\frac{m_{Id}^-}{m_{Id}^+} \right)^k - \frac{1}{m_M^+} \left(\frac{m_M^-}{m_M^+} \right)^k \right)$$

which changes sign for $n \geq 2$. This is why our method fails if V is given by (2.6). Note that the strict positivity however holds true for v_0 and for v_1 if $r, r' > 0$, which is straightforward using (6.9).

As proved in the last Section, $L_-^2(\mathbb{R}^3)$, with corresponding projectors P^- , reduces $\mathfrak{L}_Q$. We claim that the spaces $L_+^2(\mathbb{R}_+^* \times \mathbb{R}) \otimes Y_n^\sigma$, with corresponding projectors

$$P_{n,\sigma}^+ \psi(r, \varphi, z) = \left(\int_0^{2\pi} \frac{\psi(r, \varphi', z) + \psi(r, \varphi', -z)}{2} Y_n^\sigma(\varphi') d\varphi' \right) Y_n^\sigma(\varphi),$$

also reduce $\mathfrak{L}_Q$. Given that $(V \star |Q|^2) \in L_{\text{rad},+}^2(\mathbb{R}^3)$, $\frac{d^2}{d\varphi^2} Y_n^\sigma = -n^2 Y_n^\sigma$ and

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2}, \quad (6.11)$$

we have

$$[-\Delta + 1 - (V \star |Q|^2)](f Y_n^\sigma) = [-\Delta_{(n)} + 1 - (V \star |Q|^2)](f) Y_n^\sigma, \quad (6.12)$$

for any $f \in L_+^2(\mathbb{R}_+^* \times \mathbb{R})$, and so belonging to $L_+^2(\mathbb{R}_+^* \times \mathbb{R}) \otimes Y_n^\sigma$, and where

$$-\Delta_{(n)} := -\frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} - \frac{\partial^2}{\partial z^2} + \frac{n^2}{r^2}.$$

Thus the reduction property follows for $-\Delta + 1 - (V \star |Q|^2)$. Moreover, since $V \star (Q \cdot)$ and $P_{n,\sigma}^+$ are linear and using the decomposition

$$\psi(r, \varphi, z) = c^-(r, \varphi, z) + \sum_{n \geq 0, \sigma = \pm} c_{n,\sigma}^+(r, z) Y_n^\sigma(\varphi),$$

we have to prove that

$$V \star (Q P_{n',\sigma'}^+ c_{n,\sigma}^+ Y_n^\sigma) = P_{n',\sigma'}^+ (V \star (Q c_{n,\sigma}^+ Y_n^\sigma)),$$

for any $n, n' \geq 0$ and $\sigma, \sigma' = \pm$, in order to conclude. We have

$$\begin{aligned} & [V \star (Q c_{n,\sigma}^+ Y_n^\sigma)](r, \varphi, z) \\ &= \int_{\mathbb{R}_+^*} \int_{-\pi}^{\pi} \int_{\mathbb{R}} Q(r', z') c_{n,\sigma}^+(r', z') Y_n^\sigma(\varphi') V(\mathbf{r} - \mathbf{r}') r' dz' d\varphi' dr' \\ &= \left(\int_{\mathbb{R}_+^*} \int_{\mathbb{R}} Q(r', z') c_{n,\sigma}^+(r', z') v_n(r, r', z - z') r' dz' dr' \right) Y_n^\sigma(\varphi). \end{aligned} \quad (6.13)$$

Then using the parity of v_n with respect to its third variable (which is straightforward with (6.9)), we obtain $V \star (Q c_{n,\sigma}^+ Y_n^\sigma) \in L_+^2(\mathbb{R}_+^* \times \mathbb{R}) \otimes Y_n^\sigma$ and the reduction property follows. Thus we can apply [21, Lemma 2.24] which gives us that

$$\mathfrak{L}_Q = \mathfrak{L}^- \oplus \bigoplus_{n \geq 0, \sigma = \pm} \mathfrak{L}_{n,\sigma}^+,$$

with $\mathfrak{L}^- = \mathfrak{L}_Q^{z-}$ being the same operator as before and each $\mathfrak{L}_{n,\sigma}^+$ a self-adjoint operator on $L_+^2(\mathbb{R}_+^* \times \mathbb{R}) \otimes Y_n^\sigma$ with domain $P_{n,\sigma}^+ H^2(\mathbb{R}^3)$. For shortness, we omit the Q subscript in the decomposition $\mathfrak{L}_Q$.

Given (6.12) and (6.13), for any $n \geq 0$ we note $\mathfrak{L}_n^+$ the operator on $L_+^2(\mathbb{R}_+^* \times \mathbb{R})$ such that $\mathfrak{L}_{n,+}^+(f Y_n^+) = \mathfrak{L}_n^+(f) Y_n^+$ and $\mathfrak{L}_{n,-}^+(f Y_n^-) = \mathfrak{L}_n^+(f) Y_n^-$. This operator is

$$\mathfrak{L}_n^+ = -\Delta_{(n)} + 1 + \Phi + W_{(n)}$$

where Φ is the strictly negative multiplication local potential, on $\mathbb{R}_+^* \times \mathbb{R}$,

$$\Phi(r, z) = - (V \star |Q|^2)(r, z) = -\sqrt{2\pi} \int_{\mathbb{R}_+^* \times \mathbb{R}} |Q(r', z')|^2 v_0(r, r', z - z') r' dz' dr' < 0$$

and $W_{(n)}$ is the non-local operator, on $\mathbb{R}_+^* \times \mathbb{R}$,

$$(W_{(n)} f)(r, z) = -2Q(r, z) \int_{\mathbb{R}_+^* \times \mathbb{R}} Q(r', z') f(r', z') v_n(r, r', z - z') r' dz' dr'. \quad (6.14)$$

Similarly to the non-cylindrical case, we need to prove that $-W_{(n)}$ is positivity improving on $L_+^2(\mathbb{R}_+^* \times \mathbb{R})$ and this is where the result of Proposition 19 is needed.

Lemma 22. *For $n \geq 0$, the operator $-W_{(n)}$ is positivity improving on $L_+^2(\mathbb{R}_+^* \times \mathbb{R})$.*

Proof of Lemma 22. Given the definition (6.14) of $-W_{(n)}$, the fact that the v_n 's are strictly positive as soon as $r, r' > 0$ (by Proposition 19) and that $Q > 0$, it follows that $-W_{(n)}$ is positivity improving on $L_+^2(\mathbb{R}_+^* \times \mathbb{R})$ for any $n \geq 0$. $\square$

6.2.2. Perron–Frobenius property. We now prove that the $\mathfrak{L}_n^+$'s verify the Perron–Frobenius property.

Proposition 23 (Perron–Frobenius properties). *For $n > 0$, the $\mathfrak{L}_n^+$'s are essentially self-adjointness on $C_0^\infty(\mathbb{R}_+^* \times \mathbb{R})$ and bounded below.*

Moreover they have the Perron–Frobenius property: if λ_0^n denotes the lowest eigenvalue of $\mathfrak{L}_n^+$, then λ_0^n is simple and the corresponding eigenfunction ψ_0^n is strictly positive.

Proof of Proposition 23. We follow the structure of the proof of [7, Lemma 8].

Self-adjointness. We still have $V \star |Q|^2 \in L^4(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$. Moreover, defining $\mathring{f}(r, \cdot, z) = f(r, z)Y_n^+ \in L_+^2(\mathbb{R}_+^* \times \mathbb{R}) \otimes Y_n^+ \subset L^2(\mathbb{R}^3)$, for any $f \in L_+^2(\mathbb{R}_+^* \times \mathbb{R})$, we have $\langle f, g \rangle_{L_+^2(\mathbb{R}_+^* \times \mathbb{R})} = \langle \mathring{f}, \mathring{g} \rangle_{L^2(\mathbb{R}^3)}$ and, consequently, that $\Phi + 1$ is a bounded operator on $L_+^2(\mathbb{R}_+^* \times \mathbb{R})$. Then, for any $\xi \in L_+^2(\mathbb{R}_+^* \times \mathbb{R})$ and $p \in [2, \infty]$, we have

$$\|W_{(n)}\xi\|_{L^p(\mathbb{R}_+ \times \mathbb{R})} \leq \|Q\|_{L^p} \|V \star (Q\xi)\|_{L^\infty}.$$

Thus $W_{(n)}\xi \in L_+^2(\mathbb{R}_+^* \times \mathbb{R}) \cap L^\infty(\mathbb{R}_+^* \times \mathbb{R})$ and, finally, $1 + \Phi + W_{(n)}$ and, thus, $\mathfrak{L}_n^+$ are bounded below on $L_+^2(\mathbb{R}_+^* \times \mathbb{R})$. Furthermore, it is known that $-\Delta_{(n)}$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}_+^* \times \mathbb{R})$ provided that $n > 0$. Thus, given that $1 + \Phi + W_{(n)}$ is bounded (so $-\Delta_{(n)}$ -bounded of relative bound zero), symmetric (moreover self-adjoint) and that its domain contains the domain of $-\Delta_{(n)}$, we obtain by the Rellich-Kato theorem the essentially self-adjointness of $\mathfrak{L}_n^+$ on $C_0^\infty(\mathbb{R}_+^* \times \mathbb{R})$.

Positivity improving. We claim that $e^{t\Delta_{(n)}}$ is positivity improving for all $t > 0$ on $L^2(\mathbb{R}_+^* \times \mathbb{R})$. Indeed we have the explicit formula for the integral kernel $e^{t\Delta}$ on $\mathbb{R}^3$, namely,

$$e^{t\Delta}(x, y) = (4\pi t)^{-3/2} e^{-\frac{|x-y|^2}{4t}} = (4\pi t)^{-3/2} e^{-\frac{r^2+r'^2+(z-z')^2}{4t}} e^{\frac{rr'}{2t} \cos(\varphi-\varphi')}, \quad (6.15)$$

for all $x := (r, \varphi, z)$ and $y := (r', \varphi', z')$. On the other hand we have

$$e^{x \cos \theta} = \sqrt{2\pi} \sum_{m=0}^{\infty} \sqrt{2^{\delta_{\{m \geq 1\}}}} I_m(x) Y_m^+(\theta), \quad \forall x \in \mathbb{R}, \quad (6.16)$$

where $I_n(x) = \pi^{-1} \int_0^\pi \exp(x \cos \theta) \cos(n\theta) d\theta$ are the modified Bessel functions of the first kind, that are strictly positive for $n \geq 0$ and $x > 0$. From these two relations, we deduce the integral kernel $e^{t\Delta_{(n)}}$ acting on $L^2(\mathbb{R}_+ \times \mathbb{R})$ and that it is positive, which are the two points of the following lemma.

Lemma 24. *Let $f \in L^2(\mathbb{R}_+^* \times \mathbb{R})$, $r, r' > 0$ and $n \geq 0$. Then the integral kernel $e^{t\Delta_{(n)}}$ acting on $L^2(\mathbb{R}_+^* \times \mathbb{R})$ verifies*

$$e^{t\Delta_{(n)}}((r, z), (r', z')) = \frac{\sqrt{2}^{-\delta_n^0}}{4\pi t^{3/2}} e^{-\frac{r^2+r'^2+z^2+z'^2}{4t}} I_n\left(\frac{rr'}{2t}\right) \exp\left(\frac{zz'}{2t}\right) > 0. \quad (6.17)$$

Proof of Lemma 24. Let $f \in L^2(\mathbb{R}_+^* \times \mathbb{R})$. Using (6.15), for $n \geq 0$, we have

$$\begin{aligned} & (e^{t\Delta}(fY_n^\sigma))(r, \varphi, z) \\ &= (4\pi t)^{-3/2} \int_{\mathbb{R}_+ \times \mathbb{R}} e^{-\frac{r^2+r'^2+(z-z')^2}{4t}} f(r', z') \left(\int_{-\pi}^{\pi} e^{\frac{rr'}{2t} \cos(\varphi-\varphi')} Y_n^\sigma(\varphi') d\varphi' \right) r' dr' dz' \\ &= \frac{\sqrt{2}^{-\delta_n^0}}{4\pi t^{3/2}} \int_{\mathbb{R}_+ \times \mathbb{R}} e^{-\frac{r^2+r'^2+z^2+z'^2}{4t}} f(r', z') I_n \left(\frac{rr'}{2t} \right) \exp \left(\frac{zz'}{2t} \right) r' dr' dz' Y_n^\sigma(\varphi). \end{aligned}$$

Which allows to conclude the proof of Lemma 24. $\square$

So, for all $n \geq 0$, $e^{t\Delta_{(n)}}$ is positivity improving on $L^2(\mathbb{R}_+^* \times \mathbb{R})$ for all $t > 0$ and consequently on $L_+^2(\mathbb{R}_+^* \times \mathbb{R})$. Then, by functional calculus, we have

$$(-\Delta_{(n)} + \mu)^{-1} = \int_0^\infty e^{-t\mu} e^{t\Delta_{(n)}} dt, \quad \forall \mu > 0,$$

thus $(-\Delta_{(n)} + \mu)^{-1}$ is positivity improving on $L_+^2(\mathbb{R}_+^* \times \mathbb{R})$ for all $\mu > 0$.

Moreover we claim that $-(\Phi + W_{(n)})$ is positivity improving on $L_+^2(\mathbb{R}_+^* \times \mathbb{R})$ since $-\Phi$ is a positive multiplication operator and $-W_{(n)}$ is positivity improving on $L_+^2(\mathbb{R}_+^* \times \mathbb{R})$.

The end of the proof uses the exact same arguments as in the proof of the Perron–Frobenius property in the non-cylindrical case (Proposition 17) and, consequently, this ends the proof of Proposition 23. $\square$

6.2.3. Proof of Theorem 18. First, using the results of the previous Section, we have

$$\ker(\mathfrak{L}_Q)|_{L_-^2(\mathbb{R}^3)} = \ker(\mathfrak{L}_Q)|_{L_{-,+,+}^2(\mathbb{R}^3)} = \text{span}\{\partial_z Q\}$$

and $\mathfrak{L}_Q \partial_x Q \equiv \mathfrak{L}_Q \partial_y Q \equiv 0$. But now Q is furthermore *cylindrical-even*, thus $\partial_x Q = \frac{x}{r} \partial_r Q \in L_+^2(\mathbb{R}_+^* \times \mathbb{R}) \otimes Y_1^+$ and $\partial_y Q = \frac{y}{r} \partial_r Q \in L_+^2(\mathbb{R}_+^* \times \mathbb{R}) \otimes Y_1^-$, which implies that $\mathfrak{L}_1^+ \partial_r Q \equiv 0$. Then, $Q > 0$ being *cylindrical-even strictly decreasing*, $\partial_r Q < 0$ on $\mathbb{R}_+ \times \mathbb{R}$ and, by the Perron–Frobenius property (Proposition 23), it is (up to sign) the unique eigenvector associated with the lowest eigenvalue of $\mathfrak{L}_1^+$, namely $\lambda_0^1 = 0$. Consequently, $\partial_x Q$ (resp. $\partial_y Q$) is the unique eigenvector — up, in addition, to rotations in the cylindrical plane — associated with the lowest eigenvalue $\lambda_0^{1,+} = 0$ (resp. $\lambda_0^{1,-} = 0$) of $\mathfrak{L}_{1,+}^+$ (resp. $\mathfrak{L}_{1,-}^+$). To summarize, we know at this point that

$$\ker \mathfrak{L}_Q = \text{span}\{\partial_x Q, \partial_y Q, \partial_z Q\} \oplus \ker(\mathfrak{L}_Q)|_{L_{\text{rad},+}^2(\mathbb{R}^3)} \oplus \bigoplus_{\substack{n \geq 2 \\ \sigma = \pm}} \ker(\mathfrak{L}_Q)|_{L_+^2(\mathbb{R}_+^* \times \mathbb{R}) \otimes Y_n^\sigma},$$

and we have to deal with the higher orders. The end of the paper is devoted to the proof that

$$\ker \mathfrak{L}_{n,\sigma}^+ = \{0\}, \quad \forall n \geq 2, \sigma = \pm. \quad (6.18)$$

For $n \geq 2$, let $0 < \varphi^n \in L_+^2(\mathbb{R}_+^* \times \mathbb{R})$ be an eigenvector of $\mathfrak{L}_n^+$ associated with λ_0^n . Then $\varphi^n Y_n^+$ (resp. $\varphi^n Y_n^-$) is an eigenvector of $\mathfrak{L}_{n,+}^+$ (resp. $\mathfrak{L}_{n,-}^+$) associated to the eigenvalue $\lambda_0^{n,+} = \lambda_0^n$ (resp. $\lambda_0^{n,-} = \lambda_0^n$). Thus, for $n \geq 2$ and $\sigma = \pm$, we have

$$\begin{aligned} \lambda_0^{1,\sigma} - \lambda_0^{n,\sigma} &\leq \langle \varphi^n, \mathfrak{L}_1^+ \varphi^n \rangle_{L^2(\mathbb{R}_+ \times \mathbb{R})} - \langle \varphi^n, \mathfrak{L}_n^+ \varphi^n \rangle_{L^2(\mathbb{R}_+ \times \mathbb{R})} \\ &\leq - \int_{\mathbb{R}_+ \times \mathbb{R}} \frac{n^2 - 1}{r^2} (\varphi^n(r, z))^2 r dr dz \\ &\quad + 2 \iint_{(\mathbb{R}_+ \times \mathbb{R})^2} [Q\varphi^n](r, z) [Q\varphi^n](r', z') [v_n - v_1](r, r', z - z') rr' dz dz' dr dr'. \end{aligned}$$

Since $Q > 0$ and $\varphi^n > 0$ (by the Perron–Frobenius property in Proposition 23), in order to prove that $\lambda_0^{n,\sigma} > \lambda_0^{1,\sigma}$, it is sufficient to prove that $v_n < v_1$ almost everywhere on $(\mathbb{R}_+^* \times \mathbb{R})^2$. Using the explicit formula (6.8) for V , this is equivalent to prove that

$$T_n := \int_0^\pi \frac{\cos(n\theta) - \cos \theta}{\sqrt{K + 2a_2^2 r r' (1 - \cos \theta)}} d\theta < 0, \quad \text{for a.e. } (r, z), (r', z') \in (\mathbb{R}_+^* \times \mathbb{R})^2,$$

where $K = |(1 - S)^{-1}(r - r', 0, z - z')|^2$. First, let us remark that the points $\left\{2\frac{k}{n-1}\pi\right\}_{k \in \mathbb{Z}}$ and $\left\{2\frac{k}{n+1}\pi\right\}_{k \in \mathbb{Z}}$ are the zeros of $\cos(n\cdot) - \cos(\cdot)$ and that the function $g = [K + 2(1 - s_2)^{-2} r r' (1 - \cos(\cdot))]^{-1/2}$ is strictly decreasing on $]0, \pi[$. Let us define, $\theta_{2\lfloor n/2 \rfloor} := \pi$ and, for k an integer in $[0, \lfloor n/2 \rfloor - 1]$, $\theta_{2k} := 2\frac{k}{n-1}\pi$ and $\theta_{2k+1} := 2\frac{k+1}{n+1}\pi$ which are all the zeros of $\cos(n\cdot) - \cos(\cdot)$ in $[0, \pi]$, except $\theta_{2\lfloor n/2 \rfloor}$ if n is even. Then, noticing that $\cos(n\cdot) - \cos(\cdot)$ is strictly negative on intervals $]\theta_{2k}, \theta_{2k+1}[$, strictly positive on intervals $]\theta_{2k+1}, \theta_{2k+2}[$ and that $n\theta_{2k} = 2k\pi + \theta_{2k}$, we have

$$\begin{aligned} T_n &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor - 1} \int_{\theta_{2k}}^{\theta_{2k+1}} \underbrace{g(\theta)}_{> g(\theta_{2k+1}) > 0} \underbrace{(\cos(n\theta) - \cos \theta)}_{< 0} d\theta + \int_{\theta_{2k+1}}^{\theta_{2k+2}} \underbrace{g(\theta)}_{0 < g(\theta_{2k+1})} \underbrace{(\cos(n\theta) - \cos \theta)}_{> 0} d\theta \\ &< \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor - 1} g(\theta_{2k+1}) \int_{\theta_{2k}}^{\theta_{2k+2}} (\cos(n\theta) - \cos \theta) d\theta. \end{aligned}$$

If $n = 2$ or $n = 3$, this immediately leads to $T_n < 0$. Otherwise, if $n \geq 4$, we have

$$\begin{aligned} T_n &< \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor - 1} g(\theta_{2k+1}) \int_{\theta_{2k}}^{\theta_{2k+2}} (\cos(n\theta) - \cos \theta) d\theta \\ &< -\frac{n-1}{n} \left(\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor - 2} g(\theta_{2k+1}) \sin \theta_{2k+2} - \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor - 1} g(\theta_{2k+1}) \sin \theta_{2k} \right) \\ &< -\frac{n-1}{n} \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor - 1} \underbrace{[g(\theta_{2k-1}) - g(\theta_{2k+1})]}_{> 0} \underbrace{\sin \theta_{2k}}_{> 0} < 0. \end{aligned}$$

Thus we have just proved, for $n \geq 2$ and $\sigma = \pm$, that $\lambda_0^{n,\sigma} > \lambda_0^{1,\sigma} = 0$, consequently $\ker \mathfrak{L}_{n,\sigma}^+ = \{0\}$.

This concludes the proof of Theorem 18. $\square$

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