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# REMARKS ON THE INTERNAL EXPONENTIAL STABILIZATION TO A NONSTATIONARY SOLUTION FOR 1D BURGERS EQUATIONS

AXEL KRÖNER<sup>†‡</sup> AND SÉRGIO S. RODRIGUES<sup>§¶</sup>

**Abstract.** The feedback stabilization of the Burgers system to a nonstationary solution using finite-dimensional internal controls is considered. Estimates for the dimension of the controller are derived. In the particular case of no constraint on the support of the control a better estimate is derived and the possibility of getting an analogous estimate for the general case is discussed; some numerical examples are presented illustrating the stabilizing effect of the feedback control, and suggesting that the existence of an estimate in the general case analogous to that in the particular one is plausible.

**Key words.** Burgers equations, exponential stabilization, feedback control, finite elements

**AMS subject classifications.** 93B52, 93C20, 93D15, 93C50

**1. Introduction.** Let  $L > 0$  be a positive real number. We consider the controlled Burgers equations in the interval  $\Omega = (0, L) \subset \mathbb{R}$ :

$$(1.1) \quad \partial_t u + u \partial_x u - \nu \partial_{xx} u + h + \zeta = 0, \quad u|_{\Gamma} = 0.$$

Here  $u$  stands for the unknown velocity of the fluid,  $\nu > 0$  is the viscosity,  $h$  is a fixed function,  $\Gamma = \partial\Omega$  stands for the boundary  $\{0, L\}$  of  $\Omega$ , and  $\zeta$  is a control taking values in the space of square-integrable functions in  $\Omega$ , whose support, in  $x$ , is contained in a given open subset  $\omega \subset \Omega$ .

Let us be given a positive constant  $\lambda > 0$ , a continuous Lipschitz function  $\chi \in W^{1,\infty}(\Omega, \mathbb{R})$  with nonempty support, and a solution  $\hat{u} \in \mathcal{W}$  of (1.1) with  $\zeta = 0$ , in a suitable Banach space  $\mathcal{W}$ . Then, following the procedure presented in [7], we can prove that there exists an integer  $M$ , a function  $\eta = \eta(t, x)$ , defined for  $t > 0$ ,  $x \in \Omega$ , such that the solution  $u = u(t, x)$  of problem (1.1), with  $\zeta = \chi P_M \eta$ , and supplemented with the initial condition

$$(1.2) \quad u(0, x) = u_0(x)$$

is defined on  $[0, +\infty)$  and satisfies the relation  $|u(t) - \hat{u}(t)|_{L^2(\Omega, \mathbb{R})}^2 \leq C e^{-\lambda t} |u(0) - \hat{u}(0)|_{L^2(\Omega, \mathbb{R})}^2$ , provided  $|u(0) - \hat{u}(0)|_{L^2(\Omega, \mathbb{R})} < \epsilon$ , for small enough  $\epsilon$ . Here  $M$ ,  $C$ , and  $\epsilon$  can be taken depending only on  $(|\hat{u}|_{\mathcal{W}}, \lambda)$ , and  $P_M$  is the orthogonal projection in  $L^2(\Omega, \mathbb{R})$  onto the subspace  $L_M^2(\Omega, \mathbb{R}) := \text{span}\{\sin(\frac{i\pi x}{L}) \mid i \in \mathbb{N}, 1 \leq i \leq M\}$ . That is, the internal control  $\zeta = \chi P_M \eta$  stabilizes exponentially, with rate  $\frac{\lambda}{2}$ , the Burgers system to the reference trajectory  $\hat{u}$ .

Notice that the support of the control  $\zeta$  is necessarily contained in that of  $\chi$ , and that the control is finite-dimensional. Furthermore, we also know that the control can

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<sup>‡</sup>INRIA Saclay and CMAP, École Polytechnique, Route de Saclay, 91128 Palaiseau cedex, France.

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<sup>¶</sup>Johann Radon Institute for Computational and Applied Mathematics (RICAM), ÖAW, Altenbergerstraße 69, A-4040 Linz, Austria.

Emails: [axel.kroener@inria.fr](mailto:axel.kroener@inria.fr), [sergio.rodrigues@oeaw.ac.at](mailto:sergio.rodrigues@oeaw.ac.at).

be taken in feedback form,  $\zeta(t) = e^{-\lambda t} \chi P_M \chi Q_{\hat{u}}^{t, \lambda} (u(t) - \hat{u}(t))$ , for a suitable family of linear continuous operators  $Q_{\hat{u}}^{t, \lambda}: L^2(\Omega, \mathbb{R}) \rightarrow L^2(\Omega, \mathbb{R})$ ,  $t \geq 0$  (cf. [7, section 3.2]).

We can see that the dimension  $M$  of the range of the controller depends on the norm  $|\hat{u}|_{\mathcal{W}}$  of  $\hat{u}$  but, up to now no precise estimate is known. In the case  $\hat{u}$  is independent of time it is possible to give, for the case of the Navier–Stokes equations, a rather sharp description of its dimension  $M$ , though the range of the controller depends on  $\hat{u}$ ; see, for example, [2, 5, 6, 8, 39] (cf. [7, Remark 3.11(c)]). The procedure uses the spectral properties of the Oseen–Stokes system and cannot be (at least not straightforwardly) used in the time-dependent case.

The aim of this paper is to establish some first results concerning the dimension  $M$  of the range of the internal stabilizing controller, in the case of a reference time-dependent trajectory  $\hat{u}$ . Notice that this case is not less important for applications because often we are confronted with external forces  $h$  that depend on time.

In [7], the proof of existence of an  $M$ -dimensional stabilizing control uses a contradiction argument (cf. [7, proofs of Lemma A.4 and Proposition A.3]) which makes it difficult to find an estimate for  $M$ . Here we prove the existence of a stabilizing control by a more constructive procedure.

In the case we impose no restriction on the support of the control, more precisely, if we take  $\chi(x) = 1$  for all  $x \in \Omega$ , then we obtain that is it enough to take

$$(1.3) \quad M \geq \frac{L}{\pi} \left(\frac{3e}{2}\right)^{\frac{1}{2}} (\nu^{-2} |\hat{u}|_{\mathcal{W}}^2 + \nu^{-1} \lambda)^{\frac{1}{2}},$$

where  $e$  is the Napier’s constant. In the case our control is supported in a small subset  $\omega = \text{supp}(\chi)$ , we can also derive that it is “enough” to take

$$(1.4) \quad M \geq C_1 e^{C_2 \left(1 + (\nu^{-1} \lambda)^{\frac{1}{2}} + (\nu^{-1} \lambda)^{\frac{2}{3}} + \nu^{-1} |\hat{u}|_{\mathcal{W}} + \nu^{-2} |\hat{u}|_{\mathcal{W}}^2\right)}$$

where  $C_1$  and  $C_2$  are constants depending on  $\chi$  and  $\Omega$ . Estimates (1.3) and (1.4) are the main results of this paper. We easily see that the estimate in the case of the support constraint is much less reasonable, if we think about an application. The reason of the gap is that the idea used to derive (1.3) cannot be (at least not straightforwardly) used for general  $\chi(x)$ . So one question arises: can we improve (1.4)? To derive (1.4) we depart from an exact null controllability result, carrying the cost associated with the respective control. For stabilization, with a given (finite) positive rate  $\frac{\lambda}{2} > 0$ , we do not need to reach zero; that is why we believe the estimate can be improved, if we can avoid using the exact controllability result.

We have performed some numerical simulations whose results suggest that the possibility of getting, also in the general case, an estimate analogous to (1.3) is plausible. We focus on the 1D Burgers equations because the simulations are much simpler to perform in this setting. However, we believe that the difficulties to find an estimate for  $M$  will be analogous for the 2D and 3D Burgers and Navier–Stokes systems, and for a suitable class of parabolic systems.

The rest of the paper is organized as follows. In section 2 we recall some well-known results and set up our problem, in particular we recall that the problem can be reduced to the stabilization to zero of the Oseen–Burgers system. In section 3, for the linearized Oseen–Burgers system, we present the first estimates for a lower bound for the suitable dimension  $M$  of the controller; section 3.1 deals with the particular case where we impose no restriction on the support of the control and section 3.2 deals with the general case. In section 4 we consider the full nonlinear Oseen–Burgers system. The discretization of our problem is presented in section 5, and in sections 6 and 7

we present the results of some simulations we have performed. Finally, in section 8 we give a few more comments on the results.

**Notation.** We write  $\mathbb{R}$  and  $\mathbb{N}$  for the sets of real numbers and nonnegative integers, respectively, and we define  $\mathbb{R}_r := (r, +\infty)$ , for  $r \in \mathbb{R}$ , and  $\mathbb{N}_0 := \mathbb{N} \setminus \{0\}$ . We denote by  $\Omega \subset \mathbb{R}$  a bounded interval. Given a vector function  $u: (t, x) \mapsto u(t, x) \in \mathbb{R}$ , defined in an open subset of  $\mathbb{R} \times \Omega$ , its partial time derivative  $\frac{\partial u}{\partial t}$  will be denoted by  $\partial_t u$ . The partial spatial derivative  $\frac{\partial u}{\partial x}$  will be denoted by  $\partial_x u$ , and  $\frac{\partial}{\partial x} \frac{\partial}{\partial x}$  by  $\partial_{xx}$ .

Given a Banach space  $X$  and an open subset  $O \subset \mathbb{R}^n$ , let us denote by  $L^p(O, X)$ , with either  $p \in [1, +\infty)$  or  $p = \infty$ , the Bochner space of measurable functions  $f: O \rightarrow X$ , and such that  $|f|_X^p$  is integrable over  $O$ , for  $p \in [1, +\infty)$ , and such that  $\text{ess sup}_{x \in O} |f(x)|_X < +\infty$ , for  $p = \infty$ . In the case  $X = \mathbb{R}$  we recover the usual Lebesgue spaces. By  $W^{s,p}(O, \mathbb{R})$ , for  $s \in \mathbb{R}$ , denote the usual Sobolev space of order  $s$ . In the case  $p = 2$ , as usual, we denote  $H^s(O, \mathbb{R}) := W^{s,2}(O, \mathbb{R})$ . Recall that  $H^0(O, \mathbb{R}) = L^2(O, \mathbb{R})$ . For each  $s > 0$ , we recall also that  $H^{-s}(O, \mathbb{R})$  stands for the dual space of  $H_0^s(O, \mathbb{R}) = \text{closure of } \{f \in C^\infty(O, \mathbb{R}) \mid \text{supp } f \subset O\}$  in  $H^s(O, \mathbb{R})$ . Notice that  $H^{-s}(O, \mathbb{R})$  is a space of distributions.

For a normed space  $X$ , we denote by  $|\cdot|_X$  the corresponding norm, by  $X'$  its dual, and by  $\langle \cdot, \cdot \rangle_{X', X}$  the duality between  $X'$  and  $X$ . The dual space is endowed with the usual dual norm:  $|f|_{X'} := \sup\{\langle f, x \rangle_{X', X} \mid x \in X \text{ and } |x|_X = 1\}$ . In the case that  $X$  is a Hilbert space we denote the inner product by  $(\cdot, \cdot)_X$ .

Given an open interval  $I \subseteq \mathbb{R}$  and two Banach spaces  $X$  and  $Y$ , we write  $W(I, X, Y) := \{f \in L^2(I, X) \mid \partial_t f \in L^2(I, Y)\}$ , where the derivative  $\partial_t f$  is taken in the sense of distributions. This space is endowed with the natural norm  $|f|_{W(I, X, Y)} := (|f|_{L^2(I, X)}^2 + |\partial_t f|_{L^2(I, Y)}^2)^{\frac{1}{2}}$ . In the case  $X = Y$  we write  $H^1(I, X) := W(I, X, X)$ . Again, if  $X$  and  $Y$  are endowed with a scalar product, then also  $W(I, X, Y)$  is. The space of continuous linear mappings from  $X$  into  $Y$  will be denoted by  $\mathcal{L}(X \rightarrow Y)$ .

If  $\bar{I} \subset \mathbb{R}$  is a closed bounded interval,  $C(\bar{I}, X)$  stands for the space of continuous functions  $f: \bar{I} \rightarrow X$  with the norm  $|f|_{C(\bar{I}, X)} = \max_{t \in \bar{I}} |f(t)|_X$ .

$\bar{C}_{[a_1, \dots, a_k]}$  denotes a nonnegative function of nonnegative variables  $a_j$  that increases in each of its arguments.

$C, C_i, i = 1, 2, \dots$ , stand for unessential positive constants.

## 2. Preliminaries.

**2.1. Reduction to local null stabilization.** We will denote  $V := H_0^1(\Omega, \mathbb{R})$ ,  $H := L^2(\Omega, \mathbb{R})$ ,  $D(\partial_{xx}) := V \cap H^2(\Omega, \mathbb{R})$ , and  $V' := H^{-1}(\Omega, \mathbb{R})$ . The space  $H$  is supposed to be endowed with the usual  $L^2(\Omega, \mathbb{R})$ -scalar product; the space  $V$  with the scalar product  $(u, v)_V := (\partial_x u, \partial_x v)_H$ . The space  $H$  is taken a pivot space, and  $V'$  is the dual of  $V$ . The inclusions  $V \subset H \subset V'$  are dense, continuous and compact. The space  $D(\partial_{xx})$  is endowed with the scalar product  $(u, v)_{D(\partial_{xx})} := (\partial_{xx} u, \partial_{xx} v)_H$ .

Let us denote

$$(2.1) \quad \mathcal{W} := L^\infty(\mathbb{R}_0, L^\infty(\Omega, \mathbb{R}))$$

and, for given Banach spaces  $X$  and  $Y$ ,

$$\begin{aligned} L_{\text{loc}}^2(\mathbb{R}_0, X) &:= \{f \mid f|_{(0, T)} \in L^2((0, T), X) \text{ for all } T > 0\}, \\ W_{\text{loc}}(\mathbb{R}_0, X, Y) &:= \{f \mid f|_{(0, T)} \in W((0, T), X, Y) \text{ for all } T > 0\}. \end{aligned}$$

Fix a function  $h \in L_{\text{loc}}^2(\mathbb{R}_0, V')$  and suppose that  $\hat{u} \in \mathcal{W} \cap W_{\text{loc}}(\mathbb{R}_0, V, V')$  solves the Burgers system (1.1), with  $\zeta = 0$  and initial condition  $\hat{u}_0 := \hat{u}(0) \in H$ .

Let us be given a Lipschitz continuous function  $\chi \in W^{1,\infty}(\Omega, \mathbb{R})$  with nonempty support and  $\lambda > 0$ . Then, given another function  $u_0$  such that  $|u_0 - \hat{u}(0)|_H$  is small enough. Let also  $\mathcal{O} = (l_1, l_2)$  be an open interval, such that  $\text{supp}(\chi) \subseteq \overline{\mathcal{O}} \subseteq \overline{\Omega}$ .

Our goal is to find an integer  $M \in \mathbb{N}_0$  and a control  $\eta \in L^2(\mathbb{R}_0, H)$  such that the solution of the problem (1.1)–(1.2), with  $\zeta = \chi \mathbb{E}_0^\mathcal{O} P_M^\mathcal{O}(\eta|_\mathcal{O})$  is defined for all  $t > 0$  and converges exponentially to  $\hat{u}$ , that is, for some positive constant  $C > 0$  independent of  $u_0 - \hat{u}_0$ ,

$$(2.2) \quad |u(t) - \hat{u}(t)|_H^2 \leq C e^{-\lambda t} |u_0 - \hat{u}_0|_H^2 \quad \text{for } t \geq 0.$$

Here  $P_M^\mathcal{O}$  stands for the orthogonal projection in  $L^2(\mathcal{O}, \mathbb{R})$  onto the subspace spanned by the first  $M$  eigenfunctions  $\underline{s}_n$  of the Dirichlet Laplacian in  $\mathcal{O}$ , that is, onto

$$L_M^2(\mathcal{O}, \mathbb{R}) := \text{span}\{\underline{s}_n \mid n \in \mathbb{N}_0, n \leq M\}$$

where  $\mathbb{E}_0^\mathcal{O}: L^2(\mathcal{O}, \mathbb{R}) \rightarrow H$  is the extension by zero outside  $\mathcal{O}$ , defined by  $\mathbb{E}_0^\mathcal{O} f(x) := \begin{cases} f(x) & \text{if } x \in \mathcal{O} \\ 0 & \text{if } x \in \Omega \setminus \overline{\mathcal{O}} \end{cases}$ . Recall that it is well-known that the complete system of (normalized) Dirichlet eigenfunctions  $\{\underline{s}_n \mid n \in \mathbb{N}_0\} \subset D(\partial_{xx})$  and the corresponding system of eigenvalues  $\{\alpha_n \mid n \in \mathbb{N}_0\}$  are given explicitly by

$$(2.3) \quad \underline{s}_n(x) := \left(\frac{2}{l}\right)^{\frac{1}{2}} \sin\left(\frac{n\pi(x-l_1)}{l}\right), \quad \alpha_n = \left(\frac{\pi}{l}\right)^2 n^2, \quad -\partial_{xx}\underline{s}_n = \alpha_n \underline{s}_n, \quad x \in \mathcal{O},$$

where  $l = l_2 - l_1$  stands for the length of  $\mathcal{O}$ .

Let us notice that, seeking for the control  $\eta$  and considering the corresponding solution  $u$ , we find that  $v = u - \hat{u}$ , will solve the Oseen–Burgers system

$$(2.4) \quad \partial_t v - \nu \partial_{xx} v + v \partial_x v + \partial_x(\hat{u}v) + \zeta = 0, \quad v|_\Gamma = 0, \quad v(0) = v_0,$$

with  $\zeta = \chi \mathbb{E}_0^\mathcal{O} P_M^\mathcal{O}(\eta|_\mathcal{O})$  and  $v_0 = u(0) - \hat{u}(0)$ . It is now clear that to achieve (2.2) it suffices to consider the problem of local exponential stabilization to zero for solutions of (2.4), where “local” means that the property is to hold “provided  $|v_0|_H$  is small enough”.

**2.2. Weak solutions.** The existence and uniqueness of weak solutions for system (2.4) can be proved by classical arguments, where weak solutions are understood in the classical sense as in [37, chapter 1, sections 6.1 and 6.4], [42, sections 2.4 and 3.2], [43, chapter 3, section 3].

**THEOREM 2.1.** *Given  $\hat{u} \in \mathcal{W}$ ,  $\zeta \in L^2((0, T), V')$ , and  $v_0 \in H$ , then there exists a weak solution  $v \in W((0, T), V, V')$  for system (2.4), in  $(0, T) \times \Omega$ . Moreover  $v$  is unique and depends continuously on the given data  $(v_0, \eta)$ :*

$$(2.5) \quad |v|_{W((0, T), V, V')}^2 \leq \overline{C}_{[T, |\hat{u}|_{\mathcal{W}}]} \left( |v_0|_H^2 + |\zeta|_{L^2((0, T), V')}^2 \right).$$

Notice that the proof of the existence and uniqueness of a weak solution can be done following the argument in [43, chapter 3, section 3.2] by using the estimate

$$|\partial_x(wv)|_{V'}^2 \leq C |w|_{L^\infty(\Omega, \mathbb{R})}^2 |v|_{L^2(\Omega, \mathbb{R})}^2 \leq C_1 |w|_{H^1(\Omega, \mathbb{R})}^2 |v|_{L^2(\Omega, \mathbb{R})}^2 \leq C_2 |w|_V^2 |v|_H^2.$$

**DEFINITION 2.2.** *We say that  $v \in W_{\text{loc}}(\mathbb{R}_0, V, V')$  is a global weak solution for system (2.4), in  $\mathbb{R}_0 \times \Omega$ , if  $v|_{(0, T)} \in W((0, T), V, V')$  is a weak solution, for the same system, in  $(0, T) \times \Omega$ , for all  $T > 0$ .*

**COROLLARY 2.3.** *Given  $\hat{u} \in \mathcal{W}$ ,  $\zeta \in L^2_{\text{loc}}(\mathbb{R}_0, V')$ , and  $v_0 \in H$ , then there exists a weak solution  $v \in W_{\text{loc}}(\mathbb{R}_0, V, V')$  for system (2.4), in  $\mathbb{R}_0 \times \Omega$ , which is unique and there holds estimate (2.5).*

Finally notice that system (1.1)–(1.2), is a particular case of (2.4) (with  $\hat{u} = 0$ ), hence Theorem 2.1 and Corollary 2.3 also hold for (1.1)–(1.2) (with  $h + \zeta$  in the role of  $\zeta$ ).

**3. The Oseen–Burgers system. The dimension of the controller.** Here we look for a control in the form  $\zeta = \chi \mathbb{E}_0^\mathcal{O} P_M^\mathcal{O}(\eta|_\mathcal{O})$ , with  $\eta \in L^2(\mathbb{R}_0, H)$ , that stabilizes exponentially the linearized Oseen–Burgers system

$$(3.1) \quad \partial_t v - \nu \partial_{xx} v + \partial_x(\hat{u}v) + \zeta = 0, \quad v|_\Gamma = 0, \quad v(0) = v_0,$$

to zero, with a desired exponential rate  $\frac{\lambda}{2} > 0$ . We also provide some first estimates, concerning a lower bound for the integer  $M$ , depending on the triple  $(\lambda, |\hat{u}|_\mathcal{W}, \nu)$ . Later, the results will follow for system (2.4), provided  $|v_0|_H$  is small enough, by a fixed point argument.

*Remark 3.1.* *Theorem 2.1 and Corollary 2.3, also hold for system (3.1) in the role of system (2.4).*

It is well known that controllability properties for system 3.1 are closely related to observability properties for the “time-backward” adjoint Oseen–Burgers

$$(3.2) \quad -\partial_t q - \nu \partial_{xx} q - \hat{u} \partial_x q + f = 0, \quad q|_\Gamma = 0, \quad q(T) = q_1$$

for  $q_1 \in H$  and  $f \in L^2((0, T), V')$ ; below, in section 3.2, we will use some suitable observability inequalities for this adjoint system.

**3.1. The particular case  $\chi = 1_\Omega$ .** We consider the case  $\mathcal{O} = \Omega$  and  $\chi = 1_\Omega$ , with  $1_\Omega(x) := 1$  for all  $x \in \Omega$ . In particular, there is no constraint in the support of the controller.

**THEOREM 3.2.** *For given  $\hat{u} \in \mathcal{W}$  and  $\lambda > 0$ , set*

$$(3.3) \quad M \geq \frac{L}{\pi} \left(\frac{3e}{2}\right)^{\frac{1}{2}} \left(\frac{1}{\nu^2} |\hat{u}|_\mathcal{W}^2 + \frac{1}{\nu} \lambda\right)^{\frac{1}{2}},$$

where  $e$  is the Napier’s constant. Then for any given  $v_0 \in H$ , there is a control  $\eta^{\lambda, \hat{u}, \nu}(v_0) \in L^2(\mathbb{R}_0, H)$  such that the corresponding solution  $v$  of system (3.1), with  $\zeta = \chi \mathbb{E}_0^\mathcal{O} P_M^\mathcal{O}(\eta^{\lambda, \hat{u}, \nu}|_\mathcal{O})$ , satisfies the inequality

$$(3.4) \quad |v(t)|_H^2 \leq (1 + e^{\frac{1}{2}}) e^{-\lambda t} |v_0|_H^2, \quad t \geq 0.$$

Moreover, the mapping  $v_0 \mapsto \eta^{\lambda, \hat{u}, \nu}(v_0)$  is well defined, is linear, and satisfies

$$|e^{(\hat{\lambda}/2)t} \eta^{\lambda, \hat{u}, \nu}(v_0)|_{L^2(\mathbb{R}_0, H)}^2 \leq \frac{4e^{\frac{1}{2}}}{\lambda - \hat{\lambda}} \left(\frac{1}{\nu} |\hat{u}|_\mathcal{W}^2 + \lambda\right) |v_0|_H^2, \quad \text{for } 0 \leq \hat{\lambda} < \lambda.$$

*Proof.* Let  $w$  solve

$$(3.5) \quad \partial_t w = \nu \partial_{xx} w - \partial_x(\hat{u}w) + \frac{\lambda}{2} w, \quad w|_\Gamma = 0, \quad w(0) = v_0.$$

By standard arguments we can find

$$\begin{aligned} \frac{d}{dt} |w|_H^2 &\leq -2\nu |\partial_x w|_H^2 + 2|\hat{u}|_{L^\infty(\Omega, \mathbb{R})} |w|_H |\partial_x w|_H + \lambda |w|_H^2 \\ &\leq \frac{1}{2\nu} |\hat{u}|_{L^\infty(\Omega, \mathbb{R})}^2 |w|_H^2 + \lambda |w|_H^2 \end{aligned}$$

from which we can derive that

$$(3.6) \quad |w|_{L^\infty((0,T),H)}^2 \leq e^{(\frac{1}{2\nu}|\hat{u}|_{\mathcal{W}}^2 + \lambda)T} |v_0|_H^2.$$

Now let  $\varphi(t) := 1 - \frac{t}{T} \in C^1([0, T], \mathbb{R})$ , and set  $\delta := \varphi w$ . Notice that  $\delta$  solves

$$\partial_t \delta = \nu \partial_{xx} \delta - \partial_x(\hat{u} \delta) + \frac{\lambda}{2} \delta + (\partial_t \varphi) w, \quad \delta|_\Gamma = 0, \quad \delta(0) = v_0$$

with  $\delta(T) = 0$ . Let now  $M \in \mathbb{N}_0$  be a positive integer and consider the solution  $\delta_M$  for the system

$$\partial_t \delta_M = \nu \partial_{xx} \delta_M - \partial_x(\hat{u} \delta_M) + \frac{\lambda}{2} \delta_M + (\partial_t \varphi) P_M^\Omega w, \quad \delta_M|_\Gamma = 0, \quad \delta_M(0) = v_0.$$

The difference  $d := \delta - \delta_M$  solves

$$\partial_t d = \nu \partial_{xx} d - \partial_x(\hat{u} d) + \frac{\lambda}{2} d + (\partial_t \varphi)(1 - P_M^\Omega) w, \quad d|_\Gamma = 0, \quad d(0) = 0,$$

from which we can also derive

$$\begin{aligned} |d|_{L^\infty((0,T),H)}^2 &\leq e^{(\frac{3}{2\nu}|\hat{u}|_{\mathcal{W}}^2 + \lambda)T} \left( |d(0)|_H^2 + \frac{3}{4\nu} |(\partial_t \varphi)(1 - P_M^\Omega) w|_{L^2((0,T),V')}^2 \right) \\ &\leq T^{-2} e^{(\frac{3}{2\nu}|\hat{u}|_{\mathcal{W}}^2 + \lambda)T} \frac{3}{4\nu} \alpha_M^{-1} |w|_{L^2((0,T),H)}^2 \end{aligned}$$

and, from  $|w|_{L^2((0,T),H)}^2 \leq T |w|_{L^\infty((0,T),H)}^2$  and (3.6), we can arrive at

$$|d|_{L^\infty((0,T),H)}^2 \leq T^{-1} e^{2(\nu^{-1}|\hat{u}|_{\mathcal{W}}^2 + \lambda)T} \frac{3}{4\nu} \alpha_M^{-1} |v_0|_H^2.$$

Since we are interested in the stabilization of the system, we can see  $T$  as a parameter at our disposal. Minimizing the right hand side over  $T > 0$ , we can see that the minimizer  $T_*$  is defined by  $T_*^{-1} := 2(\nu^{-1}|\hat{u}|_{\mathcal{W}}^2 + \lambda)$ ; then, setting  $T = T_*$  we have that

$$(3.7) \quad |d|_{L^\infty((0,T_*),H)}^2 \leq 2(\nu^{-1}|\hat{u}|_{\mathcal{W}}^2 + \lambda) e^1 \frac{3}{4\nu} \alpha_M^{-1} |v_0|_H^2.$$

Now, from  $\alpha_M = (\frac{M\pi}{L})^2$  (cf. (2.3), with  $\mathcal{O} = \Omega$ ), setting  $M$  satisfying (3.3), and recalling that  $\delta_M(0) = v_0$  and  $\delta_M(T_*) = -d(T_*)$ , we find

$$(3.8) \quad \alpha_M \geq (\nu^{-1}|\hat{u}|_{\mathcal{W}}^2 + \lambda) \frac{3e^1}{2\nu}$$

and  $|\delta_M(T_*)|_H^2 \leq |\delta_M(0)|_H^2$ .

Further, from (3.6) and (3.7), we find  $|\delta_M|_{L^\infty((0,T_*),H)}^2 = |\delta - d|_{L^\infty((0,T_*),H)}^2 \leq C_M^\delta |\delta_M(0)|_H^2$ , with

$$\begin{aligned} C_M^\delta &:= e^{(\frac{1}{2\nu}|\hat{u}|_{\mathcal{W}}^2 + \lambda)T_*} + 2(\nu^{-1}|\hat{u}|_{\mathcal{W}}^2 + \lambda) e^1 \frac{3}{4\nu} \alpha_M^{-1} \\ &\leq e^{\frac{1}{2}} + 2(\nu^{-1}|\hat{u}|_{\mathcal{W}}^2 + \lambda) e^1 \frac{3}{4\nu} \alpha_M^{-1} \leq e^{\frac{1}{2}} + 1 =: \Upsilon_\delta. \end{aligned}$$

Now, notice that we can consider system (3.5) in  $(T_*, +\infty) \times \Omega$  with  $w(T_*) = \delta_M(T_*)$ , and repeat the arguments. Recursively, we conclude that in each interval  $J_*^i := (iT_*, (i+1)T_*)$ ,  $i \in \mathbb{N}_0$ , we have  $|\delta_M((i+1)T_*)|_H^2 \leq |\delta_M(iT_*)|_H^2$  and  $|\delta_M|_{L^\infty(J_*^i, H)}^2 \leq \Upsilon_\delta |\delta_M(iT_*)|_H^2$ . Hence, we conclude that  $|\delta_M|_{L^\infty(\mathbb{R}_0, H)}^2 \leq \Upsilon_\delta |v_0|_H^2$ .

Next we notice that  $v := e^{-\frac{\lambda}{2}t} \delta_M$  solves (3.1), in  $\mathbb{R}_0 \times \Omega$ , with the concatenated control  $\zeta = \chi P_M^\Omega (e^{-\frac{\lambda}{2}t} (-T_*^{-1}) w) = -T_*^{-1} e^{-\frac{\lambda}{2}t} \chi P_M^\Omega w$ , where  $w|_{J_*^i} =: w_i$  solves (3.5),

in  $J_*^i \times \Omega$ , with  $w_i(iT_*) = w(iT_*) = \delta_M(iT_*)$ ; from (3.6) and from the boundedness of  $\{|\delta_M(iT_*)|_H \mid i \in \mathbb{N}\}$ , we can conclude that the family  $\{|w|_{L^2(J_*^i, H)} \mid i \in \mathbb{N}\}$  is bounded; so we have that  $e^{\frac{\lambda}{2}t}\zeta \in L^2(\mathbb{R}_0, H)$  for all  $\hat{\lambda} < \lambda$ . Finally we observe that  $|v(t)|_H^2 \leq e^{-\lambda t} |\delta_M|_{L^\infty(\mathbb{R}_0, H)}^2 \leq \Upsilon_\delta e^{-\lambda t} |v_0|_H^2$ , and that for  $\eta^{\lambda, \hat{u}, \nu} := e^{-\frac{\lambda}{2}t}(-T_*^{-1})w$ ,

$$\begin{aligned} \left| e^{\frac{\lambda}{2}t} \eta^{\lambda, \hat{u}, \nu} \right|_{L^2(\mathbb{R}_0, H)}^2 &= \int_{\mathbb{R}_0} e^{(\hat{\lambda}-\lambda)s} T_*^{-2} |w(s)|_H^2 ds \\ &\leq \frac{1}{\lambda-\hat{\lambda}} T_*^{-2} e^{\left(\frac{1}{2\nu}|\hat{u}|_{\mathcal{W}}^2 + \lambda\right)T_*} |v_0|_H^2 \leq \frac{1}{\lambda-\hat{\lambda}} (2(\nu^{-1}|\hat{u}|_{\mathcal{W}}^2 + \lambda))^2 e^{\frac{1}{2}} |v_0|_H^2. \end{aligned}$$

That is,  $\left| e^{\frac{\lambda}{2}t} \eta^{\lambda, \hat{u}, \nu} \right|_{L^2(\mathbb{R}_0, H)}^2 \leq \frac{4e^{\frac{1}{2}}}{\lambda-\hat{\lambda}} (\nu^{-1}|\hat{u}|_{\mathcal{W}}^2 + \lambda)^2 |v_0|_H^2$ .  $\square$

**3.2. The general case.** Let  $w$  solve the system

$$(3.9) \quad \partial_t w = \nu \partial_{xx} w - \partial_x(\hat{u}w) + \frac{\lambda}{2}w + \chi \tilde{\eta}, \quad w|_\Gamma = 0, \quad w(0) = v_0.$$

To simplify the exposition we rescale time as  $t = \frac{\tau}{\nu}$ . Then  $\check{w}(\tau) := w(\frac{\tau}{\nu})$  solves

$$(3.10) \quad \partial_\tau \check{w} = \partial_{xx} \check{w} - \partial_x(\check{u}\check{w}) + \frac{\check{\lambda}}{2}\check{w} + \chi \check{\eta}, \quad \check{w}|_\Gamma = 0, \quad \check{w}(0) = v_0,$$

with  $(\check{u}, \check{\lambda}, \check{\eta}) = \nu^{-1}(\hat{u}, \lambda, \tilde{\eta})$ . Next, consider the adjoint system

$$(3.11) \quad -\partial_\tau q = \partial_{xx} q + \check{u} \partial_x q + \frac{\check{\lambda}}{2}q, \quad q|_\Gamma = 0, \quad q(T) = q_T$$

with  $q_T \in H$  (here with no external force; cf. system (3.2)). From, for example, [18, Theorem 2.1] and [17, Theorem 2.3] (e.g., reversing time in system (3.11)), we have that given an open set  $\omega \subseteq \Omega$ , there exists a constant  $C_{\omega, \Omega} > 0$ , depending on  $\omega$  and  $\Omega$ , such that for any time  $T > 0$ , the weak solution  $q$  for (3.11) satisfies

$$(3.12) \quad |q(0)|_H^2 \leq e^{C_{\omega, \Omega} \left(1 + \frac{1}{T} + T\check{\lambda} + \check{\lambda}^{\frac{2}{3}} + (1+T)|\check{u}|_{\mathcal{W}}^2\right)} |q|_{L^2((0, T), L^2(\omega, \mathbb{R}))}^2.$$

**PROPOSITION 3.3.** *For every  $v_0 \in H$ , we can find a control  $\check{\eta} = \bar{\eta}(v_0) \in L^2((0, T), H)$ , driving system (3.10) to  $\check{w}(T) = 0$  at time  $t = T > 0$ . Moreover, the mapping  $\bar{\eta}: v_0 \mapsto \bar{\eta}(v_0)$  is linear and continuous:  $\bar{\eta} \in \mathcal{L}(H \rightarrow L^2((0, T), H))$ , and there is a constant  $C_{\chi, \Omega}$  such that*

$$(3.13) \quad |\bar{\eta}(v_0)|_{L^2((0, T), H)}^2 \leq e^{C_{\chi, \Omega} \left(1 + \frac{1}{T} + T\check{\lambda} + \check{\lambda}^{\frac{2}{3}} + (1+T)|\check{u}|_{\mathcal{W}}^2\right)} |v_0|_H^2.$$

*Sketch of the proof.* The proof can be done following the arguments in [7]. First, from (3.12) we can derive an observability of the form

$$(3.14) \quad |q(0)|_H^2 \leq e^{C_{\chi, \Omega} \left(1 + \frac{1}{T} + T\check{\lambda} + \check{\lambda}^{\frac{2}{3}} + (1+T)|\check{u}|_{\mathcal{W}}^2\right)} |\chi q|_{L^2((0, T), H)}^2$$

for the solution  $q$  of system (3.11) (cf. [7, eq. (A.8)]). Then we can prove the null controllability considering the following minimization problem (cf. [7, Problem 3.3])

$$J_\epsilon(\check{w}, \check{\eta}) = |\check{\eta}|_{L^2}^2 + \frac{1}{\epsilon} |\check{w}(T)|_H^2 \rightarrow \min; \quad \text{with } (\check{w}, \check{\eta}) \text{ solving (3.10).}$$



Next we can consider the minimization problem

$$J_\infty(\check{w}, \check{\eta}) = |\check{\eta}|_{L^2}^2 \rightarrow \min; \quad \text{with } (\check{w}, \check{\eta}) \text{ solving (3.10) and } \check{w}(T) = 0$$

whose unique minimizer  $(\bar{w}, \bar{\eta})(v_0)$  depends linearly on  $v_0$  (cf. [7, Problem 3.4]).  $\square$

Considering the null controllability of linear parabolic equations we also refer to [23, section 2] and [46, section 5.2.2], and references therein.

**THEOREM 3.4.** *For given  $\hat{u} \in \mathcal{W}$  and  $\lambda > 0$ , set*

$$(3.15) \quad M \geq \frac{l}{\pi} C_{\chi, \Omega}^0 e^{\frac{3}{2}(1+C_{\chi, \Omega})} \left(1 + \left(\frac{\lambda}{\nu}\right)^{\frac{1}{2}} + \left(\frac{\lambda}{\nu}\right)^{\frac{2}{3}} + \frac{1}{\nu} |\hat{u}|_{\mathcal{W}} + \frac{1}{\nu^2} |\hat{u}|_{\mathcal{W}}^2\right),$$

where  $C_{\chi, \Omega}^0 = (2 + 2(\frac{l}{\pi})^2)^{\frac{1}{2}} |\chi|_{W^{1, \infty}(\Omega, \mathbb{R})}$ ,  $l$  is the length of  $\mathcal{O}$ , and  $C_{\chi, \Omega}$  is the constant from (3.13). Then for any given  $v_0 \in H$ , there is a control  $\eta^{\lambda, \hat{u}, \nu}(v_0) \in L^2(\mathbb{R}_0, H)$  such that, taking  $\zeta = \chi \mathbb{E}_0^{\mathcal{O}} P_M^{\mathcal{O}}(\eta^{\lambda, \hat{u}, \nu}|_{\mathcal{O}})$ , the corresponding solution  $v$  of system (3.1) satisfies, for  $t \geq 0$ , the inequality

$$(3.16) \quad |v(t)|_H^2 \leq K_{\chi, \Omega} e^{-\lambda t} |v_0|_H^2,$$

$$\text{with } K_{\chi, \Omega} := \left(1 + e^{\left(\frac{\lambda}{\nu} + \frac{1}{\nu^2} |\hat{u}|_{\mathcal{W}}^2\right)^{\frac{1}{2}}} + \frac{(C_{\chi, \Omega}^0)^2}{\alpha_1} e^{3(C_{\chi, \Omega} + 1)} \left(1 + \left(\frac{\lambda}{\nu}\right)^{\frac{1}{2}} + \left(\frac{\lambda}{\nu}\right)^{\frac{2}{3}} + \frac{1}{\nu} |\hat{u}|_{\mathcal{W}} + \frac{1}{\nu^2} |\hat{u}|_{\mathcal{W}}^2\right)\right).$$

Moreover, the mapping  $v_0 \mapsto \eta^{\lambda, \hat{u}, \nu}(v_0)$  is well defined, is linear, and satisfies for  $0 \leq \hat{\lambda} < \lambda$  the inequality

$$|e^{\frac{\hat{\lambda}}{2} t} \eta^{\hat{\lambda}, \lambda}(v_0)|_{L^2(\mathbb{R}_0, H)}^2 \leq \frac{\nu e^{C_{\chi, \Omega} \left(1 + 3\left(\frac{\lambda}{\nu}\right)^{\frac{1}{2}} + \left(\frac{\lambda}{\nu}\right)^{\frac{2}{3}} + 3\frac{1}{\nu} |\hat{u}|_{\mathcal{W}} + \frac{1}{\nu^2} |\hat{u}|_{\mathcal{W}}^2\right)}}{1 - e^{(\hat{\lambda} - \lambda)(2(\nu\lambda + |\hat{u}|_{\mathcal{W}}^2))^{-\frac{1}{2}}}} |v_0|_H^2.$$

*Proof.* Let  $\check{w}$  solve (3.10), for  $t \in (0, T)$ , with the control  $\check{\eta} = \bar{\eta}(v_0)$  given by Proposition 3.3, and let  $\check{w}_M$  be the solution of

$$\partial_\tau \check{w}_M = \partial_{xx} \check{w}_M - \partial_x(\check{u} \check{w}_M) + \frac{\check{\lambda}}{2} \check{w}_M + \chi \mathbb{E}_0^{\mathcal{O}} P_M^{\mathcal{O}}(\bar{\eta}(v_0)|_{\mathcal{O}}), \quad \check{w}_M|_{\Gamma} = 0, \quad \check{w}_M(0) = v_0.$$

Then, the difference  $d := \check{w} - \check{w}_M$  solves

$$\partial_\tau d = \partial_{xx} d - \partial_x(\check{u} d) + \frac{\check{\lambda}}{2} d + \chi \mathbb{E}_0^{\mathcal{O}} (1 - P_M^{\mathcal{O}})(\bar{\eta}(v_0)|_{\mathcal{O}}), \quad d|_{\Gamma} = 0, \quad d(0) = 0,$$

and taking the scalar product with  $d$ , in  $H$ , we can arrive at

$$(3.17) \quad \begin{aligned} \frac{d}{d\tau} |d|_H^2 &\leq -2 |\partial_x d|_H^2 + 2 |\check{u}|_{L^\infty(\Omega, \mathbb{R})} |d|_H |\partial_x d|_H + \check{\lambda} |d|_H^2 \\ &\quad + 2 \langle \chi \mathbb{E}_0^{\mathcal{O}} (1 - P_M^{\mathcal{O}})(\bar{\eta}(v_0)|_{\mathcal{O}}), d \rangle_{V', V}. \end{aligned}$$

For the last term we find

$$\begin{aligned} \langle \chi \mathbb{E}_0^{\mathcal{O}} (1 - P_M^{\mathcal{O}})(\bar{\eta}(v_0)|_{\mathcal{O}}), d \rangle_{V', V} &= (\chi \mathbb{E}_0^{\mathcal{O}} (1 - P_M^{\mathcal{O}})(\bar{\eta}(v_0)|_{\mathcal{O}}), d)_H \\ &\leq |\bar{\eta}(v_0)|_{\mathcal{O}}|_{L^2(\mathcal{O}, \mathbb{R})} |(1 - P_M^{\mathcal{O}})(\chi d|_{\mathcal{O}})|_{L^2(\mathcal{O}, \mathbb{R})}, \end{aligned}$$

and from

$$|(1 - P_M^{\mathcal{O}})(\chi d|_{\mathcal{O}})|_{L^2(\mathcal{O}, \mathbb{R})}^2 \leq \alpha_M^{-1} |(1 - P_M^{\mathcal{O}})(\chi d|_{\mathcal{O}})|_V^2 \leq 2\alpha_M^{-1} |\chi|_{W^{1, \infty}(\Omega, \mathbb{R})}^2 |d|_{H_0^1(\Omega, \mathbb{R})}^2$$

(which makes sense for a.e.  $t \in (0, T)$ , since from  $2\langle \chi \mathbb{E}_0^{\mathcal{O}}(1 - P_M^{\mathcal{O}})(\bar{\eta}(v_0)|_{\mathcal{O}}), d \rangle_{V', V} \leq 2|\chi \mathbb{E}_0^{\mathcal{O}}(1 - P_M^{\mathcal{O}})(\bar{\eta}(v_0)|_{\mathcal{O}})|_H |d|_H$  and (3.17), by standard arguments it follows that  $d \in L^\infty((0, T), H) \cap L^2((0, T), V)$ ) and  $|\partial_x d|_{L^2(\Omega, \mathbb{R})}^2 \geq \frac{\alpha_1^\Omega}{1 + \alpha_1^\Omega} |d|_{H_0^1(\Omega, \mathbb{R})}^2$ , where  $\alpha_1^\Omega = \frac{\pi^2}{L^2}$ , we find

$$\langle \chi \mathbb{E}_0^{\mathcal{O}}(1 - P_M^{\mathcal{O}})(\bar{\eta}(v_0)|_{\mathcal{O}}), d \rangle_{V', V} \leq \alpha_M^{-\frac{1}{2}} D_{\chi, \Omega} |\partial_x d|_{L^2(\Omega, \mathbb{R})} |\bar{\eta}(v_0)|_{\mathcal{O}}|_{L^2(\mathcal{O}, \mathbb{R})},$$

with  $D_{\chi, \Omega} = (2\frac{1 + \alpha_1^\Omega}{\alpha_1^\Omega})^{\frac{1}{2}} |\chi|_{W^{1, \infty}(\Omega, \mathbb{R})} = (2 + 2(\frac{L}{\pi})^2)^{\frac{1}{2}} |\chi|_{W^{1, \infty}(\Omega, \mathbb{R})}$ . Then, from (3.17),

$$\frac{d}{d\tau} |d|_H^2 \leq |\check{u}|_{L^\infty(\Omega, \mathbb{R})}^2 |d|_H^2 + \check{\lambda} |d|_H^2 + \alpha_M^{-1} D_{\chi, \Omega}^2 |\bar{\eta}(v_0)|_{\mathcal{O}}|_{L^2(\mathcal{O}, \mathbb{R})}^2$$

and, using (3.13), we obtain

$$(3.18) \quad |d|_{L^\infty((0, T), H)}^2 \leq \alpha_M^{-1} D_{\chi, \Omega}^2 e^{C_{\chi, \Omega}(1 + \check{\lambda}^{\frac{2}{3}} + |\check{u}|_{\mathcal{W}}^2)} e^{C_{\chi, \Omega}(\frac{1}{T} + 2(\check{\lambda} + |\check{u}|_{\mathcal{W}}^2)T)} |v_0|_H^2$$

with  $C_{\chi, \Omega}^1 = \max\{1, C_{\chi, \Omega}\}$ . Now the function  $E(T) = e^{C_{\chi, \Omega}(\frac{1}{T} + 2(\check{\lambda} + |\check{u}|_{\mathcal{W}}^2)T)}$  takes its minimum when  $T = T_*$ , with  $T_*$  defined by  $\frac{1}{T_*} = 2(\check{\lambda} + |\check{u}|_{\mathcal{W}}^2)$ . Then, choosing  $T = T_*$ , and recalling that  $\check{w}_M(T) = -d(T)$  and  $\check{w}_M(0) = v_0$ , we arrive at

$$|\check{w}_M(T_*)|_H^2 \leq \alpha_M^{-1} D_{\chi, \Omega}^2 e^{C_{\chi, \Omega}(1 + \check{\lambda}^{\frac{2}{3}} + |\check{u}|_{\mathcal{W}}^2)} e^{2\frac{2}{3} C_{\chi, \Omega}^1 (\check{\lambda} + |\check{u}|_{\mathcal{W}}^2)^{\frac{1}{2}}} |\check{w}_M(0)|_H^2.$$

Thus, choosing  $M \in \mathbb{N}_0$  satisfying (3.15) and recalling that  $\alpha_M = (\frac{M\pi}{L})^2$ , we have

$$(3.19) \quad \alpha_M \geq D_{\chi, \Omega}^2 e^{3(C_{\chi, \Omega} + 1)(1 + \check{\lambda}^{\frac{2}{3}} + |\check{u}|_{\mathcal{W}}^2 + \check{\lambda}^{\frac{1}{2}} + |\check{u}|_{\mathcal{W}})}$$

and  $|\check{w}_M(T_*)|_H^2 \leq |\check{w}_M(0)|_H^2$ . Moreover we can deduce from (3.13) and (3.18) that  $|\check{w}_M|_{L^\infty((0, T_*), H)}^2 = |\check{w} - d|_{L^\infty((0, T_*), H)}^2 \leq C_M^d |\check{w}_M(0)|_H^2$ , with

$$\begin{aligned} C_M^d &:= e^{(\check{\lambda} + |\check{u}|_{\mathcal{W}}^2)T_*} + (\alpha_1^{-1} + \alpha_M^{-1}) D_{\chi, \Omega}^2 e^{C_{\chi, \Omega}(1 + \check{\lambda}^{\frac{2}{3}} + |\check{u}|_{\mathcal{W}}^2)} e^{C_{\chi, \Omega}(\frac{1}{T_*} + 2(\check{\lambda} + |\check{u}|_{\mathcal{W}}^2)T_*)} \\ &\leq e^{(\check{\lambda} + |\check{u}|_{\mathcal{W}}^2)^{\frac{1}{2}}} + \alpha_1^{-1} D_{\chi, \Omega}^2 e^{3(C_{\chi, \Omega} + 1)(1 + \check{\lambda}^{\frac{2}{3}} + |\check{u}|_{\mathcal{W}}^2 + \check{\lambda}^{\frac{1}{2}} + |\check{u}|_{\mathcal{W}})} + 1 =: \Upsilon_d. \end{aligned}$$

Recursively, repeating the argument in the time interval  $(iT_*, +\infty)$  with  $\check{w}(iT_*) = \check{w}_M(iT_*)$  in (3.10), we can conclude that the solution  $\check{w}_M$  will remain bounded for all time  $\tau \geq 0$ . That is,  $|\check{w}_M|_{L^\infty(\mathbb{R}_0, H)} \leq \Upsilon_d |v_0|_H$ .

Next, we notice that  $v(t) := e^{-\frac{\lambda}{2}t} \check{w}_M(\nu t)$  solves (3.1), in  $\mathbb{R}_0 \times \Omega$ , with the concatenated control  $\zeta = \chi \mathbb{E}_0^{\mathcal{O}} P_M^{\mathcal{O}}(\nu e^{-\frac{\lambda}{2}t} \bar{\eta}(\check{w}_M(iT_*))(\nu t)|_{\mathcal{O}})$ , where  $\bar{\eta}(\check{w}_M(iT_*))$ ,  $i \in \mathbb{N}$ , is the control given in Proposition 3.3, when we consider system (3.10), in  $J_*^i \times \Omega$ , with  $J_*^i := (iT_*, (i+1)T_*)$ ,  $i \in \mathbb{N}_0$ , and  $\check{w}(iT_*) = \check{w}_M(iT_*)$ , in particular  $\bar{\eta}(\check{w}_M(iT_*))(\nu t)$  is defined for  $t \in (i\nu^{-1}T_*, (i+1)\nu^{-1}T_*)$ . We can also conclude from (3.13) and from the boundedness of  $\{|\check{w}_M(iT_*)|_H \mid i \in \mathbb{N}\}$  that the family  $\{|\bar{\eta}(\check{w}_M(iT_*))|_{L^2(J_*^i, H)} \mid i \in \mathbb{N}\}$  is bounded; so  $e^{\frac{\lambda}{2}t} \zeta \in L^2(\mathbb{R}_0, H)$  for all  $\hat{\lambda} < \lambda$ . Finally we observe that  $|v(t)|_H^2 \leq e^{-\lambda t} |\check{w}_M(\nu t)|_H^2 \leq \Upsilon_d e^{-\lambda t} |v_0|_H^2$ , and for  $\eta^{\lambda, \hat{\lambda}, \nu}(t) := \nu e^{-\frac{\lambda}{2}t} \bar{\eta}(\check{w}_M(iT_*))(\nu t)$ ,  $t \in J_*^i$ , it

follows

$$\begin{aligned}
\left| e^{\frac{\lambda}{2}t} \eta^{\lambda, \hat{u}, \nu} \right|_{L^2(\mathbb{R}_0, H)}^2 &= \lim_{j \rightarrow +\infty} \sum_{i=0}^j \int_{\frac{iT_*}{\nu}}^{\frac{(i+1)T_*}{\nu}} e^{(\lambda-\lambda)s} \nu^2 |\bar{\eta}(\check{w}_M(iT_*))(\nu s)|_H^2 ds \\
&\leq \lim_{j \rightarrow +\infty} \nu \sum_{i=0}^j e^{(\lambda-\lambda)\frac{iT_*}{\nu}} \int_{iT_*}^{(i+1)T_*} |\bar{\eta}(\check{w}_M(iT_*))(\tau)|_H^2 d\tau \\
&\leq \frac{\nu}{1-e^{(\lambda-\lambda)\frac{T_*}{\nu}}} e^{C_{\chi, \Omega} \left(1 + \frac{1}{T_*} + T_* \check{\lambda} + \check{\lambda}^{\frac{2}{3}} + (1+T_*)|\check{u}|_{\mathcal{W}}^2\right)} |v_0|_H^2
\end{aligned}$$

from which, using the equality  $T_* = (2(\check{\lambda} + |\check{u}|_{\mathcal{W}}^2))^{-\frac{1}{2}}$ , we can arrive to the estimate

$$\left| e^{\frac{\lambda}{2}t} \eta^{\lambda, \hat{u}, \nu} \right|_{L^2(\mathbb{R}_0, H)}^2 \leq \frac{\nu e^{C_{\chi, \Omega} \left(1 + 3(\frac{\lambda}{\nu})^{\frac{1}{2}} + (\frac{\lambda}{\nu})^{\frac{2}{3}} + 3\frac{1}{\nu}|\hat{u}|_{\mathcal{W}} + \frac{1}{\nu^2}|\hat{u}|_{\mathcal{W}}^2\right)}}{1-e^{(\lambda-\lambda)(2(\nu\lambda + |\hat{u}|_{\mathcal{W}}^2))^{-\frac{1}{2}}}} |v_0|_H^2. \quad \square$$

*Remark 3.5.* Notice that when we shrink the support of  $\chi$  the constant  $C_{\chi, \Omega}$  in (3.13) is expected to increase; we cannot expect the right hand side of (3.15) to go to 0 as the length  $l$  of  $\mathcal{O}$  does.

**3.3. The gap between (3.3) and (3.15).** Comparing estimates (3.3) and (3.15), we see that there is a big gap; the former being proportional to  $(\frac{1}{\nu^2}|\hat{u}|_{\mathcal{W}}^2 + \frac{1}{\nu}\lambda)^{\frac{1}{2}}$ , while the latter depends exponentially on both  $\frac{1}{\nu}|\hat{u}|_{\mathcal{W}}$  and  $(\frac{\lambda}{\nu})^{\frac{1}{2}}$ . For application purposes the latter is much less convenient, so one question arises naturally: can we improve (3.15)?

It seems that the idea used to derive (3.3) cannot (at least straightforwardly) be applied in the general case. On the other side to derive (3.15) we start from an exact null controllability result and carry the cost of the respective control. This means that to improve (3.15) we will probably need a different idea.

In section 6, in order to understand if it is possible to improve (3.15), say that we also have an estimate like (3.3) in the general case, we present results of some numerical simulations comparing the number of controls  $M = M_{\text{need}}$ , that we need to stabilize the system (3.1) to zero, with the following reference real numbers

$$(3.20) \quad M_{\text{ref}} := \frac{L}{\pi}(\nu^{-2}|\hat{u}|_{\mathcal{W}}^2 + \nu^{-1}\lambda)^{\frac{1}{2}}; \quad M_{\text{exp}} := \frac{L}{\pi}e^{M_{\text{ref}}}.$$

The value  $M_{\text{ref}}$  is motivated by (3.3), and the value  $M_{\text{exp}}$  by (3.15); notice that  $\frac{L}{\pi}e^{M_{\text{ref}}}$  is a lower bound for the right hand side of (3.15); we take  $\frac{L}{\pi}$  instead of  $\frac{L}{\pi}$  in front of  $e^{M_{\text{ref}}}$  in order to avoid giving the wrong idea that (3.15) goes to 0 with  $l$  (cf. Remark 3.5).

Notice that in the case  $\hat{u} = 0$ , we can see that the unstable modes of system (3.5) are those defined by the inequality  $\nu\alpha_i < \frac{\lambda}{2}$ , that is,  $i < \frac{L}{\pi}\nu^{-\frac{1}{2}}(\frac{\lambda}{2})^{\frac{1}{2}} = 2^{-\frac{1}{2}}M_{\text{ref}} < M_{\text{ref}}$ . Thus, in this case and with  $\chi = 1_{\Omega}$ , it is enough (and necessary) to take the  $M = \lfloor 2^{-\frac{1}{2}}M_{\text{ref}} \rfloor$  controls in  $\left\{ \left(\frac{2}{L}\right)^{\frac{1}{2}} \sin\left(\frac{i\pi x}{L}\right) \mid i \in \{1, 2, \dots, M\} \right\}$  (taking the family of controls considered in section 3.1). Here  $\lfloor y \rfloor \in \mathbb{N}$  stands for the biggest integer that is strictly smaller than  $y > 0$ .

**3.4. Feedback control and Riccati equation.** By the dynamic programming principle, for example following the arguments in [7, section 3.2], considering the

family of minimization problems

(3.21,  $s$ )

$$\begin{aligned} & \text{Minimize } \mathcal{J}^s(v, \eta) := \int_{\mathbb{R}_s} e^{\lambda t} (\nu |v(t)|_V^2 + |\eta(t)|_H^2) dt \quad \text{on the space } \mathcal{X} := \\ & \left\{ (v, \eta) \mid \begin{array}{l} e^{\frac{\lambda}{2}t}(v, \eta) \in W(\mathbb{R}_s, V, V') \times L^2(\mathbb{R}_s, H) \text{ and, for } t \in \mathbb{R}_s, \\ (v, \zeta) \text{ solves (3.1) with } v(s) = w \in H \text{ and } \zeta = \chi \mathbb{E}_0^\mathcal{O} P_M^\mathcal{O}(\eta|_{\mathcal{O}}) \end{array} \right\} \end{aligned}$$

where  $s$  runs over  $[0, +\infty)$  and  $\mathbb{R}_s = (s, +\infty)$ , we can derive the following result.

**THEOREM 3.6.** *The controls  $\zeta$  given in Theorems 3.2 and 3.4 can be taken in feedback form*

$$(3.22) \quad \zeta = e^{-\lambda t} \chi \mathbb{E}_0^\mathcal{O} P_M^\mathcal{O}((\chi Q_{\hat{u}}^{t, \lambda} v)|_{\mathcal{O}})$$

for a suitable family of operators  $Q_{\hat{u}}^{t, \lambda}: H \rightarrow H$ ,  $t \geq 0$ , with  $\left| Q_{\hat{u}}^{t, \lambda} \right|_{\mathcal{L}(H \rightarrow H)} \leq \overline{C}_{[\lambda, \hat{u}, \frac{\nu}{\nu}]}} e^{\lambda t}$ . Furthermore, the family  $\{Q_{\hat{u}}^{t, \lambda} \mid t \geq 0\}$  is continuous in the weak operator topology and  $Q := Q_{\hat{u}}^{t, \lambda}$  satisfies the differential Riccati equation

$$(3.23) \quad \dot{Q} - Q(-\nu \partial_{xx} + \mathcal{B}(\hat{u})) - (-\nu \partial_{xx} + \mathcal{B}(\hat{u}))^* Q - Q B_M^\mathcal{O} B_M^{\mathcal{O}*} Q - e^{\lambda t} \nu \partial_{xx} = 0$$

where  $\mathcal{B}(\hat{u})v := \partial_x(\hat{u}v)$ , and  $B_M^\mathcal{O}: H \rightarrow H$  and its adjoint  $B_M^{\mathcal{O}*}: H \rightarrow H$  given by

$$(3.24) \quad B_M^\mathcal{O} \eta := e^{-\frac{\lambda}{2}t} \chi \mathbb{E}_0^\mathcal{O} P_M^\mathcal{O}(\eta|_{\mathcal{O}}), \quad B_M^{\mathcal{O}*} \xi = e^{-\frac{\lambda}{2}t} \mathbb{E}_0^\mathcal{O} P_M^\mathcal{O}((\chi \xi)|_{\mathcal{O}}),$$

and  $(Q_{\hat{u}}^{s, \lambda} v_*^0(s), v_*^0(s))_H = \mathcal{J}^s(v_*^0|_{\mathbb{R}_s}, \eta_*^0|_{\mathbb{R}_s})$  where  $(v_*^0, \eta_*^0)$  is the unique minimizer of problem (3.21, 0). Further,  $(Q_{\hat{u}}^{s, \lambda} w, w)_H = \mathcal{J}^s(v_*^s, \eta_*^s)$ , where  $(v_*^s, \eta_*^s)$  is the unique minimizer of problem (3.21,  $s$ ).

**Remark 3.7.** Equation (3.23) can be seen as an evolutionary equation, and  $\dot{Q} := \frac{d}{dt}Q$ . A solution for (3.23), is understood in the sense that

$$\begin{aligned} (\dot{Q}w^1, w^2)_H &= (Q \mathbb{A}w_1, w^2)_H + (\mathbb{A}^* Q w_1, w^2)_H \\ &\quad + (Q B_M^\mathcal{O} B_M^{\mathcal{O}*} Q w_1, w^2)_H + (e^{\lambda t} \nu \partial_{xx} w_1, w^2)_H \end{aligned}$$

holds for all  $(w^1, w^2) \in D(\partial_{xx}) \times D(\partial_{xx})$ , with  $\mathbb{A} = \mathbb{A}(t) := -\nu \partial_{xx} + \mathcal{B}(\hat{u}(t))$ . See, for example, [15, section 5.4], [35, chapter 1, Theorem 1.4.6.4 and Corollary 1.5.3.3]. Observe also that  $(Q_{\hat{u}}^{s, \lambda} w, w)_H = \mathcal{J}^s(v_*^s, \eta_*^s) > 0$  for all  $w \neq 0$ , that is,  $Q_{\hat{u}}^{s, \lambda}$  is definite positive.

**Remark 3.8.** Notice that  $(\partial_{xx})^* = \partial_{xx}$  and  $\mathcal{B}(\hat{u})^* = -\hat{u} \partial_x$ . Notice also that from Theorems 3.2 and 3.4 (taking, e.g.,  $(2\lambda, \lambda)$  in place of  $(\lambda, \hat{\lambda})$ ), we have that the space  $\mathcal{X}$  in problem (3.21) is nonempty.

**Remark 3.9.** For any  $T > 0$  and  $w \in H$ , the function  $q := Qv_*^0$  solves the system (3.2) with  $f = -e^{\lambda t} \nu \partial_{xx} v_*^0$  and  $q(T) = Q_{\hat{u}}^{T, \lambda} v_*^0(T)$ , where  $(v_*^0, \eta_*^0) = (v_*^0, \eta_*^0)(w)$  is the minimizer of problem (3.21, 0).

We already know that  $Q$  satisfies (3.23), we can also show that it is unique in the class of operators  $e^{\lambda t} \mathcal{C}$  with

$$\mathcal{C} := \left\{ \hat{R} \in L^\infty(\mathbb{R}_0, \mathcal{L}(H \rightarrow H)) \mid \begin{array}{l} \hat{R}(t) \text{ is self-adjoint positive definite for all } t \geq 0, \\ \text{the family } \{\hat{R}(t) \mid t \geq 0\} \text{ is continuous in the} \\ \text{weak operator topology} \end{array} \right\}.$$

**LEMMA 3.10.** *If  $R$  satisfies (3.23) and  $\hat{R} := e^{-\lambda t} R \in \mathcal{C}$ , then the feedback control*

$$\zeta = e^{-\lambda t} \chi \mathbb{E}_0^\mathcal{O} P_M^\mathcal{O}((\chi R v)|_{\mathcal{O}}) = B_M^\mathcal{O} B_M^{\mathcal{O}*} R v$$

stabilizes exponentially system (3.1) to zero with rate  $\frac{\lambda}{2}$ .

*Proof.* We find that

$$\begin{aligned} \frac{d}{dt}(Rv, v)_H &= (\dot{R}v, v)_H + (R\partial_t v, v)_H + (Rv, \partial_t v)_H \\ &= ((R\mathbb{A} + \mathbb{A}^*R + RB_M^\mathcal{O}B_M^{\mathcal{O}*}R + e^{\lambda t}\nu\partial_{xx})v, v)_H \\ &\quad + (-R(\mathbb{A} + B_M^\mathcal{O}B_M^{\mathcal{O}*}R)v, v)_H + (Rv, -(\mathbb{A} + B_M^\mathcal{O}B_M^{\mathcal{O}*}R)v)_H \\ &= -e^{\lambda t}(\nu|v|_V^2 + |e^{-\frac{\lambda}{2}t}B_M^{\mathcal{O}*}Rv|_H^2). \end{aligned}$$

Notice that  $\chi\mathbb{E}_0^\mathcal{O}P_M^\mathcal{O}((\chi e^{-\frac{\lambda}{2}t}B_M^{\mathcal{O}*}Rv)|_\mathcal{O}) = B_M^\mathcal{O}B_M^{\mathcal{O}*}Rv$ . Thus we have that  $(Rv, v)_H$  is decreasing and, after integration, that  $\mathcal{J}^s(v, e^{-\frac{\lambda}{2}t}B_M^{\mathcal{O}*}Rv) = (R(s)v(s), v(s))_H - \lim_{T \rightarrow +\infty} (R(T)v(T), v(T))_H \leq (R(s)v(s), v(s))_H \leq |\hat{R}|_{L^\infty(\mathbb{R}_0, \mathcal{L}(H \rightarrow H))} e^{\lambda s} |v(s)|_H^2$ . This inequality, together with  $\partial_t(e^{\frac{\lambda}{2}t}v) = \frac{\lambda}{2}e^{\frac{\lambda}{2}t}v + e^{\frac{\lambda}{2}t}(\nu\partial_{xx}v - \partial_x(\hat{u}v) + B_M^\mathcal{O}B_M^{\mathcal{O}*}Rv)$  and  $\int_{\mathbb{R}_s} |e^{\frac{\lambda}{2}t}B_M^\mathcal{O}B_M^{\mathcal{O}*}Rv|_H^2 dt \leq C_0 \int_{\mathbb{R}_s} |B_M^{\mathcal{O}*}Rv|_H^2 dt \leq C_0 \mathcal{J}^s(v, e^{-\frac{\lambda}{2}t}B_M^{\mathcal{O}*}Rv)$ , imply that  $\partial_t(e^{\frac{\lambda}{2}t}v)$  is in  $L^2(\mathbb{R}_s, V')$ , with  $|\partial_t(e^{\frac{\lambda}{2}t}v)|_{L^2(\mathbb{R}_s, V')} \leq Ce^{\lambda s} |v(s)|_H$ . Hence it follows that  $|e^{\frac{\lambda}{2}t}v|_{C([s, +\infty), H)} \leq C_1 e^{\frac{\lambda}{2}s} |v(s)|_H$ , for suitable positive constants  $C_0, C$  and  $C_1$ . That is, the feedback control  $\zeta = B_M^\mathcal{O}B_M^{\mathcal{O}*}Rv$  stabilizes system (3.1) to zero with rate  $\frac{\lambda}{2}$ ,  $|v(t)|_H \leq C_1 e^{-\frac{\lambda}{2}(t-s)} |v(s)|_H$ .  $\square$

The uniqueness of  $Q$  will follow from the uniqueness of  $Q_1 := e^{-\lambda t}Q \in \mathcal{C}$  satisfying

$$(3.25) \quad \dot{Q}_1 - Q_1\mathbb{A} - \mathbb{A}^*Q_1 - Q_1\mathbb{B}\mathbb{B}^*Q_1 - \nu\partial_{xx} + \lambda Q_1 = 0$$

with  $\mathbb{B} = \mathbb{B}(t) := e^{\frac{\lambda}{2}t}B_M^\mathcal{O}$ . From the exponential stability, with rate  $\frac{\lambda}{2}$ , of system (3.1) with  $\zeta$  given by (3.22), it follows that

$$(3.26) \quad \partial_t z - \nu\partial_{xx}z + \partial_x(\hat{u}z) - \frac{\lambda}{2}z + \mathbb{B}\mathbb{B}^*Q_1z = 0, \quad z|_\Gamma = 0, \quad z(0) = z_0,$$

is stable, that is, there is a constant  $C > 0$  independent of  $z_0$  such that  $|z(t)|_H \leq C|z_0|_H$ , for all  $t \in \mathbb{R}_0$ . Actually we can prove that it is uniformly exponentially stable, that is, there are  $\alpha > 0$  and  $K > 0$  such that

$$(3.27) \quad |z(t)|_H \leq Ke^{-\alpha(t-t_0)} |z(t_0)|_H \text{ for all } 0 \leq t_0 \leq t.$$

Indeed, notice that we can consider the system (3.1) in the interval of time  $\mathbb{R}_{t_0} = (t_0, +\infty)$ , instead of  $\mathbb{R}_0$  and we obtain the analogous to Theorems 3.2 and 3.4, replacing the initial time  $t = 0$  by  $t = t_0$ . This means that if we denote by  $\mathcal{S}(t, t_0)w$  the solution of system (3.26) for  $t \in \mathbb{R}_{t_0}$  with initial condition  $z(t_0) = z_{t_0}$  we will have that  $|\mathcal{S}(t, t_0)w|_{L^2(\mathbb{R}_0, H)}^2 \leq C|z_{t_0}|_H^2$ , where  $C$  is given in Theorems 3.2 and 3.4 and can be taken independent of  $t_0$ . The uniform exponential stability follows then by [14, Theorem 1], see also [47, chapter 3, Theorem 3.1].

*Remark 3.11.* The operator (or family of operators)  $\mathcal{S}(t, t_0)$  is sometimes called “evolutionary process” as in [14, section 1], “Green operator” as in [36, chapter IV, section 3], or “evolution operator” as in [13, section 2].

**THEOREM 3.12.** *The solution of (3.25) is unique in  $\mathcal{C}$ .*

*Proof.* We follow ideas from [12, 13, 45], see also [35, chapter 1]. Let  $Q_2 \in \mathcal{C}$  solve (3.25). Then with  $\mathcal{A}_1 = \mathbb{A} + \mathbb{B}\mathbb{B}^*Q_1 - \frac{\lambda}{2}I$  and  $\mathcal{A}_2 = \mathbb{A} + \mathbb{B}\mathbb{B}^*Q_2 - \frac{\lambda}{2}I$ , where  $I$  is the identity operator, the difference  $D := Q_2 - Q_1$  solves

$$(3.28a) \quad \dot{D} = D\mathcal{A}_1 + \mathcal{A}_1^*D + D\mathbb{B}\mathbb{B}^*D,$$

$$(3.28b) \quad \dot{D} = D\mathcal{A}_2 + \mathcal{A}_2^*D - D\mathbb{B}\mathbb{B}^*D.$$

Let  $w \in H$  and let  $\mathcal{S}_i(t, s)w$  stand for the solution of system (3.26) in the interval of time  $t \in \mathbb{R}_s$ , with  $z(s) = w$  and with  $Q_i$  in the place of  $Q_1$ ,  $i \in \{1, 2\}$ . Then we have that  $\partial_t \mathcal{S}_i(t, s)w = -\mathcal{A}_i(t)\mathcal{S}_i(t, s)w$ , and we find  $\partial_t \mathcal{S}_i(t, s) = -\mathcal{A}_i(t)\mathcal{S}_i(t, s)$  and  $\partial_t \mathcal{S}_i(t, s)^* = -\mathcal{S}_i(t, s)^* \mathcal{A}_i(t)^*$ . For  $t \in (s, T)$  we also have  $0 = \partial_t \mathcal{S}_i(T, s)w = \partial_t (\mathcal{S}_i(T, t)\mathcal{S}_i(t, s)w)$ , which gives us  $0 = (\partial_t \mathcal{S}_i(T, t))\mathcal{S}_i(t, s) + \mathcal{S}_i(T, t)\partial_t \mathcal{S}_i(t, s)$ , that is,  $\partial_t \mathcal{S}_i(T, t) = \mathcal{S}_i(T, t)\mathcal{A}_i(t)$  and  $\partial_t \mathcal{S}_i(T, t)^* = \mathcal{A}_i(t)^* \mathcal{S}_i(T, t)^*$ .

Now we fix  $s \geq 0$  and, for  $t > s$ , set  $G_i(t) := \mathcal{S}_i(t, s)^* D(t) \mathcal{S}_i(t, s)$ . Using (3.28), it follows that

$$\dot{G}_1 = \mathcal{S}_1(t, s)^* (D\mathbb{B}\mathbb{B}^* D)(t) \mathcal{S}_1(t, s) \quad \text{and} \quad \dot{G}_2 = -\mathcal{S}_2(t, s)^* (D\mathbb{B}\mathbb{B}^* D)(t) \mathcal{S}_2(t, s).$$

Then we obtain  $(G_1(t)w, w)_H - (G_1(s)w, w)_H = \int_s^t |(\mathbb{B}^* D)(r) \mathcal{S}_1(r, s)w|_H^2 dr$  and  $(G_2(t)w, w)_H - (G_2(s)w, w)_H = -\int_s^t |(\mathbb{B}^* D)(r) \mathcal{S}_2(r, s)w|_H^2 dr$ . Then, from  $G_1(s) = D(s) = G_2(s)$ , we arrive to

$$(3.29) \quad (G_2(t)w, w)_H \leq (D(s)w, w)_H \leq (G_1(t)w, w)_H.$$

Notice that from Lemma 3.10 and (3.27) we have that

$$|\mathcal{S}_i(t, t_0)w|_H \leq K_i e^{-\alpha_i(t-t_0)} |w|_H \quad \text{for all } 0 \leq t_0 \leq t,$$

for suitable positive constants  $K_i$  and  $\alpha_i$ . Thus from (3.29), it follows

$$(3.30) \quad -\widehat{D}K_2^2 e^{-2\alpha_2(t-s)} |w|_H^2 \leq (D(s)w, w)_H \leq \widehat{D}K_1^2 e^{-2\alpha_1(t-s)} |w|_H^2.$$

with  $\widehat{D} := |D|_{L^\infty(\mathbb{R}_0, \mathcal{L}(H \rightarrow H))}$ . Letting  $t$  go to  $+\infty$  we obtain  $(D(s)w, w)_H = 0$ ; hence since  $w$  can be taken arbitrary and  $D(s)$  is self-adjoint, it follows that  $0 = (D(s)(w^1 + w^2), w^1 + w^2)_H = 2(D(s)w^1, w^2)_H$  for any  $(w^1, w^2) \in H \times H$ ; necessarily  $D(s) = 0$  and  $D = 0$  because  $s$  can be taken arbitrary.  $\square$

We know (cf. Remark 3.7) that  $Q_1(t) = e^{-\lambda t} Q(t)$  is self-adjoint and positive definite for all  $t \geq 0$ . From (3.25) we can also conclude that if, at some  $T > 0$ , we impose a final condition  $Q_1(T) = Q_1^T$  with  $Q_1^T$  self-adjoint and positive definite, then  $Q_1(t)$  remains self-adjoint and positive definite for all  $t \in [0, T]$ . Indeed from

$$(3.31) \quad \dot{Q}_1 = Q_1 \mathcal{A}_1 + \mathcal{A}_1^* Q_1 - Q_1 \mathbb{B}\mathbb{B}^* Q_1 + \nu \partial_{xx}$$

we can see that  $Q_1$  can be written as

$$(3.32) \quad Q_1(t) = \mathcal{S}_1(T, t)^* Q_1^T \mathcal{S}_1(T, t) + \int_t^T \mathcal{S}_1(s, t)^* (Q_1 \mathbb{B}\mathbb{B}^* Q_1 - \nu \partial_{xx})(s) \mathcal{S}_1(s, t) ds$$

and, for  $u \neq 0$ , we have

$$(3.33) \quad (Q_1(t)u, u)_H = (Q_1^T \mathcal{S}_1(T, t)u, \mathcal{S}_1(T, t)u)_H \\ + \int_t^T |\mathbb{B}^* Q_1 \mathcal{S}_1(s, t)u|_H^2 + \nu |\partial_x \mathcal{S}_1(s, t)u|_H^2 ds > 0.$$

Further, if  $Q_2$  also solves (3.25) with  $Q_2(T) = Q_1^T$ , then necessarily  $D(s) := Q_2(s) - Q_1(s) = 0$  for all  $s \in (0, T)$ , because from (3.29) if  $D(T) = 0$  we can derive that  $0 = (G_2(T)w, w)_H \leq (D(s)u, u)_H \leq (G_1(T)w, w)_H = 0$ .

*Remark 3.13.* Denoting  $F_1 := Q_1 \mathbb{B} \mathbb{B}^* Q_1 - \nu \partial_{xx}$  and differentiating (3.32) we find

$$\begin{aligned} \dot{Q}_1 &= \partial_t Q_1 = \mathcal{A}_1(t)^* \mathcal{S}_1(T, t)^* Q_1^T \mathcal{S}_1(T, t) + \mathcal{S}_1(T, t)^* Q_1^T \mathcal{S}_1(T, t) \mathcal{A}_1(t) - F_1(t) \\ &\quad + \int_t^T \mathcal{A}_1(t)^* \mathcal{S}_1(s, t)^* F_1(s) \mathcal{S}_1(s, t) + \mathcal{S}_1(s, t)^* F_1(s) \mathcal{S}_1(s, t) \mathcal{A}_1(t) \, ds \\ &= Q_1(t) \mathcal{A}_1(t) + \mathcal{A}_1(t)^* Q_1(t) - F_1(t), \end{aligned}$$

that is, we recover (3.31).

*Remark 3.14.* From (3.32) we see that  $Q_1(t)$  is obtained from  $Q_1^T = Q_1(T)$ . Thus, the Riccati equations (3.31), (3.23), and (3.25) must be solved backwards in time.

**4. The nonlinear system.** The next result is a corollary of Theorem 3.6. It will follow by a fixed point argument.

**THEOREM 4.1.** *Let  $M$  be the integer in Theorem 3.6 (i.e., as in either (1.3) or (1.4)). Then there are positive constants  $\Theta$  and  $\epsilon = \epsilon(\Theta)$  depending only on  $\lambda$ ,  $|\hat{u}|_{\mathcal{W}}$ , and  $\nu$  such that for  $|v_0|_H \leq \epsilon$  the solution  $v$  of system (2.4), with  $\zeta$  as in (3.22), is well defined for all  $t \geq 0$  and satisfies the inequality*

$$(4.1) \quad |v(t)|_H^2 \leq \Theta e^{-\lambda t} |v_0|_H^2 \quad \text{for } t \geq 0.$$

Notice that the feedback rule is found to globally stabilize to zero the linear Oseen–Burgers system (3.1). Then, Theorem 4.1 says that the same feedback rule also locally stabilizes to zero the bilinear system (2.4).

The proof of Theorem 4.1 will be done following the arguments in [7, section 4]. The nonlinear system (2.4) with  $\zeta$  as in (3.22) reads

$$(4.2) \quad \partial_t v - \nu \partial_{xx} v + v \partial_x v + \partial_x(\hat{u}v) + \mathcal{K}^{t, \lambda} v = 0, \quad v|_{\Gamma} = 0, \quad v(0) = v_0$$

with  $\mathcal{K}^{t, \lambda} v := e^{-\lambda t} \chi \mathbb{E}_0^{\mathcal{O}} P_M^{\mathcal{O}}((\chi Q_{\hat{u}}^{t, \lambda} v)|_{\mathcal{O}})$ . Given  $\lambda > 0$ , we denote by  $\mathcal{Z}^\lambda$  the space of functions  $z \in C([0, +\infty), H) \cap L^2(\mathbb{R}_0, V)$  such that

$$|z|_{\mathcal{Z}^\lambda} := \left( \left| e^{\frac{\lambda}{2} \cdot} z(\cdot) \right|_{L^\infty(\mathbb{R}_0, H)}^2 + \left| e^{\frac{\lambda}{2} \cdot} z(\cdot) \right|_{L^2(\mathbb{R}_0, V)}^2 \right)^{\frac{1}{2}} < \infty.$$

Let us fix a constant  $\Theta > 0$  and a function  $v_0 \in H$ , and introduce the following subset of  $\mathcal{Z}^\lambda$ :

$$\mathcal{Z}_\Theta^\lambda := \{z \in \mathcal{Z}^\lambda \mid z(0) = v_0, |z|_{\mathcal{Z}^\lambda}^2 \leq \Theta |v_0|_H^2\}.$$

We define a mapping  $\Xi : \mathcal{Z}^\lambda \rightarrow C([0, +\infty), H) \cap L_{\text{loc}}^2(\mathbb{R}_0, V)$  that takes a function  $a \in \mathcal{Z}^\lambda$  to the solution  $b$  of the problem

$$(4.3) \quad \partial_t b - \nu \partial_{xx} b + \partial_x(\hat{u}b) + \mathcal{K}^{t, \lambda} b + a \partial_x a = 0, \quad b|_{\Gamma} = 0, \quad b(0) = v_0.$$

Recall that  $|\mathcal{K}^{\cdot, \lambda}|_{L^\infty(\mathbb{R}_0, \mathcal{L}(H \rightarrow H))} \leq C_1$  and  $|a \partial_x a|_{V'} \leq C_2 |a|_H |a|_V$ .

**LEMMA 4.2.** *Let  $M$  be the integer in Theorem 3.6. Then, there exists  $\Theta = \Theta(\lambda, |\hat{u}|_{\mathcal{W}}, \nu) > 0$  such that the following property holds: for any  $\gamma \in (0, 1)$  one can find a constant  $\epsilon = \epsilon_{\Theta, \gamma} > 0$  such that for any  $v_0 \in H$  with  $|v_0|_H \leq \epsilon$  the mapping  $\Xi$  takes the set  $\mathcal{Z}_\Theta^\lambda$  into itself and satisfies the inequality*

$$(4.4) \quad |\Xi(a_1) - \Xi(a_2)|_{\mathcal{Z}^\lambda} \leq \gamma |a_1 - a_2|_{\mathcal{Z}^\lambda} \quad \text{for all } a_1, a_2 \in \mathcal{Z}_\Theta^\lambda.$$

*Proof. Step 1.* For suitable  $\Theta$  and  $\epsilon = \epsilon_\Theta$ ,  $\Xi$  maps  $\mathcal{Z}_\Theta^\lambda$  into itself. By the Duhamel formula, we can write  $b$  as

$$(4.5) \quad b(t) = \mathcal{S}(t, 0)b(0) + \int_0^t \mathcal{S}(t, s)(a\partial_x a)(s) ds$$

where  $\mathcal{S}(t, s)w$  denotes the solution of the system (4.3), for time  $t \geq s$  with initial condition  $b(s) = w$ , and  $a = 0$ . Then, we derive

$$\begin{aligned} |b(t)|_H^2 &\leq 2|\mathcal{S}(t, 0)b(0)|_H^2 + 2 \left( \int_0^t |\mathcal{S}(t, s)(a\partial_x a)(s)|_H ds \right)^2 \\ &\leq 2C_3 e^{-(\lambda+\beta)t} |b(0)|_H^2 + 2C_3 e^{-(\lambda+\beta)t} \left( \int_0^t e^{\frac{\lambda+\beta}{2}s} |(a\partial_x a)(s)|_H ds \right)^2, \end{aligned}$$

where  $\beta = \min\{\alpha, \lambda\} > 0$ , and  $\alpha$  is as in (3.27). From  $|(a\partial_x a)(s)|_H \leq C_4 |a|_V^2$ , it follows that

$$(4.6) \quad e^{(\lambda+\beta)t} |b(t)|_H^2 \leq C_5 \left( |b(0)|_H^2 + |a|_{\mathcal{Z}^\lambda}^4 \right).$$

Now, multiplying (4.3) by  $b$  and following standard arguments, we also have that

$$\begin{aligned} \nu \int_s^{s+1} |b(\tau)|_V^2 d\tau &\leq |b(s)|_H^2 + C_6 \left( |\hat{u}|_{\mathcal{W}}^2 + |\mathcal{K}^{\cdot, \lambda}|_{L^\infty(\mathbb{R}_0, \mathcal{L}(H \rightarrow H))}^2 \right) |b|_{L^2((s, s+1), H)}^2 \\ &\quad + C_7 \int_s^{s+1} |a\partial_x a(\tau)|_V^2 d\tau \end{aligned}$$

from which, using (4.6), it follows that

$$\begin{aligned} \nu \int_s^{s+1} e^{\lambda\tau} |b(\tau)|_V^2 d\tau &\leq e^{\lambda(s+1)} |b(s)|_H^2 + e^\lambda C_7 \int_s^{s+1} e^{\lambda\tau} |a\partial_x a(\tau)|_V^2 d\tau \\ &\quad + C_6 \left( |\hat{u}|_{\mathcal{W}}^2 + |\mathcal{K}^{\cdot, \lambda}|_{L^\infty(\mathbb{R}_0, \mathcal{L}(H \rightarrow H))}^2 \right) e^{\lambda(s+1)} |b|_{L^\infty((s, s+1), H)}^2 \\ &\leq e^{-\beta s} C_8 \left( |b(0)|_H^2 + |a|_{\mathcal{Z}^\lambda}^4 \right) + C_9 \int_s^{s+1} e^{\lambda\tau} |a(\tau)|_H^2 |a(\tau)|_V^2 d\tau. \end{aligned}$$

Thus, since  $e^{\lambda\tau} \leq e^{-\lambda s} e^{2\lambda\tau} \leq e^{-\beta s} e^{2\lambda\tau}$  for  $\tau \in (s, s+1)$ , summing up we obtain

$$\int_{\mathbb{R}_0} e^{\lambda\tau} |b(\tau)|_V^2 d\tau \leq C_{10} \left( |b(0)|_H^2 + |a|_{\mathcal{Z}^\lambda}^4 \right) \sum_{j=1}^{+\infty} e^{-\beta j}.$$

Then from (4.6) we arrive to  $|b|_{\mathcal{Z}^\lambda} \leq C_{11} \left( |b(0)|_H^2 + |a|_{\mathcal{Z}^\lambda}^4 \right)$ , which implies that for  $a \in \mathcal{Z}_\Theta^\lambda$  we have

$$|\Xi(a)|_{\mathcal{Z}^\lambda}^2 \leq C_{11} \left( 1 + \Theta^2 |v_0|_H^2 \right) |v_0|_H^2$$

Setting  $\Theta = 2C_{11}$  and choosing  $\epsilon_\Theta > 0$  so small that  $\Theta\epsilon_\Theta \leq 1$ , we see that if  $|v_0|_H \leq \epsilon_\Theta$ , then  $\Xi$  maps the set  $\mathcal{Z}_\Theta^\lambda$  into itself.

*Step 3.* Given  $\gamma \in (0, 1)$ ,  $\Xi$  is a  $\gamma$ -contraction for smaller  $\epsilon = \epsilon_{\Theta, \gamma}$ . Let us take two functions  $a_1, a_2 \in \mathcal{Z}_\Theta^\lambda$  and set  $a := a_1 - a_2$  and  $b := \Xi(a_1) - \Xi(a_2)$ . Then the function  $b$  satisfies (4.3) with  $b(0) = 0$  and  $a_1\partial_x a_1 - a_2\partial_x a_2$  in the place of  $a\partial_x a$ . From

$$\begin{aligned} |a_1\partial_x a_1 - a_2\partial_x a_2|_H &= |a_1\partial_x a + a\partial_x a_2|_H \leq C_4(|a_1|_V + |a_2|_V) |a|_V; \\ |a_1\partial_x a_1 - a_2\partial_x a_2|_V &\leq C_{12}(|a_1|_H + |a_2|_H) |a|_V; \end{aligned}$$



and proceeding as above we can arrive to

$$|\Xi(a_1) - \Xi(a_2)|_{\mathcal{Z}^\lambda}^2 \leq C_{13} \left( |a_1|_{\mathcal{Z}^\lambda}^2 + |a_2|_{\mathcal{Z}^\lambda}^2 \right) |a|_{\mathcal{Z}^\lambda}^2 \leq 2C_{13}\Theta |v_0|_H^2 |a_1 - a_2|_{\mathcal{Z}^\lambda}^2.$$

Choosing  $\tilde{\epsilon}_{\Theta, \gamma} > 0$  so small that  $2\Theta C_{13} \tilde{\epsilon}_{\Theta, \gamma}^2 \leq \gamma^2$ , we see that if  $|v_0|_H \leq \tilde{\epsilon}_{\Theta, \gamma}$ , then (4.4) holds. Therefore the lemma holds with  $\epsilon_{\Theta, \gamma} = \min\{\epsilon_\Theta, \tilde{\epsilon}_{\Theta, \gamma}\}$ .  $\square$

*Proof of Theorem 4.1.* If  $|v_0|_H \leq \epsilon_{\Theta, \gamma}$ , the contraction mapping principle implies that there is a unique fixed point  $v \in \mathcal{Z}_\Theta^\lambda$  for  $\Xi$ . It follows from the definition of  $\Xi$  and  $\mathcal{Z}_\Theta^\lambda$  that  $v$  is a solution of problem (4.2) and satisfies (4.1). We claim that  $v$  is the unique solution of (4.2) in the space  $C([0, +\infty), H) \cap L^2(\mathbb{R}_0, V)$ . Indeed, if  $w$  is another solution, then the difference  $z = v - w$  satisfies

$$z_t - \nu \partial_{xx} z + z \partial_x z + \partial_x(wz) + \partial_x(\hat{u}z) + \mathcal{K}^{t, \lambda} z = 0, \quad z(0) = 0.$$

Multiplying this equation by  $z$ , in  $H$ , and following a standard procedure, we arrive to  $\frac{d}{dt} |z|_H^2 + \nu |z|_V^2 \leq C_{14} \left( |w|_V^2 + |\hat{u}|_W^2 \right) |z|_H^2$  which implies  $z(t) = 0$ , for all  $t \geq 0$ .  $\square$

*Remark 4.3.* Though it would be possible to derive more precise estimates on the  $\Theta$  and  $\epsilon$  in Theorem 4.1, it would lead to a more cumbersome exposition, and these estimates are not the main focus of this work.

**5. Discretization.** To perform the simulations in order to check the stabilization of systems (1.1) and (3.1), to a reference trajectory  $\hat{u}$  and to zero respectively, we must discretize those systems with the feedback control  $\zeta$  as in (3.22).

**5.1. Discretization in space.** We use a finite-element based approach. We introduce an uniform mesh

$$(5.1) \quad \Omega_D := \left( \frac{L}{N_x}, \frac{2L}{N_x}, \dots, \frac{(N_x-1)L}{N_x} \right)$$

consisting of the interior points of  $\Omega$  that are multiples of the space step  $h = \frac{L}{N_x}$ , with  $2 \leq N_x \in \mathbb{N}$ . As basis functions we take the classical hat-functions  $\phi_i \in V$  defined for  $x \in \Omega$  and each  $i \in \{1, 2, \dots, N_x - 1\}$  by  $\phi_i(x) := \begin{cases} 1 - i + \frac{x}{h}, & \text{if } x \in [(i-1)h, ih]; \\ 1 + i - \frac{x}{h}, & \text{if } x \in [ih, (i+1)h]; \\ 0, & \text{if } x \notin [(i-1)h, (i+1)h]. \end{cases}$

Next any function  $u \in V$  can be approximated by the values it takes on  $\Omega_D$ . More precisely, we approximate  $u$  by the function  $\tilde{u}$ , defined as

$$\tilde{u} := \sum_{i=1}^{N_x-1} u(ih) \phi_i.$$

We define the evaluation vector  $\bar{u} := [u(ih)]^\top := [u(1h), u(2h), \dots, u((N_x-1)h)]^\top \in \mathcal{M}_{(N_x-1) \times 1}$ , where  $A^\top$  stands for the transpose matrix of  $A$ .

*Remark 5.1.* Notice that  $\tilde{u} := \sum_{i=1}^{N_x-1} \bar{u}_i \phi_i$ , is a piecewise (affine) linear function that takes the same values as  $u$  at the points of the mesh  $\Omega_D$ . Also notice that, since we are dealing with homogeneous Dirichlet boundary conditions, only the values at interior points are unknown for the solution of our system

The next step is the weak discretization matrix  $L_D$  of a given linear operator  $L \in \mathcal{L}(V \rightarrow V')$ . We define  $L_D$  by the formula

$$(5.2) \quad \bar{v}^\top L_D \bar{u} = \langle L\tilde{u}, \tilde{v} \rangle_{V', V} \text{ for all } u, v \in V.$$

Of key importance are the identity and Laplace operators. For the identity operator  $Iu = u$ , we find that  $I_D = [(\phi_i, \phi_j)_H] =: \mathbf{M}$  is the so-called mass matrix, while

for the Laplace operator we find  $(\partial_{xx})_D = -[(\partial_x \phi_i, \partial_x \phi_j)_H] =: -\mathbf{S}$ , where  $\mathbf{S}$  is the so-called stiffness matrix. Explicitly we have the tridiagonal matrices

$$\mathbf{M} := \frac{h}{6} \begin{bmatrix} 4 & 1 & 0 & 0 & \dots & 0 \\ 1 & 4 & 1 & 0 & \dots & 0 \\ 0 & 1 & 4 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 & 4 & 1 \\ 0 & \dots & 0 & 0 & 1 & 4 \end{bmatrix} \quad \text{and} \quad \mathbf{S} := \frac{1}{h} \begin{bmatrix} 2 & -1 & 0 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ 0 & -1 & 2 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & -1 & 2 & -1 \\ 0 & \dots & 0 & 0 & -1 & 2 \end{bmatrix}.$$

Next, we recall the reference solution  $\hat{u}$  and discretize the operator  $v \mapsto \mathcal{B}(\hat{u})v = \partial_x(\hat{u}v)$ ,  $v \in V$ . We start by noticing that, for an arbitrary  $w \in V$ ,  $(\partial_x(\hat{u}v), w)_H = -(\hat{u}v, \partial_x w)_H$ , then we consider the approximation  $\tilde{u}v = \sum_{j=1}^{N_x-1} \bar{u}_j \bar{v}_j \phi_j$  of  $\hat{u}v$ , and we find  $-(\tilde{u}v, \partial_x \tilde{w})_H = \sum_{i=1}^{N_x-1} \sum_{j=1}^{N_x-1} -\bar{u}_j \bar{v}_j \bar{w}_i (\phi_j, \partial_x \phi_i)_H$ , and

$$(\partial_x(\hat{u}v), w)_H \approx \bar{w}^\top \mathbf{B} \mathcal{D}_{\bar{u}} \bar{v},$$

with

$$\mathbf{B} := \frac{1}{2} \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ -1 & 0 & 1 & 0 & \dots & 0 \\ 0 & -1 & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & -1 & 0 & 1 \\ 0 & \dots & 0 & 0 & -1 & 0 \end{bmatrix} \quad \text{and} \quad \mathcal{D}_{\bar{u}} := \begin{bmatrix} \bar{u}_1 & 0 & 0 & \dots & 0 \\ 0 & \bar{u}_2 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \bar{u}_{N_x-2} & 0 \\ 0 & \dots & 0 & 0 & \bar{u}_{N_x-1} \end{bmatrix}.$$

Notice also that, rewriting  $v \partial_x v$  as  $\frac{1}{2} \mathcal{B}(v)v$ , we can discretize  $v \partial_x v$  as  $\frac{1}{2} \mathbf{B} \mathcal{D}_{\bar{v}} \bar{v}$ .

*Remark 5.2.* Notice that above we consider the operator  $v \mapsto \partial_x(\hat{u}v)$  as a composition  $\partial_x \circ m_{\hat{u}}$ , where  $m_{\hat{u}}$  denotes the pointwise multiplication by  $\hat{u}$ , and then we just take the product of the discretized factors. Of course, we can also discretize directly and, after some computations we can find that  $\mathcal{B}(\hat{u})_D$  is a tridiagonal matrix  $\mathbf{B}^{\hat{u}}$ :

$$\mathbf{B}_{ii}^{\hat{u}} = - \sum_{\substack{k \in \{i-1, i+1\} \\ 1 \leq k \leq N_x-1}} \bar{u}_k \int_{\Omega} (\phi_k \phi_i \partial_x \phi_i) dx = \frac{1}{6} \begin{cases} \bar{u}_2, & \text{if } i = 1 \\ \bar{u}_{i+1} - \bar{u}_{i-1}, & \text{if } i \in \{2, \dots, N_x-2\} \\ -\bar{u}_{N_x-2}, & \text{if } i = N_x-1 \end{cases};$$

$$\mathbf{B}_{ij}^{\hat{u}} = - \sum_{\substack{k \in \{i, j\} \\ 1 \leq k \leq N_x-1}} \bar{u}_k \int_{\Omega} (\phi_k \phi_j \partial_x \phi_i) dx = \frac{1}{6} \begin{cases} 2\bar{u}_{i+1} - \bar{u}_{i-1}, & \text{if } j = i+1; \\ -2\bar{u}_{i-1} + \bar{u}_i, & \text{if } j = i-1; \\ 0, & \text{if } |i-j|_{\mathbb{R}} \geq 2 \end{cases}.$$

We see that the composition based procedure leads to a simpler result. We have also performed some simulations with the direct discretization (for the nonlinear system) and, though we have noticed no substantial difference, we must say that the direct discretization could lead to better results under suitable data.

To discretize the operators in the feedback control rule in (3.22), we start by rewriting it, recalling (3.24), as

$$(5.3) \quad \mathcal{F}v := B_M^{\mathcal{O}} B_M^{\mathcal{O}*} Q_{\hat{u}}^{t, \lambda} v,$$

and we notice that what we essentially need is an approximation  $\overline{\mathcal{F}v}$  of  $\mathcal{F}v$ , when we only know the approximation  $\bar{v}$  of  $v$ .

We will construct  $\overline{\mathcal{F}v}$  in a few steps. For the multiplication operator  $v \mapsto \chi v$  we can of course take  $\mathcal{D}_{\overline{\chi}}\overline{v} = \overline{\chi v}$  as an approximation of  $\chi v$ . For the orthogonal projection  $P_M^\mathcal{O}$  we start by noticing that

$$P_M^\mathcal{O}(v|_{\mathcal{O}}) = \sum_{n=1}^M (v|_{\mathcal{O}}, \underline{\varepsilon}_n)_{L^2(\mathcal{O}, \mathbb{R})} \underline{\varepsilon}_n = \sum_{n=1}^M (v, \mathbb{E}_0^\mathcal{O} \underline{\varepsilon}_n)_{H \underline{\varepsilon}_n},$$

then we can take the approximation  $P_M^\mathcal{O}(v|_{\mathcal{O}}) \approx \sum_{n=1}^M \left( \overline{\mathbb{E}_0^\mathcal{O} \underline{\varepsilon}_n}^\top \mathbf{M} \overline{v} \right) \underline{\varepsilon}_n$ , from which we set the discrete approximation

$$\mathbf{P}_M \overline{v} \approx \mathbb{E}_0^\mathcal{O} P_M^\mathcal{O}(v|_{\mathcal{O}}) \text{ with } \mathbf{P}_M := \mathcal{S}_M \mathbf{M}, \text{ and } \mathcal{S}_M := \sum_{n=1}^M \overline{\mathbb{E}_0^\mathcal{O} \underline{\varepsilon}_n} \overline{\mathbb{E}_0^\mathcal{O} \underline{\varepsilon}_n}^\top.$$

Finally, the linear operator  $Q_{\hat{u}}^{t, \lambda}$  is, at this moment, unknown and (an approximation) has to be found. Note that denoting by  $Q_D = (Q_{\hat{u}}^{t, \lambda})_D$  the discretization of  $Q_{\hat{u}}^{t, \lambda}$ , we may take  $\mathbf{M}^{-1} Q_D \overline{v} \approx Q_{\hat{u}}^{t, \lambda} v$  and discretize the feedback rule (5.3) as follows: first we take the approximation  $B_M^{\mathcal{O}*} \xi \approx e^{-\frac{\lambda}{2} t} \mathbf{P}_M \mathcal{D}_{\overline{\chi}} \overline{\xi}$ , then from (5.2) and  $(\mathcal{F}v, w)_H = (B_M^{\mathcal{O}*} Q_{\hat{u}}^{t, \lambda} v, B_M^{\mathcal{O}*} w)_H$ , for  $(v, w) \in H \times H$ , we find

$$\begin{aligned} (\mathcal{F}v, w)_H &\approx \left( B_M^{\mathcal{O}*} \widetilde{Q_{\hat{u}}^{t, \lambda}} v, B_M^{\mathcal{O}*} w \right)_H \\ &= \overline{B_M^{\mathcal{O}*} w}^\top \overline{B_M^{\mathcal{O}*} \widetilde{Q_{\hat{u}}^{t, \lambda}} v} \\ &\approx e^{-\lambda t} (\mathbf{P}_M \mathcal{D}_{\overline{\chi}} \overline{w})^\top \mathbf{M} (\mathbf{P}_M \mathcal{D}_{\overline{\chi}} \mathbf{M}^{-1} Q_D \overline{v}) \\ &= \overline{w}^\top \mathbf{M} \mathbf{M}^{-1} e^{-\lambda t} (\mathbf{P}_M \mathcal{D}_{\overline{\chi}})^\top \mathbf{M} (\mathbf{P}_M \mathcal{D}_{\overline{\chi}} \mathbf{M}^{-1} Q_D \overline{v}) \\ &= \overline{w}^\top \mathbf{M} e^{-\lambda t} \mathbf{R} \mathbf{R}^\top Q_D \overline{v} \end{aligned}$$

where

$$(5.4) \quad \mathbf{R} := (\mathbf{M}_c \mathbf{P}_M \mathcal{D}_{\overline{\chi}} \mathbf{M}^{-1})^\top$$

and  $\mathbf{M}_c$ , satisfying  $\mathbf{M}_c^\top \mathbf{M}_c = \mathbf{M}$ , is the Cholesky factor of  $\mathbf{M}$  (notice that  $\mathbf{M}$  is positive definite). Thus we take  $\overline{\mathcal{F}v} = \overline{\mathcal{F}v}$  with

$$(5.5) \quad \overline{\mathcal{F}} := e^{-\lambda t} \mathbf{R} \mathbf{R}^\top Q_D.$$

*Remark 5.3.* Denoting  $Q = Q_{\hat{u}}^{t, \lambda}$ , the approximation  $\mathbf{M}^{-1} Q_D \overline{v} \approx Qv$  can be understood in the following formal sense: we have that  $\overline{w}^\top Q_D \overline{v} \approx (Qv, w)_H \approx \overline{w}^\top \mathbf{M} \overline{Qv}$ , thus we can write  $\overline{w}^\top \mathbf{M} \mathbf{M}^{-1} Q_D \overline{v} \approx \overline{w}^\top \mathbf{M} \overline{Qv}$ , that is,  $(\mathbf{M}^{-1} Q_D \overline{v}, \tilde{w})_H \approx (Qv, w)_H$ , where in the last expression the vector  $\rho = (\rho_1, \rho_2, \dots, \rho_{N_x-1}) = \mathbf{M}^{-1} Q_D \overline{v}$  is to be seen as an element in  $\mathcal{Y} = \text{span}\{\phi_i \mid i \in \{1, 2, \dots, N_x - 1\}\} \subset L^2(\Omega, \mathbb{R})$ , that is,  $\rho$  is to be understood as  $\sum_{i=1}^{N_x-1} \rho_i \phi_i$ . Further, notice that in this way the operator  $\mathbf{M}^{-1} Q_D$  is symmetric in  $\mathcal{Y}$ , because if  $(v, w) \in \mathcal{Y} \times \mathcal{Y}$  we have  $(v, w) = (\tilde{v}, \tilde{w})$  and  $(\mathbf{M}^{-1} Q_D \overline{v}, w)_H = \overline{w}^\top \mathbf{M} \mathbf{M}^{-1} Q_D \overline{v} = \overline{w}^\top Q_D \overline{v} = \overline{v}^\top Q_D \overline{w} = \overline{v}^\top \mathbf{M} \mathbf{M}^{-1} Q_D \overline{w} = (\mathbf{M}^{-1} Q_D \overline{w}, v)_H$ .

**5.2. Discretization in time.** For discretization in time of system (3.1), considered in a time interval  $[0, T]$ , where  $T$  is a positive real number, we introduce an uniform mesh

$$(5.6) \quad [0, T]_D := \left(0, \frac{T}{N_t}, \frac{2T}{N_t}, \dots, \frac{(N_t-1)T}{N_t}, T\right)$$

consisting of the points in  $[0, T]$  that are proportional to the time step  $k := \frac{T}{N_t}$ , with  $N_t \in \mathbb{N}_0$ . Then, any function  $u \in H^1((0, T), V)$  is approximated by the values it takes in  $[0, T]_D \times \Omega_D$ , that is, we essentially approximate  $u = u(t, x)$  by a matrix  $[u] \in \mathcal{M}_{(N_x-1) \times (N_t+1)}$  whose  $j$ th column is the vector  $\overline{u(jk, \cdot)}$ . That is,  $[u]_{ij} = u(jk, ih)$ , for  $i \in \{1, 2, \dots, N_x - 1\}$  and  $j \in \{0, 1, 2, \dots, N_t\}$

**5.3. Computation of the discretized feedback rule.** We recall the operator  $Q = Q_{\hat{u}}^{s, \lambda}$  satisfying, for  $t > 0$ , the differential Riccati equation (3.23).

**5.3.1. Discretization of the differential Riccati equation.** To construct the approximation  $Q_D$  for the operator  $Q$ , we can look for  $Q_D$  solving

$$\partial_t Q_D - Q_D X - X^\top Q_D - e^{-\lambda t} Q_D \mathbf{R} \mathbf{R}^\top Q_D + e^{\lambda t} \nu \mathbf{S} = 0, \quad t > 0,$$

with  $\mathbf{R}$  as in (5.4) and

$$(5.7) \quad X = X(t) = \mathbf{M}^{-1} \left( \nu \mathbf{S} + \mathbf{B} \mathcal{D}_{\hat{u}(t)} \right).$$

Equivalently, we can look for  $P = e^{-\lambda t} Q_D$  solving

$$(5.8) \quad \partial_t P - P X - X^\top P - \mathbf{P} \mathbf{R} \mathbf{R}^\top P + \nu \mathbf{S} + \lambda P = 0, \quad t > 0.$$

*Remark 5.4.* Notice that from the relation  $X\bar{v} \approx -\nu \partial_{xx} v + \mathcal{B}(\hat{u})v$ , we have that  $(Q(-\nu \partial_{xx} v + \mathcal{B}(\hat{u})v), w)_H \approx \bar{w}^\top Q_D X \bar{v}$ . Similarly  $((-\nu \partial_{xx} v + \mathcal{B}(\hat{u}))^* Q v, w)_H \approx \bar{w}^\top X^\top Q_D \bar{v}$ , and  $(Q \mathcal{F} v, w)_H \approx \bar{w}^\top Q_D \bar{\mathcal{F}} \bar{v}$ .

**5.3.2. Initialization of the differential Riccati equation.** Since we need to solve (3.23) backwards in time (cf. Remark 3.14), we will also solve system (5.8) backwards in time; thus the question is: how to initialize the system? Roughly speaking, it seems that we would need to know  $P(+\infty)$ , and even if we know this (limit) value it is not clear how we could use it.

Recall that, our main goal is to approach the desired solution  $\hat{u}(t)$  as time  $t$  increases but, in a real application we also want to have an effective controller that, for example, guarantees us that after some time  $t = \hat{T} > 0$  we are indeed closer than we were at initial time  $t = 0$ , say, e.g.,  $|v(\hat{T})|_H^2 \leq \frac{1}{2} |v(0)|_H^2$ . Also, in applications it is reasonable to think of a problem set for a possibly very long time range  $t \in [0, T]$ , but never for an infinite time range.

Thus we suppose we are interested in the evolution for time  $t \in [0, T]$ , then we may suppose that for time  $t > T$  our solution is stationary, that is, we may study the same problem but, now we suppose that  $\hat{u}(t) = \hat{u}(T)$ , for all  $t \geq T$ . Notice, however, that this does not reduce the full problem to the stationary case, because in the interesting time range  $t \in (0, T)$  the reference trajectory  $\hat{u}(t)$  remains unchanged.

Now we can find  $P_T$  solving the algebraic Riccati equation

$$(5.9) \quad P(-X(T) + \frac{\lambda}{2} I) + (-X(T) + \frac{\lambda}{2} I)^\top P - \mathbf{P} \mathbf{R} \mathbf{R}^\top P + \nu \mathbf{S} = 0,$$

and we can see that,  $P_T$  will solve the autonomous system (5.8) for  $t \geq T$  (under the supposition  $\hat{u}(t) = \hat{u}(T)$  for  $t \geq T$ ), see also [47, sections 1.4 and 4.4].

Then, it remains to solve (5.8) for  $t \in [0, T]$  with the final condition  $P(T) = P_T$ .

### 5.3.3. Solving the Riccati systems.

• *General procedure.* To solve the algebraic Riccati system (5.9) we use the software available from [9]; in this way we find  $P_T$ .

To solve (backwards in time) the differential system (5.8), for  $t \in [0, T]$ , with the initial condition  $P(T) = P_T$ , we proceed as follows. Recall the mesh  $[0, T]_D$  of the interval  $[0, T]$ , defined in (5.6). We have  $P^{N_t} := P(N_t k) = P(T) = P_T$ ; next, recursively, we construct  $P^j := P(jk)$  from  $P^{j+1}$ , for  $j \in \{0, 1, \dots, N_t - 1\}$ , as follows: we start by rewriting (5.8) as

$$-\partial_t P = P(-X + \frac{\lambda}{2}I) + (-X + \frac{\lambda}{2}I)^\top P - P\mathbf{R}\mathbf{R}^\top P + \nu\mathbf{S} =: R_F(P)$$

and, we use the Crank-Nicolson inspired scheme

$$-\frac{2}{k}(P^{j+1} - P^j) = R_F(P^j) + R_F(P^{j+1}),$$

from which we obtain  $R_F(P^j) - \frac{2}{k}P^j + R_F(P^{j+1}) + \frac{2}{k}P^{j+1} = 0$ , that is,

$$(5.10) \quad P^j(-X + \frac{\lambda}{2}I - \frac{1}{k}I) + (-X + \frac{\lambda}{2}I - \frac{1}{k}I)^\top P^j - P^j\mathbf{R}\mathbf{R}^\top P^j + Z^{j+1} = 0$$

with  $Z^{j+1} = R_F(P^{j+1}) + \frac{2}{k}P^{j+1} + \nu\mathbf{S}$ . Hence  $P^j$  solves again an algebraic Riccati equation and we can still use the software in [9].

• *Initial guess.* The software in [9] (see also [10]) uses a Newton method to solve an algebraic Riccati equation like (5.9). We have to provide an initial starting guess  $Y_0$  such that  $-X(T) + \frac{\lambda}{2}I - \mathbf{R}\mathbf{R}^\top Y_0^\top Y_0$  is stable. This is of course a nontrivial task (see, e.g., the discussion after (1.4) in [27]) and we look for the initial guess in three steps:

- (i) We set  $M = +\infty$  and  $\chi = 1_\Omega$ . That is, we impose no constraints neither on the dimension nor on the support of the controller. In this case we can see that  $\mathbf{R} = (\mathbf{M}_c \mathbf{M}^{-1})^\top$  and  $\mathbf{R}\mathbf{R}^\top = \mathbf{M}^{-1}$ . Then from (5.7) we can expect that

$$-X(T) + \frac{\lambda}{2}I - \mathbf{M}^{-1}Y_0^\top Y_0 = -\mathbf{M}^{-1} \left( \nu\mathbf{S} + \mathbf{B}\mathcal{D}_{\hat{u}(T)} \right) + \frac{\lambda}{2}I - \mathbf{M}^{-1}Y_0^\top Y_0$$

will be stable for  $Y_0 = \beta\mathbf{M}_c$ , with  $\sqrt{2}\beta \geq \beta_0 := (\nu^{-1}|\hat{u}(T)|_{L^\infty(\Omega, \mathbb{R})}^2 + \lambda)^{\frac{1}{2}}$ .

Notice that, proceeding as in the beginning of section 3.1 we see that a weak solution  $w$  for  $w_t = \nu\partial_{xx}w - \partial_x(\hat{u}(T)w) + \frac{\lambda}{2}w - \beta^2w$ , will satisfy the estimate  $\frac{d}{dt}|w|_H^2 \leq -\nu|\partial_x w|_H^2 + \nu^{-1}|\hat{u}(T)|_{L^\infty(\Omega, \mathbb{R})}^2|w|_H^2 + \lambda|w|_H^2 - 2\beta^2|w|_H^2$ . That is, the lower bound  $\beta_0$  works for the continuous system. However, when taking  $\beta$  strictly bigger than  $\beta_0$  we may expect that the stability is preserved for the discretized system, if  $N_x$  and  $N_t$  are big enough.

Hence we set  $\beta = (\nu^{-1}|\hat{u}(T)|_{L^\infty(\Omega, \mathbb{R})}^2 + \lambda)^{\frac{1}{2}}$  and solve (5.9), i.e.

$$P(-X(T) + \frac{\lambda}{2}I) + (-X(T) + \frac{\lambda}{2}I)^\top P - P\mathbf{M}^{-1}P + \nu\mathbf{S} = 0,$$

providing the initial guess  $Y_0 = \beta\mathbf{M}_c$ . Let us denote the solution by  $P_T^{[1]}$ .

- (ii) We set  $M = +\infty$  and the true  $\chi$ . That is, now we include the constraints on the support of the controller. In this case  $\mathbf{R} = (\mathbf{M}_c \mathcal{D}_{\bar{\chi}} \mathbf{M}^{-1})^\top$ , see (5.4). In some cases it may happen that  $P_T^{[1]}$  is not a “good” initial guess. In some cases (as we have observed in some simulations) the step from  $(+\infty, 1_\Omega)$  to  $(+\infty, \chi)$  seems to be too big, in other words  $P_T^{[1]}$  is too far from the solution corresponding to  $\mathbf{R} = (\mathbf{M}_c \mathcal{D}_{\bar{\chi}} \mathbf{M}^{-1})^\top$ . Having this in mind we connect the operators  $I$  and  $\mathcal{D}_{\bar{\chi}}$

by the homotopy  $\mathcal{H}_\tau = (1 - \tau)I + \tau\mathcal{D}_{\bar{\chi}}$ ,  $\tau \in [0, 1]$  and set  $\mathbf{H}_\tau := (\mathbf{M}_c \mathcal{H}_\tau \mathbf{M}^{-1})^\top$ . Now let us fix  $N_{\mathcal{H}} \in \mathbb{N}_0$  and set the homotopy step  $\rho = \frac{1}{N_{\mathcal{H}}}$  and solve

$$P(-X(T) + \frac{\lambda}{2}I) + (-X(T) + \frac{\lambda}{2}I)^\top P - P\mathbf{H}_\rho \mathbf{H}_\rho^\top P + \nu\mathbf{S} = 0,$$

providing the initial guess  $Y_0 = P_T^{[1]}$ . Let us denote the solution by  $P_T^{[1+\rho]}$ . Recursively we solve, for  $l \in 2, \dots, N_{\mathcal{H}}$ ,

$$P(-X(T) + \frac{\lambda}{2}I) + (-X(T) + \frac{\lambda}{2}I)^\top P - P\mathbf{H}_{l\rho} \mathbf{H}_{l\rho}^\top P + \nu\mathbf{S} = 0,$$

providing the initial guess  $Y^0 = P_T^{[1+(l-1)\rho]}$ , and denote the solution by  $P_T^{[1+l\rho]}$ . After  $N_{\mathcal{H}}$  steps we have found a solution  $P_T^{[2]}$  for

$$P(-X(T) + \frac{\lambda}{2}I) + (-X(T) + \frac{\lambda}{2}I)^\top P - P\mathbf{H}_1 \mathbf{H}_1^\top P + \nu\mathbf{S} = 0$$

with  $\mathbf{H}_1 = (\mathbf{M}_c \mathcal{D}_{\bar{\chi}} \mathbf{M}^{-1})^\top$ .

- (iii) We set the true  $M$  and the true  $\chi$ . That is, finally we include also the constraints on the dimension of the controller. In this case  $\mathbf{R}$  is given by (5.4). Analogously to step (ii) we consider the homotopy  $\mathcal{H}_\tau = (1 - \tau)\mathcal{D}_{\bar{\chi}} + \tau\mathbf{P}_M \mathcal{D}_{\bar{\chi}}$ , set  $\mathbf{H}_\tau := (\mathbf{M}_c \mathcal{H}_\tau \mathbf{M}^{-1})^\top$  and, starting with  $P_T^{[2]}$  we find, recursively after  $N_{\mathcal{H}}$  steps, a solution  $P_T^{[3]}$  for

$$P(-X(T) + \frac{\lambda}{2}I) + (-X(T) + \frac{\lambda}{2}I)^\top P - P\mathbf{H}_1 \mathbf{H}_1^\top P + \nu\mathbf{S} = 0$$

with  $\mathbf{H}_1 = (\mathbf{M}_c \mathbf{P}_M \mathcal{D}_{\bar{\chi}} \mathbf{M}^{-1})^\top = \mathbf{R}$ . That is  $P_T^{[3]}$  solves (5.9).

Of course, the number of homotopy steps  $N_{\mathcal{H}}$  may be taken different in steps (ii) and (iii). To get the convergence of the Newton method used to solve the algebraic Riccati equations at each homotopy step we may need, depending on the situations, to increase the number of homotopy steps  $N_{\mathcal{H}}$ .

Notice, however that in step (iii) increasing  $N_{\mathcal{H}}$  can be sufficient for convergence at each homotopy step only if  $M$  is big enough. Indeed we can see that the algebraic Riccati equation will have a solution up to the homotopy step before the last, because from the observability inequality (3.14) we can also derive  $|q(0)|_H^2 \leq (1 - \tau)^{-2} C |(1 - \tau)\chi q|_{L^2((0, T), H)}^2$ , and from

$$\begin{aligned} & |(1 - \tau)\chi q|_{L^2((0, T), H)}^2 \\ &= |(1 - \tau)(1 - P_M^\mathcal{O})(\chi q|_{\mathcal{O}})|_{L^2((0, T), L^2(\mathcal{O}, \mathbb{R}))}^2 + |(1 - \tau)P_M^\mathcal{O}(\chi q|_{\mathcal{O}})|_{L^2((0, T), L^2(\mathcal{O}, \mathbb{R}))}^2 \\ &\leq |(1 - \tau)(1 - P_M^\mathcal{O})(\chi q|_{\mathcal{O}})|_{L^2((0, T), L^2(\mathcal{O}, \mathbb{R}))}^2 + |P_M^\mathcal{O}(\chi q|_{\mathcal{O}})|_{L^2((0, T), L^2(\mathcal{O}, \mathbb{R}))}^2 \\ &= |(1 - \tau)(\chi q|_{\mathcal{O}}) + (1 - (1 - \tau))P_M^\mathcal{O}(\chi q|_{\mathcal{O}})|_{L^2((0, T), L^2(\mathcal{O}, \mathbb{R}))}^2, \end{aligned}$$

we arrive at  $|q(0)|_H^2 \leq (1 - \tau)^{-2} C |(1 - \tau)\chi q + \tau\mathbb{E}_0^\mathcal{O} P_M^\mathcal{O}(\chi q|_{\mathcal{O}})|_{L^2((0, T), H)}^2$ . Then, from this observability inequality it will follow that there exists a stabilizing control (for system (3.1)) of the form  $\zeta(t) = \mathbb{F}_\tau \eta := (1 - \tau)\chi \eta(t) + \tau\chi \mathbb{E}_0^\mathcal{O} P_M^\mathcal{O}(\eta(t)|_{\mathcal{O}})$ , for  $\tau < 1$ . Reasoning as in section 3.4, by the dynamic programming principle it will follow that the control can be taken in feedback form  $\zeta(t) = \mathbb{F}_\tau \mathbb{F}_\tau^* Q_\tau^t v(t)$ , where  $e^{\lambda t} Q_\tau^t$  solves the Riccati equation (3.25) with  $\mathbb{F}_\tau$  in the role of  $\mathbb{B}$ , which corresponds on the discrete level to the case we take  $\mathbf{R} = (\mathbf{M}_c [(1 - \tau)\mathcal{D}_{\bar{\chi}} + \tau\mathbf{P}_M \mathcal{D}_{\bar{\chi}}] \mathbf{M}^{-1})^\top$  in (5.8).

For the last homotopy step, that is, for  $\tau = 1$  the observability will hold if  $M$  is big enough (following the arguments in [7]). Also, from Theorem 3.4 a stabilizing control exists if  $M$  is big enough, so we cannot guarantee the existence of a stabilizing control of the form  $\zeta(t) = \chi \mathbb{E}_0^{\mathcal{O}} P_M^{\mathcal{O}}(\eta(t)|_{\mathcal{O}})$  for arbitrary (small)  $M$ , and then we cannot guarantee the existence of a solution for the algebraic Riccati equation (3.25).

In step (ii), increasing  $N_{\mathcal{H}}$  should be sufficient to get the convergence at each homotopy step, because reasoning as above we can conclude that there exists a stabilizing control of the form  $\zeta(t) = (1 - \tau)\eta(t) + \tau\chi\eta(t)$ , for all  $\tau \in [0, 1]$ .

*Therefore, if convergence is not reached at a homotopy step in (ii) or at a homotopy step before the last in (iii) we probably need either more homotopy steps or to refine our mesh; if convergence is not reached only at the last homotopy step in (iii), then probably the number of controls is not enough.*

In the simulations we present here, we have taken no more than  $N_{\mathcal{H}} = 20$  in the second step and no more than  $N_h = 10$  in the third step. Notice however that increasing the number of homotopy steps does not mean that the computational time will be much bigger because the Newton method may converge faster at each homotopy step.

Finally, in the process of solving the differential Riccati equation, to find  $P^j$  solving (5.10) we provide the natural initial guess  $P^{j+1}$ . Again, we cannot guarantee that the solution will always exist. *If this process fails at some  $j$ -step, we can try to refine the mesh (in particular, by increasing the number  $N_t$  of time steps in (5.6)); if that does not work it probably means that the number of controls  $M$  is not sufficient.*

**5.4. Solving the discretized Oseen–Burgers system.** Once we have constructed  $P$  we can simulate the evolution of the system (3.1); we look for  $\bar{v}(t) := \overline{v(t, \cdot)}$  that solves the system

$$(5.11) \quad \partial_t \bar{v} + \nu \mathbf{M}^{-1} \mathbf{S} \bar{v} + \mathbf{M}^{-1} \mathbf{B} \mathcal{D}_{\bar{u}} \bar{v} + \mathbf{R}^{\top} \mathbf{R} P(\bar{v}) = 0, \quad \bar{v}(0) = \bar{v}_0,$$

and expect  $\bar{v}$  to go exponentially to 0 as time increases, with an a priori prescribed rate  $\frac{\lambda}{2} > 0$  as time goes to infinity (cf. (3.4) and (3.16)); recall that  $P$  depends on  $\lambda$  (cf. (5.8)). Notice that from (5.5), (5.4), and  $P = e^{-\lambda t} Q_D$ , it follows  $\overline{\mathcal{F}} \bar{v} = \mathbf{R}^{\top} \mathbf{R} P \bar{v}$ .

Again, we will approximate  $\bar{v}(t) \approx [\bar{v}(jk)]$ ,  $j \in \{0, 1, \dots, N_t\}$ , and we apply a Crank-Nicolson inspired algorithm to solve system (5.11). For simplicity we denote  $\overline{\mathcal{F}}^j := \mathbf{R}^{\top} \mathbf{R} P^j$ , for  $j \in \{0, 1, \dots, N_t\}$ . Set  $\bar{v}^0 := \overline{v(0k)} = \bar{v}_0$ ; then the idea is to construct, recursively,  $\bar{v}^{j+1} := \overline{v((j+1)k)}$  from  $\bar{v}^j := \overline{v(jk)}$  by the scheme

$$\begin{aligned} \frac{2}{k}(\bar{v}^{j+1} - \bar{v}^j) &= -\nu \mathbf{M}^{-1} \mathbf{S}(\bar{v}^j + \bar{v}^{j+1}) - \mathbf{M}^{-1} (\mathbf{B} \mathcal{D}_{\bar{u}^j} \bar{v}^j + \mathbf{B} \mathcal{D}_{\bar{u}^{j+1}} \bar{v}^{j+1}) \\ &\quad - \left( \overline{\mathcal{F}}^j \bar{v}^j + \overline{\mathcal{F}}^{j+1} \bar{v}^{j+1} \right), \end{aligned}$$

with  $\bar{u}^j := \overline{\hat{u}(jk)}$ ,  $j \in \{0, 1, \dots, N_t\}$ . Then, working a little the above scheme, we can obtain

$$(5.12) \quad \begin{aligned} \bar{v}^{j+1} &= A_{\oplus}^{-1} A_{\ominus} \bar{v}^j - \frac{k}{2} A_{\oplus}^{-1} (\mathbf{B} \mathcal{D}_{\bar{u}^j} \bar{v}^j + \mathbf{B} \mathcal{D}_{\bar{u}^{j+1}} \bar{v}^{j+1}) \\ &\quad - \frac{k}{2} A_{\oplus}^{-1} \mathbf{M} \left( \overline{\mathcal{F}}^j \bar{v}^j + \overline{\mathcal{F}}^{j+1} \bar{v}^{j+1} \right), \end{aligned}$$

with  $A_{\oplus} := \mathbf{M} + \frac{k}{2} \nu \mathbf{S}$  and  $A_{\ominus} := \mathbf{M} - \frac{k}{2} \nu \mathbf{S}$ . Notice that the unknown  $\bar{v}^{j+1}$  is still present on the right hand side of (5.12). In the argument of the feedback operator we will replace  $\bar{v}^{j+1}$  by a preliminary guess  $\bar{v}_G^{j+1}$ , and approximate  $\mathbf{B} \mathcal{D}_{\bar{u}^{j+1}} \bar{v}^{j+1}$  by

$\mathbf{BD}_{\bar{u}^j} \bar{v}^j + k(\mathbf{BD}_{\bar{u}^j} \bar{v}^j - \mathbf{BD}_{\bar{u}^{j-1}} \bar{v}^{j-1})$  (where we define  $\bar{v}^{-1} := \bar{v}^0 = \bar{v}_0$ ). In this way we arrive at the scheme

$$(5.13) \quad \begin{aligned} \bar{v}^{j+1} &= A_{\oplus}^{-1} A_{\ominus} \bar{v}^j - k A_{\oplus}^{-1} \left( (1 + \frac{k}{2}) \mathbf{BD}_{\bar{u}^j} \bar{v}^j - \frac{k}{2} \mathbf{BD}_{\bar{u}^{j-1}} \bar{v}^{j-1} \right) \\ &\quad - \frac{k}{2} A_{\oplus}^{-1} \mathbf{M} \left( \bar{\mathcal{F}}^j \bar{v}^j + \bar{\mathcal{F}}^{j+1} \bar{v}_G^{j+1} \right). \end{aligned}$$

We set  $\bar{v}_G^{j+1}$  as the “uncontrolled” output

$$\bar{v}_G^{j+1} := A_{\oplus}^{-1} A_{\ominus} \bar{v}^j - k A_{\oplus}^{-1} \left( (1 + \frac{k}{2}) \mathbf{BD}_{\bar{u}^j} \bar{v}^j - \frac{k}{2} \mathbf{BD}_{\bar{u}^{j-1}} \bar{v}^{j-1} \right).$$

**5.5. Solving the discretized Burgers system.** Concerning the evolution of the system (1.1)–(1.2). We look for  $\bar{u}(t) := \bar{u}(t, \cdot)$  that solves the system

$$(5.14) \quad \partial_t \bar{u} + \nu \mathbf{M}^{-1} \mathbf{S} \bar{u} + \frac{1}{2} \mathbf{M}^{-1} \mathbf{BD}_{\bar{u}} \bar{u} + \bar{h} + \mathbf{R}^\top \mathbf{R} P(\bar{u} - \bar{u}) = 0, \quad \bar{u}(0) = \bar{u}_0,$$

and expect  $\bar{u}$  to go exponentially to  $\bar{u}$ , with an a priori prescribed rate  $\frac{\lambda}{2} > 0$ , as time increases (with  $\mathbf{R}$  as in (5.4)). However this would be meaningful if  $\bar{u}$  were a solution for the uncontrolled discrete system, which is not true. The solution of the uncontrolled discrete system

$$(5.15) \quad \partial_t \bar{u}_S + \nu \mathbf{M}^{-1} \mathbf{S} \bar{u}_S + \frac{1}{2} \mathbf{M}^{-1} \mathbf{BD}_{\bar{u}_S} \bar{u}_S + \bar{h} = 0, \quad \bar{u}_S(0) = \bar{u}_0$$

will be an approximation  $\bar{u}_S$  of  $\bar{u}$ . There is no reason to expect that  $e^{\frac{\lambda}{2}t}(\bar{u}(t) - \bar{u}_S(t))$  will remain bounded for  $t \in \mathbb{R}_0$ .

Nevertheless there is a way to check the rate of exponential stabilization  $\lambda$ . We will just have to compute the discrete (fictitious) external force  $\bar{h}_f$ , that makes  $\bar{u}$  a solution of the discrete system, that is,

$$(5.16) \quad \partial_t \bar{u} + \nu \mathbf{M}^{-1} \mathbf{S} \bar{u} + \frac{1}{2} \mathbf{M}^{-1} \mathbf{BD}_{\bar{u}} \bar{u} + \bar{h}_f = 0, \quad \bar{u}(0) = \bar{u}_0.$$

Before that we present the scheme we apply. Suppose for the moment, that we know  $\bar{h}_f$ . Then, we follow the idea in section 5.4 and arrive at the scheme

$$(5.17) \quad \begin{aligned} \bar{u}^{j+1} &= A_{\oplus}^{-1} A_{\ominus} \bar{u}^j - \frac{k}{2} A_{\oplus}^{-1} \left( (1 + \frac{k}{2}) \mathbf{BD}_{\bar{u}^j} \bar{u}^j - \frac{k}{2} \mathbf{BD}_{\bar{u}^{j-1}} \bar{u}^{j-1} \right) \\ &\quad - \frac{k}{2} A_{\oplus}^{-1} \mathbf{M} \left( \bar{h}_f^j + \bar{h}_f^{j+1} + \bar{\mathcal{F}}^j (\bar{u}^j - \hat{u}^j) + \bar{\mathcal{F}}^{j+1} (\bar{u}_G^{j+1} - \hat{u}^{j+1}) \right), \end{aligned}$$

with the preliminary “uncontrolled” guess  $\bar{u}_G^{j+1}$  given by

$$\begin{aligned} \bar{u}_G^{j+1} &:= A_{\oplus}^{-1} A_{\ominus} \bar{u}^j - \frac{k}{2} A_{\oplus}^{-1} \left( (1 + \frac{k}{2}) \mathbf{BD}_{\bar{u}^j} \bar{u}^j - \frac{k}{2} \mathbf{BD}_{\bar{u}^{j-1}} \bar{u}^{j-1} \right) \\ &\quad - \frac{k}{2} A_{\oplus}^{-1} \mathbf{M} \left( \bar{h}_f^j + \bar{h}_f^{j+1} \right). \end{aligned}$$

It remains to explain how we construct the force  $\bar{h}_f$ . Actually, from our scheme we can deduce that we only need to know the terms  $\frac{k}{2} A_{\oplus}^{-1} \mathbf{M} \left( \bar{h}_f^j + \bar{h}_f^{j+1} \right)$ , for  $j \in \{0, 1, \dots, N_t - 1\}$ , that we can easily compute as

$$\begin{aligned} &\frac{k}{2} A_{\oplus}^{-1} \mathbf{M} \left( \bar{h}_f^j + \bar{h}_f^{j+1} \right) \\ &= -\bar{u}^{j+1} + A_{\oplus}^{-1} A_{\ominus} \bar{u}^j - \frac{k}{2} A_{\oplus}^{-1} \left( (1 + \frac{k}{2}) \mathbf{BD}_{\bar{u}^j} \bar{u}^j - \frac{k}{2} \mathbf{BD}_{\bar{u}^{j-1}} \bar{u}^{j-1} \right), \end{aligned}$$

(where we define  $\bar{u}^{-1} := \bar{u}^0$ ).



**6. Numerical examples: the linear Oseen–Burgers system.** We present some results of the numerical simulations we have performed concerning the stabilization of system (3.1) to zero. Below,  $v_u$  stands for the solution of the uncontrolled (discretized) system (i.e.,  $\zeta = 0$ ), and  $v$  (or  $v_\lambda$ ) stands for the solution of the (discretized) system under the action of a (discretized) feedback controller  $\zeta = \chi\eta$ , with  $\eta = e^{-\lambda t} \mathbb{E}_0^\mathcal{O} P_M^\mathcal{O}((\chi Q_u^{t,\lambda} v)|_\mathcal{O})$  as in (3.22). If nothing is said in contrary  $\mathcal{O} = (\inf\{\Omega \cap \text{supp}(\chi)\}, \sup\{\Omega \cap \text{supp}(\chi)\})$ .

We follow a “trying and checking” procedure, we fix  $M$  and check the results of the simulations.

**6.1. Testing with a family of reference trajectories.** We set  $\nu = \frac{1}{10}$ ,  $\lambda = 2$ ,  $\Omega = (0, \pi)$ ,  $\mathcal{O} = (\frac{3}{2}, \frac{5}{2})$ , and

$$(6.1) \quad \chi(x) = \mathbb{E}_0^\mathcal{O} \left( \sin\left(\left(x - \frac{3}{2}\right)\pi\right) \middle| \mathcal{O} \right).$$

That is,  $\chi = \mathbb{E}_0^\mathcal{O} \underline{s}_1$  (cf. section 2.1). Next we set the family of reference trajectories

$$(6.2) \quad \hat{u} = \hat{u}^{(i,j)} = C_{\text{nr}} (\sin(-t) \sin(ix) - \cos(3t) \sin(jx)),$$

where the constant  $C_{\text{nr}}$  is chosen so that  $|\hat{u}|_{\mathcal{W}} = 1$ . In this case we have that  $M_{\text{ref}} = \sqrt{120} \approx 10.95$  and  $M_{\text{exp}} \approx 57208.12$ ; so our question is if the number  $M$  of needed controls stay “close” to  $M_{\text{ref}}$  or to  $M_{\text{exp}}$  (cf. section 3.3). We will test with the smaller number  $M = 4$ , and  $v_0(x) = \sin(2x)$ . The function  $\chi$  and the four controls are plotted in figure 1.

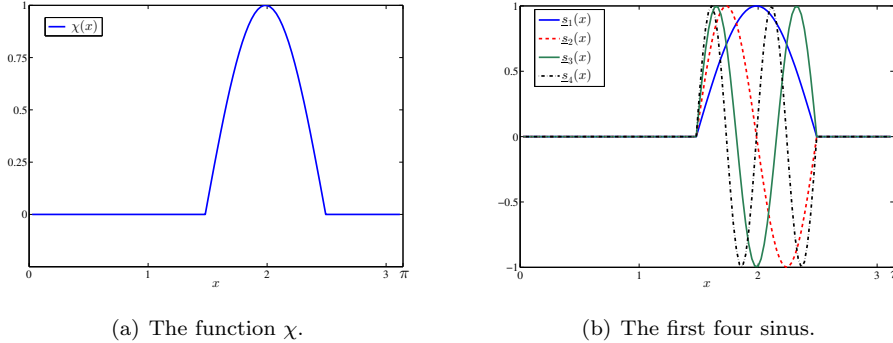


FIG. 1. Basis for the control space  $\{\chi \mathbb{E}_0^\mathcal{O} \eta \mid \eta \in \text{span}\{\underline{s}_i \mid i \in \{1, 2, 3, 4\}\}\}$ .

In figure 2 we can check that the feedback control is able to stabilize the system with the desired rate. Then, we change the initial condition to  $v_0(x) = \sin(x) - \sin(6x)$ , and test for some other reference trajectories (with higher frequencies) in the family (6.2); in figure 3 we see that the feedback control is still able to stabilize the system with the desired rate; of course, the squared norm  $|v|_H^2$  is to be understood as the discrete approximation  $\bar{v}^\top \mathbf{M} \bar{v}$  (cf. section 5.1).

*Remark 6.1.* There is no particular reason to test with  $M$  far below  $M_{\text{ref}}$ ; trivially, if  $M$  controls are enough to stabilize the system, then taking more controls we can also stabilize the system.

**Initial data in  $L^2(\Omega, \mathbb{R}) \setminus H^1(\Omega, \mathbb{R})$ .** We set  $\lambda = 4$ ,  $\nu = \frac{1}{10}$  and  $\chi$  as in (6.1). But now we set  $v_0(x) = x^{\frac{1}{2}}$  and the reference trajectory  $\hat{u} = C_{\text{nr}} \mathbf{1}_{[0, \frac{\pi}{2}]}(\sin(-t) \sin(2x) -$

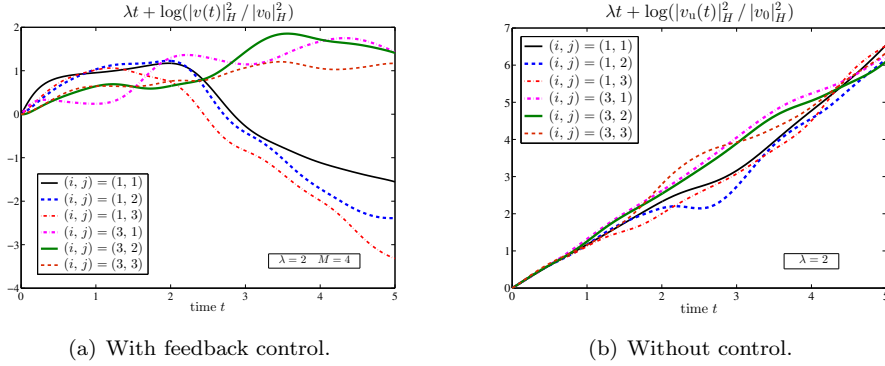


FIG. 2. Convergence rate is achieved with the feedback control.

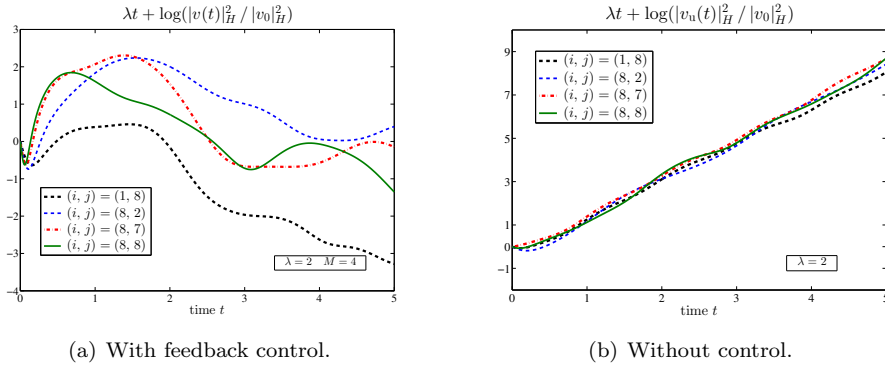


FIG. 3. Convergence rate is achieved with the feedback control.

$\cos(3t) \sin(2x)$ ), where

$$(6.3) \quad 1_{[a,b]}(x) := \begin{cases} 1, & \text{if } x \in [a, b] \\ 0, & \text{if } x \in \Omega \setminus [a, b] \end{cases}, \quad a, b \in \mathbb{R}$$

and  $C_{nr}$  is taken so that  $|\hat{u}|_{\mathcal{W}} = 1$ . We can see in figure 4 that two controls stabilize the system (3.1) to zero with the desired rate. In figure 5 we see the controls corresponding to the cases we take either two or three controls. Notice that in this case the initial condition is in  $H \setminus V$  and the support of the control is disjoint from that of  $\hat{u}$ .

**6.2. Increasing number of needed controls.** We set  $\hat{u} = 0$ ,  $\Omega = (0, \pi)$ . In this example we show that for any given  $n \in \mathbb{N}_0$  we can construct  $\chi$  supported in a subset  $\omega \subset \bar{\omega} \subset \Omega$ ,  $\lambda > 0$ , and an initial condition  $v_0$ , such that the first  $2n$  controls cannot stabilize the system (3.1) to zero with the rate  $\lambda$ . However, by increasing the number of controls we can obtain the desired stabilization. Notice that here we look for  $\chi \neq 1_\Omega$  (cf. last paragraph in section 3.3).

Let  $n \in \mathbb{N}_0$ . Set  $v_0(x) = \sin((2n+5)x)$ ,  $\mathcal{O} = (\frac{2\pi}{2n+5}, \frac{(2n+3)\pi}{2n+5})$ , and  $\chi = \mathbb{E}_0^\mathcal{O} v_0^2|_{\mathcal{O}}$ . We claim that the controls  $\chi P_{2n}^\mathcal{O} \eta$  cannot stabilize the equation with rate  $\lambda > 2\nu(2n+5)^2$ . Indeed, we can write  $(v_0, \chi P_{2n}^\mathcal{O} \eta)_H = \sum_{i=1}^{2n} \eta_i \int_{\mathcal{O}} v_0^3 s_i d\mathcal{O} = \sum_{i=1}^{2n} \eta_i (\frac{1}{2})^{\frac{3}{2}} \int_{\mathcal{O}} s_{2n+1}^3 s_i d\mathcal{O}$  and also  $\int_{\mathcal{O}} s_{2n+1}^3 s_i d\mathcal{O} = \frac{1}{4} \int_{\mathcal{O}} (1 - c_{2(2n+1)})(c_{2n+1-i} -$

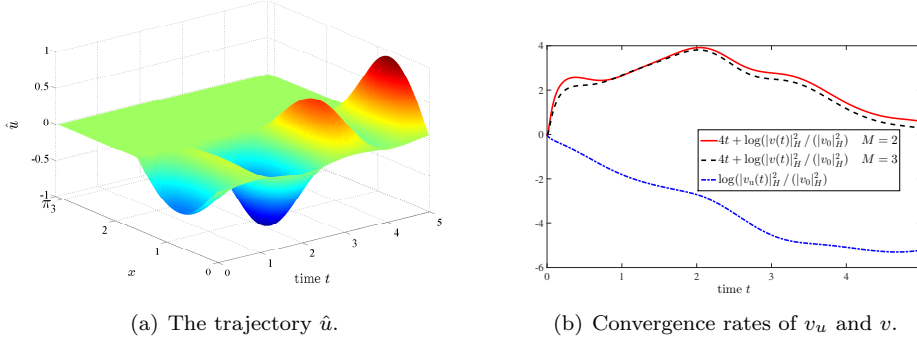


FIG. 4. Two controls stabilize the system to zero with desired rate.

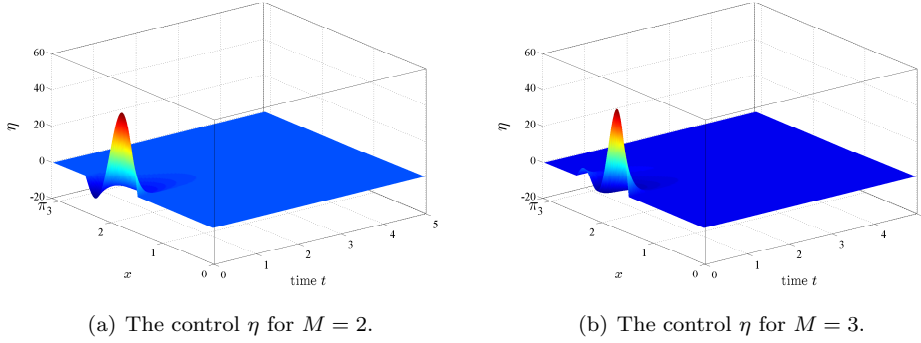


FIG. 5. The controls.

$\mathcal{C}_{2n+1+i}) d\mathcal{O}$ , with

$$\mathcal{C}_j(x) := \left(\frac{2}{l}\right)^{\frac{1}{2}} \cos\left(\frac{j\pi(x-l_1)}{l}\right), \quad x \in \mathcal{O}, \quad l := \text{length}(\mathcal{O}) = \frac{(2n+1)\pi}{2n+5}, \quad l_1 = \frac{2\pi}{2n+5}$$

(cf. definition of the functions  $\underline{s}_n$  in section 2.1). Now, notice that since  $i \leq 2n$ , we have that  $0 < 2n + 1 \pm i < 2(2n + 1)$ , thus we can conclude that  $(v_0, \chi P_{2n}^{\mathcal{O}} \eta)_H = 0$ . Therefore since the eigenspace  $\text{span}\{v_0\}$  is preserved by the Laplacian we can conclude that the control cannot change the dynamics on this space. Thus, we conclude that the rate of convergence is at most  $2\nu(2n + 5)^2$ .

Now we set  $\nu = \frac{1}{10}$ ; from above we know that for  $n \in \{1, 2\}$  the rate of convergence  $\lambda = 20 > \frac{81}{5}$  is not achieved with the first  $2n$  controls. Simulations below show that, in these examples, it is enough to add one more control to achieve the rate. In particular, we have  $M = 2n + 1 \leq 5 < \left(\frac{1}{\nu}\right)^{\frac{1}{2}} = 10\sqrt{2} = M_{\text{ref}} < M_{\text{exp}} = e^{10\sqrt{2}}$ . In Figures 6 and 7 we see the results of the simulations for the cases  $n = 1$  and  $n = 2$ ; we can check the stabilization rate to zero of the heat system (i.e., system (3.1) with  $\hat{u} = 0$ ).

**6.3. Instability of the system.** Increasing  $|\hat{u}|_{\mathcal{V}}$  and decreasing  $\nu$  brings more instability to the system, which leads to the necessity to take a bigger number  $M$  of controls. To illustrate the instability of the (uncontrolled) system (3.1), and the response of the controller, we can just take a stationary reference trajectory. The main advantage is that we do not need to solve the differential Riccati equation that is the more expensive numerical step. Notice, however that (as far as we know) an

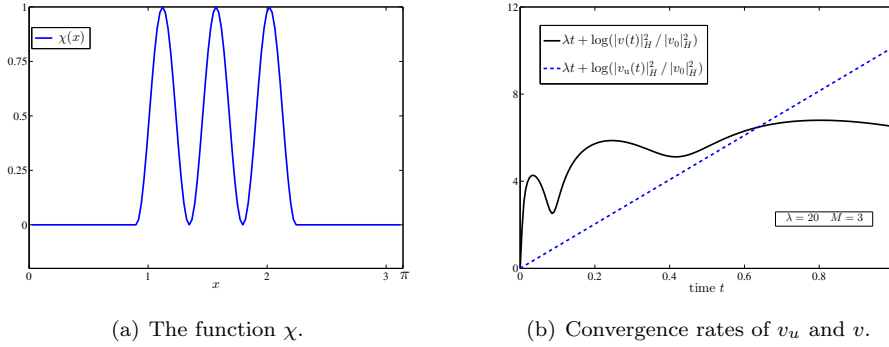


FIG. 6. Case  $n = 1$ . The first three controls can stabilize the heat system.

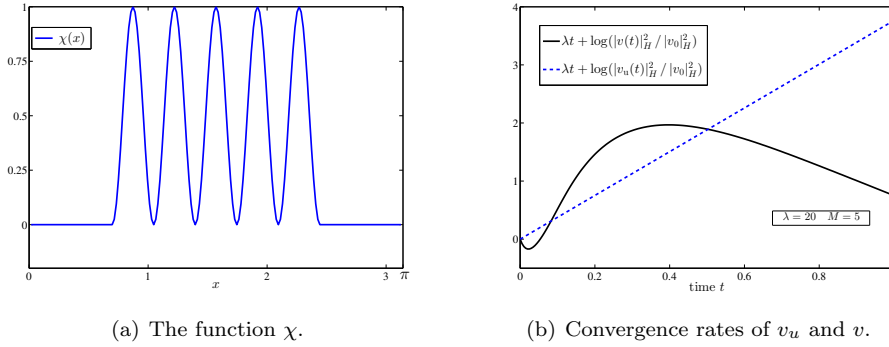


FIG. 7. Case  $n = 2$ . The first five controls can stabilize the heat system.

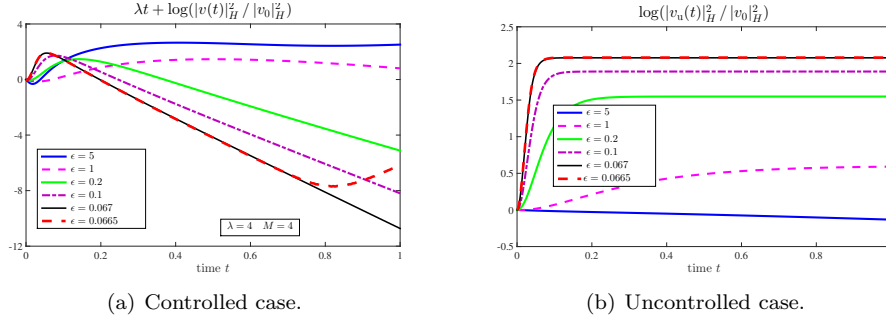
estimate depending on the norm  $|\hat{u}|_{\mathcal{W}}$  is not known also in this case; for the Oseen–Stokes system estimates are known, but depending on  $\hat{u}$  (cf. the discussion and given references in section 1). We will set

$$(6.4) \quad \hat{u}(t, x) = \epsilon^{-1} \sin(5x), \quad \text{and} \quad v_0(x) = \sin(\pi x)$$

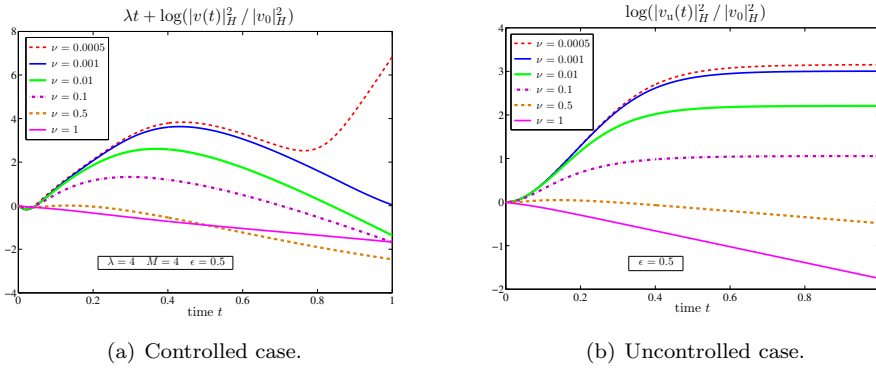
where  $\epsilon$  is a constant that we will use to change the norm  $|\hat{u}|_{\mathcal{W}}$  of  $\hat{u}$ .

**6.3.1. Changing the norm of the reference trajectory.** Here we take  $\lambda = 4$ ,  $M = 4$ , and  $\nu = \frac{1}{10}$ . In figure 8 we see that the uncontrolled system becomes more instable as  $\epsilon$  decreases, that is, as  $|\hat{u}|_{\mathcal{W}}$  increases. We can also see that the four controls work up to  $\epsilon = 0.067$ , but not for  $\epsilon = 0.0665$ . This could mean that either the number of controls is not enough anymore or that our discretization is not fine enough (notice that for smaller  $\epsilon$  the magnitudes  $|\partial_x \hat{u}(t, x)|_{\mathbb{R}}$  become bigger, we will come back to this issue hereafter in section 7.3). Notice that in all the cases we have  $M_{\text{ref}} \geq \sqrt{44} \approx 6.63$  and  $M_{\text{exp}} \geq e^{\sqrt{44}} \approx 759.95$ . In particular,  $M_{\text{ref}}^{(\epsilon=0.067)} \approx 149.49$  is already big compared to  $M$ .

**6.3.2. Changing the viscosity.** Here we take  $\lambda = 4$ ,  $M = 4$  and  $\hat{u}(t, x) = 2\sin(5x)$ . In figure 9 we see that the uncontrolled system becomes more instable as  $\nu$  decreases. We can also see that the four controls work up to  $\nu = 0.001$ , but not for  $\nu = 0.0005$ . Again, either the number of controls is not enough anymore or our

FIG. 8. *Instability increases as  $\epsilon$  decreases.*

discretization is not fine enough (notice that for smaller  $\nu$  the magnitudes  $|\partial_x \hat{u}(t, x)|_{\mathbb{R}}$  become bigger when compared to  $\nu$ ). Notice that in all the cases we have  $M_{\text{exp}} \geq e^{\sqrt{8}} \approx 16.92$ ,  $M_{\text{ref}}^{(\nu=1)} = \sqrt{8} \approx 2.83 < 4$  and  $M_{\text{ref}}^{(\nu \leq 0.5)} \geq \sqrt{22} \approx 4.69 > 4$ . In particular, notice that for  $\nu \in \{0.5, 1\}$  we have  $M \approx M_{\text{ref}}$  and the corresponding plots in figure 9(a) remain below  $0.423 \approx \log(1 + e^{\frac{1}{2}})$  which suits (3.4) better than (3.16).

FIG. 9. *Instability increases as  $\nu$  decreases.*

**6.3.3. Changing the desired decreasing rate.** Here we take  $M = 4$ ,  $\hat{u} = \sin(5x)$ , and  $\nu = \frac{1}{10}$ . In figure 10 we see that with the four controls we can get at least rate of convergence  $\lambda = 17$ . We also see that  $|v(t)|_H^2 \leq e^{C_\lambda} e^{-\lambda t} |v_0|_H^2$ , where  $C_\lambda$  is the maximum of the corresponding curves. Since in all cases we have  $\lambda > C_\lambda$  we see that the controller is effective at time  $t = 1$ , that is, the norm has been squeezed at time  $t = 1$ . On the other hand, for example for  $\lambda = 15$ , up to time  $t = 0.2$  we can guarantee that  $|v(t)|_H^2 \leq e^9 e^{-15t} |v_0|_H^2$ , in particular  $|v(0.2)|_H^2 \leq e^6 |v_0|_H^2$ , that is we cannot guarantee that the norm has been squeezed, but at time  $t = 0.8$  we find  $|v(0.8)|_H^2 \leq e^{11} e^{-15 \cdot \frac{8}{10}} |v_0|_H^2 = e^{-1} |v_0|_H^2$ , and we see that the norm is already squeezed.

**6.3.4. Changing the number of controls.** Here we take  $\hat{u} = \sin(5x)$ , and  $\nu = \frac{1}{10}$ , and  $\lambda = 4$ . In figure 11 we see that  $|v(t)|_H^2 \leq e^{C_M^\nu} e^{-\lambda t} |v_0|_H^2$ , and  $|\eta(t)|_H^2 \leq e^{C_M^\eta} e^{-\lambda t} |v_0|_H^2$ , with  $C_M^\nu$  and  $C_M^\eta$  decreasing as  $M$  increases. Figure 12(a) could explain the cusp in the control plot in figure 11; we guess that one control is not enough, because the cost function  $t \mapsto (Q(t)v(t), v(t))_H$  must be a strictly decreasing function  $(Q(s)v(s), v(s))_H = (Q(t)v(t), v(t))_H + \int_s^t e^{\lambda\tau} (|\partial_x v(\tau)|_H^2 + |\eta(\tau)|_H^2) d\tau$

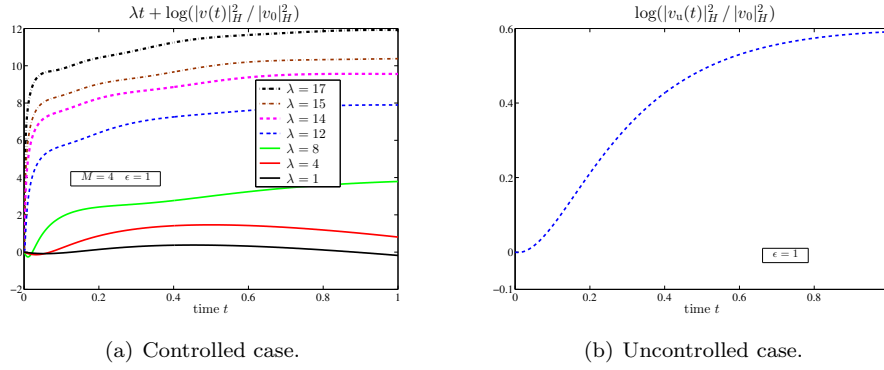


FIG. 10. Behavior as the desired exponential rate changes.

$s \leq t$  (cf. section 3.4). In figure 12(b) we can see that two controls can stabilize the system, and that the cost decreases as  $M$  increases.

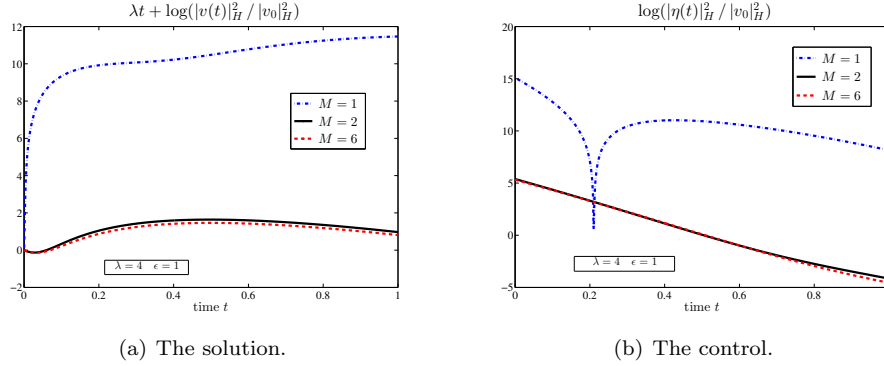
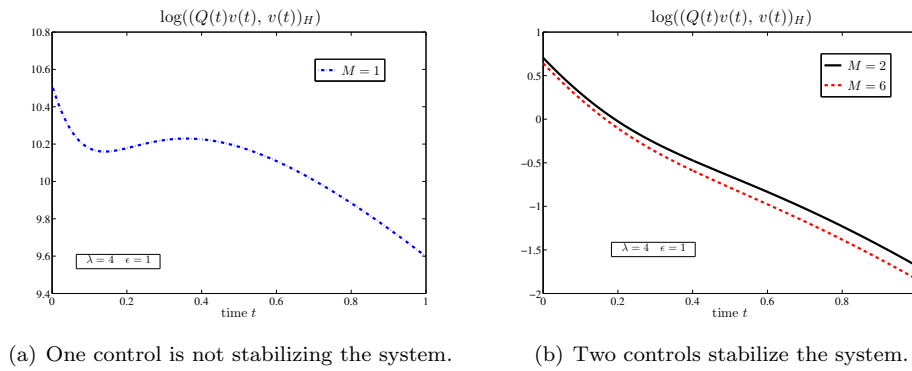
FIG. 11. Behavior as  $M$  increases.

FIG. 12. The cost decreases as the number of controls increase.

**7. Numerical examples: the Burgers system.** It remains to confirm that the feedback control stabilizes the system (1.1)–(1.2) to a given reference trajectory  $\hat{u}$ ,

provided that  $|u_0 - \hat{u}(0)|_H^2$  is “small”. We recall that  $\hat{u}$  solves (1.1) with  $\zeta = 0$  and  $\hat{u}(0) = \hat{u}_0$ . Below, we denote  $d := u - \hat{u}$  and  $d_u := u_u - \hat{u}$ , where  $u_u$  solves system (1.1)–(1.2) with  $\zeta = 0$ , and  $u$  solves system (1.1)–(1.2) with the feedback control  $\zeta$ , as in (3.22), computed to stabilize the system (3.1) to zero.

### 7.1. Local nature of the results and nonlinear nature of the equation.

As in section 6.1, we set  $\nu = \frac{1}{10}$ ,  $\chi$  as defined in (6.1),  $\lambda = 2$  and the trajectory  $\hat{u} = C_{\text{nr}}(\sin(-t)\sin(8x) - \cos(3t)\sin(8x))$  from the family (6.2). Again we set  $M = 4 < M_{\text{ref}} = \sqrt{120}$ . Next, we consider the family of initial conditions  $u_0 = u_0^\delta := d_0^\delta + \hat{u}_0$  with  $d_0 = d_0^\delta = \delta(\sin(x) - \sin(6x))$ , and  $\delta \in \mathbb{R} \setminus \{0\}$ .

In figure 13 we can see that the feedback control is able to stabilize, with the desired rate, the nonlinear system (1.1)–(1.2) to the trajectory  $\hat{u}$ , provided that  $d_0$  is small enough. We can see that, for  $|\delta|_{\mathbb{R}} > 1$ , the stabilization rate is not guaranteed; while for  $|\delta| \leq 1$  it holds. For example, for  $|\delta|_{\mathbb{R}} \leq 1$  we can see that the local maxima of the plotted curves seem either to converge to a real number or to decrease, while for  $|\delta|_{\mathbb{R}} > 1$  those local maxima seem to go to infinity. Notice that the radius 1 here is suitable for this example, for other settings the stabilization may hold only for smaller  $|\delta|_{\mathbb{R}}$ .

In figure 14 we can see that the uncontrolled systems does not go exponentially to  $\hat{u}$  (at least not with the rate  $\lambda = 2$ ); here we have plotted the curves corresponding to some of those values of  $\delta$  in figure 13 (for the other the behavior is similar).

We can also see the nonlinear nature of the equations, because changing the sign of the initial condition leads to different curves.

*Remark 7.1.* The results correspond to simulations in which we have taken a fictitious external force  $h_f$  (i.e., an approximation of  $h$ ) that makes  $\hat{u}$  a solution of the discrete system (cf. section 5.5).

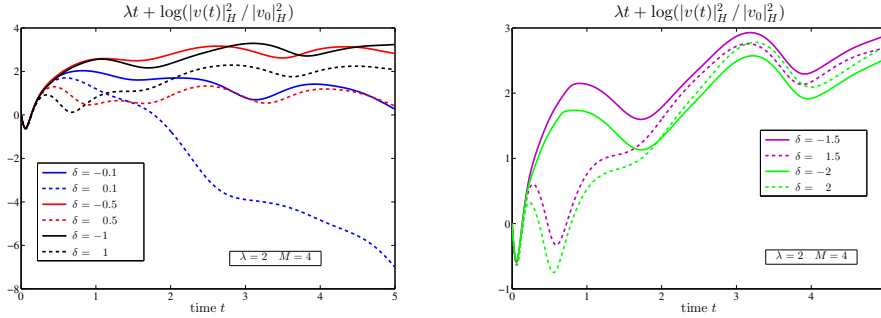
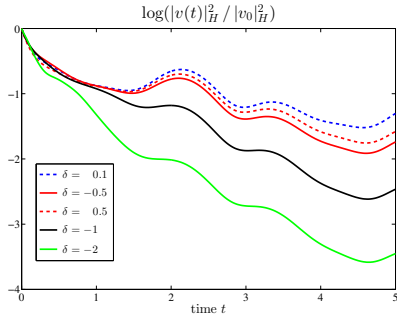
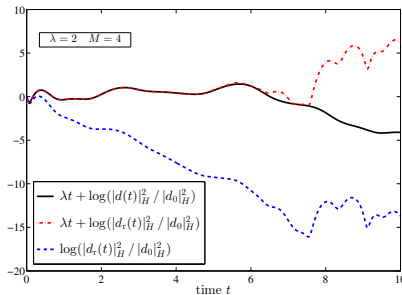


FIG. 13. Convergence rate to  $\hat{u}$  holds locally.

**7.2. Real versus fictitious external force behavior.** Here we are in the same setting as in section 7.1. But, now we fix  $\delta = 1$ , and consider  $\hat{u}$  in the longer time interval  $t \in [0, 10]$ . We compare the numerical results in the case when we take the real external force  $h = -\partial_t \hat{u} - \hat{u} \partial_x \hat{u} + \nu \partial_{xx} \hat{u}$  with those in the case when we take the fictitious external force  $h_f$  (cf. Remark 7.1). We denote  $d = \bar{u} - \hat{u}$  and  $d_r = \bar{u}_r - \hat{u}$ , where  $\bar{u}_r$  solves (5.14) (that is, with the real external force  $\bar{h}$ ), and  $\bar{u}$  solves (5.14) with  $\bar{h}_f$  in the place of  $\bar{h}$ . In figure 15, we confirm the rate of convergence of  $\bar{u}$  to  $\hat{u}$  in the entire time interval, while for  $d_r$  the rate is confirmed until time  $t = 6$ ; after time  $t = 6$  we see that  $d_r$  remains bounded, this just means that the magnitude of  $\bar{u}_r - \hat{u}$

FIG. 14. *Uncontrolled case.*

has reached that of the discretization error of our solver, and consequently we cannot expect the magnitude of  $\bar{u}_r - \hat{u}$  to decrease more.

FIG. 15. *Fictitious versus real external force.*

**7.3. On the discretization error.** Here we are in the same setting as in section 7.1 with  $\delta = 1$ . We observe that, following the scheme (5.17), with the exact external force  $h$  (in the place of  $h_f$ ), the discrete solution  $\hat{u}_S$  for (5.15) will be close to  $\hat{u}$ . Moreover,  $\hat{u}_S$  converges to  $\hat{u}$  as  $(k, h) \rightarrow (0, 0)$ .

The main goal of the presented simulations in the previous sections is to show that an estimate like (3.3) should hold, in general, instead of (3.15), and it is not our intention to compare our algorithm/discretization to solve the Burgers and Oseen-Burgers systems with existing ones. That is why we have not performed a rigorous numerical analysis concerning the convergence of the scheme. Though, we would like to say that from numerical experiments we have performed we expect linear convergence. We present some results that indeed suggest that the error  $\hat{u}_S - \hat{u}$  is proportional to  $h + k$ .

Let  $\hat{u} = \hat{u}^{(i,j)}$  be as in (6.2), with  $(i, j) = (8, 8)$ . In figure 17 the error  $\hat{u}_S - \hat{u}$  is proportional to  $h + k$ ; notice that as we squeeze  $(h, k)$  by the factor  $\frac{1}{2}$ , the plots are squeezed by a factor (not bigger than)  $\frac{1}{2}$ . Also, in the examples in figure 18, corresponding to  $\hat{u} = \hat{u}^{(i,j)}$  as in (6.2) with  $(i, j) = (1, 4)$  and  $(i, j) = (3, 2)$ , the error is proportional to  $h + k$ .

We must however recall that it is well-known that in general, when solving Burgers equation with small viscosity by finite elements, spurious oscillations may appear in the numerical solution if the convection part is dominant. They can be reduced



by a reduction of the mesh size or by stabilization of the numerical scheme (e.g., see [11,16,24]). In our simulations the reference trajectory is smooth with moderately bounded gradient. Thus we can choose moderately small mesh parameters, such that no spurious oscillations appear. When we apply our scheme to a different example where the gradient  $\partial_x u$  of the exact solution reaches big magnitudes, then we will observe spurious oscillations on a coarse mesh; the oscillations vanish by decreasing the mesh parameters, see figure 19. In 19(a) we have taken the same space step and time step and viscosity as in [1, figure 3(a)]. We see we get the same behavior near  $(t, x) = (1, 1)$ , where the gradient of the exact solution has a big magnitude  $|\partial_x u(1, 1)|_{\mathbb{R}}$ , see also [16, section 5.6.5].

For references on numerical methods for feedback control and stabilization of the Burgers equation we refer the reader to [3, 22, 31–34, 44].

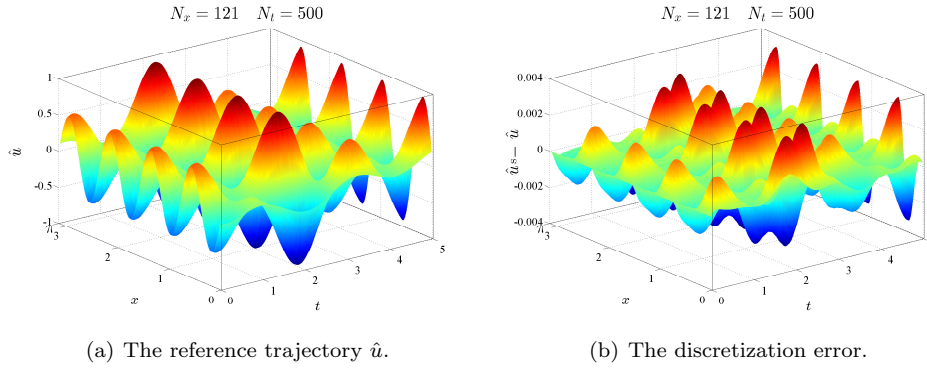


FIG. 16. The difference between the discrete  $\hat{u}_S$  and exact  $\hat{u}$  solutions.  $(h, k) = (\frac{\pi}{121}, \frac{5}{500})$ .

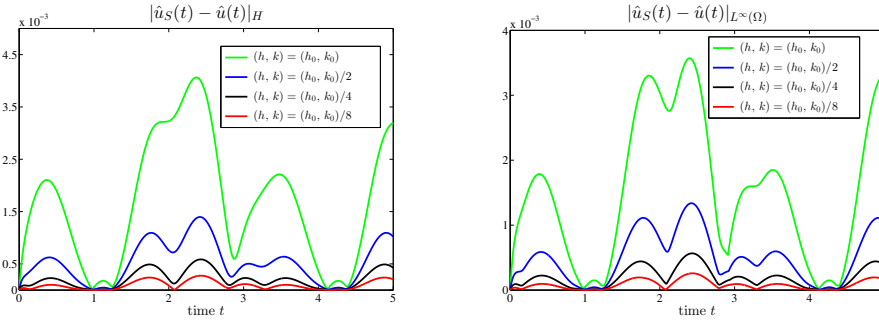
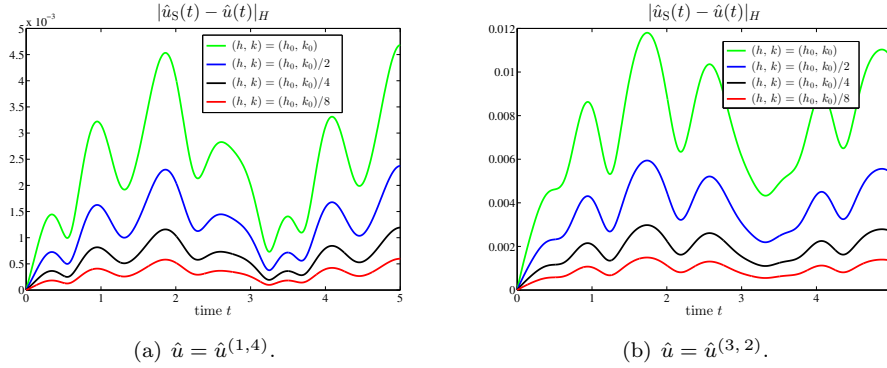
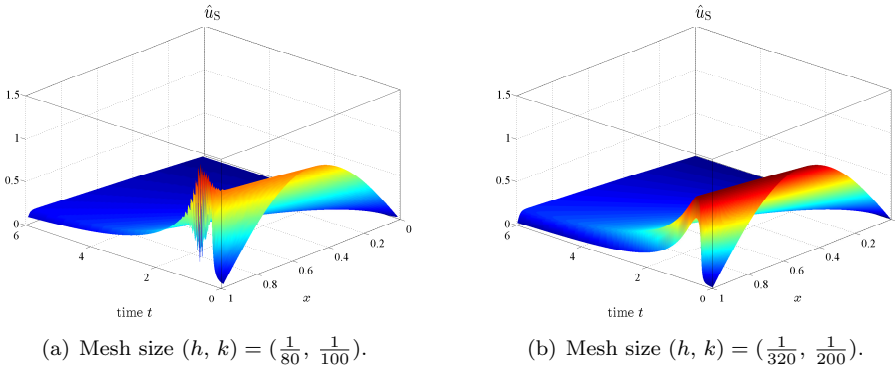


FIG. 17. The discretization error.  $\hat{u} = \hat{u}^{(8,8)}$  (cf. (6.2)),  $(h_0, k_0) = (\frac{1}{121}, \frac{1}{100})$ .

**8. Final remarks.** We have presented some estimates on the number of internal controls  $M$  we need to exponentially stabilize the Burgers system to a given reference trajectory  $\hat{u} = \hat{u}(t, x)$ . In the case we take  $\chi = 1_{\Omega}$ , in particular there is no constraint on the support of the control, we can derive a better estimate comparing with the general case (cf. sections 3.1 and 3.2), and we have presented the results of some numerical simulations that suggest that an estimate like that obtained in the case  $\chi = 1_{\Omega}$  might hold also in the general case.

FIG. 18. *The discretization error.*  $(h_0, k_0) = (\frac{1}{121}, \frac{1}{100})$ .FIG. 19. *Spurious oscillations occur on coarser meshes.*

Throughout the paper we consider controls given in the form  $\sum_{i=1}^M \eta_i(t) \chi_{\mathbb{E}_0^O} \underline{s}_i(x)$ . Usually, the controls at our disposal depend on each specific application, of course we can always consider another family of controls  $\Psi = \{\psi_i \mid i \in \mathbb{N}_0\}$ , and perform the simulations for controls like  $\sum_{i=1}^M \kappa_i(t) \psi_i(x)$ , and perhaps also derive the corresponding estimates on  $M$  following the procedure in sections 3.1 and 3.2.

We have focused on the viscous 1D Burgers system. However, we are convinced that the challenge to find an estimate for  $M$ , through a condition like (3.8) (preferable to a condition like (3.19)), will present analogous difficulties for the cases of 2D and 3D Burgers and Navier–Stokes systems, and also for a wide class of parabolic systems.

Our results do not apply to the case of nonviscous Burgers equation (i.e., to the case we take  $\nu = 0$  in (1.1)), that is a completely different problem. We do not even know if a finite number  $M$  of controls is enough to stabilize the system (in a general situation the number  $M$  of needed controls will go to  $+\infty$  as  $\nu$  goes to 0).

The existence of a finite-dimensional feedback control supported on a small subset of the boundary and stabilizing the system to reference nonstationary solution, is work still going on (see [40, 41] for some work on this direction). Also in this case, it will be interesting to have an estimate on the dimension of the controller.

The value  $\nu = \frac{1}{10}$  we use in most of the simulations is (perhaps) too big for many applications. Of course we can take smaller  $\nu$  but, in that case we may need to take also a finer mesh in order to guarantee that the stabilization observed for

the discretized system in the numerical simulations, will also hold for the continuous system. Notice that, when the numerical solution for system (1.1) goes to  $\hat{u}$  as time increases, we can extrapolate that also the evaluations  $u(t, ih)$ ,  $i \in \{1, 2, \dots, N_x - 1\}$ , of the continuous solution at the spatial mesh points will go to  $\hat{u}(t, ih)$  as time increases. Recall that if  $|u(t) - \hat{u}(t)|_H$  goes to 0 as  $t$  increases, then also  $|u(t) - \hat{u}(t)|_V$  does (provided that  $\hat{u} \in \{v \in \mathcal{W} \mid \sup_{\tau \geq 0} |\partial_x \hat{u}|_{L^2((\tau, \tau+1), L^2(\Omega, \mathbb{R}))} < +\infty\}$ , due to the smoothing property of the system (3.1), see [7, Lemma 2.1]). However, the fact that  $|u(t, ih) - \hat{u}(t, ih)|_{\mathbb{R}}$  goes to 0, for all  $i \in \{1, 2, \dots, N_x - 1\}$ , as time increases, is in general not enough to conclude that  $u$  goes to  $\hat{u}$ . Indeed, from [25, Theorem 4.2] (for the case of the Navier–Stokes system in a two-dimensional Torus) we can derive that to conclude that  $u$  goes to  $\hat{u}$ , the space step  $h$  should be taken proportional to  $\frac{\nu^2}{1-2\log(\nu)}$  (for small  $\nu$ ); and supposing that a similar estimate holds for the 1D Burgers system, it would follow that the number  $N_x$  of space points (determining nodes) should be proportional to  $\frac{1-2\log(\nu)}{\nu^2}$ . Notice that the computational effort and computational time will increase with  $N_x$ . We refer also to [26] and [19, chapter III, section 2], and references therein, concerning the estimates on the number of determining nodes.

The mathematical theory concerning stabilization to time-dependent trajectories (cf. [7]) is not so developed as for stabilization to a stationary state (cf. [2, 4–6, 8, 38, 39]). However, since these problems arise in applications, methods to solve these problems numerically have already been developed (see, e.g., [20, 28, 29] and references therein), notice that in this setting “trajectory” will often mean a suitable evolutionary discrete process  $u_0 \in Z$ ,  $u_{i+1} = S(u_i) \in Z$ ,  $i \in \mathbb{N}$ , where  $Z$  is a Hilbert space. Other approaches can be found, for example, in [30] (in particular, see section 4 concerning trajectory tracking) and in [21] (in particular, see section 7.1 concerning linear feedback control of Navier–Stokes flows).

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