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Asymptotic Control for a Class of Piecewise Deterministic Markov Processes Associated to Temperate Viruses

Dan Goreac [*] [†]

Abstract

We aim at characterizing the asymptotic behavior of value functions in the control of piecewise deterministic Markov processes (PDMP) of switch type under nonexpansive assumptions. For a particular class of processes inspired by temperate viruses, we show that uniform limits of discounted problems as the discount decreases to zero and time-averaged problems as the time horizon increases to infinity exist and coincide. The arguments allow the limit value to depend on initial configuration of the system and do not require dissipative properties on the dynamics. The approach strongly relies on viscosity techniques, linear programming arguments and coupling via random measures associated to PDMP. As an intermediate step in our approach, we present the approximation of discounted value functions when using piecewise constant (in time) open-loop policies.

AMS Classification: 49L25, 60J25, 93E20, 92C42

Keywords: Piecewise Deterministic Markov Process; Switch Process; Stochastic Control; Limit Value Function; Stochastic Gene Networks; Phage λ

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1 Introduction

We focus on the study of some asymptotic properties in the control of a particular family of piecewise deterministic Markov processes (abbreviated PDMP), non diffusive, jump processes introduced in the seminal paper [24]. Namely, we are concerned with the existence of a limit of the value functions minimizing the Cesàro-type averages of some cost functional as the time increases to infinity for controlled switch processes. The main theoretical contribution of the paper is that the arguments in our proofs are entirely independent on dissipativity properties of the PDMP and they apply under mild nonexpansivity assumptions. Concerning the potential applications, our systems are derived from the theory of stochastic gene networks (and, in particular, genetic applets modelling temperate viruses). Readers wishing to get acquainted to biological or mathematical aspects in these models are referred to [16], [37], [23], [22], [33]).

Switch processes can be described by a couple $(\gamma_\cdot^{\gamma_0,x_0,u}, X_\cdot^{\gamma_0,x_0,u})$, where the first component is a pure jump process called mode and taking its values in some finite set $\mathbb{M}$. The couple process is governed by a jump rate and a transition measure, both depending on the current state of the system. Between consecutive jumps, $X_\cdot^{\gamma_0,x_0,u}$ evolves according to some mode-dependent flow. Finally, these characteristics (rate, measure, flow) depend on an external control parameter u.

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Precise assumptions and construction make the object of Section 2. In connection to these jump systems, we consider the Abel-type (resp. Cesàro-type) average

$$v^{\delta}(\gamma_0, x_0) := \inf_u \delta \mathbb{E}\left[\int_0^{\infty} e^{-\delta t} h\left(\gamma_t^{\gamma_0,x_0,u}, X_t^{\gamma_0,x_0,u}, u_t\right) dt\right],$$
$$V_T(\gamma_0, x_0) := \inf_u \tfrac{1}{T}\mathbb{E}\left[\int_0^{T} h\left(\gamma_t^{\gamma_0,x_0,u}, X_t^{\gamma_0,x_0,u}, u_t\right) dt\right],$$

and investigate the existence of limits as the discount parameter $\delta \to 0$, respectively the time horizon $T \to \infty$.

In the context of sequences of real numbers, the first result connecting asymptotic behavior of Abel and Cesàro means goes back to Hardy and Littlewood in [36]. Their result has known several generalizations: to uncontrolled deterministic dynamics in [29, XIII.5], to controlled deterministic systems in [2], [3], etc.

Ergodic behavior of systems and asymptotic of Cesàro-type averages have made the object of several papers dealing with either deterministic or stochastic control systems. The partial differential system approach originating in [42] relies on coercitivity of the associated Hamiltonian (see also [5] for explicit criteria). Although the method generalizes to deterministic (resp. Brownian) control systems in [3] (resp. [4]), the main drawback resides in the fact that, due to the ergodic setting, the limit is independent of the initial condition of the control system. Another approach to the asymptotic behavior relies on estimations on trajectories available under controllability and dissipativity assumptions. The reader is referred to [5], [10] for the deterministic setting or [8], [11], [15], [46] for Brownian systems. Although the method is different, it presents the same drawback as the PDE one: it fails to give general limit value functions that depend on the initial data.

In the context of piecewise deterministic Markov processes, the infinite-horizon optimal control literature is quite extensive ([25], [48], [27], [1], [30], etc.). To our best knowledge, average control problems have first been considered in an impulsive control framework in [17] and [31]. The first papers dealing with long time average costs in the framework of continuous control policies are [19] and [18] (see also [21]). The problem studied in the latter papers is somewhat different, since it concerns $\inf_u \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\int_0^T h\left(\gamma_t^{\gamma_0,x_0,u}, X_t^{\gamma_0,x_0,u}, u_t\right) dt\right]$, thus leading to an inf/sup formulation, while, in our case, we deal with a sup/inf formulation. Moreover, the methods employed are substantially different. Our work should be regarded as a complement to the studies developed in the cited papers.

A nonexpansivity condition has been employed in [44] in connection to deterministic control systems allowing to obtain the existence of a general (uniform) limit value function. This method has been (partially) extended to Brownian control systems in [14]. In both these papers, convenient estimates on the trajectories in finite horizon allow to prove the uniform continuity of Cesàro averages V_T and an intuition coming from repeated games theory (inspired by [45]) gives the candidate for the limit value function. If the convergence to this limit value function is uniform, the results of [43] for deterministic systems yield the equivalence between Abel and Cesàro long-time averages. This latter assertion is still valid for controlled Brownian diffusions (see [14, Theorems 10 and 13]) and (to some extent) for piecewise deterministic Markov processes (see [32, Theorem 4.1]).

In the present paper, we generalize the results of [44] and [14] to the framework of switch piecewise deterministic Markov processes. The methods are based on viscosity solutions arguments. We deal with two specific problems. The key point is, as for Brownian systems, a uniform continuity of average value functions with respect to the average parameter (δ or T). However, the approach in [14] benefits from dynamic programming principles, which, within the framework of PDMP, are easier obtained for Abel means (discounted functions v^{δ}). This is why, results like [14, Proposition 7 and Theorem 8] are not directly applicable and we cannot make use of the already mentioned intuition on repeated games. To overcome this problem, we proceed as follows : if the system admits an invariant compact set, we prove the uniform continuity of $\left(v^{\delta}\right)_{\delta>0}$ and use the results

in [32, Theorem 4.1] to show (in Theorem 4) that this family admits a unique adherent point with respect to the topology of continuous functions and, hence, it converges uniformly. This implies the existence of $\lim_{T\to\infty} V_T(\gamma_0, x_0)$ and the limit is uniform with respect to the initial data.

The second problem is proving the uniform continuity of $\left(v^\delta\right)_{\delta>0}$ under explicit noxexpansivity conditions. In the Brownian setting, this follows from estimates on the trajectories and a natural coupling with respect to the same Brownian motion in [14, Lemma 3]. For switch PDMP, we obtain similar results (in a convenient setting) by using some reference random measure generated by the process. Although the second marginal of this coupling might not come from a controlled PDMP, it is shown to belong to a convenient class of measures by using linear programming techniques (developed in [33], [34] and inspired by Krylov [41]).

Let us now explain how the paper is organized. We begin with fixing some notations employed throughout the paper (in Subsection 2.1). We proceed by recalling the construction of controlled PDMP of switch type and present the main assumptions on the characteristics in Subsection 2.2. In Subsection 2.3, we introduce the concept of invariance with respect to PDMP dynamics, the value functions (Cesàro and Abel averages) and the occupation measures associated to controlled dynamics (taken from [34]). The main contributions of our paper are stated in Section 3. We begin with introducing a very general, yet abstract nonexpansivity condition in Subsection 3.1, Condition 2. The first main result of the paper (Theorem 4) is given in Subsection 3.2. This result states that whenever the nonexpansivity Condition 2 is satisfied, there exists a unique limit value function $\lim_{T\to\infty} V_T = \lim_{\delta\to 0} v^\delta$ independent of the average considered (Abel/ Cesàro). In subsection 3.3, we give explicit nonexpansive conditions on the dynamics and the cost functional implying Condition 2. The second main result of the paper (Subsection 3.4, Theorem 7) provides an explicit construction of (pseudo-)couplings.

We proceed with a biological framework justifying our models in Section 4. We present the foundations of Hasty's model for Phage λ inspired by [37] in Subsection 4.1. In order to give a hint to our readers less familiarized with mathematical models in systems biology, we briefly explain how a PDMP can be associated to Hasty's genetic applet in Subsection 4.2. Finally, the aim of Subsection 4.3 is to give an extensive choice of characteristics satisfying all the assumptions of the main Theorem 7.

Section 5 gives the proof of the first main result (Theorem 4). First, we prove that Condition 2 implies the equicontinuity of the family of Abel-average value functions $\left(v^\delta\right)_{\delta>0}$. Next, we recall the results in [32] on Abel-type theorems to conclude. The results of this section work in all the generality of [48] (see also [33], [34]).

The proof of the second main result (Theorem 7) is given in Section 6. The proof is based on constructing explicit couplings satisfying Condition 2 and it relies on four steps. The first step is showing that the value functions v^δ can be suitably approximated by using piecewise constant open-loop policies. This is done in Subsection 6.2. The proof combines the approach in [41] with the dynamic programming principles in [48]. We think that neither the result, nor the method are surprising but, for reader's sake, we have provided the key elements in the Appendix. The second step is to interpret the system as a stochastic differential equation (SDE) with respect to some random measure (Subsection 6.3). The third step (Subsection 6.4) is to embed the solutions of these SDE in a space of measures satisfying a suitable linear constraint via the linear programming approach. To conclude, the fourth step (given in Subsection 6.5) provides a constructive (pseudo-) coupling using SDE estimates.

2 Preliminaries

2.1 Notations

Throughout the paper, we will use the following notations.

Unless stated otherwise, the Euclidian spaces $\mathbb{R}^N$, for some $N \geq 1$ are endowed with the usual Euclidian inner product $\langle x, y \rangle = \left(y^t\right) x$, where y^t stands for the transposed, row vector and with the associated norm $|x| = \sqrt{\langle x, x \rangle}$, for all $x, y \in \mathbb{R}^N$.

For every $r > 0$, the set $\overline{B}(0, r)$ denotes the closed r-radius ball of $\mathbb{R}^N$.

The set $\mathbb{M}$ will denote some finite set. Whenever needed, the set $\mathbb{M}$ is endowed with the discrete topology.

Unless stated otherwise, $\mathbb{U}$ is a compact metric space referred to as the control space. We let $\mathcal{A}_0(\mathbb{U})$ denote the space of $\mathbb{U}$-valued Borel measurable functions defined on $\mathbb{M} \times \mathbb{R}^N \times \mathbb{R}_+$. The sequence $u = (u_1, u_2, ...)$, where $u_k \in \mathcal{A}_0(\mathbb{U})$, for all $k \geq 1$ is said to be an admissible control. The class of such sequences is denoted by $\mathcal{A}_{ad}(\mathbb{U})$ (or simply $\mathcal{A}_{ad}$ whenever no confusion is at risk concerning $\mathbb{U}$). We introduce, for every $n \geq 1$, the spaces of piecewise constant policies

$$\mathcal{A}_0^n(\mathbb{U}) = \left\{ u \in \mathcal{A}_0(\mathbb{U}) : \ u(\gamma, x, t) = u^0(\gamma, x) 1_{\{0\}}(t) + \sum_{k \geq 0} u^k(\gamma, x) 1_{\left(\frac{k}{n}, \frac{k+1}{n}\right]}(t) \right\},$$
$$\mathcal{A}_{ad}^n(\mathbb{U}) = \left\{ (u_m)_{m \geq 1} \in \mathcal{A}_{ad}(\mathbb{U}) : \ u_m \in \mathcal{A}_0^n(\mathbb{U}), \ m \geq 1 \right\}.$$

As before, we may drop the dependency on $\mathbb{U}$.

For every bounded function $\varphi : \mathbb{M} \times \mathbb{R}^N \times \mathbb{U} \longrightarrow \mathbb{R}^k$, for some $N, k \geq 1$ which is Lipschitz-continuous with respect to the $\mathbb{R}^N$ component, we let

$$\varphi_{\max} = \sup_{(\gamma, x, u) \in \mathbb{M} \times \mathbb{R}^N \times \mathbb{U}} |\varphi_\gamma(x, u)|, \ Lip(\varphi) = \sup_{\substack{(\gamma, x, y, u) \in \mathbb{M} \times \mathbb{R}^{2N} \times \mathbb{U} \\ x \neq y}}, \frac{|\varphi_\gamma(x, u) - \varphi_\gamma(y, u)|}{|x - y|},$$
$$|\varphi|_1 = \varphi_{\max} + Lip(\varphi).$$

This is the Lipschitz norm of φ.

Whenever $\mathbb{K} \subset \mathbb{R}^N$ is a closed set, we denote by $C(\mathbb{M} \times \mathbb{K}; \mathbb{R})$ the set of continuous real-valued functions defined on $\mathbb{M} \times \mathbb{K}$. The set $BUC(\mathbb{M} \times \mathbb{K}; \mathbb{R})$ stands for the family of real-valued bounded, uniformly continuous functions defined on $\mathbb{M} \times \mathbb{K}$.

The real-valued function $\varphi : \mathbb{R}^N \longrightarrow \mathbb{R}$ is said to be of class C_b^1 if it has continuous, bounded, first-order derivatives. The gradient of such functions is denoted by $\partial_x \varphi$.

The real-valued function $\varphi : \mathbb{M} \times \mathbb{R}^N \longrightarrow \mathbb{R}$ is said to be of class C_b^1 if $\varphi(\gamma, \cdot)$ is of class C_b^1, for all $\gamma \in \mathbb{M}$.

Given a generic metric space $\mathbb{A}$, we let $\mathcal{B}(\mathbb{A})$ denote the Borel subsets of $\mathbb{A}$. We also let $\mathcal{P}(\mathbb{A})$ denote the family of probability measures on $\mathbb{A}$. The distance W_1 is the usual Wasserstein distance on $\mathcal{P}(\mathbb{A})$ and $W_{1,Hausdorff}$ is the usual Pompeiu-Hausdorff distance between subsets of $\mathcal{P}(\mathbb{A})$ constructed with respect to W_1.

For a generic real vector space $\mathbb{A}$, we let $\overline{co}$ denote the closed convex hull operator.

2.2 Construction of Controlled Piecewise Deterministic Processes of Switch Type

Piecewise deterministic Markov processes have been introduced in [24] and extensively studied for the last thirty years in connection to various phenomena in biology (see [16], [23], [49], [22], [33]), reliability or storage modelling (in [12], [28]), finance (in [47]), communication networks ([35]), etc. The optimal control of these processes makes the object of several papers (e.g. [25], [48], [20], etc.). For reader's sake we will briefly recall the construction of these processes, the assumptions, as well as the type of controls we are going to employ throughout the paper.

The switch PDMP is constructed on a space $(\Omega, \mathcal{F}, \mathbb{P})$ allowing to consider a sequence of independent, $[0, 1]$ uniformly distributed random variables (e.g. the Hilbert cube starting from $[0, 1]$ endowed with its Lebesgue measurable sets and the Lebesgue measure for coordinate, see [26, Section 23]). We consider a compact metric space $\mathbb{U}$ referred to as the control space. The process is

given by a couple (γ, X), where γ is the discrete mode component and takes its values in some finite set $\mathbb{M}$ and the state component X takes its values in some Euclidian state space $\mathbb{R}^N$ $(N \geq 1)$. The process is governed by a characteristic triple :

- a family of bounded, uniformly continuous vector fields $f_\gamma : \mathbb{R}^N \times \mathbb{U} \longrightarrow \mathbb{R}^N$ such that $|f_\gamma(x,u) - f_\gamma(y,u)| \leq C|x-y|$, for some $C > 0$ and all $x, y \in \mathbb{R}^N$, $\gamma \in \mathbb{M}$ and all $u \in \mathbb{U}$,

- a family of bounded, uniformly continuous jump rates $\lambda_\gamma : \mathbb{R}^N \times \mathbb{U} \longrightarrow \mathbb{R}_+$ such that $|\lambda_\gamma(x,u) - \lambda_\gamma(y,u)| \leq C|x-y|$, for some $C > 0$ and all $x, y \in \mathbb{R}^N$, $\gamma \in \mathbb{M}$ and all $u \in \mathbb{U}$,

- a transition measure $Q : \mathbb{M} \times \mathbb{R}^N \times \mathbb{U} \longrightarrow \mathcal{P}(\mathbb{M} \times \mathbb{R}^N)$. We assume that this transition measure has the particular form

$$Q(\gamma, x, u, d\theta dy) = \delta_{x+g_\gamma(\theta,x,u)}(dy)\, Q^0(\gamma, u, d\theta),$$

for all $(\gamma, x, u) \in \mathbb{M} \times \mathbb{R}^N \times \mathbb{U}$. The bounded, uniformly continuous jump functions $g_\gamma : \mathbb{M} \times \mathbb{R}^N \times \mathbb{U} \longrightarrow \mathbb{R}^N$ are such that $|g_\gamma(\theta,x,u) - g_\gamma(\theta,y,u)| \leq C|x-y|$, for some $C > 0$ and all $x, y \in \mathbb{R}^N$, all $(\theta, \gamma) \in \mathbb{M}^2$ and all $u \in \mathbb{U}$. The transition measure for the mode component is given by $Q^0 : \mathbb{M} \times \mathbb{U} \longrightarrow \mathcal{P}(\mathbb{M})$. For every $A \subset \mathbb{M}$, the function $(\gamma, u) \mapsto Q^0(\gamma, u, A)$ is assumed to be measurable and, for every $(\gamma, u) \in \mathbb{M} \times \mathbb{U}$, $Q^0(\gamma, u, \{\gamma\}) = 0$.

These assumptions are needed in order to guarantee smoothness of value functions in this context (see also [33] for further comments). Of course, more general transition measures Q can be considered under the assumptions of [33], [34] and the results of Subsection 3.2 still hold true. However, the approach in Section 6 only holds true for these particular dynamics and it is the reason why we have chosen to work under these conditions. Whenever $u \in \mathcal{A}_0(\mathbb{U})$ and $(t_0, \gamma_0, x_0) \in \mathbb{R}_+ \times \mathbb{M} \times \mathbb{R}^N$, we consider the ordinary differential equation

$$\begin{cases} d\Phi_t^{t_0,x_0,u;\gamma_0} = f_{\gamma_0}\left(\Phi_t^{t_0,x_0,u;\gamma_0}, u(\gamma_0, x_0, t-t_0)\right) dt, \ t \geq t_0, \\ \Phi_{t_0}^{t_0,x_0,u;\gamma_0} = x_0. \end{cases}$$

Given some sequence $u := (u_1, u_2, ...) \subset \mathcal{A}_0(\mathbb{U})$, the first jump time T_1 has a jump rate $\lambda_{\gamma_0}\left(\Phi_t^{0,x_0,u_1;\gamma_0}, u_1(\gamma_0, x_0, t)\right)$, i.e. $\mathbb{P}(T_1 \geq t) = \exp\left(-\int_0^t \lambda_{\gamma_0}\left(\Phi_s^{0,x_0,u_1;\gamma_0}, u_1(\gamma_0, x_0, s)\right) ds\right)$. The controlled PDMP is defined by setting $(\Gamma_t^{\gamma_0,x_0,u}, X_t^{\gamma_0,x_0,u}) = \left(\gamma_0, \Phi_t^{0,x_0,u_1;\gamma_0}\right)$, if $t \in [0, T_1)$. The post-jump location (Υ_1, Y_1) has $Q\left(\gamma_0, \Phi_\tau^{0,x_0,u_1;\gamma_0}, u_1(\gamma_0, x_0, \tau), \cdot\right)$ as conditional distribution given $T_1 = \tau$. Starting from (Υ_1, Y_1) at time T_1, we select the inter-jump time $T_2 - T_1$ such that

$$\mathbb{P}(T_2 - T_1 \geq t \ / \ T_1, \Upsilon_1, Y_1) = \exp\left(-\int_{T_1}^{T_1+t} \lambda_{\Upsilon_1}\left(\Phi_s^{T_1,Y_1,u_2;\Upsilon_1}, u_2(\Upsilon_1, Y_1, s - T_1)\right) ds\right).$$

We set $(\Gamma_t^{\gamma_0,x_0,u}, X_t^{\gamma_0,x_0,u}) = \left(\Upsilon_1, \Phi_t^{T_1,Y_1,u_2;\Upsilon_1}\right)$, if $t \in [T_1, T_2)$. The post-jump location (Υ_2, Y_2) satisfies

$$\mathbb{P}((\Upsilon_2, Y_2) \in A \ / \ T_2, T_1, \Upsilon_1, Y_1) = Q\left(\Upsilon_1, \Phi_{T_2}^{T_1,Y_1,u_2;\Upsilon_1}, u_2(\Upsilon_1, Y_1, T_2 - T_1), A\right),$$

for all Borel set $A \subset \mathbb{M} \times \mathbb{R}^N$. And so on. For simplicity purposes, we set $(T_0, \Upsilon_0, Y_0) = (0, \gamma_0, x_0)$.

2.3 Definitions

Before stating the main assumptions and results of our paper we will need to recall some concepts : invariance with respect to PDMP dynamics, the Abel and Cesàro value functions and the embedding of trajectories into occupation measures.

2.3.1 Invariance

In order to get convenient estimates on the trajectories, we assume, whenever necessary, that the switch system admits some invariant compact set $\mathbb{K}$. For the applications we have in mind, this is not a drawback since, for biological systems, we deal either with discrete components or with normalized concentrations (hence not exceeding given limits). We recall the notion of invariance.

Definition 1 *The closed set $\mathbb{K}$ is said to be invariant with respect to the controlled PDMP with characteristics (f, λ, Q) if, for every $(\gamma, x) \in \mathbb{M} \times \mathbb{K}$ and every $u \in \mathcal{A}_{ad}$, one has $X_t^{\gamma,x,u} \in \mathbb{K}$, for all $t \geq 0$, $\mathbb{P}-a.s.$*

Explicit geometric conditions on the coefficients and the normal cone to $\mathbb{K}$ equivalent to the property of invariance are given in [33, Theorem 2.8]. Roughly speaking, these properties are derived from the sub/superjet formulation of the condition on $d_{\mathbb{K}}$ being a viscosity supersolution of some associated Hamilton-Jacobi integrodifferential system. This invariance condition is natural even for purely deterministic nonexpansive systems (cf. [44]). It can be avoided either by localization procedures or by imposing some relative compactness on reachable sets (or, equivalently, occupation measures). We prefer to work under this condition in order to focus on specific details of our method, rather than localization technicalities.

2.3.2 Value Functions

We investigate the asymptotic behavior of discounted value functions (also known as Abel-averages)

$$\begin{aligned} v^{\delta}(\gamma, x) &:= \inf_{u \in \mathcal{A}_{ad}} \delta \mathbb{E}\left[\int_0^{\infty} e^{-\delta t} h\left(\Gamma_t^{\gamma,x,u}, X_t^{\gamma,x,u}\right) dt\right] \\ &= \inf_{u=(u_n)_{n\geq 1} \in \mathcal{A}_{ad}} \delta \mathbb{E}\left[\sum_{n\geq 1} \int_{T_{n-1}}^{T_n} e^{-\delta t} h\left(\Gamma_t^{\gamma,x,u}, X_t^{\gamma,x,u}, u_n\left(\Gamma_{T_{n-1}}^{\gamma,x,u}, X_{T_{n-1}}^{\gamma,x,u}, t - T_{n-1}\right)\right) dt\right], \end{aligned}$$

$\gamma \in \mathbb{M}$, $x \in \mathbb{R}^N$, $\delta > 0$, as the discount parameter $\delta \to 0$ and Cesàro-average values

$$\begin{aligned} V_t(\gamma, x) &:= \inf_{u \in \mathcal{A}_{ad}} \frac{1}{t} \mathbb{E}\left[\int_0^t h\left(\Gamma_s^{\gamma,x,u}, X_s^{\gamma,x,u}, u_s\right) ds\right] \\ &= \inf_{u=(u_n)_{n\geq 1} \in \mathcal{A}_{ad}} \frac{1}{t} \mathbb{E}\left[\sum_{n\geq 1} \int_{T_{n-1}\wedge t}^{T_n \wedge t} h\left(\Gamma_s^{\gamma,x,u}, X_s^{\gamma,x,u}, u_n\left(\Gamma_{T_{n-1}}^{\gamma,x,u}, X_{T_{n-1}}^{\gamma,x,u}, s - T_{n-1}\right)\right) ds\right], \end{aligned}$$

$\gamma \in \mathbb{M}$, $x \in \mathbb{R}^N$, $t > 0$, as the time horizon $t \to \infty$. The cost function $h : \mathbb{M} \times \mathbb{R}^N \times \mathbb{U} \longrightarrow \mathbb{R}$ is assumed to be bounded, uniformly continuous and Lipschitz continuous w.r.t. the state component, uniformly in control and mode (i.e. $|h(\gamma, x, u) - h(\gamma, y, u)| \leq C|x - y|$, for some $C > 0$ and all $\gamma \in \mathbb{M}$, $x, y \in \mathbb{R}^N$ and all $u \in \mathbb{U}$).

2.3.3 The Infinitesimal Generator and Occupation Measures

We recall that, for regular functions φ (for example of class C_b^1), the generator of the control process is given by

$$\mathcal{L}^u \varphi(\gamma, x) = \langle f_\gamma(x, u), \partial_x \varphi(\gamma, x)\rangle + \lambda_\gamma(x, u) \int_{\mathbb{M}} \left(\varphi(\theta, x + g_\gamma(\theta, x, u)) - \varphi(\gamma, x)\right) Q^0(\gamma, u, d\theta),$$

for all $(\gamma, x) \in \mathbb{M} \times \mathbb{R}^N$ and any $u \in \mathcal{A}_{ad}(\mathbb{U})$. A complete description of the domain of this operator can be found, for instance, in [26, Theorem 26.14].

To any $(\gamma, x) \in \mathbb{M}\times\mathbb{R}^N$ and any $u \in \mathcal{A}_{ad}(\mathbb{U})$, we associate the discounted occupation measure

$$\mu^{\delta}_{\gamma,x,u}(A) = \delta\mathbb{E}\left[\int_0^{\infty} e^{-\delta t} 1_A\left(\Gamma_t^{\gamma,x,u}, X_t^{\gamma,x,u}, u_t\right) dt\right], \tag{1}$$

for all Borel subsets $A \subset \mathbb{M}\times\mathbb{R}^N \times \mathbb{U}$. The set of all discounted occupation measures is denoted by $\Theta_0^{\delta}(\gamma, x)$. We also define

$$\Theta^{\delta}(\gamma, x) = \left\{ \begin{array}{c} \mu \in \mathcal{P}\left(\mathbb{M}\times\mathbb{R}^N \times \mathbb{U}\right) \ s.t. \ \forall \phi : \mathbb{M} \longrightarrow C_b^1\left(\mathbb{R}^N\right), \\ \int_{\mathbb{M}\times\mathbb{R}^N\times\mathbb{U}} \left(\mathcal{L}^u\phi(\theta, y) + \delta\left(\phi(\gamma, x) - \phi(\theta, y)\right)\right) \mu(d\theta, dy, du) = 0 \end{array} \right\}. \tag{2}$$

Links between $\Theta_0^{\delta}(\gamma, x)$ and $\Theta^{\delta}(\gamma, x)$ will be given in Theorem 9. For further details, the reader is referred to [34].

3 Assumptions and Main Results

In this section, we present the main assumptions and results of our paper.

We begin with giving an abstract nonexpansivity condition under which the Abel means $\left(v^{\delta}\right)_{\delta>0}$ and the Cesàro means $(V_t)_{t>0}$ converge uniformly and to a common limit. It is a very general one and, in a less general form, it reads "a coupling μ can be found between a fixed controlled trajectory starting from x and another one starting from y such that the difference of costs evaluated on the two trajectories be controlled by the distance $|x - y|$". This is essential in proving ergodic behavior. In the uncontrolled dissipative case (see, for example [9]), the couplings are such that the distance between the law of X_t^x and the one of X_t^y decreases exponentially (is upper-bounded by some term $e^{-ct}|x - y|$). In particular, this implies that $\lim_{\delta\to 0} v^{\delta}(x)$ is constant (independent of x). Unlike the classical dissipative approach, our framework allows the limit to depend on the initial data. The first main result of the paper states that, under the nonexpansivity Condition 2, the Abel means $\left(v^{\delta}\right)_{\delta>0}$ and the Cesàro means $(V_t)_{t>0}$ converge uniformly and to a common limit.

The main drawback of this Condition 2 is that it is abstract (theoretical). In a setting inspired by gene networks, we give an explicit condition (Condition 5) on the characteristics of the PDMP implying the abstract nonexpansivity condition. The second main result of the paper allows to link the explicit Condition 5 and the abstract one given before. This is done by constructing suitable (pseudo-)couplings and the proof requires several steps.

3.1 An Abstract Nonexpansivity Condition

Throughout the section, we assume the following nonexpansivity condition

Condition 2 *For every $\delta > 0$, every $\varepsilon > 0$, every $(\gamma, x, y) \in \mathbb{M} \times \mathbb{R}^{2N}$ and every $u \in \mathcal{A}_{ad}(\mathbb{U})$, there exists $\mu \in \mathcal{P}\left(\left(\mathbb{M} \times \mathbb{R}^N \times \mathbb{U}\right)^2\right)$ such that*

i. $\mu\left(\cdot, \mathbb{M} \times \mathbb{R}^N \times \mathbb{U}\right) = \mu^{\delta}_{\gamma,x,u} \in \Theta_0^{\delta}(\gamma, x)$;
ii. $\mu\left(\mathbb{M} \times \mathbb{R}^N \times \mathbb{U}, \cdot\right) \in \Theta^{\delta}(\gamma, y)$;
iii. $\int\limits_{(\mathbb{M}\times\mathbb{R}^N\times\mathbb{U})^2} |h(\theta, z, w) - h(\theta', z', w')| \, \mu(d\theta, dz, dw, d\theta', dz', dw') \leq Lip(h)|x - y| + \varepsilon.$

Remark 3 *Whenever h only depends on the x component (but not on the mode γ, nor on the control u), one can impose*

$$W_1\left(\widetilde{\Theta}_0^{\delta}(\gamma, x), \widetilde{\Theta}^{\delta}(\gamma, y)\right) \leq |x - y|,$$

where $\widetilde{\Theta}_0$ (resp. $\widetilde{\Theta}$) denote the marginals $\mu(\mathbb{M},\cdot,\mathbb{U})$ of measures $\mu\in\Theta_0$ (resp. Θ). One can impose the slightly stronger conditions

$$W_1\left(\widetilde{\Theta}_0^\delta(\gamma,x),\widetilde{\Theta}_0^\delta(\gamma,y)\right)\leq|x-y|\ \text{ or } W_{1,Hausdorff}\left(\widetilde{\Theta}^\delta(\gamma,x),\widetilde{\Theta}^\delta(\gamma,y)\right)\leq|x-y|$$

and the notion of nonexpansivity is transparent in this setting.

3.2 First Main Result (Existence of Limit Values and Abel-Tauberian Results)

The first main result of the paper states that, under the nonexpansivity Condition 2, the Abel means $\left(v^\delta\right)_{\delta>0}$ and the Cesàro means $(V_t)_{t>0}$ converge uniformly and to a common limit.

Theorem 4 *Let us assume that there exists a compact set $\mathbb{K}\subset\mathbb{R}^N$ invariant with respect to the piecewise deterministic dynamics. Moreover, we assume Condition 2 to hold true for every $(\gamma,x,y)\in\mathbb{M}\times\mathbb{K}^2$. Then, $\left(v^\delta\right)_{\delta>0}$ admits a unique limit $v^*\in C(\mathbb{M}\times\mathbb{K};\mathbb{R})$ and $(V_t)_{t>0}$ converges to v^* uniformly.*

The proof is postponed to Section 5. In order to prove Theorem 4, we proceed as follows. First, we prove that Condition 2 implies the equicontinuity of the family of Abel-average value functions $\left(v^\delta\right)_{\delta>0}$. Next, we recall the results in [32] on Abel-type theorems to conclude.

3.3 An Explicit Nonexpansive Framework

For the remaining of the section, we assume that the control is given by a couple $(u,v)\in\mathbb{U}:=U\times V$ acting as follows : the jump rate (and the measure Q^0 giving the new mode) only depend on the mode component and is controlled by the parameter u. The component X is controlled both by u and by v and it behaves as in the general case. One has a vector field $f:\mathbb{M}\times\mathbb{R}^N\times U\times V\longrightarrow\mathbb{R}^N$, a jump rate $\lambda:\mathbb{M}\times\mathbb{R}^N\times U\times V\to\mathbb{R}_+$ given by

$$\lambda_\gamma(x,u,v)=\lambda_\gamma(u),$$

and the transition measure $Q:\mathbb{M}\times\mathbb{R}^N\times U\times V\to\mathcal{P}\left(\mathbb{M}\times\mathbb{R}^N\right)$ having the particular form

$$Q((\gamma,x),u,v,d\theta dy)=\delta_{x+g_\gamma(\theta,x,u,v)}(dy)\,Q^0(\gamma,u,d\theta),$$

where Q^0 governs the post-jump position of the mode component. In this case, the extended generator of (γ,X) has the form

(3)

$$\mathcal{L}^{u,v}\varphi(\gamma,x)=\langle f_\gamma(x,u,v),\partial_x\varphi(\gamma,x)\rangle+\lambda(\gamma,u)\int_{\mathbb{M}}\left(\varphi(\theta,x+g_\gamma(\theta,x,u,v))-\varphi(\gamma,x)\right)Q^0(\gamma,u,d\theta).$$

We will show that the results on convergence of the discounted value functions hold true under the following explicit condition on the dynamics.

Condition 5 *For every $\gamma\in\mathbb{M}$, every $u\in U$ and every $x,y\in\mathbb{R}^N$, the following holds true*

$$\sup_{v\in V}\inf_{w\in V}\max\left\{\begin{array}{c}\langle f_\gamma(x,u,v)-f_\gamma(y,u,w),x-y\rangle,\\ \sup_{\theta\in\mathbb{M}}|x+g_\gamma(\theta,x,u,v)-y-g_\gamma(\theta,y,u,w)|-|x-y|,\\ |h(\gamma,x,u,v)-h(\gamma,y,u,w)|-Lip(h)|x-y|\end{array}\right\}\leq 0.$$

Remark 6 *1. Whenever h does not depend on the control, the latter condition naturally follows from the Lipschitz-continuity of h.*

2. If, moreover, the post-jump position is given by a (state and control free) translation $x\mapsto x+g_\gamma(\theta)$, this condition is the usual deterministic nonexpansive one (i.e.

$$\sup_{v\in V}\inf_{w\in V}\langle f_\gamma(x,u,v)-f_\gamma(y,u,w),x-y\rangle\leq 0\ ).$$

This kind of jump (up to a slight modification guaranteeing that protein concentrations do not become negative) fits the general theory described in [23].

3.4 Second Main Result (Explicit Coupling)

The second main result of the paper allows to link the explicit Condition 5 and the abstract Condition 2 in the framework described in Subsection 3.3.

Theorem 7 *We assume Condition 5 to hold true. Moreover, we assume that there exists a compact set $\mathbb{K}$ invariant with respect to the PDMP governed by (f, λ, Q). Then the conclusion of Theorem 4 holds true (i.e. the family $\left(v^{\delta}\right)_{\delta>0}$ admits a unique limit $v^* \in C(\mathbb{M} \times \mathbb{K}; \mathbb{R})$ and $(V_t)_{t>0}$ converges to v^* uniformly).*

The proof of this result is postponed to Section 6. It is based on constructing explicit couplings satisfying Condition 2 and it relies on four steps. The first step is showing that the value functions v^{δ} can be suitably approximated by using piecewise constant open-loop policies. The second step is to interpret the system as a stochastic differential equation (SDE) with respect to some random measure. The third step is to embed the solutions of these SDE in a space of measures satisfying a suitable linear constraint via the linear programming approach. To conclude, the fourth step provides a constructive (pseudo-) coupling using SDE estimates.

4 Example of Application

4.1 Some Considerations on a Biological Model

We consider the model introduced in [37] to describe the regulation of gene expression. The model is derived from the promoter region of bacteriophage λ. The simplification proposed by the authors of [37] consists in considering a mutant system in which only two operator sites (known as OR2 and OR3) are present. The gene cI expresses repressor (CI), which dimerizes and binds to the DNA as a transcription factor in one of the two available sites. The site OR2 leads to enhanced transcription, while OR3 represses transcription. Using the notations in [37], we let X_1 stand for the repressor, X_2 for the dimer, D for the DNA promoter site, DX_2 for the binding to the OR2 site, DX_2^* for the binding to the OR3 site and DX_2X_2 for the binding to both sites. We also denote by P the RNA polymerase concentration and by n the number of proteins per mRNA transcript. The dimerization, binding, transcription and degradation reactions are summarized by

$$\begin{cases} 2X_1 & \overset{K_1(u,v)}{\rightleftarrows} X_2, \\ D + X_2 & \overset{K_2(u)}{\rightleftarrows} DX_2, \\ D + X_2 & \overset{K_3(u)}{\rightleftarrows} DX_2^*, \\ DX_2 + X_2 & \overset{K_4(u)}{\rightleftarrows} DX_2X_2, \\ DX_2 + P & \overset{K_t(u)}{\rightarrow} DX_2 + P + nX_1, \\ X_1 & \overset{K_d(u,v)}{\rightarrow} . \end{cases}$$

The capital letters K_i, $1 \leq i \leq 4$ for the reversible reactions correspond to couples of direct/reverse speed functions k_i, k_{-i}, while K_t and K_d only to direct speed functions k_t and k_d. Host DNA gyrase puts negative supercoils in the circular chromosome, causing A-T-rich regions to unwind and drive transcription. This is why, in the model written here, the binding speeds k_2 (to the promoter of the lysogenic cycle P_{RM}) , k_3 (to OR3), respectively k_4 and the reverse speeds as well as the transcription speed k_t are assumed to depend only on a control on the host E-coli (denoted by u). This control also acts on the prophage and, hence, we find it, together with some additional control v, in the dimerization speed k_1 and the degradation speed k_d.

4.2 The Associated Mathematical Model

Let us briefly explain how a mathematical model can be associated to the previous system.

(A) Discrete/continuous components

We distinguish between components that are discrete (only affected by jumps) and components that also have piecewise continuous dynamics. For the host (E-Coli), we can have the following modes: either unoccupied DNA ($D = 1$, $DX_2 = DX_2^* = DX_2X_2 = 0$), or binding to OR2 ($D = DX_2^* = DX_2X_2 = 0$, $DX_2 = 1$), or to OR3 ($D = DX_2^* = DX_2X_2 = 0$, $DX_2^* = 1$) or to both sites ($D = DX_2^* = DX_2 = 0$, $DX_2X_2 = 1$). It is obvious that these components are discrete and they belong to $\mathbb{M} = \left\{\gamma \in \{0,1\}^4 : \sum_{i=1}^4 \gamma_i = 1\right\}$. Every reaction involving at least one discrete component will be of jump-type. We then have the four jump reactions

$$\begin{aligned} &D + X_2 \overset{K_2(u)}{\rightleftarrows} DX_2,\ D + X_2 \overset{K_3(u)}{\rightleftarrows} DX_2^*, \\ &DX_2 + X_2 \overset{K_4(u)}{\rightleftarrows} DX_2X_2,\ DX_2 + P \overset{K_t(u)}{\rightarrow} DX_2 + P + nX_1, \end{aligned}$$

The couple repressor/dimer (X_1, X_2) has a different scale and is averaged (has a deterministic evolution) between jumps. Hence, we deal with a hybrid model on $\mathbb{M} \times \mathbb{R}^2$.

(B) Jump mechanism

Let us take an example. We assume that the current mode is unoccupied DNA ($\gamma = (1,0,0,0)$). The only jump reactions possible are

$$D + X_2 \overset{k_2(u)}{\rightarrow} DX_2 \text{ or } D + X_2 \overset{k_3(u)}{\rightarrow} DX_2^*.$$

The reaction $D + X_2 \overset{k_2(u)}{\rightarrow} DX_2$ means that free DNA will be occupied by one dimer at OR2 position. Therefore, we have a DNA and a dimer "consumed" and an OR2 binding "created". The system jumps

$$\text{from } (1,0,0,0,x_1,x_2) \text{ to } (0,1,0,0,x_1,x_2-1).$$

Of course, for a consistent mathematical model, since concentrations cannot be negative, to (x_1, x_2) we actually add $(0, -\min(1, x_2))$.

The parameter λ is chosen as the "propensity" function (i.e. the sum of all possible reaction speeds)

$$\lambda_{(1,0,0,0)}(u,v) = \lambda_{(1,0,0,0)}(u) = k_2(u) + k_3(u),$$

The probability for the reaction $D + X_2 \overset{k_2(u)}{\rightarrow} DX_2$ to take place is proportional to its reaction speed (i.e. $\frac{k_2(u)}{k_2(u)+k_3(u)}$).

To summarize, one constructs

$$\begin{cases} Q^0((1,0,0,0),(u,v),d\theta) \\ = Q^0((1,0,0,0),u,d\theta) = \frac{k_2(u)}{\lambda_{(1,0,0,0)}(u)}\delta_{(0,1,0,0)}(d\theta) + \frac{k_3(u)}{\lambda_{(1,0,0,0)}(u)}\delta_{(0,0,1,0)}(d\theta), \\ Q((1,0,0,0),(x_1,x_2),u,d\theta dy) = \delta_{(x_1,x_2)+g_\gamma(\theta,(x_1,x_2),u,v)}(dy)\, Q^0((1,0,0,0),u,d\theta), \text{ where} \\ g_{(1,0,0,0)}(\theta,(x_1,x_2),u,v) = (0, -\min(1,x_2))\, 1_{\theta\in\{(0,1,0,0),(0,0,1,0)\}}. \end{cases}$$

Remark 8 *A special part is played by the transcription reaction*

$$DX_2 + P \overset{K_t(u)}{\rightarrow} DX_2 + P + nX_1$$

which is a slow reaction. Details on a possible construction will be given in Subsection 4.3.

(C) Deterministic flow

The deterministic behavior is governed by the (three) reactions

$$2X_1 \overset{K_1(u,v)}{\rightleftarrows} X_2 \text{ and } X_1 \overset{K_d(u,v)}{\rightarrow}.$$

For example, the reaction $2X_1 \overset{k_1(u,v)}{\rightarrow} X_2$ needs two molecules of reactant X_1 and has $k_1(u,v)$ as speed and produces one X_2. Then, following the approach in [37], its contribution to the vector field is proportional to the product of reactants at the power molecules needed (i.e. x_1^2) and the speed $k_1(u,v)$. We get $-2x_1^2 k_1(u,v)$ for the reactant and $+1x_1^2 k_1(u,v)$ for the product. Adding the three contributions, we get

$$f_\gamma((x_1,x_2),u,v) = \left(-2k_1(u,v)x_1^2 + 2k_{-1}(u,v)x_2 - k_d(u,v)x_1, k_1(u,v)x_1^2 - k_{-1}(u,v)x_2\right).$$

For further details on constructions related to systems of chemical reactions in gene networks, the reader is referred to [23], [22], [33], etc.

4.3 A Toy Nondissipative Model

Let us exhibit a simple choice of coefficients in the study of phage λ. We consider $U = V = [0,1]$ (worst conditions, best conditions for chemical reactions). For the reaction speed, we take

$$k_i(u) = k_i(u+u_0), \text{ for } i \in \{\pm 2, \pm 3, \pm 4, t\} \text{ for the jump reactions determined by host,}$$
$$k_1(u,v) = \frac{1}{\alpha}uv,\ k_2(u,v) = uv, k_d(u,v) = uv \text{ to reflect a certain competition,}$$

for all $u,v \in [0,1]$. Here, $u_0 > 0$ corresponds to the slowest reaction speed, $k_i > 0$ are real constants and α is some maximal concentration level for repressor and dimer. We assume P to toggle between 0 and 1 and, as soon as P toggles to 1 a transcription burst takes place. To take into account the slow aspect of the transcription reaction (see [37]), we actually take

$$\mathbb{M} = \{(1,0,0,0,0),(0,1,0,0,0),(0,1,0,0,1),(0,0,1,0,0),(0,0,0,1,0)\}.$$

This means that binding to OR2 corresponds to two states $(0,1,0,0,0)$ (allowing transcription), $(0,1,0,0,1)$ (when transcription has just taken place and is no longer allowed). We set

$$\lambda_\gamma(u,v) = \lambda_\gamma(u) = \begin{cases} (k_2+k_3)(u+u_0), & \text{if } \gamma = (1,0,0,0,0), \\ (k_{-2}+k_4+k_t)(u+u_0), & \text{if } \gamma = (0,1,0,0,0), \\ (k_{-2}+k_4)(u+u_0), & \text{if } \gamma = (0,1,0,0,1), \\ k_{-3}(u+u_0), & \text{if } \gamma = (0,0,1,0,0), \\ k_{-4}(u+u_0), & \text{if } \gamma = (0,0,0,1,0). \end{cases},$$

$$Q^0(\gamma,u) = \begin{cases} \frac{k_2}{k_2+k_3}\delta_{(0,1,0,0,0)} + \frac{k_3}{k_2+k_3}\delta_{(0,0,1,0,0)}, & \text{if } \gamma = (1,0,0,0,0), \\ \frac{k_{-2}}{k_{-2}+k_4+k_t}\delta_{(1,0,0,0,0)} + \frac{k_4}{k_{-2}+k_4+k_t}\delta_{(0,0,0,1,0)} + \frac{k_t}{k_{-2}+k_4+k_t}\delta_{(0,1,0,0,1)}, & \text{if } \gamma = (0,1,0,0,0), \\ \frac{k_{-2}}{k_{-2}+k_4}\delta_{(1,0,0,0,0)} + \frac{k_4}{k_{-2}+k_4}\delta_{(0,0,0,1,0)}, & \text{if } \gamma = (0,1,0,0,1), \\ \delta_{(1,0,0,0,0)}, & \text{if } \gamma = (0,0,1,0,0), \\ \delta_{(0,1,0,0,0)}, & \text{if } \gamma = (0,0,0,1,0). \end{cases}$$

$$g_\gamma(\theta,(x_1,x_2)) = \begin{cases} (0, -\min(1,x_2)), & \text{if } \gamma = (1,0,0,0,0), \\ (0, \min(1,\alpha - x_2)), & \text{if } \gamma = (0,1,0,0,\gamma_5),\ \gamma_5 \in \{0,1\},\ \theta = (1,0,0,0,0), \\ (0, -\min(1,x_2)), & \text{if } \gamma = (0,1,0,0,\gamma_5),\ \gamma_5 \in \{0,1\},\ \theta = (0,0,0,1,0), \\ (\min(n,\alpha - x_1), 0), & \text{if } \gamma = (0,1,0,0,0),\ \theta = (0,1,0,0,1), \\ (0, \min(1,\alpha - x_2)), & \text{if } \gamma = (0,0,1,0,0),\ \theta = (1,0,0,0,0), \\ (0, \min(1,\alpha - x_2)), & \text{if } \gamma = (0,0,0,1,0), \end{cases}$$

$$\begin{aligned} f_\gamma((x_1,x_2),u,v) &= \left(f_\gamma^1((x_1,x_2),u,v), f_\gamma^2((x_1,x_2),u,v)\right) \\ &= \left(-\tfrac{2uv}{\alpha}x_1^2 + 2uvx_2 - uvx_1, \tfrac{uv}{\alpha}x_1^2 - uvx_2\right). \end{aligned}$$

The reader is invited to note that $\mathbb{M}\times[0,\alpha]^2$ is invariant with respect to the previous dynamics. One can either note that for $x_1, x_2 \in [0,\alpha]$

$$f_\gamma^1((0,x_2),u,v) = 2x_2uv \geq 0,\ f_\gamma^2((x_1,0),u,v) = \frac{x_1^2}{\alpha}uv \geq 0,$$
$$f_\gamma^1((\alpha,x_2),u,v) = (2x_2-3\alpha)uv \leq 0,\ f_\gamma^2((x_1,\alpha),u,v) = \left(\frac{x_1^2}{\alpha}-\alpha\right)uv \leq 0,$$

or, alternatively, compute the normal cone to the frontier of $[0,\alpha]^2$ (see [33, Theorem 2.8]) to deduce that $[0,\alpha]^2$ is invariant with respect to the deterministic dynamics. Moreover, the definition of g_γ guarantees that (X_1,X_2) does not leave $[0,\alpha]^2$. Let us also note that, whenever $x=(x_1,x_2)\in[0,\alpha]^2$, $y=(y_1,y_2)\in[0,\alpha]^2$,

$$\begin{aligned}&\langle f_\gamma(x,u,v)-f_\gamma(y,u,v),x-y\rangle\\ &= -uv\left(\left[\frac{2}{\alpha}(x_1+y_1)+1\right](x_1-y_1)^2-\left[2+\frac{1}{\alpha}(x_1+y_1)\right](x_1-y_1)(x_2-y_2)+(x_2-y_2)^2\right)\leq 0.\end{aligned}$$

This is a simple consequence of the fact that

$$\left[2+\frac{1}{\alpha}(x_1+y_1)\right]^2-4\left[\frac{2}{\alpha}(x_1+y_1)+1\right]=\left[2-\frac{1}{\alpha}(x_1+y_1)\right]^2-4\leq 0.$$

We deduce that

$$\sup_{v\in V}\inf_{w\in V}\langle f_\gamma(x,u,v)-f_\gamma(y,u,w),x-y\rangle\leq 0.$$

Moreover, for all $(x,y)\in[0,\alpha]^2$,

$$\sup_{v\in V}\inf_{w\in V}\langle f_\gamma(x,0,v)-f_\gamma(y,0,w),x-y\rangle=0.$$

It follows that one is not able to find any positive constant $c>0$ such that

$$\sup_{v\in V}\inf_{w\in V}\langle f_\gamma(x,0,v)-f_\gamma(y,0,w),x-y\rangle\leq -c|x-y|^2,$$

for all $(x,y)\in[0,\alpha]^2$ and we deal with a non-dissipative system (unlike, for instance, [9]).

Finally, the function $t\mapsto t-\min(1,t)$, $t\mapsto t+\min(k,\alpha-t)$ are Lipschitz continuous with Lipschitz constant 1 on $[0,\alpha]$ for all $k>0$. It follows that $\sup_{\theta\in\mathbb{M}}|x+g_\gamma(\theta,x)-y-g_\gamma(\theta,y)|\leq|x-y|$.

5 Proof of the First Main Result (Theorem 4)

In order to prove Theorem 4, we proceed as follows. First, we recall the link between the set of occupation measures $\Theta_0^\delta(\gamma,x)$ and the family $\Theta^\delta(\gamma,x)$ (taken from [34]). Next, we prove that Condition 2 implies the equicontinuity of the family of Abel-average value functions $\left(v^\delta\right)_{\delta>0}$. Finally, we recall the results in [32] on Abel-type theorems to conclude.

5.1 Step 1 : Equicontinuity of Abel-average Values

The following result corresponds to [34, Theorem 7 and Corollary 8] for this (less general) setting. It gives the link between the set of occupation measures $\Theta_0^\delta(\gamma,x)$ and the family $\Theta^\delta(\gamma,x)$.

Theorem 9 *i) For every $x\in\mathbb{R}^N$ and every $\delta>0$,*

$$v^\delta(\gamma,x)=\inf_{\mu\in\Theta^\delta(\gamma,x)}\int_{\mathbb{M}\times\mathbb{R}^N\times\mathbb{U}}h(\theta,y,u)\,\mu(d\theta,dy,du).$$

ii) For every $(\gamma,x)\in\mathbb{M}\times\mathbb{R}^N$ and every $\delta>0$, $\Theta^\delta(\gamma,x)=\overline{co}\left(\Theta_0^\delta(\gamma,x)\right)$.

We are now able to prove the following equicontinuity result.

Proposition 10 *We assume Condition 2 to hold true. Then, for every $\delta > 0$ and every $(\gamma, x, y) \in \mathbb{M} \times \mathbb{R}^{2N}$, one has*

$$\left| v^{\delta}(\gamma, x) - v^{\delta}(\gamma, y) \right| \leq Lip(h) |x - y|.$$

Proof. Let us fix $\delta > 0$ and $(\gamma, x, y) \in \mathbb{M} \times \mathbb{R}^{2N}$. We only need to prove that, for every $\varepsilon > 0$,

$$v^{\delta}(\gamma, y) \leq v^{\delta}(\gamma, x) + Lip(h) |x - y| + \varepsilon.$$

By definition of v^{δ}, for a fixed $\varepsilon > 0$, there exists some $u \in \mathcal{A}_{ad}(\mathbb{U})$ such that

$$v^{\delta}(\gamma, x) + \frac{\varepsilon}{2} \geq \delta \mathbb{E}\left[\int_0^{\infty} e^{-\delta t} h\left(\Gamma_t^{\gamma,x,u}, X_t^{\gamma,x,u}, u_t\right) dt \right] = \int_{\mathbb{M}\times\mathbb{R}^N\times\mathbb{U}} h(\theta, z, w)\, \mu^{\delta}_{\gamma,x,u}(d\theta, dz, dw).$$

If μ is the coupling measure given by Condition 2 and associated to $\frac{\varepsilon}{2}$, we deduce, using Theorem 9.i that

$$\begin{aligned} v^{\delta}(\gamma, y) &\leq \int_{(\mathbb{M}\times\mathbb{R}^N\times\mathbb{U})^2} h\left(\theta', z', w'\right) \mu\left(d\theta, dz, dw, d\theta', dz', dw'\right) \\ &\leq \int_{\mathbb{M}\times\mathbb{R}^N\times\mathbb{U}} h(\theta, z, w)\, \mu^{\delta}_{\gamma,x,u}(d\theta, dz, dw) + Lip(h) |x - y| + \frac{\varepsilon}{2} \\ &\leq v^{\delta}(\gamma, x) + Lip(h) |x - y| + \varepsilon. \end{aligned}$$

The proof of our Proposition follows by recalling that $\varepsilon > 0$ is arbitrary. ■

5.2 Step 2 : Proof of Theorem 4

The previous results on existence of a continuity modulus uniform with respect to the discount parameter $\delta > 0$ allows us to prove the existence of a limit value function as $\delta \to 0$. Before going to the proof of Theorem 4, we recall the following.

Lemma 11 *(i) [32, **Step 1 of Theorem 4.1**] Let us assume that $\left(v^{\delta}\right)_{\delta>0}$ is a relatively compact subset of $C\left(\mathbb{M}\times\mathbb{R}^N;\mathbb{R}\right)$. Then, for every $v \in C\left(\mathbb{M}\times\mathbb{R}^N;\mathbb{R}\right)$, every sequence $(\delta_m)_{m\geq 1}$ such that $\lim_{m\to\infty} \delta_m = 0$ and $\left(v^{\delta_m}\right)_{m\geq 1}$ converges uniformly to v on $\mathbb{M}\times\mathbb{R}^N$ and every $\varepsilon > 0$, there exists $T > 0$ such that*

$$V_t(\gamma, x) \geq v(\gamma, x) - \varepsilon, \text{ for all } (\gamma, x) \in \mathbb{M}\times\mathbb{R}^N \text{ and all } t \geq T.$$

*(ii) [32, **Theorem 4.1**] Let us assume that $\left(v^{\delta}\right)_{\delta>0}$ is a relatively compact subset of $C\left(\mathbb{M}\times\mathbb{R}^N;\mathbb{R}\right)$. Then, for every $v \in C\left(\mathbb{M}\times\mathbb{R}^N;\mathbb{R}\right)$ and every sequence $(\delta_m)_{m\geq 1}$ such that $\lim_{m\to\infty} \delta_m = 0$ and $\left(v^{\delta_m}\right)_{m\geq 1}$ converges uniformly to v on $\mathbb{M}\times\mathbb{R}^N$, the following equality holds true*

$$\liminf_{t\to\infty} \sup_{\gamma\in\mathbb{M},\ x\in\mathbb{R}^N} |V_t(\gamma, x) - v(\gamma, x)| = 0.$$

*(iii) [32, **Remark 4.2**] Let us assume that $\left(v^{\delta}\right)_{\delta>0}$ converges uniformly to some $v^* \in C\left(\mathbb{M}\times\mathbb{R}^N;\mathbb{R}\right)$ as $\delta \to 0$. Then the functions $(V_t)_{t>0}$ converge uniformly on $\mathbb{M}\times\mathbb{R}^N$ to v^**

$$\lim_{t\to\infty} \sup_{\gamma\in\mathbb{M},\ x\in\mathbb{R}^N} |V_t(\gamma, x) - v^*(\gamma, x)| = 0.$$

Remark 12 *In the proof of [32, **Theorem 4.1**], one gives the condition (i) in Step 1. However, in all generality, the converse is only partial. Indeed, Step 2 (see [32, **Page 174, Eq. (10)**] reads : For every $\varepsilon > 0$, there exists $m_0 \geq 1$ such that*

$$V_{\delta_m^{-1}}(\gamma, x) \leq v(\gamma, x) + \varepsilon,$$

for all $m \geq m_0$ and all $(\gamma, x) \in \mathbb{M} \times \mathbb{R}^N$. Thus, the index t of the subfamily $(V_t)_t$ converging to v depends directly on the sequence $(\delta_m)_{m \geq 1}$. Of course, whenever the limit v is independent on the choice of δ, so is t and one gets (iii).

We are now able to complete the proof of Theorem 4.

Proof of Theorem 4. Let us denote by

$$v^*(\gamma, x) := \limsup_{\delta \to 0} v^\delta(\gamma, x), \tag{4}$$

for every $(\gamma, x) \in \mathbb{M} \times \mathbb{K}$ (the pointwise lim sup). We fix, for the time being, some $(\gamma, x) \in \mathbb{M} \times \mathbb{K}$. Then, there exists some sequence $(\delta_m)_{m \geq 1}$ such that $\lim_{m \to \infty} \delta_m = 0$ and $\lim_{m \to \infty} v^{\delta_m}(\gamma, x) = v^*(\gamma, x)$. Due to Condition 2, the sequence $\left(v^{\delta_m}\right)_{m \geq 1}$ is equicontinuous (see Proposition 10) and, by definition, it is also bounded. Then, using Arzelà-Ascoli Theorem, it follows that some subsequence (still denoted $\left(v^{\delta_m}\right)_{m \geq 1}$) converges uniformly on $\mathbb{M} \times \mathbb{K}$ to some limit function $v \in C(\mathbb{M} \times \mathbb{K}; \mathbb{R})$. In particular, $w(\gamma, x) = v^*(\gamma, x)$. Using Lemma 11 (i), one gets that

$$\liminf_{t \to \infty} V_t(\gamma, x) \geq v^*(\gamma, x). \tag{5}$$

Obviously, this argument can be repeated for every $(\gamma, x) \in \mathbb{M} \times \mathbb{K}$. Let us now consider $w \in C(\mathbb{M} \times \mathbb{K}; \mathbb{R})$ to be an adherence point of the relatively compact family $\left(v^\delta\right)_{\delta > 0}$. Then, using Lemma 11 (ii), one establishes the existence of some increasing sequence $(t_n)_{n \geq 1}$ such that $\lim_{n \to \infty} t_n = \infty$ and

$$\lim_{n \to \infty} \sup_{\gamma \in \mathbb{M}, x \in \mathbb{R}^N} |V_{t_n}(\gamma, x) - w(\gamma, x)| = 0.$$

In particular, it follows that

$$w(\gamma, x) \geq \liminf_{t \to \infty} V_t(\gamma, x), \tag{6}$$

for all $(\gamma, x) \in \mathbb{M} \times \mathbb{K}$. Combining (4),(5) and (6), one deduces that the unique adherence point of $\left(v^\delta\right)_{\delta > 0}$ is v^*. The convergence of $(V_t)_{t > 0}$ follows by invoking Lemma 11 (iii). Our Theorem is now complete. ■

6 Proof of the Second Main Result (Theorem 7)

The proof of Theorem 7 is constructive and relies on four steps. We begin with recalling the Hamilton-Jacobi integrodifferential systems satisfied by the Abel-average functions and Krylov's shaking the coefficient method. The first step is showing that the value functions v^δ can be suitably approximated by using piecewise constant open-loop policies. The proof in this part strongly rely on the tools presented before. The second step is to interpret the system as a stochastic differential equation (SDE) with respect to some random measure. The third step is to embed the solutions of these SDE in a space of measures satisfying a suitable linear constraint via the linear programming approach. To conclude, the fourth step provides a constructive (pseudo-) coupling using SDE estimates.

6.1 Krylov's Shaking the Coefficients

For every $\delta > 0$, the value function v^δ is known to be the unique bounded, uniformly continuous viscosity solution of the Hamilton-Jacobi integro-differential system

$$\delta v^\delta(\gamma, x) + H\left(\gamma, x, \partial_x v^\delta(\gamma, x), v^\delta\right) = 0, \tag{7}$$

where the Hamiltonian is defined by setting

$$\begin{aligned} &H(\gamma, x, p, \varphi) \\ &:= \sup_{u \in \mathbb{U}} \left[-h(\gamma, x, u) - \langle f_\gamma(x, u), p \rangle - \lambda_\gamma(x, u) \int_{\mathbb{M}} \left(\varphi(\theta, x + g_\gamma(\theta, x, u)) - \varphi(\gamma, x)\right) Q^0(\gamma, u, d\theta) \right], \end{aligned}$$

for all $x, p \in \mathbb{R}^N$ and all bounded function $\varphi : \mathbb{M} \times \mathbb{R}^N \longrightarrow \mathbb{R}$. For further details on the subject, the reader is referred to [48].

Although uniformly continuous, the value functions v^δ are, in general, not of class C_b^1. However, adapting the method introduced in [41] (see also [7]), v^δ can be seen as the supremum over regular subsolutions of the system (7). Alternatively, one can give a variational formulation of v^δ with respect to an explicit set of constraints. We recall the following basic elements taken from [34].

We begin by perturbing the coefficients and consider an extended characteristic triple
- $\overline{f}_\gamma : \mathbb{R}^N \times \mathbb{U} \times \overline{B}(0,1) \longrightarrow \mathbb{R}^N$, $f_\gamma\left(x, u^1, u^2\right) = f_\gamma\left(x + u^2, u^1\right)$, $u_1 \in \mathbb{U}, u_2 \in \overline{B}(0,1)$, $\gamma \in \mathbb{M}$,
- $\overline{\lambda}_\gamma : \mathbb{R}^N \times \mathbb{U} \times \overline{B}(0,1) \longrightarrow \mathbb{R}^N$, $\lambda_\gamma\left(x, u^1, u^2\right) = \lambda_\gamma\left(x + u^2, u^1\right)$, $u_1 \in \mathbb{U}, u_2 \in \overline{B}(0,1)$, $\gamma \in \mathbb{M}$,
- $\overline{Q} : \mathbb{R}^N \times \mathbb{U} \times \overline{B}(0,1) \longrightarrow \mathcal{P}\left(\mathbb{R}^N\right)$, $\overline{Q}\left(\gamma, x, u^1, u^2, A\right) = Q\left(\gamma, x + u^2, u^1, A + \left(0, u^2\right)\right)$, where $A + \left(0, u^2\right) = \left\{\left(a_1, a_2 + u^2\right) : (a_1, a_2) \in A\right\}$, for all $x \in \mathbb{R}^N$, $u^1 \in U$, $u^2 \in \overline{B}(0,1)$ and all Borel set $A \subset \mathbb{M} \times \mathbb{R}^N$.

One can easily construct the process $\left(\Gamma^{\gamma,x,u^1,u^2}, X^{\gamma,x,u^1,u^2}\right)$ with $u = \left(u^1, u^2\right) \in \mathcal{A}_{ad}\left(\mathbb{U} \times \overline{B}(0,1)\right)$. The initial process associated to (f, λ, Q) can be obtained by imposing $u^2 = 0$. Let us note that, with this construction,

$$\overline{Q}\left(\gamma, x, u^1, u^2, d\theta dy\right) = \delta_{x + g_\gamma(\theta, x + u^2, u^1)}(dy)\, Q^0\left(\gamma, u^1, d\theta\right).$$

6.2 Step 1: Piecewise Constant Open-loop Policies

The aim of this subsection is to show that the value functions v^δ can be approximated by functions in which the control processes are piecewise (in time) constant. For Brownian diffusions, this type of result has been proven in [40]. In this section we adapt the method of [40] to our setting by hinting to the modifications whenever necessary. Following [40], for all $n \geq 1$, we introduce the value function

$$v^{\delta,n}(\gamma, x) = \inf_{u \in \mathcal{A}_{ad}^n} \delta \mathbb{E}\left[\int_0^\infty e^{-\delta t} h\left(\Gamma_t^{\gamma,x,u,0}, X_t^{\gamma,x,u,0}, u_t\right) dt\right],$$

for all $(\gamma, x) \in \mathbb{M} \times \mathbb{R}^N$.

The main result of the subsection is the following.

Theorem 13 *Let us assume that there exists a compact, convex set $\mathbb{K}$ which is invariant with respect to the controlled PDMP with characteristics (f, λ, Q). Then, for every $\delta > 0$, the value functions $v^{\delta,n}$ converge uniformly to v^δ as the discretization step n increases to infinity*

$$\lim_{n \to \infty} \sup_{\gamma \in \mathbb{M},\ x \in \mathbb{K}} \left| v^\delta(\gamma, x) - v^{\delta,n}(\gamma, x) \right| = 0.$$

The proof relies on the same arguments as those developed in [40] combined with dynamic programming principles. Let us briefly explain the approach. For every $n \geq 1$, one begins by proving a dynamic programming principle for $v^{\delta,n}$ and involving $T \wedge T_1$ as intermediate time, for $T \in n^{-1}\mathbb{N}$. The arguments are essentially the same as those in [48] and we only specify when the structure of $\mathcal{A}_{ad}^n$ intervenes. Next, one takes a sequence of smooth functions $\left(v_{(\varepsilon)}^{\delta,n}\right)_{\varepsilon>0}$ converging uniformly to $v^{\delta,n}$ by adapting to the present framework Krylov's shaking of coefficients method introduced in [41] (see also [7] or [33] for the PDMP case). Then, one proceeds by writing the Hamilton-Jacobi integrodifferential system satisfied by $v_{(\varepsilon)}^{\delta,n}$. This equation is $\varepsilon-$close to the one satisfied by v^{δ} (with a uniform behavior w.r.t. $n \geq 1$). Our assertion follows by integrating this subsolution condition with respect to the law of the piecewise deterministic Markov process then allowing $\varepsilon \to 0$. For our reader's convenience, we have indicated the main modifications and arguments in the Appendix.

Remark 14 *If the invariance condition holds true, then, by applying this result, one only needs to check that the nonexpansive Condition 2 holds true for all $u \in \mathcal{A}_{ad}^n(\mathbb{U})$ for all n large enough (larger than some n_ε).*

6.3 Step 2 : Associated Random Measures and Stochastic Differential Equations

Let us fix $\gamma_0 \in \mathbb{M}$, $x_0 \in \mathbb{R}^N$ and $(u,v) \in \mathcal{A}_{ad}(U \times V)$. The following construction is quite standard and makes the object of [26, Section 26] for more general PDMP (without control) and [26, Section 41] (when control is present). We let $S_0 = T_0 = 0$, $S_n = T_n - T_{n-1}$, for all $n \geq 1$ and $\xi_n = \left(S_n, \gamma_{T_n}^{\gamma_0,x_0,u,v}, X_{T_n}^{\gamma_0,x_0,u,v}\right)$. We look at the process (γ, X) under $\mathbb{P}^{\gamma_0,x_0,u,v}$ (which depends on both the initial state (γ_0, x_0) and the control couple (u,v), but, having fixed these elements and for notation purposes, this dependency will be dropped). By abuse of notation, we let

$$u_s := u_1(\gamma_0, x_0, s)\, 1_{0 \leq s \leq T_1} + \sum_{n \geq 1} u_{n+1}\left(\gamma_{T_n}^{\gamma_0,x_0,u,v}, X_{T_n}^{\gamma_0,x_0,u,v}, s - T_n\right) 1_{T_n < s \leq T_{n+1}},$$

(and similar for v). We denote by $\mathbb{F}$ the filtration $\left(\mathcal{F}_{[0,t]} := \sigma\left\{(\gamma_r^{\gamma_0,x_0,u,v}, X_r^{\gamma_0,x_0,u,v}) : r \in [0,t]\right\}\right)_{t \geq 0}$. The predictable σ-algebra will be denoted by $\mathcal{P}$ and the progressive σ-algebra by $Prog$. For the general structure of predictable processes, the reader is referred to [26, Section 26], [39, Proposition 4.2.1] or [13, Appendix A2, Theorem T34]. In particular, due to the previous notations, it follows that u and v are predictable.

As usual, we introduce the random measure $\overline{p}$ on $\Omega \times (0,\infty) \times \mathbb{M} \times \mathbb{R}^N$ by setting

$$\overline{p}(\omega, A) = \sum_{k \geq 1} 1_{\left(T_k(\omega), \left(\gamma_{T_k}^{\gamma_0,x_0,u,v}, X_{T_k}^{\gamma_0,x_0,u,v}\right)(\omega)\right) \in A}, \text{ for all } \omega \in \Omega,\ A \in \mathcal{B}(0,\infty) \times \mathcal{B}\left(\mathbb{M} \times \mathbb{R}^N\right).$$

The compensator of $\overline{p}$ is

$$\widehat{\overline{p}}(dsdyd\theta) = \lambda\left(\gamma_{s-}^{\gamma_0,x_0,u,v}, u_s\right) \delta_{X_{s-}^{\gamma_0,x_0,u,v} + g_{\gamma_{s-}^{\gamma_0,x_0,u,v}}\left(\theta, X_{s-}^{\gamma_0,x_0,u,v}, u_s, v_s\right)}(dy)\, Q^0\left(\gamma_{s-}^{\gamma_0,x_0,u,v}, u_s, d\theta\right) ds.$$

and the compensated martingale measure (see [26, Proposition 26.7]) is given by $\overline{q} := \overline{p} - \widehat{\overline{p}}$.

By construction, for our model, on $[T_{n-1}, T_n)$, $X_t^{\gamma_0,x_0,u,v}$ is a deterministic function of $X_{T_{n-1}}^{\gamma_0,x_0,u,v}$, $\gamma_{T_{n-1}}^{\gamma_0,x_0,u,v}$, $u_n\left(X_{T_{n-1}}^{\gamma_0,x_0,u,v}, \gamma_{T_{n-1}}^{\gamma_0,x_0,u,v}, \cdot - T_{n-1}\right)$ and $v_n\left(X_{T_{n-1}}^{\gamma_0,x_0,u,v}, \gamma_{T_{n-1}}^{\gamma_0,x_0,u,v}, \cdot - T_{n-1}\right)$. In this particular framework,

$$\begin{aligned} X_{T_n}^{\gamma_0,x_0,u,v} &= \Phi_{T_n - T_{n-1}}^{0, X_{T_{n-1}}^{\gamma_0,x_0,u,v}, u_n\left(\gamma_{T_{n-1}}^{\gamma_0,x_0,u,v}, X_{T_{n-1}}^{\gamma_0,x_0,u,v}, \cdot\right), v_n\left(\gamma_{T_{n-1}}^{\gamma_0,x_0,u,v}, X_{T_{n-1}}^{\gamma_0,x_0,u,v}, \cdot\right); \gamma_{T_{n-1}}^{\gamma_0,x_0,u,v}} \\ &\quad + g_{\gamma_{T_{n-1}}^{\gamma_0,x_0,u,v}}\left(\gamma_{T_n}^{\gamma_0,x_0,u,v}, u_n\left(\gamma_{T_{n-1}}^{\gamma_0,x_0,u,v}, X_{T_{n-1}}^{\gamma_0,x_0,u,v}, T_n - T_{n-1}\right), v_n\left(\gamma_{T_{n-1}}^{\gamma_0,x_0,u,v}, X_{T_{n-1}}^{\gamma_0,x_0,u,v}, T_n - T_{n-1}\right)\right), \end{aligned}$$

hence being a a deterministic function of $S_n, X_{T_{n-1}}^{\gamma_0,x_0,u,v}, \gamma_{T_{n-1}}^{\gamma_0,x_0,u,v}$. It follows that the filtration $\mathbb{F}$ is actually generated by the marked point process $\left(T_k, \gamma_{T_k}^{\gamma_0,x_0,u,v}\right)_{k\geq 0}$. As a consequence, $v_{n+1} = v_{n+1}\left(X_{T_n}^{\gamma_0,x_0,u,v}, \gamma_{T_n}^{\gamma_0,x_0,u,v}, \cdot\right)$ is a deterministic function of $T_1, ..., T_n, \gamma_{T_1}^{\gamma_0,x_0,u,v}, ..., \gamma_{T_n}^{\gamma_0,x_0,u,v}$ still denoted by $v_{n+1}\left(T_1, ..., T_n, \gamma_{T_1}^{\gamma_0,x_0,u,v}, ..., \gamma_{T_n}^{\gamma_0,x_0,u,v}, \cdot\right)$. In the case when $(u,v) \in \mathcal{A}_{ad}^m(U\times V)$ for some $m \geq 1$ are piecewise constant, v_{n+1} is of type

$$\sum_{k\geq 0} v_{n+1}^k\left(T_1, ..., T_n, \gamma_{T_1}^{\gamma_0,x_0,u,v}, ..., \gamma_{T_n}^{\gamma_0,x_0,u,v}\right) 1_{\left(\frac{k}{m},\frac{k+1}{m}\right]}(t).$$

Similar assertion hold true for u.

We now define the random measure p on $\Omega\times(0,\infty)\times\mathbb{M}$ by setting

$$p(\omega, A) = \overline{p}\left(\omega, A\times\mathbb{R}^N\right), \text{ for all } \omega\in\Omega,\ A\in\mathcal{B}(0,\infty)\times\mathcal{B}(\mathbb{M}).$$

The properties of $\overline{p}$ imply that the compensator of p is

$$\widehat{p}(dsd\theta) = \lambda\left(\gamma_{s-}^{\gamma_0,x_0,u,v}, u_s\right) Q^0\left(\gamma_{s-}^{\gamma_0,x_0,u,v}, u_s, d\theta\right) ds$$

and

$$q(dsd\theta) = p(dsd\theta) - \lambda\left(\gamma_{s-}^{\gamma_0,x_0,u,v}, u_s\right) Q^0\left(\gamma_{s-}^{\gamma_0,x_0,u,v}, u_s, d\theta\right) ds$$

is its martingale measure. Following the general theory of integration with respect to random measures (see, for example [38]), the second state component can be identified with the unique solution of the stochastic differential equation (SDE)

$$\begin{cases} dX_t^{\gamma_0,x_0,u,v} = f_{\gamma_t^{\gamma_0,x_0,u,v}}\left(X_t^{\gamma_0,x_0,u,v}, u_t, v_t\right) dt + \int_{\mathbb{M}} g_{\gamma_{t-}^{\gamma_0,x_0,u,v}}\left(\theta, X_{t-}^{\gamma_0,x_0,u,v}, u_t, v_t\right) p(dtd\theta), t\geq 0, \\ X_t^{\gamma_0,x_0,u,v} = x_0,\ \mathbb{P}-a.s. \end{cases}$$

6.4 Step 3 : Measure Embedding of Solutions

More general, whenever w is an $\mathbb{F}$-predictable process, we can consider the equation

$$\begin{cases} dY_t^{y_0,w} = f_{\gamma_t^{\gamma_0,x_0,u,v}}\left(Y_t^{y_0,w}, u_t, w_t\right) dt + \int_{\mathbb{M}} g_{\gamma_{t-}^{\gamma_0,x_0,u,v}}\left(\theta, Y_{t-}^{y_0,w}, u_t, w_t\right) p(dtd\theta), t\geq 0, \\ Y_0^{y_0,w} = y_0,\ \mathbb{P}-a.s. \end{cases}$$

The assumptions on the coefficients f and g guarantee that, for every $y_0\in\mathbb{R}^N$ and every predictable, $V-$valued process w, this equation admits a unique solution $Y^{y_0,w}$. We fix $\delta>0$ and consider some (arbitrary) regular test function $\phi\in C_b^1\left(\mathbb{M}\times\mathbb{R}^N;\mathbb{R}\right)$. Itô's formula (see [38, Chapter II, Theorem 5.1]) applied to $\delta e^{-\delta\cdot}\phi\left(\gamma_\cdot^{\gamma_0,x_0,u,v}, Y_\cdot^{y_0,w}\right)$ on $[0,T]$ yields

$$\begin{aligned} &\delta e^{-\delta T}\mathbb{E}\left[\phi\left(\gamma_T^{\gamma_0,x_0,u,v}, Y_T^{y_0,w}\right)\right] \\ &= \delta\phi(\gamma_0, y_0) \\ &+ \mathbb{E}\left[\int_0^T \delta e^{-\delta t}\left(\begin{array}{c} -\delta\phi(\gamma_t, Y_t) + \langle f_{\gamma_t}(Y_t, u_t, w_t), \partial_x\phi(\gamma_t, Y_t)\rangle \\ +\lambda(\gamma_t, u_t)\int_{\mathbb{M}}\left(\phi(\theta, Y_t + g_{\gamma_t}(\theta, Y_t, u_t, w_t)) - \phi(\gamma_t, Y_t)\right) Q^0(\gamma_t, u_t, d\theta) \end{array}\right) dt\right], \end{aligned}$$

where we have denoted by $(\gamma_t, Y_t) = (\gamma_t^{\gamma_0,x_0,u,v}, Y_t^{y_0,w})$. By letting $T\to\infty$, it follows that the occupation measure $\mu^{y_0,w}\in\mathcal{P}\left(\mathbb{M}\times\mathbb{R}^N\times U\times V\right)$ given by

$$\mu^{y_0,w}(A) = \mathbb{E}\left[\int_0^\infty \delta e^{-\delta t} 1_A\left(\gamma_t^{\gamma_0,x_0,u,v}, Y_t^{y_0,w}, u_t, w_t\right) dt\right], \text{ for } A\in\mathcal{B}\left(\mathbb{M}\times\mathbb{R}^N\times U\times V\right)$$

satisfies

$$\int_{\mathbb{M}\times\mathbb{R}^N\times\mathbb{U}}\left(\mathcal{L}^{u,v}\phi(\theta, y) + \delta\left(\phi(\gamma, x) - \phi(\theta, y)\right)\right)\mu(d\theta, dy, du, dv) = 0.$$

We recall that $\mathcal{L}^{u,v}$ is the generator given by (3).

There is no reason for the couple $(\gamma^{\gamma_0,x_0,u,v}, Y^{y_0,w})$ to be associated to a $U \times V$-valued piecewise open-loop control couple. Nevertheless, the previous arguments show that the occupation measure $\mu^{y_0,w}$ belongs to $\Theta^{\delta}(\gamma_0, y_0)$ (see (2)).

Remark 15 *Let us note that if there exists a set $\mathbb{K}$ invariant with respect to the PDMP driven by (f, λ, Q), then, for all $\gamma_0 \in \mathbb{M}$, $y_0 \in \mathbb{K}$, the occupation measures $\mu \in \Theta_0^{\delta}(\gamma_0, y_0)$ satisfy the support condition $\mu(\mathbb{M} \times \mathbb{K} \times U \times V) = 1$. Then, by Theorem 9, the same holds true for $\Theta^{\delta}(\gamma_0, y_0)$ and, hence, $Y^{y_0,w}$ takes its values in $\mathbb{K}$. Alternatively, one can use [33, Theorem 2.8 (ii)].*

6.5 Step 4 : Coupling via the Random Measure

As in the previous arguments, one can define a measure $\mu \in \mathcal{P}\left(\left(\mathbb{M} \times \mathbb{R}^N \times U \times V\right)^2\right)$ by setting

$$\mu(A \times B) = \mathbb{E}\left[\int_0^{\infty} \delta e^{-\delta t} 1_A(\gamma_t^{\gamma_0,x_0,u,v}, X_t^{\gamma_0,x_0,u,v}, u_t, v_t)\, 1_B(\gamma_t^{\gamma_0,x_0,u,v}, Y_t^{y_0,w}, u_t, w_t)\, dt\right],$$

whenever $A \in \mathcal{B}\left(\left(\mathbb{M} \times \mathbb{R}^N \times U \times V\right)^2\right)$. It is clear that

$$\begin{aligned}
&\int_{(\mathbb{M}\times\mathbb{R}^N\times\mathbb{U})^2} \left|h(\theta, z, w) - h(\theta', z', w')\right| \mu\left(d\theta, dz, dw, d\theta', dz', dw'\right) \\
&= \mathbb{E}\left[\int_0^{\infty} \delta e^{-\delta t} \left|h(\gamma_t^{\gamma_0,x_0,u,v}, X_t^{\gamma_0,x_0,u,v}, u_t, v_t) - h(\gamma_t^{\gamma_0,x_0,u,v}, Y_t^{y_0,w}, u_t, w_t)\right| dt\right],
\end{aligned}$$

$\mu\left(A \times \left(\mathbb{M} \times \mathbb{R}^N \times U \times V\right)\right) = \mu^{\delta}_{\gamma_0,x_0,u,v} \in \Theta_0^{\delta}(\gamma_0, x_{0,})$ and $\mu\left(\left(\mathbb{M} \times \mathbb{R}^N \times U \times V\right) \times B\right) = \mu^{y_0,w} \in \Theta^{\delta}(\gamma_0, y_0)$, where $\mu^{y_0,w}$ given in the previous arguments. Convenient estimates for this integral term imply the condition (2) and, hence, the results on existence of a limit value function. In fact (see Remark 14), in order to prove Theorem 7, it suffices to provide good estimates when the process is constructed with piecewise constant (in time) policies $(u, v) \in \mathcal{A}_{ad}^n(U \times V)$. This is done by the following.

Lemma 16 *We assume Condition 5 to hold true. Moreover, we assume that there exists a compact set $\mathbb{K}$ invariant with respect to the PDMP governed by (f, λ, Q). Then, there exists $\omega : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that $lim_{\varepsilon \to 0}\omega(\varepsilon) = 0$ and, for every $n \geq 1$ and every $(u, v) \in \mathcal{A}_{ad}^n(U \times V)$, there exists w predictable with respect to the filtration $\mathbb{F}^{\gamma_0,x_0,u,v}$ such that*

$$\mathbb{E}\left[\int_0^{\infty} \delta e^{-\delta t} \left|h(\gamma_t^{\gamma_0,x_0,u,v}, X_t^{\gamma_0,x_0,u,v}, u_t, v_t) - h(\gamma_t^{\gamma_0,x_0,u,v}, Y_t^{y_0,w}, u_t, w_t)\right| dt\right] \leq \omega\left(n^{-1}\right).$$

Proof. Step 0.

Let us define a set-valued function

$$\begin{aligned}
&\mathbb{M} \times \mathbb{K}^2 \times U \times V \ni (\gamma, x, y, u, v) \rightsquigarrow \Xi(\gamma, x, y, u, v) \\
&:= \left\{ \begin{array}{c} w \in V : \ \forall \theta \in \mathbb{M}, \\ \langle f_\gamma(x, u, v) - f_\gamma(y, u, w), x - y\rangle \leq 0, \\ |x + g_\gamma(\theta, x, u, v) - y - g_\gamma(\theta, y, u, w)| \leq |x - y|, \\ |h(\gamma, x, u, v) - h(\gamma, y, u, w)| \leq Lip(h)\,|x - y|, \end{array} \right\}.
\end{aligned}$$

One easily checks that the function has compact values and is upper semicontinuous. Hence, there exists some measurable selection

$$\widehat{w} : \mathbb{M} \times \mathbb{K}^2 \times U \times V \longrightarrow V, \ \widehat{w}(\gamma, x, y, u, v) \in \Xi(\gamma, x, y, u, v),$$

for all $(\gamma, x, y, u, v) \in \mathbb{M} \times \mathbb{K}^2 \times U \times V$. For further details, the reader is referred to [6, Subsection 9.2].

We construct an $\mathbb{F}$-predictable V-valued control process w as follows. We begin by fixing $T > 0$ (depending on n) and $m \geq 1$ (depending on n). The choice of T and m will be made explicit later on. Moreover, we assume that $\mathbb{K} \subset \overline{B}(0, k_0)$, for some $k_0 > 0$.

Step 1. We consider

$$w_s^{1,0} := \widehat{w}(\gamma_0, x_0, y_0, u_1(0), v_1(0)) = w_0^{1,0}, \ s \geq 0,$$

where we have denoted, by abuse of notation,

$$u_1(s) = u_1(\gamma_0, x_0, s), \ v_1(s) = v_1(\gamma_0, x_0, s).$$

We recall that if $s \leq \frac{1}{n}$, one has $u_1(s) = u_1(0, \gamma_0, x_0)$ and similar assertions hold true for v_1. By recalling that $\mathbb{K}$ is invariant with respect to the controlled piecewise deterministic dynamics, one gets

$$\begin{aligned}
&\left\langle f_{\gamma_0}\left(\Phi_s^{0,x_0,u,v;\gamma_0}, u_s, v_s\right) - f_{\gamma_0}\left(\Phi_s^{0,y_0,u,w^{1,0};\gamma_0}, u_s, w_s^{1,0}\right), \Phi_s^{0,x_0,u,v;\gamma_0} - \Phi_s^{0,y_0,u,w^{1,0};\gamma_0}\right\rangle \\
&\leq \left\langle f_{\gamma_0}(x_0, u_s, v_s) - f_{\gamma_0}\left(y_0, u_s, w_s^{1,0}\right), x_0 - y_0\right\rangle + c\left(\left|x_0 - \Phi_s^{0,x_0,u,v;\gamma_0}\right| + \left|y_0 - \Phi_s^{0,y_0,u,w^{1,0};\gamma_0}\right|\right) \\
&\leq c\left(\left|x_0 - \Phi_s^{0,x_0,u,v;\gamma_0}\right| + \left|y - \Phi_s^{0,y_0,u,w^{1,0};\gamma_0}\right|\right) \leq \frac{c}{n}, \qquad (8)
\end{aligned}$$

for all $0 \leq s \leq \frac{1}{n}$. Similarly,

(9)

$$\begin{aligned}
i. \quad \tfrac{c}{n} &\geq \left|h\left(\gamma_0 \Phi_s^{0,x_0,u,v;\gamma_0}, u_s, v_s\right) - h\left(\gamma_0 \Phi_s^{0,y_0,u,w^{1,0};\gamma_0}, u_s, w_s^{1,0}\right)\right| \\
&\quad - Lip(h)\left|\Phi_s^{0,x_0,u,v;\gamma_0} - \Phi_s^{0,y_0,u,w^{1,0};\gamma_0}\right|, \\
ii. \quad \tfrac{c}{n} &\geq \left|\Phi_s^{0,x_0,u,v;\gamma_0} + g_{\gamma_0}\left(\theta, \Phi_s^{0,x_0,u,v;\gamma_0}, u_s, v_s\right) - \Phi_s^{0,y_0,u,w^{1,0};\gamma_0} - g_{\gamma_0}\left(\theta, \Phi_s^{0,y_0,u,w^{1,0};\gamma_0}, u_s, w_s^{1,0}\right)\right| \\
&\quad - \left|\Phi_s^{0,x_0,u,v;\gamma_0} - \Phi_s^{0,y_0,u,w^{1,0};\gamma_0}\right|,
\end{aligned}$$

for all $0 \leq s \leq \frac{1}{n}$. The constant $c > 1$ is generic, independent of $\delta > 0, n, s, x, y, u$ and is allowed to change from one line to another. We define the control process $w^{1,1}$ by setting

$$w_s^{1,1} = w_s^{1,0} 1_{s \leq \frac{1}{n}} + \widehat{w}\left(\gamma_0, \Phi_{\frac{1}{n}}^{0,x_0,u,v;\gamma_0}, \Phi_{\frac{1}{n}}^{0,y_0,u,w^{1,0};\gamma_0}, u_1\left(\frac{1}{n}\right), v_1\left(\frac{1}{n}\right)\right) 1_{s > \frac{1}{n}}.$$

Then the estimates in (8) hold true for $s \in \left[0, \frac{2}{n}\right]$ if substituting $w^{1,1}$ to $w^{1,0}$. We set

$$w_s^{1,2} = w_s^{1,1} 1_{s \leq \frac{2}{n}} + \widehat{w}\left(\gamma_0, \Phi_{\frac{2}{n}}^{0,x_0,u,v;\gamma_0}, \Phi_{\frac{2}{n}}^{0,y_0,u,w^{1,1};\gamma_0}, u_1\left(\frac{2}{n}\right), v_1\left(\frac{2}{n}\right)\right) 1_{s > \frac{2}{n}},$$

and so on, to define $w^{1,3}, w^{1,4}, ..., w^{1,n([T]+1)}$. We fix some $w^0 \in V$ and let $w^1 = w_s^{1,n([T]+1)} 1_{s \leq T} + w^0 1_{s > T}$, where $[\cdot]$ denotes the floor function. As consequence, by recalling that, prior to T_1, both $X^{\gamma_0,x_0,u,v}$ and $Y^{y_0,w}$ are deterministic and can be identified with $\Phi^{0,x_0,u,v;\gamma_0}$ (resp. $\Phi_s^{0,y_0,u,w^{1,0};\gamma_0}$), we use (8) to get

$$\left|X_t^{\gamma_0,x_0,u,v} - Y_t^{y_0,w^1}\right|^2 \leq |x_0 - y_0|^2 + \frac{ct}{n}, \qquad (10)$$

for all $t \leq T$ on $t < T_1$. It follows that

$$\left|X_t^{\gamma_0,x_0,u,v} - Y_t^{y_0,w^1}\right| \leq |x_0 - y_0| + \frac{c}{2\sqrt{n}} + \frac{t}{\sqrt{n}}, \qquad (11)$$

or all $t \leq T$ on $t < T_1$. Moreover, on $T_1 \leq T$, using (9.ii) and (10) and recalling that $\mathbb{K} \subset \overline{B}(0, k_0)$ is invariant (see also Remark 15) one has

$$\begin{aligned}
\left|X_{T_1}^{\gamma_0,x_0,u,v} - Y_{T_1}^{y_0,w^1}\right| &= \Big|\Phi_{T_1-}^{0,x_0,u,v;\gamma_0} + g_{\gamma_0}\left(\gamma_{T_1}^{\gamma_0,x_0,u,v}, \Phi_{T_1}^{0,x_0,u,v;\gamma_0}, u_{T_1}, v_{T_1}\right) \\
&\quad -\Phi_{T_1-}^{0,y_0,u,w^{1,0};\gamma_0} - g_{\gamma_0}\left(\gamma_{T_1}^{\gamma_0,x_0,u,v}, \Phi_{T_1}^{0,y_0,u,w^{1,0};\gamma_0}, u_{T_1}, w_{T_1}^{1,0}\right)\Big| \\
&\leq \left|\Phi_{T_1-}^{0,x_0,u,v;\gamma_0} - \Phi_{T_1-}^{0,y_0,u,w^{1,0};\gamma_0}\right| + \frac{c}{n} \\
&\leq \min\left(\sqrt{|x_0 - y_0|^2 + \frac{cT_1}{n}}, 2k_0\right) + \frac{c}{n} \\
&\leq |x_0 - y_0| + \frac{c + T_1}{\sqrt{n}}, \qquad (12)
\end{aligned}$$

for all $n \geq 4$. (The reader is invited to recall that u_{T_1} is still $u_1(T_1, \gamma_0, x_0)$ and that $u_1 \in \mathcal{A}_0^n$ is left continuous). One gets, on $T_1 \leq T$,

$$\begin{aligned}
\left|X_{T_1}^{\gamma_0,x_0,u,v} - Y_{T_1}^{y_0,w^1}\right|^2 &\leq \left(\min\left(\sqrt{|x_0 - y_0|^2 + \frac{cT_1}{n}}, 2k_0\right) + \frac{c}{n}\right)^2 \leq |x_0 - y_0|^2 + \frac{cT_1}{n} + \frac{c^2}{n^2} + 4k_0\frac{c}{n} \\
&\leq |x_0 - y_0|^2 + \frac{c(T_1 + 4k_0 + 1)}{n}, \qquad (13)
\end{aligned}$$

whenever $n \geq c$. Finally, using (9.i) and (11), we get

$$\begin{aligned}
\left|h\left(\gamma_0, \Phi_s^{0,x_0,u,v;\gamma_0}, u_s, v_s\right) - h\left(\gamma_0, \Phi_s^{0,y_0,u,w^{1,0};\gamma_0}, u_s, w_s^1\right)\right| &\leq Lip(h)\left|\Phi_s^{0,x_0,u,v;\gamma_0} - \Phi_s^{0,y_0,u,w^{1,0};\gamma_0}\right| + \frac{c}{n} \\
&\leq Lip(h)|x_0 - y_0| + \frac{(Lip(h) + 1)c + Lip(h)t}{\sqrt{n}}.
\end{aligned}$$

for all $s < T \wedge T_1$.

Step 2. We continue the construction on $[T_1, T_2)$. By abuse of notation, we let $u_2(s) := u_2\left(\gamma_{T_1}^{\gamma_0,x_0,u,v}, X_{T_1}^{\gamma_0,x_0,u,v}, s\right)$, for $s \geq 0$ and similar for v_2. We set

$$w_s^{2,1} := w_s^1 1_{0 \leq s \leq T_1} + \widehat{w}\left(\gamma_{T_1}^{\gamma_0,x_0,u,v}, X_{T_1}^{\gamma_0,x_0,u,v}, Y_{T_1}^{y_0,w^1}, u_2(0), v_2(0)\right) 1_{s > T_1}.$$

It is clear that this control process is predictable. We apply the same method as in Step 1 (ω-wise) on the (stochastic) time interval $\left[T_1, \left(T_1 + \frac{1}{n}\right) \wedge T_2\right]$, then on $\left[T_1, \left(T_1 + \frac{2}{n}\right) \wedge T_2\right]$, etc. We construct a sequence of control processes $\left(w^{2,m}\right)_{m \geq 0}$ and, by choosing m large enough, we establish the existence of some w^2 such that

(14)

$$\begin{cases}
\left|X_t^{\gamma_0,x_0,u,v} - Y_t^{y_0,w^2}\right|^2 \leq \left|X_{T_1}^{\gamma_0,x_0,u,v} - Y_{T_1}^{y_0,w^1}\right|^2 + \frac{c(t-T_1)}{n} \leq |x_0 - y_0|^2 + \frac{c(t+4k_0+1)}{n} \\
\left|X_{T_2}^{\gamma_0,x_0,u,v} - Y_{T_2}^{y_0,w^2}\right| \leq \left|X_{T_2-}^{\gamma_0,x_0,u,v} - Y_{T_2-}^{y_0,w^1}\right| + \frac{c}{n} \\
\qquad \leq \min\left(\sqrt{|x_0 - y_0|^2 + \frac{c(T_2+4k_0+1)}{n}}, 2k_0\right) + \frac{c}{n} \leq |x_0 - y_0| + \frac{c+T_2+4k_0+1}{\sqrt{n}}, \\
\left|X_{T_2}^{\gamma_0,x_0,u,v} - Y_{T_2}^{y_0,w^2}\right|^2 \leq |x_0 - y_0|^2 + \frac{c(T_2+4k_0+1)}{n} + \frac{c^2}{n^2} + 4k_0\frac{c}{n} \leq |x_0 - y_0|^2 + \frac{c(T_2+2(4k_0+1))}{n}, \\
\left|h\left(\gamma_t^{\gamma_0,x_0,u,v}, X_t^{\gamma_0,x_0,u,v}, u_t, v_t\right) - h\left(\gamma_t^{\gamma_0,x_0,u,v}, Y_t^{y_0,w^2}, u_t, w_t^2\right)\right| \\
\leq Lip(h)\left|X_t^{\gamma_0,x_0,u,v} - Y_t^{y_0,w^1}\right| + \frac{c}{n} \leq Lip(h)|x_0 - y_0| + \frac{(Lip(h)+1)c + Lip(h)(t+4k_0+1)}{\sqrt{n}},
\end{cases}$$

for all $T_1 \leq t < T_2 \wedge T$, $\mathbb{P}-a.s.$ if $n \geq \max(4, c)$. We continue our construction on $[0, T_3 \wedge T]$, $[0, T_4 \wedge T]$ and so on to finally get a predictable process w^m such that

$$
(15) \quad \begin{cases} \left| X_t^{\gamma_0, x_0, u, v} - Y_t^{y_0, w^m} \right|^2 \leq |x_0 - y_0|^2 + \frac{c[t+(i-1)(4k_0+1)]}{n}, \\ \left| X_{T_i}^{\gamma_0, x_0, u, v} - Y_{T_i}^{y_0, w^m} \right| \leq \min\left(\sqrt{|x_0 - y_0|^2 + \frac{c[T_i+(i-1)(4k_0+1)]}{n}}, 2k_0 \right) + \frac{c}{n} \\ \qquad \leq |x_0 - y_0| + \frac{c+T_i+(i-1)(4k_0+1)}{\sqrt{n}}, \\ \left| X_{T_i}^{\gamma_0, x_0, u, v} - Y_{T_i}^{y_0, w^m} \right|^2 \leq |x_0 - y_0|^2 + \frac{c[T_i+i(4k_0+1)]}{n}, \\ \left| h\left(\gamma_t^{\gamma_0, x_0, u, v}, X_t^{\gamma_0, x_0, u, v}, u_t, v_t\right) - h\left(\gamma_t^{\gamma_0, x_0, u, v}, Y_t^{y_0, w^m}, u_t, w_t^m\right) \right| \\ \qquad \leq Lip(h) |x_0 - y_0| + \frac{(Lip(h)+1)c + Lip(h)[t+(i-1)(4k_0+1)]}{\sqrt{n}}, \end{cases}
$$

for all $i \leq m$, $T_{i-1} \leq t < T_i \wedge T$, $\mathbb{P}-a.s.$

Let us note that in the same way as [48, Inequality 3.27], one has

$$
\mathbb{E}\left[e^{-\delta T_m}\right] \leq \left(1 - \delta \int_0^\infty e^{-t(\delta+\lambda_{\max})} dt\right)^m = \left(\frac{\lambda_{\max}}{\delta + \lambda_{\max}}\right)^m,
$$

where $\lambda_{\max} = \sup_{(\gamma,x,u,v) \in \mathbb{M} \times \mathbb{R}^N \times U \times V} |\lambda_\gamma(x, u, v)|$. Then, using the estimates (15), one gets

$$
\begin{aligned}
&\mathbb{E}\left[\delta \int_0^\infty e^{-\delta t} \left| h\left(\gamma_t^{\gamma_0, x_0, u, v}, X_t^{\gamma_0, x_0, u, v}, u_t, v_t\right) - h\left(\gamma_t^{\gamma_0, x_0, u, v}, Y_t^{y_0, w^m}, u_t, w_t^m\right) \right| dt\right] \\
&\leq \mathbb{E}\left[\begin{array}{c} \delta \sum_{i=0}^{m-1} \int_{T_i \wedge T}^{T_{i+1} \wedge T} e^{-\delta t} \left(Lip(h) |x_0 - y_0| + \frac{(Lip(h)+1)c + Lip(h)[t+i(4k_0+1)]}{\sqrt{n}} \right) dt \\ +2h_{\max} 1_{T_m < T} \int_{T_m}^T \delta e^{-\delta t} dt + 2h_{\max} e^{-\delta T} \end{array} \right] \\
&\leq Lip(h) |x_0 - y_0| + \frac{(Lip(h)+1)c + mLip(h)(4k_0+1)}{\sqrt{n}} + \frac{Lip(h)}{\sqrt{n}} \int_0^\infty \delta e^{-\delta t} t dt + 2h_{\max} e^{-\delta T} \\
&+ 2h_{\max} \mathbb{E}\left[\left(e^{-\delta T_m} - e^{-\delta T}\right) 1_{T_m < T}\right] \\
&\leq Lip(h) |x_0 - y_0| + \frac{Lip(h)+1}{\sqrt{n}} \left(c + m(4k_0+1) + \frac{1}{\delta}\right) + 2h_{\max} e^{-\delta T} + 2h_{\max} \mathbb{E}\left[e^{-\delta T_m}\right] \\
&\leq Lip(h) |x_0 - y_0| + \frac{Lip(h)+1}{\sqrt{n}} \left(c + m(4k_0+1) + \frac{1}{\delta}\right) + 2h_{\max} e^{-\delta T} + 2h_{\max} \left(\frac{\lambda_{\max}}{\delta + \lambda_{\max}}\right)^m.
\end{aligned}
$$

The proof of our Lemma is now complete by picking $T = m = n^{\frac{1}{4}}$. ■

7 Appendix

We provide, in this appendix, the key elements of proof leading to Theorem 13. As we have already hinted before, the proof relies on the same arguments as those developed in [40] combined with dynamic programming principles developed in [48].

If one assumes that $\mathbb{K} \subset \overline{B}(0, k_0)$ is convex and invariant w.r.t. the PDMP governed by (f, λ, Q), then one modifies the dynamics such that for $\rho \in \{f, \lambda\}$, $\widetilde{\rho}_\gamma(x, u) = 0$, if $x \notin B(0, k_0 + 1)$, $\widetilde{\rho}_\gamma(x, u) = \rho_\gamma(x, u)$, if $x \in \mathbb{K}$ and setting, for example, $\widetilde{g}_\gamma(\theta, x, u) = \Pi_{\mathbb{K}}(x) - x + g_\gamma(\theta, \Pi_{\mathbb{K}}(x), u)$, for all $x \in \mathbb{R}^N$. Here, $\Pi_{\mathbb{K}}$ stands for the projector onto $\mathbb{K}$. In this way, all jumps $x \mapsto x + \widetilde{g}(\theta, x, u) = \Pi_K(x) + g(\theta, \Pi_K(x), u)$ take the trajectory in $\mathbb{K}$ (by invariance of this set) and, if the trajectory reaches $\overline{B}(0, k_0 + 1)$, it stays constant. For the extended dynamics (constructed from this modification as in Subsection 6.1), one gets

$$
\begin{aligned}
\overline{f}_\gamma\left(x, u^1, u^2\right) &= \widetilde{f}_\gamma\left(x + u^2, u^1\right) = 0, \text{ for all } x \in \mathbb{R}^N \text{ such that } |x| \geq k_0 + 2, \\
x + \overline{g}_\gamma\left(\theta, x, u^1, u^2\right) &= \left[x + u^2 + \widetilde{g}_\gamma\left(\theta, x + u^2, u^1\right)\right] - u^2 \in \mathbb{K} - u^2 \subset \overline{B}(0, k_0 + 1), \text{ for all } x \in \mathbb{R}^N,
\end{aligned}
$$

for all $u^1 \in \mathbb{U}$ and all $\left|u^2\right| \leq 1$.

It follows that the set $\mathbb{K}^+ := \overline{B}(0, k_0+2)$ is invariant w.r.t the extended dynamics. In fact all sets $\overline{B}(0, k_0+n)$, $n \geq 2$ are invariant. Let us emphasize that this construction is the only point in which the convexity of $\mathbb{K}$ plays a part and it can be avoided by further assumptions.

Let us fix, for the time being, $\delta > 0$ and $n \geq 1$.

7.1 Dynamic Programming Principle(s) for (Time) Piecewise Constant Policies

The first ingredient is to provide dynamic programming principles and uniform continuity for the value functions given with respect to piecewise constant policies with respect to the initial and auxiliary systems (cf. Subsection 6.1). In addition to $\mathcal{A}_0^n = \mathcal{A}_0^n(\mathbb{U})$ and $\mathcal{A}_{ad}^n := \mathcal{A}_{ad}^n(\mathbb{U})$, one introduces $\mathcal{B}_0^n = \mathcal{A}_0^n\left(\mathbb{U}\times\overline{B}(0,1)\right)$ and $\mathcal{B}_{ad}^n = \mathcal{A}_{ad}^n\left(\mathbb{U}\times\overline{B}(0,1)\right)$ and

$$v_\varepsilon^{\delta,n}(\gamma,x) = \inf_{(u^1,u^2)\in\mathcal{B}_{ad}^n} \delta\mathbb{E}\left[\int_0^\infty e^{-\delta t} h\left(\Gamma_t^{\gamma,x,u^1,\varepsilon u^2}, X_t^{\gamma,x,u^1,\varepsilon u^2} + \varepsilon u_t^2, u_t^1\right) dt\right],$$

for all initial data $\gamma \in \mathbb{M}$, $x \in \overline{B}(0, k_0+3)$.

One begins with proving the dynamic programming principles.

$$v^{\delta,n}(\gamma,x) = \inf_{u\in\mathcal{A}_0^n} \mathbb{E}\left[\int_0^{T_1\wedge T} \delta e^{-\delta t} h\left(\Gamma_t^{\gamma,x,u,0}, X_t^{\gamma,x,u,0}, u_t\right) dt + e^{-\delta(T_1\wedge T)} v^{\delta,n}\left(\Gamma_{T_1\wedge T}^{\gamma x,u,0}, X_{T_1\wedge T}^{\gamma x,u,0}\right)\right] \text{ and}$$

$$v_\varepsilon^{\delta,n}(\gamma,x) = \inf_{u\in\mathcal{B}_0^n} \mathbb{E}\left[\begin{array}{c} \int_0^{T_1\wedge T} \delta e^{-\delta t} h\left(\Gamma_t^{\gamma,x,u^1,\varepsilon u^2}, X_t^{\gamma,x,u^1,\varepsilon u^2} + \varepsilon u_t^2, u_t^1\right) dt \\ +e^{-\delta(T_1\wedge T)} v_\varepsilon^{\delta,n}\left(\Gamma_{T_1\wedge T}^{\gamma,x,u^1,\varepsilon u^2}, X_{T_1\wedge T}^{\gamma,x,u^1,\varepsilon u^2}\right) \end{array}\right].$$

The arguments are similar to those employed in [48]. We will only emphasize what changes when using controls from $\mathcal{A}_{ad}^n$ (or $\mathcal{B}_{ad}^n$) instead of the (more) classical $\mathcal{A}_{ad}$.

Following [48], we introduce

$$w^{M,n}(\gamma,x) := \inf_{u\in\mathcal{A}_0^n} J^{M,n}(\gamma,x,u),$$

where

$$J^{M,n}(\gamma,x,u) := \mathbb{E}\left[\delta\int_0^{T_1} e^{-\delta t} h\left(\Gamma_t^{\gamma,x,u,0}, X_t^{\gamma,x,u,0}, u_t\right) dt + e^{-\delta T_1} w^{M-1,n}\left(\Gamma_{T_1}^{\gamma,x,u,0}, X_{T_1}^{\gamma,x,u,0}\right)\right],$$

whenever $M \geq 1$. The initial value $w^{0,n}$ is given with respect to the deterministic control problem (with no jump) and it is standard to check that it is Hölder continuous (the Hölder exponent may be chosen $\frac{\delta}{Lip(f)}$, where $Lip(f)$ is the Lipschitz constant for f_γ for all $\gamma \in \mathbb{M}$ and the Hölder constant only depends on the Lipschitz constants and supremum norm of f and h). In particular, the continuity modulus of $w^{0,n}$ (resp. $w_\varepsilon^{0,n}$ defined w.r.t. $\mathcal{B}_0^n$) is independent of n (resp. n and ε).

Step 1. If $w^{M-1,n} \in BUC\left(\mathbb{M}\times\mathbb{R}^N;\mathbb{R}\right)$, then the dynamic programming principle holds true for $w^{M,n}$ and all $(\gamma,x) \in \mathbb{M}\times\mathbb{R}^N$, $T \in n^{-1}\mathbb{N}$:

$$w^{M,n}(\gamma,x) = \inf_{u\in\mathcal{A}_0^n} \mathbb{E}\left[\begin{array}{c} \delta\int_0^{T_1\wedge T} e^{-\delta t} h\left(\Gamma_t^{\gamma,x,u,0}, X_t^{\gamma,x,u,0}, u_t\right) dt + e^{-\delta T} w^{M,n}\left(\gamma, \Phi_T^{0,x,u;\gamma}\right) 1_{T_1>T} \\ +e^{-\delta T_1} w^{M-1,n}\left(\Gamma_{T_1}^{\gamma,x,u,0}, X_{T_1}^{\gamma,x,u,0}\right) 1_{\tau_1\leq T} \end{array}\right].$$

The proof is identical with the proof of [48, Lemma 3.1]. The reader needs only note that the control policy given by [48, Eq. (3.5)] of the form

$$\overline{u}(\theta,y,t) := u(\theta,y,t)\, 1_{[0,T]}(t) + u^*\left(\theta, \Phi_T^{0,y,u;\theta}, t-T\right) 1_{t>T}$$

belongs to $\mathcal{A}_0^n$ if u and u^* belong to $\mathcal{A}_0^n$.

Step 2. Since $\overline{B}(0,k_0+3)$ is invariant with respect to the extended PDMP, one has $w^{M,n} \in BUC\left(\mathbb{M}\times\overline{B}(0,k_0+3)\right)$ and, for every $\alpha > 0$, there exists a $\alpha-$optimal control policy $u^* \in \mathcal{A}_0^n$ such that

$$J^{M,n}(\gamma,x,u^*) \leq w^{M,n}(\gamma,x) + \alpha,$$

for all $x \in \overline{B}(0,k_0+3)$.

Again, the proof is identical with the proof of the analogous Lemma 3.3 in [48] and based on recurrence. The reader needs only note that, for $r > 0$, there exists a finite family $\{x_k : k = 1,m\}$ such that

$$\overline{B}(0,k_0+3) \subset \bigcup_{k=1}^{m} B(x_k,r).$$

Then, the control policy u defined after (3.18) in [48] belongs to $\mathcal{A}_0^n$ if u_k belong to $\mathcal{A}_0^n$, for all $k = 1,m$. We also wish to point out that the estimates leading to the continuity modulus of $w^{M,n}$ only depend on the Lipschitz constants and the supremum of h, f, g and λ but are independent of the control policies. In particular, this allows one to work with a common continuity modulus $\omega^{\delta,M}$ for all $n \geq 1$ and $\varepsilon > 0$.

One concludes using the same arguments (no particular changes needed) as those in [48, Theorem 3.4]. Due to [48, Inequality 3.27], one gets

$$\sup_{(\gamma,x)\in\mathbb{M}\times\overline{B}(0,k_0+3)} \left|v^{\delta,n}(\gamma,x) - w^{M,n}(\gamma,x)\right| \leq c\alpha^M,$$

where $c > 0$ and $0 < \alpha < 1$ are independent of n (c can be chosen as in [48] equal to $2f_{\max}$ and α as in [48, Page 1120, last line] to be $1 - \delta\int_0^\infty e^{-(\delta+\lambda_{\max})t}dt = \frac{\lambda_{\max}}{\delta+\lambda_{\max}} < 1$). The same is true for $v_\varepsilon^{\delta,n} - w_\varepsilon^{M,n}$ for $\varepsilon > 0$. In particular,

$$\left|v_\varepsilon^{\delta,n}(\gamma,x) - v_\varepsilon^{\delta,n}(\gamma,y)\right| \leq \omega^{\delta,M}(|x-y|) + 2c\left(\frac{\lambda_{\max}}{\delta+\lambda_{\max}}\right)^M,$$

i.e. the continuity modulus of $v_\varepsilon^{\delta,n}$ can also be chosen independent of $n \geq 1$ and $\varepsilon \geq 0$ (we identify $v_0^{\delta,n}$ with $v^{\delta,n}$). This common continuity modulus will be denoted by ω^δ, i.e.

$$\omega^\delta(r) = \sup_{n\geq 1,\varepsilon>0} \sup_{\substack{\gamma\in\mathbb{M}\\ |x-y|\leq r}} \left|v_\varepsilon^{\delta,n}(\gamma,x) - v_\varepsilon^{\delta,n}(\gamma,x)\right|,\ r > 0,\ \omega^\delta(0) := \lim_{\substack{r\to 0\\ r>0}} \omega^\delta(r) = 0. \tag{16}$$

The reader will note that $\omega^\delta(r) \geq cr$, for some $c > 0$, where the equality corresponds to the Lipschitz case.

7.2 Estimates and Proof of Theorem 13

We begin with the following convergence result.

Proposition 17 *For every $\delta > 0$ there exists a decreasing function $\eta^\delta : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ that satisfies $\lim_{\varepsilon\to 0}\eta^\delta(\varepsilon) = 0$ and such that*

$$\sup_{x\in\mathbb{K}^+} \left|v_\varepsilon^{\delta,n}(\gamma,x) - v^{\delta,n}(\gamma,x)\right| \leq \eta^\delta(\varepsilon), \tag{17}$$

for all $n \geq 1$ and all $\varepsilon \geq 0$.

Proof. The proof is similar to the one of [33, Theorem 3.6]. However, we present the arguments for reader's sake. Let us fix $\gamma \in \mathbb{M}$, $x \in \mathbb{K}^+$ and $\varepsilon > 0$. The definition of the value functions implies

that $v_{\varepsilon}^{\delta,n}(\gamma,x) \leq v^{\delta,n}(\gamma,x)$. Standard estimates yield the existence of some positive constant $C>0$ which is independent of γ, x, of $n \geq 1$ and $\varepsilon > 0$ such that

$$\left|\Phi_t^{0,x,u^1,\varepsilon u^2;\gamma} - \Phi_t^{0,x,u^1,0;\gamma}\right| \leq C\varepsilon, \tag{18}$$

for all $t \in [0,1]$, and all $\left(u^1,u^2\right) \in \mathcal{B}_0^n$. We recall that $\Phi_\cdot^{0,x,u^1,u^2;\gamma}$ is the unique solution of the deterministic equation

$$\begin{cases} d\Phi_t^{0,x,u^1,u^2;\gamma} = \overline{f}_\gamma\left(\Phi_t^{0,x,u^1,\varepsilon u^2;\gamma}, u_t^1, u_t^2\right)dt = f_\gamma\left(\Phi_t^{0,x,u^1,u^2;\gamma} + u_t^2, u_t^1\right)dt, \\ \Phi_0^{0,x,u^1,u^2;\gamma} = x. \end{cases}$$

The constant C in (18) is generic and may change from one line to another. We emphasize that throughout the proof, C may be chosen independent of $x \in \mathbb{R}^N$, $n \geq 1$, $\varepsilon > 0$ and of $\left(u^1,u^2\right) \in \mathcal{B}_0^n$ (it only depends on Lipschitz constants and bounds of f, λ, g and h). The dynamic programming principle written for $v^{\delta,n}$ yields

$$v^{\delta,n}(\gamma,x) \leq \mathbb{E}\left[\int_0^{T_1\wedge 1} \delta e^{-\delta t} h\left(\Gamma_t^{\gamma,x,u^1,0}, X_t^{\gamma,x,u^1,0}, u_t^1\right)dt + e^{-\delta(T_1\wedge 1)} v^{\delta,n}\left(\Gamma_{T_1\wedge 1}^{\gamma,x,u^1,0}, X_{T_1\wedge 1}^{\gamma,x,u^1,0}\right)\right], \tag{19}$$

for all $u^1 \in \mathcal{A}_0^n$. We consider an arbitrary admissible control couple $\left(u^1,u^2\right) \in \mathcal{B}_0^n$. For simplicity, we introduce the following notations:

$$\begin{aligned} u_t^i &= u^i(x,t), \ i=1,2, \\ \lambda^1(t) &= \lambda_\gamma\left(\Phi_t^{0,x,u^1,0;\gamma}, u_t^1\right), \ \Lambda^1(t) = \exp\left(-\int_0^t \lambda^1(s)\,ds\right) \\ \lambda^{1,2}(t) &= \lambda_\gamma\left(\Phi_t^{0,x,u^1,\varepsilon u^2;\gamma} + \varepsilon u_t^2, u_t^1\right), \ \Lambda^{1,2}(t) = \exp\left(-\int_0^t \lambda^{1,2}(s)\,ds\right), \end{aligned}$$

for all $t \geq 0$. We denote the right-hand member of the inequality (19) by I. Then, I is explicitly given by

$$\begin{aligned} I &= \int_0^1 \lambda^1(t)\Lambda^1(t)\int_0^t \delta e^{-\delta s} h\left(\gamma, \Phi_s^{0,x,u^1,0;\gamma}, u_s^1\right)dsdt \\ &+ \int_0^1 \lambda^1(t)\Lambda^1(t)\, e^{-\delta t}\int_{\mathbb{R}^N} v^{\delta,n}\left(\theta, \Phi_t^{0,x,u^1,0;\gamma} + g_\gamma\left(\theta, \Phi_t^{0,x,u^1,0;\gamma}, u_t^1\right)\right) Q^0\left(\gamma, u_t^1, d\theta\right)dt \\ &+ \Lambda^1(1)\int_0^1 \delta e^{-\delta t} h\left(\gamma, \Phi_t^{0,x,u^1,0;\gamma}, u_t^1\right)dt + \Lambda^1(1)\, e^{-\delta} v^{\delta,n}\left(\gamma, \Phi_1^{0,x,u^1,0;\gamma}\right) \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned}$$

Using the inequality (18), one gets

$$I_1 \leq \int_0^1 \lambda^{1,2}(t)\Lambda^{1,2}(t)\int_0^t \delta e^{-\delta s} h\left(\gamma, \Phi_s^{0,x,u^1,\varepsilon u^2;\gamma} + \varepsilon u_s^2, u_s^1\right)dsdt + C\varepsilon, \tag{20}$$

$$I_3 \leq \Lambda^{1,2}(1)\int_0^1 \delta e^{-\delta t} h\left(\gamma, \Phi_t^{0,x,u^1,\varepsilon u^2;\gamma} + \varepsilon u_t^2, u_t^1\right)dt + C\varepsilon. \tag{21}$$

For the term I_2, with the notation (16), one has

$$
\begin{aligned}
&I_2\\
&\leq \int_0^1 \lambda^{1,2}(t)\Lambda^{1,2}(t)\, e^{-\delta t} \int_{\mathbb{R}^N} v^{\delta,n}\left(\theta, \Phi_t^{0,x,u^1,\varepsilon u^2;\gamma} + g_\gamma\left(\theta, \Phi_t^{0,x,u^1,\varepsilon u^2;\gamma} + \varepsilon u_t^2, u_t^1\right)\right) Q^0\left(\gamma, u_t^1, d\theta\right) dt\\
&+ C\left(\varepsilon + \omega^\delta\left(C\varepsilon\right)\right)\\
&\leq \int_0^1 \lambda^{1,2}(t)\Lambda^{1,2}(t)\, e^{-\delta t} \int_{\mathbb{R}^N} v_\varepsilon^{\delta,n}\left(\theta, \Phi_t^{0,x,u^1,\varepsilon u^2;\gamma} + g_\gamma\left(\theta, \Phi_t^{0,x,u^1,\varepsilon u^2;\gamma} + \varepsilon u_t^2, u_t^1\right)\right) Q^0\left(\gamma, u_t^1, d\theta\right) dt\\
&+ \left(\int_0^1 \lambda^{1,2}(t)\Lambda^{1,2}(t)\, e^{-\delta t} dt\right) \sup_{\theta\in\mathbb{M}, z\in\mathbb{K}^+} \left|v^{\delta,n}(z) - v_\varepsilon^{\delta,n}(z)\right| + C\left(\varepsilon + \omega^\delta\left(C\varepsilon\right)\right). \qquad (22)
\end{aligned}
$$

Finally,

$$
\begin{aligned}
&I_4 \leq \Lambda^{1,2}(1)\, e^{-\delta} v^{\delta,n}\left(\gamma, \Phi_1^{0,x,u^1,\varepsilon u^2;\gamma}\right) + C\left(\varepsilon + \omega^\delta\left(C\varepsilon\right)\right)\\
&\leq \Lambda^{1,2}(1) e^{-\delta} v_\varepsilon^{\delta,n}\left(\gamma, \Phi_1^{0,x,u^1,\varepsilon u^2;\gamma}\right) + \Lambda^{1,2}(1) e^{-\delta} \sup_{\theta\in\mathbb{M}, z\in\mathbb{K}^+} \left|v^{\delta,n}(\theta, z) - v_\varepsilon^{\delta,n}(\theta, z)\right| + C\left(\omega^\delta\left(C\varepsilon\right) + \varepsilon\right). \qquad (23)
\end{aligned}
$$

We substitute (20)-(23) in (19). We take the infimum over the family of $\left(u^1, u^2\right) \in \mathcal{B}_0^n$ and use the dynamic programming principle to have

$$
\begin{aligned}
v^{\delta,n}(\gamma, x) &\leq v_\varepsilon^{\delta,n}(\gamma, x) + C\left(\varepsilon + \omega^\delta\left(C\varepsilon\right)\right)\\
&+ \left(\int_0^1 \lambda^{1,2}(t)\Lambda^{1,2}(t)\, e^{-\delta t} dt + \Lambda^{1,2}(1) e^{-\delta}\right) \sup_{\theta\in\mathbb{M}, z\in\mathbb{K}^+} \left|v^{\delta,n}(z) - v_\varepsilon^{\delta,n}(z)\right|.
\end{aligned}
$$

We notice that

$$
\begin{aligned}
\int_0^1 \lambda^{1,2}(t)\Lambda^{1,2}(t)\, e^{-\delta t} dt + \Lambda^{1,2}(1) e^{-\delta} &= 1 - \delta \int_0^1 e^{-\int_0^t \overline{\lambda}_\gamma\left(\Phi_s^{0,x,u^1,\varepsilon u^2;\gamma}, u_s^1, \varepsilon u_s^2\right) ds} e^{-\delta t} dt\\
&\leq \frac{\lambda_{\max}}{\lambda_{\max} + \delta} + \frac{\delta}{\lambda_{\max} + \delta} e^{-(\lambda_{\max} + \delta)}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
v^{\delta,n}(\gamma, x) - v_\varepsilon^{\delta,n}(\gamma, x) &\leq C\left(\varepsilon + \omega^\delta\left(C\varepsilon\right)\right)\\
&+ \left(\frac{\lambda_{\max}}{\lambda_{\max} + \delta} + \frac{\delta}{\lambda_{\max} + \delta} e^{-(\lambda_{\max} + \delta)}\right) \sup_{\theta\in\mathbb{M}, z\in\mathbb{R}^N} \left|v^{\delta,n}(\theta, z) - v_\varepsilon^{\delta,n}(\theta, z)\right|.
\end{aligned}
$$

Here, $\lambda_{\max} := \sup\left\{\lambda_\gamma(x, u) : \gamma \in \mathbb{M}, x \in \mathbb{R}^N, u \in \mathbb{U}\right\} < \infty$. The conclusion follows by taking the supremum over $\theta \in \mathbb{M}$ and $x \in \mathbb{K}^+$ and recalling that C is independent of x and $\varepsilon > 0$ (and $n \geq 1$).
■

We consider (ρ_ε) a sequence of standard mollifiers i.e. $\rho_\varepsilon(y) = \frac{1}{\varepsilon^N}\rho\left(\frac{y}{\varepsilon}\right)$, $y \in \mathbb{R}^N$, $\varepsilon > 0$, where $\rho \in C^\infty\left(\mathbb{R}^N\right)$ is a positive function such that

$$
Supp(\rho) \subset \overline{B}(0, 1) \text{ and } \int_{\mathbb{R}^N} \rho(x) dx = 1.
$$

We introduce the convoluted functions

$$
v_{(\varepsilon)}^{\delta,n}(\gamma, \cdot) := v_\varepsilon^{\delta,n}(\gamma, \cdot) * \rho_\varepsilon.
$$

In analogy to [40, Lemma 3.5], one gets

Proposition 18 *The value functions $v_{(\varepsilon)}^{\delta,n}$ are such that*

$$\begin{cases} \sup\limits_{x\in\mathbb{K}} \left(\left| v_{(\varepsilon)}^{\delta,n}(\gamma,x) \right| + \left| \partial_{x^i} v_{(\varepsilon)}^{\delta,n}(\gamma,x) \right| \right) \leq C^\delta \varepsilon^{-1} \left(\varepsilon + \omega^\delta(\varepsilon) \right), \\ \sup\limits_{x,y\in\mathbb{K},\ y\neq x} \frac{\left| \partial_{x^i} v_{(\varepsilon)}^{\delta,n}(\gamma,x) - \partial_{x^i} v_{(\varepsilon)}^{\delta,n}(\gamma,y) \right|}{|x-y|} \leq C^\delta \varepsilon^{-1} \omega^\delta(|x-y|) \\ \sup\limits_{x\in\mathbb{K}} \left| v_{(\varepsilon)}^{\delta,n}(\gamma,x) - v^{\delta,n}(\gamma,x) \right| \leq \omega^\delta(\varepsilon) + \eta^\delta(\varepsilon), \end{cases}$$

for all $\gamma \in \mathbb{M}$. Here, C^δ is a positive constant independent of $\varepsilon > 0$, $n \geq 1$ and $\gamma \in \mathbb{M}$.

Proof. To prove the first inequality, one recalls the definition of $v_{(\varepsilon)}^{\delta,n}$. Then, due to Proposition 17 and using the notation (16), one gets

$$\begin{aligned} \left| \partial_{x^i} v_{(\varepsilon)}^{\delta,n}(x) \right| &= \left| \varepsilon^{-1} \int_{\overline{B}(0,1)} v_\varepsilon^{\delta,n}(x-\varepsilon y)\, \partial_{x^i}\rho(y)\, dy \right| = \left| \varepsilon^{-1} \int\limits_{y\in\overline{B}(0,1)} \left(v_\varepsilon^{\delta,n}(x-\varepsilon y) - v_\varepsilon^{\delta,n}(x) \right) \partial_{x^i}\rho(y)\, dy \right| \\ &\leq C^\delta \varepsilon^{-1} \omega^\delta(\varepsilon). \end{aligned}$$

Similarly,

$$\left| \partial_{x^i} v_{(\varepsilon)}^{\delta,n}(x) - \partial_{x^i} v_{(\varepsilon)}^{\delta,n}(y) \right| = \left| \varepsilon^{-1} \int_{\overline{B}(0,1)} \left(v_\varepsilon^{\delta,n}(x-\varepsilon z) - v_\varepsilon^{\delta,n}(y-\varepsilon z) \right) \partial_{x^i}\rho(z)\, dy \right| \leq C^\delta \varepsilon^{-1} \omega^\delta(|x-y|).$$

Moreover, again with the notation (16) and the help of Proposition 17, one gets

$$\begin{aligned} \sup_{x\in\mathbb{R}^N} \left| v_{(\varepsilon)}^{\delta,n}(x) - v^{\delta,n}(x) \right| &= \left| \int\limits_{y\in\overline{B}(0,1)} \left(v_\varepsilon^{\delta,n}(x-\varepsilon y) - v_\varepsilon^{\delta,n}(x) + v_\varepsilon^{\delta,n}(x) - v^{\delta,n}(x) \right) \rho(y)\, dy \right| \\ &\leq \omega^\delta(\varepsilon) + \eta^\delta(\varepsilon). \end{aligned}$$

The proof of our proposition is now complete. ■

We now come to the proof of the main convergence result.

Proof. (of Theorem 13). Let us fix $\left(u^1, u^2\right) \in \mathbb{U} \times \overline{B}(0,1)$ .The dynamic programming principle written for $v_\varepsilon^{\delta,n}$ yields

$$v_\varepsilon^{\delta,n}(\gamma,x) \leq \mathbb{E} \left[\begin{array}{c} \int_0^{T_1\wedge n^{-1}} \delta e^{-\delta t} h\left(\Gamma_t^{\gamma,x,u^1,\varepsilon u^2}, X_t^{\gamma,x,u^1,\varepsilon u^2} + \varepsilon u_t^2, u_t^1 \right) dt \\ + e^{-\delta\left(T_1\wedge n^{-1}\right)} v_\varepsilon^{\delta,n}\left(\Gamma_{T_1\wedge n^{-1}}^{\gamma,x,u^1,\varepsilon u^2}, X_{T_1\wedge n^{-1}}^{\gamma x,u^1,\varepsilon u^2} \right) \end{array} \right],$$

where $\left(u^1, u^2\right) \in \mathcal{B}_0^n$ and $x \in \mathbb{K}$. In particular, if $\left(u_t^1, u_t^2\right) = (u,y) \in \mathbb{U} \times \overline{B}(0,1)$, for $t \in \left[0, n^{-1}\right)$, one notices that on $\left[0, T_1 \wedge n^{-1}\right)$,

$$X_t^{\gamma,x-\varepsilon y,u^1,\varepsilon u^2} = X_t^{\gamma,x,u^1,0} - \varepsilon y \text{ and } \Gamma_t^{\gamma,x,u^1,\varepsilon u^2} = \Gamma_t^{\gamma,x,u^1,0}.$$

As consequence, the (law of the) first jump time starting from $(\gamma, x-\varepsilon y)$ when the trajectory is controlled by the couple $\left(u^1, \varepsilon u^2\right)$ given above only depends on u and x (but not on ε, nor on y). To emphasize this dependence, we denote it by $T_1^{x,u}$. Similarly, $\left(\Gamma_{T_1^{x,u}\wedge n^{-1}}^{\gamma,x,u^1,\varepsilon u^2}, X_{T_1^{x,u}\wedge n^{-1}}^{\gamma,x,u^1,\varepsilon u^2} \right)$ has the same law as $\left(\Gamma_{T_1^{x,u}\wedge n^{-1}}^{\gamma,x,u^1,0}, X_{T_1^{x,u}\wedge n^{-1}}^{\gamma,x,u^1,0} - \varepsilon y \right)$. On gets

$$v_\varepsilon^{\delta,n}(\gamma, x-\varepsilon y) \leq \mathbb{E} \left[\begin{array}{c} \int_0^{T_1^{x,u}\wedge n^{-1}} \delta e^{-\delta t} h\left(\Gamma_t^{\gamma,x,u^1,0}, X_t^{\gamma,x,u^1,0}, u_t^1 \right) dt \\ + e^{-\delta\left(T_1^{x,u}\wedge n^{-1}\right)} v_\varepsilon^{\delta,n}\left(\Gamma_{T_1^{x,u}\wedge n^{-1}}^{\gamma,x,u^1,0}, X_{T_1^{x,u}\wedge n^{-1}}^{\gamma,x,u^1,0} - \varepsilon y \right) \end{array} \right].$$

It follows that

$$
\begin{aligned}
& v_{(\varepsilon)}^{\delta,n}(\gamma,x) \\
& \leq \mathbb{E}\left[\int_0^{T_1^{x,u}\wedge n^{-1}} \delta e^{-\delta t} h\left(\Gamma_t^{\gamma,x,u^1,0}, X_t^{\gamma,x,u,0}, u_t^1\right) dt + e^{-\delta\left(T_1^{x,u}\wedge n^{-1}\right)} v_{(\varepsilon)}^{\delta,n}\left(\Gamma_{T_1^{x,u}\wedge n^{-1}}^{\gamma,x,u,0}, X_{T_1^{x,u}\wedge n^{-1}}^{\gamma,x,u,0}\right)\right].
\end{aligned}
$$

Applying Itô's formula to $v_{(\varepsilon)}^{\delta,n}\left(\Gamma_t^{\gamma,x,u,0}, X_t^{\gamma,x,u,0}\right)$ on $\left[0, T_1^{x,u}\wedge n^{-1}\right]$ and recalling that $u_t^1 = u$ prior to n^{-1}, it follows that

$$
\begin{aligned}
0 &\leq \mathbb{E}\left[\int_0^{T_1^{x,u}\wedge n^{-1}} e^{-\delta t}\left[\begin{array}{c} \delta\left(h\left(\Gamma_t^{\gamma,x,u,0}, X_t^{\gamma,x,u,0}, u\right) - v_{(\varepsilon)}^{\delta,n}\left(\Gamma_t^{\gamma,x,u,0}, X_t^{\gamma x,u,0}\right)\right) \\ +\mathcal{L}^u v_{(\varepsilon)}^{\delta,n}\left(\Gamma_t^{\gamma,x,u,0}, X_t^{\gamma,x,u,0}\right)\end{array}\right]\right] dt \\
&= \mathbb{E}\left[\int_0^{T_1^{x,u}\wedge n^{-1}} e^{-\delta t}\left[\begin{array}{c} \delta\left(h\left(\gamma, \Phi_t^{0,x,u,0;\gamma}, u\right) - v_{(\varepsilon)}^{\delta,n}\left(\gamma, \Phi_t^{0,x,u,0;\gamma}\right)\right) \\ +\mathcal{L}^u v_{(\varepsilon)}^{\delta,n}\left(\gamma, \Phi_t^{0,x,u^1,0;\gamma}\right)\end{array}\right]\right] dt \\
&\leq \mathbb{E}\left[\int_0^{T_1^{x,u}\wedge n^{-1}} e^{-\delta t} dt\right]\left(\delta\left(h(\gamma,x,u) - v_{(\varepsilon)}^{\delta,n}(\gamma,x)\right) + \mathcal{L}^u v_{(\varepsilon)}^{\delta,n}(\gamma,x)\right) \\
&+ \mathbb{E}\left[\int_0^{T_1^{x,u}\wedge n^{-1}} e^{-\delta t}\left[\begin{array}{c} \delta\left(|h|_1 + C^\delta \varepsilon^{-1}\left(\varepsilon + \omega^\delta(\varepsilon)\right)\right)|f|_1 t + |f|_1^2 t C^\delta \varepsilon^{-1}\left(\varepsilon + \omega^\delta(\varepsilon)\right) \\ + |f|_1 C^\delta \varepsilon^{-1}\omega^\delta\left(\frac{|f|_1}{n}\right) + 2|h|_1 |\lambda|_1 |f|_1 t \\ + |\lambda|_1 C^\delta \varepsilon^{-1}\left(\varepsilon + \omega^\delta(\varepsilon)\right)(1 + |g|_1)|f|_1 t\end{array}\right] dt\right] \\
&\leq \mathbb{E}\left[T_1^{x,u}\wedge n^{-1}\right]\left(\left[\delta\left(h(\gamma,x,u) - v_{(\varepsilon)}^{\delta,n}(\gamma,x)\right) + \mathcal{L}^u v_{(\varepsilon)}^{\delta,n}(\gamma,x)\right]\right) \\
&+ \mathbb{E}\left[T_1^{x,u}\wedge n^{-1}\right]\widetilde{C}^\delta\left(\frac{1+\varepsilon^{-1}\omega^\delta(\varepsilon)}{n} + \varepsilon^{-1}\omega^\delta\left(\frac{|f|_1}{n}\right)\right).
\end{aligned}
$$

The generic constant $\widetilde{C}^\delta$ is independent of $x \in \mathbb{K}, \gamma \in \mathbb{M}, u \in \mathbb{U}, \varepsilon > 0$ and $n \geq 1$ and may change from one line to another. As consequence,

$$
\delta\left(h(\gamma,x,u) - v_{(\varepsilon)}^{\delta,n}(\gamma,x)\right) + \mathcal{L}^u v_{(\varepsilon)}^{\delta,n}(\gamma,x) \geq -\widetilde{C}^\delta\left(\frac{1+\varepsilon^{-1}\omega^\delta(\varepsilon)}{n} + \varepsilon^{-1}\omega^\delta\left(\frac{|f|_1}{n}\right)\right).
$$

We fix (for the time being), the initial configuration $(\gamma_0, x_0) \in \mathbb{M}\times\mathbb{K}$ and an arbitrary control $u^1 \in \mathcal{A}_{ad}$. We apply the previous inequality for $(\gamma, x) = \left(\Gamma_t^{\gamma_0,x_0,u^1,0}, X_t^{\gamma_0,x_0,u^1,0}\right)$, integrate the inequality with respect to $e^{-\delta t}dt$ on $[0,T]$ for $T > 0$ and use Itô's formula to get

$$
\begin{aligned}
\mathbb{E}\left[\int_0^T \delta e^{-\delta t} h\left(\Gamma_t^{\gamma_0,x_0,u^1,0}, X_t^{\gamma_0,x_0,u^1,0}, u_t^1\right) dt\right] &\geq v_{(\varepsilon)}^{\delta,n}(\gamma_0,x_0) - e^{-\delta T} v_{(\varepsilon)}^{\delta,n}\left(\Gamma_T^{\gamma_0,x_0,u^1,0}, X_T^{\gamma_0,x_0,u^1,0}\right) \\
&- \widetilde{C}^\delta\left(\frac{1+\varepsilon^{-1}\omega^\delta(\varepsilon)}{n} + \varepsilon^{-1}\omega^\delta\left(\frac{|f|_1}{n}\right)\right).
\end{aligned}
$$

One lets $T \to \infty$ and takes the infimum over $u^1 \in \mathcal{A}_{ad}$ to get

$$
v^\delta(\gamma_0,x_0) \geq v_{(\varepsilon)}^{\delta,n}(\gamma_0,x_0) - \widetilde{C}^\delta\left(\frac{1+\varepsilon^{-1}\omega^\delta(\varepsilon)}{n} + \varepsilon^{-1}\omega^\delta\left(\frac{|f|_1}{n}\right)\right).
$$

Finally, using the third estimate in Proposition 18, one gets

$$
v^\delta(\gamma_0,x_0) \geq v^{\delta,n}(\gamma_0,x_0) - \widetilde{C}^\delta\left(\frac{1+\varepsilon^{-1}\omega^\delta(\varepsilon)}{n} + \varepsilon^{-1}\omega^\delta\left(\frac{|f|_1}{n}\right) + \omega^\delta(\varepsilon) + \eta^\delta(\varepsilon).\right).
$$

The conclusion follows by taking $\varepsilon = \left(\omega^\delta\left(\frac{|f|_1}{n}\right)\right)^{1-\eta}$, for some $1 > \eta > 0$ (e.g. $\eta = \frac{1}{2}$). Our result is now complete. ■

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