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Asymptotic Control for a Class of Piecewise Deterministic Markov Processes Associated to Temperate Viruses

Dan Goreac [†]

Abstract

We aim at characterizing the asymptotic behavior of value functions in the control of piecewise deterministic Markov processes (PDMP) of switch type under nonexpansive assumptions. For a particular class of processes inspired by temperate viruses, we show that uniform limits of discounted problems as the discount decreases to zero and time-averaged problems as the time horizon increases to infinity exist and coincide. The arguments allow the limit value to depend on initial configuration of system and do not require dissipative properties on the dynamics. The approach strongly relies on viscosity techniques, linear programming arguments and coupling via random measures associated to PDMP. As an intermediate step in our approach, we present the approximation of discounted value functions when using piecewise constant (in time) open-loop policies.

AMS Classification: 49L25, 60J25, 93E20, 92C42

1 Introduction

We focus on the study of some asymptotic properties in the control of a particular family of piecewise deterministic Markov processes (abbreviated PDMP), non diffusive, jump processes introduced in the seminal paper [24]. Namely, we are concerned with the existence of a limit of the value functions minimizing the Cesàro-type average of some cost functional as the time increases to infinity for controlled switch processes. The main theoretical contribution of the paper is that the arguments in our proofs are entirely independent on dissipativity properties of the PDMP and they apply under mild nonexpansivity assumptions. Concerning the potential applications, our systems are derived from the theory of stochastic gene networks (and, in particular, genetic applets modelling temperate viruses). Readers wishing to get acquainted to biological or mathematical aspects in these models are referred to [16], [37], [23], [22], [33]).

The switch processes can be described by a couple $(\gamma^{\gamma_0, x_0, u}, X^{\gamma_0, x_0, u})$, where the first component is a pure jump process called mode and taking its values in some finite set \mathbb{M} . The couple process is governed by a jump rate and a transition measure, both depending on the current state of the system. Between consecutive jumps, $X^{\gamma_0, x_0, u}$ evolves according to some mode-dependent flow. Finally, these characteristics (rate, measure, flow) depend on an external control parameter u . Precise assumptions and construction make the object of Section 2. In connection to these jump systems, we consider the Abel-type (resp. Cesàro-type) average

$$v^\delta(\gamma_0, x_0) := \inf_u \delta \mathbb{E} \left[\int_0^\infty e^{-\delta t} h_{\gamma_t^{\gamma_0, x_0, u}}(X_t^{\gamma_0, x_0, u}, u_t) dt \right],$$
$$V_T(\gamma_0, x_0) := \inf_u \frac{1}{T} \mathbb{E} \left[\int_0^T h_{\gamma_t^{\gamma_0, x_0, u}}(X_t^{\gamma_0, x_0, u}, u_t) dt \right],$$

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and investigate the existence of limits as the discount parameter $\delta \rightarrow 0$, respectively the time horizon $T \rightarrow \infty$.

In the context of sequences of real numbers, the first result connecting asymptotic behavior of Abel and Cesàro means goes back to Hardy and Littlewood in [36]. Their result has known several generalizations: to uncontrolled deterministic dynamics in [29, XIII.5], to controlled deterministic systems in [2], [3], etc.

Ergodic behavior of systems and asymptotics of Cesàro-type averages have made the object of several papers dealing with either deterministic or stochastic control systems. The partial differential system approach originating in [42] relies on coercitivity of the associated Hamiltonian (see also [5] for explicit criteria). Although the method generalizes to deterministic (resp. Brownian) control systems in [3] (resp. [4]), the main drawback resides in the fact that, due to the ergodic setting, the limit is independent on the initial condition of the control system. Another approach to the asymptotic behavior relies on estimations on trajectories available under controllability and dissipativity assumptions. The reader is referred to [5], [10] for the deterministic setting or [8], [11], [15], [46] for Brownian systems. Although the method is different, it presents the same drawback as the PDE one: it fails to give general limit value functions that depend on the initial data.

In the context of piecewise deterministic Markov processes, the infinite-horizon optimal control literature is quite extensive ([25], [48], [27], [1], [30], etc.). To our best knowledge, average control problems have first been considered in an impulsive control framework in [17] and [31]. The first papers dealing with long time average costs in the framework of continuous control policies are [19] and [18] (see also [21]).

A nonexpansivity condition has been used in [44] in connection to deterministic control systems allowing to obtain the existence of a general (uniform) limit value function. This method has been (partially) extended to Brownian control systems in [14]. In both these papers, convenient estimates on the trajectories in finite horizon allow to prove the uniform continuity of Cesàro averages V_T and an intuition coming from repeated games theory (inspired by [45]) gives the candidate for the limit value function. If the convergence to this limit value function is uniform, the results of [43] for deterministic systems yield the equivalence between Abel and Cesàro long-time averages. This latter assertion is still valid for controlled Brownian diffusions (see [14, Theorems 10 and 13]) and (to some extent) for piecewise deterministic Markov processes (see [32, Theorem 4.1]).

In the present paper, we generalize the results of [44] and [14] to the framework of switch piecewise deterministic Markov processes. The methods are based on viscosity solutions arguments. We deal with two specific problems. The keypoint is, as for Brownian systems, a uniform continuity of average value functions with respect to the average parameter (δ or T). However, the approach in [14] benefits from dynamic programming principles, which, within the framework of PDMP, are easier obtained for Abel means (discounted functions v^δ). This is why, results like [14, Proposition 7 and Theorem 8] are not directly applicable and we cannot make use of the already mentioned intuition on repeated games. To overcome this problem, we proceed as follows : if the system admits an invariant compact set, we prove the uniform continuity of $(v^\delta)_{\delta>0}$ and use the results in [32, Theorem 4.1] to show (see Theorem 8) that this family admits an unique adherent point with respect to the topology of continuous functions and, hence, it converges uniformly. This implies the existence of $\lim_{T \rightarrow \infty} V_T(\gamma_0, x_0)$ and the limit is uniform with respect to the initial data.

The second problem is proving the uniform continuity of $(v^\delta)_{\delta>0}$ under explicit nonexpansivity conditions. In the Brownian setting, this follows from estimates on the trajectories and a natural coupling with respect to the same Brownian motion in [14, Lemma 3]. For switch PDMP, we obtain similar results (in a convenient setting) by using some reference random measure generated by the process. Although the second marginal of this coupling might not be issued from a controlled process, it is shown to belong to a convenient class of measures by using linear programming techniques (developed in [33], [34] and inspired by Krylov [41]).

Let us now explain how the paper is organized. In Section 2, we begin by recalling the construc-

tion of controlled PDMP of switch type and present the main assumptions on the characteristics. In Subsection 2.2, we introduce the value functions (Cesàro and Abel averages) and the Hamilton-Jacobi integrodifferential systems associated to these value functions. Moreover, we briefly present Krylov's shaking of coefficients method and the linear programming formulations of control problems (taken from [34, Theorem 7 and Corollary 8]). In Section 3, we prove that the value functions can be conveniently approximated by piecewise constant (in time) open-loop policies. This kind of policies simplifies the arguments on coupling in the following sections. The results of Section 3 are inspired by [41]. The proof combines the approach in [41] with the dynamic programming principles in [48]. We think that neither the result, nor the method are surprising but, for reader's sake, we have provided the key elements in the Appendix. Section 4 contains the main contributions of this paper. Under an abstract nonexpansivity condition, we prove that the family $(v^\delta)_{\delta>0}$ is equicontinuous in Proposition 4. As consequence, in the same framework, we prove the existence of a limit value function $\lim_{T \rightarrow \infty} V_T = \lim_{\delta \rightarrow 0} v^\delta$ independent of the average considered (Abel/ Cesàro) in Theorem 8. The aim of Section 5 is to provide a simple condition on the characteristics of the PDMP implying the (abstract) nonexpansivity condition as well as a motivation for this condition. In Subsection 5.1, we discuss a model for Phage λ inspired by [37]. Following this example, we introduce a class of switch systems satisfying an appropriate nonexpansive condition in Subsection 5.2. We recall the elements on random measures generated by PDMP in Subsection 5.3. We prove that any solution of convenient (controlled) stochastic differential equations with respect to these random measures can be embedded into a reasonable set of measures (coherent with the LP methods described in Section 2). Finally, we exhibit reasonable couplings allowing to obtain the results of the previous section (in Subsection 5.5). Subsection 5.6 presents a toy example of parameters under which Hasty's model for Phage λ is nonexpansive.

2 Preliminaries

2.1 Construction of Controlled Piecewise Deterministic Processes of Switch Type

Piecewise deterministic Markov processes have been introduced in [24] and extensively studied for the last thirty years in connection to various phenomena in biology (see [16], [23], [49], [22], [33]), reliability or storage modelling (in [12], [28]), finance (in [47]), communication networks ([35]), etc. The optimal control of these processes makes the object of several papers (e.g. [25], [48], [20], etc.). For reader's sake we will briefly recall the construction of these processes, the assumptions as well as the type of controls we are going to employ throughout the paper.

The switch PDMP is constructed on a space $(\Omega, \mathcal{F}, \mathbb{P})$ allowing to consider a sequence of independent, $[0, 1]$ uniformly distributed random variables (e.g. the Hilbert cube starting from $[0, 1]$ endowed with its Lebesgue measurable sets and the Lebesgue measure for coordinate, see [26, Section 23]). We consider a compact metric space \mathbb{U} referred to as the control space. The process is given by a couple (γ, X) , where γ is the discrete mode component and takes its values in some finite set \mathbb{M} and the state component X takes its values in some Euclidian state space \mathbb{R}^N ($N \geq 1$). The process is governed by a characteristic triple :

- a family of bounded, uniformly continuous vector fields $f_\gamma : \mathbb{R}^N \times \mathbb{U} \longrightarrow \mathbb{R}^N$ such that $|f_\gamma(x, u) - f_\gamma(y, u)| \leq C|x - y|$, for some $C > 0$ and all $x, y \in \mathbb{R}^N$, $\gamma \in \mathbb{M}$ and all $u \in \mathbb{U}$,
- a family of bounded, uniformly continuous jump rates $\lambda_\gamma : \mathbb{R}^N \times \mathbb{U} \longrightarrow \mathbb{R}_+$ such that $|\lambda_\gamma(x, u) - \lambda_\gamma(y, u)| \leq C|x - y|$, for some $C > 0$ and all $x, y \in \mathbb{R}^N$, $\gamma \in \mathbb{M}$ and all $u \in \mathbb{U}$,
- a transition measure $Q : \mathbb{M} \times \mathbb{R}^N \times \mathbb{U} \longrightarrow \mathcal{P}(\mathbb{M} \times \mathbb{R}^N)$, where $\mathcal{P}(\mathbb{A})$ denotes the family of probability measures on the metric space \mathbb{A} . We assume that this transition measure has the particular form

$$Q(\gamma, x, u, d\theta dy) = \delta_{x+g_\gamma(\theta, x, u)}(dy) Q^0(\gamma, u, d\theta),$$

for all $(\gamma, x, u) \in \mathbb{M} \times \mathbb{R}^N \times \mathbb{U}$. The bounded, uniformly continuous jump functions $g_\gamma : \mathbb{M} \times \mathbb{R}^N \times \mathbb{U} \rightarrow \mathbb{R}^N$ are such that $|g_\gamma(\theta, x, u) - g_\gamma(\theta, y, u)| \leq C|x - y|$, for some $C > 0$ and all $x, y \in \mathbb{R}^N$, all $(\theta, \gamma) \in \mathbb{M}^2$ and all $u \in \mathbb{U}$. The transition measure for the mode component is given by $Q^0 : \mathbb{M} \times \mathbb{U} \rightarrow \mathcal{P}(\mathbb{M})$. For every $A \in \mathcal{B}(\mathbb{M})$, the function $(\gamma, u) \mapsto Q^0(\gamma, u, A)$ is assumed to be measurable and, for every $(\gamma, u) \in \mathbb{M} \times \mathbb{U}$, $Q^0(\gamma, u, \{\gamma\}) = 0$.

These assumptions are needed in order to guarantee smoothness of value functions in this context (see also [33] for further comments). Of course, more general transition measures Q can be considered under the assumptions of [33], [34] and the results of Section 3 still hold true. However, the approach in Section 5 only holds true for these particular dynamics. We let $\mathcal{A}_0(\mathbb{U})$ denote the space of \mathbb{U} -valued Borel measurable functions defined on $\mathbb{M} \times \mathbb{R}^N \times \mathbb{R}_+$. Whenever $u \in \mathcal{A}_0(\mathbb{U})$ and $(t_0, \gamma_0, x_0) \in \mathbb{R}_+ \times \mathbb{M} \times \mathbb{R}^N$, we consider the ordinary differential equation

$$\begin{cases} d\Phi_t^{t_0, x_0, u; \gamma_0} = f_{\gamma_0} \left(\Phi_t^{t_0, x_0, u; \gamma_0}, u(\gamma_0, x_0, t - t_0) \right) dt, & t \geq t_0, \\ \Phi_{t_0}^{t_0, x_0, u; \gamma_0} = x_0. \end{cases}$$

Given some sequence $u := (u_1, u_2, \dots) \in \mathcal{A}_0(\mathbb{U})$, the first jump time T_1 has a jump rate $\lambda_{\gamma_0} \left(\Phi_t^{0, x_0, u_1; \gamma_0}, u_1(\gamma_0, x_0, t) \right)$, i.e. $\mathbb{P}(T_1 \geq t) = \exp \left(- \int_0^t \lambda_{\gamma_0} \left(\Phi_s^{0, x_0, u_1; \gamma_0}, u_1(\gamma_0, x_0, s) \right) ds \right)$. The controlled PDMP is defined by setting $(\Gamma_t^{\gamma_0, x_0, u}, X_t^{\gamma_0, x_0, u}) = \left(\gamma_0, \Phi_t^{0, x_0, u_1; \gamma_0} \right)$, if $t \in [0, T_1)$. The post-jump location (Υ_1, Y_1) has $Q \left(\gamma_0, \Phi_\tau^{0, x_0, u_1; \gamma_0}, u_1(\gamma_0, x_0, \tau), \cdot \right)$ as conditional distribution given $T_1 = \tau$. Starting from (Υ_1, Y_1) at time T_1 , we select the inter-jump time $T_2 - T_1$ such that

$$\mathbb{P}(T_2 - T_1 \geq t / T_1, \Upsilon_1, Y_1) = \exp \left(- \int_{T_1}^{T_1+t} \lambda_{\Upsilon_1} \left(\Phi_s^{T_1, Y_1, u_2; \Upsilon_1}, u_2(\Upsilon_1, Y_1, s - T_1) \right) ds \right).$$

We set $(\Gamma_t^{\gamma_0, x_0, u}, X_t^{\gamma_0, x_0, u}) = \left(\Upsilon_1, \Phi_t^{T_1, Y_1, u_2; \Upsilon_1} \right)$, if $t \in [T_1, T_2)$. The post-jump location (Υ_2, Y_2) satisfies

$$\mathbb{P}((\Upsilon_2, Y_2) \in A / T_2, T_1, \Upsilon_1, Y_1) = Q \left(\Upsilon_1, \Phi_{T_2}^{T_1, Y_1, u_2; \Upsilon_1}, u_2(\Upsilon_1, Y_1, T_2 - T_1), A \right),$$

for all Borel set $A \subset \mathbb{M} \times \mathbb{R}^N$ (where the topology on \mathbb{M} is the discrete one). And so on. For simplicity purposes, we set $(\Upsilon_0, Y_0) = (\gamma_0, x_0)$. The sequence $u = (u_1, u_2, \dots)$ is said to be an admissible control. The class of such processes is denoted by $\mathcal{A}_{ad}(\mathbb{U})$ (or simply \mathcal{A}_{ad} whenever no confusion is at risk).

2.2 Value Functions and Linear Programming Techniques

We investigate the asymptotic behavior of discounted value functions

$$\begin{aligned} v^\delta(\gamma, x) &:= \inf_{u \in \mathcal{A}_{ad}} \delta \mathbb{E} \left[\int_0^\infty e^{-\delta t} h(\Gamma_t^{\gamma, x, u}, X_t^{\gamma, x, u}) dt \right] \\ &= \inf_{u=(u_n)_{n \geq 1} \in \mathcal{A}_{ad}} \delta \mathbb{E} \left[\sum_{n \geq 1} \int_{T_{n-1}}^{T_n} e^{-\delta t} h \left(\Gamma_t^{\gamma, x, u}, X_t^{\gamma, x, u}, u_n \left(\Gamma_{T_{n-1}}^{\gamma, x, u}, X_{T_{n-1}}^{\gamma, x, u}, t - T_{n-1} \right) \right) dt \right], \end{aligned}$$

$\gamma \in \mathbb{M}$, $x \in \mathbb{R}^N$, $\delta > 0$, as the discount parameter $\delta \rightarrow 0$ and average values

$$\begin{aligned} V_t(\gamma, x) &:= \inf_{u \in \mathcal{A}_{ad}} \frac{1}{t} \mathbb{E} \left[\int_0^t h(\Gamma_s^{\gamma, x, u}, X_s^{\gamma, x, u}, u_s) ds \right] \\ &= \inf_{u=(u_n)_{n \geq 1} \in \mathcal{A}_{ad}} \frac{1}{t} \mathbb{E} \left[\sum_{n \geq 1} \int_{T_{n-1} \wedge t}^{T_n \wedge t} h \left(\Gamma_s^{\gamma, x, u}, X_s^{\gamma, x, u}, u_n \left(\Gamma_{T_{n-1}}^{\gamma, x, u}, X_{T_{n-1}}^{\gamma, x, u}, t - T_{n-1} \right) \right) ds \right], \end{aligned}$$

$\gamma \in \mathbb{M}$, $x \in \mathbb{R}^N$, $t > 0$, as the time horizon $t \rightarrow \infty$. The cost function $h : \mathbb{M} \times \mathbb{R}^N \times \mathbb{U} \rightarrow \mathbb{R}$ is bounded, uniformly continuous and Lipschitz continuous w.r.t. the state component, uniformly in control and mode (i.e. $|h(\gamma, x, u) - h(\gamma, y, u)| \leq C|x - y|$, for some $C > 0$ and all $\gamma \in \mathbb{M}$, $x, y \in \mathbb{R}^N$ and all $u \in \mathbb{U}$). We recall some basic tools in the discounted setting.

2.2.1 The Hamilton-Jacobi approach

For every $\delta > 0$, the value function v^δ is known to be the unique bounded, uniformly continuous viscosity solution of the Hamilton-Jacobi integro-differential system

$$(1) \quad \delta v^\delta(\gamma, x) + H(\gamma, x, \partial_x v^\delta(\gamma, x), v^\delta) = 0,$$

where the Hamiltonian is defined by setting

$$H(\gamma, x, p, \varphi) := \sup_{u \in \mathbb{U}} \left[-h(\gamma, x, u) - \langle f_\gamma(x, u), p \rangle - \lambda_\gamma(x, u) \int_{\mathbb{M}} (\varphi(\theta, x + g_\gamma(\theta, x, u)) - \varphi(\gamma, x)) Q^0(\gamma, u, d\theta) \right],$$

for all $x, p \in \mathbb{R}^N$ and all bounded function $\varphi : \mathbb{R}^N \rightarrow \mathbb{R}$. For further details on the subject, the reader is referred to [48]. For regular functions φ (for example of class C_b^1), the generator of the control process is given

$$\mathcal{L}^u \varphi(\gamma, x) = \langle f_\gamma(x, u), \partial_x \varphi(\gamma, x) \rangle + \lambda_\gamma(x, u) \int_{\mathbb{M}} (\varphi(\theta, x + g_\gamma(\theta, x, u)) - \varphi(\gamma, x)) Q^0(\gamma, u, d\theta).$$

(A complete description of the domain of this operator can be found, for instance, in [26, Theorem 26.14].)

2.2.2 Krylov's Shaking the Coefficients and Linear Programming Formulation

Although uniformly continuous, the value functions v^δ are, in general, not of class C^1 . However, adapting the method introduced in [41] (see also [7]), v^δ can be seen as the supremum over regular subsolutions of the system (1). Alternatively, one can give a variational formulation of v^δ with respect to an explicit set of constraints. We recall the basic elements taken from [34].

We begin by perturbing the coefficients and consider an extended characteristic triple

- $\bar{f}_\gamma : \mathbb{R}^N \times \mathbb{U} \times \bar{B}(0, 1) \rightarrow \mathbb{R}^N$, $f_\gamma(x, u^1, u^2) = \bar{f}_\gamma(x + u^2, u^1)$, $u_1 \in \mathbb{U}, u_2 \in \bar{B}(0, 1)$, $\gamma \in \mathbb{M}$,
- $\bar{\lambda}_\gamma : \mathbb{R}^N \times \mathbb{U} \times \bar{B}(0, 1) \rightarrow \mathbb{R}^N$, $\lambda_\gamma(x, u^1, u^2) = \bar{\lambda}_\gamma(x + u^2, u^1)$, $u_1 \in \mathbb{U}, u_2 \in \bar{B}(0, 1)$, $\gamma \in \mathbb{M}$,
- $\bar{Q} : \mathbb{R}^N \times \mathbb{U} \times \bar{B}(0, 1) \rightarrow \mathcal{P}(\mathbb{R}^N)$, $\bar{Q}(\gamma, x, u^1, u^2, A) = Q(x + u^2, u^1, A + (0, u^2))$, where $A + (0, u^2) = \{(a_1, a_2 + u^2) : (a_1, a_2) \in A\}$, for all $x \in \mathbb{R}^N$, $u^1 \in \mathbb{U}$, $u^2 \in \bar{B}(0, 1)$ and all Borel set $A \subset \mathbb{M} \times \mathbb{R}^N$.

Next, one can easily construct the process $(\Gamma^{\gamma, x, u^1, u^2}, X^{\gamma, x, u^1, u^2})$ with $u = (u^1, u^2) \in \mathcal{A}_{ad}(\mathbb{U} \times \bar{B}(0, 1))$. The initial process can be recovered by imposing $u^2 = 0$.

To any $(\gamma, x) \in \mathbb{M} \times \mathbb{R}^N$ and any $u \in \mathcal{A}_{ad}(\mathbb{U})$, we associate the discounted occupation measure

$$(2) \quad \mu_{\gamma, x, u}^\delta(A) = \delta \mathbb{E} \left[\int_0^\infty e^{-\delta t} 1_A(\Gamma_t^{\gamma, x, u}, X_t^{\gamma, x, u}, u_t) dt \right],$$

for all Borel subsets $A \subset \mathbb{M} \times \mathbb{R}^N \times \mathbb{U}$. The set of all discounted occupation measures is denoted by $\Theta_0^\delta(\gamma, x)$. We let $\mathcal{P}(\mathbb{R}^N \times \mathbb{U})$ denote the set of all probability measures on $\mathbb{R}^N \times \mathbb{U}$ and define

$$(3) \quad \Theta^\delta(\gamma, x) = \left\{ \mu \in \mathcal{P}(\mathbb{M} \times \mathbb{R}^N \times \mathbb{U}) \text{ s.t. } \forall \phi : \mathbb{M} \rightarrow C_b^1(\mathbb{R}^N), \int_{\mathbb{M} \times \mathbb{R}^N \times \mathbb{U}} (\mathcal{L}^u \phi(\theta, y) + \delta(\phi(\gamma, x) - \phi(\theta, y))) \mu(d\theta, dy, du) = 0 \right\}.$$

The following result corresponds to [34, Theorem 7 and Corollary 8] for this (less general) setting.

Theorem 1 *i) For every $x \in \mathbb{R}^N$ and every $\delta > 0$, $v^\delta(\gamma, x) = \inf_{\mu \in \Theta^\delta(\gamma, x)} \int_{\mathbb{M} \times \mathbb{R}^N \times \mathbb{U}} h(\theta, y, u) \mu(d\theta, dy, du)$.
ii) For every $(\gamma, x) \in \mathbb{M} \times \mathbb{R}^N$ and every $\delta > 0$, $\Theta^\delta(\gamma, x) = \overline{c\mathcal{O}}(\Theta_0^\delta(\gamma, x))$.*

3 Piecewise Constant Open-loop Policies

We envisage the following stochastic control scenario. Let us assume that a given control policy u is convenient as long as some constraint is satisfied. One would construct a globally admissible strategy by defining the minimal time when the constraint is not satisfied and continue by choosing another control. If the constraint is given by a smooth function and a closed set (say a constraint of type $c(X_s^{x,u}, u_s) \leq 0$), the time $\tau := \inf \{s \geq 0 : c(X_s^{x,u}, u_s) > 0\}$ will be optional. However, this might no longer be the case if the the function c is only measurable w.r.t. u (and lacking convenient continuity properties in this variable). This can be overcome if u is chosen (locally or piecewise) constant. Of course, in this context, one should check that the change in the value function is sufficiently small (controlled, for example by the time discretization step.) For Brownian diffusion, this type of result has been proven in [40]. In this section we adapt the method of [40] to our setting by hinting to the modifications whenever necessary.

We introduce the spaces of piecewise constant policies

$$\begin{aligned} \mathcal{A}_0^n(\mathbb{U}) &= \left\{ u \in \mathcal{A}_0(\mathbb{U}) : u(\gamma, x, t) = u^0(\gamma, x) 1_{\{0\}}(t) + \sum_{k \geq 0} u^k(\gamma, x) 1_{\left(\frac{k}{n}, \frac{k+1}{n}\right]}(t) \right\}, \\ \mathcal{A}_{ad}^n(\mathbb{U}) &= \left\{ (u_m)_{m \geq 1} \in \mathcal{A}_{ad}(\mathbb{U}) : u_m \in \mathcal{A}_0^n(\mathbb{U}), m \geq 1 \right\}. \end{aligned}$$

As before, we drop the dependency on \mathbb{U} . Following [40], for all $n \geq 1$, we introduce the value function

$$v^{\delta, n}(\gamma, x) = \inf_{u \in \mathcal{A}_{ad}^n} \delta \mathbb{E} \left[\int_0^\infty e^{-\delta t} h \left(\Gamma_t^{\gamma, x, u, 0}, X_t^{\gamma, x, u, 0}, u_t \right) dt \right],$$

for all $x \in \mathbb{R}^N$.

In order to get convenient estimates on the trajectories, we assume that the switch system admits some invariant compact set \mathbb{K} , i.e., for all $(\gamma, x) \in \mathbb{M} \times \mathbb{K}$ and all $u \in \mathcal{A}_{ad}$ one has $X_t^{\gamma, x, u} \in \mathbb{K}$ for all $t \geq 0$, \mathbb{P} -a.s. Explicit geometric condition on the coefficients and the normal cone to \mathbb{K} equivalent to the property of invariance are given in [33, Theorem 2.8].

The main result of the subsection is the following.

Theorem 2 *Let us assume that there exists a compact, convex set \mathbb{K} which is invariant with respect to the controlled PDMP with characteristics (f, λ, Q) . Then, for every $\delta > 0$, the value functions $v^{\delta, n}$ converge uniformly to v^δ as the discretization step n increases to infinity*

$$\lim_{n \rightarrow \infty} \sup_{\gamma \in \mathbb{M}, x \in \mathbb{K}} \left| v^\delta(\gamma, x) - v^{\delta, n}(\gamma, x) \right| = 0.$$

The proof relies on the same arguments as those developed in [40] combined with dynamic programming principles. Let us briefly explain the approach. For every $n \geq 1$, one begins by proving a dynamic programming principle for $v^{\delta, n}$ and involving $n^{-1} \wedge T_1$ as intermediate time. The arguments are essentially the same as those in [48] and we only specify when the structure of \mathcal{A}_{ad}^n intervenes. Next, one takes a sequence of smooth functions $\left(v_{(\varepsilon)}^{\delta, n} \right)_{\varepsilon > 0}$ converging uniformly to $v^{\delta, n}$ by adapting to the present framework Krylov's shaking of coefficients method introduced in [41] (see also [7] or [33] for the PDMP case). Then, one proceeds by writing the Hamilton-Jacobi integrodifferential system satisfied by $v_{(\varepsilon)}^{\delta, n}$. This equation is ε -close to the one satisfied by v^δ (with a uniform behavior w.r.t. $n \geq 1$). Our assertion follows by integrating this subsolution condition with respect to the law of the piecewise deterministic Markov process then allowing $\varepsilon \rightarrow 0$. For our reader's convenience, we have indicated the main modifications and arguments in the Appendix.

4 Existence of The Limit Value Function(s)

4.1 Nonexpansivity and Uniform Continuity Moduli

Throughout the section, we assume the following nonexpansivity condition

For every $\delta > 0$, every $\varepsilon > 0$, every $(\gamma, x, y) \in \mathbb{M} \times \mathbb{R}^{2N}$ and every $u \in \mathcal{A}_{ad}(\mathbb{U})$, there exists $\mu \in \mathcal{P}\left(\left(\mathbb{M} \times \mathbb{R}^N \times \mathbb{U}\right)^2\right)$ such that

$$(4) \quad \begin{aligned} & i. \quad \mu(\cdot, \mathbb{M} \times \mathbb{R}^N \times \mathbb{U}) = \mu_{\gamma, x, u}^\delta \\ & ii. \quad \mu(\mathbb{M} \times \mathbb{R}^N \times \mathbb{U}, \cdot) \in \Theta^\delta(\gamma, y) \\ & iii. \quad \int_{(\mathbb{M} \times \mathbb{R}^N \times \mathbb{U})^2} |h(\theta, z, w) - h(\theta', z', w')| \mu(d\theta, dz, dw, d\theta', dz', dw') \leq Lip(h) |x - y| + \varepsilon. \end{aligned}$$

Remark 3 1. Whenever h only depends on the x component (but not on the mode γ , nor on the control u), one can impose

$$W_1\left(\tilde{\Theta}_0^\delta(\gamma, x), \tilde{\Theta}^\delta(\gamma, y)\right) \leq |x - y|,$$

where W_1 is the usual Wasserstein distance on $\mathcal{P}(\mathbb{R}^N)$ and $\tilde{\Theta}_0$ (resp. $\tilde{\Theta}$) denote the marginals $\mu(\mathbb{M}, \cdot, \mathbb{U})$ of measures $\mu \in \Theta_0$ (resp. Θ). One can impose the slightly stronger conditions

$$W_1\left(\tilde{\Theta}_0^\delta(\gamma, x), \tilde{\Theta}_0^\delta(\gamma, y)\right) \leq |x - y| \quad \text{or} \quad W_{1, Hausdorff}\left(\Theta^\delta(\gamma, x), \Theta^\delta(\gamma, y)\right) \leq |x - y|$$

and the notion of nonexpansivity is transparent in this setting. Here, $W_{1, Hausdorff}$ is the usual Pompeiu-Hausdorff distance constructed with respect to W_1 .

2. In the next section we will show that this condition is satisfied for a particular subclass of switch controlled Markov processes satisfying a nonexpansive condition of type

$$\inf_{u \in \mathbb{U}} \sup_{u' \in \mathbb{U}} [\langle f_\gamma(x, u) - f_\gamma(y, u'), x - y \rangle] \leq 0$$

together with some conditions on the jumps and the cost function h (see condition 9). We will also give an example of nonexpansive switch system which is not dissipative.

Proposition 4 We assume 4 to hold true. Then, for every $\delta > 0$ and every $(\gamma, x, y) \in \mathbb{M} \times \mathbb{R}^{2N}$, one has

$$\left|v^\delta(\gamma, x) - v^\delta(\gamma, y)\right| \leq Lip(h) |x - y|.$$

Proof. Let us fix $\delta > 0$ and $(\gamma, x, y) \in \mathbb{M} \times \mathbb{R}^{2N}$. We only need to prove that, for every $\varepsilon > 0$,

$$v^\delta(\gamma, y) \leq v^\delta(\gamma, x) + Lip(h) |x - y| + \varepsilon.$$

By definition of v^δ , for a fixed $\varepsilon > 0$, there exists some $u \in \mathcal{A}_{ad}(\mathbb{U})$ such that

$$v^\delta(\gamma, x) + \frac{\varepsilon}{2} \geq \delta \mathbb{E} \left[\int_0^\infty e^{-\delta t} h\left(\Gamma_t^{\gamma, x, u, 0}, X_t^{\gamma, x, u, 0}, u_t\right) dt \right] = \int_{\mathbb{M} \times \mathbb{R}^N \times \mathbb{U}} h(\theta, z, w) \mu_{\gamma, x, u}^\delta(d\theta, dz, dw).$$

If μ is the measure given by 4 and associated to $\frac{\varepsilon}{2}$, we deduce, using Theorem 1.i that

$$\begin{aligned} v^\delta(\gamma, y) & \leq \int_{(\mathbb{M} \times \mathbb{R}^N \times \mathbb{U})^2} h(\theta', z', w') \mu(d\theta, dz, dw, d\theta', dz', dw') \\ & \leq \int_{\mathbb{M} \times \mathbb{R}^N \times \mathbb{U}} h(\theta, z, w) \mu_{\gamma, x, u}^\delta(d\theta, dz, dw) + Lip(h) |x - y| + \frac{\varepsilon}{2} \\ & \leq v^\delta(\gamma, x) + Lip(h) |x - y| + \varepsilon. \end{aligned}$$

The proof of our Proposition follows by recalling that $\varepsilon > 0$ is arbitrary. ■

In particular, it follows that the family of functions $(v^\delta)_{\delta > 0}$ is equicontinuous and uniformly bounded.

Remark 5 *If the invariance conditions of Theorem 2 holds true, one only needs to check that the nonexpansive condition (4) holds true for all $u \in \mathcal{A}_{ad}^n(\mathbb{U})$ for all n large enough (larger than some n_ε).*

4.2 A Limit Value Function

The previous results on existence of a continuity modulus uniform with respect to the discount parameter $\delta > 0$ allows us to prove the existence of a limit value function as $\delta \rightarrow 0$. Before going to the main result of the section, we recall the following.

Lemma 6 *(i) [32, Step 1 of Theorem 4.1] Let us assume that $(v^\delta)_{\delta>0}$ is a relatively compact subset of $C(\mathbb{M} \times \mathbb{R}^N; \mathbb{R})$. Then, for every $v \in C(\mathbb{M} \times \mathbb{R}^N; \mathbb{R})$, every sequence $(\delta_m)_{m \geq 1}$ such that $\lim_{m \rightarrow \infty} \delta_m = 0$ and $(v^{\delta_m})_{m \geq 1}$ converges uniformly to v on $\mathbb{M} \times \mathbb{R}^N$ and every $\varepsilon > 0$, there exists $T > 0$ such that*

$$V_t(\gamma, x) \geq v(\gamma, x) - \varepsilon, \text{ for all } (\gamma, x) \in \mathbb{M} \times \mathbb{R}^N \text{ and all } t \geq T.$$

(ii) [32, Theorem 4.1] Let us assume that $(v^\delta)_{\delta>0}$ is a relatively compact subset of $C(\mathbb{M} \times \mathbb{R}^N; \mathbb{R})$. Then, for every $v \in C(\mathbb{M} \times \mathbb{R}^N; \mathbb{R})$ and every sequence $(\delta_m)_{m \geq 1}$ such that $\lim_{m \rightarrow \infty} \delta_m = 0$ and $(v^{\delta_m})_{m \geq 1}$ converges uniformly to v on $\mathbb{M} \times \mathbb{R}^N$, the following equality holds true

$$\liminf_{t \rightarrow \infty} \sup_{\gamma \in \mathbb{M}, x \in \mathbb{R}^N} |V_t(\gamma, x) - v(\gamma, x)| = 0.$$

(iii) [32, Remark 4.2] Let us assume that $(v^\delta)_{\delta>0}$ converges uniformly to some $v^ \in C(\mathbb{M} \times \mathbb{R}^N; \mathbb{R})$ as $\delta \rightarrow 0$. Then the functions $(V_t)_{t>0}$ converge uniformly on $\mathbb{M} \times \mathbb{R}^N$ to v^**

$$\lim_{t \rightarrow \infty} \sup_{\gamma \in \mathbb{M}, x \in \mathbb{R}^N} |V_t(\gamma, x) - v^*(\gamma, x)| = 0.$$

Remark 7 *In the proof of [32, Theorem 4.1], one gives the condition (i) in Step 1. However, in all generality, the converse is only partial. Indeed, Step 2 (see [32, Page 174, Eq. (10)]) reads : For every $\varepsilon > 0$, there exists $m_0 \geq 1$ such that*

$$V_{\delta_m^{-1}}(\gamma, x) \leq v(\gamma, x) + \varepsilon,$$

for all $m \geq m_0$ and all $(\gamma, x) \in \mathbb{M} \times \mathbb{R}^N$. Thus, the index t of the subfamily $(V_t)_t$ converging to v depends directly on the sequence $(\delta_m)_{m \geq 1}$. Of course, whenever the limit v is independent on the choice of δ , so is t and one gets (iii).

The main result of the section is the following.

Theorem 8 *Let us assume that there exists a compact set $\mathbb{K} \subset \mathbb{R}^N$ invariant with respect to the piecewise deterministic dynamics. Moreover, we assume (4) to hold true for every $(\gamma, x, y) \in \mathbb{M} \times \mathbb{K}^2$. Then, $(v^\delta)_{\delta>0}$ admits a unique limit $v^* \in C(\mathbb{M} \times \mathbb{K}; \mathbb{R})$ and $(V_t)_{t>0}$ converges to v^* uniformly.*

Proof. Let us denote by

$$(5) \quad v^*(\gamma, x) := \limsup_{\delta \rightarrow 0} v^\delta(\gamma, x),$$

for every $(\gamma, x) \in \mathbb{M} \times \mathbb{K}$ the pointwise limsup. We fix, for the time being some $(\gamma, x) \in \mathbb{M} \times \mathbb{K}$. Then, there exists some sequence $(\delta_m)_{m \geq 1}$ such that $\lim_{m \rightarrow \infty} \delta_m = 0$ and $\lim_{m \rightarrow \infty} v^{\delta_m}(\gamma, x) = v^*(\gamma, x)$. Under the assumption (4), the sequence $(v^{\delta_m})_{m \geq 1}$ is equicontinuous and, by definition, it is also

bounded. Then, using Arzelà-Ascoli Theorem, it follows that some subsequence (still denoted $(v^{\delta_m})_{m \geq 1}$) converges uniformly on $\mathbb{M} \times \mathbb{K}$ to some limit function $v \in C(\mathbb{M} \times \mathbb{K}; \mathbb{R})$. In particular, $w(\gamma, x) = v^*(\gamma, x)$. Using Lemma 6 (i), one gets that

$$(6) \quad \liminf_{t \rightarrow \infty} V_t(\gamma, x) \geq v^*(\gamma, x).$$

Obviously, this argument can be repeated for every $(\gamma, x) \in \mathbb{M} \times \mathbb{K}$. Let us now consider $w \in C(\mathbb{M} \times \mathbb{K}; \mathbb{R})$ to be an adherence point of the relatively compact family $(v^\delta)_{\delta > 0}$. Then, using Lemma 6 (ii), one establishes the existence of some increasing sequence $(t_n)_{n \geq 1}$ such that $\lim_{n \rightarrow \infty} t_n = \infty$ and

$$\lim_{n \rightarrow \infty} \sup_{\gamma \in \mathbb{M}, x \in \mathbb{R}^N} |V_{t_n}(\gamma, x) - w(\gamma, x)| = 0.$$

In particular, it follows that

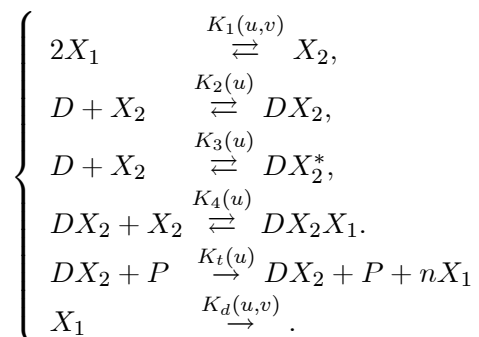
$$(7) \quad w(\gamma, x) \geq \liminf_{t \rightarrow \infty} V_t(\gamma, x),$$

for all $(\gamma, x) \in \mathbb{M} \times \mathbb{K}$. Combining (5, 6 and 7), one deduces that the unique adherence point of $(v^\delta)_{\delta > 0}$ is v^* . The convergence of $(V_t)_{t > 0}$ follows by invoking Lemma 6 (iii). Our Theorem is now complete. ■

5 A Switch Nonexpansive Model

5.1 Some Considerations on a Biological Model

We consider the model introduced in [37] to describe the regulation of gene expression. The model is derived from the promoter region of bacteriophage λ . The simplification proposed by the authors of [37] consists in considering a mutant system in which only two operator sites (known as OR2 and OR3) are present. The gene *cI* expresses repressor (CI), which dimerizes and binds to the DNA as a transcription factor in one of the two available sites. The site OR2 leads to enhanced transcription, while OR3 represses transcription. Using the notations in [37], we let X_1 stand for the repressor, X_2 for the dimer, D for the DNA promoter site, DX_2 for the binding to the OR2 site, DX_2^* for the binding to the OR3 site and DX_2X_1 for the binding to both sites. We also denote by P the RNA polymerase concentration and by n the number of proteins per mRNA transcript. The dimerization, binding, transcription and degradation reactions are summarized by

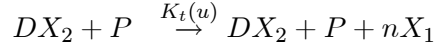


The capital letters K_i , $1 \leq i \leq 4$ for the reversible reactions correspond to couples of direct/reverse speeds functions k_i, k_{-i} , while K_t and K_d only to direct speed functions k_t and k_d . Host DNA gyrase puts negative supercoils in the circular chromosome, causing A-T-rich regions to unwind and drive transcription. This is why, in the model written here, the binding speeds k_2 (to the promoter of the lysogenic cycle P_{RM}), k_3 (to OR3), respectively k_4 and the reverse speeds as well as the transcription speed k_t are assumed to depend only on a control on the host E-coli (denoted by u). This control also acts on the prophage and, hence, we find it, together with some

additional control v , in the dimerization speed k_1 and the degradation speed k_d . The mode is given, in this setting by $\mathbb{M} = \left\{ \nu \in \{0, 1\}^5 : \sum_{i=1}^4 \nu_i = 1 \right\}$ corresponding to $(D, DX_2, DX_2^*, DX_2X_1, P)$. The couple repressor/dimer are given with respect to their concentrations (X, X_2) . Hence, we deal with a hybrid model on $\mathbb{M} \times \mathbb{R}^2$. The jump mechanism is governed by the control parameter u and (essentially) concerns \mathbb{M} . Each jump induces a jump on X . For example, if the current mode is $\gamma = (1, 0, 0, 0)$, two kind of jumps are possible $D + X_2 \xrightarrow{k_2(u)} DX_2$ or $D + X_2 \xrightarrow{k_3(u)} DX_2^*$. The reaction $D + X_2 \xrightarrow{k_2(u)} DX_2$ means going from $(1, 0, 0, 0)$ to $(0, 1, 0, 0)$ and this affects $x = (X_1, X_2) \in \mathbb{R}^2$ by adding $(0, -1)$, etc. Of course, for a consistent mathematical model, since concentrations cannot be negative, to (x_1, x_2) we actually add $(0, -\min(1, x_2))$. One constructs

$$\left\{ \begin{array}{l} \lambda_{(1,0,0,0)}(u, v) = \lambda_{(1,0,0,0)}(u) = k_2(u) + k_3(u), \\ Q^0((1, 0, 0, 0), (u, v), d\theta) = Q^0((1, 0, 0, 0), u, d\theta) = \frac{k_2(u)}{\lambda_{(1,0,0,0)}(u)} \delta_{(0,1,0,0)}(d\theta) + \frac{k_3(u)}{\lambda_{(1,0,0,0)}(u)} \delta_{(0,0,1,0)}(d\theta), \\ Q((1, 0, 0, 0), (x_1, x_2), u, d\theta dy) = \delta_{(x_1, x_2) + g_\gamma(\theta, (x_1, x_2), u, v)}(dy) Q^0((1, 0, 0, 0), u, d\theta), \text{ where} \\ g_{(1,0,0,0)}(\theta, (x_1, x_2), u, v) = (0, -\min(1, x_2)) \mathbb{1}_{\theta \in \{(0,1,0,0), (0,0,1,0)\}}. \end{array} \right.$$

A special part is played by the transcription reaction



which is a slow reaction. Details on a possible construction will be given in Subsection 5.6. Following the approach in [37], the deterministic behavior is governed by

$$f_\gamma((x_1, x_2), u, v) = (-2k_1(u, v)x_1^2 + 2k_{-1}(u, v)x_2 - k_d(u, v)x_1, k_1(u, v)x_1^2 - k_{-1}(u, v)x_2).$$

For further details on constructions related to systems of chemical reactions in gene networks, the reader is referred to [23], [22], [33], etc. An example of coefficients is provided in Subsection 5.6.

5.2 A Simple Nonexpansive Framework

In the previous example, the control is given by a couple $(u, v) \in \mathbb{U} := U \times V$ acting as follows : the jump rate (and the measure Q^0 giving the new mode) only depend on the mode component and is controlled by the parameter u . The component X is controlled both by u and by v and it behaves as in the general case. One has a vector field $f : \mathbb{M} \times \mathbb{R}^N \times U \times V \rightarrow \mathbb{R}^N$, a jump rate $\lambda : \mathbb{M} \times \mathbb{R}^N \times U \times V \rightarrow \mathbb{R}_+$ given by

$$\lambda_\gamma(x, u, v) = \lambda_\gamma(u),$$

and the transition measure $Q : \mathbb{M} \times \mathbb{R}^N \times U \times V \rightarrow \mathcal{P}(\mathbb{M} \times \mathbb{R}^N)$ having the particular form

$$Q((\gamma, x), u, v, d\theta dy) = \delta_{x+g(\gamma, x, u, v, \theta)}(dy) Q^0(\gamma, u, d\theta),$$

where Q^0 governs the post-jump position of the mode component. In this case, the extended generator of (γ, X) has the form

$$(8) \quad \mathcal{L}^{u,v} \varphi(\gamma, x) = \langle f_\gamma(x, u, v), \partial_x \varphi(\gamma, x) \rangle + \lambda(\gamma, u) \int_{\mathbb{M}} (\varphi(\theta, x + g(\gamma, x, u, v, \theta)) - \varphi(\gamma, x)) Q_0(\gamma, u, d\theta).$$

We will show that the results on convergence of the discounted value functions hold true under the following explicit condition on the dynamics.

Condition 9 For every $\gamma \in \mathbb{M}$, every $u \in U$ and every $x, y \in \mathbb{R}^N$, the following holds true

$$\sup_{v \in V} \inf_{w \in V} \max \left\{ \begin{array}{l} \langle f_\gamma(x, u, v) - f_\gamma(y, u, w), x - y \rangle, \\ \sup_{\theta \in \mathbb{M}} |x + g_\gamma(\theta, x, u, v) - y - g_\gamma(\theta, y, u, w)| - |x - y|, \\ |h(\gamma, x, u, v) - h(\gamma, y, u, w)| - Lip(h) |x - y|, \end{array} \right\} \leq 0.$$

Remark 10 1. Whenever h does not depend on the control, the latter conditions naturally follow from the Lipschitz-continuity of h .

2. If, moreover, the post-jump position is given by a (state and control free) translation $x \mapsto x + g_\gamma(\theta)$, this condition is the usual deterministic nonexpansive one (i.e.

$$\sup_{w \in V} \inf_{v \in V} \langle f_\gamma(x, u, v) - f_\gamma(y, u, w), x - y \rangle \leq 0).$$

This kind of jump (up to a slight modification guaranteeing that protein concentrations do not become negative) fits the general theory described in [23].

5.3 Associated Random Measures and Stochastic Differential Equations

Let us fix $\gamma_0 \in \mathbb{M}$, $x_0 \in \mathbb{R}^N$ and $(u, v) \in \mathcal{A}_{ad}(U \times V)$. The following construction is quite standard and makes the object of [26, Section 26] for more general PDMP (without control) and [26, Section 41] (when control is present). We let $S_0 = T_0 = 0$, $S_n = T_n - T_{n-1}$, for all $n \geq 1$ and $\xi_n = (S_n, \gamma_{T_n}^{\gamma_0, x_0, u, v}, X_{T_n}^{\gamma_0, x_0, u, v})$. We look at the process (γ, X) under $\mathbb{P}^{\gamma_0, x_0, u, v}$ (which depends on both the initial state (γ_0, x_0) and the control couple (u, v) , but, having fixed these elements and for notation purposes, this dependency will be dropped). By abuse of notation, we let

$$u_s := u_1(s, \gamma_0, x_0) 1_{0 \leq s \leq T_1} + \sum_{n \geq 1} u_{n+1}(\gamma_{T_n}^{\gamma_0, x_0, u, v}, X_{T_n}^{\gamma_0, x_0, u, v}, s - T_n) 1_{T_n < s \leq T_{n+1}},$$

(and similar for v). We denote by \mathbb{F} the filtration $(\mathcal{F}_{[0,t]} := \sigma\{(\gamma_r^{\gamma_0, x_0, u, v}, X_r^{\gamma_0, x_0, u, v}) : r \in [0, t]\})_{t \geq 0}$. The predictable σ -algebra will be denoted by \mathcal{P} and the progressive σ -algebra by $Prog$. For the general structure of predictable processes, the reader is referred to [26, Section 26], [39, Proposition 4.2.1] or [13, Appendix A2, Theorem T34]. In particular, due to the previous notations, it follows that u and v are predictable.

As usual, we introduce the random measure \bar{p} on $\Omega \times (0, \infty) \times \mathbb{M} \times \mathbb{R}^N$ by setting

$$\bar{p}(\omega, A) = \sum_{k \geq 1} 1_{(T_k(\omega), (\gamma_{T_k}^{\gamma_0, x_0, u, v}, X_{T_k}^{\gamma_0, x_0, u, v}))(\omega) \in A}, \text{ for all } \omega \in \Omega, A \in \mathcal{B}(0, \infty) \times \mathcal{B}(\mathbb{M} \times \mathbb{R}^N).$$

The compensator of \bar{p} is

$$\widehat{\bar{p}}(ds dy d\theta) = \lambda(\gamma_{s-}^{\gamma_0, x_0, u, v}, u_s) \delta_{X_{s-}^{\gamma_0, x_0, u, v} + g(\theta, \gamma_{s-}^{\gamma_0, x_0, u, v}, X_{s-}^{\gamma_0, x_0, u, v}, u_s, v_s)}(dy) Q^0(\gamma_{s-}^{\gamma_0, x_0, u, v}, v_s, d\theta) ds.$$

and the compensated martingale measure (see [26, Proposition 26.7]) is given by $\bar{q} := \bar{p} - \widehat{\bar{p}}$.

By construction, for our model, on $[T_{n-1}, T_n)$, $X_t^{\gamma_0, x_0, u, v}$ is a deterministic function of $X_{T_{n-1}}^{\gamma_0, x_0, u, v}$, $\gamma_{T_{n-1}}^{\gamma_0, x_0, u, v}$, $u_n(X_{T_{n-1}}^{\gamma_0, x_0, u, v}, \gamma_{T_{n-1}}^{\gamma_0, x_0, u, v}, \cdot - T_{n-1})$ and $v_n(X_{T_{n-1}}^{\gamma_0, x_0, u, v}, \gamma_{T_{n-1}}^{\gamma_0, x_0, u, v}, \cdot - T_{n-1})$. In this particular framework,

$$X_{T_n}^{\gamma_0, x_0, u, v} = \Phi_{T_n - T_{n-1}}^{0, X_{T_{n-1}}^{\gamma_0, x_0, u, v}, u_n(\gamma_{T_{n-1}}^{\gamma_0, x_0, u, v}, X_{T_{n-1}}^{\gamma_0, x_0, u, v}, \cdot), v_n(\gamma_{T_{n-1}}^{\gamma_0, x_0, u, v}, X_{T_{n-1}}^{\gamma_0, x_0, u, v}, \cdot); \gamma_{T_{n-1}}^{\gamma_0, x_0, u, v}} + g_{\gamma_{T_{n-1}}^{\gamma_0, x_0, u, v}}(\gamma_{T_n}^{\gamma_0, x_0, u, v}, u_n(\gamma_{T_{n-1}}^{\gamma_0, x_0, u, v}, X_{T_{n-1}}^{\gamma_0, x_0, u, v}, T_n - T_{n-1}), v_n(\gamma_{T_{n-1}}^{\gamma_0, x_0, u, v}, X_{T_{n-1}}^{\gamma_0, x_0, u, v}, T_n - T_{n-1})),$$

hence being a deterministic function of $S_n, X_{T_{n-1}}^{\gamma_0, x_0, u, v}, \gamma_{T_{n-1}}^{\gamma_0, x_0, u, v}, \gamma_{T_{n-1}}^{\gamma_0, x_0, u, v}$. It follows that $\mathcal{F}_{[0,t]} = \sigma\{\gamma_r^{\gamma_0, x_0, u, v} : r \in [0, t]\}$. As a consequence, $v_{n+1} = v_{n+1}(X_{T_n}^{\gamma_0, x_0, u, v}, \gamma_{T_n}^{\gamma_0, x_0, u, v}, \cdot)$ is a deterministic function still denoted by $v_{n+1}(T_1, \dots, T_n, \gamma_{T_1}^{\gamma_0, x_0, u, v}, \dots, \gamma_{T_n}^{\gamma_0, x_0, u, v}, \cdot)$. In the case when $(u, v) \in \mathcal{A}_{ad}^m(U \times V)$ for some $m \geq 1$ are piecewise constant, v_{n+1} is of type

$$\sum_{k \geq 0} v_{n+1}^k(T_1, \dots, T_n, \gamma_{T_1}^{\gamma_0, x_0, u, v}, \dots, \gamma_{T_n}^{\gamma_0, x_0, u, v}) 1_{[\frac{k}{m}, \frac{k+1}{m})}(t).$$

Similar assertion hold true for u .

We now define the random measure p on $\Omega \times (0, \infty) \times \mathbb{M}$ by setting

$$p(\omega, A) = \bar{p}(\omega, A \times \mathbb{R}^N), \text{ for all } \omega \in \Omega, A \in \mathcal{B}(0, \infty) \times \mathcal{B}(\mathbb{M}).$$

The properties of \bar{p} imply that the compensator of p is

$$\hat{p}(dsd\theta) = \lambda(\gamma_{s-}^{\gamma_0, x_0, u, v}, u_s) Q^0(\gamma_{s-}^{\gamma_0, x_0, u, v}, v_s, d\theta) ds$$

and

$$q(ds d\theta) = p(ds d\theta) - \lambda(\gamma_{s-}^{\gamma_0, x_0, u, v}, u_s) Q^0(\gamma_{s-}^{\gamma_0, x_0, u, v}, v_s, d\theta) ds$$

is its martingale measure. Following the general theory of integration with respect to point processes (see, for example [38]), the second state component can be identified with the unique solution of the stochastic differential equation (SDE)

$$\begin{cases} dX_t^{\gamma_0, x_0, u, v} = f_{\gamma_t^{\gamma_0, x_0, u, v}}(X_t^{\gamma_0, x_0, u, v}, u_t, v_t) dt + \int_{\mathbb{M}} g_{\gamma_t^{\gamma_0, x_0, u, v}}(\theta, X_t^{\gamma_0, x_0, u, v}, u_t, v_t) p(dt d\theta), t \geq 0, \\ X_t^{\gamma_0, x_0, u, v} = x_0, \mathbb{P} - a.s. \end{cases}$$

5.4 Measure Embedding of Solutions

More general, whenever w is an \mathbb{F} -predictable process, we can consider the equation

$$\begin{cases} dY_t^{y_0, w} = f_{\gamma_t^{\gamma_0, x_0, u, v}}(Y_t^{y_0, w}, u_t, w_t) dt + \int_{\mathbb{M}} g_{\gamma_t^{\gamma_0, x_0, u, v}}(\theta, Y_t^{y_0, w}, u_t, w_t) p(dt d\theta), t \geq 0, \\ Y_0^{y_0, w} = y_0, \mathbb{P} - a.s. \end{cases}$$

The assumptions on the coefficients f and g guarantee that, for every $y_0 \in \mathbb{R}^N$ and every predictable, V -valued process w , this equation admits a unique solution $Y^{y_0, w}$. We fix $\delta > 0$ and consider some (arbitrary) regular test function $\phi \in C_b^2(\mathbb{M} \times \mathbb{R}^N; \mathbb{R})$. Itô's formula (see [38, Chapter II, Theorem 5.1]) applied to $\delta e^{-\delta \cdot} \phi(\gamma^{\gamma_0, x_0, u, v}, Y^{y_0, w})$ on $[0, T]$ yields

$$\begin{aligned} & \delta e^{-\delta T} \mathbb{E} [\phi(\gamma_T^{\gamma_0, x_0, u, v}, Y_T^{y_0, w})] \\ &= \delta \phi(\gamma_0, y_0) \\ &+ \mathbb{E} \left[\int_0^T \delta e^{-\delta t} \left(-\delta \phi(\gamma_t, Y_t) + \langle f_{\gamma_t}(Y_t, u_t, w_t), \partial_x \phi(\gamma_t, Y_t) \rangle \right. \right. \\ &\quad \left. \left. + \lambda(\gamma_t, u_t) \int_{\mathbb{M}} (\phi(\theta, Y_t + g_{\gamma_t}(\theta, Y_t, u_t, w_t)) - \phi(\gamma_t, Y_t)) Q_0(\gamma_t, u_t, d\theta) \right) dt \right], \end{aligned}$$

where we have denoted by $(\gamma_t, Y_t) = (\gamma_t^{\gamma_0, x_0, u, v}, Y_t^{y_0, w})$. By letting $T \rightarrow \infty$, it follows that the occupation measure $\mu^{y_0, w} \in \mathcal{P}(\mathbb{M} \times \mathbb{R}^N \times U \times V)$ given by

$$\mu^{y_0, w}(A) = \mathbb{E} \left[\int_0^\infty \delta e^{-\delta t} 1_A(\gamma_t^{\gamma_0, x_0, u, v}, Y_t^{y_0, w}, u_t, w_t) dt \right], \text{ for } A \in \mathcal{B}(\mathbb{M} \times \mathbb{R}^N \times U \times V)$$

satisfies

$$\int_{\mathbb{M} \times \mathbb{R}^N \times U} (\mathcal{L}^{u, v} \phi(\theta, y) + \delta(\phi(\gamma, x) - \phi(\theta, y))) \mu(d\theta, dy, du, dv) = 0.$$

We recall that $\mathcal{L}^{u, v}$ is the generator given by (8).

There is no reason for the couple $(\gamma^{\gamma_0, x_0, u, v}, Y^{y_0, w})$ to be a Markov process, or, a fortiori, that it should be associated to a $U \times V$ -valued piecewise open-loop control couple. Nevertheless, the previous arguments show that the occupation measure $\mu^{y_0, w}$ belongs to $\Theta^\delta(\gamma_0, y_0)$ (see (3)).

Remark 11 *Let us note that if there exists a set \mathbb{K} invariant with respect to the PDMP driven by (f, λ, Q) , then, for all $\gamma_0 \in \mathbb{M}$, $y_0 \in \mathbb{K}$, the occupation measures $\mu \in \Theta_0^\delta(\gamma_0, y_0)$ satisfy the support condition $\mu(\mathbb{M} \times \mathbb{K} \times U \times V) = 1$. Then, by Theorem 1, the same holds true for $\Theta^\delta(\gamma_0, y_0)$ and, hence, $Y^{y_0, w}$ takes its values in \mathbb{K} . Alternatively, one can use [33, Theorem 2.8 (ii)].*

5.5 Coupling via the Random Measure

As in the previous arguments, one can define a measure $\mu \in \mathcal{P} \left((\mathbb{M} \times \mathbb{R}^N \times U \times V)^2 \right)$ by setting

$$\mu(A \times B) = \mathbb{E} \left[\int_0^\infty \delta e^{-\delta t} 1_A(\gamma_t^{\gamma_0, x_0, u, v}, X_t^{\gamma_0, x_0, u, v}, u_t, v_t) 1_B(\gamma_t^{\gamma_0, x_0, u, v}, Y_t^{y_0, w}, u_t, w_t) dt \right],$$

whenever $A \in \mathcal{B} \left((\mathbb{M} \times \mathbb{R}^N \times U \times V)^2 \right)$. It is clear that

$$\begin{aligned} & \int_{(\mathbb{M} \times \mathbb{R}^N \times U)^2} |h(\theta, z, w) - h(\theta', z', w')| \mu(d\theta, dz, dw, d\theta', dz', dw') \\ &= \mathbb{E} \left[\int_0^\infty \delta e^{-\delta t} |h(\gamma_t^{\gamma_0, x_0, u, v}, X_t^{\gamma_0, x_0, u, v}, u_t, v_t) - h(\gamma_t^{\gamma_0, x_0, u, v}, Y_t^{y_0, w}, u_t, w_t)| dt \right], \end{aligned}$$

$\mu(A \times (\mathbb{M} \times \mathbb{R}^N \times U \times V)) = \mu_{\gamma_0, x_0, u, v}^\delta \in \Theta_0^\delta(\gamma_0, x_0)$ and $\mu((\mathbb{M} \times \mathbb{R}^N \times U \times V) \times B) = \mu^{y_0, w} \in \Theta^\delta(\gamma_0, y_0)$ given in the previous arguments. Convenient estimates for this integral term imply the condition 4 and, hence, the results on existence of a limit value function. In fact (see Remark 5), it suffices to provide good estimates when the process is constructed with piecewise constant (in time) policies $(u, v) \in \mathcal{A}_{ad}^n(U \times V)$.

Lemma 12 *We assume Condition 9 to hold true. Moreover, we assume that there exists a compact set \mathbb{K} invariant with respect to the PDMP governed by (f, λ, Q) . Then, there exists $\omega : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $\lim_{\varepsilon \rightarrow 0} \omega(\varepsilon) = 0$ and, for every $n \geq 1$ and every $(u, v) \in \mathcal{A}_{ad}^n(U \times V)$, there exists w predictable with respect to the filtration $\mathbb{F}^{\gamma_0, x_0, u, v}$ such that*

$$\mathbb{E} \left[\int_0^\infty \delta e^{-\delta t} |h(\gamma_t^{\gamma_0, x_0, u, v}, X_t^{\gamma_0, x_0, u, v}, u_t, v_t) - h(\gamma_t^{\gamma_0, x_0, u, v}, Y_t^{y_0, w}, u_t, w_t)| dt \right] \leq \omega(n^{-1}).$$

Proof. Step 0.

Let us define a set-valued function

$$\begin{aligned} & \mathbb{M} \times \mathbb{K}^2 \times U \times V \ni (\gamma, x, y, u, v) \rightsquigarrow \Xi((\gamma, x, y, u, v)) \\ & := \left\{ \begin{array}{l} w \in V : \forall \theta \in \mathbb{M}, \\ \langle f_\gamma(x, u, v) - f_\gamma(y, u, w), x - y \rangle \leq 0, \\ |x + g_\gamma(\theta, x, u, v) - y - g_\gamma(\theta, y, u, w)| \leq |x - y|, \\ |h(\gamma, x, u, v) - h(\gamma, y, u, w)| \leq Lip(h) |x - y|, \end{array} \right\}. \end{aligned}$$

One easily checks that the function has compact values and is upper semicontinuous. Hence, there exists some measurable selection

$$\widehat{w} : \mathbb{M} \times \mathbb{K}^2 \times U \times V \rightarrow V, \quad \widehat{w}(\gamma, x, y, u, v) \in \Xi(\gamma, x, y, u, v),$$

for all $(\gamma, x, y, u, v) \in \mathbb{M} \times \mathbb{K}^2 \times U \times V$. For further details, the reader is referred to [6, Subsection 9.2].

We construct an \mathbb{F} -predictable V -valued control process w as follows. We begin by fixing $T > 0$ (depending on n) and $m \geq 1$ (depending on n). The choice of T and m will be made explicit later on. Moreover, we assume that $\mathbb{K} \subset \overline{B}(0, k_0)$, for some $k_0 > 0$.

Step 1. We consider

$$w_s^{1,0} := \widehat{w}(\gamma_0, x_0, y_0, u_1(0), v_1(0)) = w_0^{1,0}, \quad s \geq 0,$$

where we have denoted, by abuse of notation,

$$u_1(s) = u_1(\gamma_0, x_0, s), \quad v_1(s) = v_1(\gamma_0, x_0, s).$$

We recall that if $s \leq \frac{1}{n}$, one has $u_1(s) = u_1(0, \gamma_0, x_0)$ and similar assertions hold true for v_1 . By recalling that \mathbb{K} is invariant with respect to the controlled piecewise dynamics, one gets

$$\begin{aligned} & \left\langle f_{\gamma_0}(\Phi_s^{0,x_0,u,v;\gamma_0}, u_s, v_s) - f_{\gamma_0}(\Phi_s^{0,y_0,u,w^{1,0};\gamma_0}, u_s, w_s^{1,0}), \Phi_s^{0,x_0,u,v;\gamma_0} - \Phi_s^{0,y_0,u,w^{1,0};\gamma_0} \right\rangle \\ & \leq \left\langle f_{\gamma_0}(x_0, u_s, v_s) - f_{\gamma_0}(y_0, u_s, w_s^{1,0}), x_0 - y_0 \right\rangle + c \left(|x_0 - \Phi_s^{0,x_0,u,v;\gamma_0}| + |y_0 - \Phi_s^{0,y_0,u,w^{1,0};\gamma_0}| \right) \\ (9) \quad & \leq c \left(|x_0 - \Phi_s^{0,x_0,u,v;\gamma_0}| + |y_0 - \Phi_s^{0,y_0,u,w^{1,0};\gamma_0}| \right) \leq \frac{c}{n}, \end{aligned}$$

for all $0 \leq s \leq \frac{1}{n}$. Similarly,

$$(10) \quad \begin{aligned} i. \quad & \frac{c}{n} \geq \left| h_{\gamma_0}(\Phi_s^{0,x_0,u,v;\gamma_0}, u_s, v_s) - h_{\gamma_0}(\Phi_s^{0,y_0,u,w^{1,0};\gamma_0}, u_s, w_s^{1,0}) \right| - Lip(h) \left| \Phi_s^{0,x_0,u,v;\gamma_0} - \Phi_s^{0,y_0,u,w^{1,0};\gamma_0} \right|, \\ ii. \quad & \frac{c}{n} \geq \left| \Phi_s^{0,x_0,u,v;\gamma_0} + g_{\gamma_0}(\theta, \Phi_s^{0,x_0,u,v;\gamma_0}, u_s, v_s) - \Phi_s^{0,y_0,u,w^{1,0};\gamma_0} - g_{\gamma_0}(\theta, \Phi_s^{0,y_0,u,w^{1,0};\gamma_0}, u_s, w_s^{1,0}) \right| \\ & - \left| \Phi_s^{0,x_0,u,v;\gamma_0} - \Phi_s^{0,y_0,u,w^{1,0};\gamma_0} \right|, \end{aligned}$$

for all $0 \leq s \leq \frac{1}{n}$. The constant $c > 1$ is generic, independent of $\delta > 0, n, s, x, y, u$ and is allowed to change from one line to another. We define the control process $w^{1,1}$ by setting

$$w_s^{1,1} = w_s^{1,0} \mathbf{1}_{s \leq \frac{1}{n}} + \widehat{w} \left(\gamma_0, \Phi_{\frac{1}{n}}^{0,x_0,u,v;\gamma_0}, \Phi_{\frac{1}{n}}^{0,y_0,u,w^{1,0};\gamma_0}, u_1 \left(\frac{1}{n} \right), v_1 \left(\frac{1}{n} \right) \right) \mathbf{1}_{s > \frac{1}{n}}.$$

Then the estimates in (9) hold true for $s \in [0, \frac{2}{n}]$ if substituting $w^{1,1}$ to $w^{1,0}$. We set

$$w_s^{1,2} = w_s^{1,1} \mathbf{1}_{s \leq \frac{2}{n}} + \widehat{w} \left(\gamma_0, \Phi_{\frac{2}{n}}^{0,x_0,u,v;\gamma_0}, \Phi_{\frac{2}{n}}^{0,y_0,u,w^{1,1};\gamma_0}, u_1 \left(\frac{2}{n} \right), v_1 \left(\frac{2}{n} \right) \right) \mathbf{1}_{s > \frac{2}{n}},$$

and so on, to define $w^{1,3}, w^{1,4}, \dots, w^{1,n([T]+1)}$. We fix some $w^0 \in V$ and let $w^1 = w_s^{1,n([T]+1)} \mathbf{1}_{s \leq T} + w^0 \mathbf{1}_{s > T}$. As consequence, by recalling that, prior to T_1 , both $X^{\gamma_0, x_0, u, v}$ and $Y^{y_0, w}$ are deterministic and can be identified with $\Phi^{0, x_0, u, v; \gamma_0}$ (resp. $\Phi_s^{0, y_0, u, w^{1,0}; \gamma_0}$), we use (9) to get

$$(11) \quad \left| X_t^{\gamma_0, x_0, u, v} - Y_t^{y_0, w^1} \right|^2 \leq |x_0 - y_0|^2 + \frac{ct}{n},$$

for all $t \leq T$ on $t < T_1$. It follows that

$$(12) \quad \left| X_t^{\gamma_0, x_0, u, v} - Y_t^{y_0, w^1} \right| \leq |x_0 - y_0| + \frac{c}{2\sqrt{n}} + \frac{t}{\sqrt{n}},$$

or all $t \leq T$ such that $t < T_1$. Moreover, on $T_1 \leq T$, using (10.ii) and (11) and recalling that $\mathbb{K} \subset \overline{B}(0, k_0)$ is invariant (see also Remark 11) one has

$$\begin{aligned} (13) \quad \left| X_{T_1}^{\gamma_0, x_0, u, v} - Y_{T_1}^{y_0, w^1} \right| &= \left| \Phi_{T_1-}^{0, x_0, u, v; \gamma_0} + g_{\gamma_0} \left(\gamma_{T_1}^{\gamma_0, x_0, u, v}, \Phi_{T_1}^{0, x_0, u, v; \gamma_0}, u_{T_1}, v_{T_1} \right) \right. \\ & \quad \left. - \Phi_{T_1-}^{0, y_0, u, w^{1,0}; \gamma_0} - g_{\gamma_0} \left(\gamma_{T_1}^{\gamma_0, x_0, u, v}, \Phi_{T_1}^{0, y_0, u, w^{1,0}; \gamma_0}, u_{T_1}, w_{T_1}^{1,0} \right) \right| \\ & \leq \left| \Phi_{T_1-}^{0, x_0, u, v; \gamma_0} - \Phi_{T_1-}^{0, y_0, u, w^{1,0}; \gamma_0} \right| + \frac{c}{n} \\ & \leq \min \left(\sqrt{|x_0 - y_0|^2 + \frac{cT_1}{n}}, 2k_0 \right) + \frac{c}{n} \\ & \leq |x_0 - y_0| + \frac{c + T_1}{\sqrt{n}}, \end{aligned}$$

for all $n \geq 4$. (The reader is invited to recall that u_{T_1} is still $u_1(T_1, \gamma_0, x_0)$ and that $u_1 \in \mathcal{A}_{ad}^0$ is left continuous). One gets, on $T_1 \leq T$,

$$(14) \quad \begin{aligned} \left| X_{T_1}^{\gamma_0, x_0, u, v} - Y_{T_1}^{y_0, w^1} \right|^2 &\leq \left(\min \left(\sqrt{|x_0 - y_0|^2 + \frac{cT_1}{n}}, 2k_0 \right) + \frac{c}{n} \right)^2 \leq |x_0 - y_0|^2 + \frac{cT_1}{n} + \frac{c^2}{n^2} + 4k_0 \frac{c}{n} \\ &\leq |x_0 - y_0|^2 + \frac{c(T_1 + 4k_0 + 1)}{n}, \end{aligned}$$

whenever $n \geq c$. Finally, using (10.i) and (12), we get

$$\begin{aligned} \left| h_{\gamma_0} \left(\Phi_s^{0, x_0, u, v; \gamma_0}, u_s, v_s \right) - h_{\gamma_0} \left(\Phi_s^{0, y_0, u, w^1; \gamma_0}, u_s, w_s^1 \right) \right| &\leq Lip(h) \left| \Phi_s^{0, x_0, u, v; \gamma_0} - \Phi_s^{0, y_0, u, w^1; \gamma_0} \right| + \frac{c}{n} \\ &\leq Lip(h) |x_0 - y_0| + \frac{(Lip(h) + 1)c + Lip(h)t}{\sqrt{n}}. \end{aligned}$$

for all $s < T \wedge T_1$.

Step 2. We continue the construction on $[T_1, T_2]$. By abuse of notation, we let $u_2(s) := u_2 \left(s, \gamma_{T_1}^{\gamma_0, x_0, u, v}, X_{T_1}^{\gamma_0, x_0, u, v} \right)$, for $s \geq 0$ and similar for v_2 . We set

$$w_s^{2,1} := w_s^1 \mathbf{1}_{0 \leq s \leq T_1} + \widehat{w} \left(\gamma_{T_1}^{\gamma_0, x_0, u, v}, X_{T_1}^{\gamma_0, x_0, u, v}, Y_{T_1}^{y_0, w^1}, u_2(0), v_2(0) \right) \mathbf{1}_{s > T_1}.$$

It is clear that this control process is predictable. We apply the same method as in Step 1 (ω -wise) on the (stochastic) time interval $[T_1, (T_1 + \frac{1}{n}) \wedge T_2]$, then on $[T_1, (T_1 + \frac{2}{n}) \wedge T_2]$, etc. We construct a sequence of control processes $(w^{2,m})_{m \geq 0}$ and, by choosing m large enough, we establish the existence of some w^2 such that

$$(15) \quad \left\{ \begin{array}{l} \left| X_t^{\gamma_0, x_0, u, v} - Y_t^{y_0, w^2} \right|^2 \leq \left| X_{T_1}^{\gamma_0, x_0, u, v} - Y_{T_1}^{y_0, w^1} \right|^2 + \frac{c(t-T_1)}{n} \leq |x_0 - y_0|^2 + \frac{c(t+4k_0+1)}{n} \\ \left| X_{T_2}^{\gamma_0, x_0, u, v} - Y_{T_2}^{y_0, w^2} \right| \leq \left| X_{T_2^-}^{\gamma_0, x_0, u, v} - Y_{T_2^-}^{y_0, w^1} \right| + \frac{c}{n} \\ \leq \min \left(\sqrt{|x_0 - y_0|^2 + \frac{c(T_2+4k_0+1)}{n}}, 2k_0 \right) + \frac{c}{n} \leq |x_0 - y_0| + \frac{c+T_2+4k_0+1}{\sqrt{n}}, \\ \left| X_{T_2}^{\gamma_0, x_0, u, v} - Y_{T_2}^{y_0, w^2} \right|^2 \leq |x_0 - y_0|^2 + \frac{c(T_2+4k_0+1)}{n} + \frac{c^2}{n^2} + 4k_0 \frac{c}{n} \leq |x_0 - y_0|^2 + \frac{c(T_2+2(4k_0+1))}{n}, \\ \left| h_{\gamma_t} \left(X_t^{\gamma_0, x_0, u, v}, u_t, v_t \right) - h_{\gamma_t} \left(Y_t^{y_0, w^2}, u_t, w_t^2 \right) \right| \\ \leq Lip(h) \left| X_t^{\gamma_0, x_0, u, v} - Y_t^{y_0, w^1} \right| + \frac{c}{n} \leq Lip(h) |x_0 - y_0| + \frac{(Lip(h)+1)c + Lip(h)(t+4k_0+1)}{\sqrt{n}}, \end{array} \right.$$

for all $T_1 \leq t < T_2 \wedge T$, \mathbb{P} -a.s. if $n \geq \max(4, c)$. We continue our construction on $[0, T_3 \wedge T]$, $[0, T_4 \wedge T]$ and so on to finally get a predictable process w^m such that

$$(16) \quad \left\{ \begin{array}{l} \left| X_t^{\gamma_0, x_0, u, v} - Y_t^{y_0, w^m} \right|^2 \leq |x_0 - y_0|^2 + \frac{c[t+(i-1)(4k_0+1)]}{n}, \\ \left| X_{T_i}^{\gamma_0, x_0, u, v} - Y_{T_i}^{y_0, w^m} \right| \leq \min \left(\sqrt{|x_0 - y_0|^2 + \frac{c[T_i+(i-1)(4k_0+1)]}{n}}, 2k_0 \right) + \frac{c}{n} \\ \leq |x_0 - y_0| + \frac{c+T_i+(i-1)(4k_0+1)}{\sqrt{n}}, \\ \left| X_{T_i}^{\gamma_0, x_0, u, v} - Y_{T_i}^{y_0, w^m} \right|^2 \leq |x_0 - y_0|^2 + \frac{c[T_i+i(4k_0+1)]}{n}, \\ \left| h_{\gamma_t} \left(X_t^{\gamma_0, x_0, u, v}, u_t, v_t \right) - h_{\gamma_t} \left(Y_t^{y_0, w^m}, u_t, w_t^m \right) \right| \\ \leq Lip(h) |x_0 - y_0| + \frac{(Lip(h)+1)c + Lip(h)[t+(i-1)(4k_0+1)]}{\sqrt{n}}, \end{array} \right.$$

for all $i \leq m$, $T_{i-1} \leq t < T_i \wedge T$, \mathbb{P} -a.s.

Let us note that in the same way as [48, Inequality 3.27], one has

$$\mathbb{E} \left[e^{-\delta T_m} \right] \leq \left(\frac{\lambda_{\max}}{\delta + \lambda_{\max}} \right)^m,$$

where $\lambda_{\max} = \sup_{(\gamma, x, u, v) \in \mathbb{M} \times \mathbb{R}^N \times U \times V} |\lambda_\gamma(x, u, v)|$. Then, using the estimates (16), one gets

$$\begin{aligned} & \left| \mathbb{E} \left[\delta \int_0^\infty e^{-\delta t} h_{\gamma_t^{\gamma_0, x_0, u, v}} (X_t^{\gamma_0, x_0, u, v}, u_t, v_t) dt \right] - \mathbb{E} \left[\delta \int_0^\infty e^{-\delta t} h_{\gamma_t^{\gamma_0, x_0, u, v}} (Y_t^{y_0, w^m}, u_t, w_t^m) dt \right] \right| \\ & \leq \mathbb{E} \left[\delta \sum_{i=0}^{m-1} \int_{T_i \wedge T}^{T_{i+1} \wedge T} e^{-\delta t} \left(Lip(h) |x_0 - y_0| + \frac{(Lip(h)+1)c + Lip(h)[t + (i-1)(4k_0+1)]}{\sqrt{n}} \right) dt \right] \\ & \quad + 2h_{\max} 1_{T_m < T} \int_{T_m}^T \delta e^{-\delta t} dt + 2h_{\max} e^{-\delta T} \\ & \leq Lip(h) |x_0 - y_0| + \frac{(Lip(h) + 1)c + mLip(h)(4k_0 + 1)}{\sqrt{n}} + \frac{Lip(h)}{\sqrt{n}} \int_0^\infty \delta e^{-\delta t} t dt + 2h_{\max} e^{-\delta T} \\ & \quad + 2h_{\max} \mathbb{E} \left[\left(e^{-\delta T_m} - e^{-\delta T} \right) 1_{T_m < T} \right] \\ & \leq Lip(h) |x_0 - y_0| + \frac{Lip(h) + 1}{\sqrt{n}} \left(c + m(4k_0 + 1) + \frac{1}{\delta} \right) + 2h_{\max} e^{-\delta T} + 2h_{\max} \mathbb{E} \left[e^{-\delta T_m} \right] \\ & \leq Lip(h) |x_0 - y_0| + \frac{Lip(h) + 1}{\sqrt{n}} \left(c + m(4k_0 + 1) + \frac{1}{\delta} \right) + 2h_{\max} e^{-\delta T} + 2h_{\max} \left(\frac{\lambda_{\max}}{\delta + \lambda_{\max}} \right)^m. \end{aligned}$$

The proof of our Lemma is now complete by picking $T = m = n^{\frac{1}{4}}$. ■

5.6 A Toy Nondissipative Model

Let us exhibit a simple choice of coefficients in the study of phage λ . We consider $U = V = [0, 1]$ (worst conditions, best conditions for chemical reactions). For the reaction speed, we take

$$\begin{aligned} k_i(u) &= k_i(u + u_0), \text{ for } i \in \{\pm 2, \pm 3, \pm 4, t\} \text{ for the jump reactions determined by host,} \\ k_1(u, v) &= \frac{1}{\alpha} uv, \quad k_2(u, v) = uv, \quad k_d(u, v) = uv \text{ to reflect a certain competition,} \end{aligned}$$

for all $u, v \in [0, 1]$. Here, $u_0 > 0$ corresponds to the slowest reaction speed, $k_i > 0$ are real constants and α is some maximal concentration level for repressor and dimer. We assume P to toggle between 0 and 1 and, as soon as P toggles to 1 a transcription burst takes place. To take into account the slow aspect of the transcription reaction (see [37]), we actually take

$$\mathbb{M} = \{(1, 0, 0, 0, 0), (0, 1, 0, 0, 0), (0, 1, 0, 0, 1), (0, 0, 1, 0, 0), (0, 0, 1, 0, 0)\}.$$

This means that binding to OC2 corresponds to two states $(0, 1, 0, 0, 0)$ (allowing transcription), $(0, 1, 0, 0, 1)$ (when transcription has just taken place and is no longer allowed). We set

$$\left\{ \begin{array}{l} \lambda_\gamma(u, v) = \lambda_\gamma(u) = \begin{cases} (k_2 + k_3)(u + u_0), & \text{if } \gamma = (1, 0, 0, 0, 0), \\ (k_{-2} + k_4 + k_t)(u + u_0), & \text{if } \gamma = (0, 1, 0, 0, 0), \\ (k_{-2} + k_4)(u + u_0), & \text{if } \gamma = (0, 1, 0, 0, 1), \\ k_{-3}(u + u_0), & \text{if } \gamma = (0, 0, 1, 0, 0), \\ k_{-4}(u + u_0), & \text{if } \gamma = (0, 0, 0, 1, 0). \end{cases} \\ \\ Q^0(\gamma, u) = \begin{cases} \frac{k_2}{k_2+k_3}\delta_{(0,1,0,0,0)} + \frac{k_3}{k_2+k_3}\delta_{(0,0,1,0,0)}, & \text{if } \gamma = (1, 0, 0, 0, 0), \\ \frac{k_{-2}}{k_{-2}+k_4+k_t}\delta_{(1,0,0,0,0)} + \frac{k_4}{k_{-2}+k_4+k_t}\delta_{(0,0,0,1,0)} + \frac{k_t}{k_{-2}+k_4+k_t}\delta_{(0,1,0,0,1)}, & \text{if } \gamma = (0, 1, 0, 0, 0), \\ \frac{k_{-2}}{k_{-2}+k_4}\delta_{(1,0,0,0,0)} + \frac{k_4}{k_{-2}+k_4}\delta_{(0,0,0,1,0)}, & \text{if } \gamma = (0, 1, 0, 0, 1), \\ \delta_{(1,0,0,0,0)}, & \text{if } \gamma = (0, 0, 1, 0, 0), \\ \delta_{(0,1,0,0,0)}, & \text{if } \gamma = (0, 0, 1, 0, 0). \end{cases} \\ \\ g_\gamma(\theta, (x_1, x_2)) = \begin{cases} (0, -\min(1, x_2)), & \text{if } \gamma = (1, 0, 0, 0, 0), \\ (0, \min(1, \alpha - x_2)), & \text{if } \gamma = (0, 1, 0, 0, \gamma_5), \gamma_5 \in \{0, 1\}, \theta = (1, 0, 0, 0, 0), \\ (0, -\min(1, x_2)), & \text{if } \gamma = (0, 1, 0, 0, \gamma_5), \gamma_5 \in \{0, 1\}, \theta = (0, 0, 0, 1, 0), \\ (\min(n, \alpha - x_1), 0), & \text{if } \gamma = (0, 1, 0, 0, 0), \theta = (0, 1, 0, 0, 1), \\ (0, \min(1, \alpha - x_2)), & \text{if } \gamma = (0, 0, 1, 0, 0), \theta = (1, 0, 0, 0, 0), \\ (0, \min(1, \alpha - x_2)), & \text{if } \gamma = (0, 0, 1, 0, 0), \end{cases} \\ \\ f_\gamma((x_1, x_2), u, v) = (f_\gamma^1((x_1, x_2), u, v), f_\gamma^2((x_1, x_2), u, v)) \\ = \left(-\frac{2uv}{\alpha}x_1^2 + 2uvx_2 - uvx_1, \frac{uv}{\alpha}x_1^2 - uvx_2\right). \end{array} \right.$$

The reader is invited to note that $\mathbb{M} \times [0, \alpha]^2$ is invariant with respect to the previous dynamics. One can either note that for $x_1, x_2 \in [0, \alpha]$

$$\begin{aligned} f_\gamma^1((0, x_2), u, v) &= 2x_2uv \geq 0, \quad f_\gamma^2((x_1, 0), u, v) = \frac{x_1^2}{\alpha}uv \geq 0, \\ f_\gamma^1((\alpha, x_2), u, v) &= (2x_2 - 3\alpha)uv \leq 0, \quad f_\gamma^2((x_1, \alpha), u, v) = \left(\frac{x_1^2}{\alpha} - \alpha\right)uv \leq 0, \end{aligned}$$

or, alternatively, compute the normal cone to the frontier of $[0, \alpha]^2$ (see [33, Theorem 2.8]) to deduce that $[0, \alpha]^2$ is invariant with respect to the deterministic dynamics. Moreover, the definition of g_γ guarantees that (X_1, X_2) does not leave $[0, \alpha]^2$. Let us also note that, whenever $x = (x_1, x_2) \in [0, \alpha]^2$, $y = (y_1, y_2) \in [0, \alpha]^2$,

$$\begin{aligned} &\langle f_\gamma(x, u, v) - f_\gamma(y, u, v), x - y \rangle \\ &= -uv \left(\left[\frac{2}{\alpha}(x_1 + y_1) + 1 \right] (x_1 - y_1)^2 - \left[2 + \frac{1}{\alpha}(x_1 + y_1) \right] (x_1 - y_1)(x_2 - y_2) + (x_2 - y_2)^2 \right) \leq 0. \end{aligned}$$

This is a simple consequence of the fact that $\left[2 + \frac{1}{\alpha}(x_1 + y_1) \right]^2 - 4 \left[\frac{2}{\alpha}(x_1 + y_1) + 1 \right] = \left[2 - \frac{1}{\alpha}(x_1 + y_1) \right]^2 - 4 \leq 0$. We deduce that

$$\sup_{v \in V} \inf_{w \in V} \langle f_\gamma(x, u, v) - f_\gamma(y, u, w), x - y \rangle \leq 0.$$

Moreover, for all $(x, y) \in [0, \alpha]^2 \times [0, \alpha]^2$,

$$\sup_{v \in V} \inf_{w \in V} \langle f_\gamma(x, 0, v) - f_\gamma(y, 0, w), x - y \rangle = 0.$$

It follows that one is not able to find any positive constant $c > 0$ such that

$$\sup_{v \in V} \inf_{w \in V} \langle f_\gamma(x, 0, v) - f_\gamma(y, 0, w), x - y \rangle \leq -c|x - y|^2,$$

for all $u \in U$ and we deal with a non-dissipative system (unlike, for instance, [9]).

Finally, the function $t \mapsto t - \min(1, t)$, $t \mapsto t + \min(k, \alpha - t)$ are Lipschitz continuous with Lipschitz constant 1 on $[0, \alpha]$ for all $k > 0$. It follows that $\sup_{\theta \in \mathbb{M}} |x + g_\gamma(\theta, x) - y - g_\gamma(\theta, y)| \leq |x - y|$.

6 Appendix

We provide, in this appendix, the key elements of proof leading to Theorem 2. As we have already hinted before, the proof relies on the same arguments as those developed in [40] combined with dynamic programming principles developed in [48].

If one assumes that $\mathbb{K} \subset \overline{B}(0, k_0)$ is convex and invariant w.r.t. the PDMP governed by (f, λ, Q) , then one modifies the dynamics such that for $\rho \in \{f, \lambda\}$, $\tilde{\rho}_\gamma(x, u) = 0$, if $x \notin B(0, k_0 + 1)$, $\tilde{\rho}_\gamma(x, u) = \rho_\gamma(x, u)$, if $x \in \mathbb{K}$ and setting, for example, $\tilde{g}(\theta, x, u) = \Pi_{\mathbb{K}}(x) - x + g(\theta, \Pi_{\mathbb{K}}(x), u)$. Here, $\Pi_{\mathbb{K}}$ stands for the projector onto \mathbb{K} . In this way, all jumps $x \mapsto x + \tilde{g}(\theta, x, u) = \Pi_{\mathbb{K}}(x) + g(\theta, \Pi_{\mathbb{K}}(x), u)$ take the trajectory in \mathbb{K} (by invariance of this set) and, if the trajectory reaches $\overline{B}(0, k_0 + 1)$, it stays constant. It follows that the set $\mathbb{K}^+ := \overline{B}(0, k_0 + 2)$ is invariant w.r.t the extended dynamics introduced as in Subsection 2.2.2. In fact all sets $\overline{B}(0, k_0 + n)$, $n \geq 2$ are invariant. Let us emphasize that this construction is the only point at which the convexity of \mathbb{K} plays a part and it can be avoided by further assumptions.

Let us fix, for the time being, $\delta > 0$ and $n \geq 1$.

6.1 Dynamic Programming Principle(s) for (Time) Piecewise Constant Policies

The first ingredient is to provide dynamic programming principles and uniform continuity for the value functions given with respect to piecewise constant policies with respect to the initial and auxiliary systems (cf. Section 2.2.2). In addition to $\mathcal{A}_{ad}^n := \mathcal{A}_{ad}^n(\mathbb{U})$, one introduces $\mathcal{B}_{ad}^n = \mathcal{A}_{ad}^n(\mathbb{U} \times \overline{B}(0, 1))$ and

$$v_\varepsilon^{\delta, n}(\gamma, x) = \inf_{(u^1, u^2) \in \mathcal{B}_{ad}^n} \delta \mathbb{E} \left[\int_0^\infty e^{-\delta t} h \left(\Gamma_t^{\gamma, x, u^1, \varepsilon u^2}, X_t^{\gamma, x, u^1, \varepsilon u^2} + \varepsilon u_t^2, u_t^1 \right) dt \right],$$

for all initial data $\gamma \in \mathbb{M}$, $x \in \overline{B}(0, k_0 + 3)$.

One begins with proving the dynamic programming principles.

$$v^{\delta, n}(\gamma, x) = \inf_{u \in \mathcal{A}_0^n} \mathbb{E} \left[\int_0^{T_1 \wedge T} \delta e^{-\delta t} h \left(\Gamma_t^{\gamma, x, u, 0}, X_t^{\gamma, x, u, 0}, u_t \right) dt + e^{-\delta(T_1 \wedge T)} v^{\delta, n} \left(\Gamma_{T_1 \wedge T}^{\gamma, x, u, 0}, X_{T_1 \wedge T}^{\gamma, x, u, 0} \right) \right] \text{ and}$$

$$v_\varepsilon^{\delta, n}(\gamma, x) = \inf_{u \in \mathcal{B}_0^n} \mathbb{E} \left[\int_0^{T_1 \wedge T} \delta e^{-\delta t} h \left(\Gamma_t^{\gamma, x, u^1, \varepsilon u^2}, X_t^{\gamma, x, u^1, \varepsilon u^2} + \varepsilon u_t^2, u_t^1 \right) dt \right. \\ \left. + e^{-\delta(T_1 \wedge T)} v_\varepsilon^{\delta, n} \left(\Gamma_{T_1 \wedge T}^{\gamma, x, u^1, \varepsilon u^2}, X_{T_1 \wedge T}^{\gamma, x, u^1, \varepsilon u^2} \right) \right].$$

The arguments are similar to those employed in [48]. We will only emphasize what changes when using controls from \mathcal{A}_{ad}^n (or \mathcal{B}_{ad}^n) instead of the (more) classical \mathcal{A}_{ad} .

Following [48], we introduce

$$w^{M, n}(\gamma, x) := \inf_{u \in \mathcal{A}_0^n} J^{M, n}(\gamma, x, u),$$

where

$$J^{M, n}(\gamma, x, u) := \mathbb{E} \left[\delta \int_0^{T_1} e^{-\delta t} h \left(\Gamma_t^{\gamma, x, u, 0}, X_t^{\gamma, x, u, 0}, u_t \right) dt + e^{-\delta T_1} w^{M-1, n} \left(\Gamma_{T_1}^{\gamma, x, u, 0}, X_{T_1}^{\gamma, x, u, 0} \right) \right],$$

whenever $M \geq 1$. The initial value $w^{0, n}$ is given with respect to the deterministic control problem (with no jump) and it is standard to check that it is Hölder continuous (the Hölder exponent may be chosen $\frac{\delta}{Lip(f)}$, where $Lip(f)$ is the Lipschitz constant for f_γ for all $\gamma \in \mathbb{M}$ and the Hölder constant only depends on the Lipschitz constants and supremum norm of f and h). In particular, the continuity modulus of $w^{0, n}$ (resp. $w_\varepsilon^{0, n}$ defined w.r.t. \mathcal{B}_0^n) is independent of n (resp. n and ε). Then, one has the following.

Step 1. If $w^{M-1,n} \in BUC(\mathbb{R}^N)$ for all $\gamma \in \mathbb{M}$ and all $x, y \in \mathbb{R}^N$, then the dynamic programming principle holds true for $w^{M,n}$ and all $(\gamma, x) \in \mathbb{M} \times \mathbb{R}^N$, $T \in n^{-1}\mathbb{N}$:

$$w^{M,n}(\gamma, x) = \inf_{u \in \mathcal{A}_0^n} \mathbb{E} \left[\delta \int_0^{T_1 \wedge T} e^{-\delta t} h \left(\Gamma_t^{\gamma, x, u, 0}, X_t^{\gamma, x, u, 0}, u_t \right) dt + e^{-\delta T} w^{M,n} \left(\gamma, \Phi_T^{0, x, u; \gamma} \right) 1_{T_1 > T} \right. \\ \left. + e^{-\delta T_1} w^{M-1,n} \left(\Gamma_{T_1}^{\gamma, x, u, 0}, X_{T_1}^{\gamma, x, u, 0} \right) 1_{T_1 \leq T} \right].$$

The proof is identical with the proof of Lemma 3.1 in [48]. The reader needs only note that the control policy given by (3.5) in [48] of the form

$$\bar{u}(\theta, y, t) := u(\theta, y, t) 1_{[0, T]}(t) + u^* \left(\theta, \Phi_T^{0, y, u; \theta}, t - T \right) 1_{t > T}$$

belongs to \mathcal{A}_0^n if u and u^* belong to \mathcal{A}_0^n .

Step 2. Since $\bar{B}(0, k_0 + 3)$ is invariant with respect to the extended PDMP, one has $w^{M,n} \in BUC(\bar{B}(0, k_0 + 3))$ and, for every $\alpha > 0$, there exists a α -optimal control policy $u^* \in \mathcal{A}_0^n$ such that

$$J^{M,n}(x, u^*) \leq w^{M,n}(x) + \alpha,$$

for all $x \in \bar{B}(0, k_0 + 3)$.

Again, the proof is identical with the proof of the analogous Lemma 3.3 in [48] and based on recurrence. The reader needs only note that there exists a finite family $\{x_k : k = 1, m\}$ such that

$$\bar{B}(0, k_0 + 3) \subset \bigcup_{k=1}^m B(x_k, r).$$

Then, the control policy u defined after (3.18) in [48] belongs to \mathcal{A}_0^n if u_k belong to \mathcal{A}_0^n , for all $k = 1, m$. We also wish to point out that the estimates leading to the continuity modulus of $w^{M,n}$ only depend on the Lipschitz constants and the supremum of h, f, g and λ but are independent of the control policies. In particular, this allows one to work with a common continuity modulus $\omega^{\delta, M}$ for all $n \geq 1$ and $\varepsilon > 0$.

One concludes using the same arguments (no particular changes needed) as those in [48, Theorem 3.4]. Due to [48, Inequality 3.27], one gets

$$\sup_{(\gamma, x) \in \mathbb{M} \times \bar{B}(0, k_0 + 3)} \left| v^{\delta, n}(\gamma, x) - w^{M,n}(\gamma, x) \right| \leq c\alpha^M,$$

where $c > 0$ and $0 < \alpha < 1$ are independent of n (c can be chosen as in [48] equal to $2f_{\max}$ and α as in [48, Page 1120, last line] to be $1 - \int_0^\infty e^{-(\delta + \lambda_{\max})t} dt = \frac{\lambda_{\max}}{\delta + \lambda_{\max}} < 1$). The same is true for $v_\varepsilon^{\delta, n} - w_\varepsilon^{M, n}$ for $\varepsilon > 0$. In particular,

$$\left| v_\varepsilon^{\delta, n}(\gamma, x) - v_\varepsilon^{\delta, n}(\gamma, y) \right| \leq \omega^{\delta, M}(|x - y|) + 2c \left(\frac{\lambda_{\max}}{\delta + \lambda_{\max}} \right)^M,$$

i.e. the continuity modulus of $v_\varepsilon^{\delta, n}$ can also be chosen independent of $n \geq 1$ and $\varepsilon \geq 0$ (we identify $v_0^{\delta, n}$ with $v^{\delta, n}$). This common continuity modulus will be denoted by ω^δ , i.e.

$$(17) \quad \omega^\delta(r) = \sup_{n \geq 1, \varepsilon > 0} \sup_{\substack{\gamma \in \mathbb{M} \\ |x - y| \leq r}} \left| v_\varepsilon^{\delta, n}(\gamma, x) - v_\varepsilon^{\delta, n}(\gamma, y) \right|, \quad r > 0, \quad \omega^\delta(0) := \lim_{\substack{r \rightarrow 0 \\ r > 0}} \omega^\delta(r) = 0.$$

6.2 Estimates and Proof of Theorem 2

We begin with the following convergence result.

Proposition 13 For every $\delta > 0$ there exists a decreasing function $\eta^\delta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ that satisfies $\lim_{\varepsilon \rightarrow 0} \eta^\delta(\varepsilon) = 0$ and such that

$$(18) \quad \sup_{x \in \mathbb{K}^+} \left| v_\varepsilon^{\delta,n}(\gamma, x) - v^{\delta,n}(\gamma, x) \right| \leq \eta^\delta(\varepsilon),$$

for all $n \geq 1$ and all $\varepsilon \geq 0$.

Proof. The proof is similar to the one of Theorem 3.6 in [33]. However, we present the arguments for reader's sake. Let us fix $\gamma \in \mathbb{M}$, $x \in \mathbb{K}^+$ and $\varepsilon > 0$. The definition of the value functions implies that $v_\varepsilon^{\delta,n}(\gamma, x) \leq v^{\delta,n}(\gamma, x)$. Standard estimates yield the existence of some positive constant $C > 0$ which is independent of γ, x , of $n \geq 1$ and $\varepsilon > 0$ such that

$$(19) \quad \left| \Phi_t^{0,x,u^1,\varepsilon u^2;\gamma} - \Phi_t^{0,x,u^1,0;\gamma} \right| \leq C\varepsilon,$$

for all $t \in [0, 1]$, and all $(u^1, u^2) \in \mathcal{B}_0^n$. We recall that $\Phi_t^{0,x,u^1,\varepsilon u^2;\gamma}$ is the unique solution of the deterministic equation

$$\begin{cases} \dot{\Phi}_t^{0,x,u^1,\varepsilon u^2;\gamma} = \bar{f}_\gamma \left(\Phi_t^{0,x,u^1,\varepsilon u^2;\gamma}, u_t^1, \varepsilon u_t^2 \right) = f_\gamma \left(\Phi_t^{0,x,u^1,\varepsilon u^2;\gamma} + \varepsilon u_t^2, u_t^1 \right), \\ \Phi_0^{0,x,u^1,\varepsilon u^2;\gamma} = x. \end{cases}$$

The constant C in (19) is generic and may change from one line to another. We emphasize that throughout the proof, C may be chosen independent of $x \in \mathbb{R}^N$, $n \geq 1$, $\varepsilon > 0$ and of $(u^1, u^2) \in \mathcal{B}_0^n$ (it only depends on the Lipschitz constant of f). The dynamic programming principle written for $v^{\delta,n}$ yields

$$(20) \quad v^{\delta,n}(\gamma, x) \leq \mathbb{E} \left[\int_0^{T_1 \wedge 1} \delta e^{-\delta t} h \left(\Gamma_t^{\gamma,x,u^1,0}, X_t^{\gamma,x,u^1,0}, u_t^1 \right) dt + e^{-\delta(T_1 \wedge 1)} v^{\delta,n} \left(\Gamma_{T_1 \wedge 1}^{\gamma,x,u^1,0}, X_{T_1 \wedge 1}^{\gamma,x,u^1,0} \right) \right],$$

for all $u^1 \in \mathcal{A}_0^n$. We consider an arbitrary admissible control couple $(u^1, u^2) \in \mathcal{B}_0^n$. For simplicity, we introduce the following notations:

$$\begin{aligned} u_t^i &= u^i(x, t), \quad i = 1, 2, \\ \lambda^1(t) &= \lambda_\gamma \left(\Phi_t^{0,x,u^1,0;\gamma}, u_t^1 \right), \quad \Lambda^1(t) = \exp \left(- \int_0^t \lambda^1(s) ds \right) \\ \lambda^{1,2}(t) &= \lambda_\gamma \left(\Phi_t^{0,x,u^1,\varepsilon u^2;\gamma} + \varepsilon u_t^2, u_t^1 \right), \quad \Lambda^{1,2}(t) = \exp \left(- \int_0^t \lambda^{1,2}(s) ds \right), \end{aligned}$$

for all $t \geq 0$. We denote the right-hand member of the inequality (20) by I . Then, I is explicitly given by

$$\begin{aligned} I &= \int_0^1 \lambda^1(t) \Lambda^1(t) \int_0^t \delta e^{-\delta s} h \left(\gamma, \Phi_s^{0,x,u^1,0;\gamma}, u_s^1 \right) ds dt \\ &+ \int_0^1 \lambda^1(t) \Lambda^1(t) e^{-\delta t} \int_{\mathbb{R}^N} v^{\delta,n} \left(\theta, \Phi_t^{0,x,u^1,0;\gamma} + g_\gamma \left(\theta, \Phi_t^{0,x,u^1,0;\gamma}, u_t^1 \right) \right) Q^0(\gamma, u_t^1, d\theta) dt \\ &+ \Lambda^1(1) \int_0^1 \delta e^{-\delta t} h \left(\gamma, \Phi_t^{0,x,u^1,0;\gamma}, u_t^1 \right) dt + \Lambda^1(1) e^{-\delta} v^{\delta,n} \left(\gamma, \Phi_1^{0,x,u^1,0;\gamma} \right) \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned}$$

Using the inequality (19), one gets

$$(21) \quad I_1 \leq \int_0^1 \lambda^{1,2}(t) \Lambda^{1,2}(t) \int_0^t \delta e^{-\delta s} h\left(\gamma, \Phi_s^{0,x,u^1,\varepsilon u^2;\gamma} + \varepsilon u_s^2, u_s^1\right) ds dt + C\varepsilon,$$

$$(22) \quad I_3 \leq \Lambda^{1,2}(1) \int_0^1 \delta e^{-\delta t} h\left(\gamma, \Phi_t^{0,x,u^1,\varepsilon u^2;\gamma} + \varepsilon u_t^2, u_t^1\right) dt + C\varepsilon.$$

For the term I_2 , with the notation (17), one has

$$\begin{aligned} I_2 &\leq \int_0^1 \lambda^{1,2}(t) \Lambda^{1,2}(t) e^{-\delta t} \int_{\mathbb{R}^N} v^{\delta,n}\left(\theta, \Phi_t^{0,x,u^1,\varepsilon u^2;\gamma} + g_\gamma\left(\theta, \Phi_t^{0,x,u^1,\varepsilon u^2;\gamma}, u_t^1\right)\right) Q^0(\gamma, u_t^1, d\theta) dt \\ &\quad + C\left(\varepsilon + \omega^\delta(C\varepsilon)\right) \\ &\leq \int_0^1 \lambda^{1,2}(t) \Lambda^{1,2}(t) e^{-\delta t} \int_{\mathbb{R}^N} v_\varepsilon^{\delta,n}\left(\theta, \Phi_t^{0,x,u^1,\varepsilon u^2;\gamma} + \varepsilon u_t^2 + g_\gamma\left(\theta, \Phi_t^{0,x,u^1,\varepsilon u^2;\gamma} + \varepsilon u_t^2, u_t^1\right)\right) Q^0(\gamma, u_t^1, d\theta) dt \\ (23) \quad &+ \left(\int_0^1 \lambda^{1,2}(t) \Lambda^{1,2}(t) e^{-\delta t} dt\right) \sup_{\theta \in \mathbb{M}, z \in \mathbb{K}^+} \left|v^{\delta,n}(z) - v_\varepsilon^{\delta,n}(z)\right| + C\left(\varepsilon + \omega^\delta(C\varepsilon)\right). \end{aligned}$$

Finally,

$$\begin{aligned} I_4 &\leq \Lambda^{1,2}(1) e^{-\delta} v^{\delta,n}\left(\gamma, \Phi_1^{0,x,u^1,\varepsilon u^2;\gamma}\right) + C\left(\varepsilon + \omega^\delta(C\varepsilon)\right) \\ (24) \quad &\leq \Lambda^{1,2}(1) e^{-\delta} v_\varepsilon^{\delta,n}\left(\gamma, \Phi_1^{0,x,u^1,\varepsilon u^2;\gamma}\right) + \Lambda^{1,2}(1) e^{-\delta} \sup_{\theta \in \mathbb{M}, z \in \mathbb{K}^+} \left|v^{\delta,n}(\theta, z) - v_\varepsilon^{\delta,n}(\theta, z)\right| + C\left(\omega^\delta(C\varepsilon) + \varepsilon\right). \end{aligned}$$

We substitute (21)-(24) in (20). We take the infimum over the family of $(u^1, u^2) \in \mathcal{B}_0^n$ and use the dynamic programming principle to have

$$\begin{aligned} v^{\delta,n}(\gamma, x) &\leq v_\varepsilon^{\delta,n}(\gamma, x) + C\left(\varepsilon + \omega^\delta(C\varepsilon)\right) \\ &\quad + \left(\int_0^1 \lambda^{1,2}(t) \Lambda^{1,2}(t) e^{-\delta t} dt + \Lambda^{1,2}(1) e^{-\delta}\right) \sup_{\theta \in \mathbb{M}, z \in \mathbb{K}^+} \left|v^{\delta,n}(z) - v_\varepsilon^{\delta,n}(z)\right|. \end{aligned}$$

We notice that

$$\begin{aligned} \int_0^1 \lambda^{1,2}(t) \Lambda^{1,2}(t) e^{-\delta t} dt + \Lambda^{1,2}(1) e^{-\delta} &= 1 - \delta \int_0^1 e^{-\int_0^t \bar{\lambda}_\gamma\left(\Phi_s^{0,x,u^1,\varepsilon u^2;\gamma}, u_s^1, u_s^2\right) ds} e^{-\delta t} dt \\ &\leq \frac{\lambda_{\max}}{\lambda_{\max} + \delta} + \frac{\delta}{\lambda_{\max} + \delta} e^{-(\lambda_{\max} + \delta)}. \end{aligned}$$

Thus,

$$\begin{aligned} v^{\delta,n}(\gamma, x) - v_\varepsilon^{\delta,n}(\gamma, x) &\leq C\left(\varepsilon + \omega^\delta(C\varepsilon)\right) \\ &\quad + \left(\frac{\lambda_{\max}}{\lambda_{\max} + \delta} + \frac{\delta}{\lambda_{\max} + \delta} e^{-(\lambda_{\max} + \delta)}\right) \sup_{\theta \in \mathbb{M}, z \in \mathbb{R}^N} \left|v^{\delta,n}(\theta, z) - v_\varepsilon^{\delta,n}(\theta, z)\right|. \end{aligned}$$

Here, $\lambda_{\max} := \sup\{\lambda_\gamma(x, u) : \gamma \in \mathbb{M}, x \in \mathbb{R}^N, u \in \mathbb{U}\} < \infty$. The conclusion follows by taking the supremum over $\theta \in \mathbb{M}$ and $x \in \mathbb{R}^N$ and recalling that C is independent of x and $\varepsilon > 0$ (and $n \geq 1$).

■

We consider (ρ_ε) a sequence of standard mollifiers i.e. $\rho_\varepsilon(y) = \frac{1}{\varepsilon^N} \rho\left(\frac{y}{\varepsilon}\right)$, $y \in \mathbb{R}^N$, $\varepsilon > 0$, where $\rho \in C^\infty(\mathbb{R}^N)$ is a positive function such that

$$\text{Supp}(\rho) \subset \overline{B}(0, 1) \text{ and } \int_{\mathbb{R}^N} \rho(x) dx = 1.$$

We introduce the convoluted functions

$$v_{(\varepsilon)}^{\delta, n}(\gamma, \cdot) := v_\varepsilon^{\delta, n}(\gamma, \cdot) * \rho_\varepsilon.$$

In analogy to [40, Lemma 3.5], one gets

Proposition 14 *The value functions $v_{(\varepsilon)}^{\delta, n}$ are such that*

$$\left\{ \begin{array}{l} \sup_{x \in \mathbb{K}} \left(\left| v_{(\varepsilon)}^{\delta, n}(\gamma, x) \right| + \left| \partial_{x^i} v_{(\varepsilon)}^{\delta, n}(\gamma, x) \right| \right) \leq C^\delta \varepsilon^{-1} (\varepsilon + \omega^\delta(\varepsilon)), \\ \sup_{x, y \in \mathbb{K}, y \neq x} \frac{\left| \partial_{x^i} v_{(\varepsilon)}^{\delta, n}(\gamma, x) - \partial_{x^i} v_{(\varepsilon)}^{\delta, n}(\gamma, y) \right|}{|x - y|} \leq C^\delta \varepsilon^{-1} \omega^\delta(|x - y|) \\ \sup_{x \in \mathbb{K}} \left| v_{(\varepsilon)}^{\delta, n}(\gamma, x) - v^{\delta, n}(\gamma, x) \right| \leq \omega^\delta(\varepsilon) + \eta^\delta(\varepsilon), \end{array} \right.$$

for all $\gamma \in \mathbb{M}$. Here, C^δ is a positive constant independent of $\varepsilon > 0$, $n \geq 1$ and $\gamma \in \mathbb{M}$.

Proof. To prove the first inequality, one recalls the definition of $v_{(\varepsilon)}^{\delta, n}$. Then, due to Proposition 13 and using the notation (17), one gets

$$\begin{aligned} \left| \partial_{x^i} v_{(\varepsilon)}^{\delta, n}(x) \right| &= \left| \varepsilon^{-1} \int_{\overline{B}(0, 1)} v_\varepsilon^{\delta, n}(x - \varepsilon y) \partial_{x^i} \rho(y) dy \right| = \left| \varepsilon^{-1} \int_{y \in \overline{B}(0, 1)} \left(v_\varepsilon^{\delta, n}(x - \varepsilon y) - v_\varepsilon^{\delta, n}(x) \right) \partial_{x^i} \rho(y) dy \right| \\ &\leq C^\delta \varepsilon^{-1} \omega^\delta(\varepsilon). \end{aligned}$$

Similarly,

$$\left| \partial_{x^i} v_{(\varepsilon)}^{\delta, n}(x) - \partial_{x^i} v_{(\varepsilon)}^{\delta, n}(y) \right| = \left| \varepsilon^{-1} \int_{\overline{B}(0, 1)} \left(v_\varepsilon^{\delta, n}(x - \varepsilon z) - v_\varepsilon^{\delta, n}(y - \varepsilon z) \right) \partial_{x^i} \rho(z) dz \right| \leq C^\delta \varepsilon^{-1} \omega^\delta(|x - y|).$$

Moreover, again with the notation (17) and the help of Proposition 13, one gets

$$\begin{aligned} \sup_{x \in \mathbb{R}^N} \left| v_{(\varepsilon)}^{\delta, n}(x) - v^{\delta, n}(x) \right| &= \left| \int_{y \in \overline{B}(0, 1)} \left(v_\varepsilon^{\delta, n}(x - \varepsilon y) - v_\varepsilon^{\delta, n}(x) + v_\varepsilon^{\delta, n}(x) - v^{\delta, n}(x) \right) \rho(y) dy \right| \\ &\leq \omega^\delta(\varepsilon) + \eta^\delta(\varepsilon). \end{aligned}$$

The proof of our proposition is now complete. ■

We now come to the proof of the main convergence result.

Proof. (of Theorem 2). Let us fix $(u^1, u^2) \in \mathbb{U} \times \overline{B}(0, 1)$. The dynamic programming principle written for $v_\varepsilon^{\delta, n}$ yields

$$v_\varepsilon^{\delta, n}(\gamma, x) \leq \mathbb{E} \left[\int_0^{T_1 \wedge n^{-1}} \delta e^{-\delta t} h \left(\Gamma_t^{\gamma, x, u^1, \varepsilon u^2}, X_t^{\gamma, x, u^1, \varepsilon u^2} + \varepsilon u_t^2, u_t^1 \right) dt \right] + e^{-\delta(T_1 \wedge n^{-1})} v_\varepsilon^{\delta, n} \left(\Gamma_{T_1 \wedge n^{-1}}^{\gamma, x, u^1, \varepsilon u^2}, X_{T_1 \wedge n^{-1}}^{\gamma, x, u^1, \varepsilon u^2} \right),$$

where $(u^1, u^2) \in \mathcal{B}_{ad}^n$ and $x \in \mathbb{K}$. In particular, if $(u_t^1, u_t^2) = (u, y) \in \mathbb{U} \times \overline{B}(0, 1)$, for $t \in [0, n^{-1}]$, one notices that on $[0, T_1 \wedge n^{-1}]$,

$$X_t^{\gamma, x - \varepsilon y, u^1, \varepsilon u^2} = X_t^{\gamma, x, u^1, 0} - \varepsilon y \text{ and } \Gamma_t^{\gamma, x, u^1, \varepsilon u^2} = \Gamma_t^{\gamma, x, u^1, 0}.$$

As consequence, the (law of the) first jump time starting from $(\gamma, x - \varepsilon y)$ when the trajectory is controlled by the couple (u^1, u^2) given above only depends on u and x (but not on ε , nor on y). To emphasize this dependence, we denote it by $T_1^{x, u}$. Similarly, the law of $(\Gamma_{T_1^{x, u} \wedge n^{-1}}^{\gamma, x, u^1, \varepsilon u^2}, X_{T_1^{x, u} \wedge n^{-1}}^{\gamma, x, u^1, \varepsilon u^2})$ is the same as the law of $(\Gamma_{T_1^{x, u} \wedge n^{-1}}^{\gamma, x, u^1, 0}, X_{T_1^{x, u} \wedge n^{-1}}^{\gamma, x, u^1, 0} - \varepsilon y)$. One gets

$$v_{\varepsilon}^{\delta, n}(\gamma, x - \varepsilon y) \leq \mathbb{E} \left[\int_0^{T_1^{x, u} \wedge n^{-1}} \delta e^{-\delta t} h(\Gamma_t^{\gamma, x, u^1, 0}, X_t^{\gamma, x, u^1, 0}, u_t^1) dt + e^{-\delta(T_1^{x, u} \wedge n^{-1})} v_{\varepsilon}^{\delta, n}(\Gamma_{T_1^{x, u} \wedge n^{-1}}^{\gamma, x, u^1, 0}, X_{T_1^{x, u} \wedge n^{-1}}^{\gamma, x, u^1, 0} - \varepsilon y) \right].$$

It follows that

$$v_{(\varepsilon)}^{\delta, n}(\gamma, x) \leq \mathbb{E} \left[\int_0^{T_1^{x, u} \wedge n^{-1}} \delta e^{-\delta t} h(\Gamma_t^{\gamma, x, u^1, 0}, X_t^{\gamma, x, u^1, 0}, u_t^1) dt + e^{-\delta(T_1^{x, u} \wedge n^{-1})} v_{(\varepsilon)}^{\delta, n}(\Gamma_{T_1^{x, u} \wedge n^{-1}}^{\gamma, x, u^1, 0}, X_{T_1^{x, u} \wedge n^{-1}}^{\gamma, x, u^1, 0}) \right].$$

Applying Itô's formula to $v_{(\varepsilon)}^{\delta, n}(\Gamma_t^{\gamma, x, u, 0}, X_t^{\gamma, x, u, 0})$ on $[0, T_1^{x, u} \wedge n^{-1}]$ and recalling that $u_t^1 = u$ prior to n^{-1} , it follows that

$$\begin{aligned} 0 &\leq \mathbb{E} \left[\int_0^{T_1^{x, u} \wedge n^{-1}} e^{-\delta t} \left[\delta \left(h(\Gamma_t^{\gamma, x, u, 0}, X_t^{\gamma, x, u, 0}, u) - v_{(\varepsilon)}^{\delta, n}(\Gamma_t^{\gamma, x, u, 0}, X_t^{\gamma, x, u, 0}) \right) + \mathcal{L}^u v_{(\varepsilon)}^{\delta, n}(\Gamma_t^{\gamma, x, u, 0}, X_t^{\gamma, x, u, 0}) \right] dt \right] \\ &= \mathbb{E} \left[\int_0^{T_1^{x, u} \wedge n^{-1}} e^{-\delta t} \left[\delta \left(h(\gamma, \Phi_t^{0, x, u^1, 0; \gamma}, u) - v_{(\varepsilon)}^{\delta, n}(\gamma, \Phi_t^{0, x, u^1, 0; \gamma}) \right) + \mathcal{L}^u v_{(\varepsilon)}^{\delta, n}(\gamma, \Phi_t^{0, x, u^1, 0; \gamma}) \right] dt \right] \\ &\leq \mathbb{E} \left[\int_0^{T_1^{x, u} \wedge n^{-1}} e^{-\delta t} dt \right] \left(\delta \left(h(\gamma, x, u) - v_{(\varepsilon)}^{\delta, n}(\gamma, x) \right) + \mathcal{L}^u v_{(\varepsilon)}^{\delta, n}(\gamma, x) \right) \\ &+ \mathbb{E} \left[\int_0^{T_1^{x, u} \wedge n^{-1}} e^{-\delta t} \left[\delta \left(|h|_1 + C^\delta \varepsilon^{-1} (\varepsilon + \omega^\delta(\varepsilon)) |f|_1 t + |f|_1^2 t C^\delta \varepsilon^{-1} (\varepsilon + \omega^\delta(\varepsilon)) \right) \right. \right. \\ &\quad \left. \left. + |f|_1 C^\delta \varepsilon^{-1} \omega^\delta \left(\frac{|f|_1}{n} \right) + 2|h|_1 |\lambda|_1 |f|_1 t + |\lambda|_1 C^\delta \varepsilon^{-1} (\varepsilon + \omega^\delta(\varepsilon)) (1 + |g|_1) |f|_1 t \right] dt \right] \\ &\leq \mathbb{E} [T_1^{x, u} \wedge n^{-1}] \left(\left[\delta \left(h(\gamma, x, u) - v_{(\varepsilon)}^{\delta, n}(\gamma, x) \right) + \mathcal{L}^u v_{(\varepsilon)}^{\delta, n}(\gamma, x) \right] \right) \\ &+ \mathbb{E} [T_1^{x, u} \wedge n^{-1}] \tilde{C}^\delta \left(\frac{1 + \varepsilon^{-1} \omega^\delta(\varepsilon)}{n} + \varepsilon^{-1} \omega^\delta \left(\frac{|f|_1}{n} \right) \right). \end{aligned}$$

The generic constant \tilde{C}^δ is independent of $x \in \mathbb{K}, \gamma \in \mathbb{M}, u \in \mathbb{U}, \varepsilon > 0$ and $n \geq 1$ and may change from one line to another. As consequence,

$$\delta \left(h(\gamma, x, u) - v_{(\varepsilon)}^{\delta, n}(\gamma, x) \right) + \mathcal{L}^u v_{(\varepsilon)}^{\delta, n}(\gamma, x) \geq -\tilde{C}^\delta \left(\frac{1 + \varepsilon^{-1} \omega^\delta(\varepsilon)}{n} + \varepsilon^{-1} \omega^\delta \left(\frac{|f|_1}{n} \right) \right).$$

We fix (for the time being), the initial configuration (γ_0, x_0) and an arbitrary control $u^1 \in \mathcal{A}_{ad}$. We apply the previous inequality for $(\gamma, x) = (\Gamma_t^{\gamma_0, x_0, u^1, 0}, X_t^{\gamma_0, x_0, u^1, 0})$ and integrate the inequality with respect to $e^{-\delta t} dt$ on $[0, T]$ for $T > 0$ and use Itô's formula to get

$$\begin{aligned} \mathbb{E} \left[\int_0^T \delta e^{-\delta t} h(\Gamma_t^{\gamma_0, x_0, u^1, 0}, X_t^{\gamma_0, x_0, u^1, 0}, u_t^1) dt \right] &\geq v_{(\varepsilon)}^{\delta, n}(\gamma_0, x_0) - e^{-\delta T} v_{(\varepsilon)}^{\delta, n}(\Gamma_T^{\gamma_0, x_0, u^1, 0}, X_T^{\gamma_0, x_0, u^1, 0}) \\ &\quad - \tilde{C}^\delta \left(\frac{1 + \varepsilon^{-1} \omega^\delta(\varepsilon)}{n} + \varepsilon^{-1} \omega^\delta \left(\frac{|f|_1}{n} \right) \right). \end{aligned}$$

One lets $T \rightarrow \infty$ and takes the infimum over $u^1 \in \mathcal{A}_{ad}$ to get

$$v^\delta(\gamma_0, x_0) \geq v_{(\varepsilon)}^{\delta, n}(\gamma_0, x_0) - \tilde{C}^\delta \left(\frac{1 + \varepsilon^{-1} \omega^\delta(\varepsilon)}{n} + \varepsilon^{-1} \omega^\delta \left(\frac{|f|_1}{n} \right) \right).$$

Finally, using the third estimate in Proposition 14, one gets

$$v^\delta(\gamma_0, x_0) \geq v^{\delta, n}(\gamma_0, x_0) - \tilde{C}^\delta \left(\frac{1 + \varepsilon^{-1} \omega^\delta(\varepsilon)}{n} + \varepsilon^{-1} \omega^\delta \left(\frac{|f|_1}{n} \right) + \omega^\delta(\varepsilon) + \eta^\delta(\varepsilon) \right).$$

The conclusion follows by taking $\varepsilon = \left(\omega^\delta \left(\frac{|f|_1}{n} \right) \right)^{1-\eta}$, for some $1 > \eta > 0$ (e.g. $\eta = \frac{1}{2}$). The reader will note that $\omega^\delta(r) \geq cr$ where the equality corresponds to the Lipschitz case). Our result is now complete. ■

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