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# Rigidity results with applications to best constants and symmetry of Caffarelli-Kohn-Nirenberg and logarithmic Hardy inequalities

Jean Dolbeault · Maria J. Esteban ·  
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**Abstract** We take advantage of a rigidity result for the equation satisfied by an extremal function associated with a special case of the Caffarelli-Kohn-Nirenberg inequalities to get a symmetry result for a larger set of inequalities. The main ingredient is a reparametrization of the solutions to the Euler-Lagrange equations and estimates based on the rigidity result. The symmetry results cover a range of parameters which go well beyond the one that can be achieved by symmetrization methods or comparison techniques so far.

**Keywords** Caffarelli-Kohn-Nirenberg inequalities; Hardy-Sobolev inequality; extremal functions; ground state; bifurcation; branches of solutions; Emden-Fowler transformation; radial symmetry; symmetry breaking; rigidity; Keller-Lieb-Thirring inequalities

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## 1 Introduction and main results

Let  $2^* := \infty$  if  $d = 1, 2$ , and  $2^* := 2d/(d-2)$  if  $d \geq 3$ . Define

$$\vartheta(p, d) := \frac{d(p-2)}{2p}, \quad a_c := \frac{d-2}{2},$$

and consider the space  $\mathcal{D}_a^{1,2}(\mathbb{R}^d)$  obtained by completion of  $\mathcal{D}(\mathbb{R}^d \setminus \{0\})$  with respect to the norm  $v \mapsto \| |x|^{-a} \nabla v \|_{L^2(\mathbb{R}^d)}^2$ . We will be concerned with the following two families of inequalities

**Caffarelli-Kohn-Nirenberg Inequalities (CKN)** [2] Let  $d \geq 1$ . For any  $p \in [2, 2^*]$  if  $d \geq 3$  or  $p \in [2, 2^*)$  if  $d = 1, 2$ , for any  $\theta \in [\vartheta(p, d), 1]$  with  $\theta > 1/2$  if  $d = 1$ , there exists a positive constant  $C_{\text{CKN}}(\theta, p, a)$  such that

$$\left( \int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{bp}} dx \right)^{\frac{2}{p}} \leq C_{\text{CKN}}(\theta, p, a) \left( \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2a}} dx \right)^{\theta} \left( \int_{\mathbb{R}^d} \frac{|v|^2}{|x|^{2(a+1)}} dx \right)^{1-\theta} \quad (1)$$

holds true for any  $v \in \mathcal{D}_a^{1,2}(\mathbb{R}^d)$ . Here  $a, b$  and  $p$  are related by  $b = a - a_c + d/p$ , with the restrictions  $a \leq b \leq a + 1$  if  $d \geq 3$ ,  $a < b \leq a + 1$  if  $d = 2$  and  $a + 1/2 < b \leq a + 1$  if  $d = 1$ . Moreover, the constants  $C_{\text{CKN}}(\theta, p, a)$  are uniformly bounded outside a neighborhood of  $a = a_c$ .

In [4], a new class of inequalities, called *weighted logarithmic Hardy inequalities*, was considered. These inequalities can be obtained from (1) by taking  $\theta = \gamma(p-2)$  and passing to the limit as  $p \rightarrow 2_+$ .

**Weighted Logarithmic Hardy Inequalities (WLH)**[4] Let  $d \geq 1, a < a_c, \gamma \geq d/4$  and  $\gamma > 1/2$  if  $d = 2$ . Then there exists a positive constant  $C_{\text{WLH}}(\gamma, a)$  such that, for any  $v \in \mathcal{D}_a^{1,2}(\mathbb{R}^d)$  normalized by

$$\int_{\mathbb{R}^d} |x|^{-2(a+1)} |v|^2 dx = 1,$$

we have

$$\int_{\mathbb{R}^d} \frac{|v|^2}{|x|^{2(a+1)}} \log \left( |x|^{2(a_c-a)} |v|^2 \right) dx \leq 2\gamma \log \left[ C_{\text{WLH}}(\gamma, a) \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2a}} dx \right]. \quad (2)$$

Moreover, the constants  $C_{\text{WLH}}(\gamma, a)$  are uniformly bounded outside a neighborhood of  $a = a_c$ .

It is very convenient to reformulate the Caffarelli-Kohn-Nirenberg inequality in cylindrical variables as in [3]. By means of the Emden-Fowler transformation

$$s = \log |x| \in \mathbb{R}, \quad \omega = \frac{x}{|x|} \in \mathbb{S}^{d-1}, \quad y = (s, \omega), \quad u(y) = |x|^{a_c-a} v(x),$$

Inequality (1) for  $v$  is equivalent to a Gagliardo-Nirenberg-Sobolev inequality for the function  $u$  on the cylinder  $\mathcal{C} := \mathbb{R} \times \mathbb{S}^{d-1}$ :

$$\mathsf{K}_{\text{CKN}}(\theta, p, \Lambda) \|u\|_{L^p(\mathcal{C})}^2 \leq \left( \|\nabla u\|_{L^2(\mathcal{C})}^2 + \Lambda \|u\|_{L^2(\mathcal{C})}^2 \right)^\theta \|u\|_{L^2(\mathcal{C})}^{2(1-\theta)} \quad \forall u \in \mathsf{H}^1(\mathcal{C}) . \quad (3)$$

Here and throughout the rest of the work we set

$$\Lambda := (a_c - a)^2 .$$

Similarly, with  $u(y) = |x|^{a_c - a} v(x)$ , Inequality (2) is equivalent to

$$\int_{\mathcal{C}} |u|^2 \log |u|^2 \, dy \leq 2\gamma \log \left[ \frac{1}{\mathsf{K}_{\text{WLH}}(\gamma, \Lambda)} \left( \|\nabla u\|_{L^2(\mathcal{C})}^2 + \Lambda \right) \right] , \quad (4)$$

for any  $u \in \mathsf{H}^1(\mathcal{C})$  such that  $\|u\|_{L^2(\mathcal{C})} = 1$ . In both cases, we consider on  $\mathcal{C}$  the measure  $d\mu = |\mathbb{S}^{d-1}|^{-1} d\omega \, ds$  obtained by normalizing the surface of  $\mathbb{S}^{d-1}$  to 1 (that is, the uniform probability measure), tensorized with the usual Lebesgue measure on the axis of the cylinder.

We are interested in *symmetry* and *symmetry breaking* issues: when do we know that equality in (1) and (2) is achieved by radial functions or, alternatively, by functions depending only on  $s$  in (3) and (4)? Related with inequality (3) is the Rayleigh quotient:

$$\mathcal{Q}_\Lambda^\theta[u] := \frac{\left( \|\nabla u\|_2^2 + \Lambda \|u\|_2^2 \right)^\theta \|u\|_2^{2(1-\theta)}}{\|u\|_p^2} .$$

Here  $\|u\|_q := \left( \int_{\mathcal{C}} |u|^q \, d\mu \right)^{1/q}$ . Then (3) and (4) are equivalent to state that

$$\begin{aligned} \mathsf{K}_{\text{CKN}}(\theta, p, \Lambda) &= \inf_{u \in \mathsf{H}^1(\mathcal{C}) \setminus \{0\}} \mathcal{Q}_\Lambda^\theta[u] , \\ \mathsf{K}_{\text{WLH}}(\gamma, \Lambda) &= \inf_{\substack{u \in \mathsf{H}^1(\mathcal{C}) \setminus \{0\} \\ \|u\|_2 = 1}} \left( \|\nabla u\|_2^2 + \Lambda \right) e^{-\frac{1}{2\gamma} \int_{\mathcal{C}} |u|^2 \log |u|^2 \, d\mu} . \end{aligned}$$

Let  $\mathsf{K}_{\text{CKN}}^*(\theta, p, \Lambda)$  and  $\mathsf{K}_{\text{WLH}}^*(\gamma, \Lambda)$  be the corresponding values of the infimum when the set of minimization is restricted to functions depending only on  $s$ . The main interest of introducing the measure  $d\mu$  is that  $\mathsf{K}_{\text{CKN}}^*(\theta, p, \Lambda)$  and  $\mathsf{K}_{\text{WLH}}^*(\gamma, \Lambda)$  are independent of the dimension and can be computed for  $d = 1$  by solving the problem on the real line  $\mathbb{R}$ .

Radial symmetry of  $v = v(x)$  means that  $u = u(s, \omega)$  is independent of  $\omega$ . Up to translations in  $s$  and a multiplication by a constant, the optimal functions in the class of functions depending only on  $s \in \mathbb{R}$  solve the equation

$$-u_*'' + \Lambda u_* = u_*^{p-1} \quad \text{in } \mathbb{R}$$

if  $\theta = 1$ . See Section 2 if  $\theta < 1$ . Up to translations in  $s$ , non-negative solutions of this equation are all equal to the function

$$u_*(s) := \frac{A}{\left[ \cosh(Bs) \right]^{\frac{2}{p-2}}} \quad \forall s \in \mathbb{R} , \quad (5)$$

with  $A^{p-2} = \frac{p}{2} \Lambda$  and  $B = \frac{1}{2} \sqrt{\Lambda} (p-2)$ . The uniqueness up to translations is a standard result (see for instance [11, Proposition B.2] for a proof).

The symmetry breaking issue is now reduced to the question of knowing whether the inequalities

$$\mathsf{K}_{\text{CKN}}(\theta, p, \Lambda) \leq \mathsf{K}_{\text{CKN}}^*(\theta, p, \Lambda) \quad \text{and} \quad \mathsf{K}_{\text{WLH}}(\gamma, \Lambda) \leq \mathsf{K}_{\text{WLH}}^*(\gamma, \Lambda) \quad (6)$$

are strict or not, when  $d \geq 2$ . Symmetry breaking occurs if the inequality is strict and then optimal functions *are not* symmetric (symmetric means: depending only on  $s$  in the setting of the cylinder, or on  $|x|$  in the case of the Euclidean space). In [4, pp. 2048 and 2057], the values of the symmetric constants have been computed. They are given by

$$\mathsf{K}_{\text{CKN}}^*(\theta, p, \Lambda) := \left[ \frac{2p\theta+2-p}{(p-2)^2} \right]^{\frac{p-2}{2p}} \left[ \frac{2p\theta}{2p\theta+2-p} \right]^\theta \left[ \frac{p+2}{4} \right]^{\frac{6-p}{2p}} \left[ \frac{\sqrt{\pi} \Gamma(\frac{2}{p-2})}{\Gamma(\frac{2}{p-2} + \frac{1}{2})} \right]^{\frac{p-2}{p}} \Lambda^{\theta - \frac{p-2}{2p}} \quad (7)$$

and

$$\begin{aligned} \mathsf{K}_{\text{WLH}}^*(\gamma, \Lambda) &= \frac{\gamma (8\pi^{d+1} e)^{\frac{1}{4\gamma}}}{\Gamma(\frac{d}{2})^{2\gamma}} \left( \frac{4\Lambda}{4\gamma-1} \right)^{\frac{4\gamma-1}{4\gamma}} \quad \text{if } \gamma > \frac{1}{4}, \\ \mathsf{K}_{\text{WLH}}^*(\gamma, \Lambda) &= \frac{2\pi^{d+1} e}{\Gamma(\frac{d}{2})^2} \quad \text{if } \gamma = \frac{1}{4}. \end{aligned}$$

Let

$$\Lambda_{\text{FS}}(\theta, p, d) := 4 \frac{d-1}{p^2-4} \frac{(2\theta-1)p+2}{p+2} \quad \text{and} \quad \Lambda_*(1, p, d) := \frac{1}{4} (d-1) \frac{6-p}{p-2}. \quad (8)$$

We will define  $\Lambda_*(\theta, p, d)$  for  $\theta < 1$  later in the Introduction. Symmetry breaking occurs for any  $\Lambda > \Lambda_{\text{FS}}$  according to a result of V. Felli and M. Schneider in [15] for  $\theta = 1$  and in [4] for  $\theta < 1$  (also see [3] for previous results and [14] if  $d = 2$  and  $\theta = 1$ ). This symmetry breaking is a straightforward consequence of the fact that for  $\Lambda > \Lambda_{\text{FS}}$ , the symmetric *optimals* are saddle points of an energy functional, and thus cannot be even local minima. As a consequence, we know that  $\mathsf{K}_{\text{CKN}}(\theta, p, \Lambda) < \mathsf{K}_{\text{CKN}}^*(\theta, p, \Lambda)$  if  $\Lambda > \Lambda_{\text{FS}}(\theta, p, d)$ .

Concerning the log Hardy inequality, it was shown in [4] that symmetry breaking occurs, that is,  $\mathsf{K}_{\text{WLH}}(\gamma, \Lambda) < \mathsf{K}_{\text{WLH}}^*(\gamma, \Lambda)$ , when either  $d = 2$  and  $\gamma > 1/2$  or  $d \geq 3$  and  $\gamma \geq d/4$  provided that

$$\Lambda > (d-1) \left( \gamma - \frac{1}{4} \right).$$

Concerning symmetry, if  $\theta = 1$ , from [12], we know that symmetry holds for CKN for any  $\Lambda \leq \Lambda_*(1, p, d)$ . The precise statement goes as follows.

**Theorem 1** [12] *Let  $d \geq 2$ . For any  $p \in [2, 2^*]$  if  $d \geq 3$  or  $p \in [2, \infty)$  if  $d = 2$ , under the conditions*

$$0 < \mu \leq \Lambda_*(1, p, d) \quad \text{and} \quad \mathcal{Q}_\mu^1[u] \leq \mathsf{K}_{\text{CKN}}^*(1, p, \mu),$$

the solution of

$$-\Delta u + \mu u = u^{p-1} \quad \text{on } \mathcal{C} \quad (9)$$

is given by the one-dimensional equation, written on  $\mathbb{R}$ . It is unique, up to translations.

Theorem 1 is a *rigidity result*. In [12], the proof is given for a minimizer of  $\mathcal{Q}_\mu^1$ , which therefore satisfies  $\mathcal{Q}_\mu^1[u] \leq \mathcal{K}_{\text{CKN}}^*(1, p, \mu)$ , but the reader is invited to check that only the latter condition is used in the proof. The proof is based on a chain of estimates which involve optimal interpolation inequalities on the sphere and the Keller-Lieb-Thirring inequality. These inequalities turn out to be equalities, and equality in each of the inequalities is shown to imply that the solution only depends on  $s$  (no angular dependence). The result of Theorem 1 gives a sufficient condition for symmetry when  $\theta = 1$ . We shall say that *any minimizer is symmetric* if it is given by (5), up to multiplications by constants and translations.

**Theorem 2** [12] *Let  $d \geq 2$ . For any  $p \in [2, 2^*]$  if  $d \geq 3$  or any  $p \in [2, \infty)$  if  $d = 2$ , if  $0 < \Lambda \leq \Lambda_*(1, p, d)$ , then  $\mathcal{K}_{\text{CKN}}(1, p, \Lambda) = \mathcal{K}_{\text{CKN}}^*(1, p, \Lambda)$  and any minimizer is symmetric.*

In [12], the case  $\theta < 1$  is also considered. According to [12, Theorem 9], for any  $d \geq 3$ , any  $p \in (2, 2^*)$  and any  $\theta \in [\vartheta(p, d), 1)$ , we have the estimate

$$\mathfrak{C}(\theta, p)^{-\frac{2\theta}{q+2}} \mathcal{K}_{\text{CKN}}^*(\theta, \Lambda, p) \leq \mathcal{K}_{\text{CKN}}(\theta, \Lambda, p) \leq \mathcal{K}_{\text{CKN}}^*(\theta, \Lambda, p) \quad (10)$$

where  $q := \frac{2(p-2)}{(2\theta-1)p+2}$  and

$$\mathfrak{C}(\theta, p) := \frac{(p+2)^{\frac{p+2}{(2\theta-1)p+2}}}{(2\theta-1)^{\frac{p+2}{p+2}}} \left(2 - \frac{p}{2}(1-\theta)\right)^{1-\frac{q}{2}} \cdot \left(\frac{\Gamma(\frac{p}{p-2})}{\Gamma(\frac{\theta p}{p-2})}\right)^{2q} \left(\frac{\Gamma(\frac{2\theta p}{p-2})}{\Gamma(\frac{2p}{p-2})}\right)^q$$

under the condition  $a_c^2 < \Lambda \leq \frac{(d-1)}{\mathfrak{C}(\theta, p)} \frac{(2\theta-3)p+6}{4(p-2)}$ . If  $\theta = 1$ , the equality case in the last inequality characterizes  $\Lambda_*(1, p, d)$  as defined in (8). However (10) does not give a range for symmetry unless  $\theta = 1$ .

Much more is known. According to [13, 5], there is a continuous curve  $p \mapsto \Lambda_s(\theta, p, d)$  with  $\lim_{p \rightarrow 2^+} \Lambda_s(\theta, p, d) = \infty$  and  $\Lambda_s(\theta, p, d) > a_c^2$  for any  $p \in (2, 2^*)$  such that symmetry holds for any  $\Lambda \leq \Lambda_s(1, p, d)$  and there is symmetry breaking if  $\Lambda > \Lambda_s(1, p, d)$ , for any  $\theta \in [\vartheta(p, d), 1)$ . Additionally, we have that  $\lim_{p \rightarrow 2^+} \Lambda_s(1, p, d) = a_c^2$  if  $d \geq 3$  and, if  $d = 2$ ,  $\lim_{p \rightarrow \infty} \Lambda_s(1, p, d) = 0$  and  $\lim_{p \rightarrow \infty} p^2 \Lambda_s(1, p, d) = 4$ . The existence of this function  $\Lambda_s$  has been proven in an indirect way, and it is not explicitly known. It has been a long-standing question to decide whether the curves  $p \rightarrow \Lambda_s(\theta, p, d)$  and the curve  $p \rightarrow \Lambda_{\text{FS}}(\theta, p, d)$  coincide or not. This is still an open question, at least for  $\theta = 1$ . For  $\theta < 1$ , and for some specific values of  $p$ , it has been shown that, in some cases,  $\Lambda_s(\theta, p, d) < \Lambda_{\text{FS}}(\theta, p, d)$ ; see [5] for more details, as well as some symmetry results based on symmetrization techniques. A scenario based on numerical computations and asymptotic expansions at the point where non-symmetric positive solutions bifurcate from the symmetric ones has been proposed; see [7, 9, 10] for details.

Our interest in this work is to establish symmetry of the minimizers of CKN for  $\theta < 1$  as well as of the log Hardy inequalities, thus identifying the corresponding sharp constants.

Our first result is an extension of Theorem 2 to the case  $\theta < 1$ . Our goal is to give explicit estimates of the range for which symmetry holds. This requires some notations and a preliminary result. We set

$$\Pi^*(\theta, p, q) := \left( \frac{\mathsf{K}_{\text{CKN}}^*(\theta, p, 1)}{\mathsf{K}_{\text{CKN}}^*(1, q, 1)^{\frac{q(p-2)}{p(q-2)}}} \right)^{\frac{1}{\theta - \frac{q(p-2)}{p(q-2)}}}. \quad (11)$$

Next we define

$$q^* = q^*(\theta, p) := \frac{2p\theta}{2 - p(1 - \theta)}. \quad (12)$$

The condition  $\theta > \frac{q(p-2)}{p(q-2)}$  is equivalent to  $q > q^*(\theta, p)$  and we can notice that  $p < q^*(\theta, p) < 2^*$  for any  $\theta \in (\vartheta(p, d), 1)$ . For  $d \geq 3$  we define

$$\Lambda_1(\theta, p, d) := \max_{q \in (q^*, 2^*)} \min \left\{ \Lambda_\star(1, q, d), \frac{\theta \Lambda_\star(1, p, d)}{(1 - \theta) \Pi^*(\theta, p, q) + \theta} \right\},$$

whereas for  $d = 2$

$$\Lambda_1(\theta, p, 2) := \max_{q \in (q^*, 6)} \min \left\{ \Lambda_\star(1, q, 2), \frac{\theta \Lambda_\star(1, p, 2)}{(1 - \theta) \Pi^*(\theta, p, q) + \theta} \right\}.$$

Next, we can also define

$$\mathsf{N}(\theta, p) := \frac{(\mathsf{K}_{\text{CKN}}^*(\theta, p, 1))^{1/\theta}}{\mathsf{K}_{\text{CKN}}^*(1, q^*(\theta, p), 1)}. \quad (13)$$

We refer to Section 3 for an explicit expression of  $\mathsf{N}(\theta, p)$ . We introduce the exponent

$$\beta = \beta(\theta, p) := 1 - \frac{p-2}{2p\theta}. \quad (14)$$

For  $2 < p < 6$  and  $\theta \in (\vartheta(p, 3), 1)$  we denote by  $x^* = x^*(\theta, p)$  the unique root of the equation

$$\theta(6-p)(x^\beta - \mathsf{N})x - (2p\theta - 3(p-2))(\theta(x^\beta - \mathsf{N}) + (1-\theta)(x-1)\mathsf{N}) = 0,$$

in the interval  $(\mathsf{N}^{1/\beta}, \infty)$  for  $\mathsf{N} = \mathsf{N}(\theta, p)$ , see Lemma 2 in Section 3. Next we define

$$\Lambda_2(\theta, p, d) := \frac{\Lambda_\star(1, q^*, d)}{x^*(\theta, p)} = \frac{1}{4}(d-1) \frac{2p\theta - 3(p-2)}{(p-2)x^*(\theta, p)},$$

and

$$\Lambda_\star(\theta, p, d) := \max \left\{ \Lambda_1(\theta, p, d), \Lambda_2(\theta, p, d) \right\}.$$

**Theorem 3** *Suppose that either  $d = 2$  and  $p \in (2, 6)$  or else  $d \geq 3$  and  $p \in (2, 2^*)$ . Then*

$$\mathsf{K}_{\text{CKN}}(\theta, p, \Lambda) = \mathsf{K}_{\text{CKN}}^*(\theta, p, \Lambda),$$

*and any minimizer of CKN (3) is symmetric provided that one of the following conditions is satisfied:*

- (i)  $d = 2$ ,  $\theta \in (\vartheta(p, 2), 1)$  and  $0 < \Lambda \leq \Lambda_1(\theta, p, 2)$ .
- (ii)  $d = 2$ ,  $\theta \in (\vartheta(p, 3), 1)$  and  $0 < \Lambda \leq \Lambda_\star(\theta, p, 2)$ ,
- (iii)  $d \geq 3$ ,  $\theta = \vartheta(p, d)$  and  $0 < \Lambda \leq \Lambda_2(\theta, p, d)$ ,
- (iv)  $d \geq 3$ ,  $\theta \in (\vartheta(p, d), 1)$  and  $0 < \Lambda \leq \Lambda_\star(\theta, p, d)$  .

Our definition of  $\Lambda_\star(\theta, p, d)$  for  $\theta < 1$  is consistent with the definition of  $\Lambda_\star(1, p, d)$  given in (8) because

$$\lim_{\theta \rightarrow 1} \Lambda_1(\theta, p, d) = \lim_{\theta \rightarrow 1} \Lambda_2(\theta, p, d) = \Lambda_\star(1, p, d) .$$

One of the drawbacks in the definition of  $\Lambda_2(\theta, p, d)$  is that  $x^\star(\theta, p)$  given by Lemma 2 is not explicit. For an explicit estimate of  $\Lambda_2(\theta, p, d)$  see Proposition 2 in Section 5.

By passing to the limit as  $p \rightarrow 2_+$  in the criterion  $\Lambda \leq \Lambda_2(\theta, p, d)$ , we also obtain an explicit condition for symmetry in the weighted logarithmic Hardy inequalities. For any  $N_0 > 1$ , consider the smallest root  $x > N_0^{1/\beta_0}$  of

$$4\gamma x^{\beta_0+1} - (8\gamma - 3)N_0 x + (4\gamma - 3)N_0 = 0 \quad \text{with} \quad \beta_0 = 1 - \frac{1}{4\gamma}$$

and denote it by  $x_0^\star(\gamma)$  if  $N_0 = N_0(\gamma) := \lim_{p \rightarrow 2_+} N(\gamma(p-2), p)$ . An elementary but tedious computation shows that

$$N_0(\gamma) = 2^{1-\frac{3}{4\gamma}} e^{\frac{1}{4\gamma}} \frac{(2\gamma - 1)^{1-\frac{1}{\gamma}}}{(4\gamma - 1)^{1-\frac{3}{4\gamma}}} \left( \frac{\Gamma(2\gamma - \frac{1}{2})}{\Gamma(2\gamma - 1)} \right)^{\frac{1}{2\gamma}} . \quad (15)$$

Let us define

$$\Lambda_0(\gamma, d) := \frac{(d-1)(\gamma - 3/4)}{x_0^\star(\gamma)} . \quad (16)$$

We then have

**Theorem 4** *Assume that either  $d = 2$  or  $3$  and  $\gamma > 3/4$ , or  $d \geq 4$  and  $\gamma \geq d/4$ . Then*

$$\mathbf{K}_{\text{WLH}}(\gamma, \Lambda) = \mathbf{K}_{\text{WLH}}^\star(\gamma, \Lambda) ,$$

and any minimizer of (4) is symmetric provided that

$$0 < \Lambda \leq \Lambda_0(\gamma, d) .$$

For an explicit estimate of  $\Lambda_0(\gamma, d)$  see Proposition 3 in Section 5.

Theorem 3 provides us with a *rigidity result*, which is stronger than a simple symmetry result. As a consequence, our estimates of Theorem 3 for the symmetry region cannot be optimal.

**Theorem 5** *Suppose that either  $d = 2$  and  $p \in (2, 6)$  or else  $d \geq 3$  and  $p \in (2, 2^*)$ . If  $\theta > \vartheta(p, \min\{3, d\})$ , then*

$$\Lambda_\star(\theta, p, d) < \Lambda_s(\theta, p, d) \leq \Lambda_{\text{FS}}(\theta, p, d) .$$

If either  $d = 3$  and  $\theta = \vartheta(p, 3)$ , or  $d = 2$  and  $\theta > 0$ , then

$$\Lambda_2(\theta, p, d) < \Lambda_s(\theta, p, d) \leq \Lambda_{\text{FS}}(\theta, p, d) .$$



It can be conjectured that  $\Lambda_s(\theta, p, d) = \Lambda_{\text{FS}}(\theta, p, d)$  holds in the limit case  $\theta = 1$ , and probably also for  $\theta$  close enough to 1, on the basis of the numerical results of [9] and the formal computations of [10]. On the other hand, it is known from [5] that  $\Lambda_s(\theta, p, d) < \Lambda_{\text{FS}}(\theta, p, d)$  when  $\theta - \vartheta(p, d)$  is small enough, at least for some values of  $p$  and  $d$ .

The expressions involved in the statement of Theorem 3 look quite technical, but they are interesting for two reasons:

- Theorem 3 determines a range for symmetry which goes well beyond what can be achieved using standard methods and is somewhat unexpected in view of the estimate of [12, Theorem 9]. It is a striking observation that the reparametrization method which has been extensively used in [9, 10] allows us to extend to  $\theta < 1$  results which were known only for  $\theta = 1$ .
- Even if they cannot be optimal as shown in Theorem 5, the estimates of Theorem 3 are rather accurate from the numerical point of view, as will be illustrated in Section 5.

This paper is organized as follows. Section 2 is devoted to the reparametrization and the proof of symmetry when  $\Lambda \leq \Lambda_1(\theta, p, d)$  in the subcritical case  $\vartheta(p, d) < \theta < 1$ . To the price of some additional technicalities, the range  $\Lambda \leq \Lambda_2(\theta, p, d)$  and  $\vartheta(p, \min\{3, d\}) \leq \theta < 1$  is covered in Section 3. The proofs of Theorems 3 and 5 are established in Section 4. The last section is devoted to an explicit approximation of  $\Lambda_0$  and  $\Lambda_2$ , and some numerical results which illustrate Theorems 3 and 5. The reader interested in the strategy of the proofs as well as the origin of the expressions of  $\Lambda_1(\theta, p, d)$  and  $\Lambda_2(\theta, p, d)$  is invited to read first Section 2 and the proof of Lemma 5 in Section 3.

## 2 Reparametrization and a first symmetry result

We begin by a reparametrization of the branches of the solutions which allows us to reduce the case corresponding to  $\theta < 1$  and  $\Lambda$  to the case corresponding to  $\theta = 1$  and some related  $\mu$ , as in Theorem 1. Consider an optimal function  $u$  for (3), which therefore satisfies

$$\mathcal{K}_{\text{CKN}}(\theta, p, \Lambda) = \mathcal{Q}_\Lambda^\theta[u] = (t + \Lambda)^\theta \frac{\|u\|_2^2}{\|u\|_p^2} \quad \text{with} \quad t := \frac{\|\nabla u\|_2^2}{\|u\|_2^2}.$$

According to [5, Theorem 1], such a function  $u$  exists for any  $\theta > \vartheta(p, d)$ . As a critical point of  $\mathcal{Q}_\Lambda^\theta$ ,  $u$  solves (9) with

$$\theta \mu = (1 - \theta) t + \Lambda$$

if it has been normalized by the condition

$$\|\nabla u\|_2^2 + \Lambda \|u\|_2^2 = \theta \|u\|_p^p.$$

Because of the zero-homogeneity of  $\mathcal{Q}_\Lambda^\theta$ , such a condition can be imposed without restriction and is equivalent to

$$\|u\|_2^2 = \frac{\theta}{t + \Lambda} \|u\|_p^p. \quad (17)$$

**Proposition 1** *Let us assume that  $u$  is a solution of (9), satisfying  $\mathcal{Q}_\Lambda^\theta[u] = \mathcal{K}_{\text{CKN}}(\theta, p, \Lambda)$  and (17), with  $\theta\mu = (1 - \theta)t + \Lambda$ . Then we have*

$$\mathcal{Q}_\mu^1[u] \leq \mathcal{K}_{\text{CKN}}^*(1, p, \mu). \quad (18)$$

*Proof* From (6) we know that

$$(t + \Lambda)^\theta \frac{\|u\|_2^2}{\|u\|_p^2} \leq \mathcal{K}_{\text{CKN}}^*(\theta, p, \Lambda).$$

Using (17), we rewrite this estimate as

$$\theta(t + \Lambda)^{\theta-1} \|u\|_p^{p-2} \leq \mathcal{K}_{\text{CKN}}^*(\theta, p, \Lambda).$$

Using (17) again and the expression of  $\mu$ , we obtain

$$\mathcal{Q}_\mu^1[u] = \frac{\theta(t + \mu)}{t + \Lambda} \|u\|_p^{p-2} = \|u\|_p^{p-2} \leq f(t, \theta, \Lambda, p) \mathcal{K}_{\text{CKN}}^*(1, p, \mu)$$

with

$$f(t, \theta, \Lambda, p) := \frac{1}{\theta(t + \Lambda)^{\theta-1}} \frac{\mathcal{K}_{\text{CKN}}^*(\theta, p, \Lambda)}{\mathcal{K}_{\text{CKN}}^*(1, p, \mu)}.$$

Using the expression of  $\mu$  and (7), we find that

$$f(t, \theta, \Lambda, p) = \frac{(p+2)^{\frac{p+2}{2p}}}{(2p)^{1-\theta}} \left( \frac{\Lambda\theta}{2+(2\theta-1)p} \right)^{\theta - \frac{p-2}{2p}} (t + \Lambda)^{1-\theta} ((1-\theta)t + \Lambda)^{-\frac{p+2}{2p}}$$

achieves its maximum at  $t_0 := \Lambda \left( \frac{2p\theta}{p-2} - 1 \right)^{-1} > 0$ . Hence  $f(t) \leq f(t_0) = 1$ , which concludes the proof.

Using the notations (11) and (12), we obtain our first symmetry result, which goes as follows.

**Lemma 1** *Suppose that either  $d = 2$  and  $p \in (2, 6)$  or else  $d \geq 3$ ,  $p \in (2, 2^*)$ . If  $\theta \in (\vartheta(p, d), 1)$  and*

$$\Lambda \leq \min \left\{ \Lambda_\star(1, q, d), \frac{\theta \Lambda_\star(1, p, d)}{(1-\theta) \Pi^\star(\theta, p, q) + \theta} \right\}$$

*for some  $q \in (q^\star(\theta, p), 6)$  when  $d = 2$ , or for some  $q \in (q^\star(\theta, p), 2^*)$  when  $d \geq 3$ , then any optimal function for (3) is symmetric.*

*Proof* Let  $u$  be a solution as in Proposition 1. From (6), we know that

$$\mathsf{K}_{\text{CKN}}^*(\theta, p, \Lambda) \geq (t + \Lambda)^\theta \frac{\|u\|_2^2}{\|u\|_p^2}.$$

For  $p < q < \min\{6, 2^*\}$  we have by Hölder's inequality,  $\|u\|_p \leq \|u\|_2^\delta \|u\|_q^{1-\delta}$  provided  $\delta = \frac{2}{p} \frac{q-p}{q-2}$ , and thus  $1 - \delta = \frac{q}{p} \frac{p-2}{q-2}$ . Hence

$$\mathsf{K}_{\text{CKN}}^*(\theta, p, \Lambda) \geq (t + \Lambda)^\theta \left( \frac{\|u\|_2^2}{\|u\|_q^2} \right)^{1-\delta}.$$

Now, for any  $\lambda \in (0, \Lambda_*(1, q, d)]$ , we know from Theorem 2 that

$$\|u\|_q^2 \leq \frac{\|\nabla u\|_2^2 + \lambda \|u\|_2^2}{\mathsf{K}_{\text{CKN}}^*(1, q, \lambda)},$$

which shows that

$$\mathsf{K}_{\text{CKN}}^*(\theta, p, \Lambda) \geq (t + \Lambda)^\theta \left( \frac{\mathsf{K}_{\text{CKN}}^*(1, q, \lambda)}{t + \lambda} \right)^{1-\delta}.$$

Summarizing, we have found that

$$\frac{(t + \Lambda)^\theta}{(t + \lambda)^{1-\delta}} \leq \frac{\mathsf{K}_{\text{CKN}}^*(\theta, p, \Lambda)}{(\mathsf{K}_{\text{CKN}}^*(1, q, \lambda))^{1-\delta}} \quad \text{if } \lambda \leq \Lambda_*(1, q, d). \quad (19)$$

Next we can make the ansatz  $\lambda = \Lambda$ . Provided  $\Lambda \leq \Lambda_*(1, q, d)$ , we get that

$$(t + \Lambda)^{\theta+\delta-1} \leq \frac{\mathsf{K}_{\text{CKN}}^*(\theta, p, \Lambda)}{(\mathsf{K}_{\text{CKN}}^*(1, q, \Lambda))^{1-\delta}} = (\mathcal{H}^*(\theta, p, q) \Lambda)^{\theta+\delta-1},$$

so that  $t \leq (\mathcal{H}^*(\theta, p, q) - 1) \Lambda$ . According to Theorem 1,  $u$  is symmetric if

$$\frac{1}{\theta} ((1 - \theta)t + \Lambda) = \mu \leq \Lambda_*(1, p, d), \quad (20)$$

because (18) holds by Proposition 1. This completes the proof.

In the next section we shall consider an alternative ansatz for which  $\lambda \neq \Lambda$ .

### 3 Another symmetry result

In this section we establish an estimate similar to the one of Lemma 1 but based on a different ansatz, which moreover covers the critical case  $\theta = \vartheta(p, d)$ . We recall that  $\beta = \beta(\theta, p) = 1 - \frac{p-2}{2p\theta}$  has been defined in (14). The proof is slightly more technical than the one of Lemma 1. We start with an auxiliary result.

**Lemma 2** For any  $N > 1$ ,  $p < 6$  and  $\theta \in (\vartheta(p, 3), 1)$ , if  $\beta = \beta(\theta, p)$  is given by (14), the equation

$$\theta(6-p)(x^\beta - N)x - (2p\theta - 3(p-2))(\theta(x^\beta - N) + (1-\theta)(x-1)N) = 0,$$

has a unique root in the interval  $(N^{1/\beta}, \infty)$ .

When  $N = N(\theta, p) > 1$  is given by (13), we denote this root by  $x^* = x^*(\theta, p)$ .

*Proof* Consider the function

$$f(x) := \theta(6-p)(x^\beta - N)x - (2p\theta - 3(p-2))[\theta(x^\beta - N) + (1-\theta)(x-1)N],$$

and notice first that  $f(N^{1/\beta}) < 0$  because  $\theta > \vartheta(p, 3)$  and  $N^{1/\beta} > 1$ . Next we observe that  $\alpha := 2p\theta - 3(p-2) = 2p(\theta - \vartheta(p, 3)) = 6 - p - 2p(1-\theta)$  and compute

$$f'(x) = (6-p)\theta[(1+\beta)x^\beta - N] - 2p(\theta - \vartheta(p, 3))[\beta\theta x^{\beta-1} + (1-\theta)N]$$

and

$$f''(x) = \beta\theta x^{\beta-2}[(6-p)(1+\beta)x - (\beta-1)(6-p-2p(1-\theta))] > 0$$

for any  $x > 1$ . Using the fact that  $N > 1$ , we find that

$$f'(N^{1/\beta}) \geq 2(p-2)(1-\theta)N > 0.$$

It follows that the function  $f(x)$  is increasing and convex for  $x > N^{1/\beta}$ . Since  $f(N^{1/\beta}) < 0$  we conclude that  $f(x)$  has a unique root for  $x > N^{1/\beta}$ .

When  $N = N(\theta, p)$  we only need to check that  $N(\theta, p) > 1$ . This is shown in Lemma 4. Before, we need a preliminary estimate. Consider the *Digamma* function  $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$ .

**Lemma 3** For all  $z > 0$ , we have

$$\frac{1}{2z} < \psi\left(z + \frac{1}{2}\right) - \psi(z) < \ln\left(1 + \frac{1}{2z}\right) + \frac{1}{z} - \frac{2}{2z+1}.$$

*Proof* We use the following representation formula (cf. [1, § 6.3.21, p. 259]):

$$\psi(z) = \int_0^\infty \left( \frac{e^{-t}}{t} - \frac{e^{-zt}}{1-e^{-t}} \right) dt$$

and elementary manipulations to get the lower bound

$$\psi\left(z + \frac{1}{2}\right) - \psi(z) = \int_0^\infty \frac{e^{-zt}}{1+e^{-t/2}} dt > \frac{1}{2} \int_0^\infty e^{-zt} dt = \frac{1}{2z}.$$

As for the upper bound, we have the equivalences

$$\begin{aligned} & \ln\left(1 + \frac{1}{2z}\right) + \frac{1}{z} - \frac{2}{2z+1} - \int_0^\infty \frac{e^{-zt}}{1+e^{-t/2}} dt > 0 \\ \iff & \ln\left(1 + \frac{1}{2z}\right) + \int_0^\infty e^{-zt} dt - \frac{2}{2z+1} - \int_0^\infty \frac{e^{-zt}}{1+e^{-t/2}} dt > 0 \\ \iff & \ln\left(1 + \frac{1}{2z}\right) + \int_0^\infty \frac{e^{-t/2} e^{-zt}}{1+e^{-t/2}} dt - \frac{2}{2z+1} > 0. \end{aligned}$$

The result follows from

$$\int_0^\infty \frac{e^{-(z+\frac{1}{2})t}}{1+e^{-t/2}} dt > \frac{1}{2} \int_0^\infty e^{-(z+\frac{1}{2})t} dt = \frac{1}{2z+1}$$

and, by monotonicity of the function  $z \mapsto \ln\left(1 + \frac{1}{2z}\right) - \frac{1}{2z+1}$ ,

$$\ln\left(1 + \frac{1}{2z}\right) > \frac{1}{2z+1}.$$

**Lemma 4** *Assume that  $2 < p < 6$  and  $\vartheta(p, 2) < \theta \leq 1$ . Then the function  $\theta \mapsto \mathbf{N}(\theta, p)$  is decreasing and  $\mathbf{N}(1, p) = 1$ .*

*Proof*  $\mathbf{N}(1, p) = 1$  is a consequence of the definition of  $\mathbf{N}$ . Using the precise value  $\text{pf } \mathbf{K}_{\text{CKN}}^*(\theta, p, \Lambda)$ , we obtain the following explicit expression of the function  $\mathbf{N}(\theta, p)$ , namely

$$\left(\frac{2}{2-p(1-\theta)}\right)^{\frac{p-2}{2p\theta}} \left(\frac{p+2}{4}\right)^{\frac{6-p}{2p\theta}} \left(\frac{2(2-p(1-\theta))}{(2\theta-1)p+2}\right)^{\frac{2p\theta-3(p-2)}{2p\theta}} \left[\frac{\Gamma\left(\frac{2}{p-2}\right)\Gamma\left(\frac{2-p(1-\theta)}{p-2} + \frac{1}{2}\right)}{\Gamma\left(\frac{2}{p-2} + \frac{1}{2}\right)\Gamma\left(\frac{2-p(1-\theta)}{p-2}\right)}\right]^{\frac{p-2}{p\theta}}.$$

Let us define  $G := \mathbf{N}^\theta$  and compute

$$\begin{aligned} \frac{1}{G} \frac{\partial G}{\partial \theta} &= \frac{p\theta - 2(p-2)}{2-p(1-\theta)} - \frac{2p\theta - 3(p-2)}{(2\theta-1)p+2} + \ln\left(\frac{2(2-p(1-\theta))}{(2\theta-1)p+2}\right) \\ &\quad + \frac{\Gamma'\left(\frac{2-p(1-\theta)}{p-2} + \frac{1}{2}\right)}{\Gamma\left(\frac{2-p(1-\theta)}{p-2} + \frac{1}{2}\right)} - \frac{\Gamma'\left(\frac{2-p(1-\theta)}{p-2}\right)}{\Gamma\left(\frac{2-p(1-\theta)}{p-2}\right)}. \end{aligned}$$

By Lemma 3 we get that

$$\begin{aligned} \frac{1}{G} \frac{\partial G}{\partial \theta} &< \frac{p\theta - 2(p-2)}{2-p(1-\theta)} - \frac{2p\theta - 3(p-2)}{(2\theta-1)p+2} + \ln\left(\frac{2(2-p(1-\theta))}{(2\theta-1)p+2}\right) \\ &\quad + \ln\left(\frac{(2\theta-1)p+2}{2(2-p(1-\theta))}\right) + \frac{p-2}{2-p(1-\theta)} - \frac{2(p-2)}{(2\theta-1)p+2} = 0. \end{aligned}$$

Since

$$\frac{1}{G} \frac{\partial G}{\partial \theta} = \ln \mathbf{N} + \frac{\theta}{\mathbf{N}} \frac{\partial \mathbf{N}}{\partial \theta} < 0,$$

for  $\theta \in (\vartheta(p, 2), 1]$  and  $\mathbf{N}(1, p) = 1$ , it follows that  $\frac{\partial}{\partial \theta} \mathbf{N} < 0$ .

After these preliminaries, we can now state the main result of this section.

**Lemma 5** *Assume that*

$$\begin{aligned} 2 < p < 6 \quad \text{and} \quad \vartheta(p, 3) < \theta < 1 \quad \text{if} \quad d = 2 \quad \text{or} \quad 3, \\ 2 < p < 2^* \quad \text{and} \quad \vartheta(p, d) \leq \theta < 1 \quad \text{if} \quad d \geq 4. \end{aligned}$$

*Then any optimal function for (3) is symmetric if  $\Lambda < \Lambda_2(\theta, p, d)$ . Moreover, we have  $\lim_{\theta \rightarrow 1} \Lambda_2(\theta, p, d) = \Lambda_*(1, p, d)$ .*

*Proof* As in the proof of Lemma 1, the starting point of our estimate is inequality (19), which becomes

$$\frac{t + \Lambda}{t + \lambda} \leq \mathbf{N}(\theta, p) \left( \frac{\Lambda}{\lambda} \right)^\beta \quad \text{if} \quad \lambda < \Lambda_*(1, q, d),$$

under the restriction that we choose  $q = q^*(\theta, p)$  given by (12), that is  $1 - \delta = \theta$  with  $\delta$  as in (11). Remarkably, we observe that, for this specific value of  $q$ , we have

$$\theta - \frac{p-2}{2p} = \left( 1 - \frac{q-2}{2q} \right) (1 - \delta)$$

and, as a consequence,

$$\frac{t + \Lambda}{t + \lambda} \leq \mathbf{N} \left( \frac{\Lambda}{\lambda} \right)^\beta$$

where  $\beta := 1 - \frac{p-2}{2p\theta}$  and  $\mathbf{N} = \mathbf{N}(\theta, p)$ . Hence we get that

$$t \leq \frac{\mathbf{N} \Lambda^\beta \lambda - \Lambda \lambda^\beta}{\lambda^\beta - \mathbf{N} \Lambda^\beta} =: \bar{t}.$$

As in the proof of Lemma 1, we can apply Theorem 1 if

- Condition (20) holds and a sufficient condition is therefore given by the condition

$$(1 - \theta) \bar{t} + \Lambda \leq \theta \Lambda_*(1, p, d),$$

that is,

$$(\theta \Lambda_*(1, p, d) - \Lambda) (\lambda^\beta - \mathbf{N} \Lambda^\beta) \geq (1 - \theta) (\mathbf{N} \Lambda^\beta \lambda - \Lambda \lambda^\beta).$$

- Condition  $\lambda < \Lambda_*(1, q, d)$ , which is required to get (19), holds, *i.e.*,

$$\lambda < \Lambda_*(1, q, d) = \frac{1}{4} (d-1) \frac{6-q}{q-2} = \frac{1}{4} (d-1) \frac{2p\theta - 3(p-2)}{p-2}.$$

For a suitable  $x = \lambda/\Lambda > \mathbf{N}^{1/\beta}$ , to be chosen, these two conditions amount to

$$\begin{aligned} \Lambda \leq \phi(x) &:= \frac{\theta \Lambda_\star(1, p, d) (x^\beta - \mathbf{N})}{\theta (x^\beta - \mathbf{N}) + (1 - \theta) (x - 1) \mathbf{N}} , \\ \Lambda < \chi(x) &:= \frac{1}{4} (d - 1) \frac{2p\theta - 3(p - 2)}{p - 2} \frac{1}{x} . \end{aligned}$$

After replacing  $\Lambda_\star(1, p, d)$  by its value according to (8), we get that  $\phi(x) - \chi(x)$  has the sign of  $f(x)$  as defined in the proof of Lemma 2. By Corollary 4, we know that  $\mathbf{N} \geq 1$  and conclude henceforth that any minimizer is *symmetric* if  $\Lambda < \chi(x^*(\theta, p)) = \Lambda_2(\theta, p, d)$ .

In the limiting regime corresponding to as  $\theta \rightarrow 1_-$ , we observe that  $\phi(x) = \Lambda_\star(1, p, d)$  and  $\chi(x) = \Lambda_\star(1, p, d)/x$ , so that  $\lim_{\theta \rightarrow 1} \Lambda_2(\theta, p, d) = \chi(1) = \Lambda_\star(1, p, d)$ .

#### 4 Proof of the main results

*Proof (Theorem 3)* It is a straightforward consequence of Lemma 1 and Lemma 5. Notice that  $\lim_{\theta \rightarrow 1} \Lambda_1(\theta, p, d) = \Lambda_\star(1, p, d)$  because

$$\lim_{\theta \rightarrow 1} \frac{\theta \Lambda_\star(1, p, d)}{(1 - \theta) \Pi^*(\theta, p, q) + \theta} = \Lambda_\star(1, p, d) .$$

*Proof (Theorem 5)* The function  $q \mapsto \Lambda_1(1, q, d)$  is monotone decreasing and

$$q^*(\theta, p) - p = \frac{p(p - 2)(1 - \theta)}{2 - p(1 - \theta)} \geq 0$$

so that, for  $i = 1, 2$ ,

$$\Lambda_i(\theta, p, d) \leq \Lambda_\star(1, q^*(\theta, p), d) \leq \Lambda_\star(1, p, d) < \Lambda_{\text{FS}}(\theta, p, d) .$$

By definition of  $\Lambda_s(\theta, p, d)$ , we know that  $\Lambda_\star(\theta, p, d) \leq \Lambda_s(\theta, p, d)$ . By Theorem 3, if  $\Lambda = \Lambda_\star(\theta, p, d)$  any minimizer for  $\mathcal{K}_{\text{CKN}}(\theta, p, \Lambda)$  is symmetric. On the other hand, by continuity, we know that

$$\mathcal{K}_{\text{CKN}}(\theta, p, \Lambda_s(\theta, p, d)) = \mathcal{K}_{\text{CKN}}^*(\theta, p, \Lambda_s(\theta, p, d)) .$$

Let us assume that  $\Lambda_s(\theta, p, d) < \Lambda_{\text{FS}}(\theta, p, d)$  and consider a sequence  $(\lambda_n)_{n \in \mathbb{N}}$  converging to  $\Lambda_s(\theta, p, d)$  with  $\lambda_n > \Lambda_s(\theta, p, d)$ . If  $u_n$  is a non-symmetric minimizer of  $\mathcal{K}_{\text{CKN}}(\theta, p, \lambda_n)$ , we can pass to the limit: up to the extraction of a subsequence,  $(u_n)_{n \in \mathbb{N}}$  converges in  $H^1(\mathcal{C})$  towards a minimizer  $u$  for  $\mathcal{K}_{\text{CKN}}(\theta, p, \Lambda_s(\theta, p, d))$ . The function  $u$  cannot only depend on  $s$ , because any symmetric minimizer for  $\mathcal{K}_{\text{CKN}}^*(\theta, p, \Lambda)$  is a strict local minimum in  $H^1(\mathcal{C})$  due to the fact that  $\Lambda_s(\theta, p, d) < \Lambda_{\text{FS}}(\theta, p, d)$ . Hence, for  $\Lambda = \Lambda_s(\theta, p, d)$  there are two distinct minimizers for  $\mathcal{K}_{\text{CKN}}(\theta, p, \Lambda)$ : one is symmetric and the other one is *not* symmetric. This proves that  $\Lambda_\star(\theta, p, d) < \Lambda_s(\theta, p, d)$  if  $\theta > \vartheta(p, \min\{3, d\})$ .

In the other cases, that is, if either  $d = 3$  and  $\theta = \vartheta(p, 3)$ , or  $d = 2$  and  $\theta > 0$ , the same method applies if we replace  $\Lambda_\star(\theta, p, d)$  by  $\Lambda_2(\theta, p, d)$ .

*Proof (Theorem 4)* Let us consider  $f(x)$  as in the proof of Lemma 2 and assume that  $\theta = \gamma(p-2)$ . As  $p \rightarrow 2_+$ ,  $f(x)/(p-2)$  converges towards

$$f_0(x) := 4\gamma x^{\beta_0+1} - (8\gamma - 3)\mathbf{N}_0 x + (4\gamma - 3)\mathbf{N}_0 \quad \text{with} \quad \beta_0 = 1 - \frac{1}{4\gamma} .$$

We easily check that the function  $f_0(x)$  is convex for  $x > 0$ ,  $f_0(\mathbf{N}_0^{1/\beta_0}) < 0$  and  $f_0'(\mathbf{N}_0^{1/\beta_0}) = 2\mathbf{N}_0 > 0$ . We conclude that  $f_0(x)$  has a unique root for  $x > \mathbf{N}_0^{1/\beta_0}$ . We denote this unique root by  $x_0^* = x_0^*(\gamma)$ . It follows that  $x^*(\gamma(p-2), p)$  converges to  $x_0^*(\gamma)$  as  $p \rightarrow 2_+$ . Symmetry then is established by passing to the limit for any  $\Lambda \in (0, \Lambda_0(\gamma, d))$  with  $\Lambda_0(\gamma, d)$  given by (16).

## 5 An approximation and some numerical results

The functions  $x^*(\theta, p)$  and  $x_0^*(\gamma)$  which enter in the results of Theorem 3 and Theorem 4 are not explicit but easy to estimate, which in turn gives explicit estimates of  $\Lambda_2(\theta, p, d)$  and  $\Lambda_0(\gamma, d)$ . Let

$$\begin{aligned} \alpha &= 2p(\theta - \vartheta(p, 3)) = 2p\theta - 3(p-2) , \\ \beta &= \beta(\theta, p) = 1 - \frac{p-2}{2p\theta} , \end{aligned}$$

and

$$\Lambda_{2,\text{approx}}(\theta, p, d) := \frac{(d-1)\alpha}{4(p-2)} \frac{\beta\theta(6-p) - \alpha(1-\theta + \beta\theta\mathbf{N}^{-1/\beta})}{\beta\theta(6-p)\mathbf{N}^{1/\beta} - \alpha(\beta\theta + 1 - \theta)} .$$

**Proposition 2** *Suppose that either  $d = 2$  and  $p \in (2, 6)$  or else  $d \geq 3$  and  $p \in (2, 2^*)$ . Then for any  $\theta \in (\vartheta(p, 3), 1)$ , we have the estimate*

$$\Lambda_2(\theta, p, d) > \Lambda_{2,\text{approx}}(\theta, p, d) .$$

*Proof* Let us consider the function  $f$  defined in the proof of Lemma 2 and recall that  $f''(x)$  is positive for any  $x \geq \mathbf{N}^{1/\beta} > 1$ . Moreover we verify that

$$\begin{aligned} f(\mathbf{N}^{1/\beta}) &= -(1-\theta)\alpha\mathbf{N}(\mathbf{N}^{1/\beta} - 1) < 0 , \\ f'(\mathbf{N}^{1/\beta}) &= \mathbf{N} \left[ \beta\theta(6-p) - \alpha(1-\theta + \beta\theta\mathbf{N}^{-1/\beta}) \right] > 0 . \end{aligned}$$

which provides the estimate

$$x^*(\theta, p) < \mathbf{N}^{1/\beta} - \frac{f(\mathbf{N}^{1/\beta})}{f'(\mathbf{N}^{1/\beta})} = \frac{\beta\theta(6-p)\mathbf{N}^{1/\beta} - \alpha(\beta\theta + 1 - \theta)}{\beta\theta(6-p) - \alpha(1-\theta + \beta\theta\mathbf{N}^{-1/\beta})} ,$$

and the result follows.



Next we give an estimate of  $\Lambda_0(\gamma, d)$  in Theorem 4. Let

$$\Lambda_{0,\text{approx}}(\gamma, d) := \frac{(d-1)(\gamma - \frac{3}{4})}{2(\gamma - \frac{1}{4})\mathbf{N}_0^{\frac{4\gamma}{4\gamma-1}} - 2(\gamma - \frac{3}{4})},$$

with  $\mathbf{N}_0(\gamma)$  as defined by (15).

**Proposition 3** *Assume that  $d \geq 2$  and  $\gamma > 3/4$ . Then*

$$\Lambda_0(\gamma, d) > \Lambda_{0,\text{approx}}(\gamma, d).$$

*Proof* Recall that  $\beta_0 = 1 - \frac{1}{4\gamma}$ . Let us consider the function  $f_0$  defined in the proof of Theorem 4. We note that  $f_0''(x)$  is positive for  $x > 0$ . Moreover we verify that  $f_0'(\mathbf{N}_0^{1/\beta_0}) = 2\mathbf{N}_0 > 0$  and  $f_0(\mathbf{N}_0^{1/\beta_0}) = -(4\gamma-3)\mathbf{N}_0(\mathbf{N}_0^{1/\beta_0}-1) < 0$ , which provides the estimates

$$x_0^*(\gamma) < \mathbf{N}_0^{1/\beta_0} - \frac{f(\mathbf{N}_0^{1/\beta_0})}{f'(\mathbf{N}_0^{1/\beta_0})} = 2(\gamma - \frac{1}{4})\mathbf{N}_0^{\frac{4\gamma}{4\gamma-1}} - 2(\gamma - \frac{3}{4}),$$

and the result follows.

To conclude this paper, let us illustrate Theorems 3 and 5 with some numerical results. First we address the case of subcritical  $\theta \in (\vartheta(p, d), 1)$  and compare  $\Lambda_\star$  with  $\Lambda_{\text{FS}}$ : Fig. 1 corresponds to the particular case  $d = 5$  and  $\theta = 0.5$ .

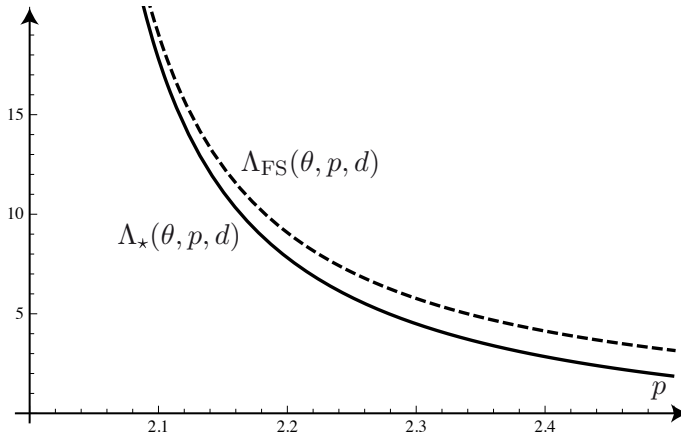
The expression of  $\Lambda_\star(\theta, p, d)$  is not explicit but easy to compute numerically. We recall that  $\Lambda_\star$  is the maximum of  $\Lambda_1$  and  $\Lambda_2$ , both of them being non-explicit. In practice, for low values of the dimension  $d$ , the relative difference of  $\Lambda_1$  and  $\Lambda_2$  is in the range of a fraction of a percent to a few percents, depending on  $\theta$  and on the exponent  $p$ . Moreover, we numerically observe that  $\Lambda_1 \leq \Lambda_2$ , at least for the values of the parameters considered in Fig. 1. The estimate  $\Lambda_{2,\text{approx}}(\theta, p, d)$  of Proposition 2 is remarkably good.

In Fig. 2, we consider the critical case  $\theta = \vartheta(p, d)$ . The plot corresponds to  $d = 5$  and all  $p$  in the interval  $(2, 10/3)$ . The exponent  $\vartheta(p, d)$  is the one which enters in the Gagliardo-Nirenberg inequality

$$\|u\|_{L^p(\mathbb{R}^d)}^2 \leq C_{\text{GN}}(p, d) \|\nabla u\|_{L^2(\mathbb{R}^d)}^{2\vartheta(p, d)} \|u\|_{L^2(\mathbb{R}^d)}^{2(1-\vartheta(p, d))} \quad \forall u \in H^1(\mathbb{R}^d)$$

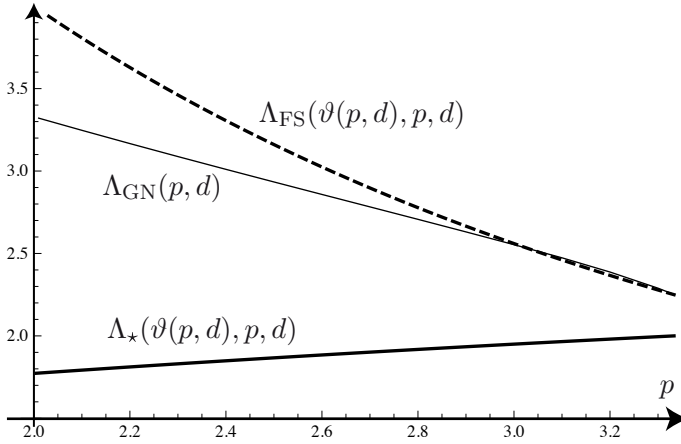
on the Euclidean space  $\mathbb{R}^d$ , *without weights*. Here  $C_{\text{GN}}(p, d)$  denotes the optimal constant and  $p \in (2, \infty)$  if  $d = 1$  or  $2$ ,  $p \in (2, 2^*]$  if  $d \geq 3$ . The optimizers are radially symmetric but not known explicitly.

It has been shown in [8, Theorem 1.4] that optimal functions for (1) exist if  $C_{\text{GN}}(p, d) < C_{\text{CKN}}(\theta, p, a)$ . On the other hand, optimal functions cannot be symmetric  $C_{\text{GN}}(p, d) > C_{\text{CKN}}^*(\theta, p, a)$ : see [5, Section 5] for further details and consequences. This symmetry breaking condition determines a curve  $p \mapsto \Lambda_{\text{GN}}(p, d)$  which has been computed numerically in [6, 7]: there are values of  $p$  and  $d$  for which the condition  $\Lambda > \Lambda_{\text{GN}}(p, d)$ , which guarantees symmetry breaking (but not existence), is weaker than the condition  $\Lambda > \Lambda_{\text{FS}}(\theta, p, d)$ ,



**Fig. 1** Curves  $p \mapsto \Lambda_*(\theta, p, d)$  and  $\Lambda \mapsto \Lambda_{\text{FS}}(\theta, p, d)$  with  $\theta = 0.5$  and  $d = 5$ . Symmetry holds for  $\Lambda \leq \Lambda_*(\theta, p, d)$ , while symmetry is broken for  $\Lambda \geq \Lambda_{\text{FS}}(\theta, p, d)$ . The relative difference of  $\Lambda_1$  and  $\Lambda_2$ , i.e.,  $\Lambda_2(\theta, p, d)/\Lambda_1(\theta, p, d) - 1$ , is below 4%. The estimate of Proposition 2 is such that  $1 - \Lambda_{2,\text{approx}}(\theta, p, d)/\Lambda_2(\theta, p, d)$  is of the order of  $5 \times 10^{-3}$ .

that is  $\Lambda_{\text{GN}}(p, d) < \Lambda_{\text{FS}}(\theta, p, d)$ . See Fig. 2. A rather complete scenario of explanations, based on numerical computations and some formal expansions, has been established in [9, 10]. As it had to be expected, we numerically observe that  $\Lambda_*(\theta, p, d) \leq \min\{\Lambda_{\text{FS}}(\theta, p, d), \Lambda_{\text{GN}}(p, d)\}$  when  $\theta = \vartheta(p, d)$ , for any  $p \in (2, 2^*)$ .



**Fig. 2** With  $\theta = \vartheta(p, d)$ , the curve  $p \mapsto \Lambda_*(\theta, p, d)$  is always below the curves  $p \mapsto \Lambda_{\text{FS}}(\theta, p, d)$  and  $p \mapsto \Lambda_{\text{GN}}(p, d)$  for any  $p \in (2, 2^*)$ , although  $\Lambda_{\text{FS}}$  and  $\Lambda_{\text{GN}}$  are not ordered. The plot corresponds to  $d = 5$  and we may notice that  $\Lambda_{\text{GN}}(p, d) < \Lambda_{\text{FS}}$  if  $p$  is small enough.

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