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SHARP REGULARITY PROPERTIES FOR THE NON-CUTOFF SPATIALLY HOMOGENEOUS BOLTZMANN EQUATION

L. GLANGETAS, H.-G. LI AND C.-J. XU

ABSTRACT. In this work, we study the Cauchy problem for the spatially homogeneous non-cutoff Boltzmann equation with Maxwellian molecules. We prove that this Cauchy problem enjoys Gelfand-Shilov's regularizing effect, meaning that the smoothing properties are the same as the Cauchy problem defined by the evolution equation associated to a fractional harmonic oscillator. The power of the fractional exponent is exactly the same as the singular index of the non-cutoff collisional kernel of the Boltzmann equation. Therefore, we get the sharp regularity of solutions in the Gevrey class and also the sharp decay of solutions with an exponential weight. We also give a method to construct the solution of the Boltzmann equation by solving an infinite system of ordinary differential equations. The key tool is the spectral decomposition of linear and non-linear Boltzmann operators.

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1. INTRODUCTION

In this work, we consider the spatially homogeneous Boltzmann equation

$$(1.1) \quad \begin{cases} \partial_t f = Q(f, f), \\ f|_{t=0} = f_0, \end{cases}$$

where $f = f(t, v)$ is the density distribution function depending on the variables $t \geq 0$ and $v \in \mathbb{R}^3$. The Boltzmann bilinear collision operator is given by

$$Q(g, f)(v) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - v_*, \sigma) (g(v'_*)f(v') - g(v_*)f(v)) dv_* d\sigma,$$

where for $\sigma \in \mathbb{S}^2$, the symbols v'_* and v' are abbreviations for the expressions,

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma, \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2} \sigma,$$

which are obtained in such a way that collision preserves momentum and kinetic energy, namely

$$v'_* + v' = v + v_*, \quad |v'_*|^2 + |v'|^2 = |v|^2 + |v_*|^2.$$

The non-negative cross section $B(z, \sigma)$ depends only on $|z|$ and the scalar product $\frac{z}{|z|} \cdot \sigma$. For physical models, it usually takes the form

$$B(v - v_*, \sigma) = \Phi(|v - v_*|) b(\cos \theta), \quad \cos \theta = \frac{v - v_*}{|v - v_*|} \cdot \sigma, \quad 0 \leq \theta \leq \frac{\pi}{2}.$$

In this paper, we consider only the Maxwellian molecules case which corresponds to the case $\Phi \equiv 1$, and we focus our attention on the angular part b satisfying

$$(1.2) \quad \beta(\theta) = 2\pi b(\cos 2\theta) |\sin 2\theta| \approx |\theta|^{-1-2s}, \quad \text{when } \theta \rightarrow 0^+,$$

for some $0 < s < 1$. Without loss of generality, we may assume that $b(\cos \theta)$ is supported on the set $\cos \theta \geq 0$. See for instance [11] for more details on $\beta(\cdot)$ and [22] for a general collision kernel.

We linearize the Boltzmann equation near the absolute Maxwellian distribution

$$\mu(v) = (2\pi)^{-\frac{3}{2}} e^{-\frac{|v|^2}{2}}.$$

We introduce $g(t, v)$ such that $f(t, v) = \mu(v) + \sqrt{\mu}(v)g(t, v)$ and obtain

$$\frac{\partial g}{\partial t} + \mathcal{L}[g] = \Gamma(g, g)$$

with

$$\Gamma(g, h) = \frac{1}{\sqrt{\mu}} Q(\sqrt{\mu}g, \sqrt{\mu}h), \quad \mathcal{L}(g) = -\frac{1}{\sqrt{\mu}} [Q(\sqrt{\mu}g, \mu) + Q(\mu, \sqrt{\mu}g)].$$

Therefore the Cauchy problem (1.1) can be re-written in the form

$$(1.3) \quad \begin{cases} \partial_t g + \mathcal{L}(g) = \Gamma(g, g), \\ g|_{t=0} = g_0. \end{cases}$$

The linear operator $\mathcal{L}$ is nonnegative ([11, 12, 13]), with the null space

$$\mathcal{N} = \text{span} \left\{ \sqrt{\mu}, \sqrt{\mu}v_1, \sqrt{\mu}v_2, \sqrt{\mu}v_3, \sqrt{\mu}|v|^2 \right\}.$$

In the present work, we study the smoothing effect for the Cauchy problem associated with the spatially homogeneous non-cutoff Boltzmann equation in the case of Maxwellian molecules. It is well known that the non-cutoff spatially homogeneous Boltzmann equation enjoys an $\mathcal{S}(\mathbb{R}^3)$ -regularizing effect for the weak solutions to the Cauchy problem (see [4, 16]). Regarding the Gevrey regularity, Ukai showed in [21] that the Cauchy problem for the Boltzmann equation has a unique local solution in Gevrey classes. Then Desvillettes, Furioli and Terraneo proved in [3] the propagation of Gevrey regularity for solutions of the Boltzmann equation with Maxwellian molecules. For mild singularities, Morimoto and Ukai proved in [15] the Gevrey regularity of smooth Maxwellian decay solutions to the Cauchy problem of the spatially homogeneous Boltzmann equation with a modified kinetic factor (see also [26] for the non-modified case). On the other hand, Lekrine and Xu proved in [10] the property of Gevrey smoothing effect for the weak solutions to the Cauchy problem associated to the radially symmetric spatially homogeneous Boltzmann equation with Maxwellian molecules for $0 < s < 1/2$. This result was then completed by Glangetas and Najeme who established in [6] the analytic smoothing effect in the case when $1/2 \leq s < 1$. In [11], for the radially symmetric case, the linearized Boltzmann operator was shown to behave, essentially as a fractional harmonic oscillator $\mathcal{H}^s$, with $\mathcal{H} = -\Delta + \frac{|v|^2}{4}$ and $0 < s < 1$. For the non-radial case, the linearized non-cutoff Boltzmann operator behaves as

$$\mathcal{L} = a(\mathcal{H}, \Delta_{\mathbb{S}^2}) \mathcal{L}_L^s$$

where $\Delta_{\mathbb{S}^2}$ is the Laplace-Beltrami operator on the sphere, $\mathcal{L}_L$ is the linearized Landau operator and $a(\mathcal{H}, \Delta_{\mathbb{S}^2})$ is an isomorphism on $L^2(\mathbb{R}^3)$. Here, the fractional power $\mathcal{L}_L^s$ is defined through functional calculus. We can refer to [12]. The solutions of the following Cauchy problem

$$\begin{cases} \partial_t g + \mathcal{L}(g) = 0, \\ g|_{t=0} = g_0 \in L^2, \end{cases}$$

belong to the symmetric Gelfand-Shilov space $S_{1/2s}^{1/2s}(\mathbb{R}^3)$ for any positive time and

$$\forall t > 0, \quad \|e^{ct\mathcal{H}^s} g(t)\|_{L^2} \leq C \|g_0\|_{L^2},$$

where the Gelfand-Shilov space $S_\nu^\mu(\mathbb{R}^3)$, with $\mu, \nu > 0, \mu + \nu \geq 1$, is the space of smooth functions $f \in C^{+\infty}(\mathbb{R}^3)$ satisfying:

$$\exists A > 0, C > 0, \sup_{v \in \mathbb{R}^3} |v^\beta \partial_v^\alpha f(v)| \leq CA^{|\alpha|+|\beta|} (\alpha!)^\mu (\beta!)^\nu, \quad \forall \alpha, \beta \in \mathbb{N}^3.$$

This Gelfand-Shilov space can also be characterized as the sub-space of Schwartz functions $f \in \mathcal{S}(\mathbb{R}^3)$ such that,

$$\exists C > 0, \epsilon > 0, |f(v)| \leq Ce^{-\epsilon|v|^\frac{1}{\mu}}, \quad v \in \mathbb{R}^3 \text{ and } |\hat{f}(\xi)| \leq Ce^{-\epsilon|\xi|^\frac{1}{\nu}}, \quad \xi \in \mathbb{R}^3.$$

The symmetric Gelfand-Shilov space $S_\nu^\nu(\mathbb{R}^3)$ with $\nu \geq \frac{1}{2}$ can also be identified with

$$S_\nu^\nu(\mathbb{R}^3) = \left\{ f \in C^\infty(\mathbb{R}^3); \exists \tau > 0, \|e^{\tau \mathcal{H}^\frac{1}{2\nu}} f\|_{L^2} < +\infty \right\}.$$

See Appendix 7 for more properties of Gelfand-Shilov spaces.

From a historical point of view, the spectral analysis is an important method to study the linear Boltzmann operator (see [2]). In [11], the linearized non-cutoff radially symmetric Boltzmann operator is shown to be diagonal in the Hermite basis. This property has been used to prove in the continued work [13] that the Cauchy problem of the non-cutoff spatially homogeneous Boltzmann equation with a radial initial datum $g_0 \in L^2(\mathbb{R}^3)$, has a unique global radial solution which belongs to the Gelfand-Shilov class $S_{1/2s}^{1/2s}(\mathbb{R}^3)$.

The main theorem of this paper is given in the following.

Theorem 1.1. *Assume that the Maxwellian collision cross-section $b(\cdot)$ is given in (1.2) with $0 < s < 1$, then there exists $\varepsilon_0 > 0$ such that for any initial datum $g_0 \in L^2(\mathbb{R}^3) \cap \mathcal{N}^\perp$ with $\|g_0\|_{L^2(\mathbb{R}^3)}^2 \leq \varepsilon_0$, the Cauchy problem (1.3) admits a weak solution, which belongs to the Gelfand-Shilov space $S_{1/2s}^{1/2s}(\mathbb{R}^3)$ for any $t > 0$. Moreover, there exists $c_0 > 0$, such that, for any $t \geq 0$,*

$$(1.4) \quad \|e^{c_0 t \mathcal{H}^s} g(t)\|_{L^2(\mathbb{R}^3)} \leq C e^{-\frac{\lambda_{2,0}}{4} t} \|g_0\|_{L^2(\mathbb{R}^3)},$$

where

$$\lambda_{2,0} = \int_{-\pi/4}^{\pi/4} \beta(\theta)(1 - \sin^4 \theta - \cos^4 \theta) d\theta > 0.$$

Remark 1.2. *We have proved that for the Cauchy problem (1.1), if the initial data is a small perturbation of Maxwellian in L^2 , then the global solution returns to the equilibrium with an exponential rate with respect to Gelfand-Shilov norm, which is an exponentially weighted norm of both the solution and the Fourier transformation of the solution.*

The rest of the paper is arranged as follows: In Section 2, we introduce the spectral analysis of the linear and nonlinear Boltzmann operators, and transform the nonlinear Cauchy problem of Boltzmann equation to an infinite system of ordinary differential equations which can be solved explicitly. Then we derive the formal solution of the Cauchy problem for Boltzmann equation. In Section 3, we establish an upper bounded estimates of some nonlinear operators with an exponential weighted norm, which is crucial to obtain the convergence of the formal solution in Gelfand-Shilov spaces. The proof of the main Theorem 1.1 will be presented in Section 4. Finally, Section 5 and Section 6 are devoted to the proof of some propositions used in Section 4. In Section 5, we study the spectral representation of the non linear Boltzmann operator, and prove that it can be represented by an “inferior triangular matrix” of infinite dimension with three indices, so that the presentation and the computations are very complicated. This inferior triangular property is essential for the construction of the formal solution by solving an infinite system of ordinary differential equations. In Section 6, we prove some estimates on the entries of the triangular matrix obtained in Section 5, and this is a key point in proving the convergence of the formal solution with respect to Gelfand-Shilov norms.

2. THE SPECTRAL ANALYSIS OF THE BOLTZMANN OPERATORS

2.1. Diagonalization of the linear operators. We first recall the spectral decomposition of the linear Boltzmann operator. In the cutoff case, that is, when $b(\cos \theta) \sin \theta \in L^1([0, \frac{\pi}{2}])$, it was shown in [23] that

$$\mathcal{L}(\varphi_{n,l,m}) = \lambda_{n,l} \varphi_{n,l,m}, \quad n, l \in \mathbb{N}, \quad m \in \mathbb{Z}, \quad |m| \leq l.$$

This diagonalization of the linearized Boltzmann operator with Maxwellian molecules holds as well in the non-cutoff case, (see [1, 2, 5, 11, 12]). The eigenvalues are

$$\lambda_{n,l} = \int_{|\theta| \leq \frac{\pi}{4}} \beta(\theta) \left(1 + \delta_{n,0} \delta_{l,0} - (\sin \theta)^{2n+l} P_l(\sin \theta) - (\cos \theta)^{2n+l} P_l(\cos \theta) \right) d\theta,$$

the eigenfunctions are

$$(2.1) \quad \varphi_{n,l,m}(v) = \left(\frac{n!}{\sqrt{2}\Gamma(n+l+3/2)} \right)^{1/2} \left(\frac{|v|}{\sqrt{2}} \right)^l e^{-\frac{|v|^2}{4}} L_n^{(l+1/2)} \left(\frac{|v|^2}{2} \right) Y_l^m \left(\frac{v}{|v|} \right),$$

where $\Gamma(\cdot)$ is the standard Gamma function, for any $x > 0$,

$$\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} dt.$$

The l^{th} -Legendre polynomial P_l and the Laguerre polynomial $L_n^{(\alpha)}$ of order α , degree n read,

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2 - 1)^l, \text{ where } |x| \leq 1;$$

$$L_n^{(\alpha)}(x) = \sum_{r=0}^n (-1)^{n-r} \frac{\Gamma(\alpha + n + 1)}{r!(n-r)!\Gamma(\alpha + n - r + 1)} x^{n-r}.$$

We refer the properties of these special functions to the classical book [20] (see (7) of Sec. 1 in Chap. III and (3₁) of Sec. 1 in Chap. IV). For any unit vector

$$\sigma = (\cos \theta, \sin \theta \cos \phi, \sin \theta \sin \phi)$$

with $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$, the orthonormal basis of spherical harmonics $Y_l^m(\sigma)$ is

$$Y_l^m(\sigma) = N_{l,m} P_l^{|m|}(\cos \theta) e^{im\phi}, \quad |m| \leq l,$$

where the normalisation factor is given by

$$N_{l,m} = \sqrt{\frac{2l+1}{4\pi} \cdot \frac{(l-|m|)!}{(l+|m|)!}}$$

and $P_l^{|m|}$ is the associated Legendre functions of the first kind of order l and degree $|m|$ with

$$(2.2) \quad P_l^{|m|}(x) = (1-x^2)^{\frac{|m|}{2}} \left(\frac{d}{dx} \right)^{|m|} P_l(x).$$

The family $(Y_l^m(\sigma))_{l \geq 0, |m| \leq l}$ constitutes an orthonormal basis of the space $L^2(\mathbb{S}^2, d\sigma)$ with $d\sigma$ being the surface measure on $\mathbb{S}^2$ (see (16) of Chap. 1 in the book [9]). Noting that $\{\varphi_{n,l,m}(v)\}$ consist an orthonormal basis of $L^2(\mathbb{R}^3)$ composed of eigenvectors of the harmonic oscillator (see [1], [12])

$$\mathcal{H}(\varphi_{n,l,m}) = (2n + l + \frac{3}{2}) \varphi_{n,l,m}.$$

As a special case, $\{\varphi_{n,0,0}(v)\}$ is an orthonormal basis of $L_{\text{rad}}^2(\mathbb{R}^3)$, the radially symmetric function space (see [13]). We have that, for suitable functions g ,

$$\mathcal{L}(g) = \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=-l}^l \lambda_{n,l} g_{n,l,m} \varphi_{n,l,m},$$

where $g_{n,l,m} = (g, \varphi_{n,l,m})_{L^2(\mathbb{R}^3)}$, and

$$\mathcal{H}(g) = \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=-l}^l (2n + l + \frac{3}{2}) g_{n,l,m} \varphi_{n,l,m}.$$

Using this spectral decomposition, the definition of $\mathcal{H}^s$, $e^{c\mathcal{H}^s}$, $e^{c\mathcal{L}}$ is then classical.

2.2. Triangular effect of the non linear operators. We now study the algebra property of the nonlinear terms

$$\Gamma(\varphi_{n,l,m}, \varphi_{\tilde{n},\tilde{l},\tilde{m}}).$$

We have the following triangular effect for the nonlinear Boltzmann operators on the basis $\{\varphi_{n,l,m}\}$.

Proposition 2.1. *The following algebraic identities hold,*

$$(2.3) \quad \begin{aligned} (i_1) \quad & \Gamma(\varphi_{0,0,0}, \varphi_{\tilde{n},\tilde{l},\tilde{m}}) = \lambda_{\tilde{n},\tilde{l}}^1 \varphi_{\tilde{n},\tilde{l},\tilde{m}}; \\ (i_2) \quad & \Gamma(\varphi_{n,l,m}, \varphi_{0,0,0}) = \lambda_{n,l}^2 \varphi_{n,l,m}; \\ (ii_1) \quad & \Gamma(\varphi_{n,0,0}, \varphi_{\tilde{n},\tilde{l},\tilde{m}}) = \lambda_{n,\tilde{n},\tilde{l}}^{rad,1} \varphi_{n+\tilde{n},\tilde{l},\tilde{m}}, \text{ for } n \geq 1; \\ (ii_2) \quad & \Gamma(\varphi_{n,l,m}, \varphi_{\tilde{n},0,0}) = \lambda_{n,\tilde{n},l}^{rad,2} \varphi_{n+\tilde{n},l,m}, \text{ for } n \in \mathbb{N}, l \geq 1; \\ (iii) \quad & \Gamma(\varphi_{n,l,m}, \varphi_{\tilde{n},\tilde{l},\tilde{m}}) = \sum_{k=0}^{k_0(l,\tilde{l},m,\tilde{m})} \mu_{n,\tilde{n},l,\tilde{l},k}^{m,\tilde{m},m+\tilde{m}} \varphi_{n+\tilde{n}+k,l+\tilde{l}-2k,m+\tilde{m}} \text{ with} \\ & k_0(l,\tilde{l},m,\tilde{m}) = \min \left(\left\lfloor \frac{l+\tilde{l}-|m+\tilde{m}|}{2} \right\rfloor, l, \tilde{l} \right) \\ & \text{for } l \geq 1, \tilde{l} \geq 1, |m| \leq l, |\tilde{m}| \leq \tilde{l}. \end{aligned}$$

The notations in the above Proposition are as following:

$$\begin{aligned} \lambda_{\tilde{n},\tilde{l}}^1 &= \int_{|\theta| \leq \frac{\pi}{4}} \beta(\theta) ((\cos \theta)^{2\tilde{n}+\tilde{l}} P_{\tilde{l}}(\cos \theta) - 1) d\theta; \\ \lambda_{n,l}^2 &= \int_{|\theta| \leq \frac{\pi}{4}} \beta(\theta) ((\sin \theta)^{2n+l} P_l(\sin \theta) - \delta_{0,n} \delta_{0,l}) d\theta; \\ \lambda_{n,\tilde{n},\tilde{l}}^{rad,1} &= \frac{1}{\sqrt{4\pi}} \left(\frac{2\pi^{\frac{3}{2}}(n+\tilde{n})! \Gamma(n+\tilde{n}+\tilde{l}+\frac{3}{2})}{\tilde{n}! \Gamma(\tilde{n}+\tilde{l}+\frac{3}{2}) n! \Gamma(n+\frac{3}{2})} \right)^{\frac{1}{2}} \\ &\quad \times \int_{|\theta| \leq \frac{\pi}{4}} \beta(\theta) (\sin \theta)^{2n} (\cos \theta)^{2\tilde{n}+\tilde{l}} P_{\tilde{l}}(\cos \theta) d\theta; \\ \lambda_{n,\tilde{n},l}^{rad,2} &= \frac{1}{\sqrt{4\pi}} \left(\frac{2\pi^{\frac{3}{2}}(n+\tilde{n})! \Gamma(n+\tilde{n}+l+\frac{3}{2})}{\tilde{n}! \Gamma(\tilde{n}+\frac{3}{2}) n! \Gamma(n+l+\frac{3}{2})} \right)^{\frac{1}{2}} \\ &\quad \times \int_{|\theta| \leq \frac{\pi}{4}} \beta(\theta) (\sin \theta)^{2n+l} (\cos \theta)^{2\tilde{n}} P_l(\sin \theta) d\theta \end{aligned}$$

and for $|m^*| \leq l + \tilde{l} - 2k$,

$$\begin{aligned} \mu_{n,\tilde{n},l,\tilde{l},k}^{m,\tilde{m},m^*} &= (-1)^k \left(\frac{2\pi^{\frac{3}{2}}(n+\tilde{n}+k)! \Gamma(n+\tilde{n}+l+\tilde{l}-k+\frac{3}{2})}{\tilde{n}! \Gamma(\tilde{n}+\tilde{l}+\frac{3}{2}) n! \Gamma(n+l+\frac{3}{2})} \right)^{\frac{1}{2}} \\ &\quad \times \int_{\mathbb{S}_k^2} \left[\int_{\mathbb{S}^2} b(\kappa \cdot \sigma) \left(\frac{|\kappa - \sigma|}{2} \right)^{2n+l} \left(\frac{|\kappa + \sigma|}{2} \right)^{2\tilde{n}+\tilde{l}} \right. \\ &\quad \left. \times Y_l^m \left(\frac{\kappa - \sigma}{|\kappa - \sigma|} \right) Y_{\tilde{l}}^{\tilde{m}} \left(\frac{\kappa + \sigma}{|\kappa + \sigma|} \right) d\sigma \right] \overline{Y_{l+\tilde{l}-2k}^{m^*}}(\kappa) d\kappa. \end{aligned}$$

We remark that $\mu_{n,\tilde{n},l,\tilde{l},k}^{m,\tilde{m},m^*}$ vanishes to 0 if $m + \tilde{m} \neq m^*$, so

$$(2.4) \quad \mu_{n,\tilde{n},l,\tilde{l},k}^{m,\tilde{m},m^*} = \mu_{n,\tilde{n},l,\tilde{l},k}^{m,\tilde{m},m^*} \delta_{m^*, m+\tilde{m}}.$$

The coefficient $\mu_{n,\tilde{n},l,\tilde{l},k}^{m,\tilde{m},m^*}$ satisfies the following orthogonal property.

Proposition 2.2. *For any integers $0 \leq k_1, k_2 \leq \min(l, \tilde{l})$, $|m_1^*| \leq l + \tilde{l} - 2k_1$, $|m_2^*| \leq l + \tilde{l} - 2k_2$, we have*

$$(2.5) \quad \sum_{|m| \leq l} \sum_{|\tilde{m}| \leq \tilde{l}} \mu_{n,\tilde{n},l,\tilde{l},k_1}^{m,\tilde{m},m_1^*} \overline{\mu_{n,\tilde{n},l,\tilde{l},k_2}^{m,\tilde{m},m_2^*}} = \sum_{|m| \leq l} \sum_{|\tilde{m}| \leq \tilde{l}} \left| \mu_{n,\tilde{n},l,\tilde{l},k_1}^{m,\tilde{m},m_1^*} \right|^2 \delta_{k_1,k_2} \delta_{m_1^*,m_2^*}.$$

We prove Proposition 2.1, Proposition 2.2 and the claim (2.4) in Section 5.

Remark 2.3. 1) *Similar to the radially symmetric case, the property (iii) of Proposition 2.1 and the above Proposition 2.2 imply that we have also a “triangular effect” but with a noise of order $k_0(l, \tilde{l}, m, \tilde{m})$. In the 3-dimensional case, this effect is not easy to understand. For more details, we refer to subsection 2.3.*

2) *We have also*

$$(2.6) \quad \lambda_{n,l} = -\lambda_{n,l}^1 - \lambda_{n,l}^2.$$

It is trivial to obtain that $\lambda_{0,0} = \lambda_{1,0} = \lambda_{0,1} = 0$ and the others are strictly positive, since when $l \neq 0$, and for $n \neq 0, 1$,

$$\lambda_{n,l} \geq 2 \int_0^{\frac{\pi}{4}} (1 - \sin^{2n} \theta - \cos^{2n} \theta) \beta(\theta) d\theta = \lambda_{n,0} > 0.$$

Moreover, from Theorem 2.2 in [12] (see also Theorem 2.3 in [14]), there exists a constant $0 < c_1 < 1$ dependent on s such that, for any $n, l \in \mathbb{N}$ and $n + l \geq 2$,

$$(2.7) \quad c_1 \left((2n + l + \frac{3}{2})^s + l^{2s} \right) \leq \lambda_{n,l} \leq \frac{1}{c_1} \left((2n + l + \frac{3}{2})^s + l^{2s} \right).$$

2.3. Formal and explicit solution of the Cauchy problem. Now we solve explicitly the Cauchy problem associated to the non-cutoff spatially homogeneous Boltzmann equation with Maxwellian molecules. Consider the solution of the Cauchy problem (1.3) in the form

$$g(t) = \sum_{n=0}^{+\infty} \sum_{l=0}^{+\infty} \sum_{|m| \leq l} g_{n,l,m}(t) \varphi_{n,l,m},$$

with initial data

$$g(0) = \sum_{n=0}^{+\infty} \sum_{l=0}^{+\infty} \sum_{|m| \leq l} g_{n,l,m}^0 \varphi_{n,l,m} \in L^2(\mathbb{R}^3),$$

where

$$g_{n,l,m}(t) = (g(t), \varphi_{n,l,m})_{L^2(\mathbb{R}^3)}, \quad g_{n,l,m}^0 = (g_0, \varphi_{n,l,m})_{L^2(\mathbb{R}^3)}.$$

In the following, we will use the short notation

$$\sum_{n,l,m} = \sum_{n=0}^{+\infty} \sum_{l=0}^{+\infty} \sum_{|m| \leq l}.$$

This summation is divided into three terms, which are

$$(2.8) \quad \sum_{n=0}^{+\infty} \sum_{l=0}^{+\infty} \sum_{|m| \leq l} f_{n,l,m} = f_{0,0,0} + \sum_{n=1}^{+\infty} f_{n,0,0} + \sum_{n=0}^{+\infty} \sum_{l=1}^{+\infty} \sum_{|m| \leq l} f_{n,l,m}.$$

It follows from $\Gamma(\varphi_{0,0,0}, \varphi_{0,0,0}) = \Gamma(\sqrt{\mu}, \sqrt{\mu}) = 0$, Proposition 2.1 and the above decomposition (2.8)

that, for suitable function g ,

$$\begin{aligned}
\Gamma(g, g) = & \sum_{\tilde{n}=0}^{+\infty} \sum_{\tilde{l}=0}^{+\infty} \sum_{|\tilde{m}| \leq \tilde{l}} g_{0,0,0}(t) g_{\tilde{n},\tilde{l},\tilde{m}}(t) (\lambda_{\tilde{n},\tilde{l}}^1 + \lambda_{\tilde{n},\tilde{l}}^2) \varphi_{\tilde{n},\tilde{l},\tilde{m}} \\
& + \sum_{n=1}^{+\infty} \sum_{\tilde{n}=1}^{+\infty} g_{n,0,0}(t) g_{\tilde{n},0,0}(t) \lambda_{n,\tilde{n},0}^{rad,1} \varphi_{\tilde{n}+n,0,0} \\
& + \sum_{n=1}^{+\infty} \sum_{\tilde{n}=0}^{+\infty} \sum_{\tilde{l}=1}^{+\infty} \sum_{|\tilde{m}| \leq \tilde{l}} g_{n,0,0}(t) g_{\tilde{n},\tilde{l},\tilde{m}}(t) \lambda_{n,\tilde{n},\tilde{l}}^{rad,1} \varphi_{\tilde{n}+n,\tilde{l},\tilde{m}} \\
& + \sum_{n=0}^{+\infty} \sum_{l=1}^{+\infty} \sum_{\tilde{n}=1}^{+\infty} \sum_{|m| \leq l} g_{n,l,m}(t) g_{\tilde{n},0,0}(t) \lambda_{n,\tilde{n},l}^{rad,2} \varphi_{\tilde{n}+n,l,m} \\
& + \sum_{n=0}^{+\infty} \sum_{l=1}^{+\infty} \sum_{\tilde{n}=0}^{+\infty} \sum_{\tilde{l}=1}^{+\infty} \sum_{|m| \leq l} \sum_{|\tilde{m}| \leq \tilde{l}} \sum_{k=0}^{k_0(l,\tilde{l},m,\tilde{m})} g_{n,l,m}(t) g_{\tilde{n},\tilde{l},\tilde{m}}(t) \mu_{n,\tilde{n},l,\tilde{l},k}^{m,\tilde{m},m+\tilde{m}} \varphi_{n+\tilde{n}+k,l+\tilde{l}-2k,m+\tilde{m}},
\end{aligned}$$

where $k_0(l, \tilde{l}, m, \tilde{m})$ was given in (2.3). Since for fixed $l, \tilde{l} \in \mathbb{N}$,

$$\begin{aligned}
& \{(m, \tilde{m}, k) \in \mathbb{Z}^2 \times \mathbb{N}; |m| \leq l, |\tilde{m}| \leq \tilde{l}, 0 \leq k \leq k_0(l, \tilde{l}, m, \tilde{m})\} \\
& = \{(m, \tilde{m}, k) \in \mathbb{Z}^2 \times \mathbb{N}; |m| \leq l, |\tilde{m}| \leq \tilde{l}, 0 \leq k \leq \min(l, \tilde{l}), |m + \tilde{m}| \leq l + \tilde{l} - 2k\},
\end{aligned}$$

we obtain by changing the order of the summation

$$(2.9) \quad \sum_{|m| \leq l} \sum_{|\tilde{m}| \leq \tilde{l}} \sum_{k=0}^{k_0(l,\tilde{l},m,\tilde{m})} = \sum_{k=0}^{\min(l,\tilde{l})} \sum_{\substack{|m| \leq l, |\tilde{m}| \leq \tilde{l} \\ |m+\tilde{m}| \leq l+\tilde{l}-2k}}$$

where $\sum_{\substack{|m| \leq l, |\tilde{m}| \leq \tilde{l} \\ |m+\tilde{m}| \leq l+\tilde{l}-2k}}$ is the double summation of m and $\tilde{m}$ with constraints $|m + \tilde{m}| \leq l + \tilde{l} - 2k$.

Using the equality (2.6) (that is $\lambda_{\tilde{n},\tilde{l}}^1 + \lambda_{\tilde{n},\tilde{l}}^2 = -\lambda_{\tilde{n},\tilde{l}}$) and (2.9), $\Gamma(g, g)$ can be rewritten as

$$\begin{aligned}
\Gamma(g, g) = & -g_{0,0,0}(t) \sum_{\tilde{n}=0}^{+\infty} \sum_{\tilde{l}=0}^{+\infty} \sum_{|\tilde{m}| \leq \tilde{l}} g_{\tilde{n},\tilde{l},\tilde{m}}(t) \lambda_{\tilde{n},\tilde{l}} \varphi_{\tilde{n},\tilde{l},\tilde{m}} \\
& + \sum_{n=1}^{+\infty} \sum_{\tilde{n}=1}^{+\infty} g_{n,0,0}(t) g_{\tilde{n},0,0}(t) \lambda_{n,\tilde{n},0}^{rad,1} \varphi_{\tilde{n}+n,0,0} \\
& + \sum_{n=1}^{+\infty} \sum_{\tilde{n}=0}^{+\infty} \sum_{\tilde{l}=1}^{+\infty} \sum_{|\tilde{m}| \leq \tilde{l}} g_{n,0,0}(t) g_{\tilde{n},\tilde{l},\tilde{m}}(t) (\lambda_{n,\tilde{n},\tilde{l}}^{rad,1} + \lambda_{\tilde{n},n,\tilde{l}}^{rad,2}) \varphi_{\tilde{n}+n,\tilde{l},\tilde{m}} \\
& + \sum_{n=0}^{+\infty} \sum_{l=1}^{+\infty} \sum_{\tilde{n}=0}^{+\infty} \sum_{\tilde{l}=1}^{+\infty} \sum_{k=0}^{\min(l,\tilde{l})} \sum_{\substack{|m| \leq l, |\tilde{m}| \leq \tilde{l} \\ |m+\tilde{m}| \leq l+\tilde{l}-2k}} g_{n,l,m}(t) g_{\tilde{n},\tilde{l},\tilde{m}}(t) \mu_{n,\tilde{n},l,\tilde{l},k}^{m,\tilde{m},m+\tilde{m}} \varphi_{n+\tilde{n}+k,l+\tilde{l}-2k,m+\tilde{m}}.
\end{aligned}$$

For suitable function g , we also have

$$\mathcal{L}g = \sum_{n,l,m}^{+\infty} \lambda_{n,l} g_{n,l,m}(t) \varphi_{n,l,m}.$$

Formally, taking the inner product with $\overline{\varphi_{n^*, l^*, m^*}}$ on both sides of (1.3), we find that the functions $\{g_{n^*, l^*, m^*}(t)\}$ satisfy the following infinite system of the differential equations

$$\begin{aligned} & \partial_t g_{n^*, l^*, m^*}(t) + \lambda_{n^*, l^*}(1 + g_{0,0,0}(t))g_{n^*, l^*, m^*}(t) \\ &= \delta_{l^*, 0} \sum_{\substack{n+\tilde{n}=n^* \\ n \geq 1, \tilde{n} \geq 1}} \lambda_{n, \tilde{n}, 0}^{rad, 1} g_{n, 0, 0}(t) g_{\tilde{n}, 0, 0}(t) \\ &+ \sum_{\substack{n+\tilde{n}=n^* \\ n \geq 1, \tilde{n} \geq 0}} \sum_{\tilde{l} \geq 1} \delta_{\tilde{l}, l^*} (\lambda_{n, \tilde{n}, l^*}^{rad, 1} + \lambda_{\tilde{n}, n, l^*}^{rad, 2}) g_{\tilde{n}, l^*, m^*}(t) g_{n, 0, 0}(t) \\ &+ \sum_{(n, \tilde{n}, l, \tilde{l}, k, m, \tilde{m}) \in \Delta_{n^*, l^*, m^*}} \mu_{n, \tilde{n}, l, \tilde{l}, k}^{m, \tilde{m}, m^*} g_{n, l, m}(t) g_{\tilde{n}, \tilde{l}, \tilde{m}}(t) \end{aligned}$$

with initial data

$$(2.10) \quad g_{n^*, l^*, m^*}(0) = g_{n^*, l^*, m^*}^0$$

and where

$$\begin{aligned} \Delta_{n^*, l^*, m^*} = & \{(n, \tilde{n}, l, \tilde{l}, k, m, \tilde{m}) \in \mathbb{N}^5 \times \mathbb{Z}^2; \\ & l \geq 1, \tilde{l} \geq 1, 0 \leq k \leq \min(l, \tilde{l}), |m| \leq l, |\tilde{m}| \leq \tilde{l} \\ & \text{and } n + \tilde{n} + k = n^*, l + \tilde{l} - 2k = l^*, m + \tilde{m} = m^*\}, \end{aligned}$$

which is a subset of a hyperplane of dimension 4.

Remark 2.4. The summation in the last term of the previous equation for $\partial_t g_{n^*, l^*, m^*}(t)$ is a bit complicated. For the sake of simplicity, it will be convenient to abuse the notation in this summation and write

$$\begin{aligned} & \partial_t g_{n^*, l^*, m^*}(t) + \lambda_{n^*, l^*}(1 + g_{0,0,0}(t))g_{n^*, l^*, m^*}(t) \\ &= \delta_{l^*, 0} \sum_{\substack{n+\tilde{n}=n^* \\ n \geq 1, \tilde{n} \geq 1}} \lambda_{n, \tilde{n}, 0}^{rad, 1} g_{n, 0, 0}(t) g_{\tilde{n}, 0, 0}(t) \\ &+ \sum_{\substack{n+\tilde{n}=n^* \\ n \geq 1, \tilde{n} \geq 0}} \sum_{\tilde{l} \geq 1} \delta_{\tilde{l}, l^*} (\lambda_{n, \tilde{n}, l^*}^{rad, 1} + \lambda_{\tilde{n}, n, l^*}^{rad, 2}) g_{\tilde{n}, l^*, m^*}(t) g_{n, 0, 0}(t) \\ &+ \sum_{\substack{n+\tilde{n}+k=n^* \\ n \geq 0, \tilde{n} \geq 0}} \sum_{\substack{l+\tilde{l}-2k=l^* \\ l \geq 1, \tilde{l} \geq 1}} \sum_{\substack{m+\tilde{m}=m^* \\ |m| \leq l, |\tilde{m}| \leq \tilde{l} \\ 0 \leq k \leq \min(l, \tilde{l})}} \mu_{n, \tilde{n}, l, \tilde{l}, k}^{m, \tilde{m}, m^*} g_{n, l, m}(t) g_{\tilde{n}, \tilde{l}, \tilde{m}}(t). \end{aligned} \quad (2.11)$$

Here and after, we always use this notation.

Theorem 2.5. For any initial data $\{g_{n^*, l^*, m^*}^0; n^*, l^* \in \mathbb{N}, |m^*| \leq l^*\}$ with

$$(2.12) \quad g_{0,0,0}^0 = g_{1,0,0}^0 = g_{0,1,0}^0 = g_{0,1,1}^0 = g_{0,1,-1}^0 = 0,$$

the system (2.11) admits a global solution $\{g_{n^*, l^*, m^*}(t); n^*, l^* \in \mathbb{N}, |m^*| \leq l^*\}$.

Remark 2.6. Using the triangular effect property of Proposition 2.1, we solve explicitly the infinite system of differential equations (2.11) with any initial data $\{g_{n^*, l^*, m^*}^0; n^*, l^* \in \mathbb{N}, |m^*| \leq l^*\}$. Note that the initial data doesn't need to belong to ℓ^2 . That means we can construct the formal solution for any initial data $g_0 \in \mathcal{S}'(\mathbb{R}^3)$.

Proof. Formally, the system (2.11) is non linear of quadratic form, but the infinite matrix of this quadratic form is in fact inferior triangular (see [13] for the radially symmetric case involving a simple index). Since the sequence is defined by multi-indices, we prove this property by the following different case, and in each case by induction.

Induction on $n^* \in \mathbb{N}$:

(1) the case : $n^* = 0$. Now we prove the existence of $\{g_{0,l^*,m^*}(t); l^* \in \mathbb{N}, |m^*| \leq l^*\}$ by induction on $l^* \in \mathbb{N}$. From the assumption (2.12), and $\lambda_{0,0} = \lambda_{0,1} = \lambda_{1,0} = 0$, one gets that

$$g_{0,0,0}(t) = g_{1,0,0}(t) = g_{0,1,0}(t) = g_{0,1,1}(t) = g_{0,1,-1}(t) = 0.$$

Let now $l^* \geq 1$, we put the following induction assumption:

(H-1) : For any $l \leq l^* - 1$, $|m| \leq l$, the functions $g_{0,l,m}(t)$ solve the equation (2.11) with initial data (2.10).

We consider the following equation

$$\partial_t g_{0,l^*,m^*}(t) + \lambda_{0,l^*} g_{0,l^*,m^*}(t) = \sum_{\substack{l+\tilde{l}=l^* \\ l \geq 1, \tilde{l} \geq 1}} \sum_{\substack{m+\tilde{m}=m^* \\ |m| \leq l, |\tilde{m}| \leq \tilde{l}}} \mu_{0,0,l,\tilde{l},0}^{m,\tilde{m},m^*} g_{0,l,m}(t) g_{0,\tilde{l},\tilde{m}}(t).$$

This differential equation can be solved since the functions $g_{0,l,m}(t)$ on the right hand side are only involving the functions $\{g_{0,l,m}(t)\}_{l \leq l^*-1}$ which have been already known by induction assumption **(H-1)**.

In particular, for any $|m| \leq 2$,

$$(2.13) \quad g_{0,2,m}(t) = e^{-\lambda_{0,2}t} g_{0,2,m}(0)$$

and we compute easily from the identity $P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$

$$\lambda_{0,2} = 3 \int_{|\theta| \leq \frac{\pi}{4}} \beta(\theta) \sin^2 \theta \cos^2 \theta d\theta > 0.$$

(2) the case : $n^* \geq 1$. Now we put the following induction assumption:

(H-2) : For any $n \leq n^* - 1$, $l \in \mathbb{N}$ and $|m| \leq l$, the functions $\{g_{n,l,m}(t)\}$ solve the equation (2.11) with initial data (2.10).

First, we want to solve the function $g_{n^*,0,0}$ in (2.11). Since $l^* = m^* = 0$, (2.11) can be written as

$$\begin{aligned} & \partial_t g_{n^*,0,0}(t) + \lambda_{n^*,0} g_{n^*,0,0}(t) \\ &= \sum_{\substack{n+\tilde{n}=n^* \\ n \geq 1, \tilde{n} \geq 1}} \lambda_{n,\tilde{n},0}^{rad,1} g_{n,0,0}(t) g_{\tilde{n},0,0}(t) \\ &+ \sum_{\substack{n+\tilde{n}+k=n^* \\ n \geq 0, \tilde{n} \geq 0}} \sum_{\substack{l+\tilde{l}-2k=0 \\ l \geq 1, \tilde{l} \geq 1}} \sum_{\substack{m+\tilde{m}=0 \\ |m| \leq l, |\tilde{m}| \leq \tilde{l} \\ 0 \leq k \leq \min(l,\tilde{l})}} \mu_{n,\tilde{n},l,\tilde{l},k}^{m,\tilde{m},0} g_{n,l,m}(t) g_{\tilde{n},\tilde{l},\tilde{m}}(t). \end{aligned}$$

In the last term, when $k = 0$, we have

$$l + \tilde{l} = 0 \text{ with constraints } l \geq 1, \tilde{l} \geq 1,$$

which is impossible. Therefore, the equation (2.11) is:

$$\begin{aligned} \partial_t g_{n^*,0,0}(t) + \lambda_{n^*,0} g_{n^*,0,0}(t) \\ = \sum_{\substack{n+\tilde{n}=n^* \\ n \geq 1, \tilde{n} \geq 1}} \lambda_{n,\tilde{n},0}^{rad,1} g_{n,0,0}(t) g_{\tilde{n},0,0}(t) \\ + \sum_{\substack{n+\tilde{n}+k=n^* \\ 0 \leq n, \tilde{n} \leq n^*-1}} \sum_{\substack{l+\tilde{l}-2k=0 \\ l \geq 1, \tilde{l} \geq 1}} \sum_{\substack{m+\tilde{m}=0 \\ |m| \leq l, |\tilde{m}| \leq \tilde{l} \\ 1 \leq k \leq \min(l, \tilde{l})}} \mu_{n,\tilde{n},l,\tilde{l},k}^{m,\tilde{m},0} g_{n,l,m}(t) g_{\tilde{n},\tilde{l},\tilde{m}}(t). \end{aligned}$$

This equation can be also solved since the functions on the right hand side are only involving $\{g_{n,l,m}(t)\}_{n \leq n^*-1, l \in \mathbb{N}, |m| \leq l}$, which have been already given in the induction assumption **(H-2)**.

Finally, let $l^* \geq 1$, we can improve the induction assumption as following:

(H-3): For any $n \leq n^* - 1$, $l \in \mathbb{N}$, $|m| \leq l$ or $n = n^*$, $l \leq l^* - 1$, $|m| \leq l$, the functions $\{g_{n,l,m}(t)\}$ solve the equation (2.11) with initial data (2.10).

We want to solve the functions $g_{n^*,l^*,m^*}(t)$ for all $|m^*| \leq l^*$ in (2.11), which is

$$\begin{aligned} \partial_t g_{n^*,l^*,m^*}(t) + \lambda_{n^*,l^*} g_{n^*,l^*,m^*}(t) \\ = \sum_{\substack{n+\tilde{n}=n^* \\ n \geq 1, \tilde{n} \geq 0}} (\lambda_{n,\tilde{n},l^*}^{rad,1} + \lambda_{\tilde{n},n,l^*}^{rad,2}) g_{n,l^*,m^*}(t) g_{\tilde{n},0,0}(t) \\ + \sum_{\substack{n+\tilde{n}=n^* \\ n \geq 0, \tilde{n} \geq 0}} \sum_{\substack{l+\tilde{l}=l^* \\ l \geq 1, \tilde{l} \geq 1}} \sum_{\substack{m+\tilde{m}=m^* \\ |m| \leq l, |\tilde{m}| \leq \tilde{l}}} \mu_{n,\tilde{n},l,\tilde{l},0}^{m,\tilde{m},m^*} g_{n,l,m}(t) g_{\tilde{n},\tilde{l},\tilde{m}}(t) \\ + \sum_{\substack{n+\tilde{n}+k=n^* \\ n \geq 0, \tilde{n} \geq 0}} \sum_{\substack{l+\tilde{l}-2k=l^* \\ l \geq 1, \tilde{l} \geq 1}} \sum_{\substack{m+\tilde{m}=m^* \\ |m| \leq l, |\tilde{m}| \leq \tilde{l} \\ 1 \leq k \leq \min(l, \tilde{l})}} \mu_{n,\tilde{n},l,\tilde{l},k}^{m,\tilde{m},m^*} g_{n,l,m}(t) g_{\tilde{n},\tilde{l},\tilde{m}}(t). \end{aligned}$$

Here the summation in the last two terms is understanding as Remark 2.4. This equation can be also solved since the functions on the right hand side are only involving $\{g_{n,l,m}(t)\}_{n \leq n^*-1, l \in \mathbb{N}}$ and $\{g_{n,l,m}(t)\}_{n=n^*, l \leq l^*-1}$ which is given by the improved induction assumption **(H-3)**. $\square$

Now the proof of Theorem 1.1 is reduced to prove the convergence of following series

$$(2.14) \quad g(t) = \sum_{n,l,m}^{+\infty} g_{n,l,m}(t) \varphi_{n,l,m}$$

in the suitable function space.

3. THE ESTIMATE OF THE NON LINEAR OPERATORS

3.1. The estimate of the trilinear term. To prove the convergence of the formal solution obtained in the precedent section, we need to estimate the following trilinear terms

$$(\mathbf{\Gamma}(f, g), h)_{L^2(\mathbb{R}^3)}, \quad f, g, h \in \mathcal{S}(\mathbb{R}^3) \cap \mathcal{N}^\perp.$$

Using the spectral representation of $\mathbf{\Gamma}(\cdot, \cdot)$ given in Proposition 2.1, we need to estimate their coefficients.

Proposition 3.1.

1) For $n \geq 1, \tilde{n}, \tilde{l} \in \mathbb{N}$, we have,

$$|\lambda_{n,\tilde{n},\tilde{l}}^{rad,1}|^2 \lesssim \tilde{n}^s (\tilde{n} + \tilde{l})^s n^{-\frac{5}{2}-2s}.$$

2) For all $\tilde{n} \geq 1, n, l \in \mathbb{N}, n + l \geq 2$, we have

$$|\lambda_{n,\tilde{n},l}^{rad,2}|^2 \lesssim \frac{\tilde{n}^{2s}}{(n+1)^s (n+l)^{\frac{5}{2}+s}}.$$

3) For any $n^*, l^* \in \mathbb{N}, |m^*| \leq l^*$, we have also

$$\sum_{\substack{n+\tilde{n}+k=n^* \\ n+l \geq 2, \tilde{n}+\tilde{l} \geq 2 \\ n \geq 0, \tilde{n} \geq 0}} \sum_{\substack{l+\tilde{l}-2k=l^* \\ l \geq 1, \tilde{l} \geq 1 \\ 0 \leq k \leq \min(l, \tilde{l})}} \sum_{|m| \leq l} \sum_{|\tilde{m}| \leq \tilde{l}} \frac{|\mu_{n,\tilde{n},l,l,k}^{m,\tilde{m},m^*}|^2}{\lambda_{\tilde{n},\tilde{l}}} \lesssim \lambda_{n^*,l^*}.$$

The constraint of the above summation is

$$(3.1) \quad \Delta_{n^*,l^*} = \left\{ (n, \tilde{n}, l, \tilde{l}, k, m, \tilde{m}) \in \mathbb{N}^5 \times \mathbb{Z}^2; n + l \geq 2, \tilde{n} + \tilde{l} \geq 2, \right. \\ \left. l \geq 1, \tilde{l} \geq 1, |m| \leq l, |\tilde{m}| \leq \tilde{l}, 0 \leq k \leq \min(l, \tilde{l}) \right. \\ \left. \text{and } n + \tilde{n} + k = n^*, l + \tilde{l} - 2k = l^* \right\}.$$

Following Remark 2.4, we will always write the complicated summation

$$\sum_{(n,\tilde{n},l,\tilde{l},k,m,\tilde{m}) \in \Delta_{n^*,l^*}}$$

in the simplified form :

$$\sum_{\substack{n+\tilde{n}+k=n^* \\ n+l \geq 2, \tilde{n}+\tilde{l} \geq 2 \\ n \geq 0, \tilde{n} \geq 0}} \sum_{\substack{l+\tilde{l}-2k=l^* \\ l \geq 1, \tilde{l} \geq 1 \\ 0 \leq k \leq \min(l, \tilde{l})}} \sum_{|m| \leq l} \sum_{|\tilde{m}| \leq \tilde{l}}.$$

Since the proof of Proposition 3.1 is technical, we prove it in Section 6. Now we prove the following trilinear estimates for the non linear Boltzmann operator.

Proposition 3.2. For all $f, g, h \in \mathcal{S}(\mathbb{R}^3) \cap \mathcal{N}^\perp$,

$$\left| (\mathbf{\Gamma}(f, g), h)_{L^2(\mathbb{R}^3)} \right| \lesssim \|f\|_{L^2(\mathbb{R}^3)} \|\mathcal{L}^{\frac{1}{2}} g\|_{L^2(\mathbb{R}^3)} \|\mathcal{L}^{\frac{1}{2}} h\|_{L^2(\mathbb{R}^3)}.$$

Proof. For any $f, g, h \in \mathcal{S}(\mathbb{R}^3) \cap \mathcal{N}^\perp$, we use the following spectral decomposition,

$$f = \sum_{\substack{n+l \geq 2 \\ n \geq 0, l \geq 0}} \sum_{|m| \leq l} f_{n,l,m} \varphi_{n,l,m}, \quad g = \sum_{\substack{n+l \geq 2 \\ n \geq 0, l \geq 0}} \sum_{|m| \leq l} g_{n,l,m} \varphi_{n,l,m}, \quad h = \sum_{\substack{n+l \geq 2 \\ n \geq 0, l \geq 0}} \sum_{|m| \leq l} h_{n,l,m} \varphi_{n,l,m}.$$

Using the orthogonality of basis $\{\varphi_{n,l,m}\}$ and the formula (2.9), we deduce from Proposition 2.1 that,

$$\begin{aligned}
& (\mathbf{\Gamma}(f, g), h)_{L^2(\mathbb{R}^3)} \\
&= \sum_{\substack{n^* \geq 0, l^* \geq 0 \\ n^* + l^* \geq 2}} \sum_{|m^*| \leq l^*} \overline{h_{n^*, l^*, m^*}} \left(\delta_{l^*, 0} \sum_{\substack{n+\tilde{n}=n^* \\ n \geq 2, \tilde{n} \geq 2}} \lambda_{n, \tilde{n}, 0}^{rad, 1} f_{n, 0, 0} g_{\tilde{n}, 0, 0} \right) \\
&+ \sum_{\substack{n^* \geq 0, l^* \geq 0 \\ n^* + l^* \geq 2}} \sum_{|m^*| \leq l^*} \overline{h_{n^*, l^*, m^*}} \left(\sum_{\substack{n+\tilde{n}=n^*, \tilde{l} \geq 1 \\ n \geq 2, \tilde{n} \geq 0, \tilde{n} + \tilde{l} \geq 2}} \delta_{\tilde{l}, l^*} \lambda_{n, \tilde{n}, l^*}^{rad, 1} f_{n, 0, 0} g_{\tilde{n}, l^*, m^*} \right) \\
&+ \sum_{\substack{n^* \geq 0, l^* \geq 0 \\ n^* + l^* \geq 2}} \sum_{|m^*| \leq l^*} \overline{h_{n^*, l^*, m^*}} \left(\sum_{\substack{n+\tilde{n}=n^*, l \geq 1 \\ n \geq 0, \tilde{n} \geq 2, n + l \geq 2}} \delta_{l, l^*} \lambda_{n, \tilde{n}, l^*}^{rad, 2} f_{n, l^*, m^*} g_{\tilde{n}, 0, 0} \right) \\
&+ \sum_{\substack{n^* \geq 0, l^* \geq 0 \\ n^* + l^* \geq 2}} \sum_{|m^*| \leq l^*} \overline{h_{n^*, l^*, m^*}} \left(\sum_{\substack{n+l \geq 2 \\ n \geq 0, l \geq 1}} \sum_{\substack{\tilde{n} + \tilde{l} \geq 2 \\ \tilde{n} \geq 0, \tilde{l} \geq 1}} \sum_{k=0}^{\min(l, \tilde{l})} \sum_{\substack{|m| \leq l, |\tilde{m}| \leq \tilde{l} \\ |m+\tilde{m}| \leq l+\tilde{l}-2k}} \mu_{n, \tilde{n}, l, \tilde{l}, k}^{m, \tilde{m}, m+\tilde{m}} f_{n, m} g_{\tilde{n}, \tilde{l}, \tilde{m}} \right) \\
&\times \delta_{n^*, n+\tilde{n}+k} \delta_{l^*, l+\tilde{l}-2k} \delta_{m^*, m+\tilde{m}}.
\end{aligned}$$

For brevity, using the formula (2.4), we have

$$\begin{aligned}
& (\mathbf{\Gamma}(f, g), h)_{L^2(\mathbb{R}^3)} \\
&= \sum_{n^*=4}^{+\infty} \overline{h_{n^*, 0, 0}} \left(\sum_{\substack{n+\tilde{n}=n^* \\ n \geq 2, \tilde{n} \geq 2}} \lambda_{n, \tilde{n}, 0}^{rad, 1} f_{n, 0, 0} g_{\tilde{n}, 0, 0} \right) \\
&+ \sum_{\substack{n^* \geq 0, l^* \geq 1 \\ n^* + l^* \geq 2}} \sum_{|m^*| \leq l^*} \overline{h_{n^*, l^*, m^*}} \left(\sum_{\substack{n+\tilde{n}=n^* \\ n \geq 2, \tilde{n} \geq 0, \tilde{n} \geq 2-l^*}} \lambda_{n, \tilde{n}, l^*}^{rad, 1} f_{n, 0, 0} g_{\tilde{n}, l^*, m^*} \right) \\
&+ \sum_{\substack{n^* \geq 0, l^* \geq 1 \\ n^* + l^* \geq 2}} \sum_{|m^*| \leq l^*} \overline{h_{n^*, l^*, m^*}} \left(\sum_{\substack{n+\tilde{n}=n^* \\ n \geq 0, \tilde{n} \geq 2, n \geq 2-l^*}} \lambda_{n, \tilde{n}, l^*}^{rad, 2} f_{n, l^*, m^*} g_{\tilde{n}, 0, 0} \right) \\
&+ \sum_{\substack{n \geq 0, l \geq 1 \\ n+l \geq 2}} \sum_{\substack{\tilde{n} \geq 0, \tilde{l} \geq 1 \\ \tilde{n} + \tilde{l} \geq 2}} \sum_{|m| \leq l} \sum_{|\tilde{m}| \leq \tilde{l}} \sum_{k=0}^{\min(l, \tilde{l})} \sum_{|m^*| \leq l+\tilde{l}-2k} \mu_{n, \tilde{n}, l, \tilde{l}, k}^{m, \tilde{m}, m^*} f_{n, m} g_{\tilde{n}, \tilde{l}, \tilde{m}} \overline{h_{n+\tilde{n}+k, l+\tilde{l}-2k, m^*}} \\
&= I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

For the term I_1 , since $\lambda_{n,0} \approx n^s$ in (2.7), we deduce from Cauchy-Schwarz inequality and 1) of Proposition 3.1 that,

$$\begin{aligned}
|I_1| &\leq \sum_{n^* \geq 4} |h_{n^*,0,0}| \left(\sum_{\substack{n+\tilde{n}=n^* \\ n \geq 2, \tilde{n} \geq 2}} |\lambda_{n,\tilde{n},0}^{rad,1}| |f_{n,0,0}| |g_{\tilde{n},0,0}| \right) \\
&\leq \|f\|_{L^2} \|\mathcal{L}^{\frac{1}{2}} g\|_{L^2} \left(\sum_{\tilde{n}=2}^{+\infty} \sum_{n=2}^{+\infty} |h_{n+\tilde{n},0,0}|^2 \frac{|\lambda_{n,\tilde{n},0}^{rad,1}|^2}{\lambda_{\tilde{n},0}} \right)^{\frac{1}{2}} \\
&\lesssim \|f\|_{L^2} \|\mathcal{L}^{\frac{1}{2}} g\|_{L^2} \left[\sum_{n^*=2}^{\infty} |h_{n^*,0,0}|^2 \left(\sum_{\substack{n+\tilde{n}=n^* \\ n \geq 2, \tilde{n} \geq 2}} \frac{\tilde{n}^s}{n^{\frac{s}{2}+2s}} \right) \right]^{\frac{1}{2}} \lesssim \|f\|_{L^2} \|\mathcal{L}^{\frac{1}{2}} g\|_{L^2} \|\mathcal{L}^{\frac{1}{2}} h\|_{L^2}.
\end{aligned}$$

For the term I_2 , we use Cauchy-Schwarz inequality,

$$\begin{aligned}
|I_2| &\leq \sum_{\substack{n^* \geq 0, l^* \geq 1 \\ n^* + l^* \geq 2}} \sum_{m^* = -l^*}^{l^*} |h_{n^*,l^*,m^*}| \left(\sum_{\substack{n+\tilde{n}=n^* \\ n \geq 2, \tilde{n} \geq \max(0, 2-l^*)}} |\lambda_{n,\tilde{n},l^*}^{rad,1}| |f_{n,0,0}| |g_{\tilde{n},l^*,m^*}| \right) \\
&\leq \sum_{\substack{\tilde{n} \geq 0, l^* \geq 1 \\ \tilde{n} + l^* \geq 2}} \sum_{|m^*| \leq l^*} \sum_{n \geq 2} |\lambda_{n,\tilde{n},l^*}^{rad,1}| |f_{n,0,0}| |g_{\tilde{n},l^*,m^*}| |h_{n+\tilde{n},l^*,m^*}| \\
&\leq \|\mathcal{L}^{\frac{1}{2}} g\|_{L^2} \left(\sum_{\substack{\tilde{n} \geq 0, l^* \geq 1 \\ \tilde{n} + l^* \geq 2}} \sum_{|m^*| \leq l^*} \frac{1}{\lambda_{\tilde{n},l^*}} \left| \sum_{n \geq 2} |\lambda_{n,\tilde{n},l^*}^{rad,1}| |f_{n,0,0}| |h_{n+\tilde{n},l^*,m^*}| \right|^2 \right)^{\frac{1}{2}} \\
&\leq \|f\|_{L^2} \|\mathcal{L}^{\frac{1}{2}} g\|_{L^2} \left(\sum_{\substack{\tilde{n} \geq 0, l^* \geq 1 \\ \tilde{n} + l^* \geq 2}} \sum_{|m^*| \leq l^*} \sum_{n \geq 2} |h_{n+\tilde{n},l^*,m^*}|^2 \frac{|\lambda_{n,\tilde{n},l^*}^{rad,1}|^2}{\lambda_{\tilde{n},l^*}} \right)^{\frac{1}{2}} \\
&\leq \|f\|_{L^2} \|\mathcal{L}^{\frac{1}{2}} g\|_{L^2} \left[\sum_{n^* \geq 2, l^* \geq 1} \sum_{|m^*| \leq l^*} |h_{n^*,l^*,m^*}|^2 \left(\sum_{\substack{n+\tilde{n}=n^* \\ n \geq 2, \tilde{n} \geq \max(0, 2-l^*)}} \frac{1}{\lambda_{\tilde{n},l^*}} |\lambda_{n,\tilde{n},l^*}^{rad,1}|^2 \right) \right]^{\frac{1}{2}}.
\end{aligned}$$

Since $\tilde{n} + l^* \geq 2$, we can deduce from the formula (2.7) that,

$$\lambda_{\tilde{n},l^*} \gtrsim (\tilde{n} + l^*)^s + (l^*)^{2s}.$$

It follows from 1) of Proposition 3.1 and the formula (2.7) that,

$$(3.2) \quad \sum_{\substack{n+\tilde{n}=n^* \\ n \geq 2, \tilde{n} \geq \max(0, 2-l^*)}} \frac{|\lambda_{n,\tilde{n},l^*}^{rad,1}|^2}{\lambda_{\tilde{n},l^*}} \lesssim \sum_{\substack{n+\tilde{n}=n^* \\ n \geq 2, \tilde{n} \geq 0}} \tilde{n}^s n^{-\frac{s}{2}-2s} \lesssim (n^*)^s \lesssim \lambda_{n^*,l^*}.$$

Therefore,

$$|I_2| \leq \|f\|_{L^2} \|\mathcal{L}^{\frac{1}{2}} g\|_{L^2} \|\mathcal{L}^{\frac{1}{2}} h\|_{L^2}.$$

Similarly, for the term I_3 , by using Cauchy-Schwarz inequality, we obtain,

$$\begin{aligned}
|I_3| &\leq \sum_{\substack{n^* \geq 0, l^* \geq 1 \\ n^* + l^* \geq 2}} \sum_{|m^*| \leq l^*} |h_{n^*, l^*, m^*}| \left(\sum_{\substack{n + \tilde{n} = n^* \\ \tilde{n} \geq 2, n \geq \max(0, 2 - l^*)}} |\lambda_{n, \tilde{n}, l^*}^{rad, 2}| |f_{n, l^*, m^*}| |g_{\tilde{n}, 0, 0}| \right) \\
&\leq \sum_{\tilde{n} \geq 2} \sum_{\substack{n \geq 0, l^* \geq 1 \\ n + l^* \geq 2}} \sum_{|m^*| \leq l^*} |\lambda_{n, \tilde{n}, l^*}^{rad, 2}| |f_{n, l^*, m^*}| |g_{\tilde{n}, 0, 0}| |h_{n + \tilde{n}, l^*, m^*}| \\
&\leq \|\mathcal{L}^{\frac{1}{2}} g\|_{L^2} \left(\sum_{\tilde{n} \geq 2} \frac{1}{\lambda_{\tilde{n}, 0}} \left| \sum_{\substack{n \geq 0, l^* \geq 1 \\ n + l^* \geq 2}} \sum_{|m^*| \leq l^*} |\lambda_{n, \tilde{n}, l^*}^{rad, 2}| |f_{n, l^*, m^*}| |h_{n + \tilde{n}, l^*, m^*}| \right|^2 \right)^{\frac{1}{2}} \\
&\leq \|\mathcal{L}^{\frac{1}{2}} g\|_{L^2} \|f\|_{L^2} \left(\sum_{\tilde{n} \geq 2} \sum_{\substack{n \geq 0, l^* \geq 1 \\ n + l^* \geq 2}} \sum_{|m^*| \leq l^*} \frac{|\lambda_{n, \tilde{n}, l^*}^{rad, 2}|^2}{\lambda_{\tilde{n}, 0}} |h_{n + \tilde{n}, l^*, m^*}|^2 \right)^{\frac{1}{2}} \\
&\leq \|\mathcal{L}^{\frac{1}{2}} g\|_{L^2} \|f\|_{L^2} \left(\sum_{\substack{n^* \geq 2, l^* \geq 1 \\ n^* + l^* \geq 2}} \sum_{|m^*| \leq l^*} |h_{n^*, l^*, m^*}|^2 \left[\sum_{\substack{n + \tilde{n} = n^* \\ \tilde{n} \geq 2, n \geq \max(0, 2 - l^*)}} \frac{|\lambda_{n, \tilde{n}, l^*}^{rad, 2}|^2}{\lambda_{\tilde{n}, 0}} \right] \right)^{\frac{1}{2}}.
\end{aligned}$$

We use 2) of Proposition 3.1 and $\lambda_{\tilde{n}, 0} \approx \tilde{n}^s$ in (2.7)

$$\begin{aligned}
\sum_{\substack{n + \tilde{n} = n^* \\ \tilde{n} \geq 2, n \geq \max(0, 2 - l^*)}} \frac{|\lambda_{n, \tilde{n}, l^*}^{rad, 2}|^2}{\lambda_{\tilde{n}, 0}} &\lesssim \sum_{\substack{n + \tilde{n} = n^* \\ \tilde{n} \geq 2, n \geq \max(0, 2 - l^*)}} \frac{\tilde{n}^s}{(n + 1)^s (n + l^*)^{\frac{s}{2} + s}} \\
(3.3) \quad &\lesssim \sum_{\substack{n + \tilde{n} = n^* \\ n = 0}} (n^*)^s + \sum_{\substack{n + \tilde{n} = n^* \\ \tilde{n} \geq 2, n \geq 1}} \frac{(n^*)^s}{n^2} \lesssim (n^*)^s \leq \lambda_{n^*, l^*},
\end{aligned}$$

which gives

$$|I_3| \lesssim \|f\|_{L^2} \|\mathcal{L}^{\frac{1}{2}} g\|_{L^2} \|\mathcal{L}^{\frac{1}{2}} h\|_{L^2}.$$

For the term I_4 , we note that $l \geq 1, \tilde{l} \geq 1$,

$$\begin{aligned}
|I_4| &\leq \sum_{\substack{\tilde{n} \geq 0, \tilde{l} \geq 1 \\ \tilde{n} + \tilde{l} \geq 2}} \sum_{|\tilde{m}| \leq \tilde{l}} |g_{\tilde{n}, \tilde{l}, \tilde{m}}| \sum_{\substack{n \geq 0, l \geq 1 \\ n + l \geq 2}} \sum_{|m| \leq l} |f_{n, l, m}| \\
&\quad \times \left| \sum_{k=0}^{\min(l, \tilde{l})} \sum_{|m^*| \leq l + \tilde{l} - 2k} \mu_{n, \tilde{n}, l, \tilde{l}, k}^{m, \tilde{m}, m^*} \overline{h_{n + \tilde{n} + k, l + \tilde{l} - 2k, m^*}} \right|.
\end{aligned}$$

Applying the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
|I_4| &\leq \left(\sum_{\substack{\tilde{n}+\tilde{l} \geq 2 \\ \tilde{n} \geq 0, \tilde{l} \geq 1}} \sum_{|\tilde{m}| \leq \tilde{l}} \lambda_{\tilde{n}, \tilde{l}} |g_{\tilde{n}, \tilde{l}, \tilde{m}}|^2 \right)^{\frac{1}{2}} \left[\sum_{\substack{\tilde{n}+\tilde{l} \geq 2 \\ \tilde{n} \geq 0, \tilde{l} \geq 1}} \sum_{|\tilde{m}| \leq \tilde{l}} \frac{1}{\lambda_{\tilde{n}, \tilde{l}}} \right. \\
&\quad \left. \left(\sum_{\substack{n+l \geq 2 \\ n \geq 0, l \geq 1}} \sum_{|m| \leq l} |f_{n, l, m}| \left| \sum_{k=0}^{\min(l, \tilde{l})} \sum_{|m^*| \leq l+\tilde{l}-2k} \mu_{n, \tilde{n}, l, \tilde{l}, k}^{m, \tilde{m}, m^*} \overline{h_{n+\tilde{n}+k, l+\tilde{l}-2k, m^*}} \right| \right)^2 \right]^{\frac{1}{2}} \\
&\leq \|\mathcal{L}^{\frac{1}{2}} g\|_{L^2} \|f\|_{L^2} \\
&\quad \times \left[\sum_{\substack{\tilde{n}+\tilde{l} \geq 2 \\ \tilde{n} \geq 0, \tilde{l} \geq 1}} \sum_{\substack{n+l \geq 2 \\ n \geq 0, l \geq 1}} \frac{1}{\lambda_{\tilde{n}, \tilde{l}}} \sum_{|m| \leq l} \sum_{|\tilde{m}| \leq \tilde{l}} \left| \sum_{k=0}^{\min(l, \tilde{l})} \sum_{|m^*| \leq l+\tilde{l}-2k} \mu_{n, \tilde{n}, l, \tilde{l}, k}^{m, \tilde{m}, m^*} \overline{h_{n+\tilde{n}+k, l+\tilde{l}-2k, m^*}} \right|^2 \right]^{\frac{1}{2}}.
\end{aligned}$$

We observe the summation

$$\begin{aligned}
&\sum_{|m| \leq l} \sum_{|\tilde{m}| \leq \tilde{l}} \left| \sum_{k=0}^{\min(l, \tilde{l})} \sum_{|m^*| \leq l+\tilde{l}-2k} \mu_{n, \tilde{n}, l, \tilde{l}, k}^{m, \tilde{m}, m^*} \overline{h_{n+\tilde{n}+k, l+\tilde{l}-2k, m^*}} \right|^2 \\
&= \sum_{k_1=0}^{\min(l, \tilde{l})} \sum_{|m_1^*| \leq l+\tilde{l}-2k_1} \sum_{k_2=0}^{\min(l, \tilde{l})} \sum_{|m_2^*| \leq l+\tilde{l}-2k_2} \overline{h_{n+\tilde{n}+k_1, l+\tilde{l}-2k_1, m_1^*}} h_{n+\tilde{n}+k_2, l+\tilde{l}-2k_2, m_2^*} \\
&\quad \times \left(\sum_{|m| \leq l} \sum_{|\tilde{m}| \leq \tilde{l}} \mu_{n, \tilde{n}, l, \tilde{l}, k_1}^{m, \tilde{m}, m_1^*} \overline{\mu_{n, \tilde{n}, l, \tilde{l}, k_2}^{m, \tilde{m}, m_2^*}} \right).
\end{aligned}$$

By using the formula (2.5) in Proposition 2.2, we obtain

$$\sum_{|m| \leq l} \sum_{|\tilde{m}| \leq \tilde{l}} \mu_{n, \tilde{n}, l, \tilde{l}, k_1}^{m, \tilde{m}, m_1^*} \overline{\mu_{n, \tilde{n}, l, \tilde{l}, k_2}^{m, \tilde{m}, m_2^*}} = \sum_{|m| \leq l} \sum_{|\tilde{m}| \leq \tilde{l}} \left| \mu_{n, \tilde{n}, l, \tilde{l}, k_1}^{m, \tilde{m}, m_1^*} \right|^2 \delta_{k_1, k_2} \delta_{m_1^*, m_2^*}.$$

Therefore, the summation can be re-written as

$$\begin{aligned}
&\sum_{|m| \leq l} \sum_{|\tilde{m}| \leq \tilde{l}} \left| \sum_{k=0}^{\min(l, \tilde{l})} \sum_{|m^*| \leq l+\tilde{l}-2k} \mu_{n, \tilde{n}, l, \tilde{l}, k}^{m, \tilde{m}, m^*} \overline{h_{n+\tilde{n}+k, l+\tilde{l}-2k, m^*}} \right|^2 \\
&= \sum_{k=0}^{\min(l, \tilde{l})} \sum_{|m^*| \leq l+\tilde{l}-2k} \left| h_{n+\tilde{n}+k, l+\tilde{l}-2k, m^*} \right|^2 \sum_{|m| \leq l} \sum_{|\tilde{m}| \leq \tilde{l}} \left| \mu_{n, \tilde{n}, l, \tilde{l}, k}^{m, \tilde{m}, m^*} \right|^2.
\end{aligned}$$

It follows that

$$\begin{aligned}
|I_4| &\leq \|f\|_{L^2} \|\mathcal{L}^{\frac{1}{2}} g\|_{L^2} \left(\sum_{\substack{\tilde{n}+\tilde{l} \geq 2 \\ \tilde{n} \geq 0, \tilde{l} \geq 1}} \sum_{\substack{n+l \geq 2 \\ n \geq 0, l \geq 1}} \sum_{0 \leq k \leq \min(l, \tilde{l})} \sum_{|m^*| \leq l+\tilde{l}-2k} |h_{n+\tilde{n}+k, l+\tilde{l}-2k, m^*}|^2 \right. \\
&\quad \times \left. \frac{1}{\lambda_{\tilde{n}, \tilde{l}}} \sum_{|m| \leq l} \sum_{|\tilde{m}| \leq \tilde{l}} |\mu_{n, \tilde{n}, l, \tilde{l}, k}^{m, \tilde{m}, m^*}|^2 \right)^{\frac{1}{2}} \\
&\leq \|f\|_{L^2} \|\mathcal{L}^{\frac{1}{2}} g\|_{L^2} \left[\sum_{n^*=0}^{+\infty} \sum_{l^*=0}^{+\infty} \sum_{|m^*| \leq l^*} |h_{n^*, l^*, m^*}|^2 \right. \\
&\quad \times \left. \left(\sum_{\substack{n+\tilde{n}+k=n^* \\ n+l \geq 2, \tilde{n}+\tilde{l} \geq 2 \\ n \geq 0, \tilde{n} \geq 0}} \sum_{\substack{l+\tilde{l}-2k=l^* \\ l \geq 1, \tilde{l} \geq 1 \\ 0 \leq k \leq \min(l, \tilde{l})}} \sum_{|m| \leq l} \sum_{|\tilde{m}| \leq \tilde{l}} \frac{1}{\lambda_{\tilde{n}, \tilde{l}}} |\mu_{n, \tilde{n}, l, \tilde{l}, k}^{m, \tilde{m}, m^*}|^2 \right) \right]^{\frac{1}{2}},
\end{aligned}$$

where the last summation is understanding as (3.1). Using 3) of Proposition 3.1, we have

$$\sum_{\substack{n+\tilde{n}+k=n^* \\ n+l \geq 2, \tilde{n}+\tilde{l} \geq 2 \\ n \geq 0, \tilde{n} \geq 0}} \sum_{\substack{l+\tilde{l}-2k=l^* \\ l \geq 1, \tilde{l} \geq 1 \\ 0 \leq k \leq \min(l, \tilde{l})}} \sum_{|m| \leq l} \sum_{|\tilde{m}| \leq \tilde{l}} \frac{1}{\lambda_{\tilde{n}, \tilde{l}}} |\mu_{n, \tilde{n}, l, \tilde{l}, k}^{m, \tilde{m}, m^*}|^2 \leq C \lambda_{n^*, l^*}.$$

We get then

$$|I_4| \lesssim \|f\|_{L^2} \|\mathcal{L}^{\frac{1}{2}} g\|_{L^2} \left[\sum_{n^*=0}^{+\infty} \sum_{l^*=0}^{+\infty} \sum_{|m^*| \leq l^*} \lambda_{n^*, l^*} |h_{n^*, l^*, m^*}|^2 \right]^{\frac{1}{2}},$$

which ends the proof of Proposition 3.2. $\square$

3.2. The estimate of the trilinear term with exponential weight. To prove the regularity in Gelfand-Shilov spaces, we need an upper bound on non linear operators with exponential weight.

Proposition 3.3. *For any $f, g, h \in \mathcal{S}(\mathbb{R}^3) \cap \mathcal{N}^\perp$, any $N \geq 0$, and for any $c > 0$, we have*

$$\begin{aligned}
(3.4) \quad &|(\mathbf{T}(f, g), e^{c\mathcal{H}^s} \mathbf{S}_N h)_{L^2}| \\
&\leq C \|e^{\frac{s}{2}\mathcal{H}^s} \mathbf{S}_{N-2} f\|_{L^2} \|e^{\frac{s}{2}\mathcal{H}^s} \mathcal{L}^{\frac{1}{2}} \mathbf{S}_{N-2} g\|_{L^2} \|e^{\frac{s}{2}\mathcal{H}^s} \mathcal{L}^{\frac{1}{2}} \mathbf{S}_N h\|_{L^2},
\end{aligned}$$

where C is a positive constant only dependent on s , and $\mathbf{S}_N$ is the orthogonal projector such that,

$$\begin{aligned}
\mathbf{S}_N f &= \sum_{\substack{2n+l \leq N \\ n \geq 0, l \geq 0}} \sum_{|m| \leq l} (f, \varphi_{n, l, m})_{L^2} \varphi_{n, l, m}, \\
e^{c\mathcal{H}^s} \mathbf{S}_N f &= \sum_{\substack{2n+l \leq N \\ n \geq 0, l \geq 0}} \sum_{|m| \leq l} e^{c(2n+l+\frac{3}{2})^s} (f, \varphi_{n, l, m})_{L^2} \varphi_{n, l, m}.
\end{aligned}$$

Remark 3.4. 1) For $h \in \mathcal{S}(\mathbb{R}^3)$, we can't use $e^{\frac{s}{2}\mathcal{H}^s} h$ as test function, since we don't know if it belongs to $\mathcal{S}(\mathbb{R}^3)$. However, for any $h \in \mathcal{S}'(\mathbb{R}^3)$, we have $e^{\frac{s}{2}\mathcal{H}^s} \mathbf{S}_N h \in \mathcal{S}(\mathbb{R}^3)$.

2) In the right hand side of (3.4), the projector of f and g with $\mathbf{S}_{N-2}$ shows more clearly the triangular effect of $\mathbf{T}(\cdot, \cdot)$.

Proof. Since $f, g, h \in \mathcal{S}(\mathbb{R}^3) \cap \mathcal{N}^\perp$, similarly to Proposition 3.2, we have

$$\begin{aligned}
& \left(\Gamma(f, g), e^{c\mathcal{H}^s} \mathbf{S}_N h \right)_{L^2(\mathbb{R}^3)} \\
&= \sum_{n^*=2}^{\lfloor \frac{N}{2} \rfloor} e^{c(2n^* + \frac{3}{2})^s} \overline{h_{n^*,0,0}} \left(\sum_{\substack{n+\tilde{n}=n^* \\ n \geq 2, \tilde{n} \geq 2}} \lambda_{n,\tilde{n},0}^{rad,1} f_{n,0,0} g_{\tilde{n},0,0} \right) \\
&+ \sum_{\substack{2n^*+l^* \leq N \\ n^* \geq 0, l^* \geq 1, n^*+l^* \geq 2}} \sum_{|m^*| \leq l^*} e^{c(2n^*+l^* + \frac{3}{2})^s} \overline{h_{n^*,l^*,m^*}} \left(\sum_{\substack{n+\tilde{n}=n^* \\ n \geq 2, \tilde{n} \geq 0, \tilde{n} \geq 2-l^*}} \lambda_{n,\tilde{n},l^*}^{rad,1} f_{n,0,0} g_{\tilde{n},l^*,m^*} \right) \\
&+ \sum_{\substack{2n^*+l^* \leq N \\ n^* \geq 0, l^* \geq 1, n^*+l^* \geq 2}} \sum_{|m^*| \leq l^*} e^{c(2n^*+l^* + \frac{3}{2})^s} \overline{h_{n^*,l^*,m^*}} \left(\sum_{\substack{n+\tilde{n}=n^* \\ n \geq 0, \tilde{n} \geq 2, n \geq 2-l^*}} \lambda_{n,\tilde{n},l^*}^{rad,2} f_{n,l^*,m^*} g_{\tilde{n},0,0} \right) \\
&+ \sum_{\substack{2n+2\tilde{n}+l+\tilde{l} \leq N \\ n+l \geq 2, \tilde{n}+\tilde{l} \geq 2 \\ n \geq 0, l \geq 1, \tilde{n} \geq 0, \tilde{l} \geq 1}} \sum_{|m| \leq l} \sum_{|\tilde{m}| \leq \tilde{l}} f_{n,l,m} g_{\tilde{n},\tilde{l},\tilde{m}} e^{c(2n+2\tilde{n}+l+\tilde{l} + \frac{3}{2})^s} \\
&\quad \times \left(\sum_{k=0}^{\min(l,\tilde{l})} \sum_{|m^*| \leq l+\tilde{l}-2k} \mu_{n,\tilde{n},l,\tilde{l},k}^{m,\tilde{m},m^*} \overline{h_{n+\tilde{n}+k,l+\tilde{l}-2k,m^*}} \right) \\
&= J_1 + J_2 + J_3 + J_4.
\end{aligned}$$

For the term J_1 , by using $\lambda_{n,0} \approx n^s$ and Cauchy-Schwarz inequality, we obtain,

$$\begin{aligned}
|J_1| &\leq \sum_{\substack{n+\tilde{n} \leq \lfloor \frac{N}{2} \rfloor \\ n \geq 2, \tilde{n} \geq 2}} e^{c(2(n+\tilde{n}) + \frac{3}{2})^s} |\lambda_{n,\tilde{n},0}^{rad,1}| |f_{n,0,0}| |g_{\tilde{n},0,0}| |h_{n+\tilde{n},0,0}| \\
&\leq \left(\sum_{\tilde{n}=2}^{\lfloor \frac{N}{2} \rfloor - 2} e^{c(2\tilde{n} + \frac{3}{2})^s} \lambda_{\tilde{n},0} |g_{\tilde{n},0,0}|^2 \right)^{1/2} \\
&\quad \times \left(\sum_{\tilde{n}=2}^{\lfloor \frac{N}{2} \rfloor - 2} e^{-c(2\tilde{n} + \frac{3}{2})^s} \frac{1}{\lambda_{\tilde{n},0}} \left| \sum_{n=2}^{\lfloor \frac{N}{2} \rfloor - \tilde{n}} e^{c(2(n+\tilde{n}) + \frac{3}{2})^s} |\lambda_{n,\tilde{n},0}^{rad,1}| |f_{n,0,0}| |h_{n+\tilde{n},0,0}|^2 \right) \right)^{\frac{1}{2}} \\
&\leq \|e^{\frac{s}{2}\mathcal{H}^s} \mathcal{L}^{\frac{1}{2}} \mathbf{S}_{N-2} g\|_{L^2} \|e^{\frac{s}{2}\mathcal{H}^s} \mathbf{S}_{N-2} f\|_{L^2} \\
&\quad \times \left(\sum_{\tilde{n}=2}^{\lfloor \frac{N}{2} \rfloor - 2} \sum_{n=2}^{\lfloor \frac{N}{2} \rfloor - \tilde{n}} e^{2c(2n+2\tilde{n} + \frac{3}{2})^s - c(2\tilde{n} + \frac{3}{2})^s - c(2n + \frac{3}{2})^s} |h_{n+\tilde{n},0,0}|^2 \frac{|\lambda_{n,\tilde{n},0}^{rad,1}|^2}{\lambda_{\tilde{n},0}} \right)^{\frac{1}{2}}.
\end{aligned}$$

For any $0 < s < 1, n, \tilde{n} \in \mathbb{N}$,

$$(2n + 2\tilde{n} + \frac{3}{2})^s \leq (2\tilde{n} + \frac{3}{2})^s + (2n + \frac{3}{2})^s.$$

Then

$$|J_1| \leq \|e^{\frac{s}{2}\mathcal{H}^s} \mathcal{L}^{\frac{1}{2}} \mathbf{S}_{N-2} g\|_{L^2} \|e^{\frac{s}{2}\mathcal{H}^s} \mathbf{S}_{N-2} f\|_{L^2} \left(\sum_{\tilde{n}=2}^{\lfloor \frac{N}{2} \rfloor - 2} \sum_{n=2}^{\lfloor \frac{N}{2} \rfloor - \tilde{n}} e^{c(2n+2\tilde{n} + \frac{3}{2})^s} |h_{n+\tilde{n},0,0}|^2 \frac{|\lambda_{n,\tilde{n},0}^{rad,1}|^2}{\lambda_{\tilde{n},0}} \right)^{\frac{1}{2}}.$$

It follows by the same estimate as I_1 ,

$$\begin{aligned} |J_1| &\lesssim \|e^{\frac{\epsilon}{2}\mathcal{H}^s} \mathcal{L}^{\frac{1}{2}} \mathbf{S}_{N-2} g\|_{L^2} \|e^{\frac{\epsilon}{2}\mathcal{H}^s} \mathbf{S}_{N-2} f\|_{L^2} \left(\sum_{n^*=4}^{\lfloor \frac{N}{2} \rfloor} e^{c(2n^*+\frac{3}{2})^s} |h_{n^*,0,0}|^2 \left(\sum_{\substack{n+\tilde{n}=n^* \\ n \geq 2, \tilde{n} \geq 2}} \frac{\tilde{n}^s}{n^{\frac{s}{2}+2s}} \right) \right)^{\frac{1}{2}} \\ &\lesssim \|e^{\frac{\epsilon}{2}\mathcal{H}^s} \mathcal{L}^{\frac{1}{2}} \mathbf{S}_{N-2} g\|_{L^2} \|e^{\frac{\epsilon}{2}\mathcal{H}^s} \mathbf{S}_{N-2} f\|_{L^2} \|e^{\frac{\epsilon}{2}\mathcal{H}^s} \mathcal{L}^{\frac{1}{2}} \mathbf{S}_N h\|_{L^2}. \end{aligned}$$

For the term J_2 ,

$$\begin{aligned} |J_2| &\leq \sum_{\substack{2\tilde{n}+l^* \leq N-4 \\ \tilde{n} \geq 0, l^* \geq 1, \tilde{n}+l^* \geq 2}} \sum_{|m^*| \leq l^*} e^{\frac{\epsilon}{2}(2\tilde{n}+l^*+\frac{3}{2})^s} |g_{\tilde{n},l^*,m^*}| \sum_{\substack{n \geq 2 \\ 2n+2\tilde{n}+l^* \leq N}} e^{\frac{\epsilon}{2}(2n+\frac{3}{2})^s} |f_{n,0,0}| \\ &\quad \times e^{c(2n+2\tilde{n}+l^*+\frac{3}{2})^s - \frac{c(2n+\frac{3}{2})^s}{2} - \frac{c(2\tilde{n}+l^*+\frac{3}{2})^s}{2}} |\lambda_{n,\tilde{n},l^*}^{rad,1} h_{n+\tilde{n},l^*,m^*}|. \end{aligned}$$

Since for any $0 < s < 1$, $n, \tilde{n}, l^* \in \mathbb{N}$,

$$(2n + 2\tilde{n} + l^* + \frac{3}{2})^s \leq (2n + \frac{3}{2})^s + (2\tilde{n} + l^* + \frac{3}{2})^s,$$

we can deduce from Cauchy-Schwarz inequality that,

$$\begin{aligned} |J_2| &\leq \left(\sum_{\substack{2\tilde{n}+l^* \leq N-4 \\ \tilde{n} \geq 0, l^* \geq 1, \tilde{n}+l^* \geq 2}} \sum_{|m^*| \leq l^*} e^{c(2\tilde{n}+l^*+\frac{3}{2})^s} |\lambda_{\tilde{n},l^*,m^*}|^2 \right)^{\frac{1}{2}} \\ &\quad \times \left(\sum_{\substack{2\tilde{n}+l^* \leq N-4 \\ \tilde{n} \geq 0, l^* \geq 1, \tilde{n}+l^* \geq 2}} \sum_{|m^*| \leq l^*} \frac{1}{|\lambda_{\tilde{n},l^*}|} \left| \sum_{\substack{n \geq 2 \\ 2n+2\tilde{n}+l^* \leq N}} e^{\frac{\epsilon}{2}(2n+\frac{3}{2})^s} |f_{n,0,0}| \right. \right. \\ &\quad \left. \left. \times e^{\frac{\epsilon}{2}(2n+2\tilde{n}+l^*+\frac{3}{2})^s} |\lambda_{n,\tilde{n},l^*}^{rad,1} h_{n+\tilde{n},l^*,m^*}| \right|^2 \right)^{\frac{1}{2}} \\ &\leq \|e^{\frac{\epsilon}{2}\mathcal{H}^s} \mathcal{L}^{\frac{1}{2}} \mathbf{S}_{N-2} g\|_{L^2} \|e^{\frac{\epsilon}{2}\mathcal{H}^s} \mathbf{S}_{N-2} f\|_{L^2} \\ &\quad \times \left(\sum_{\substack{2\tilde{n}+l^* \leq N-4 \\ \tilde{n} \geq 0, l^* \geq 1, \tilde{n}+l^* \geq 2}} \sum_{|m^*| \leq l^*} \sum_{\substack{n \geq 2 \\ 2n+2\tilde{n}+l^* \leq N}} e^{c(2n+2\tilde{n}+l^*+\frac{3}{2})^s} |h_{n+\tilde{n},l^*,m^*}|^2 \frac{|\lambda_{n,\tilde{n},l^*}^{rad,1}|^2}{|\lambda_{\tilde{n},l^*}|} \right)^{\frac{1}{2}} \\ &\leq \|e^{\frac{\epsilon}{2}\mathcal{H}^s} \mathcal{L}^{\frac{1}{2}} \mathbf{S}_{N-2} g\|_{L^2} \|e^{\frac{\epsilon}{2}\mathcal{H}^s} \mathbf{S}_{N-2} f\|_{L^2} \\ &\quad \times \left[\sum_{\substack{4 \leq 2n^*+l^* \leq N \\ n^* \geq 0, l^* \geq 1}} \sum_{|m^*| \leq l^*} e^{c(2n^*+l^*+\frac{3}{2})^s} |h_{n^*,l^*,m^*}|^2 \left(\sum_{\substack{n+\tilde{n}=n^* \\ n \geq 2, \tilde{n} \geq \max(0, 2-l^*)}} \frac{1}{|\lambda_{\tilde{n},l^*}|} |\lambda_{n,\tilde{n},l^*}^{rad,1}|^2 \right) \right]^{\frac{1}{2}}. \end{aligned}$$

Recall the estimate (3.2) that,

$$\sum_{\substack{n+\tilde{n}=n^* \\ n \geq 2, \tilde{n} \geq \max(0, 2-l^*)}} \frac{1}{|\lambda_{\tilde{n},l^*}|} |\lambda_{n,\tilde{n},l^*}^{rad,1}|^2 \lesssim (n^*) \lesssim \lambda_{n^*,l^*},$$

we obtain,

$$|J_2| \lesssim \|e^{\frac{\epsilon}{2}\mathcal{H}^s} \mathcal{L}^{\frac{1}{2}} \mathbf{S}_{N-2} g\|_{L^2} \|e^{\frac{\epsilon}{2}\mathcal{H}^s} \mathbf{S}_{N-2} f\|_{L^2} \|e^{\frac{\epsilon}{2}\mathcal{H}^s} \mathcal{L}^{\frac{1}{2}} \mathbf{S}_N h\|_{L^2}.$$

Similarly, for the third term J_3 ,

$$|J_3| \leq \sum_{\tilde{n}=2}^{\frac{N}{2}-1} e^{\frac{c}{2}(2\tilde{n}+\frac{3}{2})^s} |g_{\tilde{n},0,0}| \sum_{\substack{n \geq 0, l^* \geq 1, n+l^* \geq 2 \\ 2n+2\tilde{n}+l^* \leq N}} \sum_{|m^*| \leq l^*} e^{\frac{c}{2}(2n+l^*+\frac{3}{2})^s} |f_{n,l^*,m^*}| \\ \times e^{c(2n+2\tilde{n}+l^*+\frac{3}{2})^s - \frac{c(2\tilde{n}+\frac{3}{2})^s}{2} - \frac{c(2n+l^*+\frac{3}{2})^s}{2}} |\lambda_{n,\tilde{n},l^*}^{rad,2} h_{n+\tilde{n},l^*,m^*}|.$$

By using Cauchy-Schwarz inequality and the inequality

$$(2n + 2\tilde{n} + l^* + \frac{3}{2})^s \leq (2\tilde{n} + \frac{3}{2})^s + (2n + l^* + \frac{3}{2})^s, \forall 0 < s < 1,$$

we obtain,

$$|J_3| \leq \|e^{\frac{c}{2}\mathcal{H}^s} \mathcal{L}^{\frac{1}{2}} \mathbf{S}_{N-2} g\|_{L^2} \|e^{\frac{c}{2}\mathcal{H}^s} \mathbf{S}_{N-2} f\|_{L^2} \\ \times \left[\sum_{\substack{4 \leq 2n^*+l^* \leq N \\ n^* \geq 0, l^* \geq 1}} \sum_{|m^*| \leq l^*} e^{c(2n^*+l^*+\frac{3}{2})^s} |h_{n^*,l^*,m^*}|^2 \left(\sum_{\substack{n+\tilde{n}=n^* \\ \tilde{n} \geq 2, n \geq \max(0, 2-l^*)}} \frac{|\lambda_{n,\tilde{n},l^*}^{rad,2}|^2}{\lambda_{\tilde{n},0}} \right) \right]^{\frac{1}{2}}.$$

By using the estimate (3.3), one has,

$$|J_3| \lesssim \|e^{\frac{c}{2}\mathcal{H}^s} \mathcal{L}^{\frac{1}{2}} \mathbf{S}_{N-2} g\|_{L^2} \|e^{\frac{c}{2}\mathcal{H}^s} \mathbf{S}_{N-2} f\|_{L^2} \|e^{\frac{c}{2}\mathcal{H}^s} \mathcal{L}^{\frac{1}{2}} \mathbf{S}_N h\|_{L^2}.$$

Consider the fourth term J_4 ,

$$|J_4| \leq \sum_{\substack{2\tilde{n}+\tilde{l} \leq N-2 \\ \tilde{n}+\tilde{l} \geq 2, \tilde{n} \geq 0, \tilde{l} \geq 1}} \sum_{|\tilde{m}| \leq \tilde{l}} (\lambda_{\tilde{n},\tilde{l}})^{\frac{1}{2}} e^{\frac{c(2\tilde{n}+\tilde{l}+\frac{3}{2})^s}{2}} |g_{\tilde{n},\tilde{l},\tilde{m}}| \sum_{\substack{2n+l \leq N-2 \\ n+l \geq 2, l \geq 1, n \geq 0 \\ 2n+2\tilde{n}+l+\tilde{l} \leq N}} \sum_{|m| \leq l} e^{\frac{c(2n+l+\frac{3}{2})^s}{2}} |f_{n,l,m}| \\ \times e^{c(2n+2\tilde{n}+l+\tilde{l}+\frac{3}{2})^s - \frac{c(2n+l+\frac{3}{2})^s}{2} - \frac{c(2\tilde{n}+\tilde{l}+\frac{3}{2})^s}{2}} \\ \times \left| \sum_{k=0}^{\min(l,\tilde{l})} \sum_{|m^*| \leq l+\tilde{l}-2k} \mu_{n,\tilde{n},l,\tilde{l},k}^{m,\tilde{m},m^*} (\lambda_{\tilde{n},\tilde{l}})^{-\frac{1}{2}} \overline{h_{n+\tilde{n}+k,l+\tilde{l}-2k,m^*}} \right|.$$

Using the fact that for any $0 < s < 1$, $n, \tilde{n}, l, \tilde{l} \in \mathbb{N}$,

$$(2n + l + 2\tilde{n} + \tilde{l} + \frac{3}{2})^s \leq (2n + l + \frac{3}{2})^s + (2\tilde{n} + \tilde{l} + \frac{3}{2})^s,$$

we can deduce from Cauchy-Schwarz inequality that,

$$\begin{aligned}
|J_4| &\leq \|e^{\frac{\epsilon}{2}\mathcal{H}^s} \mathcal{L}^{\frac{1}{2}} \mathbf{S}_{N-2} g\|_{L^2} \|e^{\frac{\epsilon}{2}\mathcal{H}^s} \mathbf{S}_{N-2} f\|_{L^2} \\
&\times \left[\sum_{\substack{4 \leq 2n+2\tilde{n}+l+\tilde{l} \leq N \\ 2 \leq n+l, 2 \leq \tilde{n}+\tilde{l} \\ n \geq 0, l \geq 1, \tilde{n} \geq 0, \tilde{l} \geq 1}} \sum_{k=0}^{\min(l, \tilde{l})} \sum_{|m^*| \leq l+\tilde{l}-2k} e^{c(2n+2\tilde{n}+l+\tilde{l}+\frac{3}{2})^s} \right. \\
&\quad \left. \times |h_{n+\tilde{n}+k, l+\tilde{l}-2k, m^*}|^2 \sum_{|\tilde{m}| \leq \tilde{l}} \sum_{|m| \leq l} \frac{|\mu_{n, \tilde{n}, l, \tilde{l}, k}^{m, \tilde{m}, m^*}|^2}{\lambda_{\tilde{n}, \tilde{l}}} \right]^{\frac{1}{2}} \\
&\leq \|e^{\frac{\epsilon}{2}\mathcal{H}^s} \mathcal{L}^{\frac{1}{2}} \mathbf{S}_{N-2} g\|_{L^2} \|e^{\frac{\epsilon}{2}\mathcal{H}^s} \mathbf{S}_{N-2} f\|_{L^2} \\
&\times \left[\sum_{4 \leq 2n^*+l^* \leq N} \sum_{|m^*| \leq l+\tilde{l}-2k} e^{c(2n^*+l^*+\frac{3}{2})^s} |h_{n^*, l^*, m^*}|^2 \right. \\
&\quad \left. \times \left(\sum_{\substack{n+\tilde{n}+k=n^* \\ n+l \geq 2, \tilde{n}+\tilde{l} \geq 2 \\ n \geq 0, \tilde{n} \geq 0}} \sum_{\substack{l+\tilde{l}-2k=l^* \\ l \geq 1, \tilde{l} \geq 1 \\ 0 \leq k \leq \min(l, \tilde{l})}} \sum_{|m| \leq l} \sum_{|\tilde{m}| \leq \tilde{l}} \frac{|\mu_{n, \tilde{n}, l, \tilde{l}, k}^{m, \tilde{m}, m^*}|^2}{\lambda_{\tilde{n}, \tilde{l}}} \right) \right]^{\frac{1}{2}},
\end{aligned}$$

The last summation is understanding as (3.1). We use again 3) of Proposition 3.1,

$$\sum_{\substack{n+\tilde{n}+k=n^* \\ n+l \geq 2, \tilde{n}+\tilde{l} \geq 2 \\ n \geq 0, \tilde{n} \geq 0}} \sum_{\substack{l+\tilde{l}-2k=l^* \\ l \geq 1, \tilde{l} \geq 1 \\ 0 \leq k \leq \min(l, \tilde{l})}} \sum_{|m| \leq l} \sum_{|\tilde{m}| \leq \tilde{l}} \frac{|\mu_{n, \tilde{n}, l, \tilde{l}, k}^{m, \tilde{m}, m^*}|^2}{\lambda_{\tilde{n}, \tilde{l}}} \leq C \lambda_{n^*, l^*}.$$

We can finish the proof exactly as that of Proposition 3.2. $\square$

4. THE PROOF OF THE MAIN THEOREM

In this section, we study the convergence of the formal solutions obtained on Section 2 with small L^2 initial data and this ends the proof of Theorem 1.1.

4.1. The uniform estimate. Let $\{g_{n,l,m}(t)\}$ be the solution of (2.11). For any $N \in \mathbb{N}$, set

$$(4.1) \quad \mathbf{S}_N g(t) = \sum_{\substack{2n+l \leq N \\ n \geq 0, l \geq 0}} \sum_{|m| \leq l} g_{n,l,m}(t) \varphi_{n,l,m}.$$

Then $\mathbf{S}_N g(t), e^{c_0 t \mathcal{H}^s} \mathbf{S}_N g(t) \in \mathcal{S}(\mathbb{R}^3) \cap \mathcal{N}^\perp$,

Multiplying $e^{c_0 t(2n^*+l^*+\frac{3}{2})^s} \overline{g_{n^*, l^*, m^*}}(t)$ on both sides of (2.11) and taking summation for $2n^*+l^* \leq N$, then Proposition 2.1 and the orthogonality of the basis $(\varphi_{n,l,m})_{n,l \geq 0, |m| \leq l}$ imply

$$\begin{aligned}
& \left(\partial_t (\mathbf{S}_N g)(t), e^{c_0 t \mathcal{H}^s} \mathbf{S}_N g(t) \right)_{L^2(\mathbb{R}^3)} + \left(\mathcal{L}(\mathbf{S}_N g)(t), e^{c_0 t \mathcal{H}^s} \mathbf{S}_N g(t) \right)_{L^2(\mathbb{R}^3)} \\
&= \left(\Gamma(\mathbf{S}_N g, \mathbf{S}_N g), e^{c_0 t \mathcal{H}^s} \mathbf{S}_N g(t) \right)_{L^2(\mathbb{R}^3)}.
\end{aligned}$$

Since $\mathbf{S}_N g(t) \in \mathcal{S}(\mathbb{R}^3) \cap \mathcal{N}^\perp$, we have

$$\left(\mathcal{L}(\mathbf{S}_N g)(t), e^{c_0 t \mathcal{H}^s} \mathbf{S}_N g(t) \right)_{L^2(\mathbb{R}^3)} = \|e^{\frac{c_0 t}{2} \mathcal{H}^s} \mathcal{L}^{\frac{1}{2}} \mathbf{S}_N g(t)\|_{L^2(\mathbb{R}^3)},$$

we then obtain that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|e^{\frac{c_0 t}{2} \mathcal{H}^s} \mathbf{S}_N g(t)\|_{L^2}^2 - \frac{c_0}{2} \|e^{\frac{c_0 t}{2} \mathcal{H}^s} \mathcal{H}^{\frac{s}{2}} \mathbf{S}_N g\|_{L^2}^2 + \|e^{\frac{c_0 t}{2} \mathcal{H}^s} \mathcal{L}^{\frac{1}{2}} \mathbf{S}_N g(t)\|_{L^2}^2 \\ = \left(\Gamma((\mathbf{S}_N g), (\mathbf{S}_N g)), e^{c_0 t \mathcal{H}^s} \mathbf{S}_N g(t) \right)_{L^2}. \end{aligned}$$

It follows from (2.7) and Proposition 3.3 that, for $0 \leq c_0 \leq c_1$ and any $N \geq 2$, $t \geq 0$,

$$(4.2) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \|e^{\frac{c_0 t}{2} \mathcal{H}^s} \mathbf{S}_N g(t)\|_{L^2}^2 + \frac{1}{2} \|e^{\frac{c_0 t}{2} \mathcal{H}^s} \mathcal{L}^{\frac{1}{2}} \mathbf{S}_N g\|_{L^2}^2 \\ \leq C \|e^{\frac{c_0 t}{2} \mathcal{H}^s} \mathbf{S}_{N-2} g\|_{L^2} \|e^{\frac{c_0 t}{2} \mathcal{H}^s} \mathcal{L}^{\frac{1}{2}} \mathbf{S}_N g\|_{L^2}^2. \end{aligned}$$

Proposition 4.1. *There exist $\epsilon_0 > 0$ and $\tilde{c}_0 > 0$ such that for all $0 < \epsilon \leq \epsilon_0$, $0 \leq c_0 \leq \tilde{c}_0$, $g_0 \in L^2 \cap \mathcal{N}^\perp$ with $\|g_0\|_{L^2} \leq \epsilon$, then,*

$$\|e^{\frac{c_0 t}{2} \mathcal{H}^s} \mathbf{S}_N g(t)\|_{L^2(\mathbb{R}^3)}^2 + \frac{1}{2} \int_0^t \|e^{\frac{c_0 \tau}{2} \mathcal{H}^s} \mathcal{L}^{\frac{1}{2}} \mathbf{S}_N g(\tau)\|_{L^2}^2 d\tau \leq \|g_0\|_{L^2(\mathbb{R}^3)}^2,$$

for any $t \geq 0$, $N \geq 0$.

Proof. We prove the Proposition by induction on N .

1). For $N \leq 2$. we have $\|e^{\frac{c_0 t}{2} \mathcal{H}^s} \mathbf{S}_0 g\|_{L^2}^2 = |g_{0,0,0}(t)|^2 = 0$,

$$\|e^{\frac{c_0 t}{2} \mathcal{H}^s} \mathbf{S}_1 g\|_{L^2}^2 = |g_{0,0,0}(t)|^2 + \sum_{|m| \leq 1} e^{c_0 t} |g_{0,1,m}(t)|^2 = 0,$$

and

$$\|e^{\frac{c_0 t}{2} \mathcal{H}^s} \mathbf{S}_2 g\|_{L^2}^2 = e^{c_0 t(2+\frac{3}{2})^s} |g_{1,0,0}(t)|^2 + \sum_{|m| \leq 2} e^{c_0 t(2+\frac{2}{3})^s} |g_{0,2,m}(t)|^2.$$

Recalling from (2.13) in Section 2.2 that for all $t > 0$

$$g_{0,2,m}(t) = e^{-\lambda_{0,2} t} g_{0,2,m}(0),$$

we choose $0 < \tilde{c}_0$ small such that $\tilde{c}_0(2 + 3/2)^s - 2\lambda_{0,2} \leq 0$. Therefore

$$\|e^{\frac{c_0 t}{2} \mathcal{H}^s} \mathbf{S}_2 g\|_{L^2}^2 \leq \sum_{|m| \leq 2} |g_{0,2,m}(0)|^2 \leq \|g_0\|_{L^2}^2 \leq \epsilon^2$$

for all $0 \leq c_0 \leq \tilde{c}_0$, $0 < \epsilon \leq \epsilon_0$.

2). For $N > 2$. We want to prove that

$$\|e^{\frac{c_0 t}{2} \mathcal{H}^s} \mathbf{S}_{N-1} g\|_{L^2} \leq \epsilon \leq \epsilon_0$$

implies

$$\|e^{\frac{c_0 t}{2} \mathcal{H}^s} \mathbf{S}_N g\|_{L^2} \leq \epsilon.$$

Take now $\epsilon_0 > 0$ such that

$$0 < \epsilon_0 \leq \frac{1}{4C}.$$

Then we deduce from (4.2) that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|e^{\frac{c_0 t}{2} \mathcal{H}^s} \mathbf{S}_N g(t)\|_{L^2}^2 + \frac{1}{2} \|e^{\frac{c_0 t}{2} \mathcal{H}^s} \mathcal{L}^{\frac{1}{2}} \mathbf{S}_N g\|_{L^2}^2 \\ \leq C \|e^{\frac{c_0 t}{2} \mathcal{H}^s} \mathbf{S}_{N-2} g\|_{L^2} \|e^{\frac{c_0 t}{2} \mathcal{H}^s} \mathcal{L}^{\frac{1}{2}} \mathbf{S}_N g\|_{L^2}^2 \\ \leq \frac{1}{4} \|e^{\frac{c_0 t}{2} \mathcal{H}^s} \mathcal{L}^{\frac{1}{2}} \mathbf{S}_N g\|_{L^2}^2. \end{aligned}$$

Therefore

$$(4.3) \quad \frac{d}{dt} \|e^{\frac{c_0 t}{2} \mathcal{H}^s} \mathbf{S}_N g(t)\|_{L^2}^2 + \frac{1}{2} \|e^{\frac{c_0 t}{2} \mathcal{H}^s} \mathcal{L}^{\frac{1}{2}} \mathbf{S}_N g\|_{L^2}^2 \leq 0,$$

and this ends the proof of the Proposition. $\square$

4.2. Existence of the weak solution. Now we prove the convergence of the sequence

$$g(t) = \sum_{n,l \geq 0} \sum_{|m| \leq l} g_{n,l,m}(t) \varphi_{n,l,m}$$

defined in (2.14).

Multiplying $\varphi_{n^*,l^*,m^*}(v)$ on both sides of (2.11) and taking the summation for $2n^* + l^* \leq N$, then for all $N \geq 2$, $\mathbf{S}_N g(t)$ satisfies the following Cauchy problem

$$(4.4) \quad \begin{cases} \partial_t \mathbf{S}_N g + \mathcal{L}(\mathbf{S}_N g) = \mathbf{S}_N \Gamma(\mathbf{S}_N g, \mathbf{S}_N g), \\ \mathbf{S}_N g(0) = \sum_{\substack{n \geq 0, l \geq 0 \\ 2n+l \leq N}} \sum_{|m| \leq l} g_{n,l,m}^0 \varphi_{n,l,m}. \end{cases}$$

By Proposition 4.1 with $c_0 = 0$, we have for all $t > 0$, $N \in \mathbb{N}$,

$$\|\mathbf{S}_N g(t)\|_{L^2(\mathbb{R}^3)}^2 + \frac{1}{2} \int_0^t \|\mathcal{L}^{\frac{1}{2}} \mathbf{S}_N g(\tau)\|_{L^2}^2 d\tau \leq \|g_0\|_{L^2(\mathbb{R}^3)}^2.$$

The orthogonality of the basis $(\varphi_{n,l,m})_{n,l \geq 0, |m| \leq l}$ implies that

$$\|\mathbf{S}_N g(t)\|_{L^2(\mathbb{R}^3)}^2 = \sum_{\substack{2n+l \leq N \\ n \geq 0, l \geq 0}} \sum_{|m| \leq l} |g_{n,l,m}(t)|^2.$$

By using the monotone convergence theorem, the sequence

$$g(t) = \sum_{n,l \geq 0} \sum_{|m| \leq l} g_{n,l,m}(t) \varphi_{n,l,m}$$

is convergent, for any $t \geq 0$,

$$\lim_{N \rightarrow \infty} \|\mathbf{S}_N g - g\|_{L^\infty([0,t]; L^2(\mathbb{R}^3))} = 0$$

and

$$\lim_{N \rightarrow \infty} \|\mathcal{L}^{\frac{1}{2}}(\mathbf{S}_N g - g)\|_{L^2([0,t]; L^2(\mathbb{R}^3))} = 0.$$

For any $\phi(t) \in C^1(\mathbb{R}_+, \mathcal{S}(\mathbb{R}^3))$, we have

$$\begin{aligned} & \left| \int_0^t (\mathbf{S}_N \Gamma(\mathbf{S}_N g, \mathbf{S}_N g) - \Gamma(g, g), \phi(\tau))_{L^2(\mathbb{R}^3)} d\tau \right| \\ &= \left| \int_0^t (\Gamma(\mathbf{S}_N g, \mathbf{S}_N g), \mathbf{S}_N \phi(\tau))_{L^2(\mathbb{R}^3)} - (\Gamma(g, g), \phi(\tau))_{L^2(\mathbb{R}^3)} d\tau \right| \\ &\leq \left| \int_0^t (\Gamma(\mathbf{S}_N g, \mathbf{S}_N g), \mathbf{S}_N \phi(\tau) - \phi(\tau))_{L^2} d\tau \right| \\ &\quad + \left| \int_0^t (\Gamma(\mathbf{S}_N g - g, \mathbf{S}_N g), \phi(\tau))_{L^2} d\tau \right| + \left| \int_0^t (\Gamma(g, \mathbf{S}_N g - g), \phi(\tau))_{L^2} d\tau \right|. \end{aligned}$$

By Proposition 3.2, one can verify that

$$\begin{aligned}
& \left| \int_0^t (\mathbf{S}_N \Gamma(\mathbf{S}_N g, \mathbf{S}_N g) - \Gamma(g, g), \phi(\tau))_{L^2(\mathbb{R}^3)} d\tau \right| \\
& \leq C \int_0^t \|\mathbf{S}_N g\|_{L^2} \|\mathcal{L}^{\frac{1}{2}} \mathbf{S}_N g\|_{L^2} \left\| \mathcal{L}^{\frac{1}{2}} (\mathbf{S}_N \phi - \phi) \right\|_{L^2(\mathbb{R}^3)} dt \\
& \quad + C \int_0^t \|\mathbf{S}_N g - g\|_{L^2} \|\mathcal{L}^{\frac{1}{2}} \mathbf{S}_N g\|_{L^2} \|\mathcal{L}^{\frac{1}{2}} \phi\|_{L^2(\mathbb{R}^3)} dt \\
& \quad + C \int_0^t \|g\|_{L^2} \|\mathcal{L}^{\frac{1}{2}} (\mathbf{S}_N g - g)\|_{L^2(\mathbb{R}^3)} \|\mathcal{L}^{\frac{1}{2}} \phi\|_{L^2(\mathbb{R}^3)} dt.
\end{aligned}$$

Using Proposition 4.1 with $c_0 = 0$

$$\begin{aligned}
& \left| \int_0^t (\mathbf{S}_N \Gamma(\mathbf{S}_N g, \mathbf{S}_N g) - \Gamma(g, g), \phi(\tau))_{L^2(\mathbb{R}^3)} d\tau \right| \\
& \leq C \|\mathbf{S}_N g\|_{L^\infty([0,t];L^2)} \|\mathcal{L}^{\frac{1}{2}} \mathbf{S}_N g\|_{L^2([0,t];L^2)} \left\| \mathcal{L}^{\frac{1}{2}} (\mathbf{S}_N \phi - \phi) \right\|_{L^2([0,t];L^2(\mathbb{R}^3))} \\
& \quad + C \|\mathbf{S}_N g - g\|_{L^\infty([0,t];L^2)} \|\mathcal{L}^{\frac{1}{2}} \mathbf{S}_N g\|_{L^2([0,t];L^2)} \|\mathcal{L}^{\frac{1}{2}} \phi\|_{L^2([0,t];L^2)} \\
& \quad + C \|g\|_{L^\infty([0,t];L^2)} \|\mathcal{L}^{\frac{1}{2}} (\mathbf{S}_N g - g)\|_{L^2([0,t];L^2)} \|\mathcal{L}^{\frac{1}{2}} \phi\|_{L^2([0,t];L^2)},
\end{aligned}$$

then

$$\begin{aligned}
& \left| \int_0^t (\mathbf{S}_N \Gamma(\mathbf{S}_N g, \mathbf{S}_N g) - \Gamma(g, g), \phi(\tau))_{L^2(\mathbb{R}^3)} d\tau \right| \\
& \leq \sqrt{2} C \|g_0\|_{L^2}^2 \left\| \mathcal{L}^{\frac{1}{2}} (\mathbf{S}_N \phi - \phi) \right\|_{L^2([0,t];L^2(\mathbb{R}^3))} \\
& \quad + \sqrt{2} C \|\mathbf{S}_N g - g\|_{L^\infty([0,t];L^2)} \|g_0\|_{L^2} \|\mathcal{L}^{\frac{1}{2}} \phi\|_{L^2([0,t];L^2)} \\
& \quad + C \|g_0\|_{L^2} \|\mathcal{L}^{\frac{1}{2}} (\mathbf{S}_N g - g)\|_{L^2([0,t];L^2)} \|\mathcal{L}^{\frac{1}{2}} \phi\|_{L^2([0,t];L^2)}.
\end{aligned}$$

Letting $N \rightarrow +\infty$ in (4.4), we conclude that, for any $\phi(t) \in C^1(\mathbb{R}_+, \mathcal{S}(\mathbb{R}^3))$,

$$\begin{aligned}
& (g(t), \phi(t))_{L^2(\mathbb{R}^3)} - (g(0), \phi(0))_{L^2(\mathbb{R}^3)} - \int_0^t (g(\tau), \partial_t \phi(\tau))_{L^2(\mathbb{R}^3)} d\tau \\
& = - \int_0^t (\mathcal{L}g(\tau), \phi(\tau))_{L^2(\mathbb{R}^3)} d\tau + \int_0^t (\Gamma(g(\tau), g(\tau)), \phi(\tau))_{L^2(\mathbb{R}^3)} d\tau,
\end{aligned}$$

which shows that $g \in L^\infty([0, +\infty[; L^2(\mathbb{R}^3))$ is a global weak solution of the Cauchy problem (1.3).

4.3. Regularity of the solution. For $\mathbf{S}_N g$ defined in (4.1), since

$$\lambda_{n,l} \geq \lambda_{2,0} > 0, \quad \forall n + l \geq 2,$$

we deduce from the formulas (4.3) that

$$\begin{aligned}
& \frac{d}{dt} \|e^{\frac{\lambda_{2,0}t}{2}} \mathbf{S}_N g(t)\|_{L^2}^2 + \frac{\lambda_{2,0}}{2} \|e^{\frac{\lambda_{2,0}t}{2}} \mathbf{S}_N g\|_{L^2}^2 \\
& \leq \frac{d}{dt} \|e^{\frac{\lambda_{2,0}t}{2}} \mathbf{S}_N g(t)\|_{L^2}^2 + \frac{1}{2} \|e^{\frac{\lambda_{2,0}t}{2}} \mathcal{L}^{\frac{1}{2}} \mathbf{S}_N g\|_{L^2}^2 \leq 0.
\end{aligned}$$

We then have

$$\frac{d}{dt} \left(e^{\frac{\lambda_{2,0}t}{2}} \|e^{\frac{\lambda_{2,0}t}{2}} \mathbf{S}_N g(t)\|_{L^2}^2 \right) \leq 0,$$

and we derive that for any $t > 0$, and $N \in \mathbb{N}$,

$$\|e^{\frac{c_0 t}{2} \mathcal{H}^s} \mathbf{S}_N g(t)\|_{L^2(\mathbb{R}^3)} \leq e^{-\frac{\lambda_2 0 t}{4}} \|g_0\|_{L^2(\mathbb{R}^3)}.$$

The orthogonality of the basis $(\varphi_{n,l,m})_{n,l \geq 0, |m| \leq l}$ implies that

$$\|e^{\frac{c_0 t}{2} \mathcal{H}^s} \mathbf{S}_N g(t)\|_{L^2(\mathbb{R}^3)}^2 = \sum_{\substack{2n+l \leq N \\ n \geq 0, l \geq 0}} \sum_{|m| \leq l} e^{c_0 t(2n+l+\frac{3}{2})^s} |g_{n,l,m}(t)|^2.$$

Using the monotone convergence theorem, we conclude that

$$\|e^{\frac{c_0 t}{2} \mathcal{H}^s} g(t)\|_{L^2(\mathbb{R}^3)} \leq e^{-\frac{\lambda_2 0 t}{4}} \|g_0\|_{L^2(\mathbb{R}^3)}.$$

This ends the proof of Theorem 1.1.

5. THE SPECTRAL REPRESENTATION

This section is devoted to the proof of Proposition 2.1, Proposition 2.2 and some Propositions used in section 6.

5.1. Harmonic identities. We prepare some technical computation. In all this section, $n, l, \tilde{n}, \tilde{l}$ will be fixed integers of $\mathbb{N}$, $m \in \mathbb{Z}$, $|m| \leq l$ and we will use the following notation in this section :

$$a_{l,m} = \frac{(l - |m|)!}{(l + |m|)!}.$$

For any unit vector

$$\sigma = (\sigma_1, \sigma_2, \sigma_3) = (\cos \theta, \sin \theta \cos \phi, \sin \theta \sin \phi) \in \mathbb{S}^2$$

with $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$, the orthonormal basis of spherical harmonics $Y_l^m(\sigma)$ ($|m| \leq l$) is (see the definition (2.2) of $P_l^{[k]}$)

$$\begin{aligned} Y_l^m(\sigma) &= \sqrt{\frac{2l+1}{4\pi} \frac{(l-|m|)!}{(l+|m|)!}} P_l^{|m|}(\cos \theta) e^{im\phi}, \\ (5.1) \quad &= \sqrt{\frac{2l+1}{4\pi} \frac{(l-|m|)!}{(l+|m|)!}} \left(\frac{d^{|m|} P_l}{dx^{|m|}} \right) (\sigma_1) (\sigma_2 + i \operatorname{sgn}(m) \sigma_3)^{|m|}. \end{aligned}$$

We recall the following properties (see (7 – 34) of Chapter 7 in the book [19], (VIII) of Sec.19, Chap.III in the book [20] and Theorem 1 of Sec.4, Chap 1 in the book [17]):

- Addition theorem: For any integer $l \geq 0$ and α_1, α_2 in $\mathbb{S}^2$,

$$(5.2) \quad P_l(\alpha_1 \cdot \alpha_2) = \frac{4\pi}{2l+1} \sum_{m=-l}^l Y_l^m(\alpha_1) Y_l^{-m}(\alpha_2).$$

If we set the following coordinates

$$\alpha_j = (\cos \theta_j, \sin \theta_j \cos \phi_j, \sin \theta_j \sin \phi_j)$$

for $j = 1, 2$, the previous addition theorem reads as follows

$$\begin{aligned} &P_l(\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 \cos(\phi_1 - \phi_2)) \\ (5.3) \quad &= \sum_{m=-l}^l \frac{(l-|m|)!}{(l+|m|)!} P_l^{|m|}(\cos \theta_1) P_l^{|m|}(\cos \theta_2) e^{im(\phi_1 - \phi_2)}. \end{aligned}$$

- Integral form of the addition theorem: For any integer $l \geq 0$ and $m, |m| \leq l$, any $\sigma \in \mathbb{S}^2$,

$$(5.4) \quad Y_l^m(\sigma) = \frac{2l+1}{4\pi} \int_{\mathbb{S}_\eta^2} P_l(\sigma \cdot \eta) Y_l^m(\eta) d\eta.$$

- Funk-Hecke Formula: For any continuous function $f \in C([-1, 1])$, any $\sigma \in \mathbb{S}^2$ and integers $l \geq 0, |m| \leq l$,

$$(5.5) \quad \int_{\mathbb{S}_\eta^2} f(\sigma \cdot \eta) Y_l^m(\eta) d\eta = \left(2\pi \int_{-1}^1 f(x) P_l(x) dx \right) Y_l^m(\sigma).$$

For $\kappa \in \mathbb{S}^2$ fixed, we can find $\theta_0 \in [0, \pi], \phi_0 \in [0, 2\pi]$ such that

$$\kappa = (\cos \theta_0, \sin \theta_0 \cos \phi_0, \sin \theta_0 \sin \phi_0).$$

Construct the orthogonal vectors with respect to κ

$$(5.6) \quad \kappa^1 = (-\sin \theta_0, \cos \theta_0 \cos \phi_0, \cos \theta_0 \sin \phi_0), \quad \kappa^2 = (0, \sin \phi_0, -\cos \phi_0),$$

and for $\phi \in \mathbb{R}$

$$(5.7) \quad \kappa^\perp(\phi) = \kappa^1 \cos \phi + \kappa^2 \sin \phi.$$

Then $\kappa, \kappa^1, \kappa^2$ constitute an orthonormal frame in $\mathbb{R}^3$. For any unit vector σ , we can find $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$ such that

$$(5.8) \quad \sigma = \kappa \cos \theta + \kappa^1 \sin \theta \cos \phi + \kappa^2 \sin \theta \sin \phi.$$

It is easy to verify

$$(5.9) \quad \frac{\kappa + \sigma}{|\kappa + \sigma|} = \kappa \cos \frac{\theta}{2} + \sin \frac{\theta}{2} (\kappa^1 \cos \phi + \kappa^2 \sin \phi),$$

$$(5.10) \quad \frac{\kappa - \sigma}{|\kappa - \sigma|} = \kappa \sin \frac{\theta}{2} - \cos \frac{\theta}{2} (\kappa^1 \cos \phi + \kappa^2 \sin \phi).$$

In the proof of Proposition 2.1, we need the following lemma.

Lemma 5.1. For any function f in $C([-1, 1])$ any $\kappa \in \mathbb{S}^2, l \in \mathbb{N}$ and $|m| \leq l$, we have

$$(i) \quad \int_{\mathbb{S}^2} f(\kappa \cdot \sigma) Y_l^m \left(\frac{\kappa + \sigma}{|\kappa + \sigma|} \right) d\sigma = \left(2\pi \int_0^\pi f(\cos \theta) \sin \theta P_l \left(\cos \frac{\theta}{2} \right) d\theta \right) Y_l^m(\kappa),$$

$$(ii) \quad \int_{\mathbb{S}^2} f(\kappa \cdot \sigma) Y_l^m \left(\frac{\kappa - \sigma}{|\kappa - \sigma|} \right) d\sigma = \left(2\pi \int_0^\pi f(\cos \theta) \sin \theta P_l \left(\sin \frac{\theta}{2} \right) d\theta \right) Y_l^m(\kappa).$$

Proof. For $\kappa \in \mathbb{S}^2$ fixed, we can find $\theta_0 \in [0, \pi], \phi_0 \in [0, 2\pi]$ such that

$$\kappa = (\cos \theta_0, \sin \theta_0 \cos \phi_0, \sin \theta_0 \sin \phi_0).$$

In the orthonormal frame $(\kappa, \kappa^1, \kappa^2)$ constructed in (5.6), for any $\sigma \in \mathbb{S}^2$ we have

$$\sigma = \kappa \cos \theta + \kappa^1 \sin \theta \cos \phi + \kappa^2 \sin \theta \sin \phi$$

with $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$. Therefore, $\kappa \cdot \sigma = \cos \theta$, and for any $\eta \in \mathbb{S}^2$ with

$$\eta = \kappa \cos \theta_1 + \kappa^1 \sin \theta_1 \cos \phi_1 + \kappa^2 \sin \theta_1 \sin \phi_1.$$

we deduce from (5.9)-(5.10)

$$(5.11) \quad \frac{\kappa - \sigma}{|\kappa - \sigma|} \cdot \eta = \sin \frac{\theta}{2} \cos \theta_1 - \cos \frac{\theta}{2} \sin \theta_1 \cos(\phi - \phi_1),$$

$$(5.12) \quad \frac{\kappa + \sigma}{|\kappa + \sigma|} \cdot \eta = \cos \frac{\theta}{2} \cos \theta_1 + \sin \frac{\theta}{2} \sin \theta_1 \cos(\phi - \phi_1).$$

Proof of (i). Applying the formula (5.4) for $Y_l^m\left(\frac{\kappa+\sigma}{|\kappa+\sigma|}\right)$, we have

$$\begin{aligned} \int_{\mathbb{S}_\sigma^2} f(\kappa \cdot \sigma) Y_l^m\left(\frac{\kappa+\sigma}{|\kappa+\sigma|}\right) d\sigma \\ = \frac{2l+1}{4\pi} \int_{\mathbb{S}_\sigma^2} f(\kappa \cdot \sigma) \int_{\mathbb{S}_\eta^2} P_l\left(\frac{\kappa+\sigma}{|\kappa+\sigma|} \cdot \eta\right) Y_l^m(\eta) d\eta d\sigma \\ = \frac{2l+1}{4\pi} \int_{\mathbb{S}_\eta^2} Y_l^m(\eta) A(\eta) d\eta \end{aligned}$$

where

$$A(\eta) = \int_{\mathbb{S}^2} f(\kappa \cdot \sigma) P_l\left(\frac{\kappa+\sigma}{|\kappa+\sigma|} \cdot \eta\right) d\sigma.$$

Then, applying the addition theorem (5.3) and (5.12)

$$\begin{aligned} P_l\left(\frac{\kappa+\sigma}{|\kappa+\sigma|} \cdot \eta\right) &= P_l(\cos \frac{\theta}{2} \cos \theta_1 + \sin \theta_1 \sin \frac{\theta}{2} \cos(\phi - \phi_1)) \\ &= \sum_{m=-l}^l \frac{(l-|m|)!}{(l+|m|)!} P_l^{|m|}(\cos \frac{\theta}{2}) P_l^{|m|}(\cos \theta_1) e^{im(\phi-\phi_1)}, \end{aligned}$$

direct calculation shows that

$$A(\eta) = \left(2\pi \int_0^\pi f(\cos \theta) \sin \theta P_l\left(\cos \frac{\theta}{2}\right) d\theta\right) P_l(\kappa \cdot \eta).$$

Henceforth, we get that

$$\begin{aligned} \int_{\mathbb{S}_\sigma^2} f(\kappa \cdot \sigma) Y_l^m\left(\frac{\kappa+\sigma}{|\kappa+\sigma|}\right) d\sigma \\ = \left(2\pi \int_0^\pi f(\cos \theta) \sin \theta P_l\left(\cos \frac{\theta}{2}\right) d\theta\right) \frac{2l+1}{4\pi} \int_{\mathbb{S}_\eta^2} Y_l^m(\eta) P_l(\kappa \cdot \eta) d\eta \end{aligned}$$

and we conclude by formula (5.4).

The proof of (ii) is similar by using (5.11). □

As a direct consequence of part (i) of the previous lemma, we have :

Corollary 5.2. For $\tilde{l}, \tilde{m} \in \mathbb{N}$ and $|\tilde{m}| \leq \tilde{l}$, we have for the cross section $b(\cos \theta)$ satisfying (1.2) with $\cos \theta \geq 0$,

$$\begin{aligned} (5.13) \quad \int_{\mathbb{S}^2} b(\kappa \cdot \sigma) \left(Y_{\tilde{l}}^{\tilde{m}}\left(\frac{\kappa+\sigma}{|\kappa+\sigma|}\right) \left(\frac{1+\kappa \cdot \sigma}{2}\right)^{\frac{2\tilde{n}+\tilde{l}}{2}} - Y_{\tilde{l}}^{\tilde{m}}(\kappa) \right) d\sigma \\ = \left[\int_{|\theta| \leq \frac{\pi}{4}} \beta(\theta) \left((\cos \theta)^{2\tilde{n}+\tilde{l}} P_{\tilde{l}}(\cos \theta) - 1 \right) d\theta \right] Y_{\tilde{l}}^{\tilde{m}}(\kappa). \end{aligned}$$

Lemma 5.3. Let $\kappa \in \mathbb{S}^2$ and the cross section $b(\cos \theta)$ satisfies (1.2) with $\cos \theta \geq 0$. Assume also that $n, l, \tilde{n}, \tilde{l} \in \mathbb{N}$ with $l \geq 1, \tilde{l} \geq 1, |m| \leq l, |\tilde{m}| \leq \tilde{l}$. Then there exist some constants $c_{n,l,m,\tilde{n},\tilde{l},\tilde{m}}^k$ such that

$$\begin{aligned} (5.14) \quad \int_{\mathbb{S}^2} b(\kappa \cdot \sigma) Y_l^m\left(\frac{\kappa-\sigma}{|\kappa-\sigma|}\right) Y_{\tilde{l}}^{\tilde{m}}\left(\frac{\kappa+\sigma}{|\kappa+\sigma|}\right) \left(\frac{1-\kappa \cdot \sigma}{2}\right)^{\frac{2n+l}{2}} \left(\frac{1+\kappa \cdot \sigma}{2}\right)^{\frac{2\tilde{n}+\tilde{l}}{2}} d\sigma \\ = \sum_{k=0}^{k_0(l,\tilde{l},m,\tilde{m})} c_{n,l,m,\tilde{n},\tilde{l},\tilde{m}}^k Y_{l+\tilde{l}-2k}^{m+\tilde{m}}(\kappa). \end{aligned}$$

Proof. Without loss of generality, we can assume that $\min(l, \tilde{l}) = \tilde{l}$ (the integral in (5.14) is invariant if σ is changed to $-\sigma$ and the integers (l, m) and $(\tilde{l}, \tilde{m})$ are exchanged).

- *Step 1.* We first claim that there exist some constants $C_{\tilde{l}, l_1, l_2}$ such that

$$(5.15) \quad Y_{\tilde{l}}^{\tilde{m}} \left(\frac{\kappa + \sigma}{|\kappa + \sigma|} \right) = \sum_{l_1 + l_2 = \tilde{l}} C_{\tilde{l}, l_1, l_2} \frac{\left(\frac{1 - \kappa \cdot \sigma}{2} \right)^{\frac{l_2}{2}}}{\left(\frac{1 + \kappa \cdot \sigma}{2} \right)^{\frac{l_1}{2}}} \\ \times \sum_{m_1 = -l_1}^{l_1} \sum_{m_2 = -l_2}^{l_2} \left(\int_{\mathbb{S}_{\eta}^2} Y_{\tilde{l}}^{\tilde{m}}(\eta) Y_{l_1}^{-m_1}(\eta) Y_{l_2}^{-m_2}(\eta) d\eta \right) Y_{l_1}^{m_1}(\kappa) Y_{l_2}^{m_2} \left(\frac{\kappa - \sigma}{|\kappa - \sigma|} \right)$$

where, in all the sequel of the proof, l_1 and l_2 in the summation will be non-negative integers.

Proof of (5.15) : From the integral addition theorem (5.4) we have

$$Y_{\tilde{l}}^{\tilde{m}} \left(\frac{\kappa + \sigma}{|\kappa + \sigma|} \right) = \frac{2\tilde{l} + 1}{4\pi} \int_{\mathbb{S}_{\eta}^2} P_{\tilde{l}} \left(\frac{\kappa + \sigma}{|\kappa + \sigma|} \cdot \eta \right) Y_{\tilde{l}}^{\tilde{m}}(\eta) d\eta.$$

From the formula (see (43) in Chapter III in [20]), there is some real constants $c_{\tilde{l}, q}$ such that

$$(5.16) \quad x^{\tilde{l}} = \sum_{0 \leq q \leq \tilde{l}/2} c_{\tilde{l}, q} P_{\tilde{l}-2q}(x)$$

where

$$c_{\tilde{l}, 0} = \frac{\tilde{l}!}{1 \times 3 \times 5 \times \dots \times (2\tilde{l} - 1)} \neq 0.$$

Writing $P_{\tilde{l}}(x) = (1/c_{\tilde{l}, 0})x^{\tilde{l}} - (1/c_{\tilde{l}, 0}) \sum_{1 \leq q \leq \tilde{l}/2} c_{\tilde{l}, q} P_{\tilde{l}-2q}(x)$ in the previous integral, we obtain

$$(5.17) \quad Y_{\tilde{l}}^{\tilde{m}} \left(\frac{\kappa + \sigma}{|\kappa + \sigma|} \right) = \frac{2\tilde{l} + 1}{4\pi c_{\tilde{l}, 0}} \int_{\mathbb{S}_{\eta}^2} \left(\frac{\kappa + \sigma}{|\kappa + \sigma|} \cdot \eta \right)^{\tilde{l}} Y_{\tilde{l}}^{\tilde{m}}(\eta) d\eta + R_1$$

where

$$R_1 = -\frac{2\tilde{l} + 1}{4\pi c_{\tilde{l}, 0}} \sum_{1 \leq q < \tilde{l}/2} c_{\tilde{l}, q} \int_{\mathbb{S}_{\eta}^2} P_{\tilde{l}-2q} \left(\frac{\kappa + \sigma}{|\kappa + \sigma|} \cdot \eta \right) Y_{\tilde{l}}^{\tilde{m}}(\eta) d\eta.$$

We observe that $R_1 = 0$. Indeed, for any $q \neq \tilde{l}$ and $\gamma \in \mathbb{S}^2$, we have from the Funck-Hecke Formula (5.5) and the orthogonality of the polynomials $(P_l)_l$,

$$\int_{\mathbb{S}_{\eta}^2} P_q(\gamma \cdot \eta) Y_{\tilde{l}}^{\tilde{m}}(\eta) d\eta = \left(2\pi \int_{-1}^1 P_q(x) P_{\tilde{l}}(x) dx \right) Y_{\tilde{l}}^{\tilde{m}}(\gamma) = 0.$$

Set $\gamma = \frac{\kappa - \sigma}{|\kappa - \sigma|}$. Therefore we have

$$\kappa + \sigma = 2\kappa - 2(\kappa \cdot \gamma)\gamma, \quad |\kappa + \sigma| = 2\sqrt{1 - (\kappa \cdot \gamma)^2}$$

and

$$\left(\frac{\kappa + \sigma}{|\kappa + \sigma|} \cdot \eta \right)^{\tilde{l}} = \frac{((\kappa \cdot \eta) - (\kappa \cdot \gamma)(\gamma \cdot \eta))^{\tilde{l}}}{(\sqrt{1 - (\kappa \cdot \gamma)^2})^{\tilde{l}}}.$$

Expanding this identity thanks to the binomial formula and plugging it in (5.17), we get

$$(5.18) \quad Y_{\tilde{l}}^{\tilde{m}} \left(\frac{\kappa + \sigma}{|\kappa + \sigma|} \right) = \frac{2\tilde{l} + 1}{4\pi c_{\tilde{l}, 0}} \sum_{l_1 + l_2 = \tilde{l}} \frac{\tilde{l}!}{l_1! l_2!} \frac{(-\kappa \cdot \gamma)^{l_2}}{(\sqrt{1 - (\kappa \cdot \gamma)^2})^{\tilde{l}}} \int_{\mathbb{S}_{\eta}^2} (\kappa \cdot \eta)^{l_1} (\gamma \cdot \eta)^{l_2} Y_{\tilde{l}}^{\tilde{m}}(\eta) d\eta.$$

Using the expansion (5.16) for x^{l_1} and x^{l_2} , we express the previous integrals :

$$\int_{\mathbb{S}_\eta^2} (\kappa \cdot \eta)^{l_1} (\gamma \cdot \eta)^{l_2} Y_{\tilde{l}}^{\tilde{m}}(\eta) d\eta = \sum_{\substack{0 \leq q_1 \leq l_1/2 \\ 0 \leq q_2 \leq l_2/2}} c_{l_1, q_1} c_{l_2, q_2} \int_{\mathbb{S}_\eta^2} P_{l_1-2q_1}(\kappa \cdot \eta) P_{l_2-2q_2}(\gamma \cdot \eta) Y_{\tilde{l}}^{\tilde{m}}(\eta) d\eta.$$

From the addition Theorem (5.2) applied to both $P_{l_1-2q_1}(\kappa \cdot \eta)$ and $P_{l_2-2q_2}(\gamma \cdot \eta)$, we derive

$$\begin{aligned} \int_{\mathbb{S}_\eta^2} P_{l_1-2q_1}(\kappa \cdot \eta) P_{l_2-2q_2}(\gamma \cdot \eta) Y_{\tilde{l}}^{\tilde{m}}(\eta) d\eta &= \frac{4\pi}{2(l_1-2q_1)+1} \frac{4\pi}{2(l_2-2q_2)+1} \\ &\times \sum_{m_1=-(l_1-2q_1)}^{l_1-2q_1} \sum_{m_2=-(l_2-2q_2)}^{l_2-2q_2} \left(\int_{\mathbb{S}_\eta^2} Y_{l_1-2q_1}^{-m_1}(\eta) Y_{l_2-2q_2}^{-m_2}(\eta) Y_{\tilde{l}}^{\tilde{m}}(\eta) d\eta \right) Y_{l_1-2q_1}^{m_1}(\kappa) Y_{l_2-2q_2}^{m_2}(\gamma). \end{aligned}$$

From (7.6), the integral $(\int_{\mathbb{S}^2} Y_{l_1-2q_1}^{-m_1} Y_{l_2-2q_2}^{-m_2} Y_{\tilde{l}}^{\tilde{m}})$ is equal to 0 when the parameters do not satisfy

$$\tilde{m} = m_1 + m_2 \text{ and } \tilde{l} = (l_1 - 2q_1) + (l_2 - 2q_2) - 2j \text{ with } 0 \leq j \leq \min(l_1 - 2q_1, l_2 - 2q_2),$$

consequently for $(q_1, q_2) \neq (0, 0)$ (recall that $l_1 + l_2 = \tilde{l}$). Therefore we obtain

$$\begin{aligned} \int_{\mathbb{S}_\eta^2} (\kappa \cdot \eta)^{l_1} (\gamma \cdot \eta)^{l_2} Y_{\tilde{l}}^{\tilde{m}}(\eta) d\eta &= \frac{4\pi c_{l_1,0}}{2l_1+1} \frac{4\pi c_{l_2,0}}{2l_2+1} \\ &\times \sum_{m_1=-l_1}^{l_1} \sum_{m_2=-l_2}^{l_2} \left(\int_{\mathbb{S}_\eta^2} Y_{l_1}^{-m_1}(\eta) Y_{l_2}^{-m_2}(\eta) Y_{\tilde{l}}^{\tilde{m}}(\eta) d\eta \right) Y_{l_1}^{m_1}(\kappa) Y_{l_2}^{m_2}(\gamma). \end{aligned}$$

Plugging the previous identity into (5.18), we have for some constants $C_{\tilde{l}, l_1, l_2}$

$$\begin{aligned} Y_{\tilde{l}}^{\tilde{m}} \left(\frac{\kappa + \sigma}{|\kappa + \sigma|} \right) &= \sum_{l_1+l_2=\tilde{l}} C_{\tilde{l}, l_1, l_2} \frac{(-\kappa \cdot \gamma)^{l_2}}{(\sqrt{1 - (\kappa \cdot \gamma)^2})^{\tilde{l}}} \\ &\times \sum_{m_1=-l_1}^{l_1} \sum_{m_2=-l_2}^{l_2} \left(\int_{\mathbb{S}_\eta^2} Y_{\tilde{l}}^{\tilde{m}}(\eta) Y_{l_1}^{-m_1}(\eta) Y_{l_2}^{-m_2}(\eta) d\eta \right) Y_{l_1}^{m_1}(\kappa) Y_{l_2}^{m_2}(\gamma). \end{aligned}$$

Remembering that $\gamma = \frac{\kappa - \sigma}{|\kappa - \sigma|}$ and checking that $\kappa \cdot \gamma = (\frac{1 - \kappa \cdot \sigma}{2})^{\frac{1}{2}}$, $\sqrt{1 - (\kappa \cdot \gamma)^2} = (\frac{1 + \kappa \cdot \sigma}{2})^{\frac{1}{2}}$, we conclude the proof of (5.15).

- *Step 2.* We now prove (5.14). We note

$$\mathbf{N} = \int_{\mathbb{S}^2} b(\kappa \cdot \sigma) Y_l^m \left(\frac{\kappa - \sigma}{|\kappa - \sigma|} \right) Y_{\tilde{l}}^{\tilde{m}} \left(\frac{\kappa + \sigma}{|\kappa + \sigma|} \right) \left(\frac{1 - \kappa \cdot \sigma}{2} \right)^{\frac{2n+l}{2}} \left(\frac{1 + \kappa \cdot \sigma}{2} \right)^{\frac{2\tilde{n}+\tilde{l}}{2}} d\sigma.$$

From step 1, we derive

$$\begin{aligned} \mathbf{N} &= \sum_{l_1+l_2=\tilde{l}} C_{\tilde{l}, l_1, l_2} \sum_{m_1=-l_1}^{l_1} \sum_{m_2=-l_2}^{l_2} \left(\int_{\mathbb{S}^2} Y_{\tilde{l}}^{\tilde{m}} Y_{l_1}^{-m_1} Y_{l_2}^{-m_2} \right) \\ &\times \left(\int_{\mathbb{S}^2} b(\kappa \cdot \sigma) \left(\frac{1 - \kappa \cdot \sigma}{2} \right)^{\frac{2n+l+l_2}{2}} \left(\frac{1 + \kappa \cdot \sigma}{2} \right)^{\frac{2\tilde{n}}{2}} Y_l^m \left(\frac{\kappa - \sigma}{|\kappa - \sigma|} \right) Y_{l_2}^{m_2} \left(\frac{\kappa - \sigma}{|\kappa - \sigma|} \right) d\sigma \right) Y_{l_1}^{m_1}(\kappa). \end{aligned}$$

Moreover we have from (7.5) and (7.6)

$$Y_l^m \left(\frac{\kappa - \sigma}{|\kappa - \sigma|} \right) Y_{l_2}^{m_2} \left(\frac{\kappa - \sigma}{|\kappa - \sigma|} \right) = \sum_{l'} \sum_{m'=-l'}^{l'} \left(\int_{\mathbb{S}^2} Y_l^m Y_{l_2}^{m_2} \overline{Y_{l'}^{m'}} \right) Y_{l'}^{m'} \left(\frac{\kappa - \sigma}{|\kappa - \sigma|} \right)$$

where l' is defined by $l' = l + l_2 - 2j_1$ with $0 \leq j_1 \leq \min(l, l_2)$. Therefore

$$\begin{aligned} \mathbf{N} &= \sum_{l_1+l_2=\tilde{l}} C_{\tilde{l}, l_1, l_2} \sum_{m_1=-l_1}^{l_1} \sum_{m_2=-l_2}^{l_2} \left(\int_{\mathbb{S}^2} Y_{\tilde{l}}^{\tilde{m}} Y_{l_1}^{-m_1} Y_{l_2}^{-m_2} \right) \sum_{l'} \sum_{m'=-l'}^{l'} \left(\int_{\mathbb{S}^2} Y_l^m Y_{l_2}^{m_2} \overline{Y_{l'}^{m'}} \right) \\ &\quad \times \left(\int_{\mathbb{S}^2} b(\kappa \cdot \sigma) \left(\frac{1 - \kappa \cdot \sigma}{2} \right)^{\frac{2n+l+l_2}{2}} \left(\frac{1 + \kappa \cdot \sigma}{2} \right)^{\frac{2\tilde{n}}{2}} Y_{l'}^{m'} \left(\frac{\kappa - \sigma}{|\kappa - \sigma|} \right) d\sigma \right) Y_{l_1}^{m_1}(\kappa). \end{aligned}$$

We apply the part (ii) of lemma 5.1 with $f(x) = b(x) \left(\frac{1-x}{2} \right)^{\frac{2n+l+l_2}{2}} \left(\frac{1+x}{2} \right)^{\frac{2\tilde{n}}{2}}$, the assumption $\cos \theta \geq 0$ and the notation $\beta(\theta) = 2\pi b(\cos 2\theta) \sin 2\theta$,

$$\begin{aligned} &\int_{\mathbb{S}^2} b(\kappa \cdot \sigma) \left(\frac{1 - \kappa \cdot \sigma}{2} \right)^{\frac{2n+l+l_2}{2}} \left(\frac{1 + \kappa \cdot \sigma}{2} \right)^{\frac{2\tilde{n}}{2}} Y_{l'}^{m'} \left(\frac{\kappa - \sigma}{|\kappa - \sigma|} \right) d\sigma \\ &= \left(2\pi \int_0^{\frac{\pi}{2}} b(\cos \theta) (\sin(\theta/2))^{2n+l+l_2} (\cos(\theta/2))^{2\tilde{n}} P_{l'}(\sin(\theta/2)) \sin \theta d\theta \right) Y_{l'}^{m'}(\kappa) \\ &= \left(4\pi \int_0^{\frac{\pi}{4}} b(\cos 2\theta) \sin 2\theta (\sin \theta)^{2n+l+l_2} (\cos \theta)^{2\tilde{n}} P_{l'}(\sin \theta) d\theta \right) Y_{l'}^{m'}(\kappa) \\ &= \left(2 \int_0^{\frac{\pi}{4}} \beta(\theta) (\sin \theta)^{2n+l+l_2} (\cos \theta)^{2\tilde{n}} P_{l'}(\sin \theta) d\theta \right) Y_{l'}^{m'}(\kappa). \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbf{N} &= \sum_{l_1+l_2=\tilde{l}} C_{\tilde{l}, l_1, l_2} \sum_{m_1=-l_1}^{l_1} \sum_{m_2=-l_2}^{l_2} \left(\int_{\mathbb{S}^2} Y_{\tilde{l}}^{\tilde{m}} Y_{l_1}^{-m_1} Y_{l_2}^{-m_2} \right) \sum_{l'} \sum_{m'=-l'}^{l'} \left(\int_{\mathbb{S}^2} Y_l^m Y_{l_2}^{m_2} \overline{Y_{l'}^{m'}} \right) \\ &\quad \times \left(\left(2 \int_0^{\frac{\pi}{4}} \beta(\theta) (\sin \theta)^{2n+l+l_2} (\cos \theta)^{2\tilde{n}} P_{l'}(\sin \theta) d\theta \right) Y_{l'}^{m'}(\kappa) \right) Y_{l_1}^{m_1}(\kappa). \end{aligned}$$

We again derive from (7.5) and (7.6)

$$Y_{l'}^{m'}(\kappa) Y_{l_1}^{m_1}(\kappa) = \sum_{l''} \sum_{m''=-l''}^{l''} \left(\int_{\mathbb{S}^2} Y_{l'}^{m'} Y_{l_1}^{m_1} \overline{Y_{l''}^{m''}} \right) Y_{l''}^{m''}(\kappa)$$

where l'' is defined by $l'' = l' + l_1 - 2j_2$ with $0 \leq j_2 \leq \min(l', l_1)$. We conclude to

$$\begin{aligned} \mathbf{N} &= \sum_{l_1+l_2=\tilde{l}} C_{\tilde{l}, l_1, l_2} \sum_{m_1=-l_1}^{l_1} \sum_{m_2=-l_2}^{l_2} \left(\int_{\mathbb{S}^2} Y_{\tilde{l}}^{\tilde{m}} Y_{l_1}^{-m_1} Y_{l_2}^{-m_2} \right) \sum_{l'} \sum_{m'=-l'}^{l'} \left(\int_{\mathbb{S}^2} Y_l^m Y_{l_2}^{m_2} \overline{Y_{l'}^{m'}} \right) \\ &\quad \times \left(2 \int_0^{\frac{\pi}{4}} \beta(\theta) (\sin \theta)^{2n+l+l_2} (\cos \theta)^{2\tilde{n}} P_{l'}(\sin \theta) d\theta \right) \sum_{l''} \sum_{m''=-l''}^{l''} \left(\int_{\mathbb{S}^2} Y_{l'}^{m'} Y_{l_1}^{m_1} \overline{Y_{l''}^{m''}} \right) Y_{l''}^{m''}(\kappa) \end{aligned}$$

which is nonzero from (7.6) when

$$m'' = m' + m_1 = m + m_2 + m_1 = m + \tilde{m}.$$

From the previous expressions of l'' and l' ,

$$l'' = l + l_1 + l_2 - 2(j_1 + j_2) = \tilde{l} + l - 2(j_1 + j_2)$$

and $0 \leq j_1 + j_2 \leq \min(l, l_2) + \min(l', l_1)$. From $l_1 + l_2 = \tilde{l}$ and the assumption $\tilde{l} \leq l$, we get

$$\min(l, l_2) + \min(l', l_1) = l_2 + \min(l', l_1) = \min(l_2 + l', \tilde{l}) \leq \tilde{l}.$$

We derive $0 \leq j_1 + j_2 \leq \tilde{l} = \min(l, \tilde{l})$. For $l, \tilde{l}, m, \tilde{m}$ fixed, we can define $l'' = l + \tilde{l} - 2k$ with $0 \leq k \leq \min(l, \tilde{l})$. Then the coefficient of $Y_{l''}^{m''}(\kappa)$ is nonzero when

$$|m + \tilde{m}| \leq l + \tilde{l} - 2k.$$

Therefore,

$$k \leq \frac{l + \tilde{l} - |m + \tilde{m}|}{2}.$$

In conclusion,

$$0 \leq k \leq k_0(l, \tilde{l}, m, \tilde{m})$$

where $k_0(l, \tilde{l}, m, \tilde{m})$ was given in (2.3). This ends the proof of (5.14). $\square$

5.2. The proof of Proposition 2.1. The spectral representation will be based on the Bobylev formula, which is the Fourier transform of the Boltzmann operator in the Maxwellian molecules case:

$$\mathcal{F}(Q(g, f))(\xi) = \int_{\mathbb{S}^2} b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) [\hat{g}(\xi^-) \hat{f}(\xi^+) - \hat{g}(0) \hat{f}(\xi)] d\sigma,$$

where

$$\xi^- = \frac{\xi - |\xi|\sigma}{2} = \frac{|\xi|}{2}(\kappa - \sigma), \quad \xi^+ = \frac{\xi + |\xi|\sigma}{2} = \frac{|\xi|}{2}(\kappa + \sigma)$$

with $\kappa = \frac{\xi}{|\xi|}$. Remark that

$$\kappa \cdot \sigma = \cos \theta, \quad |\xi^-| = |\xi| |\sin(\theta/2)|, \quad |\xi^+| = |\xi| \cos(\theta/2).$$

Let $\varphi_{n,l,m}$ be the functions defined in (2.1), then for $n, l \in \mathbb{N}$, $|m| \leq l$, we have (see Lemma 7.2)

$$(5.19) \quad \widehat{\sqrt{\mu}\varphi_{n,l,m}}(\xi) = A_{n,l} \left(\frac{|\xi|}{\sqrt{2}}\right)^{2n+l} e^{-\frac{|\xi|^2}{2}} Y_l^m\left(\frac{\xi}{|\xi|}\right),$$

where

$$A_{n,l} = (-i)^l (2\pi)^{\frac{3}{4}} \left(\frac{1}{\sqrt{2}n! \Gamma(n + l + \frac{3}{2})} \right)^{\frac{1}{2}}.$$

In the special case $l = 0$, this is the Hermite function,

$$(5.20) \quad \widehat{\sqrt{\mu}\varphi_{n,0,0}}(\xi) = \frac{1}{\sqrt{(2n+1)!}} |\xi|^{2n} e^{-\frac{|\xi|^2}{2}}.$$

We deduce from the Bobylev formula that, $\forall n, l, m, \tilde{n}, \tilde{l}, \tilde{m} \in \mathbb{N}$, with $|m| \leq l, |\tilde{m}| \leq \tilde{l}$,

$$(5.21) \quad \begin{aligned} \mathcal{F}\left(\sqrt{\mu}\Gamma(\varphi_{n,l,m}, \varphi_{\tilde{n},\tilde{l},\tilde{m}})\right)(\xi) &= \mathcal{F}(Q(\sqrt{\mu}\varphi_{n,l,m}, \sqrt{\mu}\varphi_{\tilde{n},\tilde{l},\tilde{m}}))(\xi) \\ &= \int_{\mathbb{S}^2} b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) [\widehat{\sqrt{\mu}\varphi_{n,l,m}}(\xi^-) \widehat{\sqrt{\mu}\varphi_{\tilde{n},\tilde{l},\tilde{m}}}(\xi^+) - \widehat{\sqrt{\mu}\varphi_{n,l,m}}(0) \widehat{\sqrt{\mu}\varphi_{\tilde{n},\tilde{l},\tilde{m}}}(\xi)] d\sigma. \end{aligned}$$

In the next propositions, we will compute the terms $\Gamma(\varphi_{n,l,m}, \varphi_{\tilde{n},\tilde{l},\tilde{m}})$ and proposition 2.1 will follows.

Proposition 5.4. *The following algebraic identities hold,*

$$\begin{aligned}
(i) \quad & \Gamma(\varphi_{0,0,0}, \varphi_{\tilde{n},\tilde{l},\tilde{m}}) \\
&= \left(\int_{|\theta| \leq \frac{\pi}{4}} \beta(\theta) \left((\cos \theta)^{2\tilde{n}+\tilde{l}} P_{\tilde{l}}(\cos \theta) - 1 \right) d\theta \right) \varphi_{\tilde{n},\tilde{l},\tilde{m}}; \\
(ii) \quad & \Gamma(\varphi_{n,l,m}, \varphi_{0,0,0}) \\
&= \left(\int_{|\theta| \leq \frac{\pi}{4}} \beta(\theta) \left((\sin \theta)^{2n+l} P_l(\sin \theta) - \delta_{0,n} \delta_{0,l} \right) d\theta \right) \varphi_{n,l,m}.
\end{aligned}$$

This is exactly (i_1) and (i_2) of Proposition 2.1.

Proof. Since

$$\widehat{\sqrt{\mu}\varphi_{n,l,m}}(0) = \delta_{n,0} \delta_{l,0},$$

then when $n = 0, l = 0$, by using (5.19) and (5.20), recall that $\kappa = \xi/|\xi|$, we have

$$\begin{aligned}
& \mathcal{F}(Q(\sqrt{\mu}\varphi_{0,0,0}, \sqrt{\mu}\varphi_{\tilde{n},\tilde{l},\tilde{m}}))(\xi) \\
&= \int_{\mathbb{S}^2} b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) \left[\widehat{\sqrt{\mu}\varphi_{0,0,0}}(\xi^-) \widehat{\sqrt{\mu}\varphi_{\tilde{n},\tilde{l},\tilde{m}}}(\xi^+) - \widehat{\sqrt{\mu}\varphi_{0,0,0}}(0) \widehat{\sqrt{\mu}\varphi_{\tilde{n},\tilde{l},\tilde{m}}}(\xi) \right] d\sigma. \\
&= A_{\tilde{n},\tilde{l}} e^{-\frac{|\xi|^2}{2}} \left(\frac{|\xi|}{\sqrt{2}} \right)^{2\tilde{n}+\tilde{l}} \int_{\mathbb{S}^2} b(\kappa \cdot \sigma) \left[Y_{\tilde{l}}^{\tilde{m}} \left(\frac{\kappa + \sigma}{|\kappa + \sigma|} \right) \left(\frac{|\kappa + \sigma|}{2} \right)^{2\tilde{n}+\tilde{l}} - Y_{\tilde{l}}^{\tilde{m}}(\kappa) \right] d\sigma.
\end{aligned}$$

Apply now the identity (5.13) of Corollary 5.2, one can find that,

$$\begin{aligned}
& \mathcal{F}(Q(\sqrt{\mu}\varphi_{0,0,0}, \sqrt{\mu}\varphi_{\tilde{n},\tilde{l},\tilde{m}}))(\xi) \\
&= \int_{|\theta| \leq \frac{\pi}{4}} \beta(\theta) \left[(\cos \theta)^{2\tilde{n}+\tilde{l}} P_{\tilde{l}}(\cos \theta) - 1 \right] d\theta \widehat{\sqrt{\mu}\varphi_{\tilde{n},\tilde{l},\tilde{m}}}(\xi).
\end{aligned}$$

Hence by the inverse Fourier transform

$$\begin{aligned}
\Gamma(\varphi_{0,0,0}, \varphi_{\tilde{n},\tilde{l},\tilde{m}}) &= \frac{1}{\sqrt{\mu}} Q(\sqrt{\mu}\varphi_{0,0,0}, \sqrt{\mu}\varphi_{\tilde{n},\tilde{l},\tilde{m}}) \\
&= \left(\int_{|\theta| \leq \frac{\pi}{4}} \beta(\theta) \left((\cos \theta)^{2\tilde{n}+\tilde{l}} P_{\tilde{l}}(\cos \theta) - 1 \right) d\theta \right) \varphi_{\tilde{n},\tilde{l},\tilde{m}}.
\end{aligned}$$

The result of (i) follows. Similar arguments apply to the case (ii) , and this ends the proof of Proposition 5.4. $\square$

Proposition 5.5. *The following algebraic identities hold,*

$$\begin{aligned}
(i) \quad & \Gamma(\varphi_{n,0,0}, \varphi_{\tilde{n},\tilde{l},\tilde{m}}) \\
&= \frac{1}{\sqrt{4\pi}} \frac{A_{\tilde{n},\tilde{l}} A_{n,0}}{A_{n+\tilde{n},\tilde{l}}} \left(\int_{|\theta| \leq \frac{\pi}{4}} \beta(\theta) (\sin \theta)^{2n} (\cos \theta)^{2\tilde{n}+\tilde{l}} P_{\tilde{l}}(\cos \theta) d\theta \right) \varphi_{n+\tilde{n},\tilde{l},\tilde{m}}, \quad n \geq 1; \\
(ii) \quad & \Gamma(\varphi_{n,l,m}, \varphi_{\tilde{n},0,0}) \\
&= \frac{1}{\sqrt{4\pi}} \frac{A_{\tilde{n},0} A_{n,l}}{A_{n+\tilde{n},l}} \left(\int_{|\theta| \leq \frac{\pi}{4}} \beta(\theta) (\sin \theta)^{2n+l} P_l(\sin \theta) (\cos \theta)^{2\tilde{n}} d\theta \right) \varphi_{n+\tilde{n},l,m}, \quad l \geq 1.
\end{aligned}$$

This is exactly (ii_1) and (ii_2) of Proposition 2.1.

Proof. For $n \geq 1$, using (5.20), we obtain

$$\widehat{\sqrt{\mu}\varphi_{n,0,0}}(0) = 0.$$

By using (5.19) for (5.21), recall that $\kappa = \xi/|\xi|$, $Y_0^0 = 1/\sqrt{4\pi}$, one gets

$$\begin{aligned} \mathcal{F}(Q(\sqrt{\mu}\varphi_{n,0,0}, \sqrt{\mu}\varphi_{\tilde{n},\tilde{l},\tilde{m}}))(\xi) &= A_{\tilde{n},\tilde{l}}A_{n,0}e^{-\frac{|\xi|^2}{2}}\left(\frac{|\xi|}{\sqrt{2}}\right)^{2(n+\tilde{n})+\tilde{l}} \\ &\times \frac{1}{\sqrt{4\pi}} \int_{\mathbb{S}^2} b(\kappa \cdot \sigma) \left(\frac{1-\kappa \cdot \sigma}{2}\right)^n \left(\frac{|\kappa + \sigma|}{2}\right)^{2\tilde{n}+\tilde{l}} Y_{\tilde{l}}^{\tilde{m}}\left(\frac{\kappa + \sigma}{|\kappa + \sigma|}\right) d\sigma. \end{aligned}$$

We then apply the identity (5.13) of corollary 5.2 and again (5.19) and derive

$$\begin{aligned} \mathcal{F}(Q(\sqrt{\mu}\varphi_{n,0,0}, \sqrt{\mu}\varphi_{\tilde{n},\tilde{l},\tilde{m}}))(\xi) \\ = \frac{1}{\sqrt{4\pi}} \frac{A_{\tilde{n},\tilde{l}}A_{n,0}}{A_{n+\tilde{n},\tilde{l}}} \left[\int_{|\theta| \leq \frac{\pi}{4}} \beta(\theta) \left((\sin \theta)^{2n} (\cos \theta)^{2\tilde{n}+\tilde{l}} P_{\tilde{l}}(\cos \theta) \right) d\theta \right] \widehat{\sqrt{\mu}\varphi_{\tilde{n}+n,\tilde{l},\tilde{m}}}(\xi). \end{aligned}$$

We obtain that by the inverse Fourier transform

$$\begin{aligned} \Gamma(\varphi_{n,0,0}, \varphi_{\tilde{n},\tilde{l},\tilde{m}}) &= \frac{1}{\sqrt{\mu}} Q(\sqrt{\mu}\varphi_{n,0,0}, \sqrt{\mu}\varphi_{\tilde{n},\tilde{l},\tilde{m}}) \\ &= \frac{1}{\sqrt{4\pi}} \frac{A_{\tilde{n},\tilde{l}}A_{n,0}}{A_{n+\tilde{n},\tilde{l}}} \left[\int_{|\theta| \leq \frac{\pi}{4}} \beta(\theta) \left((\sin \theta)^{2n} (\cos \theta)^{2\tilde{n}+\tilde{l}} P_{\tilde{l}}(\cos \theta) \right) d\theta \right] \varphi_{\tilde{n}+n,\tilde{l},\tilde{m}}. \end{aligned}$$

Thus (i) follows. Analogously, (ii) holds true. This ends the proof of Proposition 5.5. $\square$

Proposition 5.6. *The following algebraic identities hold for $l \geq 1$, $\tilde{l} \geq 1$:*

$$\begin{aligned} \Gamma(\varphi_{n,l,m}, \varphi_{\tilde{n},\tilde{l},\tilde{m}}) \\ = \sum_{k=0}^{k_0(l,\tilde{l},m,\tilde{m})} \frac{A_{\tilde{n},\tilde{l}}A_{n,l}}{A_{n+\tilde{n}+k,l+\tilde{l}-2k}} \left(\int_{\mathbb{S}_\kappa^2} G_{n,\tilde{n},l,\tilde{l}}^{m,\tilde{m}}(\kappa) \overline{Y_{l+\tilde{l}-2k}^{m+\tilde{m}}(\kappa)} d\kappa \right) \varphi_{n+\tilde{n}+k,l+\tilde{l}-2k,m+\tilde{m}}, \end{aligned}$$

where $k_0(l,\tilde{l},m,\tilde{m})$ is given in (2.3) and $G_{n,\tilde{n},l,\tilde{l}}^{m,\tilde{m}}$ is defined by

$$\begin{aligned} (5.22) \quad G_{n,\tilde{n},l,\tilde{l}}^{m,\tilde{m}}(\kappa) &= \int_{\mathbb{S}^2} b(\kappa \cdot \sigma) \left(|\kappa - \sigma|/2 \right)^{2n+l} \left(|\kappa + \sigma|/2 \right)^{2\tilde{n}+\tilde{l}} \\ &\times Y_l^m\left(\frac{\kappa - \sigma}{|\kappa - \sigma|}\right) Y_{\tilde{l}}^{\tilde{m}}\left(\frac{\kappa + \sigma}{|\kappa + \sigma|}\right) d\sigma. \end{aligned}$$

This is exactly (iii) of Proposition 2.1.

Proof. Now we consider the case when $l \geq 1$ and $\tilde{l} \geq 1$. Since $\widehat{\sqrt{\mu}\varphi_{n,l,m}}(0) = 0$, we get from (5.19)-(5.21)

$$\begin{aligned} \mathcal{F}(Q(\sqrt{\mu}\varphi_{n,l,m}, \sqrt{\mu}\varphi_{\tilde{n},\tilde{l},\tilde{m}}))(\xi) &= A_{\tilde{n},\tilde{l}}A_{n,l}e^{-\frac{|\xi|^2}{2}}\left(\frac{|\xi|}{\sqrt{2}}\right)^{2(n+\tilde{n})+\tilde{l}+l} \\ &\times \int_{\mathbb{S}^2} b(\kappa \cdot \sigma) Y_l^m\left(\frac{\kappa - \sigma}{|\kappa - \sigma|}\right) Y_{\tilde{l}}^{\tilde{m}}\left(\frac{\kappa + \sigma}{|\kappa + \sigma|}\right) \left(\frac{|\kappa - \sigma|}{2}\right)^{2n+l} \left(\frac{|\kappa + \sigma|}{2}\right)^{2\tilde{n}+\tilde{l}} d\sigma \\ &= A_{\tilde{n},\tilde{l}}A_{n,l}e^{-\frac{|\xi|^2}{2}}\left(\frac{|\xi|}{\sqrt{2}}\right)^{2(n+\tilde{n})+\tilde{l}+l} G_{\tilde{n},n,\tilde{l},l}^{m,\tilde{m}}(\kappa). \end{aligned}$$

From Lemma 5.3, $G_{n,\tilde{n},l,\tilde{l}}^{m,\tilde{m}}(\kappa)$ can be decomposed as a finite Laplace series

$$G_{n,\tilde{n},l,\tilde{l}}^{m,\tilde{m}}(\kappa) = \sum_{k=0}^{k_0(l,\tilde{l},m,\tilde{m})} \left(\int_{\mathbb{S}_\kappa^2} G_{n,\tilde{n},l,\tilde{l}}^{m,\tilde{m}}(\kappa) \overline{Y_{l+\tilde{l}-2k}^{m+\tilde{m}}} d\kappa \right) Y_{l+\tilde{l}-2k}^{m+\tilde{m}}(\kappa),$$

where $k_0(l, \tilde{l}, m, \tilde{m})$ was given in (2.3). By using this expansion, we derive

$$\begin{aligned} & \mathcal{F}(Q(\sqrt{\mu}\varphi_{n,l,m}, \sqrt{\mu}\varphi_{\tilde{n},\tilde{l},\tilde{m}}))(\xi) \\ &= \sum_{k=0}^{k_0(l,\tilde{l},m,\tilde{m})} \frac{A_{\tilde{n},\tilde{l}} A_{n,l}}{A_{n+\tilde{n}+k,l+\tilde{l}-2k}} \left(\int_{\mathbb{S}_\kappa^2} G_{n,\tilde{n},l,\tilde{l}}^{m,\tilde{m}}(\kappa) \overline{Y_{l+\tilde{l}-2k}^{m+\tilde{m}}}(\kappa) d\kappa \right) \mathcal{F}(\sqrt{\mu}\varphi_{n+\tilde{n}+k,l+\tilde{l}-2k,m+\tilde{m}}) \end{aligned}$$

and we conclude by taking the inverse Fourier transform. This ends the proof of Proposition 5.6. $\square$

5.3. The proof of Proposition 2.2. We now prove the following identity of Proposition 2.2

$$\sum_{|m| \leq l} \sum_{|\tilde{m}| \leq \tilde{l}} \mu_{n,\tilde{n},l,\tilde{l},k_1}^{m,\tilde{m},m'_1} \overline{\mu_{n,\tilde{n},l,\tilde{l},k_2}^{m,\tilde{m},m'_2}} = \sum_{|m| \leq l} \sum_{|\tilde{m}| \leq \tilde{l}} \left| \mu_{n,\tilde{n},l,\tilde{l},k_1}^{m,\tilde{m},m'_1} \right|^2 \delta_{k_1,k_2} \delta_{m'_1,m'_2}.$$

We state it in the following proposition with the notations $G_{n,\tilde{n},l,\tilde{l}}^{m,\tilde{m}}(\kappa)$ given in (5.22), since from Proposition 2.1, we have, for $|m'| \leq l + \tilde{l} - 2k$

$$\begin{aligned} \mu_{n,\tilde{n},l,\tilde{l},k}^{m,\tilde{m},m'} &= (-1)^k \left(\frac{2\pi^{\frac{3}{2}}(n+\tilde{n}+k)! \Gamma(n+\tilde{n}+l+\tilde{l}-k+\frac{3}{2})}{\tilde{n}! \Gamma(\tilde{n}+\tilde{l}+\frac{3}{2}) n! \Gamma(n+l+\frac{3}{2})} \right)^{\frac{1}{2}} \\ &\quad \times \int_{\mathbb{S}_\kappa^2} G_{n,\tilde{n},l,\tilde{l}}^{m,\tilde{m}}(\kappa) \overline{Y_{l+\tilde{l}-2k}^{m'}}(\kappa) d\kappa. \end{aligned}$$

Proposition 5.7. For $G_{n,\tilde{n},l,\tilde{l}}^{m,\tilde{m}}(\kappa)$ given in (5.22) and any integers $n, \tilde{n} \geq 0$, $|m'| \leq l'$, $|m^*| \leq l^*$, we have

$$\begin{aligned} & \sum_{|m| \leq l} \sum_{|\tilde{m}| \leq \tilde{l}} \left(\int_{\mathbb{S}_\kappa^2} G_{n,\tilde{n},l,\tilde{l}}^{m,\tilde{m}}(\kappa) \overline{Y_{l'}^{m'}}(\kappa) d\kappa \right) \left(\int_{\mathbb{S}_\kappa^2} \overline{G_{n,\tilde{n},l,\tilde{l}}^{m,\tilde{m}}(\kappa)} Y_{l^*}^{m^*}(\kappa) d\kappa \right) \\ &= \sum_{|m| \leq l} \sum_{|\tilde{m}| \leq \tilde{l}} \left| \int_{\mathbb{S}_\kappa^2} G_{n,\tilde{n},l,\tilde{l}}^{m,\tilde{m}}(\kappa) \overline{Y_{l'}^{m'}}(\kappa) d\kappa \right|^2 \delta_{l',l^*} \delta_{m',m^*}. \end{aligned}$$

Proof. We recall the definition (5.22) of $G_{n,\tilde{n},l,\tilde{l}}^{m,\tilde{m}}(\kappa)$

$$\begin{aligned} G_{n,\tilde{n},l,\tilde{l}}^{m,\tilde{m}}(\kappa) &= \int_{\mathbb{S}^2} b(\kappa \cdot \sigma) (|\kappa - \sigma|/2)^{2n+l} (|\kappa + \sigma|/2)^{2\tilde{n}+\tilde{l}} \\ &\quad \times Y_l^m \left(\frac{\kappa - \sigma}{|\kappa - \sigma|} \right) Y_{\tilde{l}}^{\tilde{m}} \left(\frac{\kappa + \sigma}{|\kappa + \sigma|} \right) d\sigma \end{aligned}$$

and we consider the transform (5.6)-(5.8) for an unit vector σ

$$\begin{aligned} \sigma &= \kappa \cos \theta + \kappa^1 \sin \theta \cos \phi + \kappa^2 \sin \theta \sin \phi \\ &= \kappa \cos \theta + \kappa^\perp(\phi) \sin \theta \end{aligned}$$

with $\theta \in [0, \frac{\pi}{2}]$ and $\phi \in [0, 2\pi]$. Therefore, using (5.9)-(5.10), the change of variable $\theta = 2\theta_1$, the odd-even parity of P_l^m and the definition (1.2) of β we find

$$(5.23) \quad G_{n,\tilde{n},l,\tilde{l}}^{m,\tilde{m}}(\kappa) = \int_{|\theta_1| \leq \frac{\pi}{4}} \beta(\theta_1)(\sin \theta_1)^{2n+l}(\cos \theta_1)^{2\tilde{n}+\tilde{l}} \times \\ \int_0^{2\pi} Y_l^m(\kappa \sin \theta_1 - \kappa^\perp(\phi_1) \cos \theta_1) Y_{\tilde{l}}^{\tilde{m}}(\kappa \cos \theta_1 + \kappa^\perp(\phi_1) \sin \theta_1) \frac{d\phi_1}{2\pi} d\theta_1$$

and

$$(5.24) \quad \int_{\mathbb{S}_\kappa^2} G_{n,\tilde{n},l,\tilde{l}}^{m,\tilde{m}}(\kappa) \overline{Y_{l'}^{m'}}(\kappa) d\kappa \\ = \int_{|\theta_1| \leq \frac{\pi}{4}} \beta(\theta_1)(\sin \theta_1)^{2n+l}(\cos \theta_1)^{2\tilde{n}+\tilde{l}} \times \int_0^{2\pi} \\ \left(\int_{\mathbb{S}_\kappa^2} Y_l^m(\kappa \sin \theta_1 - \kappa^\perp(\phi_1) \cos \theta_1) Y_{\tilde{l}}^{\tilde{m}}(\kappa \cos \theta_1 + \kappa^\perp(\phi_1) \sin \theta_1) \overline{Y_{l'}^{m'}}(\kappa) d\kappa \right) \frac{d\phi_1}{2\pi} d\theta_1.$$

Using equivalent notations of (5.6)-(5.8) for a unit vector σ expanded in another orthonormal frame $(\gamma, \gamma^1, \gamma^2)$,

$$\begin{aligned} \sigma &= \gamma \cos 2\theta_2 + \gamma^1 \sin 2\theta_2 \cos \phi_2 + \gamma^2 \sin 2\theta_2 \sin \phi_2 \\ &= \gamma \cos 2\theta_2 + \gamma^\perp(\phi_2) \sin 2\theta_2 \end{aligned}$$

with $\theta_2 \in [0, \pi/4]$, $\phi_2 \in [0, 2\pi]$, we have also

$$\begin{aligned} &\int_{\mathbb{S}_\gamma^2} \overline{G_{n,\tilde{n},l,\tilde{l}}^{m,\tilde{m}}(\gamma)} Y_{l'}^{m*}(\gamma) d\gamma \\ &= \int_{|\theta_2| \leq \frac{\pi}{4}} \beta(\theta_2)(\sin \theta_2)^{2n+l}(\cos \theta_2)^{2\tilde{n}+\tilde{l}} \times \int_0^{2\pi} \int_{\mathbb{S}_\gamma^2} \\ &\quad \overline{Y_l^m}(\gamma \sin \theta_2 - \gamma^\perp(\phi_2) \cos \theta_2) \overline{Y_{\tilde{l}}^{\tilde{m}}}(\gamma \cos \theta_2 + \gamma^\perp(\phi_1) \sin \theta_2) Y_{l'}^{m*}(\gamma) d\gamma \frac{d\phi_2}{2\pi} d\theta_2. \end{aligned}$$

The goal of this proposition is to compute the following term :

$$\mathbf{A} = \sum_{|m| \leq l} \sum_{|\tilde{m}| \leq \tilde{l}} \left(\int_{\mathbb{S}_\kappa^2} G_{n,\tilde{n},l,\tilde{l}}^{m,\tilde{m}}(\kappa) \overline{Y_{l'}^{m'}}(\kappa) d\kappa \right) \left(\int_{\mathbb{S}_\gamma^2} \overline{G_{n,\tilde{n},l,\tilde{l}}^{m,\tilde{m}}(\gamma)} Y_{l'}^{m*}(\gamma) d\gamma \right)$$

From the addition theorem (5.2), we find

$$\begin{aligned} &\sum_{|m| \leq l} Y_l^m(\kappa \sin \theta_1 - \kappa^\perp(\phi_1) \cos \theta_1) \overline{Y_l^m}(\gamma \sin \theta_2 - \gamma^\perp(\phi_2) \cos \theta_2) \\ &= \frac{2l+1}{4\pi} P_l((\kappa \sin \theta_1 - \kappa^\perp(\phi_1) \cos \theta_1) \cdot (\gamma \sin \theta_2 - \gamma^\perp(\phi_2) \cos \theta_2)) \end{aligned}$$

and

$$\begin{aligned} &\sum_{|\tilde{m}| \leq \tilde{l}} Y_{\tilde{l}}^{\tilde{m}}(\kappa \cos \theta_1 + \kappa^\perp(\phi_1) \sin \theta_1) \overline{Y_{\tilde{l}}^{\tilde{m}}}(\gamma \cos \theta_2 + \gamma^\perp(\phi_2) \sin \theta_2) \\ &= \frac{2\tilde{l}+1}{4\pi} P_{\tilde{l}}((\kappa \cos \theta_1 + \kappa^\perp(\phi_1) \sin \theta_1) \cdot (\gamma \cos \theta_2 + \gamma^\perp(\phi_2) \sin \theta_2)). \end{aligned}$$

We then plug the two previous identities into the expression of $\mathbf{A}$ and we directly derive

$$(5.25) \quad \begin{aligned} \mathbf{A} &= \frac{2l+1}{4\pi} \frac{2\tilde{l}+1}{4\pi} \int_{|\theta_1| \leq \frac{\pi}{4}} \left(\beta(\theta_1) (\sin \theta_1)^{2n+l} (\cos \theta_1)^{2\tilde{n}+\tilde{l}} \right) \\ &\quad \times \int_{|\theta_2| \leq \frac{\pi}{4}} \left(\beta(\theta_2) (\sin \theta_2)^{2n+l} (\cos \theta_2)^{2\tilde{n}+\tilde{l}} \right) \mathbf{B}_1(\theta_1, \theta_2) d\theta_1 d\theta_2 \end{aligned}$$

where

$$(5.26) \quad \mathbf{B}_1(\theta_1, \theta_2) = \int_{\mathbb{S}_\gamma^2} \int_{\mathbb{S}_\kappa^2} \int_0^{2\pi} \mathbf{B}_2(\kappa, \gamma, \theta_1, \theta_2, \phi_2) \frac{d\phi_2}{2\pi} \overline{Y_{l'}^{m'}}(\kappa) Y_{l^*}^{m^*}(\gamma) d\kappa d\gamma$$

and

$$\begin{aligned} \mathbf{B}_2(\kappa, \gamma, \theta_1, \theta_2, \phi_2) &= \int_0^{2\pi} P_l \left((\kappa \sin \theta_1 - \kappa^\perp(\phi_1) \cos \theta_1) \cdot \gamma^- \right) \\ &\quad \times P_{\tilde{l}} \left((\kappa \cos \theta_1 + \kappa^\perp(\phi_1) \sin \theta_1) \cdot \gamma^+ \right) \frac{d\phi_1}{2\pi}. \end{aligned}$$

Here, γ^+ and γ^- are defined by (and depend on γ, θ_2, ϕ_2)

$$(5.27) \quad \gamma^+ = \gamma \cos \theta_2 + \gamma^\perp(\phi_2) \sin \theta_2, \quad \gamma^- = \gamma \sin \theta_2 - \gamma^\perp(\phi_2) \cos \theta_2.$$

From lemma 5.9 (proved after the Proposition), we have

$$\mathbf{B}_2(\kappa, \gamma, \theta_1, \theta_2, \phi_2) = \sum_{0 \leq q \leq l} \sum_{0 \leq \tilde{q} \leq \tilde{l}} b_{l, \tilde{l}}^{q, \tilde{q}}(\theta_1) P_q(\kappa \cdot \gamma^-) P_{\tilde{q}}(\kappa \cdot \gamma^+)$$

where $b_{l, \tilde{l}}^{q, \tilde{q}}(\theta_1)$ is a continuous function dependent on θ_1 .

Therefore from (5.26) we deduce

$$(5.28) \quad \mathbf{B}_1(\theta_1, \theta_2) = \sum_{0 \leq q \leq l} \sum_{0 \leq \tilde{q} \leq \tilde{l}} b_{l, \tilde{l}}^{q, \tilde{q}}(\theta_1) \mathbf{B}_{q, \tilde{q}}(\theta_2)$$

where

$$(5.29) \quad \mathbf{B}_{q, \tilde{q}}(\theta_2) = \int_{\mathbb{S}_\gamma^2} \int_{\mathbb{S}_\kappa^2} \left(\int_0^{2\pi} P_q(\kappa \cdot \gamma^-) P_{\tilde{q}}(\kappa \cdot \gamma^+) \frac{d\phi_2}{2\pi} \right) \overline{Y_{l'}^{m'}}(\kappa) Y_{l^*}^{m^*}(\gamma) d\kappa d\gamma.$$

Since $(\gamma, \gamma^1, \gamma^2)$ is an orthonormal basis in $\mathbb{R}^3$, then for any unit vector κ , we can find $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$ such that

$$\kappa \cdot \gamma = \cos \theta; \kappa \cdot \gamma^1 = \sin \theta \cos \phi; \kappa \cdot \gamma^2 = \sin \theta \sin \phi.$$

Therefore (recall that $\gamma^\pm$ defined in (5.27) depend on γ, θ_2, ϕ_2),

$$\begin{aligned} \kappa \cdot \gamma^- &= (\kappa \cdot \gamma) \sin \theta_2 - \kappa \cdot \gamma^1 \cos \theta_2 \cos \phi_2 - \kappa \cdot \gamma^2 \cos \theta_2 \sin \phi_2 \\ &= \cos \theta \sin \theta_2 + \sin \theta \cos \theta_2 \cos(\phi - \phi_2 - \pi); \\ \kappa \cdot \gamma^+ &= (\kappa \cdot \gamma) \cos \theta_2 + \kappa \cdot \gamma^1 \sin \theta_2 \cos \phi_2 + \kappa \cdot \gamma^2 \sin \theta_2 \sin \phi_2 \\ &= \cos \theta \cos \theta_2 + \sin \theta \sin \theta_2 \cos(\phi - \phi_2) \end{aligned}$$

We use again two times the addition theorem (5.3):

$$\begin{aligned}
P_q(\kappa \cdot \gamma^-) &= \sum_{|m| \leq q} a_{q,m} P_q^{|m|}(\cos \theta) P_q^{|m|}(\sin \theta_2) e^{im(\phi - \phi_2 - \pi)} \\
&= \sum_{|m| \leq q} a_{q,m} P_q^{|m|}(\kappa \cdot \gamma) P_q^{|m|}(\sin \theta_2) e^{im(\phi - \phi_2 - \pi)} \\
P_{\tilde{q}}(\kappa \cdot \gamma^+) &= \sum_{|\tilde{m}| \leq \tilde{q}} a_{\tilde{q},\tilde{m}} P_{\tilde{q}}^{|\tilde{m}|}(\kappa \cdot \gamma) P_{\tilde{q}}^{|\tilde{m}|}(\cos \theta_2) e^{i\tilde{m}(\phi - \phi_2)},
\end{aligned}$$

where $a_{q,m}$, $a_{\tilde{q},\tilde{m}}$ was defined in the beginning of this section. Then

$$\begin{aligned}
&\int_0^{2\pi} P_q(\kappa \cdot \gamma^-(\phi_2)) P_{\tilde{q}}(\kappa \cdot \gamma^+(\phi_2)) \frac{d\phi_2}{2\pi} \\
&= \sum_{|k| \leq \min(q, \tilde{q})} a_{q,k} a_{\tilde{q},k} P_q^{|k|}(\kappa \cdot \gamma) P_q^{|k|}(\sin \theta_2) (-1)^k P_{\tilde{q}}^{|k|}(\kappa \cdot \gamma) P_{\tilde{q}}^{|k|}(\cos \theta_2).
\end{aligned}$$

Since $P_q^{|k|}(\kappa \cdot \gamma) P_{\tilde{q}}^{|k|}(\kappa \cdot \gamma)$ is a continuous function of $\kappa \cdot \gamma$, we apply the Funk-Hecke Formula (5.5) and obtain

$$\int_{\mathbb{S}^2} P_q^{|k|}(\kappa \cdot \gamma) P_{\tilde{q}}^{|k|}(\kappa \cdot \gamma) Y_{l^*}^{m^*}(\gamma) d\gamma = \left(2\pi \int_{-1}^1 P_q^{|k|}(x) P_{\tilde{q}}^{|k|}(x) P_{l^*}(x) dx \right) Y_{l^*}^{m^*}(\kappa).$$

Combining the two previous relations into (5.29), we obtain

$$\begin{aligned}
\mathbf{B}_{q,\tilde{q}}(\theta_2) &= \sum_{|k| \leq \min(q, \tilde{q})} a_{q,k} a_{\tilde{q},k} P_q^{|k|}(\sin \theta_2) (-1)^k P_{\tilde{q}}^{|k|}(\cos \theta_2) \\
(5.30) \quad &\times \left(2\pi \int_{-1}^1 P_q^{|k|}(x) P_{\tilde{q}}^{|k|}(x) P_{l^*}(x) dx \right) \int_{\mathbb{S}_\kappa^2} Y_{l^*}^{m^*}(\kappa) \overline{Y_{l'}^{m'}}(\kappa) d\kappa.
\end{aligned}$$

Finally, if $(l^*, m^*) \neq (l', m')$, the orthogonality of the spherical harmonics implies that $\mathbf{B}_{q,\tilde{q}} = 0$ for all q and $\tilde{q}$, and so on for $\mathbf{B}_1$ and $\mathbf{A}$. This concludes the proof of Proposition 5.7. $\square$

Remark 5.8. From the previous proof, in the special case $(l^*, m^*) = (l', m')$, we have from (5.25), (5.28) and (5.30)

$$\begin{aligned}
&\sum_{|m| \leq l} \sum_{|\tilde{m}| \leq \tilde{l}} \left| \int_{\mathbb{S}_\kappa^2} G_{n,\tilde{n},l,\tilde{l}}^{m,\tilde{m}}(\kappa) \overline{Y_{l'}^{m'}}(\kappa) d\kappa \right|^2 \\
&= \frac{2l+1}{4\pi} \frac{2\tilde{l}+1}{4\pi} \int_{|\theta_1| \leq \frac{\pi}{4}} (\beta(\theta_1) (\sin \theta_1)^{2n+l} (\cos \theta_1)^{2\tilde{n}+\tilde{l}}) \\
&\quad \times \int_{|\theta_2| \leq \frac{\pi}{4}} (\beta(\theta_2) (\sin \theta_2)^{2n+l} (\cos \theta_2)^{2\tilde{n}+\tilde{l}}) B_{l,\tilde{l},l'}(\theta_1, \theta_2) d\theta_1 d\theta_2
\end{aligned}$$

where $B_{l,\tilde{l},l'}(\theta_1, \theta_2)$ is defined by

$$\begin{aligned}
B_{l,\tilde{l},l'}(\theta_1, \theta_2) &= \sum_{0 \leq q \leq l} \sum_{0 \leq \tilde{q} \leq \tilde{l}} b_{l,\tilde{l}}^{q,\tilde{q}}(\theta_1) \times \\
&\sum_{|k| \leq \min(q, \tilde{q})} a_{q,k} a_{\tilde{q},k} P_q^{|k|}(\sin \theta_2) (-1)^k P_{\tilde{q}}^{|k|}(\cos \theta_2) \left(2\pi \int_{-1}^1 P_q^{|k|}(x) P_{\tilde{q}}^{|k|}(x) P_{l'}(x) dx \right).
\end{aligned}$$

We now prove the following lemma used in the previous proposition 5.7.

Lemma 5.9. For any $l, \tilde{l}, q, \tilde{q} \in \mathbb{N}$, there exists a continuous function $b_{l,\tilde{l}}^{q,\tilde{q}}(\theta)$ such that for any real θ, ϕ and unit vectors $\kappa, \eta^+, \eta^-, (\eta^+, \eta^-)$ being orthogonal, we have

$$\begin{aligned} & \int_0^{2\pi} P_l \left((\kappa \sin \theta - \kappa^\perp(\phi) \cos \theta) \cdot \eta^- \right) \\ & \quad \times P_{\tilde{l}} \left((\kappa \cos \theta + \kappa^\perp(\phi) \sin \theta) \cdot \eta^+ \right) \frac{d\phi}{2\pi} \\ & = \sum_{0 \leq q \leq l} \sum_{0 \leq \tilde{q} \leq \tilde{l}} b_{l,\tilde{l}}^{q,\tilde{q}}(\theta) P_q(\kappa \cdot \eta^-) P_{\tilde{q}}(\kappa \cdot \eta^+) \end{aligned}$$

where the coefficients $b_{l,\tilde{l}}^{q,\tilde{q}} \equiv 0$ if $(q, \tilde{q}) \neq (l - 2q_1, \tilde{l} - 2q_2)$ for integers q_1, q_2 satisfying $0 \leq 2q_1 \leq l, 0 \leq 2q_2 \leq \tilde{l}$.

Proof. We apply the addition theorem (5.3) in the frame $(\kappa, \kappa^1, \kappa^2)$ and the relation (5.1).

$$\begin{aligned} P_{\tilde{l}} \left((\kappa \cos \theta + \kappa^\perp(\phi) \sin \theta) \cdot \eta^+ \right) & = \sum_{|\tilde{k}| \leq \tilde{l}} a_{\tilde{l},\tilde{k}} P_{\tilde{l}}^{|\tilde{k}|}(\cos \theta) \left(\frac{d^{|\tilde{k}|} P_{\tilde{l}}}{dx^{|\tilde{k}|}} \right) (\kappa \cdot \eta^+) e^{i\tilde{k}\phi} U_{\tilde{k}}^+, \\ P_l \left((\kappa \sin \theta - \kappa^\perp(\phi) \cos \theta) \cdot \eta^- \right) & = P_l \left((\kappa \sin(\theta) + \kappa^\perp(\phi + \pi) \cos(\theta)) \cdot \eta^- \right) \\ & = \sum_{|k| \leq l} a_{l,k} P_l^{|k|}(\sin \theta) \left(\frac{d^{|k|} P_l}{dx^{|k|}} \right) (\kappa \cdot \eta^-) e^{ik(\phi + \pi)} U_k^- \end{aligned}$$

where

$$U_k^+ = \left((\kappa^1 \cdot \eta^+) - i \operatorname{sgn}(k) (\kappa^2 \cdot \eta^+) \right)^{|k|}, \quad U_k^- = \left((\kappa^1 \cdot \eta^-) - i \operatorname{sgn}(k) (\kappa^2 \cdot \eta^-) \right)^{|k|}.$$

Since the integrals are zero when $\tilde{k} \neq -k$, we derive

$$\begin{aligned} \mathbf{M} & = \int_0^{2\pi} P_l \left((\kappa \sin \theta - \kappa^\perp(\phi) \cos \theta) \cdot \eta^- \right) P_{\tilde{l}} \left((\kappa \cos \theta + \kappa^\perp(\phi) \sin \theta) \cdot \eta^+ \right) \frac{d\phi}{2\pi} \\ & = \sum_{|k| \leq \min(l, \tilde{l})} c_1^k(\theta) \left(\frac{d^{|k|} P_l}{dx^{|k|}} \right) (\kappa \cdot \eta^-) \left(\frac{d^{|k|} P_{\tilde{l}}}{dx^{|k|}} \right) (\kappa \cdot \eta^+) (-1)^k U_{-k}^- U_k^+ \end{aligned}$$

where

$$c_1^k(\theta) = a_{l,k} a_{\tilde{l},-k} P_l^{|k|}(\sin \theta) P_{\tilde{l}}^{|k|}(\cos \theta).$$

We then write

$$(5.31) \quad \mathbf{M} = \sum_{0 \leq k \leq \min(l, \tilde{l})} c_1^k(\theta) \left(\frac{d^k P_l}{dx^k} \right) (\kappa \cdot \eta^-) \left(\frac{d^k P_{\tilde{l}}}{dx^k} \right) (\kappa \cdot \eta^+) V_k$$

where $V_0 = 1$ and V_k is defined for $k \geq 1$ by

$$\begin{aligned} V_k & = \left\{ \left[(\kappa^1 \cdot \eta^+ - i\kappa^2 \cdot \eta^+) (-\kappa^1 \cdot \eta^- - i\kappa^2 \cdot \eta^-) \right]^k \right. \\ & \quad \left. + \left[(\kappa^1 \cdot \eta^+ + i\kappa^2 \cdot \eta^+) (-\kappa^1 \cdot \eta^- + i\kappa^2 \cdot \eta^-) \right]^k \right\}. \end{aligned}$$

We claim that V_k is polynomial type : $V_k = p_k((\kappa \cdot \eta^+), (\kappa \cdot \eta^-))$ where

$$\begin{aligned} p_k(x, y) & = \left(xy + i\sqrt{1-x^2-y^2} \right)^k + \left(xy - i\sqrt{1-x^2-y^2} \right)^k \\ & = 2 \sum_{0 \leq 2r \leq k} (-1)^r \binom{k}{2r} x^{k-2r} y^{k-2r} (1-x^2-y^2)^r. \end{aligned}$$

Indeed, we observe that, there exist two real numbers A, B such that

$$(\kappa^1 \cdot \eta^+ - i\kappa^2 \cdot \eta^+)(-\kappa^1 \cdot \eta^- - i\kappa^2 \cdot \eta^-) = A + iB,$$

$$(\kappa^1 \cdot \eta^+ + i\kappa^2 \cdot \eta^+)(-\kappa^1 \cdot \eta^- + i\kappa^2 \cdot \eta^-) = A - iB.$$

If we can show that

$$(5.32) \quad A = (\kappa \cdot \eta^-)(\kappa \cdot \eta^+)$$

$$(5.33) \quad B^2 = 1 - (\kappa \cdot \eta^-)^2 - (\kappa \cdot \eta^+)^2,$$

then it follows that

$$\begin{aligned} V_k &= [A + iB]^k + [A - iB]^k \\ &= 2 \sum_{0 \leq 2r \leq k} (-1)^r \binom{k}{2r} A^{k-2r} B^{2r} \\ (5.34) \quad &= 2 \sum_{0 \leq 2r \leq k} (-1)^r \binom{k}{2r} (\kappa \cdot \eta^+)^{k-2r} (\kappa \cdot \eta^-)^{k-2r} (1 - (\kappa \cdot \eta^+)^2 - (\kappa \cdot \eta^-)^2)^r. \end{aligned}$$

Let us now prove (5.32), (5.33). There exist reals ϕ^+, ϕ^-, θ^+ and θ^- such that

$$\eta^+ = \cos \theta^+ \kappa + \sin \theta^+ \cos \phi^+ \kappa^1 + \sin \theta^+ \sin \phi^+ \kappa^2$$

$$\eta^- = \cos \theta^- \kappa + \sin \theta^- \cos \phi^- \kappa^1 + \sin \theta^- \sin \phi^- \kappa^2.$$

The orthogonality of η^+ and η^- implies that

$$(5.35) \quad \cos \theta^+ \cos \theta^- + \sin \theta^+ \sin \theta^- \cos(\phi^+ - \phi^-) = 0.$$

We then have

$$\begin{aligned} &(\kappa^1 \cdot \eta^+ - i\kappa^2 \cdot \eta^+)(-\kappa^1 \cdot \eta^- - i\kappa^2 \cdot \eta^-) \\ &= (\sin \theta^+ \cos \phi^+ - i \sin \theta^+ \sin \phi^+)(-\sin \theta^- \cos \phi^- - i \sin \theta^- \sin \phi^-) \\ &= A + iB. \end{aligned}$$

- Computation of A :

$$\begin{aligned} A &= -\sin \theta^+ \cos \phi^+ \sin \theta^- \cos \phi^- - \sin \theta^+ \sin \phi^+ \sin \theta^- \sin \phi^- \\ &= -\sin \theta^+ \sin \theta^- \cos(\phi^+ - \phi^-) \\ &= \cos \theta^+ \cos \theta^- \quad (\text{from (5.35)}) \\ &= (\kappa \cdot \eta^+)(\kappa \cdot \eta^-). \end{aligned}$$

- Computation of B :

$$\begin{aligned} B &= -\sin \theta^+ \cos \phi^+ \sin \theta^- \sin \phi^- + \sin \theta^+ \sin \phi^+ \sin \theta^- \cos \phi^- \\ &= \sin \theta^+ \sin \theta^- \sin(\phi^+ - \phi^-). \end{aligned}$$

Using again (5.35)

$$\begin{aligned} B^2 &= \sin^2 \theta^+ \sin^2 \theta^- \sin^2(\phi^+ - \phi^-) \\ &= \sin^2 \theta^+ \sin^2 \theta^- - \sin^2 \theta^+ \sin^2 \theta^- \cos^2(\phi^+ - \phi^-) \\ &= (1 - \cos^2 \theta^+)(1 - \cos^2 \theta^-) - \cos^2 \theta^+ \cos^2 \theta^- \\ &= 1 - (\kappa \cdot \eta^+)^2 - (\kappa \cdot \eta^-)^2. \end{aligned}$$

This ends the proof of (5.32), (5.33).

From the expression of (5.34), V_k can be re-written as

$$V_k = \sum_{0 \leq 2r_1, 2r_2 \leq k} a_{k,r_1,r_2} (\kappa \cdot \eta^-)^{k-2r_1} (\kappa \cdot \eta^+)^{k-2r_2}$$

where a_{k,r_1,r_2} are constants dependent on k, r_1, r_2 . Recall that $P_l(x)$ is a 1-order polynomial of x in Sec. 1 of Chap. III in [20],

$$P_l(x) = \sum_{m=0}^{\lfloor \frac{l}{2} \rfloor} (-1)^m \frac{1}{2^l} \binom{l}{m} \binom{2l-2m}{l} x^{l-2m},$$

we obtain

$$\begin{aligned} \frac{d^k P_l}{dx^k} (\kappa \cdot \eta^-) &= \sum_{0 \leq 2m_1 \leq l-k} b_{l,k,m_1} (\kappa \cdot \eta^-)^{l-k-2m_1}; \\ \frac{d^k P_{\bar{l}}}{dx^k} (\kappa \cdot \eta^+) &= \sum_{0 \leq 2m_2 \leq \bar{l}-k} b_{\bar{l},k,m_2} (\kappa \cdot \eta^+)^{\bar{l}-k-2m_2}, \end{aligned}$$

where $b_{l,k,m_1}, b_{\bar{l},k,m_2}$ are constants. We plug the expression of $V_k, \frac{d^k P_l}{dx^k}, \frac{d^k P_{\bar{l}}}{dx^k}$ into (5.31),

$$\begin{aligned} \mathbf{M} &= \sum_{0 \leq k \leq \min(l, \bar{l})} c_1^k(\theta) \sum_{0 \leq 2r_1, 2r_2 \leq k} a_{k,r_1,r_2} \\ &\quad \times \sum_{0 \leq 2m_1 \leq l-k} \sum_{0 \leq 2m_2 \leq \bar{l}-k} b_{l,k,m_1} b_{\bar{l},k,m_2} (\kappa \cdot \eta^-)^{l-2r_1-2m_1} (\kappa \cdot \eta^+)^{\bar{l}-2r_2-2m_2}. \end{aligned}$$

Exchanging the order of the summation of $\mathbf{M}$, one can verify that

$$\begin{aligned} \mathbf{M} &= \sum_{0 \leq 2j_1 \leq l} \sum_{0 \leq 2j_2 \leq \bar{l}} (\kappa \cdot \eta^-)^{l-2j_1} (\kappa \cdot \eta^+)^{\bar{l}-2j_2} \\ &\quad \times \left[\sum_{0 \leq k \leq \min(l, \bar{l})} \sum_{\substack{r_1+m_1=j_1 \\ 0 \leq 2r_1 \leq k, 0 \leq 2m_1 \leq l-k}} \sum_{\substack{r_2+m_2=j_2 \\ 0 \leq 2r_2 \leq k, 0 \leq 2m_2 \leq \bar{l}-k}} c_1^k(\theta) a_{k,r_1,r_2} b_{l,k,m_1} b_{\bar{l},k,m_2} \right]. \end{aligned}$$

We use again the formula (5.16) for $(\kappa \cdot \eta^-)^{l-2j_1}, (\kappa \cdot \eta^+)^{\bar{l}-2j_2}$ that,

$$\begin{aligned} (\kappa \cdot \eta^-)^{l-2j_1} &= \sum_{0 \leq p_1 \leq \frac{l}{2}-j_1} c_{l-2j_1,p_1} P_{l-2j_1-2p_1}(\kappa \cdot \eta^-); \\ (\kappa \cdot \eta^+)^{\bar{l}-2j_2} &= \sum_{0 \leq p_2 \leq \frac{\bar{l}}{2}-j_2} c_{\bar{l}-2j_2,p_2} P_{\bar{l}-2j_2-2p_2}(\kappa \cdot \eta^+), \end{aligned}$$

and exchange the order of the summation of $\mathbf{M}$ again, there exists a continuous coefficients $b_{l,\bar{l}}^{q,\bar{q}}(\theta)$ such that

$$\mathbf{M} = \sum_{0 \leq 2q_1 \leq l} \sum_{0 \leq 2q_2 \leq \bar{l}} b_{l,\bar{l}}^{l-2q_1, \bar{l}-2q_2}(\theta) P_{l-2q_1}(\kappa \cdot \eta^-) P_{\bar{l}-2q_2}(\kappa \cdot \eta^+).$$

This conclude the proof of Lemma 5.9 and Proposition 5.7.

□

5.4. Reduction of the expression of the non-linear eigenvalues. We derive in the following Propositions 5.10 and 5.12 some simplifications of the expression of the non-linear eigenvalue $\mu_{n,\tilde{n},l,\tilde{l},k_1}^{m,\tilde{m},m'}$, which will be used in the next Section 6.

Proposition 5.10. *For $G_{n,\tilde{n},l,\tilde{l}}^{m,\tilde{m}}(\kappa)$ given in (5.22) and any integers $n, \tilde{n} \geq 0$, $|m| \leq l$, $|m'| \leq l'$, we have*

$$(5.36) \quad \sum_{|m| \leq l} \sum_{|\tilde{m}| \leq \tilde{l}} \left| \int_{\mathbb{S}_\kappa^2} G_{n,\tilde{n},l,\tilde{l}}^{m,\tilde{m}}(\kappa) \overline{Y_{l'}^{m'}(\kappa)} d\kappa \right|^2 = \frac{2l+1}{4\pi} \frac{2\tilde{l}+1}{4\pi} \int_{|\theta_1| \leq \frac{\pi}{4}} \beta(\theta_1) (\sin \theta_1)^{2n+l} (\cos \theta_1)^{2\tilde{n}+\tilde{l}} \int_{|\theta_2| \leq \frac{\pi}{4}} \beta(\theta_2) (\sin \theta_2)^{2n+l} (\cos \theta_2)^{2\tilde{n}+\tilde{l}} \times \left(2\pi \int_{-1}^1 F_{l,\tilde{l}}(x, \theta_1, \theta_2) P_{l'}(x) dx \right) d\theta_2 d\theta_1$$

where

$$(5.37) \quad F_{l,\tilde{l}}(x, \theta_1, \theta_2) = \int_0^{2\pi} \int_0^{2\pi} P_l(\tau^1(\theta_2, \phi_2) J(x) (\tau^1(\theta_1, \phi_1))^T) \times P_{\tilde{l}}(\tau(\theta_2, \phi_2) J(x) (\tau(\theta_1, \phi_1))^T) \frac{d\phi_1}{2\pi} \frac{d\phi_2}{2\pi},$$

$J(x)$ is the matrix function

$$J(x) = \begin{pmatrix} x & -\sqrt{1-x^2} & 0 \\ \sqrt{1-x^2} & x & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and $\tau(\theta, \phi)$, $\tau^1(\theta, \phi)$ are the vectors

$$\tau(\theta, \phi) = (\cos \theta, \sin \theta \cos \phi, \sin \theta \sin \phi),$$

$$\tau^1(\theta, \phi) = (\sin \theta, -\cos \theta \cos \phi, -\cos \theta \sin \phi),$$

the column vector X^T is the transposition of the row vector $X = (x_1, x_2, x_3)$.

Remark 5.11. *We remark that in the formula (5.36), the right hand side is independent of m' . Therefore this implies*

$$(5.38) \quad \sum_{|m| \leq l} \sum_{|\tilde{m}| \leq \tilde{l}} \left| \int_{\mathbb{S}_\kappa^2} G_{n,\tilde{n},l,\tilde{l}}^{m,\tilde{m}}(\kappa) \overline{Y_{l'}^{m'}(\kappa)} d\kappa \right|^2 = \sum_{|m| \leq l} \sum_{|\tilde{m}| \leq \tilde{l}} \left| \int_{\mathbb{S}_\kappa^2} G_{n,\tilde{n},l,\tilde{l}}^{m,\tilde{m}}(\kappa) \overline{Y_{l'}^0(\kappa)} d\kappa \right|^2 = \sum_{|q| \leq \min(l,\tilde{l})} \left| \int_{\mathbb{S}_\kappa^2} G_{n,\tilde{n},l,\tilde{l}}^{q,-q}(\kappa) \overline{Y_{l'}^0(\kappa)} d\kappa \right|^2,$$

since from (5.14) the integral vanishes if $m + \tilde{m} \neq 0$.

Proof. We will prove that

$$2\pi \int_{-1}^1 F_{l,\tilde{l}}(x, \theta_1, \theta_2) P_{l'}(x) dx = B_{l,\tilde{l},l'}(\theta_1, \theta_2)$$

where $B_{l,\tilde{l},l'}(\theta_1, \theta_2)$ is given in the Remark 5.8 and then conclude.

We express the terms of $F_{l,\tilde{l}}(x, \theta_1, \theta_2)$ given in (5.37). Constructing an orthonormal frame in $\mathbb{R}^3$ with respect to x ($|x| \leq 1$) such that

$$\kappa_x = (x, \sqrt{1-x^2}, 0), \quad \kappa_x^1 = (-\sqrt{1-x^2}, x, 0), \quad \kappa_x^2 = (0, 0, 1),$$

we note from (5.7)

$$\kappa_x^\perp(\phi) = \kappa_x^1 \cos \phi + \kappa_x^2 \sin \phi = (-\sqrt{1-x^2} \cos \phi, x \cos \phi, \sin \phi).$$

One can verify that

$$\begin{aligned} & J(x)(\tau^1(\theta_1, \phi_1))^T \\ &= \begin{pmatrix} x & -\sqrt{1-x^2} & 0 \\ \sqrt{1-x^2} & x & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sin \theta_1 \\ -\cos \theta_1 \cos \phi_1 \\ -\cos \theta_1 \sin \phi_1 \end{pmatrix} \\ &= \begin{pmatrix} x \sin \theta_1 + \sqrt{1-x^2} \cos \theta_1 \cos \phi_1 \\ \sqrt{1-x^2} \sin \theta_1 - x \cos \theta_1 \cos \phi_1 \\ -\cos \theta_1 \sin \phi_1 \end{pmatrix} \\ &= \sin \theta_1 \kappa_x^T - \cos \theta_1 (\kappa_x^\perp(\phi_1))^T. \end{aligned}$$

Similary, we have

$$J(x)(\tau(\theta_1, \phi_1))^T = \cos \theta_1 \kappa_x^T + \sin \theta_1 (\kappa_x^\perp(\phi_1))^T.$$

Therefore, we have

$$\begin{aligned} (5.39) \quad & \tau^1(\theta_2, \phi_2) J(x)(\tau^1(\theta_1, \phi_1))^T = (\sin \theta_1 \kappa_x - \cos \theta_1 \kappa_x^\perp(\phi_1)) \cdot \tau^1(\theta_2, \phi_2) \\ & \tau(\theta_2, \phi_2) J(x)(\tau(\theta_1, \phi_1))^T = (\cos \theta_1 \kappa_x + \sin \theta_1 \kappa_x^\perp(\phi_1)) \cdot \tau(\theta_2, \phi_2). \end{aligned}$$

From lemma 5.9 and the above equality (5.39), we deduce

$$\begin{aligned} & \int_0^{2\pi} P_l(\tau^1(\theta_2, \phi_2) J(x)(\tau^1(\theta_1, \phi_1))^T) P_{\tilde{l}}(\tau(\theta_2, \phi_2) J(x)(\tau(\theta_1, \phi_1))^T) \frac{d\phi_1}{2\pi} \\ &= \sum_{0 \leq q \leq l} \sum_{0 \leq \tilde{q} \leq \tilde{l}} b_{l, \tilde{l}}^{q, \tilde{q}}(\theta_1) P_q(\kappa_x \cdot \tau^1(\theta_2, \phi_2)) P_{\tilde{q}}(\kappa_x \cdot \tau(\theta_2, \phi_2)). \end{aligned}$$

Therefore using (5.37)

$$\begin{aligned} & 2\pi \int_{-1}^1 F_{l, \tilde{l}}(x, \theta_1, \theta_2) P_{l'}(x) dx = \sum_{0 \leq q \leq l} \sum_{0 \leq \tilde{q} \leq \tilde{l}} b_{l, \tilde{l}}^{q, \tilde{q}}(\theta_1) \times \\ & 2\pi \int_{-1}^1 \int_0^{2\pi} P_q(\kappa_x \cdot \tau^1(\theta_2, \phi_2)) P_{\tilde{q}}(\kappa_x \cdot \tau(\theta_2, \phi_2)) P_{l'}(x) \frac{d\phi_2}{2\pi} dx. \end{aligned}$$

Applying the addition theorem (5.3)

$$\begin{aligned} P_{\tilde{q}}(\kappa_x \cdot \tau(\theta_2, \phi_2)) &= \sum_{|\tilde{k}| \leq \tilde{q}} a_{\tilde{q}, k} P_{\tilde{q}}^{|\tilde{k}|}(x) P_{\tilde{q}}^{|\tilde{k}|}(\cos \theta_2) e^{i\tilde{k}\phi_2}, \\ P_q(\kappa_x \cdot \tau^1(\theta_2, \phi_2)) &= P_q(x \sin \theta_2 + \sqrt{1-x^2} \cos \theta_2 \cos(\phi_2 + \pi)) \\ &= \sum_{|k| \leq q} a_{q, k} P_q^{|k|}(x) P_q^{|k|}(\sin \theta_2) e^{ik(\phi_2 + \pi)} \\ &= \sum_{|k| \leq q} a_{q, k} P_q^{|k|}(x) P_q^{|k|}(\sin \theta_2) e^{ik\phi_2} (-1)^k \end{aligned}$$

where we use $e^{ik\pi} = (\cos \pi + i \sin \pi)^k = (-1)^k$ in the last equality, from the remark 5.8, we find

$$\begin{aligned} 2\pi \int_{-1}^1 F_{l,\tilde{l}}(x, \theta_1, \theta_2) P_{l'}(x) dx &= \sum_{0 \leq q \leq l} \sum_{0 \leq \tilde{q} \leq \tilde{l}} b_{l,\tilde{l}}^{q,\tilde{q}}(\theta_1) \times \\ &\sum_{|k| \leq \min(q,\tilde{q})} a_{q,k} a_{\tilde{q},k} (-1)^k P_q^{[k]}(\sin \theta_2) P_{\tilde{q}}^{[k]}(\cos \theta_2) 2\pi \int_{-1}^1 P_q^{[k]}(x) P_{\tilde{q}}^{[k]}(x) P_{l'}(x) dx, \\ &= B_{l,\tilde{l},l'}(\theta_1, \theta_2). \end{aligned}$$

This ends the proof of the formula (5.36). $\square$

The following Proposition will provide a convenient expression to estimate the nonlinear eigenvalue $\mu_{n,\tilde{n},l,\tilde{l},k_1}^{m,\tilde{m},m'_1}$ in Section 6.

Proposition 5.12. *For $G_{n,\tilde{n},l,\tilde{l}}^{m,\tilde{m}}(\kappa)$ given in (5.22), and any integers $n, \tilde{n} \geq 0$, $|m| \leq l$, $|m'| \leq l'$, we have*

$$\begin{aligned} (5.40) \quad &\sum_{|m| \leq l} \sum_{|\tilde{m}| \leq \tilde{l}} \left| \int_{\mathbb{S}_\kappa^2} G_{n,\tilde{n},l,\tilde{l}}^{m,\tilde{m}}(\kappa) \overline{Y_{l'}^{m'}}(\kappa) d\kappa \right|^2 \\ &= \sum_{|q| \leq \min(l,\tilde{l})} \left(\left(\frac{4\pi}{2l'+1} \right)^{\frac{1}{2}} G_{n,\tilde{n},l,\tilde{l}}^{q,-q}(e_1) \right) \left(\int_{\mathbb{S}_\kappa^2} G_{n,\tilde{n},l,\tilde{l}}^{q,-q}(\kappa) \overline{Y_{l'}^0}(\kappa) d\kappa \right), \end{aligned}$$

where $e_1 = (1, 0, 0)$ and

$$\begin{aligned} G_{n,\tilde{n},l,\tilde{l}}^{q,-q}(e_1) &= (-1)^q \left(\frac{2l+1}{4\pi} \right)^{\frac{1}{2}} \left(\frac{2\tilde{l}+1}{4\pi} \right)^{\frac{1}{2}} \left(\frac{(l-|q|)!}{(l+|q|)!} \right)^{\frac{1}{2}} \left(\frac{(\tilde{l}-|q|)!}{(\tilde{l}+|q|)!} \right)^{\frac{1}{2}} \\ &\times \int_{|\theta| \leq \frac{\pi}{4}} \beta(\theta) (\sin \theta)^{2n+l} (\cos \theta)^{2\tilde{n}+\tilde{l}} P_l^{|q|}(\sin \theta) P_{\tilde{l}}^{|q|}(\cos \theta) d\theta. \end{aligned}$$

Proof. For $0 \leq k \leq \min(l, \tilde{l})$ and $|m'| \leq l'$, we deduce from (5.36) that,

$$\begin{aligned} (5.41) \quad &\mathbf{I} = \sum_{|m| \leq l} \sum_{|\tilde{m}| \leq \tilde{l}} \left| \int_{\mathbb{S}_\kappa^2} G_{n,\tilde{n},l,\tilde{l}}^{m,\tilde{m}}(\kappa) \overline{Y_{l'}^{m'}}(\kappa) d\kappa \right|^2 = \frac{2l+1}{4\pi} \frac{2\tilde{l}+1}{4\pi} \times \\ &\int_{|\theta_1| \leq \frac{\pi}{4}} \beta(\theta_1) (\sin \theta_1)^{2n+l} (\cos \theta_1)^{2\tilde{n}+\tilde{l}} \times \\ &\int_{|\theta_2| \leq \frac{\pi}{4}} \beta(\theta_2) (\sin \theta_2)^{2n+l} (\cos \theta_2)^{2\tilde{n}+\tilde{l}} \left(2\pi \int_{-1}^1 F_{l,\tilde{l}}(x, \theta_1, \theta_2) P_{l'}(x) dx \right) d\theta_2 d\theta_1, \end{aligned}$$

where $F_{l,\tilde{l}}(x, \theta_1, \theta_2)$ was defined in (5.37), such that

$$\begin{aligned} F_{l,\tilde{l}}(x, \theta_1, \theta_2) &= \int_0^{2\pi} \int_0^{2\pi} P_l(\tau^1(\theta_2, \phi_2) J(x) (\tau^1(\theta_1, \phi_1))^T) \\ &\times P_{\tilde{l}}(\tau(\theta_2, \phi_2) J(x) (\tau(\theta_1, \phi_1))^T) \frac{d\phi_1}{2\pi} \frac{d\phi_2}{2\pi}. \end{aligned}$$

We inherit the notation in Proposition 5.10, apply the formula (5.39) and the addition theorem (5.2), then

$$P_l(\tau^1(\theta_2, \phi_2)J(x)(\tau^1(\theta_1, \phi_1))^T) = \frac{4\pi}{2l+1} \sum_{q=-l}^l Y_l^q(\sin \theta_1 \kappa_x - \cos \theta_1 \kappa_x^\perp(\phi_1)) Y_l^{-q}(\tau^1(\theta_2, \phi_2))$$

$$P_{\tilde{l}}(\tau(\theta_2, \phi_2)J(x)(\tau(\theta_1, \phi_1))^T) = \frac{4\pi}{2\tilde{l}+1} \sum_{\tilde{q}=-\tilde{l}}^{\tilde{l}} Y_{\tilde{l}}^{\tilde{q}}(\cos \theta_1 \kappa_x + \sin \theta_1 \kappa_x^\perp(\phi_1)) Y_{\tilde{l}}^{-\tilde{q}}(\tau(\theta_2, \phi_2)).$$

Since

$$Y_l^{-q}(\tau^1(\theta_2, \phi_2)) = \left(\frac{2l+1}{4\pi}\right)^{\frac{1}{2}} \left(\frac{(l-|q|)!}{(l+|q|)!}\right)^{\frac{1}{2}} P_l^{|q|}(\sin \theta_2) e^{-iq(\phi_2+\pi)}$$

$$Y_{\tilde{l}}^{-\tilde{q}}(\tau(\theta_2, \phi_2)) = \left(\frac{2\tilde{l}+1}{4\pi}\right)^{\frac{1}{2}} \left(\frac{(\tilde{l}-|\tilde{q}|)!}{(\tilde{l}+|\tilde{q}|)!}\right)^{\frac{1}{2}} P_{\tilde{l}}^{|\tilde{q}|}(\cos \theta_2) e^{-i\tilde{q}\phi_2}$$

we find that

$$F_{l,\tilde{l}}(x, \theta_1, \theta_2)$$

$$= \sum_{|q| \leq \min(l, \tilde{l})} \left(\frac{4\pi}{2l+1}\right)^{\frac{1}{2}} \left(\frac{4\pi}{2\tilde{l}+1}\right)^{\frac{1}{2}} \left(\frac{(l-|q|)!}{(l+|q|)!}\right)^{\frac{1}{2}} \left(\frac{(\tilde{l}-|q|)!}{(\tilde{l}+|q|)!}\right)^{\frac{1}{2}} P_l^{|q|}(\sin \theta_2) (-1)^q$$

$$\times P_{\tilde{l}}^{|q|}(\cos \theta_2) \int_0^{2\pi} Y_l^q(\sin \theta_1 \kappa_x - \cos \theta_1 \kappa_x^\perp(\phi_1)) Y_{\tilde{l}}^{-q}(\cos \theta_1 \kappa_x + \sin \theta_1 \kappa_x^\perp(\phi_1)) \frac{d\phi_1}{2\pi}.$$

We plug the previous relation into (5.41) and we get

$$\mathbf{I} = \sum_{|q| \leq \min(l, \tilde{l})} \int_{|\theta_2| \leq \frac{\pi}{4}} \beta(\theta_2) (\sin \theta_2)^{2n+l} (\cos \theta_2)^{2\tilde{n}+\tilde{l}} P_l^{|q|}(\sin \theta_2) P_{\tilde{l}}^{|q|}(\cos \theta_2) d\theta_2$$

$$\times (-1)^q \left(\frac{2l+1}{4\pi}\right)^{\frac{1}{2}} \left(\frac{2\tilde{l}+1}{4\pi}\right)^{\frac{1}{2}} \left(\frac{(l-|q|)!}{(l+|q|)!}\right)^{\frac{1}{2}} \left(\frac{(\tilde{l}-|q|)!}{(\tilde{l}+|q|)!}\right)^{\frac{1}{2}}$$

$$\times \int_{|\theta_1| \leq \frac{\pi}{4}} \beta(\theta_1) (\sin \theta_1)^{2n+l} (\cos \theta_1)^{2\tilde{n}+\tilde{l}} \times \int_0^{2\pi} 2\pi \int_{-1}^1$$

$$Y_l^q(\sin \theta_1 \kappa_x - \cos \theta_1 \kappa_x^\perp(\phi_1)) Y_{\tilde{l}}^{-q}(\cos \theta_1 \kappa_x + \sin \theta_1 \kappa_x^\perp(\phi_1)) P_{l'}(x) dx \frac{d\phi_1}{2\pi} d\theta_1.$$

On one hand, from (5.23)

$$G_{n,\tilde{n},l,\tilde{l}}^{q,-q}(e_1) = \int_{|\theta_2| \leq \frac{\pi}{4}} \beta(\theta_2) (\sin \theta_2)^{2n+l} (\cos \theta_2)^{2\tilde{n}+\tilde{l}} \times$$

$$\int_0^{2\pi} Y_l^q(e_1 \sin \theta_2 - e_1^\perp(\phi_2) \cos \theta_2) Y_{\tilde{l}}^{-q}(e_1 \cos \theta_2 + e_1^\perp(\phi_2) \sin \theta_2) \frac{d\phi_2}{2\pi} d\theta_2$$

$$= (-1)^q \left(\frac{2l+1}{4\pi}\right)^{\frac{1}{2}} \left(\frac{2\tilde{l}+1}{4\pi}\right)^{\frac{1}{2}} \left(\frac{(l-|q|)!}{(l+|q|)!}\right)^{\frac{1}{2}} \left(\frac{(\tilde{l}-|q|)!}{(\tilde{l}+|q|)!}\right)^{\frac{1}{2}}$$

$$\times \int_{|\theta_2| \leq \frac{\pi}{4}} \beta(\theta_2) (\sin \theta_2)^{2n+l} (\cos \theta_2)^{2\tilde{n}+\tilde{l}} P_l^{|q|}(\sin \theta_2) P_{\tilde{l}}^{|q|}(\cos \theta_2) d\theta_2.$$

On the other hand, from (5.24) and from (5.42) of the next lemma 5.13,

$$\begin{aligned}
\int_{\mathbb{S}_x^2} G_{n,\tilde{n},l,\tilde{l}}^{q,-q}(\kappa) \overline{Y_{\nu'}^0}(\kappa) d\kappa &= \int_{|\theta_1| \leq \frac{\pi}{4}} \beta(\theta_1) (\sin \theta_1)^{2n+l} (\cos \theta_1)^{2\tilde{n}+\tilde{l}} \int_0^{2\pi} \int_{\mathbb{S}_x^2} \\
&Y_l^q(\kappa \sin \theta_1 - \kappa^\perp(\phi_1) \cos \theta_1) Y_{\tilde{l}}^{-q}(\kappa \cos \theta_1 + \kappa^\perp(\phi_1) \sin \theta_1) \overline{Y_{\nu'}^0}(\kappa) d\kappa \frac{d\phi_1}{2\pi} d\theta_1 \\
&= \left(\frac{2l' + 1}{4\pi} \right)^{\frac{1}{2}} \int_{|\theta_1| \leq \frac{\pi}{4}} \beta(\theta_1) (\sin \theta_1)^{2n+l} (\cos \theta_1)^{2\tilde{n}+\tilde{l}} \times \int_0^{2\pi} 2\pi \int_{-1}^1 \\
&Y_l^q(\sin \theta_1 \kappa_x - \cos \theta_1 \kappa_x^\perp(\phi_1)) Y_{\tilde{l}}^{-q}(\cos \theta_1 \kappa_x + \sin \theta_1 \kappa_x^\perp(\phi_1)) P_{l'}(x) dx \frac{d\phi_1}{2\pi} d\theta_1.
\end{aligned}$$

Combining the three previous relations leads to (5.40), and this concludes the proof of the Proposition. $\square$

We now prove the following technical lemma.

Lemma 5.13. *For any integers $l, \tilde{l}, l' \geq 0$ and $|q| \leq l$, we have*

$$\begin{aligned}
(5.42) \quad &\left(\frac{4\pi}{2l' + 1} \right)^{\frac{1}{2}} \int_{\mathbb{S}_x^2} Y_l^q(\kappa \sin \theta_1 - \kappa^\perp(\phi_1) \cos \theta_1) Y_{\tilde{l}}^{-q}(\kappa \cos \theta_1 + \kappa^\perp(\phi_1) \sin \theta_1) \overline{Y_{\nu'}^0}(\kappa) d\kappa \\
&= 2\pi \int_{-1}^1 Y_l^q(\sin \theta_1 \kappa_x - \cos \theta_1 \kappa_x^\perp(\phi_1)) Y_{\tilde{l}}^{-q}(\cos \theta_1 \kappa_x + \sin \theta_1 \kappa_x^\perp(\phi_1)) P_{l'}(x) dx.
\end{aligned}$$

Proof. We consider

$$\mathbf{I} = \int_{\mathbb{S}_x^2} Y_l^q(\kappa \sin \theta_1 - \kappa^\perp(\phi_1) \cos \theta_1) Y_{\tilde{l}}^{-q}(\kappa \cos \theta_1 + \kappa^\perp(\phi_1) \sin \theta_1) \overline{Y_{\nu'}^0}(\kappa) d\kappa.$$

From (5.1) we have

$$\begin{aligned}
Y_l^q(\kappa \sin \theta_1 - \kappa^\perp(\phi_1) \cos \theta_1) &= N_{l,q} \left(\frac{d^{|q|} P_l}{dx^{|q|}} \right) (\sigma_1^-) (\sigma_2^- + i \operatorname{sgn}(q) \sigma_3^-)^{|q|}, \\
Y_{\tilde{l}}^{-q}(\kappa \cos \theta_1 + \kappa^\perp(\phi_1) \sin \theta_1) &= N_{\tilde{l},q} \left(\frac{d^{|q|} P_{\tilde{l}}}{dx^{|q|}} \right) (\sigma_1^+) (\sigma_2^+ - i \operatorname{sgn}(q) \sigma_3^+)^{|q|},
\end{aligned}$$

where

$$\begin{aligned}
\sigma^- &= \kappa \sin \theta_1 - \kappa^\perp(\phi_1) \cos \theta_1 = (\sigma_1^-, \sigma_2^-, \sigma_3^-) \\
\sigma^+ &= \kappa \cos \theta_1 + \kappa^\perp(\phi_1) \sin \theta_1 = (\sigma_1^+, \sigma_2^+, \sigma_3^+).
\end{aligned}$$

Noting $\kappa = (\cos \theta, \sin \theta \cos \phi, \sin \theta \sin \phi)$ with $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$, and

$$\kappa^1 = (-\sin \theta, \cos \theta \cos \phi, \cos \theta \sin \phi), \quad \kappa^2 = (0, \sin \phi, -\cos \phi),$$

and using the definition (5.7) of $\kappa^\perp(\phi_1)$, we have

$$\begin{aligned}
\sigma_1^- &= \cos \theta \sin \theta_1 + \sin \theta \cos \theta_1 \cos \phi_1, \\
\sigma_2^- &= \sin \theta \cos \phi \sin \theta_1 - \cos \theta \cos \phi \cos \theta_1 \cos \phi_1 - \sin \phi \cos \theta_1 \sin \phi_1, \\
\sigma_3^- &= \sin \theta \sin \phi \sin \theta_1 - \cos \theta \sin \phi \cos \theta_1 \cos \phi_1 + \cos \phi \cos \theta_1 \sin \phi_1,
\end{aligned}$$

and

$$\begin{aligned}\sigma_1^+ &= \cos \theta \cos \theta_1 - \sin \theta \sin \theta_1 \cos \phi_1, \\ \sigma_2^+ &= \sin \theta \cos \phi \cos \theta_1 + \cos \theta \cos \phi \sin \theta_1 \cos \phi_1 + \sin \phi \sin \theta_1 \sin \phi_1, \\ \sigma_3^+ &= \sin \theta \sin \phi \cos \theta_1 + \cos \theta \sin \phi \sin \theta_1 \cos \phi_1 - \cos \phi \sin \theta_1 \sin \phi_1.\end{aligned}$$

Therefore

$$\begin{aligned}(\sigma_2^- + i \sigma_3^-) &= (\sin \theta \cos \phi \sin \theta_1 - \cos \theta \cos \phi \cos \theta_1 \cos \phi_1 - \sin \phi \cos \theta_1 \sin \phi_1) \\ &\quad + i(\sin \theta \sin \phi \sin \theta_1 - \cos \theta \sin \phi \cos \theta_1 \cos \phi_1 + \cos \phi \cos \theta_1 \sin \phi_1) \\ &= (\sin \theta \sin \theta_1 - \cos \theta \cos \theta_1 \cos \phi_1) e^{i\phi} + i \cos \theta_1 \sin \phi_1 e^{i\phi} \\ &= (\sin \theta \sin \theta_1 - \cos \theta \cos \theta_1 \cos \phi_1 + i \cos \theta_1 \sin \phi_1) e^{i\phi}; \\ (\sigma_2^+ - i \sigma_3^+) &= (\sin \theta \cos \phi \cos \theta_1 + \cos \theta \cos \phi \sin \theta_1 \cos \phi_1 + \sin \phi \sin \theta_1 \sin \phi_1) \\ &\quad - i(\sin \theta \sin \phi \cos \theta_1 + \cos \theta \sin \phi \sin \theta_1 \cos \phi_1 - \cos \phi \sin \theta_1 \sin \phi_1) \\ &= (\sin \theta \cos \theta_1 + \cos \theta \sin \theta_1 \cos \phi_1 + i \sin \theta_1 \sin \phi_1) e^{-i\phi}.\end{aligned}$$

Direct computations lead to

$$\begin{aligned}(\sigma_2^- + i \sigma_3^-)(\sigma_2^+ - i \sigma_3^+) &= (\sin \theta \sin \theta_1 - \cos \theta \cos \theta_1 \cos \phi_1 + i \cos \theta_1 \sin \phi_1) \\ &\quad \times (\sin \theta \cos \theta_1 + \cos \theta \sin \theta_1 \cos \phi_1 + i \sin \theta_1 \sin \phi_1),\end{aligned}$$

which does not depend of ϕ . Since $\sigma_1^\pm$ do not depend also on ϕ , we get with the change of variable $x = \cos \theta$

$$\begin{aligned}\mathbf{I} &= 2\pi \int_0^\pi Y_l^q(\kappa_{\theta,0} \sin \theta_1 - \kappa_{\theta,0}^\perp(\phi_1) \cos \theta_1) \\ &\quad Y_l^{-q}(\kappa_{\theta,0} \cos \theta_1 + \kappa_{\theta,0}^\perp(\phi_1) \sin \theta_1) \overline{Y_{l'}^0(\kappa_{\theta,0})} \sin \theta d\theta \\ &= 2\pi \int_{-1}^1 Y_l^q(\sin \theta_1 \kappa_x - \cos \theta_1 \kappa_x^\perp(\phi_1)) Y_l^{-q}(\cos \theta_1 \kappa_x + \sin \theta_1 \kappa_x^\perp(\phi_1)) \overline{Y_{l'}^0(\kappa_x)} dx \\ &= 2\pi \left(\frac{2l'+1}{4\pi} \right)^{\frac{1}{2}} \int_{-1}^1 Y_l^q(\sin \theta_1 \kappa_x - \cos \theta_1 \kappa_x^\perp(\phi_1)) Y_l^{-q}(\cos \theta_1 \kappa_x + \sin \theta_1 \kappa_x^\perp(\phi_1)) P_{l'}(x) dx.\end{aligned}$$

This concludes the proof of Lemma 5.13 and Proposition 5.12. $\square$

6. THE ESTIMATES OF THE NON LINEAR EIGENVALUES

In this section, we prove Proposition 3.1, we need the following fundamental result of Gamma function. It is well known of the stirling's formula (see 12.33 of Chap. XII in [25], [18]) that,

$$\Gamma(x+1) = \sqrt{2\pi x} \left(\frac{x}{e} \right)^x e^{\frac{\nu(x)}{12x}}, \text{ for } x \geq 1,$$

where $0 < \nu(x) < 1$. Therefore we derive directly the following useful estimate.

Let a, b be two fixed constant, for any $x > 0$, with $|b-a| \leq x+b, x+a \geq 1, x+b \geq 1$, we have

$$(6.1) \quad \frac{\Gamma(x+a+1)}{\Gamma(x+b+1)} \leq C_{a,b} (x+a)^{a-b},$$

where $C_{a,b}$ is dependent only on a, b . We also recall the definition of the Beta function

$$(6.2) \quad B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

6.1. The estimate for the radially symmetric terms. We first give the estimate of $|\lambda_{n,\tilde{n},\tilde{l}}^{rad,1}|^2$, and $|\lambda_{n,\tilde{n},l}^{rad,2}|^2$, which is 1), 2) in Proposition 3.1. Recall that

$$\begin{aligned} \lambda_{n,\tilde{n},\tilde{l}}^{rad,1} &= \frac{1}{\sqrt{4\pi}} \frac{A_{\tilde{n},\tilde{l}} A_{n,0}}{A_{n+\tilde{n},\tilde{l}}} \int_{|\theta| \leq \frac{\pi}{4}} \beta(\theta) (\sin \theta)^{2n} (\cos \theta)^{2\tilde{n}+\tilde{l}} P_{\tilde{l}}(\cos \theta) d\theta, \\ \lambda_{n,\tilde{n},l}^{rad,2} &= \frac{1}{\sqrt{4\pi}} \frac{A_{\tilde{n},0} A_{n,l}}{A_{n+\tilde{n},l}} \int_{|\theta| \leq \frac{\pi}{4}} \beta(\theta) (\sin \theta)^{2n+l} (\cos \theta)^{2\tilde{n}} P_l(\sin \theta) d\theta \end{aligned}$$

where

$$A_{n,l} = (-i)^l (2\pi)^{\frac{3}{4}} \left(\frac{1}{\sqrt{2} n! \Gamma(n+l+\frac{3}{2})} \right)^{\frac{1}{2}}.$$

Lemma 6.1. For $n \geq 1, \tilde{n}, \tilde{l} \in \mathbb{N}$,

$$(6.3) \quad |\lambda_{n,\tilde{n},\tilde{l}}^{rad,1}|^2 \lesssim \tilde{n}^s (\tilde{n} + \tilde{l})^s n^{-\frac{5}{2}-2s}.$$

For all $\tilde{n} \geq 1, n, l \in \mathbb{N}, n+l \geq 2$,

$$(6.4) \quad |\lambda_{n,\tilde{n},l}^{rad,2}|^2 \lesssim \frac{\tilde{n}^{2s}}{(n+1)^s (n+l)^{\frac{5}{2}+s}}.$$

Proof. We estimate $|\lambda_{n,\tilde{n},\tilde{l}}^{rad,1}|^2$. From the assumption on $\beta(\theta)$

$$\beta(\theta) \approx |\sin \theta|^{-1-2s},$$

and the inequality $|P_{\tilde{l}}(x)| \leq 1$ for any $|x| \leq 1$, we have,

$$|\lambda_{n,\tilde{n},\tilde{l}}^{rad,1}|^2 \lesssim \frac{(n+\tilde{n})! \Gamma(n+\tilde{n}+\tilde{l}+\frac{3}{2})}{n! \tilde{n}! \Gamma(n+\frac{3}{2}) \Gamma(\tilde{n}+\tilde{l}+\frac{3}{2})} \left(\int_0^{\frac{\pi}{4}} (\sin \theta)^{2n-1-2s} (\cos \theta)^{2\tilde{n}+\tilde{l}} d\theta \right)^2.$$

Using the Cauchy-Schwarz inequality and the Beta Function (6.2), we derive that

$$\begin{aligned} & \left(\int_0^{\frac{\pi}{4}} (\sin \theta)^{2n-1-2s} (\cos \theta)^{2\tilde{n}+\tilde{l}} d\theta \right)^2 = \frac{1}{4} \left(\int_0^{\frac{1}{2}} t^{n-1-s} (1-t)^{\tilde{n}+\frac{\tilde{l}}{2}-\frac{1}{2}} dt \right)^2 \\ & \leq \frac{1}{4} \left(\int_0^{\frac{1}{2}} t^{n-1-s} (1-t)^{\tilde{n}+s} dt \right) \times \left(\int_0^{\frac{1}{2}} t^{n-1-s} (1-t)^{\tilde{n}+\tilde{l}-1-s} dt \right) \\ & \leq \frac{1}{4} \left(\int_0^{\frac{1}{2}} t^{n-1-s} (1-t)^{\tilde{n}+s} dt \right) \times 2^{\frac{3}{2}+2s} \left(\int_0^{\frac{1}{2}} t^{n-1-s} (1-t)^{\tilde{n}+\tilde{l}+\frac{1}{2}+s} dt \right) \\ & \lesssim \frac{(\Gamma(n-s))^2 \Gamma(\tilde{n}+1+s) \Gamma(\tilde{n}+\tilde{l}+\frac{3}{2}+s)}{(n+\tilde{n})! \Gamma(n+\tilde{n}+\tilde{l}+\frac{3}{2})}. \end{aligned}$$

Then,

$$\begin{aligned}
|\lambda_{n,\tilde{n},\tilde{l}}^{rad,1}|^2 &\lesssim \frac{(n+\tilde{n})!\Gamma(n+\tilde{n}+\tilde{l}+\frac{3}{2})}{n!\tilde{n}!\Gamma(n+\frac{3}{2})\Gamma(\tilde{n}+\tilde{l}+\frac{3}{2})} \\
(6.5) \quad &\times \frac{(\Gamma(n-s))^2\Gamma(\tilde{n}+1+s)\Gamma(\tilde{n}+\tilde{l}+\frac{3}{2}+s)}{(n+\tilde{n})!\Gamma(n+\tilde{n}+\tilde{l}+\frac{3}{2})} \\
&\lesssim \frac{\Gamma(n-s)\Gamma(n-s)\Gamma(\tilde{n}+1+s)\Gamma(\tilde{n}+\tilde{l}+\frac{3}{2}+s)}{n!\Gamma(n+\frac{3}{2})\tilde{n}!\Gamma(\tilde{n}+\tilde{l}+\frac{3}{2})}.
\end{aligned}$$

We deduce from the formula (6.1) with $x = n$, $a = -s$, $b = 0$,

$$\frac{\Gamma(n-s+1)}{n!} \lesssim \frac{1}{(n-s)^s}.$$

From the following recurrence formula for the Gamma function

$$\Gamma(n+1-s) = (n-s)\Gamma(n-s),$$

we obtain

$$\frac{\Gamma(n-s)}{n!} = \frac{1}{(n-s)} \frac{\Gamma(n+1-s)}{\Gamma(n+1)} \lesssim \frac{1}{(n-s)^{s+1}} \lesssim \frac{1}{n^{1+s}}.$$

Using $x = n$, $a = -s$, $b = \frac{1}{2}$ in (6.1),

$$\frac{\Gamma(n+1-s)}{\Gamma(n+\frac{3}{2})} \lesssim \frac{1}{n^{\frac{1}{2}+s}},$$

and the recurrence formula $\Gamma(n+1-s) = (n-s)\Gamma(n-s)$,

$$\frac{\Gamma(n-s)}{\Gamma(n+\frac{3}{2})} = \frac{1}{n-s} \frac{\Gamma(n+1-s)}{\Gamma(n+\frac{3}{2})} \lesssim \frac{1}{(n-s)n^{\frac{1}{2}+s}} \lesssim \frac{1}{n^{\frac{3}{2}+s}}.$$

Using $x = \tilde{n}+1$, $a = s$, $b = 0$ in (6.1), we have

$$\frac{\Gamma(\tilde{n}+1+s)}{\tilde{n}!} = \frac{\tilde{n}+1}{\tilde{n}+1+s} \frac{\Gamma(\tilde{n}+2+s)}{(\tilde{n}+1)!} \lesssim \tilde{n}^s.$$

Using $x = \tilde{n}+\tilde{l}+\frac{1}{2}$, $a = s$, $b = 0$,

$$\frac{\Gamma(\tilde{n}+\tilde{l}+\frac{3}{2}+s)}{\Gamma(\tilde{n}+\tilde{l}+\frac{3}{2})} \lesssim (\tilde{n}+\tilde{l}+\frac{1}{2}+s)^s \lesssim (\tilde{n}+\tilde{l})^s.$$

Substitute the previous estimates into (6.5), we obtain

$$|\lambda_{n,\tilde{n},\tilde{l}}^{rad,1}|^2 \lesssim \tilde{n}^s (\tilde{n}+\tilde{l})^s n^{-\frac{5}{2}-2s}.$$

This is the formula of (6.3)

Analogously, for the term $|\lambda_{n,\tilde{n},l}^{rad,2}|^2$, we use the Cauchy-Schwarz inequality

$$\begin{aligned}
|\lambda_{n,\tilde{n},l}^{rad,2}|^2 &\approx \frac{(n+\tilde{n})!\Gamma(n+\tilde{n}+l+\frac{3}{2})}{n!\tilde{n}!\Gamma(n+l+\frac{3}{2})\Gamma(\tilde{n}+\frac{3}{2})} \left(\int_0^{\frac{\pi}{4}} (\sin \theta)^{2n+l-1-2s} (\cos \theta)^{2\tilde{n}} d\theta \right)^2 \\
&= \frac{1}{4} \frac{(n+\tilde{n})!\Gamma(n+\tilde{n}+l+\frac{3}{2})}{n!\tilde{n}!\Gamma(n+l+\frac{3}{2})\Gamma(\tilde{n}+\frac{3}{2})} \left(\int_0^{\frac{1}{2}} t^{n+\frac{l}{2}-1-s} (1-t)^{\tilde{n}-\frac{1}{2}} dt \right)^2 \\
&\lesssim \frac{(n+\tilde{n})!\Gamma(n+\tilde{n}+l+\frac{3}{2})}{n!\tilde{n}!\Gamma(n+l+\frac{3}{2})\Gamma(\tilde{n}+\frac{3}{2})} \\
&\quad \times \left(\int_0^{\frac{1}{2}} t^{n+l-2-s} (1-t)^{\tilde{n}+\frac{3}{2}+s} dt \right) \times \left(\int_0^{\frac{1}{2}} t^{n-s} (1-t)^{\tilde{n}+s} dt \right) \\
&\leq \frac{(n+\tilde{n})!\Gamma(n+\tilde{n}+l+\frac{3}{2})}{n!\tilde{n}!\Gamma(n+l+\frac{3}{2})\Gamma(\tilde{n}+\frac{3}{2})} \\
&\quad \times \left(\frac{\Gamma(n+l-1-s)\Gamma(\tilde{n}+\frac{5}{2}+s)}{\Gamma(n+\tilde{n}+l+\frac{3}{2})} \right) \times \left(\frac{\Gamma(n+1-s)\Gamma(\tilde{n}+1+s)}{\Gamma(\tilde{n}+n+2)} \right) \\
&\lesssim \frac{\Gamma(n+1-s)\Gamma(n+l-1-s)\Gamma(\tilde{n}+1+s)\Gamma(\tilde{n}+\frac{5}{2}+s)}{(n+\tilde{n}+1)n!\Gamma(n+l+\frac{3}{2})\tilde{n}!\Gamma(\tilde{n}+\frac{3}{2})}.
\end{aligned}$$

We deduce from the formula (6.1) with $x = n+1 \geq 2$, $a = -s$, $b = 0$,

$$\frac{\Gamma(n+2-s)}{(n+1)!} \lesssim \frac{1}{(n+1-s)^s} \lesssim \frac{1}{(n+1)^s}.$$

This inequality is also right for $n = 0$ (indeed, $\frac{\Gamma(n+2-s)}{(n+1)!} \big|_{n=0} \lesssim 1$). By using the recurrence formula $\Gamma(n+2-s) = (n+1-s)\Gamma(n+1-s)$, we conclude that

$$\frac{\Gamma(n+1-s)}{n!} = \frac{n+1}{n+1-s} \frac{\Gamma(n+2-s)}{(n+1)!} \lesssim \frac{1}{(n+1)^s}.$$

Consider the assumption that $n, l \in \mathbb{N}$, $n+l \geq 2$, using the formula (6.1) with $x = n+l$, $a = -s$, $b = \frac{1}{2}$,

$$\frac{\Gamma(n+l+1-s)}{\Gamma(n+l+\frac{3}{2})} \lesssim \frac{1}{(n+l-s)^{\frac{1}{2}+s}}.$$

By using the recurrence formula

$$\Gamma(n+l+1-s) = (n+l-s)(n+l-1-s)\Gamma(n+l-1-s),$$

we obtain,

$$\frac{\Gamma(n+l-1-s)}{\Gamma(n+l+\frac{3}{2})} = \frac{1}{(n+l-s)(n+l-1-s)} \frac{\Gamma(n+l+1-s)}{\Gamma(n+l+\frac{3}{2})} \lesssim \frac{1}{(n+l)^{\frac{5}{2}+s}}.$$

Finally, under the assumption of $\tilde{n} \geq 1$, we use again (6.1) with $x = \tilde{n}$, $a = s$, $b = 0$, then

$$\frac{\Gamma(\tilde{n}+1+s)}{\tilde{n}!} \lesssim (\tilde{n}+s)^s \lesssim \tilde{n}^s.$$

Using the formula (6.1) with $x = \tilde{n}$, $a = \frac{3}{2} + s$, $b = \frac{1}{2}$, we obtain,

$$\frac{\Gamma(\tilde{n}+\frac{5}{2}+s)}{\Gamma(\tilde{n}+\frac{3}{2})} \lesssim (\tilde{n}+\frac{3}{2}+s)^{1+s} \lesssim \tilde{n}^{1+s}.$$

Therefore, under the assumption of $\tilde{n} \geq 1, n, l \in \mathbb{N}, n + l \geq 2$, we conclude

$$|\lambda_{n,\tilde{n},l}^{rad,2}|^2 \lesssim \frac{\tilde{n}^{1+2s}}{(n+1)^s(n+l)^{\frac{s}{2}+s}(n+\tilde{n}+1)} \leq \frac{\tilde{n}^{2s}}{(n+1)^s(n+l)^{\frac{s}{2}+s}}.$$

This ends the proof of (6.4). $\square$

6.2. The estimate for the general terms. In the proof of 3) in Proposition 3.1, we need the following technical Lemma. Recall that

$$A_{n,l} = (-i)^l (2\pi)^{\frac{3}{4}} \left(\frac{1}{\sqrt{2}n! \Gamma(n+l+\frac{3}{2})} \right)^{\frac{1}{2}},$$

and

$$\begin{aligned} G_{n,\tilde{n},l,\tilde{l}}^{m,\tilde{m}}(\kappa) &= \int_{\mathbb{S}^2} b(\kappa \cdot \sigma) \left(|\kappa - \sigma|/2 \right)^{2n+l} \left(|\kappa + \sigma|/2 \right)^{2\tilde{n}+\tilde{l}} \\ &\quad \times Y_l^m \left(\frac{\kappa - \sigma}{|\kappa - \sigma|} \right) Y_{\tilde{l}}^{\tilde{m}} \left(\frac{\kappa + \sigma}{|\kappa + \sigma|} \right) d\sigma. \end{aligned}$$

Then recalling the notation in Proposition 2.1, we have

$$\mu_{n,\tilde{n},l,\tilde{l},k}^{m,\tilde{m},m'} = \frac{A_{\tilde{n},\tilde{l}} A_{n,l}}{A_{n+\tilde{n}+k,l+\tilde{l}-2k}} \left(\int_{S_k^2} G_{n,\tilde{n},l,\tilde{l}}^{m,\tilde{m}}(\kappa) \overline{Y_{l+\tilde{l}-2k}^{m'}(\kappa)} d\kappa \right),$$

It follows that,

$$\begin{aligned} \sum_{|m| \leq l} \sum_{|\tilde{m}| \leq \tilde{l}} |\mu_{n,\tilde{n},l,\tilde{l},k}^{m,\tilde{m},m'}|^2 &= \left| \frac{A_{\tilde{n},\tilde{l}} A_{n,l}}{A_{n+\tilde{n}+k,l+\tilde{l}-2k}} \right|^2 \\ (6.6) \quad &\times \sum_{|m| \leq l} \sum_{|\tilde{m}| \leq \tilde{l}} \left| \int_{S_k^2} G_{n,\tilde{n},l,\tilde{l}}^{m,\tilde{m}}(\kappa) \overline{Y_{l+\tilde{l}-2k}^{m'}(\kappa)} d\kappa \right|^2. \end{aligned}$$

In the next Lemma we estimate

$$\sum_{|m| \leq l} \sum_{|\tilde{m}| \leq \tilde{l}} \left| \int_{\mathbb{S}_k^2} G_{n,\tilde{n},l,\tilde{l}}^{m,\tilde{m}}(\kappa) \overline{Y_{l+\tilde{l}-2k}^{m'}(\kappa)} d\kappa \right|^2.$$

Lemma 6.2. For $0 \leq k \leq \min(l, \tilde{l})$, $|m'| \leq l + \tilde{l} - 2k$, we have

$$\begin{aligned} \sum_{|m| \leq l} \sum_{|\tilde{m}| \leq \tilde{l}} \left| \int_{\mathbb{S}_k^2} G_{n,\tilde{n},l,\tilde{l}}^{m,\tilde{m}}(\kappa) \overline{Y_{l+\tilde{l}-2k}^{m'}(\kappa)} d\kappa \right|^2 \\ (6.7) \quad \lesssim \frac{\tilde{l} \sqrt{\tilde{l}}}{l + \tilde{l} - 2k + 1} \left(\int_0^{\frac{\pi}{4}} \beta(\theta) (\sin \theta)^{2n+l} (\cos \theta)^{2\tilde{n}+\tilde{l}} d\theta \right)^2. \end{aligned}$$

Proof. For $0 \leq k \leq \min(l, \tilde{l})$ and $|m'| \leq l + \tilde{l} - 2k$, we deduce from Proposition 5.12 that

$$\begin{aligned} \sum_{|m| \leq l} \sum_{|\tilde{m}| \leq \tilde{l}} \left| \int_{\mathbb{S}_k^2} G_{n,\tilde{n},l,\tilde{l}}^{m,\tilde{m}}(\kappa) \overline{Y_{l+\tilde{l}-2k}^{m'}(\kappa)} d\kappa \right|^2 \\ = \sum_{|q| \leq \min(l, \tilde{l})} \left(\left(\frac{4\pi}{2(l + \tilde{l} - 2k) + 1} \right)^{\frac{1}{2}} G_{n,\tilde{n},l,\tilde{l}}^{q,-q}(e_1) \right) \left(\int_{\mathbb{S}_k^2} G_{n,\tilde{n},l,\tilde{l}}^{q,-q}(\kappa) \overline{Y_{l+\tilde{l}-2k}^0(\kappa)} d\kappa \right), \end{aligned}$$

where $e_1 = (1, 0, 0)$ and

$$\begin{aligned} G_{n,\tilde{n},l,\tilde{l}}^{q,-q}(e_1) &= (-1)^q \left(\frac{2l+1}{4\pi} \right)^{\frac{1}{2}} \left(\frac{2\tilde{l}+1}{4\pi} \right)^{\frac{1}{2}} \left(\frac{(l-|q|)!}{(l+|q|)!} \right)^{\frac{1}{2}} \left(\frac{(\tilde{l}-|q|)!}{(\tilde{l}+|q|)!} \right)^{\frac{1}{2}} \\ &\quad \times \int_{|\theta| \leq \frac{\pi}{4}} \beta(\theta) (\sin \theta)^{2n+l} (\cos \theta)^{2\tilde{n}+\tilde{l}} P_l^{|q|}(\sin \theta) P_{\tilde{l}}^{|q|}(\cos \theta) d\theta. \end{aligned}$$

It follows from the Cauchy-Schwarz inequality that

$$\begin{aligned} &\sum_{|m| \leq l} \sum_{|\tilde{m}| \leq \tilde{l}} \left| \int_{\mathbb{S}_k^2} G_{n,\tilde{n},l,\tilde{l}}^{m,\tilde{m}}(\kappa) \overline{Y_{l+\tilde{l}-2k}^{m'}}(\kappa) d\kappa \right|^2 \\ &\leq \left(\sum_{|q| \leq \min(l,\tilde{l})} \frac{4\pi}{2(l+\tilde{l}-2k)+1} \left| G_{n,\tilde{n},l,\tilde{l}}^{q,-q}(e_1) \right|^2 \right)^{\frac{1}{2}} \left(\sum_{|q| \leq \min(l,\tilde{l})} \left| \int_{\mathbb{S}_k^2} G_{n,\tilde{n},l,\tilde{l}}^{q,-q}(\kappa) \overline{Y_{l+\tilde{l}-2k}^0}(\kappa) d\kappa \right|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

We observe from (5.38) that, for any $|m'| \leq l + \tilde{l} - 2k$,

$$\sum_{|m| \leq l} \sum_{|\tilde{m}| \leq \tilde{l}} \left| \int_{\mathbb{S}_k^2} G_{n,\tilde{n},l,\tilde{l}}^{m,\tilde{m}}(\kappa) \overline{Y_{l+\tilde{l}-2k}^{m'}}(\kappa) d\kappa \right|^2 = \sum_{|q| \leq \min(l,\tilde{l})} \left| \int_{\mathbb{S}_k^2} G_{n,\tilde{n},l,\tilde{l}}^{q,-q}(\kappa) \overline{Y_{l+\tilde{l}-2k}^0}(\kappa) d\kappa \right|^2,$$

then

$$\begin{aligned} &\sum_{|m| \leq l} \sum_{|\tilde{m}| \leq \tilde{l}} \left| \int_{\mathbb{S}_k^2} G_{n,\tilde{n},l,\tilde{l}}^{m,\tilde{m}}(\kappa) \overline{Y_{l+\tilde{l}-2k}^{m'}}(\kappa) d\kappa \right|^2 \\ &\leq \sum_{|q| \leq \min(l,\tilde{l})} \frac{4\pi}{2(l+\tilde{l}-2k)+1} \left| G_{n,\tilde{n},l,\tilde{l}}^{q,-q}(e_1) \right|^2 \\ &= \frac{(2l+1)(2\tilde{l}+1)}{4\pi[2(l+\tilde{l}-2k)+1]} \sum_{|q| \leq \min(l,\tilde{l})} \frac{(l-|q|)!}{(l+|q|)!} \frac{(\tilde{l}-|q|)!}{(\tilde{l}+|q|)!} \\ &\quad \times \left(\int_{|\theta| \leq \frac{\pi}{4}} \beta(\theta) (\sin \theta)^{2n+l} (\cos \theta)^{2\tilde{n}+\tilde{l}} P_l^{|q|}(\sin \theta) P_{\tilde{l}}^{|q|}(\cos \theta) d\theta \right)^2 \end{aligned}$$

Using the Cauchy-Schwarz inequality, we have

$$\begin{aligned} &\sum_{|q| \leq \min(l,\tilde{l})} \frac{(l-|q|)!}{(l+|q|)!} \frac{(\tilde{l}-|q|)!}{(\tilde{l}+|q|)!} \left(\int_{|\theta| \leq \frac{\pi}{4}} \beta(\theta) (\sin \theta)^{2n+l} (\cos \theta)^{2\tilde{n}+\tilde{l}} P_l^{|q|}(\sin \theta) P_{\tilde{l}}^{|q|}(\cos \theta) d\theta \right)^2 \\ &\leq \sum_{|q| \leq \min(l,\tilde{l})} \frac{(l-|q|)!}{(l+|q|)!} \frac{(\tilde{l}-|q|)!}{(\tilde{l}+|q|)!} \left(\int_{|\theta| \leq \frac{\pi}{4}} \left| \beta(\theta) (\sin \theta)^{2n+l} (\cos \theta)^{2\tilde{n}+\tilde{l}} \right| d\theta \right) \\ &\quad \times \left(\int_{|\theta| \leq \frac{\pi}{4}} \left| \beta(\theta) (\sin \theta)^{2n+l} (\cos \theta)^{2\tilde{n}+\tilde{l}} \right| \left[\left| P_l^{|q|}(\sin \theta) \right|^2 \left| P_{\tilde{l}}^{|q|}(\cos \theta) \right|^2 \right] d\theta \right) \\ &= 4 \left(\int_0^{\frac{\pi}{4}} \beta(\theta) (\sin \theta)^{2n+l} (\cos \theta)^{2\tilde{n}+\tilde{l}} d\theta \right) \times \left(\int_0^{\frac{\pi}{4}} \beta(\theta) (\sin \theta)^{2n+l} (\cos \theta)^{2\tilde{n}+\tilde{l}} \right. \\ &\quad \times \left[\sum_{|q| \leq \min(l,\tilde{l})} \frac{(l-|q|)!}{(l+|q|)!} \frac{(\tilde{l}-|q|)!}{(\tilde{l}+|q|)!} \left| P_l^{|q|}(\sin \theta) \right|^2 \left| P_{\tilde{l}}^{|q|}(\cos \theta) \right|^2 \right] d\theta \Big) \end{aligned}$$

For $\theta \in [0, \frac{\pi}{4}]$, by using the addition theorem (5.3) twice times,

$$\begin{aligned} P_l((\sin \theta)^2 + (\cos \theta)^2 \cos \phi) &= \sum_{|q| \leq l} \frac{(l - |q|)!}{(l + |q|)!} P_l^{|q|}(\sin \theta) P_l^{|q|}(\sin \theta) e^{iq\phi}; \\ P_{\tilde{l}}((\cos \theta)^2 + (\sin \theta)^2 \cos \phi) &= \sum_{|\tilde{q}| \leq \tilde{l}} \frac{(\tilde{l} - |\tilde{q}|)!}{(\tilde{l} + |\tilde{q}|)!} P_{\tilde{l}}^{|\tilde{q}|}(\cos \theta) P_{\tilde{l}}^{|\tilde{q}|}(\cos \theta) e^{i\tilde{q}\phi}, \end{aligned}$$

we obtain that,

$$\begin{aligned} &\int_0^{2\pi} P_l((\sin \theta)^2 + (\cos \theta)^2 \cos \phi) P_{\tilde{l}}((\cos \theta)^2 + (\sin \theta)^2 \cos \phi) \frac{d\phi}{2\pi} \\ &= \sum_{|q| \leq \min(l, \tilde{l})} \frac{(l - |q|)!}{(l + |q|)!} \frac{(\tilde{l} - |q|)!}{(\tilde{l} + |q|)!} |P_l^{|q|}(\sin \theta)|^2 |P_{\tilde{l}}^{|q|}(\cos \theta)|^2. \end{aligned}$$

It follows that

$$\begin{aligned} &\sum_{|m| \leq l} \sum_{|\tilde{m}| \leq \tilde{l}} \left| \int_{\mathbb{S}_k^2} G_{n, \tilde{n}, l, \tilde{l}}^{m, \tilde{m}}(\kappa) \overline{Y_{l+\tilde{l}-2k}^{m'}}(\kappa) d\kappa \right|^2 \\ &\leq \frac{(2l+1)(2\tilde{l}+1)}{\pi[2(l+\tilde{l}-2k)+1]} \\ &\quad \times \left(\int_0^{\frac{\pi}{4}} \beta(\theta) (\sin \theta)^{2n+l} (\cos \theta)^{2\tilde{n}+\tilde{l}} d\theta \right) \times \left(\int_0^{\frac{\pi}{4}} \beta(\theta) (\sin \theta)^{2n+l} (\cos \theta)^{2\tilde{n}+\tilde{l}} d\theta \right) \\ (6.8) \quad &\times \left[\int_0^{2\pi} P_l((\sin \theta)^2 + (\cos \theta)^2 \cos \phi) P_{\tilde{l}}((\cos \theta)^2 + (\sin \theta)^2 \cos \phi) \frac{d\phi}{2\pi} d\theta \right]. \end{aligned}$$

From the formula (14) of Sec.10.3 in Chap.III in [20]

$$|\sqrt{l} \sqrt[4]{1-x^2} P_l(x)| \leq 4 \sqrt{\frac{2}{\pi}}, \quad \forall -1 \leq x \leq 1,$$

then for $l \geq 1$, we have

$$|P_l((\sin \theta)^2 + (\cos \theta)^2 \cos \phi)| \leq 4 \sqrt{\frac{2}{\pi l}} \frac{1}{\sqrt[4]{1 - ((\sin \theta)^2 + (\cos \theta)^2 \cos \phi)^2}}.$$

Since

$$\begin{aligned} &1 - ((\sin \theta)^2 + (\cos \theta)^2 \cos \phi)^2 \\ &= (1 + (\sin \theta)^2 + (\cos \theta)^2 \cos \phi) (1 - ((\sin \theta)^2 + (\cos \theta)^2 \cos \phi)) \\ &\geq ((\cos \theta)^2 (1 + \cos \phi)) ((\cos \theta)^2 (1 - \cos \phi)) \\ &= (\cos \theta)^4 (1 - (\cos \phi)^2), \end{aligned}$$

we can estimate

$$|P_l((\sin \theta)^2 + (\cos \theta)^2 \cos \phi)| \leq 4 \sqrt{\frac{2}{\pi l}} \frac{1}{|\cos \theta|} \frac{1}{\sqrt[4]{1 - (\cos \phi)^2}}.$$

Recall that $|P_l(x)| \leq 1$ for $|x| \leq 1$, we can derive

$$\begin{aligned} & \int_0^{2\pi} P_l((\sin \theta)^2 + (\cos \theta)^2 \cos \phi) P_{\tilde{l}}((\cos \theta)^2 + (\sin \theta)^2 \cos \phi) \frac{d\phi}{2\pi} \\ & \lesssim \frac{1}{\sqrt{\tilde{l}}} \frac{1}{|\cos \theta|} \int_0^{2\pi} \frac{1}{\sqrt[4]{1 - (\cos \phi)^2}} d\phi \\ & \lesssim \frac{1}{\sqrt{\tilde{l}}} \frac{1}{|\cos \theta|}. \end{aligned}$$

For $0 \leq \theta \leq \frac{\pi}{4}$, we have $\cos \theta \geq \frac{\sqrt{2}}{2}$, substitute the above result into the formula (6.8), we conclude that, for $l \geq 1, \tilde{l} \geq 1$ with $0 \leq k \leq \min(l, \tilde{l})$,

$$\begin{aligned} & \sum_{|m| \leq l} \sum_{|\tilde{m}| \leq \tilde{l}} \left| \int_{\mathbb{S}_\kappa^2} G_{n, \tilde{n}, l, \tilde{l}}^{m, \tilde{m}}(\kappa) \overline{Y_{l+\tilde{l}-2k}^{m'}}(\kappa) d\kappa \right|^2 \\ & \lesssim \frac{(2l+1)(2\tilde{l}+1)}{[2(l+\tilde{l}-2k)+1]} \frac{1}{\sqrt{\tilde{l}}} \left(\int_0^{\frac{\pi}{4}} \beta(\theta) (\sin \theta)^{2n+l} (\cos \theta)^{2\tilde{n}+\tilde{l}} d\theta \right)^2 \\ & \lesssim \frac{\tilde{l} \sqrt{\tilde{l}}}{l + \tilde{l} - 2k + 1} \left(\int_0^{\frac{\pi}{4}} \beta(\theta) (\sin \theta)^{2n+l} (\cos \theta)^{2\tilde{n}+\tilde{l}} d\theta \right)^2. \end{aligned}$$

This ends the proof of (6.7). $\square$

For $l \geq 1, \tilde{l} \geq 1$ with $0 \leq k \leq \min(l, \tilde{l})$, we denote $\lambda_{n, \tilde{n}, \tilde{l}, l}^k$

$$\begin{aligned} (6.9) \quad \lambda_{n, \tilde{n}, \tilde{l}, l}^k &= \frac{\tilde{l} \sqrt{\tilde{l}}}{l + \tilde{l} - 2k + 1} \left(\frac{A_{\tilde{n}, \tilde{l}} A_{n, l}}{A_{n+\tilde{n}+k, l+\tilde{l}-2k}} \right)^2 \\ & \times \left(\int_0^{\frac{\pi}{4}} \beta(\theta) (\sin \theta)^{2n+l} (\cos \theta)^{2\tilde{n}+\tilde{l}} d\theta \right)^2. \end{aligned}$$

It follows from (6.6) and (6.7) that, for $0 \leq k \leq \min(l, \tilde{l})$, with $|m'| \leq l + \tilde{l} - 2k$,

$$\sum_{|m| \leq l} \sum_{|\tilde{m}| \leq \tilde{l}} \left| \mu_{n, \tilde{n}, l, \tilde{l}, k}^{m, \tilde{m}, m'} \right|^2 \lesssim \lambda_{n, \tilde{n}, \tilde{l}, l}^k.$$

Then we obtain

$$\begin{aligned} & \sum_{\substack{n+\tilde{n}+k=n^* \\ n+l \geq 2, \tilde{n}+\tilde{l} \geq 2 \\ n \geq 0, \tilde{n} \geq 0}} \sum_{\substack{l+l-2k=l^* \\ l \geq 1, \tilde{l} \geq 1 \\ 0 \leq k \leq \min(l, \tilde{l})}} \left(\sum_{|m| \leq l} \sum_{|\tilde{m}| \leq \tilde{l}} \frac{|\mu_{n, \tilde{n}, l, \tilde{l}, k}^{m, \tilde{m}, m^*}|^2}{\lambda_{\tilde{n}, \tilde{l}}} \right) \\ & \lesssim \sum_{\substack{n+\tilde{n}+k=n^* \\ n+l \geq 2, \tilde{n}+\tilde{l} \geq 2 \\ n \geq 0, \tilde{n} \geq 0}} \sum_{\substack{l+l-2k=l^* \\ l \geq 1, \tilde{l} \geq 1 \\ 0 \leq k \leq \min(l, \tilde{l})}} \frac{\lambda_{n, \tilde{n}, \tilde{l}, l}^k}{\lambda_{\tilde{n}, \tilde{l}}}. \end{aligned}$$

The proof of 3) in Proposition 3.1 is reduced to prove

$$(6.10) \quad \sum_{\substack{n+\tilde{n}+k=n^* \\ n+l \geq 2, \tilde{n}+\tilde{l} \geq 2 \\ n \geq 0, \tilde{n} \geq 0}} \sum_{\substack{l+l-2k=l^* \\ l \geq 1, \tilde{l} \geq 1 \\ 0 \leq k \leq \min(l, \tilde{l})}} \frac{\lambda_{n, \tilde{n}, \tilde{l}, l}^k}{\lambda_{\tilde{n}, \tilde{l}}} \leq C \lambda_{n^*, l^*}.$$

Lemma 6.3. For $n, \tilde{n}, \tilde{l}, l \in \mathbb{N}$ with $n + l \geq 2$ and $\tilde{n} + \tilde{l} \geq 2$, let $s_0 = \min(1 - s, s)$, we have

$$(6.11) \quad \lambda_{n, \tilde{n}, l, \tilde{l}}^0 \lesssim \frac{\tilde{l} \sqrt{l}}{l + \tilde{l} + 1} \frac{(\tilde{n} + \tilde{l})^{2s+s_0}}{(\tilde{n} + 1)^{s_0} (n + 1)^{1-s_0} (n + l)^{\frac{3}{2}+2s+s_0}}.$$

In addition, for $1 \leq k \leq \min(l, \tilde{l})$, we have the following estimate

$$(6.12) \quad \lambda_{n, \tilde{n}, l, \tilde{l}}^k \lesssim \frac{\tilde{l} \sqrt{l}}{l + \tilde{l} - 2k + 1} \frac{\tilde{n}^s (\tilde{n} + \tilde{l})^s}{(n + 1)^s (n + l + 1)^{\frac{s}{2}+s}}.$$

Remark 6.4. We divide the proof of the estimate of $\lambda_{n, \tilde{n}, l, \tilde{l}}^k$ into two cases: $k = 0$ and $k \geq 1$. The reason is that, when we estimate $\lambda_{n, \tilde{n}, l, \tilde{l}}^k$ in case $k \geq 1$, there is a term

$$\frac{(n + \tilde{n} + k)! \Gamma(n + \tilde{n} + l + \tilde{l} - k + \frac{3}{2})}{(n + \tilde{n} + 1)! \Gamma(n + \tilde{n} + l + \tilde{l} + \frac{1}{2})} \leq 1;$$

when $k = 0$ and $l + \tilde{l} \gg n + \tilde{n}$, by using the recurrence formula $\Gamma(x + 1) = x\Gamma(x)$, this term satisfies

$$\left. \frac{(n + \tilde{n} + k)! \Gamma(n + \tilde{n} + l + \tilde{l} - k + \frac{3}{2})}{(n + \tilde{n} + 1)! \Gamma(n + \tilde{n} + l + \tilde{l} + \frac{1}{2})} \right|_{k=0} = \frac{n + \tilde{n} + l + \tilde{l} + \frac{1}{2}}{n + \tilde{n} + 1} \gg 1.$$

Proof. Firstly, we consider the case $k \geq 1$. By using the Cauchy-Schwarz inequality and the Beta Function (6.2) we derive that, for $n + l \geq 2$,

$$\begin{aligned} & \left| \int_0^{\frac{\pi}{4}} \beta(\theta) (\sin \theta)^{2n+l} (\cos \theta)^{2\tilde{n}+\tilde{l}} d\theta \right|^2 \approx \left| \int_0^{\frac{1}{2}} t^{n+\frac{l}{2}-1-s} (1-t)^{\tilde{n}+\frac{\tilde{l}}{2}+s} dt \right|^2 \\ & \leq \frac{\Gamma(n+1-s) \Gamma(\tilde{n}+1+s)}{(n+\tilde{n}+1)!} \frac{\Gamma(n+l-1-s) \Gamma(\tilde{n}+\tilde{l}+\frac{3}{2}+s)}{\Gamma(n+\tilde{n}+l+\tilde{l}+\frac{1}{2})}. \end{aligned}$$

Then, we can express

$$\begin{aligned} \lambda_{n, \tilde{n}, l, \tilde{l}}^k & \lesssim \frac{\tilde{l} \sqrt{l}}{l + \tilde{l} - 2k + 1} \frac{(n + \tilde{n} + k)! \Gamma(n + \tilde{n} + l + \tilde{l} - k + \frac{3}{2})}{n! \Gamma(n + l + \frac{3}{2}) \tilde{n}! \Gamma(\tilde{n} + \tilde{l} + \frac{3}{2})} \\ & \quad \times \frac{\Gamma(n+1-s) \Gamma(\tilde{n}+1+s)}{(n+\tilde{n}+1)!} \frac{\Gamma(n+l-1-s) \Gamma(\tilde{n}+\tilde{l}+\frac{3}{2}+s)}{\Gamma(n+\tilde{n}+l+\tilde{l}+\frac{1}{2})} \\ & = \frac{\tilde{l} \sqrt{l}}{l + \tilde{l} - 2k + 1} \frac{(n + \tilde{n} + k)! \Gamma(n + \tilde{n} + l + \tilde{l} - k + \frac{3}{2})}{(n + \tilde{n} + 1)! \Gamma(n + \tilde{n} + l + \tilde{l} + \frac{1}{2})} \\ & \quad \times \frac{\Gamma(n+1-s) \Gamma(\tilde{n}+1+s) \Gamma(n+l-1-s) \Gamma(\tilde{n}+\tilde{l}+\frac{3}{2}+s)}{n! \Gamma(n + l + \frac{3}{2}) \tilde{n}! \Gamma(\tilde{n} + \tilde{l} + \frac{3}{2})}. \end{aligned}$$

We deduce from the formula (6.1) with $x = n + 1$, $a = -s$, $b = 0$,

$$\frac{\Gamma(n+1-s+1)}{(n+1)!} \lesssim \frac{1}{(n+1-s)^s},$$

and the recurrence formula $\Gamma(n+1-s) = \frac{1}{n+1-s} \Gamma(n+2-s)$,

$$\frac{\Gamma(n+1-s)}{n!} = \frac{n+1}{(n+1-s)} \frac{\Gamma(n+1-s+1)}{(n+1)!} \lesssim \frac{1}{(n+1)^s}.$$

Using $x = n + l - 1$, $a = -s$, $b = \frac{3}{2}$ in (6.1),

$$\frac{\Gamma(n + l - s)}{\Gamma(n + l + \frac{3}{2})} = \frac{\Gamma(n + l - 1 - s + 1)}{\Gamma(n + l - 1 + \frac{3}{2} + 1)} \lesssim \frac{1}{(n + l - 1)^{\frac{3}{2} + s}},$$

and the recurrence formula $\Gamma(n + l - 1 - s) = \frac{\Gamma(n + l - s)}{n + l - 1 - s}$,

$$\frac{\Gamma(n + l - 1 - s)}{\Gamma(n + l + \frac{3}{2})} = \frac{1}{n + l - 1 - s} \frac{\Gamma(n + l - s)}{\Gamma(n + l + \frac{3}{2})} \lesssim \frac{1}{(n + l + 1)^{\frac{5}{2} + s}}.$$

Using $x = \tilde{n} + 1$, $a = s$, $b = 0$ in (6.1), we have

$$\frac{\Gamma(\tilde{n} + 2 + s)}{(\tilde{n} + 1)!} \lesssim (\tilde{n} + 1 + s)^s.$$

By using the recurrence formula $\Gamma(x + 1) = x\Gamma(x)$,

$$\frac{\Gamma(\tilde{n} + 1 + s)}{\tilde{n}!} = \frac{\tilde{n} + 1}{\tilde{n} + 1 + s} \frac{\Gamma(\tilde{n} + 2 + s)}{(\tilde{n} + 1)!} \lesssim \tilde{n}^s.$$

Using $x = \tilde{n} + \tilde{l} + \frac{1}{2}$, $a = s$, $b = 0$ in (6.1), and $\tilde{n} + \tilde{l} \geq 2$,

$$\frac{\Gamma(\tilde{n} + \tilde{l} + \frac{3}{2} + s)}{\Gamma(\tilde{n} + \tilde{l} + \frac{3}{2})} \lesssim (\tilde{n} + \tilde{l} + \frac{1}{2} + s)^s \lesssim (\tilde{n} + \tilde{l})^s.$$

Therefore, we obtain that, $n + l \geq 2$, $\tilde{n} + \tilde{l} \geq 2$

$$\begin{aligned} \lambda_{n, \tilde{n}, l, \tilde{l}}^k &\lesssim \frac{\tilde{l} \sqrt{l}}{l + \tilde{l} - 2k + 1} \frac{\tilde{n}^s (\tilde{n} + \tilde{l})^s}{(n + 1)^s (n + l + 1)^{\frac{5}{2} + s}} \\ &\quad \times \frac{(n + \tilde{n} + k)! \Gamma(n + \tilde{n} + l + \tilde{l} - k + \frac{3}{2})}{(n + \tilde{n} + 1)! \Gamma(n + \tilde{n} + l + \tilde{l} + \frac{1}{2})}. \end{aligned}$$

When $k = 1$, we observe that,

$$\frac{(n + \tilde{n} + k)! \Gamma(n + \tilde{n} + l + \tilde{l} - k + \frac{3}{2})}{(n + \tilde{n} + 1)! \Gamma(n + \tilde{n} + l + \tilde{l} + \frac{1}{2})} = 1.$$

Now consider the case $k \geq 2$, we use again the recurrence formula $\Gamma(x + 1) = x\Gamma(x)$,

$$\begin{aligned} &\frac{(n + \tilde{n} + k)! \Gamma(n + \tilde{n} + l + \tilde{l} - k + \frac{3}{2})}{(n + \tilde{n} + 1)! \Gamma(n + \tilde{n} + l + \tilde{l} + \frac{1}{2})} \\ &= \frac{(n + \tilde{n} + k)(n + \tilde{n} + k - 1) \times \cdots \times (n + \tilde{n} + 2)}{(n + \tilde{n} + l + \tilde{l} - \frac{1}{2})(n + \tilde{n} + l + \tilde{l} - \frac{3}{2}) \times \cdots \times (n + \tilde{n} + l + \tilde{l} - k + \frac{3}{2})} \\ &= \prod_{j=2}^k \frac{n + \tilde{n} + j}{n + \tilde{n} + l + \tilde{l} - k + j - \frac{1}{2}}. \end{aligned}$$

Since $2 \leq k \leq \min(l, \tilde{l})$, one has,

$$l + \tilde{l} - k - \frac{1}{2} \geq k - \frac{1}{2} \geq \frac{3}{2}.$$

This shows that, for any $2 \leq j \leq k$,

$$\frac{n + \tilde{n} + j}{n + \tilde{n} + l + \tilde{l} - k + j - \frac{1}{2}} \leq \frac{n + \tilde{n} + j}{n + \tilde{n} + j + \frac{3}{2}} < 1.$$

Therefore, we conclude, for $k \geq 1$,

$$\frac{(n + \tilde{n} + k)! \Gamma(n + \tilde{n} + l + \tilde{l} - k + \frac{3}{2})}{(n + \tilde{n} + 1)! \Gamma(n + \tilde{n} + l + \tilde{l} + \frac{1}{2})} \leq 1.$$

We obtain the formula (6.12).

For the estimate (6.11), we assume that $n + l \geq 2, \tilde{n} + \tilde{l} \geq 2$ and $s_0 = \min(1 - s, s)$. By using the Cauchy-Schwarz inequality and the Beta Function (6.2), we obtain

$$\begin{aligned} & \left| \int_0^{\frac{\pi}{4}} \beta(\theta) (\sin \theta)^{2n+l} (\cos \theta)^{2\tilde{n}+\tilde{l}} d\theta \right|^2 \approx \left| \int_0^{\frac{1}{2}} t^{n+\frac{l}{2}-1-s} (1-t)^{\tilde{n}+\frac{\tilde{l}}{2}+\frac{1}{4}+s} dt \right|^2 \\ & \leq \int_0^{\frac{1}{2}} t^{n-1+s_0} (1-t)^{\tilde{n}-s_0} dt \times \int_0^{\frac{1}{2}} t^{n+l-2s-1-s_0} (1-t)^{\tilde{n}+\tilde{l}+\frac{1}{2}+2s+s_0} dt \\ & \leq \frac{\Gamma(n+s_0)\Gamma(\tilde{n}+1-s_0)}{(n+\tilde{n})!} \frac{\Gamma(n+l-2s-s_0)\Gamma(\tilde{n}+\tilde{l}+\frac{3}{2}+2s+s_0)}{\Gamma(n+\tilde{n}+l+\tilde{l}+\frac{3}{2})}. \end{aligned}$$

Therefore,

$$\begin{aligned} \lambda_{n,\tilde{n},l,\tilde{l}}^0 &= \frac{\tilde{l}\sqrt{\tilde{l}}}{l+\tilde{l}+1} \frac{(n+\tilde{n})! \Gamma(n+\tilde{n}+l+\tilde{l}+\frac{3}{2})}{n!\tilde{n}! \Gamma(n+l+\frac{3}{2}) \Gamma(\tilde{n}+\tilde{l}+\frac{3}{2})} \\ &\quad \times \frac{\Gamma(n+s_0)\Gamma(\tilde{n}+1-s_0)}{(n+\tilde{n})!} \frac{\Gamma(n+l-2s-s_0)\Gamma(\tilde{n}+\tilde{l}+\frac{3}{2}+2s+s_0)}{\Gamma(n+\tilde{n}+l+\tilde{l}+\frac{3}{2})} \\ &\lesssim \frac{\tilde{l}\sqrt{\tilde{l}}}{l+\tilde{l}+1} \frac{\Gamma(n+s_0)\Gamma(\tilde{n}+1-s_0)\Gamma(n+l-2s-s_0)\Gamma(\tilde{n}+\tilde{l}+\frac{3}{2}+2s+s_0)}{n!\tilde{n}! \Gamma(n+l+\frac{3}{2}) \Gamma(\tilde{n}+\tilde{l}+\frac{3}{2})}. \end{aligned}$$

Applying the estimate (6.1) with $x = n + 1, a = s_0 - 1, b = 0$,

$$\frac{\Gamma(n+1+s_0)}{(n+1)!} \lesssim \frac{1}{(n+s_0)^{1-s_0}},$$

by using the recurrence formula $\Gamma(x+1) = x\Gamma(x)$,

$$\frac{\Gamma(n+s_0)}{n!} = \frac{n+1}{n+s_0} \frac{\Gamma(n+1+s_0)}{(n+1)!} \lesssim \frac{n+1}{n+s_0} \frac{1}{(n+s_0)^{1-s_0}} \lesssim \frac{1}{(n+1)^{1-s_0}}.$$

Applying the estimate (6.1) with $x = \tilde{n} + 1, a = -s_0, b = 0$,

$$\frac{\Gamma(\tilde{n}+2-s_0)}{(\tilde{n}+1)!} \lesssim \frac{1}{(\tilde{n}+1-s_0)^{s_0}},$$

by using the recurrence formula $\Gamma(x+1) = x\Gamma(x)$,

$$\frac{\Gamma(\tilde{n}+1-s_0)}{\tilde{n}!} = \frac{\tilde{n}+1}{\tilde{n}+1-s_0} \frac{\Gamma(\tilde{n}+2-s_0)}{(\tilde{n}+1)!} \lesssim \frac{1}{(\tilde{n}+1-s_0)^{s_0}} \lesssim \frac{1}{(\tilde{n}+1)^{s_0}}.$$

Since $s_0 = \min(s, 1-s)$, we have $2s + s_0 \leq 1 - s + 2s = 1 + s \leq 2$. Using the estimate (6.1) with $x = n + l, a = -2s - s_0 + 1, b = \frac{1}{2}$, considering that $n + l \geq 2$, then $x + a \geq 1$, $x + b \geq 1$, we can derive that,

$$\frac{\Gamma(n+l+2-2s-s_0)}{\Gamma(n+l+\frac{3}{2})} \lesssim \frac{1}{(n+l+1-2s-s_0)^{2s+s_0-\frac{1}{2}}}.$$

By using the recurrence formula $\Gamma(x+1) = x\Gamma(x)$ that

$$\begin{aligned} & \frac{\Gamma(n+l-2s-s_0)}{\Gamma(n+l+\frac{3}{2})} \\ &= \frac{1}{(n+l-2s-s_0)(n+l+1-2s-s_0)} \frac{\Gamma(n+l+2-2s-s_0)}{\Gamma(n+l+\frac{3}{2})} \\ &\lesssim \frac{1}{(n+l+1-2s-s_0)^{2s+s_0+\frac{3}{2}}} \lesssim \frac{1}{(n+l)^{2s+s_0+\frac{3}{2}}}. \end{aligned}$$

Finally, applying the estimate (6.1) with $x = \tilde{n} + \tilde{l} + \frac{1}{2}$, $a = 2s + s_0$, $b = 0$

$$\frac{\Gamma(\tilde{n} + \tilde{l} + \frac{3}{2} + 2s + s_0)}{\Gamma(\tilde{n} + \tilde{l} + \frac{3}{2})} \lesssim (\tilde{n} + \tilde{l})^{2s+s_0}.$$

Combining the above four estimates together, we derive

$$\lambda_{n,\tilde{n},l,\tilde{l}}^0 \lesssim \frac{\tilde{l}\sqrt{l}}{l+\tilde{l}+1} \frac{(\tilde{n} + \tilde{l})^{2s+s_0}}{(\tilde{n}+1)^{s_0}(n+1)^{1-s_0}(n+l)^{\frac{3}{2}+2s+s_0}}.$$

This concludes the proof of Lemma 6.3. $\square$

The estimate in (6.12) is not enough accurate in proof of 3) in Proposition 3.1. To this end, we provide a more optimal estimate of $\lambda_{n,\tilde{n},l,\tilde{l}}^k$ for large k in the following Lemma.

Lemma 6.5. *For any $0 < \omega < 1$ and $k \geq 20$, we have the following estimates*

$$\begin{aligned} (i) \quad \lambda_{n,\tilde{n},l,\tilde{l}}^k &\lesssim \frac{\tilde{l}\sqrt{l}}{l+\tilde{l}-2k+1} \frac{(\tilde{n} + \tilde{l})^{2s}}{(n+l+1)^{\frac{3}{2}+2s}} e^{-\frac{1}{8}k\omega} \quad \text{when } n+l \leq k^{1-\omega}l; \\ (ii) \quad \lambda_{n,\tilde{n},l,\tilde{l}}^k &\lesssim \frac{\tilde{l}\sqrt{l}}{l+\tilde{l}-2k+1} \frac{(\tilde{n} + \tilde{l})^{2s}}{(n+l+1)^{\frac{3}{2}+2s}} \quad \text{when } n+l \geq k^{1-\omega}l. \end{aligned}$$

Proof. Recall from (6.9) that

$$\begin{aligned} \lambda_{n,\tilde{n},l,\tilde{l}}^k &\approx \frac{\tilde{l}\sqrt{l}}{l+\tilde{l}-2k+1} \frac{(n+\tilde{n}+k)!\Gamma(n+\tilde{n}+l+\tilde{l}-k+\frac{3}{2})}{n!\Gamma(n+l+\frac{3}{2})\tilde{n}!\Gamma(\tilde{n}+\tilde{l}+\frac{3}{2})} \\ &\quad \times \left(\int_0^{\frac{1}{2}} t^{n+\frac{l}{2}-1-s} (1-t)^{\tilde{n}+\frac{\tilde{l}}{2}} dt \right)^2. \end{aligned}$$

By using the Beta Function (6.2) and the Cauchy-Schwarz inequality,

$$\begin{aligned} & \left(\int_0^{\frac{1}{2}} t^{n+\frac{l}{2}-1-s} (1-t)^{\tilde{n}+\frac{\tilde{l}}{2}} dt \right)^2 \leq 2^{\frac{1}{2}+2s} \left(\int_0^{\frac{1}{2}} t^{n+\frac{l}{2}-1-s} (1-t)^{\tilde{n}+\frac{\tilde{l}}{2}+\frac{1}{4}+s} dt \right)^2 \\ & \leq 2^{\frac{1}{2}+2s} \frac{\Gamma(n+\frac{k}{2})\Gamma(\tilde{n}+\frac{k}{2}+1)}{\Gamma(n+\tilde{n}+k+1)} \times \frac{\Gamma(n+l-\frac{k}{2}-2s)\Gamma(\tilde{n}+\tilde{l}-\frac{k}{2}+\frac{3}{2}+2s)}{\Gamma(n+\tilde{n}+l+\tilde{l}-k+\frac{3}{2})}, \end{aligned}$$

we obtain

$$\begin{aligned}\lambda_{n,\tilde{n},l,\tilde{l}}^k &\lesssim \frac{\tilde{l}\sqrt{\tilde{l}}}{l+\tilde{l}-2k+1} \frac{(n+\tilde{n}+k)!\Gamma(n+\tilde{n}+l+\tilde{l}-k+\frac{3}{2})}{n!\Gamma(n+l+\frac{3}{2})\tilde{n}!\Gamma(\tilde{n}+\tilde{l}+\frac{3}{2})} \\ &\quad \times \frac{\Gamma(n+\frac{k}{2})\Gamma(\tilde{n}+\frac{k}{2}+1)}{\Gamma(n+\tilde{n}+k+1)} \frac{\Gamma(n+l-\frac{k}{2}-2s)\Gamma(\tilde{n}+\tilde{l}-\frac{k}{2}+\frac{3}{2}+2s)}{\Gamma(n+\tilde{n}+l+\tilde{l}-k+\frac{3}{2})} \\ &= \frac{\sqrt{\tilde{l}}\Gamma(n+\frac{k}{2})\Gamma(\tilde{n}+\frac{k}{2}+1)\Gamma(n+l-\frac{k}{2}-2s)\Gamma(\tilde{n}+\tilde{l}-\frac{k}{2}+\frac{3}{2}+2s)}{(l+\tilde{l}-2k+1)n!\Gamma(n+l+\frac{3}{2})\tilde{n}!\Gamma(\tilde{n}+\tilde{l}+\frac{3}{2})}.\end{aligned}$$

In case $k \geq 20$, remind that $l \geq \min(l, \tilde{l}) \geq k$, we have,

$$n+l-\frac{k}{2}-1 \geq \frac{k}{2}-1 \geq 9,$$

Let $x = n+l-\frac{k}{2}-1$, $a = -2s$, $b = \frac{5}{2}$ in formula (6.1), we then derive

$$\frac{\Gamma(n+l-\frac{k}{2}-2s)}{\Gamma(n+l-\frac{k}{2}+\frac{5}{2})} \lesssim \frac{1}{(n+l-\frac{k}{2}-1-2s)^{\frac{5}{2}+2s}} \lesssim \frac{1}{(n+l+1)^{\frac{5}{2}+2s}}.$$

When we choose $x = \tilde{n}+\tilde{l}-\frac{k}{2}+\frac{1}{2}$, $a = 2s$, $b = 0$ in formula (6.1), then

$$\frac{\Gamma(\tilde{n}+\tilde{l}-\frac{k}{2}+\frac{3}{2}+2s)}{\Gamma(\tilde{n}+\tilde{l}-\frac{k}{2}+\frac{3}{2})} \lesssim (\tilde{n}+\tilde{l})^{2s}.$$

Therefore, we can verify that for $k \geq 20$

$$\begin{aligned}\lambda_{n,\tilde{n},l,\tilde{l}}^k &\lesssim \frac{\tilde{l}\sqrt{\tilde{l}}}{l+\tilde{l}-2k+1} \frac{(\tilde{n}+\tilde{l})^{2s}}{(n+l+1)^{\frac{5}{2}+2s}} \frac{\Gamma(n+\frac{k}{2})\Gamma(n+l-\frac{k}{2}+\frac{5}{2})}{n!\Gamma(n+l+\frac{3}{2})} \\ &\quad \times \frac{\Gamma(\tilde{n}+\frac{k}{2}+1)\Gamma(\tilde{n}+\tilde{l}-\frac{k}{2}+\frac{3}{2})}{\tilde{n}!\Gamma(\tilde{n}+\tilde{l}+\frac{3}{2})}.\end{aligned}$$

We claim that, for $k \geq 20$,

$$(6.13) \quad \frac{\Gamma(\tilde{n}+\frac{k}{2}+1)\Gamma(\tilde{n}+\tilde{l}-\frac{k}{2}+\frac{3}{2})}{\tilde{n}!\Gamma(\tilde{n}+\tilde{l}+\frac{3}{2})} \leq 1.$$

Indeed, if k is even, by using the recurrence formula $\Gamma(x+1) = x\Gamma(x)$ and $\Gamma(\tilde{n}+\frac{k}{2}+1) = (\tilde{n}+\frac{k}{2})!$, one has,

$$\begin{aligned}&\frac{\Gamma(\tilde{n}+\frac{k}{2}+1)\Gamma(\tilde{n}+\tilde{l}-\frac{k}{2}+\frac{3}{2})}{\tilde{n}!\Gamma(\tilde{n}+\tilde{l}+\frac{3}{2})} \\ &= \frac{(\tilde{n}+\frac{k}{2})(\tilde{n}+\frac{k}{2}-1) \times \cdots \times (\tilde{n}+1)}{(\tilde{n}+\tilde{l}+\frac{1}{2})(\tilde{n}+\tilde{l}-\frac{1}{2}) \times \cdots \times (\tilde{n}+\tilde{l}-\frac{k}{2}+\frac{3}{2})} \\ &= \prod_{j=1}^{\lfloor \frac{k}{2} \rfloor} \frac{\tilde{n}+\frac{k}{2}+1-j}{\tilde{n}+\tilde{l}+\frac{3}{2}-j}.\end{aligned}$$

For any $1 \leq j \leq \lfloor \frac{k}{2} \rfloor$, recall that $\tilde{l} \geq k > \frac{k}{2}$, we obtain,

$$\frac{\tilde{n}+\frac{k}{2}+1-j}{\tilde{n}+\tilde{l}+\frac{3}{2}-j} \leq 1.$$

Then the formula (6.13) holds. If k is odd, by using the recurrence formula $\Gamma(x+1) = x\Gamma(x)$ and $\Gamma(\tilde{n} + \tilde{l} - \frac{k}{2} + \frac{3}{2}) = (\tilde{n} + \tilde{l} - \frac{k}{2} + \frac{1}{2})!$,

$$\begin{aligned} & \frac{\Gamma(\tilde{n} + \frac{k}{2} + 1)\Gamma(\tilde{n} + \tilde{l} - \frac{k}{2} + \frac{3}{2})}{\tilde{n}!\Gamma(\tilde{n} + \tilde{l} + \frac{3}{2})} \\ &= \frac{(\tilde{n} + \tilde{l} - \frac{k}{2} + \frac{1}{2})(\tilde{n} + \tilde{l} - \frac{k}{2} - \frac{1}{2}) \times \cdots \times (\tilde{n} + 1)}{(\tilde{n} + \tilde{l} + \frac{1}{2})(\tilde{n} + \tilde{l} - \frac{1}{2}) \times \cdots \times (\tilde{n} + \frac{k}{2} + 1)} \\ &= \prod_{j=1}^{\tilde{l} - \frac{k}{2} + \frac{1}{2}} \frac{\tilde{n} + \tilde{l} - \frac{k}{2} + \frac{3}{2} - j}{\tilde{n} + \tilde{l} + \frac{3}{2} - j}. \end{aligned}$$

We can observe directly that

$$\frac{\tilde{n} + \tilde{l} - \frac{k}{2} + \frac{3}{2} - j}{\tilde{n} + \tilde{l} + \frac{3}{2} - j} \leq 1,$$

Therefore, we conclude that the formula (6.13) holds for all $k \geq 20$. Thus, we derive

$$(6.14) \quad \lambda_{n, \tilde{n}, l, \tilde{l}}^k \lesssim \frac{\tilde{l}\sqrt{l}}{l + \tilde{l} - 2k + 1} \frac{(\tilde{n} + \tilde{l})^{2s}}{(n + l + 1)^{\frac{5}{2} + 2s}} \frac{\Gamma(n + \frac{k}{2})\Gamma(n + l - \frac{k}{2} + \frac{5}{2})}{n!\Gamma(n + l + \frac{3}{2})}.$$

Now consider the formula

$$\frac{\Gamma(n + \frac{k}{2})\Gamma(n + l - \frac{k}{2} + \frac{5}{2})}{n!\Gamma(n + l + \frac{3}{2})},$$

we claim that, for $k \geq 20$,

$$(6.15) \quad \frac{\Gamma(n + \frac{k}{2})\Gamma(n + l - \frac{k}{2} + \frac{5}{2})}{n!\Gamma(n + l + \frac{3}{2})} \leq e^{-\frac{k}{8(n+l)}}.$$

Firstly, we assume k is even. Reminding that $k \leq \min(l, \tilde{l})$, we can set $l = l_1 + k$ with $l_1 \geq 0$. Using the recurrence formula $\Gamma(x+1) = x\Gamma(x)$ for $x > 0$ and

$$\Gamma(n + \frac{k}{2}) = (n + \frac{k}{2} - 1)!$$

we obtain

$$\begin{aligned} & \frac{\Gamma(n + \frac{k}{2})\Gamma(n + l - \frac{k}{2} + \frac{5}{2})}{n!\Gamma(n + l + \frac{3}{2})} \\ &= \frac{(n + \frac{k}{2} - 1)!\Gamma(n + l_1 + \frac{k}{2} + \frac{5}{2})}{n!\Gamma(n + l_1 + k + \frac{3}{2})} \\ &= \frac{(n + \frac{k}{2} - 1)(n + \frac{k}{2} - 2) \times \cdots \times (n + 1)}{(n + l_1 + k + \frac{1}{2})(n + l_1 + k - \frac{1}{2}) \times \cdots \times (n + l_1 + \frac{k}{2} + \frac{5}{2})} \\ &= \prod_{j=1}^{\frac{k}{2}-1} \frac{n + j}{n + l_1 + \frac{k}{2} + \frac{3}{2} + j}. \end{aligned}$$

Applying the elementary inequality, for any $1 \leq j \leq \frac{k}{2} - 1$,

$$\frac{n + j}{n + l_1 + \frac{k}{2} + \frac{3}{2} + j} \leq \frac{n + \frac{k}{2}}{n + l_1 + k + \frac{3}{2}} \leq \frac{n + \frac{k}{2}}{n + l_1 + k}$$

and

$$\frac{k}{2} - 1 \geq \frac{k}{4} \text{ for } k \geq 20,$$

we get

$$\prod_{j=1}^{\frac{k}{2}-1} \frac{n+j}{n+l_1+\frac{k}{2}+\frac{3}{2}+j} \leq \left(\frac{n+\frac{k}{2}}{n+l}\right)^{\frac{k}{2}-1} \leq \left(1 - \frac{\frac{k}{2}+l_1}{n+l}\right)^{\frac{k}{2}} = e^{\frac{k}{4} \log(1 - \frac{\frac{k}{2}+l_1}{n+l})}.$$

Using the inequality,

$$\log(1+x) \leq x, \forall x > -1,$$

and recalling that $l = l_1 + k$ with $l_1 \geq 0$,

$$\frac{l}{2} \leq l_1 + \frac{k}{2} < l \leq n+l,$$

we have

$$\frac{k}{4} \log(1 - \frac{\frac{k}{2}+l_1}{n+l}) \leq -\frac{k}{4} \frac{l_1 + \frac{k}{2}}{n+l} \leq -\frac{kl}{8(n+l)}.$$

It follows that

$$\prod_{j=1}^{\frac{k}{2}-1} \frac{n+j}{n+l_1+\frac{k}{2}+\frac{3}{2}+j} \leq e^{\frac{k}{4} \log(1 - \frac{\frac{k}{2}+l_1}{n+l})} \leq e^{-\frac{kl}{8(n+l)}}.$$

Therefore, the formula (6.15) follows when k is even.

Analogously, when k is odd, we set $l = l_1 + k$ with $l_1 \geq 0$. Using the recurrence formula $\Gamma(x+1) = x\Gamma(x)$ for $x > 0$ and

$$\Gamma(n+l - \frac{k}{2} + \frac{5}{2}) = (n+l - \frac{k}{2} + \frac{3}{2})! = (n+l_1 + \frac{k}{2} + \frac{3}{2})!$$

we obtain

$$\begin{aligned} & \frac{\Gamma(n+\frac{k}{2})\Gamma(n+l-\frac{k}{2}+\frac{5}{2})}{n!\Gamma(n+l+\frac{3}{2})} \\ &= \frac{\Gamma(n+\frac{k}{2})(n+l_1+\frac{k}{2}+\frac{3}{2})!}{n!\Gamma(n+l_1+k+\frac{3}{2})} \\ &= \frac{(n+l_1+\frac{k}{2}+\frac{3}{2})(n+l_1+\frac{k}{2}+\frac{1}{2}) \times \cdots \times (n+1)}{(n+l_1+k+\frac{1}{2})(n+l_1+k-\frac{1}{2}) \times \cdots \times (n+\frac{k}{2})} \\ &= \prod_{j=1}^{l_1+\frac{k}{2}+\frac{3}{2}} \frac{n+j}{n+\frac{k}{2}-1+j}. \end{aligned}$$

Since $k \geq 20$, we have the elementary inequality : for any $1 \leq j \leq l_1 + \frac{k}{2} + \frac{3}{2}$,

$$\frac{n+j}{n+\frac{k}{2}-1+j} \leq \frac{n+l_1+\frac{k}{2}+\frac{3}{2}}{n+l_1+k+\frac{1}{2}} \leq \frac{n+l_1+\frac{3k}{4}}{n+l_1+k}.$$

Therefore

$$\prod_{j=1}^{l_1+\frac{k}{2}+\frac{3}{2}} \frac{n+j}{n+\frac{k}{2}-1+j} \leq \left(\frac{n+l_1+\frac{3k}{4}}{n+l_1+k}\right)^{l_1+\frac{k}{2}+\frac{3}{2}} \leq \left(1 - \frac{\frac{k}{4}}{n+l}\right)^{l_1+\frac{k}{2}+\frac{3}{2}} = e^{(l_1+\frac{k}{2}+\frac{3}{2}) \log(1 - \frac{\frac{k}{4}}{n+l})}.$$

Using the inequality,

$$\log(1+x) \leq x, \forall x > -1,$$

and recalling that $l = l_1 + k$ with $l_1 \geq 0, k \geq 20$

$$\frac{l}{2} \leq l_1 + \frac{k}{2} + \frac{3}{2} < l \leq n + l,$$

we have

$$(l_1 + \frac{k}{2} + \frac{3}{2}) \log(1 - \frac{k}{4(n+l)}) \leq -\frac{k}{4(n+l)}(l_1 + \frac{k}{2} + \frac{3}{2}) \leq -\frac{kl}{8(n+l)}.$$

Therefore, the formula (6.15) holds true for k is odd. This ends the proof of the formula (6.15). For $0 < \omega < 1$, if

$$\frac{k^{1-\omega}l}{n+l} > 1,$$

then

$$\frac{\Gamma(n + \frac{k}{2})\Gamma(n + l - \frac{k}{2} + \frac{5}{2})}{n!\Gamma(n + l + \frac{3}{2})} \lesssim e^{-\frac{1}{8}k^\omega}.$$

This implies that, when $n + l < k^{1-\omega}l$,

$$\lambda_{n,\tilde{n},l,\tilde{l}}^k \lesssim \frac{\tilde{l}\sqrt{l}}{l + \tilde{l} - 2k + 1} \frac{(\tilde{n} + \tilde{l})^{2s}}{(n + l + 1)^{\frac{5}{2} + 2s}} e^{-\frac{1}{8}k^\omega}.$$

This is the result of the estimate (i).

If $n + l \geq k^{1-\omega}l$, we deduce from the same estimate as (6.13) that,

$$\frac{\Gamma(n + \frac{k}{2})\Gamma(n + l - \frac{k}{2} + \frac{5}{2})}{n!\Gamma(n + l + \frac{3}{2})} \leq 1.$$

The estimate (ii) follows from the estimate (6.14). This ends the proof of Lemma 6.5. $\square$

6.3. The proof of 3) in Proposition 3.1.

Proof. For $\lambda_{n,\tilde{n},l,\tilde{l}}^k$ defined in (6.9), by the analysis in (6.10), we only need to prove

$$\sum_{\substack{n+\tilde{n}+k=n^* \\ n+l \geq 2, \tilde{n}+\tilde{l} \geq 2 \\ n \geq 0, \tilde{n} \geq 0}} \sum_{\substack{l+\tilde{l}-2k=l^* \\ l \geq 1, \tilde{l} \geq 1 \\ 0 \leq k \leq \min(l, \tilde{l})}} \frac{\lambda_{n,\tilde{n},l,\tilde{l}}^k}{\lambda_{\tilde{n},\tilde{l}}} \leq C\lambda_{n^*,l^*}.$$

The constraint of the above summation is

$$\Lambda_{n^*,l^*} = \left\{ (n, \tilde{n}, l, \tilde{l}, k) \in \mathbb{N}^5; n + \tilde{n} + k = n^*, l + \tilde{l} - 2k = l^*, \right. \\ \left. l \geq 1, \tilde{l} \geq 1, 0 \leq k \leq \min(l, \tilde{l}), n + l \geq 2, \tilde{n} + \tilde{l} \geq 2 \right\},$$

which is a subset of a hyperplane of dimension 3. By using Lemma 6.5 with $\omega = \frac{1}{4}$, we divide the sets into four parts,

$$\Lambda_{n^*,l^*} = \Lambda_{n^*,l^*}^1 \cup \Lambda_{n^*,l^*}^2 \cup \Lambda_{n^*,l^*}^3 \cup \Lambda_{n^*,l^*}^4$$

with the sets

$$\begin{aligned}
\Lambda_{n^*, l^*}^1 &= \{(n, \tilde{n}, l, \tilde{l}, k) \in \mathbb{N}^5; n + \tilde{n} = n^*, l + \tilde{l} = l^*, k = 0 \\
&\quad l \geq 1, \tilde{l} \geq 1, n + l \geq 2, \tilde{n} + \tilde{l} \geq 2\}; \\
\Lambda_{n^*, l^*}^2 &= \{(n, \tilde{n}, l, \tilde{l}, k) \in \mathbb{N}^5; n + \tilde{n} + k = n^*, l + \tilde{l} - 2k = l^*, \\
&\quad l \geq 1, \tilde{l} \geq 1, 1 \leq k \leq \min(19, l, \tilde{l}), n + l \geq 2, \tilde{n} + \tilde{l} \geq 2\}; \\
\Lambda_{n^*, l^*}^3 &= \{(n, \tilde{n}, l, \tilde{l}, k) \in \mathbb{N}^5; n + \tilde{n} + k = n^*, l + \tilde{l} - 2k = l^*, n + l < k^{\frac{3}{4}}l, \\
&\quad l \geq 1, \tilde{l} \geq 1, 20 \leq k \leq \min(l, \tilde{l}), n + l \geq 2, \tilde{n} + \tilde{l} \geq 2\}; \\
\Lambda_{n^*, l^*}^4 &= \{(n, \tilde{n}, l, \tilde{l}, k) \in \mathbb{N}^5; n + \tilde{n} + k = n^*, l + \tilde{l} - 2k = l^*, n + l \geq k^{\frac{3}{4}}l, \\
&\quad l \geq 1, \tilde{l} \geq 1, 20 \leq k \leq \min(l, \tilde{l}), n + l \geq 2, \tilde{n} + \tilde{l} \geq 2\}.
\end{aligned}$$

Then the summation can be divided into four terms corresponding to the sets.

$$\begin{aligned}
&\sum_{(n, \tilde{n}, l, \tilde{l}, k) \in \Lambda_{n^*, l^*}^1} \frac{\lambda_{n, \tilde{n}, l, \tilde{l}}^k}{\lambda_{\tilde{n}, \tilde{l}}} \\
&= \sum_{(n, \tilde{n}, l, \tilde{l}) \in \Lambda_{n^*, l^*}^1} \frac{\lambda_{n, \tilde{n}, l, \tilde{l}}^0}{\lambda_{\tilde{n}, \tilde{l}}} + \sum_{(n, \tilde{n}, l, \tilde{l}, k) \in \Lambda_{n^*, l^*}^2} \frac{\lambda_{n, \tilde{n}, l, \tilde{l}}^k}{\lambda_{\tilde{n}, \tilde{l}}} + \sum_{(n, \tilde{n}, l, \tilde{l}, k) \in \Lambda_{n^*, l^*}^3} \frac{\lambda_{n, \tilde{n}, l, \tilde{l}}^k}{\lambda_{\tilde{n}, \tilde{l}}} + \sum_{(n, \tilde{n}, l, \tilde{l}, k) \in \Lambda_{n^*, l^*}^4} \frac{\lambda_{n, \tilde{n}, l, \tilde{l}}^k}{\lambda_{\tilde{n}, \tilde{l}}} \\
&= \mathbf{K}_1 + \mathbf{K}_2 + \mathbf{K}_3 + \mathbf{K}_4.
\end{aligned}$$

By using (6.11) in Lemma 6.3 with $s_0 = \min(s, 1 - s)$, one can estimate $\mathbf{K}_1$ as follows,

$$\mathbf{K}_1 \lesssim \sum_{\substack{n+\tilde{n}=n^* \\ n+l \geq 2, \tilde{n}+\tilde{l} \geq 2 \\ n \geq 0, \tilde{n} \geq 0}} \sum_{\substack{l+\tilde{l}=l^* \\ l \geq 1, \tilde{l} \geq 1}} \frac{\tilde{l} \sqrt{l}}{l + \tilde{l} + 1} \frac{(\tilde{n} + \tilde{l})^{2s+s_0}}{(\tilde{n} + 1)^{s_0} (n + 1)^{1-s_0} (n + l)^{\frac{3}{2}+2s+s_0} \lambda_{\tilde{n}, \tilde{l}}}.$$

We claim that, for $(\tilde{n}, \tilde{l}) \in \mathbb{N}^2$, $\tilde{n} + \tilde{l} \geq 2$,

$$(6.16) \quad (\tilde{n} + \tilde{l})^{2s} \leq 2(\tilde{n} + 1)^s (\tilde{n}^s + \tilde{l}^{2s}).$$

Indeed, the formula (6.16) holds for $\tilde{n} = 0$ or $\tilde{l} = 0$ under the assumption of $\tilde{n} + \tilde{l} \geq 2$. Now we assume $\tilde{n} \geq 1$ and $\tilde{l} \geq 1$. In fact, for $0 < s < 1$,

$$(\tilde{n} + \tilde{l})^{2s} \leq 2(\tilde{n}^{2s} + \tilde{l}^{2s}) \leq 2(\tilde{n} + 1)^s (\tilde{n}^s + \tilde{l}^{2s}).$$

This ends the proof of the formula (6.16). By using the inequality $\lambda_{\tilde{n}, \tilde{l}} \gtrsim \tilde{n}^s + \tilde{l}^{2s}$ in (2.7) and (6.16), we have

$$\frac{(\tilde{n} + \tilde{l})^{2s}}{\lambda_{\tilde{n}, \tilde{l}}} \lesssim (\tilde{n} + 1)^s.$$

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Then

$$\begin{aligned}
\mathbf{K}_1 &\lesssim \sum_{\substack{n+\tilde{n}=n^* \\ n+l \geq 2, \tilde{n}+\tilde{l} \geq 2 \\ n \geq 0, \tilde{n} \geq 0}} \sum_{\substack{l+\tilde{l}=l^* \\ l \geq 1, \tilde{l} \geq 1}} \frac{\tilde{l} \sqrt{l}}{l + \tilde{l} + 1} \frac{(\tilde{n} + 1)^s (\tilde{n} + \tilde{l})^{s_0}}{(\tilde{n} + 1)^{s_0} (n + 1)^{1-s_0} (n + l)^{\frac{3}{2}+2s+s_0}} \\
&\lesssim \sum_{\substack{n+\tilde{n}=n^* \\ n \geq 0, \tilde{n} \geq 0}} \sum_{\substack{l+\tilde{l}=l^* \\ l \geq 1, \tilde{l} \geq 1}} \frac{(\tilde{n} + 1)^s}{(n + 1)^{1-s_0} (n + l)^{1+2s+s_0}} \left(1 + \frac{\tilde{l}}{\tilde{n} + 1}\right)^{s_0} \\
&\lesssim \sum_{\substack{n+\tilde{n}=n^* \\ n \geq 0, \tilde{n} \geq 0}} \sum_{\substack{l+\tilde{l}=l^* \\ l \geq 1, \tilde{l} \geq 1}} \frac{(\tilde{n} + 1)^s}{(n + 1)^{1+s} (n + l)^{1+s}} \left(1 + \frac{\tilde{l}}{\tilde{n} + 1}\right)^{s_0}.
\end{aligned}$$

Considering $0 < s_0 = \min(1 - s, s) < 1$ and using the elementary inequality,

$$\left(1 + \frac{\tilde{l}}{\tilde{n} + 1}\right)^{s_0} \leq 1 + \left(\frac{\tilde{l}}{\tilde{n} + 1}\right)^{s_0},$$

we have

$$\mathbf{K}_1 \lesssim \sum_{\substack{n+\tilde{n}=n^* \\ n \geq 0, \tilde{n} \geq 0}} \sum_{\substack{l+\tilde{l}=l^* \\ l \geq 1, \tilde{l} \geq 1}} \frac{(\tilde{n} + 1)^s}{(n + 1)^{1+s} (n + l)^{1+s}} \left(1 + \left(\frac{\tilde{l}}{\tilde{n} + 1}\right)^{s_0}\right)$$

Recall that $s_0 = \min(s, 1 - s)$, if $s_0 = s$, one gets

$$(\tilde{n} + 1)^s \left(1 + \left(\frac{\tilde{l}}{\tilde{n} + 1}\right)^{s_0}\right) = (\tilde{n} + 1)^s + (\tilde{l})^s;$$

if $s_0 < s$, from Young's inequality,

$$\tilde{l}^{s_0} (\tilde{n} + 1)^{s-s_0} \leq \frac{s-s_0}{s} (\tilde{n} + 1)^s + \frac{s_0}{s} \tilde{l}^s,$$

we conclude that,

$$(\tilde{n} + 1)^s \left(1 + \left(\frac{\tilde{l}}{\tilde{n} + 1}\right)^{s_0}\right) = (\tilde{n} + 1)^s + \tilde{l}^{s_0} (\tilde{n} + 1)^{s-s_0} \lesssim (\tilde{n} + 1)^s + \tilde{l}^s.$$

Therefore,

$$\begin{aligned}
\mathbf{K}_1 &\lesssim \sum_{\substack{n+\tilde{n}=n^* \\ n \geq 0, \tilde{n} \geq 0}} \sum_{\substack{l+\tilde{l}=l^* \\ l \geq 1, \tilde{l} \geq 1}} \frac{(\tilde{n} + 1)^s + \tilde{l}^s}{(n + 1)^{1+s} (n + l)^{1+s}} \\
&\lesssim (n^* + l^*)^s \left[\sum_{\substack{n+\tilde{n}=n^* \\ n \geq 0, \tilde{n} \geq 0}} \frac{1}{(n + 1)^{1+s}} \sum_{\substack{l+\tilde{l}=l^* \\ l \geq 1, \tilde{l} \geq 1}} \frac{1}{(n + l)^{1+s}} \right] \\
&\lesssim (n^* + l^*)^s \left[\sum_{\substack{n+\tilde{n}=n^* \\ n \geq 0, \tilde{n} \geq 0}} \frac{1}{(n + 1)^{1+s}} \sum_{\substack{l+\tilde{l}=l^* \\ l \geq 1, \tilde{l} \geq 1}} \frac{1}{l^{1+s}} \right]
\end{aligned}$$

The set $\{(n, \tilde{n}) \in \mathbb{N}^2, n + \tilde{n} = n^*\}$ is a subset of hyperplane of dimension 1,

$$\sum_{\substack{n+\tilde{n}=n^* \\ n \geq 0, \tilde{n} \geq 0}} \frac{1}{(n + 1)^{1+s}} = \sum_{n=0}^{n^*} \frac{1}{(n + 1)^{1+s}} \leq \sum_{n=1}^{+\infty} \frac{1}{n^{1+s}} < +\infty.$$

Similarly,

$$\sum_{\substack{l+\tilde{l}=l^* \\ l \geq 1, \tilde{l} \geq 1}} \frac{1}{l^{1+s}} = \sum_{l=1}^{l^*-1} \frac{1}{l^{1+s}} \leq \sum_{l=1}^{+\infty} \frac{1}{l^{1+s}} < +\infty.$$

Therefore,

$$\mathbf{K}_1 \lesssim (n^* + l^*)^s.$$

The estimate of the second term $\mathbf{K}_2$:

By using (6.12) in Lemma 6.3, we have

$$\mathbf{K}_2 \lesssim \sum_{k=1}^{\min(19, n^*)} \sum_{\substack{n+\tilde{n}=n^*-k \\ n+l \geq 2, \tilde{n}+\tilde{l} \geq 2 \\ n \geq 0, \tilde{n} \geq 0}} \sum_{\substack{l+\tilde{l}=l^*+2k \\ l \geq k, \tilde{l} \geq k}} \frac{\tilde{l}\sqrt{l}}{l+\tilde{l}-2k+1} \frac{\tilde{n}^s(\tilde{n}+\tilde{l})^s}{(n+1)^s(n+l+1)^{\frac{s}{2}+s}} \lambda_{\tilde{n}, \tilde{l}}$$

Applying the inequality (2.7) that $\lambda_{\tilde{n}, \tilde{l}} \gtrsim (\tilde{n} + \tilde{l})^s$, one can re-estimated $\mathbf{K}_2$ as

$$\begin{aligned} \mathbf{K}_2 &\lesssim \sum_{k=1}^{\min(19, n^*)} \sum_{\substack{n+\tilde{n}=n^*-k \\ n+l \geq 2, \tilde{n}+\tilde{l} \geq 2 \\ n \geq 0, \tilde{n} \geq 0}} \sum_{\substack{l+\tilde{l}=l^*+2k \\ l \geq k, \tilde{l} \geq k}} \frac{\tilde{l}\sqrt{l}}{l+\tilde{l}-2k+1} \frac{\tilde{n}^s}{(n+1)^s(n+l+1)^{\frac{s}{2}+s}} \\ &\lesssim (n^*)^s \sum_{k=1}^{19} \sum_{n=0}^{n^*-k} \sum_{l=k}^{l^*+k} \frac{l^*+2k-l}{l^*+1} \frac{1}{(n+1)^s(n+l+1)^{2+s}} \\ &\lesssim (n^*)^s \sum_{k=1}^{19} \sum_{n=0}^{n^*-k} \sum_{l=k}^{l^*+k} \frac{l^*+19}{l^*+1} \frac{1}{(n+1)^s(n+l+1)^{2+s}}. \end{aligned}$$

Since

$$\begin{aligned} &\sum_{l=k}^{l^*+k} \frac{1}{(n+l+1)^{2+s}} \\ &\leq \sum_{l=k}^{l^*+k} \int_{l-1}^l \frac{1}{(n+x+1)^{2+s}} dx = \int_{k-1}^{l^*+k} \frac{1}{(n+x+1)^{2+s}} dx \\ &= \frac{1}{1+s} \left(\frac{1}{(n+k)^{1+s}} - \frac{1}{(n+k+l^*+1)^{1+s}} \right) \lesssim \frac{1}{(n+k)^{1+s}}, \end{aligned}$$

we can estimate that

$$\begin{aligned} \mathbf{K}_2 &\lesssim (n^*)^s \sum_{k=1}^{19} \sum_{n=0}^{n^*-k} \frac{1}{(n+k)^{1+s}} \\ &\lesssim (n^*)^s \sum_{n=0}^{n^*} \frac{1}{(n+1)^{1+s}} \lesssim (n^*)^s. \end{aligned}$$

Now we consider $k \geq 20$. In this case, we assume $n^* \geq 20$. Or else,

$$\Lambda_{n^*, l^*}^3 = \emptyset, \Lambda_{n^*, l^*}^4 = \emptyset.$$

For the third term $\mathbf{K}_3$, by using (i) of Lemma 6.5 with $\omega = \frac{1}{4}$, we have

$$\mathbf{K}_3 \lesssim \sum_{k=20}^{n^*} \sum_{\substack{n+\tilde{n}=n^*-k \\ 2 \leq n+l < k^{\frac{3}{4}}, \tilde{n}+\tilde{l} \geq 2 \\ n \geq 0, \tilde{n} \geq 0}} \sum_{\substack{l+\tilde{l}=l^*+2k \\ l \geq k, \tilde{l} \geq k}} \frac{\tilde{l}\sqrt{l}}{l+\tilde{l}-2k+1} \frac{(\tilde{n}+\tilde{l})^{2s}}{(n+l+1)^{\frac{s}{2}+2s}} \lambda_{\tilde{n},\tilde{l}} e^{-\frac{1}{8}k^{\frac{1}{4}}}$$

Applying the inequality (2.7) that $\lambda_{\tilde{n},\tilde{l}} \gtrsim (\tilde{n}+\tilde{l})^s$, we estimate that,

$$\begin{aligned} \mathbf{K}_3 &\lesssim \sum_{k=20}^{n^*} \sum_{\substack{n+\tilde{n}=n^*-k \\ 2 \leq n+l < k^{\frac{3}{4}}, \tilde{n}+\tilde{l} \geq 2 \\ n \geq 0, \tilde{n} \geq 0}} \sum_{\substack{l+\tilde{l}=l^*+2k \\ l \geq k, \tilde{l} \geq k}} \frac{\tilde{l}\sqrt{l}}{l+\tilde{l}-2k+1} \frac{(\tilde{n}+\tilde{l})^s}{(n+l+1)^{\frac{s}{2}+2s}} e^{-\frac{1}{8}k^{\frac{1}{4}}} \\ &\lesssim \sum_{k=20}^{n^*} \sum_{n=0}^{n^*-k} \sum_{l=k}^{l^*+k} \frac{l^*+k}{l^*+1} \frac{(n^*+l^*)^s}{(n+l+1)^{2+2s}} e^{-\frac{1}{4}k^{\frac{1}{4}}} \\ &= (n^*+l^*)^s \sum_{k=20}^{n^*} e^{-\frac{1}{4}k^{\frac{1}{4}}} \frac{l^*+k}{l^*+1} \sum_{n=0}^{n^*-k} \sum_{l=k}^{l^*+k} \frac{1}{(n+l+1)^{2+2s}}. \end{aligned}$$

Use the estimate

$$\begin{aligned} &\sum_{l=k}^{l^*+k} \frac{1}{(n+l+1)^{2+2s}} \\ &\leq \sum_{l=k}^{l^*+k} \int_{l-1}^l \frac{1}{(n+x+1)^{2+2s}} dx = \int_{k-1}^{l^*+k} \frac{1}{(n+x+1)^{2+2s}} dx \\ &= \frac{1}{1+2s} \left(\frac{1}{(n+k)^{1+2s}} - \frac{1}{(n+k+l^*+1)^{1+2s}} \right) \\ &= \frac{1}{1+2s} \frac{(n+k+l^*+1)^{1+2s} - (n+k)^{1+2s}}{(n+k)^{1+2s}(n+k+l^*+1)^{1+2s}}, \end{aligned}$$

and the mean value theorem for $f(x) = x^{1+2s}$,

$$(n+k+l^*+1)^{1+2s} - (n+k)^{1+2s} = (1+2s)(n+k+\tau(l^*+1))^{2s}(l^*+1), \quad (0 < \tau < 1),$$

we obtain,

$$\begin{aligned} \mathbf{K}_3 &\lesssim (n^*+l^*)^s \sum_{k=1}^{n^*} e^{-\frac{1}{8}k^{\frac{1}{4}}} \sum_{n=0}^{n^*-k} \frac{l^*+1}{(n+k)^{1+2s}(n+l^*+k)} \times \frac{l^*+k}{l^*+1} \\ &\lesssim (n^*+l^*)^s \sum_{k=1}^{n^*} e^{-\frac{1}{4}k^{\frac{1}{4}}} \sum_{n=0}^{n^*} \frac{1}{(n+1)^{1+2s}} \\ &\lesssim (n^*+l^*)^s. \end{aligned}$$

Finally, we estimate the remaining term $\mathbf{K}_4$. By using (ii) of Lemma 6.5 with $\omega = \frac{1}{4}$, we have

$$\mathbf{K}_4 \lesssim \sum_{k=20}^{n^*} \sum_{\substack{n+\tilde{n}=n^*-k \\ n+l \geq k^{\frac{3}{4}}, \tilde{n}+\tilde{l} \geq 2 \\ n \geq 0, \tilde{n} \geq 0}} \sum_{\substack{l+\tilde{l}=l^*+2k \\ l \geq k, \tilde{l} \geq k}} \frac{\tilde{l}\sqrt{l}}{l+\tilde{l}-2k+1} \frac{(\tilde{n}+\tilde{l})^{2s}}{(n+l+1)^{\frac{s}{2}+2s}} \lambda_{\tilde{n},\tilde{l}}$$

Applying the inequality (2.7) that $\lambda_{\tilde{n}, \tilde{l}} \gtrsim (\tilde{n} + \tilde{l})^s$, we estimate that

$$\mathbf{K}_4 \lesssim \sum_{k=20}^{n^*} \sum_{\substack{n+\tilde{n}=n^*-k \\ n+l \geq k^{\frac{3}{4}}, \tilde{n}+\tilde{l} \geq 2 \\ n \geq 0, \tilde{n} \geq 0}} \sum_{\substack{l+\tilde{l}=l^*+2k \\ l \geq k, \tilde{l} \geq k}} \frac{\tilde{l} \sqrt{l}}{l + \tilde{l} - 2k + 1} \frac{(\tilde{n} + \tilde{l})^s}{(n + l + 1)^{\frac{5}{2} + 2s}}$$

Considering the condition

$$n + l \geq k^{\frac{3}{4}} l,$$

we obtain

$$\frac{1}{(n + l)^{\frac{3}{2}}} \leq \frac{1}{k^{\frac{9}{8}} l^{\frac{3}{2}}}.$$

Then $\mathbf{K}_4$ can be rewritten as

$$\begin{aligned} \mathbf{K}_4 &\lesssim \sum_{k=20}^{n^*} \sum_{\substack{n+\tilde{n}=n^*-k \\ n+l \geq k^{\frac{3}{4}}, \tilde{n}+\tilde{l} \geq 2 \\ n \geq 0, \tilde{n} \geq 0}} \sum_{\substack{l+\tilde{l}=l^*+2k \\ l \geq k, \tilde{l} \geq k}} \frac{\tilde{l} \sqrt{l}}{l + \tilde{l} - 2k + 1} \frac{(\tilde{n} + \tilde{l})^s}{(n + l)^{1+2s} k^{\frac{9}{8}} l^{\frac{3}{2}}} \\ &\lesssim (n^* + l^*)^s \sum_{k=20}^{n^*} \frac{1}{k^{\frac{9}{8}}} \sum_{n=0}^{n^*-k} \sum_{l=k}^{l^*+k} \frac{1}{(n + l)^{1+2s} l} \frac{l^* + k}{l^* + 1} \\ &\lesssim (n^* + l^*)^s \sum_{k=20}^{n^*} \frac{1}{k^{\frac{9}{8}}} \left[\sum_{n=0}^{n^*-k} \sum_{l=k}^{l^*+k} \frac{1}{(n + l)^{1+2s} l} + \sum_{n=0}^{n^*-k} \sum_{l=k}^{l^*+k} \frac{1}{(n + l)^{1+2s} l} \frac{k}{l^* + 1} \right] \end{aligned}$$

It is obviously that, for $k \geq 1$,

$$\sum_{n=0}^{n^*-k} \sum_{l=k}^{l^*+k} \frac{1}{(n + l)^{1+2s} l} \leq \sum_{n=0}^{n^*} \frac{1}{(n + 1)^{1+s}} \sum_{l=k}^{l^*+k} \frac{1}{l^{1+s}} < +\infty.$$

At the same time,

$$\begin{aligned} \sum_{n=0}^{n^*-k} \sum_{l=k}^{l^*+k} \frac{1}{(n + l)^{1+2s} l} \frac{k}{l^* + 1} &\leq \sum_{n=0}^{n^*-k} \frac{1}{(n + 1)^{1+2s}} \sum_{l=k}^{l^*+k} \frac{1}{l^* + 1} \\ &\leq \sum_{n=0}^{n^*-k} \frac{1}{(n + 1)^{1+2s}} < +\infty. \end{aligned}$$

Substitute these two estimates into the above inequality of $\mathbf{K}_4$, we have

$$\mathbf{K}_4 \lesssim (n^* + l^*)^s \sum_{k=20}^{n^*} \frac{1}{k^{\frac{9}{8}}} \lesssim (n^* + l^*)^s.$$

Combining the estimates of $\mathbf{K}_1$, $\mathbf{K}_2$, $\mathbf{K}_3$ and $\mathbf{K}_4$, using again (2.7)

$$(n^* + l^*)^s + (l^*)^{2s} \lesssim \lambda_{n^*, l^*},$$

we derive

$$\sum_{\substack{n+\tilde{n}+k=n^* \\ n+l \geq 2, \tilde{n}+\tilde{l} \geq 2 \\ n \geq 0, \tilde{n} \geq 0}} \sum_{\substack{l+\tilde{l}-2k=l^* \\ l \geq 1, \tilde{l} \geq 1 \\ 0 \leq k \leq \min(l, \tilde{l})}} \frac{\lambda_{n, \tilde{n}, \tilde{l}, l}^k}{\lambda_{\tilde{n}, \tilde{l}}} \leq C \lambda_{n^*, l^*}.$$

This ends the proof of 3) in Proposition 3.1. □

7. APPENDIX

The important known results but really needed for this paper are presented in this section. For the self-content of paper, we will present some proof of those properties.

7.1. Gelfand-Shilov spaces. The symmetric Gelfand-Shilov space $S_v^\nu(\mathbb{R}^3)$ can be characterized through the decomposition into the Hermite basis $\{H_\alpha\}_{\alpha \in \mathbb{N}^3}$ and the harmonic oscillator $\mathcal{H} = -\Delta + \frac{|v|^2}{4}$. For more details, see Theorem 2.1 in the book [7]

$$\begin{aligned} f \in S_v^\nu(\mathbb{R}^3) &\Leftrightarrow f \in C^\infty(\mathbb{R}^3), \exists \tau > 0, \|e^{\tau \mathcal{H}^{\frac{1}{2\nu}}} f\|_{L^2} < +\infty; \\ &\Leftrightarrow f \in L^2(\mathbb{R}^3), \exists \epsilon_0 > 0, \left\| \left(e^{\epsilon_0 |\alpha|^{\frac{1}{2\nu}}} (f, H_\alpha)_{L^2} \right)_{\alpha \in \mathbb{N}^3} \right\|_{l^2} < +\infty; \\ &\Leftrightarrow \exists C > 0, A > 0, \|(-\Delta + \frac{|v|^2}{4})^{\frac{k}{2}} f\|_{L^2(\mathbb{R}^3)} \leq AC^k (k!)^\nu, \quad k \in \mathbb{N} \end{aligned}$$

where

$$H_\alpha(v) = H_{\alpha_1}(v_1)H_{\alpha_2}(v_2)H_{\alpha_3}(v_3), \quad \alpha \in \mathbb{N}^3,$$

and for $x \in \mathbb{R}$,

$$H_n(x) = \frac{(-1)^n}{\sqrt{2^n n! \pi}} e^{\frac{x^2}{2}} \frac{d^n}{dx^n} (e^{-x^2}) = \frac{1}{\sqrt{2^n n! \pi}} \left(x - \frac{d}{dx} \right)^n (e^{-\frac{x^2}{2}}).$$

For the harmonic oscillator $\mathcal{H} = -\Delta + \frac{|v|^2}{4}$ of 3-dimension and $s > 0$, we have

$$\mathcal{H}^{\frac{k}{2}} H_\alpha = (\lambda_\alpha)^{\frac{k}{2}} H_\alpha, \quad \lambda_\alpha = \sum_{j=1}^3 \left(\alpha_j + \frac{1}{2} \right), \quad k \in \mathbb{N}, \quad \alpha \in \mathbb{N}^3.$$

7.2. Fourier Transform of special functions. For the eigenvector $\varphi_{n,l,m}$ given in (2.1), Lerner, Morimoto, Pravda-Starov and Xu in [13] shows in Lemma 3.2 the Fourier transform of $\sqrt{\mu} \varphi_{n,0,0}$. Following this work, we provide the Fourier transform of $\sqrt{\mu} \varphi_{n,l,m}$.

Lemma 7.1. *Let $\alpha, \kappa \in S^2$ and $r > 0$, then*

$$(7.1) \quad \int_{S_k^2} e^{ir\kappa \cdot \alpha} Y_l^m(\kappa) d\kappa = (2\pi)^{\frac{3}{2}} i^l \left(\frac{1}{r} \right)^{\frac{1}{2}} J_{l+\frac{1}{2}}(r) Y_l^m(\alpha),$$

where $\kappa \cdot \alpha$ denote the scalar product and $J_{l+\frac{1}{2}}$ is the Bessel function of $l + \frac{1}{2}$ order.

Proof. Since for any real r and $|z| \leq 1$,

$$(7.2) \quad \sqrt{2r} e^{irz} = \sqrt{\pi} \sum_{k=0}^{\infty} (2k+1) J_{k+\frac{1}{2}}(r) (i)^k P_k(z) \quad (\text{cf. (1) of Section 11.5 in [24]}).$$

Substituting $z = \kappa \cdot \alpha$ into (7.2),

$$e^{ir\kappa \cdot \alpha} = \sqrt{\frac{\pi}{2r}} \sum_{k=0}^{\infty} (2k+1) J_{k+\frac{1}{2}}(r) i^k P_k(\kappa \cdot \alpha).$$

By using the definition of the Bessel function $J_{k+\frac{1}{2}}(r)$, see (8) of Sec. 3.1, Chap.III in [24],

$$J_{k+\frac{1}{2}}(r) = \sum_{m=0}^{\infty} \frac{(-1)^m (r/2)^{2m+k+\frac{1}{2}}}{m! \Gamma(m+k+\frac{3}{2})}.$$

Then

$$|J_{k+\frac{1}{2}}(r)| \leq (|r|/2)^{k+\frac{1}{2}} \sum_{m=0}^{\infty} \frac{(|r|/2)^{2m}}{m! k!} \leq (|r|/2)^{k+\frac{1}{2}} \frac{1}{k!} e^{\frac{|r|^2}{4}}$$

Now since $|P_k(\kappa \cdot \alpha)| \leq 1$ and for every $r > 0$,

$$\sqrt{\frac{1}{r}} \sum_{k=0}^{\infty} (2k+1) |J_{k+\frac{1}{2}}(r)| \leq \sum_{k=0}^{\infty} \frac{(\frac{r}{2})^k}{k!} e^{\frac{r^2}{4}} \leq e^{\frac{r}{2} + \frac{r^2}{4}},$$

we obtain that

$$(7.3) \quad \int_{S_\kappa^2} e^{ir\kappa \cdot \alpha} Y_l^m(\kappa) d\kappa = \sqrt{\frac{\pi}{2r}} \sum_{k=0}^{\infty} (2k+1) J_{k+\frac{1}{2}}(r) i^k \int_{S_\kappa^2} P_k(\kappa \cdot \alpha) Y_l^m(\kappa) d\kappa.$$

Consider the addition theorem of spherical harmonics (7 – 34) in Chapter 7 of [19] that

$$P_k(\kappa \cdot \alpha) = \frac{4\pi}{2k+1} \sum_{m=-k}^k [Y_k^m(\kappa)]^* Y_k^m(\alpha),$$

where $[Y_k^m(\kappa)]^*$ is the conjugate of $Y_k^m(\kappa)$. Then

$$\int_{S_\kappa^2} P_k(\kappa \cdot \alpha) Y_l^m(\kappa) d\kappa = \frac{4\pi}{2l+1} Y_l^m(\alpha) \delta_{k,l}.$$

Substitute this addition formula into (7.3), the formula (7.1) follows. $\square$

Lemma 7.1 is the basis for calculating the Fourier transform of $\sqrt{\mu} \varphi_{n,l,m}$.

Lemma 7.2. *Let $\varphi_{n,l,m}$ be the functions defined in (2.1), then for $n, l \in \mathbb{N}$, $|m| \leq l$, we have*

$$(7.4) \quad \widehat{\sqrt{\mu} \varphi_{n,l,m}}(\xi) = (-i)^l (2\pi)^{\frac{3}{4}} \left(\frac{1}{\sqrt{2} n! \Gamma(n+l+\frac{3}{2})} \right)^{\frac{1}{2}} \left(\frac{|\xi|}{\sqrt{2}} \right)^{2n+l} e^{-\frac{|\xi|^2}{2}} Y_l^m \left(\frac{\xi}{|\xi|} \right).$$

Proof. Define $H(\xi) = \left(\frac{|\xi|}{\sqrt{2}} \right)^{2n+l} e^{-\frac{|\xi|^2}{2}} Y_l^m \left(\frac{\xi}{|\xi|} \right)$, and by Lemma 7.1 with $r = |v||\xi|$, $\alpha = \frac{v}{|v|}$, $\kappa = \frac{\xi}{|\xi|}$, we can compute the inverse Fourier transform of H ,

$$\begin{aligned} \mathcal{F}^{-1}(H)(v) &= \left(\frac{1}{2\pi} \right)^3 \int_{\mathbb{R}^3} e^{iv \cdot \xi} \left(\frac{|\xi|}{\sqrt{2}} \right)^{2n+l} e^{-\frac{|\xi|^2}{2}} Y_l^m \left(\frac{\xi}{|\xi|} \right) d\xi \\ &= \frac{1}{4\pi^3} \int_0^\infty \left(\frac{\rho}{\sqrt{2}} \right)^{2n+l+2} e^{-\frac{\rho^2}{2}} \left(\int_{S_\kappa^2} e^{i|v|\rho \kappa \cdot \alpha} Y_l^m(\kappa) d\kappa \right) d\rho \\ &= i^l \left(\frac{1}{2\pi} \right)^{\frac{3}{2}} Y_l^m \left(\frac{v}{|v|} \right) \left(\frac{2\sqrt{2}}{|v|} \right)^{\frac{1}{2}} \int_0^\infty \left(\frac{\rho}{\sqrt{2}} \right)^{2n+l+\frac{3}{2}} e^{-\frac{\rho^2}{2}} J_{l+\frac{1}{2}}(|v|\rho) d\rho. \end{aligned}$$

By using the standard formula, see (6.2.15) in [8],

$$L_n^{(l+\frac{1}{2})}(x) = \frac{e^x x^{-\frac{l+\frac{1}{2}}{2}}}{n!} \int_0^{+\infty} t^{n+\frac{l+\frac{1}{2}}{2}} J_{l+\frac{1}{2}}(2\sqrt{xt}) e^{-t} dt,$$

we have

$$L_n^{(l+\frac{1}{2})} \left(\frac{|v|^2}{2} \right) = \sqrt{2} \frac{e^{\frac{|v|^2}{2}} \left(\frac{|v|}{\sqrt{2}} \right)^{-(l+\frac{1}{2})}}{n!} \int_0^\infty \left(\frac{\rho}{\sqrt{2}} \right)^{2n+l+\frac{3}{2}} J_{l+\frac{1}{2}}(|v|\rho) e^{-\frac{\rho^2}{2}} d\rho.$$

Therefore,

$$\mathcal{F}^{-1}(H)(v) = \left(\frac{1}{2\pi} \right)^{\frac{3}{2}} n! i^l e^{-\frac{|v|^2}{2}} \left(\frac{|v|}{\sqrt{2}} \right)^l L_n^{(l+\frac{1}{2})} \left(\frac{|v|^2}{2} \right) Y_l^m \left(\frac{v}{|v|} \right).$$

Recall the expression of (2.1), one can verify

$$\widehat{\sqrt{\mu} \varphi_{n,l,m}}(v) = (-i)^l (2\pi)^{\frac{3}{4}} \left(\frac{1}{\sqrt{2} n! \Gamma(n+l+\frac{3}{2})} \right)^{\frac{1}{2}} \mathcal{F}^{-1}(H)(v).$$

Henceforth, (7.4) yields. $\square$

7.3. Spherical Harmonics. The following results with respect to the spherical harmonics is significant. For $l, \tilde{l} \in \mathbb{N}, |m| \leq l, |\tilde{m}| \leq \tilde{l}$,

$$(7.5) \quad Y_l^m Y_{\tilde{l}}^{\tilde{m}} = \sum_{l'} \sum_{m'} \left(\int_{\mathbb{S}^2} Y_l^m(\kappa) Y_{\tilde{l}}^{\tilde{m}}(\kappa) Y_{l'}^{-m'}(\kappa) d\kappa \right) Y_{l'}^{m'}$$

where $|m'| \leq l'$ and $\tilde{l} - l \leq l' \leq \tilde{l} + l$. More explicitly, in order to have a non-vanishing integral, the parameters m', l' satisfy

$$(7.6) \quad m' = m + \tilde{m}, \quad l' = l + \tilde{l} - 2j \text{ with } 0 \leq j \leq \min(l, \tilde{l}).$$

For more details, see (86) in Chap. 3 in [9] or Theorem 2.1 of Sec.2, Chap.IV in [17].

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LÉO GLANGETAS,
 UNIVERSITÉ DE ROUEN, CNRS UMR 6085, MATHÉMATIQUES
 76801 SAINT-ETIENNE DU ROUVRAY, FRANCE
E-mail address: leo.glangetas@univ-rouen.fr

HAO-GUANG LI,
 SCHOOL OF MATHEMATICS AND STATISTICS, WUHAN UNIVERSITY 430072, WUHAN, P. R. CHINA
 SCHOOL OF MATHEMATICS AND STATISTICS, SOUTH-CENTRAL UNIVERSITY FOR NATIONALITIES 430074,
 WUHAN, P. R. CHINA
E-mail address: lihaoguang@mail.scuec.edu.cn

CHAO-JIANG XU,
 SCHOOL OF MATHEMATICS AND STATISTICS, WUHAN UNIVERSITY 430072, WUHAN, P. R. CHINA
 UNIVERSITÉ DE ROUEN, CNRS UMR 6085, MATHÉMATIQUES
 76801 SAINT-ETIENNE DU ROUVRAY, FRANCE
E-mail address: chao-jiang.xu@univ-rouen.fr