The Page-Rényi parking process

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Abstract

In the Page parking (or packing) model on a discrete interval (also known as the *discrete Rényi packing problem* or the *unfriendly seating problem*), cars of length two successively park uniformly at random on pairs of adjacent places, until only isolated places remain. We give a probabilistic proof of the (known) fact that the proportion of the interval occupied by cars goes to $1 - e^{-2}$, when the length of the interval goes to infinity.

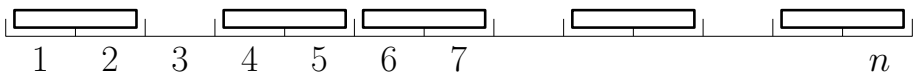
We obtain some new consequences, and also study a version of this process defined on the infinite line.

1 The Page parking

1.1 The model

For $n \geq 2$, we consider a sequence of *parking configurations* $x^t = \left(x_i^t\right)_{1 \leq i \leq n}$ in $\{0,1\}^n$, given by the following construction. Initially the parking is empty: $x^0 = 0^n$. Given x^t one draws uniformly at random (and independently from the past) a number i in $\{1, 2, \ldots, n-1\}$ and, if possible, a car of size 2 *parks* at i: if $x_i^t = x_{i+1}^t = 0$ then $x_i^{t+1} = x_{i+1}^{t+1} = 1$; $n-2$ other coordinates remain unchanged.

After some random time T_n (which is dominated by a coupon collector process with n coupons, see Section (3.2) below) parking is no more possible in x^{T_n}, in the sense that there is no adjacent coordinates $(i, i+1)$ such that $x_i^{T_n} = x_{i+1}^{T_n} = 0$. We set $X_n = x^{T_n}$ and $X_n(i) = x_i^{T_n}$. Below is an example where $X_n = (1,1,0,1,1,1,1,0,1,1,0,1,1)$.



We are mainly interested in the numbers of places occupied by a car.

$$M_n = \text{card}\left\{1 \leq i \leq n,\ X_n(i) = 1\right\}.$$

We obviously have $n/2 \leq M_n \leq n$ (the worst case being $n = 4$ and $X_n = (0,1,1,0)$) and we expect M_n/n to converge, at least in some sense.

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Theorem 1 (Page (1959))**.**

$$\lim_{n\to+\infty} \frac{\mathbb{E}[M_n]}{n} = 1 - e^{-2} = 0.8646647\ldots$$

A nice heuristic for the limit $1 - e^{-2}$ is given by Page, it is based on a recursion on $\mathbb{P}(X_n(1) = 0)$. The proof of Theorem 1 is essentially obtained on conditioning on the position i of the first car. This gives the recursion identity

$$M_n \overset{(\mathrm{d})}{=} M_{I-1} + M'_{n-I-1} + 2,$$

where M, M', I are independent, I is uniform in $\{1, 2, \ldots, n-1\}$, and we have set $M_0 = M_1 = 0$ a.s. This gives a recursion for the moments of M_n, which can be handled using generating functions.

The Page parking problem has a long story, it has been studied by many people and under different names. It is equivalent to the *unfriendly seating* problem [5], and sometimes also called the *discrete Rényi Packing* model [7], more generally, it is a toy model for *random deposition*. We refer to [10, 9] for some interpretations of the model in polymer chemistry. Theorem 1 can also be found in [3, 5, 11] with similar proofs, Page also obtained a variance estimate (see also [1, 13]) which proves that M_n/n converges to $1 - e^{-2}$ in probability.

We also mention that much is known also when cars have size $\ell > 2$, we refer to [8, 11]. There are also several similar models which are time-continuous (see for instance [6, 7]).

The aim of the present paper is to present a probabilistic (and apparently new) proof of Theorem 1 and to study the asymptotic behavior of T_n (the numbers of cars that have tried to park). We also study a version of this process defined on the whole line.

1.2 The probabilistic construction

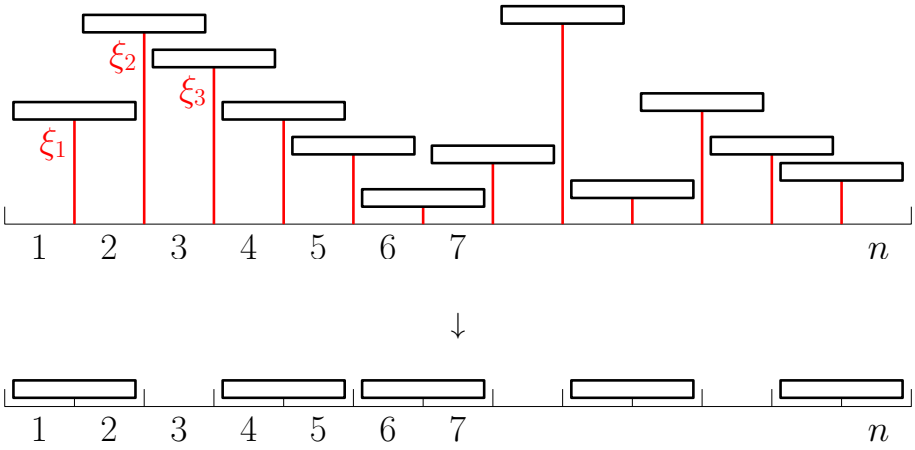
An alternative way of defininining M_n is the following. Let $\xi = (\xi_i)_{1\le i\le n-1}$ be i.i.d. random variables with continuous distribution function F (hereafter we will take $F(t) = 1 - e^{-t}$). Then the order statistics $\left(\xi_{(1)} < \cdots < \xi_{(n-1)}\right)$ give the order in which the cars park:

- the first car parks at $(\xi_{(1)}, \xi_{(1)} + 1)$,
- the second one parks (if possible) at $(\xi_{(2)}, \xi_{(2)} + 1)$,
- ...

It is easy to see that we obtain the same distribution, and was already observed by previous authors (see [13]). Let us collect for further use some obvious features of this contruction.

Remark.
- *The configuration X_n only depends on the ordering fo the ξ_i's.*
- *If i is a local minimum of ξ (*i.e. *$\xi_{i-1} > \xi_i < \xi_{i+1}$, with $\xi_0 = \xi_n = +\infty$) then places i and $i+1$ are empty at time ξ_i and a car parks at $(i, i+1)$.*
- *To define X_n from the ξ_i's, one can treat separately the intervals defined by two successive local minima.*

Here is a sample of ξ (here $\xi_{(1)} = \xi_6$) and the corresponding configuration X_n:



It does not seem that this construction was used to its full extent, yet it gives a very simple way to characterize positions i occupied by a car. We need a few definitions.

We say that there is a *rise* of length ℓ at i if $i > \ell$ and

$$\xi_{i-\ell-1} > \xi_{i-\ell} < \xi_{i-\ell+1} < \xi_{i-\ell+2} < \cdots < \xi_{i-1}$$

or if $i = \ell$ and

$$\xi_1 < \xi_2 < \cdots < \xi_{i-1}.$$

There is a *descent* of length ℓ at i if $i < n - \ell$ and

$$\xi_i > \xi_{i+1} > \cdots > \xi_{i+\ell-1} < \xi_{i+\ell}$$

or if $i = n - \ell$ and

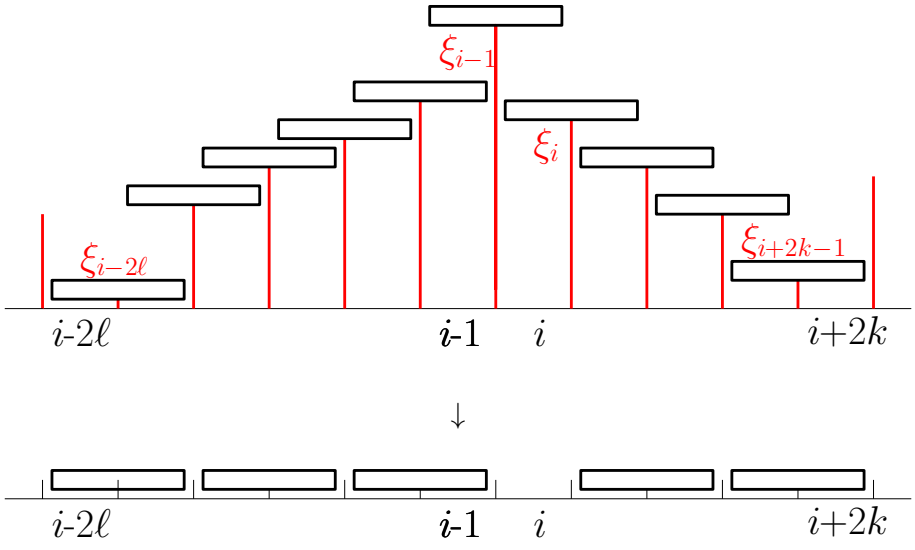
$$\xi_i > \xi_{i+1} > \cdots > \xi_{n-1}.$$

Consistently we say that there is a rise (resp. a descent) of length 1 at i if $\xi_{i-2} > \xi_{i-1}$ (resp. if $\xi_i < \xi_{i+1}$), so that for every i there is a rise and a descent at i.

Note that by construction the events { rise of length ℓ at i } and { descent of length ℓ' at i } are independent for every i, ℓ, ℓ'.

Lemma 1. *There is no car at i (*i.e. *$X_n(i) = 0$) if and only if there is rise of even length at i and a descent of even length at i.*

An example of a rise of length 6 at i and a descent of length 4:



Proof of Lemma 1. As already said, if i is a local minimum then a car parks at $(i, i+1)$. Consider first the case where i is inside a rise: $\xi_{r-1} > \xi_r < \xi_{r+1} < \xi_{r+2} < \cdots < \xi_i < \xi_{i+1}$ for some r. Then cars successively park at $(r, r+1)$, $(r+2, r+3)$, ..., until $(i-1, i)$ or $(i, i+1)$ (depending on the parity of $i-r$). Then $X_n(i) = 1$, and the same goes in the case where i is inside a descent.

Therefore the only case where $X_n(i)$ could be zero is if i is a local maximum for ξ. Define m_i (resp. m_i') as the closest local minimum of ξ on the left of i (resp. on the right). A rise begins at m_i and a descent ends at m_i' and $m_i = i - s$, $m' - i = i + s' - 1$ where s, s' are the lengths of these rise and descent.

Cars successively try to fill places of the rise $(m_i, m_i + 1)$, $(m_i + 2, m_i + 3)$, ... from left to right and places of the descent $(m_i', m_i' + 1)$, $(m_i' - 2, m_i' - 1)$, ... from right to left. If only s is odd (resp. only s') then i is occupied by a car of the rise (resp. descent). If both are odd a car parks at $(i-1, i)$ or $(i, i+1)$ depending on wether $\xi_{i-1} < \xi_i$ or not. If both are even the rightmost car of the rise parks at $(i-2, i-1)$ and the leftmost car of the descent parks at $(i+1, i+2)$, leaving i unoccupied.

□

2 The infinite parking

An interesting feature of the probabilistic construction is that it allows to define the model $(X_\infty(i))_{i\in\mathbb{Z}}$ on $\mathbb{Z}$, by considering an infinite sequence $(\xi_i)_{i\in\mathbb{Z}}$. Since $\inf\{i \in \mathbb{Z}\}\, \xi_i = 0$, "*the* first car" does not make sense but we still can define the model: we first set $X_\infty(i) = X_\infty(i+1) = 1$ for every i such that ξ_i is a local minimum, and then we define $X_\infty(i)$ as before, using only the ξ's between m_i, m_i'. (Note that with probability one, for all i one has $m_i' - m_i < +\infty$.) Lemma 1 also holds for the infinite parking.

Theorem 2 (The density of the parking on $\mathbb{Z}$)**.** *For every i,*

$$\mathbb{P}(X_\infty(i) = 0) = 1 - e^{-2}.$$

In fact Theorem 2 follows from Theorem 3 below, but we give here a short proof, whose interest is to give a natural interpretation of the limit.

Proof. By construction the rise at i and the descent at i are independent. By Lemma 1 we have

$$\begin{aligned}
\mathbb{P}(X_n(i) = 0) &= \mathbb{P}(\text{ even rise at } i\ )\mathbb{P}(\text{ event descent at } i\ ) \\
&= \mathbb{P}(\text{ even descent at } i)^2 \\
&= \left(\sum_{\ell\geq 1} \mathbb{P}\left(\xi_i > \xi_{i+1} > \cdots > \xi_{i+2\ell-1} < \xi_{i+2\ell}\right)\right)^2 \\
&= \left(\sum_{\ell\geq 1} \mathbb{P}\left(\sigma_1 > \sigma_2 > \cdots > \sigma_{2\ell} < \sigma_{2\ell+1}\right)\right)^2
\end{aligned}$$

where σ is a uniform permutation of $2\ell+1$ elements. There are 2ℓ permutations such that $\sigma_1 > \sigma_2 > \cdots > \sigma_{2\ell} < \sigma_{2\ell+1}$, we get

$$\mathbb{P}(X_\infty(i)=0) = \left(\sum_{\ell\geq 1} \frac{2\ell}{(2\ell+1)!}\right)^2 = (1/e)^2. \tag{1}$$

(The sum is the derivative of $\sum_{\ell\geq 1} \frac{z^{2\ell}}{(2\ell+1)!} = \sinh(z)/z - 1$ at $z=1$.) □

Obviously $X_\infty(i)$ are not independent but it is possible to prove with this construction that the doubly infinite sequence $(X_\infty(i))_{i\in\mathbb{Z}}$ is *strongly mixing*. Thus we could prove (using for instance [12]) a Central Limit Theorem for the sequence $(X_\infty(i))_{i\geq 1}$. This was done by [13] for the parking on a finite interval.

2.1 Evolution of the density

We now consider the process given by the time arrivals of cars. As above $X_\infty(i)$ is the indicator that there is eventually a car at i. We define the process $\left(X^t_\infty\right)_{t\geq 0}$ with values in $\{0,1\}^{\mathbb{Z}}$ by

$$X^t_\infty(i) = \begin{cases} 1 & \text{if } X_\infty(i)=1 \text{ and } \tau_i \leq t, \\ 0 & \text{otherwise.} \end{cases}$$

Here, $\tau_i = \xi_{i-1}$ if the car parked at i is parked at $(i-1,i)$ and $\tau_i = \xi_i$ if this car is at $(i,i+1)$. Then τ_i is indeed the time arrival of the corresponding car, we set $\tau_i = +\infty$ if there is no car at i.

Recall that F is the distribution function of the variables ξ. Note that in the case where $F(t) = 1-e^{-t}$ then $(X^t_\infty)_{t\geq 0}$ defines a homogeneous Markov process.

Theorem 3 (Evolution of the density of cars)**.**
Let τ_i be the arrival time of the car i,

$$\mathbb{E}[X^t_\infty(i)] = \mathbb{P}(\tau_i \leq t) = 1 - e^{-2F(t)}.$$

Of course we recover $\mathbb{P}(X_\infty = 1) = \mathbb{P}(\tau_i < +\infty) = \lim_{t\to+\infty} 1-e^{-2F(t)} = 1-e^{-2}$. Note that the particular case of Theorem 3 with $F(t) = 1-e^{-t}$ was proved in ([7], eq. (19)), using Markovianity.

Proof. By translation-invariance we assume $i=0$. Lemma 1 gives that $\tau_i \leq t$ if and only if

- there is an odd rise at 0 and $\xi_{-1} \leq t$,
- or there is an odd descent at 0 and $\xi_0 \leq t$.

These two events being independent we have

$$\mathbb{P}(\tau_i \leq t) = 2f(t) - f(t)^2 = f(t)\,(2-f(t))$$

where

$$f(t) = \mathbb{P}(\xi_0 \leq t;\ \text{odd descent at } 0\ ).$$

Now,

$$\begin{aligned}
f(t) &= \mathbb{P}\left(t \geq \xi_0 < \xi_1\right) + \sum_{k\geq 1} \mathbb{P}\left(t \geq \xi_0 > \xi_1 > \cdots > \xi_{2k} < \xi_{2k+1}\right) \\
&= \int_0^t (1-F(r))dF(r) + \sum_{k\geq 1} \int_0^t \mathbb{P}\left(t \geq \xi_0 > \xi_1 > \cdots > \xi_{2k-1} > r \ ; \ r < \xi_{2k+1}\right) dF(r), \\
&= \int_0^t (1-F(r))dF(r) + \sum_{k\geq 1} \int_0^t \mathbb{P}\left(t \geq \xi_0 > \xi_1 > \cdots > \xi_{2k-1} > r\right) \mathbb{P}\left(r < \xi_{2k+1}\right) dF(r),
\end{aligned}$$

at second line we have conditioned respectively on $\{\xi_0 = r\}$ and on $\{\xi_{2k} = r\}$.

Set $A = \{\xi_0, \xi_1, \ldots, \xi_{2k-1} \in (r,t)\}$, then $\mathbb{P}(A) = (F(t)-F(r))^{2k}$ and conditional on A these random variables are ordered as a uniform permutation:

$$\begin{aligned}
f(t) &= \int_{r=0}^t (1-F(r))dF(r) + \sum_{k\geq 1} \int_0^t \frac{1}{(2k)!}(F(t)-F(r))^{2k}(1-F(r))dF(r) \\
&= \int_0^t \sum_{k\geq 0} \frac{1}{(2k)!}(F(t)-F(r))^{2k}(1-F(r))dF(r) \\
&= \int_0^{F(t)} \sum_{k\geq 0} \frac{1}{(2k)!}(F(t)-s)^{2k}(1-s)ds \\
&= \int_0^{F(t)} \cosh\left(F(t)-s\right)(1-s)ds \\
&= 1 - \exp\big(-F(t)\big).
\end{aligned}$$

□

3 Parking on an interval

3.1 The convergence

We obtain the following refinement of Theorem 1:

Theorem 4 (The density of cars on an interval)**.** *Let $n \geq 2$,*

$$\left|\mathbb{E}[M_n] - n(1-e^{-2})\right| \leq 12. \tag{2}$$

This estimate is not as tight as $\mathbb{E}[M_n] = n(1-e^{-2}) + (1-3e^{-2}) + \mathrm{o}(1)$ which has been proved by Friedman [5] (see also [4]). The proof we provide here is more probabilist, we use ξ_i's to make a coupling between finite and infinite parkings. This provides a simple estimate on $\mathbb{P}(X_n(i) = 0)$ which will be useful later.

Proof. Let $(\xi_i)_{i\in\mathbb{Z}}$ be as before a sequence of i.i.d. continuous random variables of common distribution function F, we use the same ξ_i's to define $(X_n(i))_{1\leq i\leq n}$ and $(X_\infty(i))_{i\in\mathbb{Z}}$. Let m (resp. m') be the leftmost (resp. rightmost) local minimum of ξ in $\{2,\ldots,n-2\}$. We set $m = m' = +\infty$ if there is no local minimum.

By contruction, $X_n(i)$ and $X_\infty(i)$ coincide for every $i \in [m, m'+1]$. Therefore

$$\begin{aligned}\left|\mathbb{E}[M_n] - n(1-e^{-2})\right| &= \left|\sum_{i=1}^{n} \mathbb{E}[X_n(i)] - \mathbb{E}[X_\infty(i)]\right| \\ &\leq \mathbb{E}\left[\operatorname{card}\left\{i;\ X_n(i) \neq X_\infty(i)\right\}\right] \\ &\leq \mathbb{E}[m\mathbf{1}_{m<+\infty}] + \mathbb{E}[m'\mathbf{1}_{m'<+\infty}] + n\mathbb{P}(m = +\infty).\end{aligned}$$

There are different ways to bound $\mathbb{E}[m\mathbf{1}_{m<+\infty}]$, an easy one is to observe that local minima appear independently at $2, 5, 8, \ldots$, with probability $\mathbb{P}(\xi_1 > \xi_2 < \xi_3) = 1/3$.

Thus $\mathbb{P}\left(m \geq i\right) \leq (2/3)^{\lfloor i/3 \rfloor} \leq (2/3)^{i/3-1}$ and $\mathbb{E}[m\mathbf{1}_{m<+\infty}] \leq 11$ for every n. Finally, we have $\mathbb{P}(m = +\infty) = 2/(n-1)!$ and we get the desired result. □

Note that as a by-product of the proof we get that

$$\left|\mathbb{P}(X_n(i) = 0) - e^{-2}\right| \leq \mathbb{P}\left(m > i \cup m' < i-1\right) \leq 2\varepsilon_i, \tag{3}$$

where $\varepsilon_i = \max\left\{(2/3)^{i/3-1}, (2/3)^{(n-i)/3-1}\right\}$.

3.2 Number of trials: Poissonization

Let T_n be the number of cars that have tried to park before the parking process is over. It is clear that T_n is stochastically smaller than the number of trials needed to pick each number in $\{1, \ldots, n-1\}$ at least once, *i.e.* stochastically smaller than a coupon collector with n coupons. Thus, the lim sup (in probability) of $T_n/(n \log n)$ is less than one.

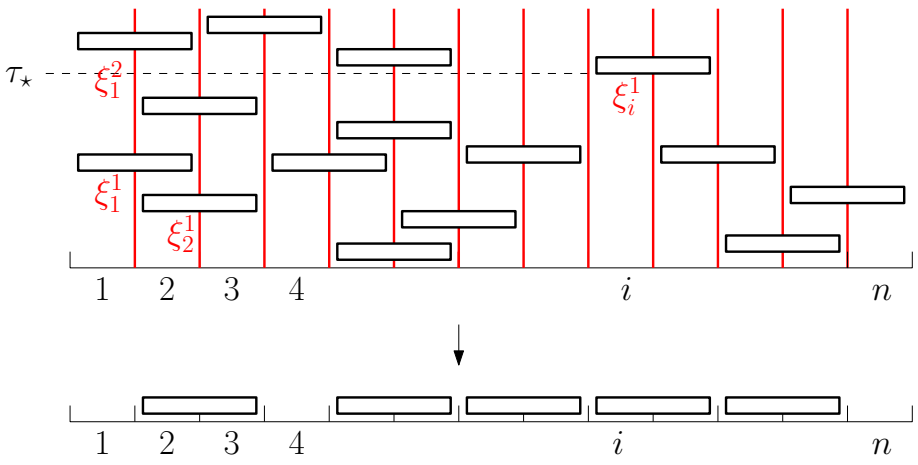
In order to estimate T_n we use another construction of the arrival process, in order to take into account the arrivals of cars that did not succeed to park. We now are given for each $i \in \{1, \ldots, n\}$ a sequence of random variables $(\xi_i^j)_{j \geq 1}$, the family $\left\{\xi_i^j\right\}_{i,j}$ being i.i.d. exponentially distributed with mean one. At $(i, i+1)$, cars try to park at times

$$\xi_i^1, \xi_i^1 + \xi_i^2, \xi_i^1 + \xi_i^2 + \xi_i^3, \ldots$$

For simplicity, we write as before $\xi_i = \xi_i^1$ for the first arrival of a car at i. Let $\tau_\star$ be the arrival of the last car that succeeds in parking:

$$\tau_\star = \max\left\{\tau_i, \tau_i < +\infty\right\}.$$

Here is a picture that sums up notations (here the last car parks at $(i, i+1)$, $T_n = 11$, note that $\tau_\star \neq \max \xi_i^1$):



By construction and by Markovianity we have

$$T_n \overset{(d)}{=} \sum_{i=1}^{n} \max\left\{j;\ \xi_i^1 + \cdots + \xi_i^j \leq \tau_\star\right\},$$

since $\max\left\{j;\ \xi_i^1 + \cdots + \xi_i^j \leq t\right\}$ is the number of cars that tried to park at $(i, i+1)$ before time t.

Theorem 5 (Number of trials)**.**

$$\frac{T_n}{n\log(n)} \overset{prob.}{\to} 1.$$

Proof. **Upper bound.** As noted above, T_n is stochastically smaller than a coupon collector with n coupons. It is classical (see for instance [2] Example 2.2.3) that for each $\varepsilon > 0$ we have

$$\mathbb{P}\left(\frac{T_n}{n\log(n)} \geq 1+\varepsilon\right) \to 0$$

Lower bound. The strategy is the following: Theorem 3 suggests that, as long as i is bounded away from 0 and n, we have $\mathbb{P}(u \leq \tau_i < +\infty) \approx e^{-2(1-e^{-u})} - e^{-2} \sim_{u\to+\infty} 2e^{-2}e^{-u}$. The τ_i's being weakly dependent, we expect $\tau_\star = \max\{\tau_i;\ \tau_i < +\infty\}$ to be of order $\log(n)$. To conclude, we will use the fact that $T_n \approx n \times \tau_\star$.

Lemma 2. *For every $\delta > 0$,*

$$\mathbb{P}(\tau_\star \geq (1-\delta)\log(n)) \overset{n\to+\infty}{\to} 1.$$

Proof of Lemma 2. For two integers i, ℓ such that $[i-\ell, i+\ell] \subset [1, n]$, let $A_{i,\ell}(u)$ be the event $E_{i,\ell}(u) \cup F_{i,\ell}(u) \cup F_{i,\ell}(u)$ where

$$\begin{aligned}
E_{i,\ell}(u) &= \{ \text{ odd rise of length} \leq \ell-2 \text{ at } i,\ \xi_{i-1} \geq u,\ \text{even descent of length} \leq \ell-2 \text{ at } i\}\\
F_{i,\ell}(u) &= \{ \text{ even rise of length} \leq \ell-2 \text{ at } i,\ \xi_i \geq u,\ \text{odd descent of length} \leq \ell-2 \text{ at } i\}\\
G_{i,\ell}(u) &= \{ \text{ odd rise of length} \leq \ell-2 \text{ at } i,\ \xi_{i-1} \geq u, \xi_i \geq u,\ \text{odd descent of length} \leq \ell-2 \text{ at } i\}
\end{aligned}$$

Event $A_{i,\ell}$ only depends on $\{\xi_{i'}, i-\ell+1 \leq i' \leq i+\ell-1\}$. Again by Lemma 1 we have

$$\{u \leq \tau_i < +\infty\} \supset A_{i,\ell}(u) \supset \begin{cases} u \leq \tau_i < +\infty, \\ \textbf{and } \text{there is a local minimum among } \xi_{i-\ell+2}, \ldots, \xi_{i-1} \\ \textbf{and } \text{there is a local minimum among } \xi_i, \ldots, \xi_{i+\ell-2}. \end{cases}$$

Then

$$\begin{aligned}
0 \leq \mathbb{P}(u \leq \tau_i < +\infty) - \mathbb{P}(A_{i,\ell}(u)) &\leq \mathbb{P}\Big(\text{ no local min. between } i-\ell+2 \text{ and } i-1 \\
&\qquad \textbf{or } \text{no local min. between } i \text{ and } i+\ell-2\Big)\\
&\leq 2(2/3)^{\lfloor(\ell-2)/3\rfloor} \leq 2(2/3)^{\ell/4}
\end{aligned}$$

for large ℓ. (Here we have re-used the fact that local minima appear independently at $i+1, i+4, i+7, \ldots, i+\ell$ with probability 1/3.)

Besides,

$$\begin{aligned}\mathbb{P}(u \le \tau_i < +\infty) &= 1 - \mathbb{P}(\tau_i = +\infty) - \mathbb{P}(\tau_i < u) \\ &= 1 - \mathbb{P}(\tau_i = +\infty) - \mathbb{P}\Big(\big\{ \xi_{i-1} < u;\ \text{odd rise at } i \ \big\} \cup \big\{ \xi_i < u;\ \text{odd descent at } i \ \big\} \Big) \end{aligned} \tag{4}$$

Aside from the boundary effets, the argument of the proof of Theorem 3 are still valid and we get for $i \ge 2$

$$\begin{aligned}\mathbb{P}\big(\xi_{i-1} < u;\ \text{odd rise at } i \ \big) &= \sum_{k=0}^{\lfloor i/2 \rfloor - 1} \mathbb{P}\left(\xi_{i-(2k+2)} > \xi_{i-(2k+1)} < \xi_{i-2k} < \cdots < \xi_{i-1} \le u\right) \\ &= \int_0^{F(u)} \sum_{k=0}^{\lfloor i/2 \rfloor - 1} \frac{1}{(2k)!}(s - F(t))^{2k}(1-s)ds \\ &= \delta_i + \int_0^{F(u)} \cosh\left(s - F(t)\right)(1-s)ds \\ &= 1 - \exp\big(-F(u)\big) + \delta_i,\end{aligned}$$

where δ_i does not depend on n and $|\delta_i| \le 2/i!$ (δ_i is obtained by bounding the remainder of the Taylor series of cosh). By symmetry $i \leftrightarrow n - i$ we have the symmetric estimate on $\mathbb{P}\big(\xi_i \le u;\ \text{odd descent at } i \ \big)$.

Plugging this into (4) and combining with our estimate (3) on $\mathbb{P}(\tau_i = +\infty)$ we obtain for $i \ge 2$, since $\varepsilon_i \ge \delta_i$

$$\mathbb{P}(u \le \tau_i < +\infty) \ge e^{-2F(u)} - e^{-2} - 10\eta_i \ge 2e^{-2}(1 - F(u)) - 10\eta_i = 2e^{-2}e^{-u} - 10\eta_i.$$

where $|\eta_i| \le \max\left\{(2/3)^{i/3-1}, (2/3)^{(n-i)/3-1}\right\}$.

Now, events

$$A_{\ell,\ell}(u), A_{3\ell,\ell}(u), A_{5\ell,\ell}(u), \dots A_{\lfloor n/\ell - 1 \rfloor \ell, \ell}(u)$$

are independent and (we skip integer parts in order to lighten notations):

$$\begin{aligned}\mathbb{P}(\tau_\star \le u) &\le \mathbb{P}\Big(\text{not } A_{\ell,\ell}(u), \text{not } A_{3\ell,\ell}(u), \text{not } A_{5\ell,\ell}(u), \dots, \text{not } A_{n/\ell \times \ell, \ell}(u)\Big) \\ &\le \prod_{j=1}^{n/\ell} \left(1 - 2e^{-2}e^{-u} + 10\eta_j + 2(2/3)^{\ell/4}\right) \\ &\le \prod_{j=\log(n)}^{n/\ell - \log(n)} \left(1 - 2e^{-2}e^{-u} + 10\eta_j + 2(2/3)^{\ell/4}\right).\end{aligned}$$

Choose now $\ell = 50\log(n)$ and take $u = (1-\delta)\log(n)$, so that for large n the inner is less than $1 - e^{-2}e^{-u}$,

$$\mathbb{P}(\tau_\star \le (1-\delta)\log(n)) \le \left(1 - \frac{e^{-2}}{n^{1-\delta}}\right)^{n/50\log(n) - 2\log(n)} \le \exp(-n^{\delta/2}),$$

for large n. □

We now conclude the lower bound:

$$\begin{aligned}\mathbb{P}(T_n \leq (1-\varepsilon)n\log(n)) &\leq \mathbb{P}(\tau_\star \leq (1-\delta)\log(n)) + \mathbb{P}(T_n \leq (1-\varepsilon)n\log(n); \tau_\star > (1-\delta)\log(n)) \\ &\leq \mathbb{P}(\tau_\star \leq (1-\delta)\log(n)) \\ &\quad + \mathbb{P}\left(\sum_{i=1}^{n} \max\left\{j;\ \xi_i^1 + \cdots + \xi_i^j \leq (1-\delta)\log(n)\right\} \leq (1-\varepsilon)n\log(n)\right) \\ &\leq \mathbb{P}(\tau_\star \leq (1-\delta)\log(n)) \\ &\quad + \mathbb{P}\left(\sum_{i=1}^{n} \mathrm{Poiss}_i\left((1-\delta)\log(n)\right) \leq (1-\varepsilon)n\log(n)\right),\end{aligned}$$

where $\mathrm{Poiss}_i(\lambda)$ are i.i.d. Poisson with mean λ. The first term in the right-hand side goes to zero thanks to Lemma 2, so does the second one by taking $\delta = \varepsilon/2$ and using Chebyshev's inequality. □

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