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The Page-Rényi parking process

Lucas Gerin

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Abstract

In the Page parking (or packing) model on a discrete interval (also known as the *discrete Rényi packing problem* or the *unfriendly seating problem*), cars of length two successively park uniformly at random on pairs of adjacent places, until only isolated places remain. We give a probabilistic proof of the (known) fact that the proportion of the interval covered by cars goes to $1 - e^{-2}$, when the length of the interval goes to infinity.

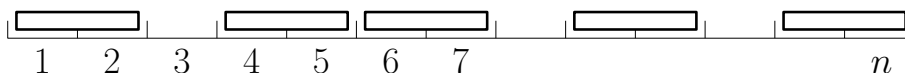
We obtain some new consequences, and also study a version of this process defined on the infinite line.

1 The Page parking

1.1 The model

For $n \geq 2$, we consider a sequence of *parking configurations* $x^t = (x_i^t)_{1 \leq i \leq n}$ in $\{0, 1\}^n$, given by the following construction. Initially the parking is empty: $x^0 = 0^n$. Given x^t one draws uniformly at random (and independently from the past) a number i in $\{1, 2, \dots, n-1\}$ and, if possible, a car of size 2 *parks* at i : if $x_i^t = x_{i+1}^t = 0$ then $x_i^{t+1} = x_{i+1}^{t+1} = 1$; $n-2$ other coordinates remain unchanged.

After some random time T_n (which is dominated by a coupon collector process with n coupons, see Section (3.2) below) parking is no more possible in x^{T_n} , in the sense that there is no adjacent coordinates $(i, i+1)$ such that $x_i^{T_n} = x_{i+1}^{T_n} = 0$. We set $X_n = x^{T_n}$ and $X_n(i) = x_i^{T_n}$. Below is an example where $X_n = (1, 1, 0, 1, 1, 1, 1, 0, 1, 1, 0, 1, 1)$.



We are mainly interested in the numbers of places i occupied by a car.

$$M_n = \text{card} \{1 \leq i \leq n, X_n(i) = 1\}.$$

We obviously have $n/2 \leq M_n \leq n$ (the worst case being $n = 4$ and $X_n = (0, 1, 1, 0)$) and we expect M_n/n to converge, at least in some sense.

Theorem 1 (Page (1959)).

$$\lim_{n \rightarrow +\infty} \frac{\mathbb{E}[M_n]}{n} = 1 - e^{-2} = 0.8646647 \dots$$

A nice heuristic for the limit $1 - e^{-2}$ is given by Page, it is based on a recursion on $\mathbb{P}(X_n(1) = 0)$. The proof of Theorem 1 is essentially obtained on conditioning on the position i of the first car. This gives the recursion identity

$$M_n \stackrel{(d)}{=} M_{I-1} + M'_{n-I-1} + 2,$$

where M, M', I are independent, I is uniform in $\{1, 2, \dots, n-1\}$, and we have set $M_0 = M_1 = 0$ a.s. This gives a recursion for the moments of M_n , which can be handled using generating functions.

The Page parking problem has a long story, it has been studied by many people and under different names. It is equivalent to the *unfriendly seating* problem [5], and sometimes also called the *discrete Rényi Packing* model [7], more generally, it is a toy model for *random deposition*. We refer to [10, 9] for some interpretations of the model in polymer chemistry. Theorem 1 can also be found in [3, 5, 13] with similar proofs, Page also obtained a variance estimate (see also [1, 12]) which proves that M_n/n converges to $1 - e^{-2}$ in probability.

We also mention that much is known also when cars have size $\ell > 2$, we refer to [8, 13]. There are also several similar models which are time-continuous (see for instance [6]).

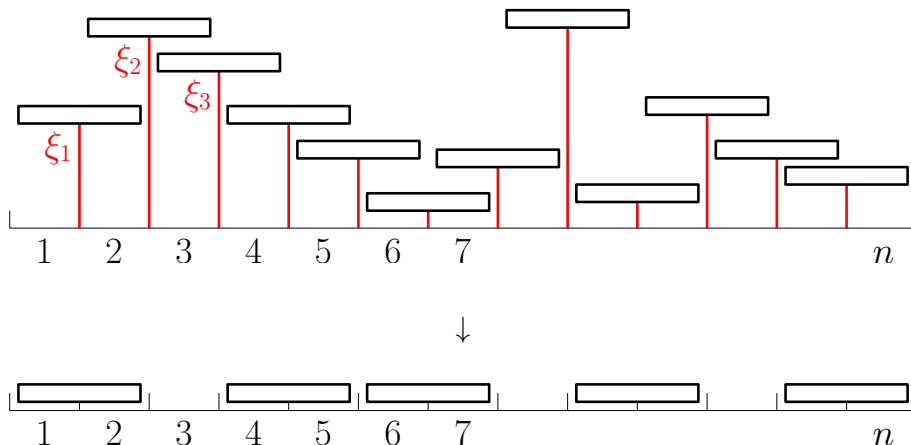
The aim of the present paper is to present a probabilistic (and apparently new) proof of Theorem 1 and to study the asymptotic behavior of T_n (the numbers of cars that have tried to park). We also study a version of this process defined on the infinite line.

1.2 The probabilistic construction

An alternative way of defining M_n is the following. Let $\xi = (\xi_i)_{1 \leq i \leq n-1}$ be i.i.d. random variables with continuous distribution function F (hereafter we will take $F(t) = 1 - e^{-t}$). Then the order statistics $(\xi_{(1)} < \dots < \xi_{(n-1)})$ give the order in which the cars park:

- the first car parks at $(\xi_{(1)}, \xi_{(1)} + 1)$,
- the second one parks (if possible) at $(\xi_{(2)}, \xi_{(2)} + 1)$,
- ...

Here is a sample of ξ (here $\xi_{(1)} = \xi_6$) and the corresponding configuration X_n :



It is easy to see that we obtain the same distribution, and was already observed by previous authors (see [12]). It does not seem that this construction was used to its full extent, yet it gives a very nice way to characterize positions i covered by a car. We need a few definitions.

We say that there is a *rise* of length ℓ at i if $i > \ell$ and

$$\xi_{i-\ell-1} > \xi_{i-\ell} < \xi_{i-\ell+1} < \xi_{i-\ell+2} < \cdots < \xi_{i-1}$$

or if $i = \ell$ and

$$\xi_1 < \xi_2 < \cdots < \xi_{i-1}.$$

There is a *descent* of length ℓ at i if $i < n - \ell$ and

$$\xi_i > \xi_{i+1} > \cdots > \xi_{i+\ell-1} < \xi_{i+\ell}$$

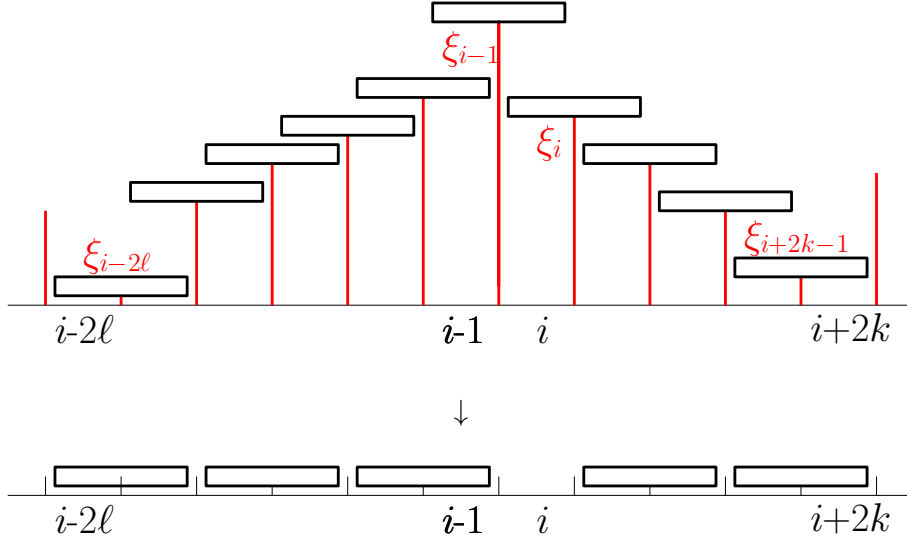
or if $i = n - \ell$ and

$$\xi_i > \xi_{i+1} > \cdots > \xi_{n-1}.$$

Note that by construction the events $\{ \text{rise of length } \ell \text{ at } i \}$ and $\{ \text{descent of length } \ell' \text{ at } i \}$ are independent for every i, ℓ, ℓ' .

Lemma 1. *There is no car at i (i.e. $X_n(i) = 0$) if and only if there is rise of even length at i and a descent of even length at i .*

An example of a rise of length 6 at i and a descent of length 4, we see that $X_n(i) = 0$:



Proof of Lemma 1. By construction, if i is a local minimum¹ of ξ (i.e. $\xi_{i-1} > \xi_i < \xi_{i+1}$) then a car parks at $(i, i+1)$. If i is not a local minimum then the value $X_n(i)$ only depends on the ξ 's between m_i, m'_i where m_i (resp. m'_i) is the closest local minimum of ξ on the left of i (resp. on the right). On the picture above $m_i = i - 2\ell$, $m'_i = i + 2k - 1$.

In this case a rise begins at m_i and a descent ends at m'_i . Cars successively fill places of the rise $(m_i, m_i + 1)$, $(m_i + 2, m_i + 3)$, \dots from left to right and places of the descent $(m_i, m_i + 1)$, $(m_i - 2, m_i - 1)$ from right to left. This process fills all the interval $[m_i, m'_i]$ unless these rise and descent are of even length. \square

¹Here we put $\xi_0 = \xi_n = +\infty$.

2 The infinite parking

An interesting feature of the probabilistic construction is that it allows to define the model $(X_\infty(i))_{i \in \mathbb{Z}}$ on \mathbb{Z} , by considering an infinite sequence $(\xi_i)_{i \in \mathbb{Z}}$. Since $\inf_{i \in \mathbb{Z}} \xi_i = 0$, "first car" does not make sense but we still can define the model by using the proof of Lemma Lem:RiseDescent.

We first set $X_\infty(i) = 1$ for every i such that ξ_i is a local minimum, and then we define $X_\infty(i)$ as before, using only the ξ 's between m_i, m'_i . (Note that with probability one, for all i one has $m'_i - m_i < +\infty$.)

Theorem 2 (The density of the parking on \mathbb{Z}). *For every i ,*

$$\mathbb{P}(X_\infty(i) = 0) = 1 - e^{-2}.$$

In fact Theorem 2 follows from Theorem 3 below, but we give here a short proof, whose interest is to give a natural interpretation of the limit.

Proof. By construction the rise at i and the descent at i are independent. Lemma 1 also holds for the infinite parking and gives

$$\begin{aligned} \mathbb{P}(X_n(i) = 0) &= \mathbb{P}(\text{even rise at } i) \mathbb{P}(\text{event descent at } i) \\ &= \mathbb{P}(\text{descent at } i)^2 \\ &= \left(\sum_{\ell \geq 1} \mathbb{P}(\xi_i > \xi_{i+1} > \cdots > \xi_{i+2\ell-1} < \xi_{i+2\ell}) \right)^2 \\ &= \left(\sum_{\ell \geq 1} \mathbb{P}(\sigma_1 > \sigma_2 > \cdots > \sigma_{2\ell} < \sigma_{2\ell+1}) \right)^2 \end{aligned}$$

where σ is a uniform permutation of $2\ell + 1$ elements. There are 2ℓ permutations such that $\sigma_1 > \sigma_2 > \cdots > \sigma_{2\ell} < \sigma_{2\ell+1}$, we get

$$\mathbb{P}(X_\infty(i) = 0) = \left(\sum_{\ell \geq 1} \frac{2\ell}{(2\ell + 1)!} \right)^2 = (1/e)^2. \quad (1)$$

(The sum is the derivative of $\sum_{\ell \geq 1} \frac{z^{2\ell}}{(2\ell+1)!} = \sinh(z)/z - 1$ at $z = 1$.) \square

Obviously $X_\infty(i)$ are not independent but it is possible to prove with this construction that the doubly infinite sequence $(X_\infty(i))_{i \in \mathbb{Z}}$ is *strongly mixing*. Thus we can prove (using for instance [11]) that we have the Central Limit Theorem

$$\sqrt{k} \left(\frac{X_\infty(1) + X_\infty(2) + \cdots + X_\infty(k)}{k} - (1 - e^{-2}) \right) \xrightarrow{(d)} \mathcal{N}(0, e^{-2}(1 - e^{-2})).$$

This result was shown by [12] for the model on the interval.

2.1 Evolution of the density

We now consider the process given by the time arrivals of cars. As above $X_\infty(i)$ is the indicator that there is eventually a car at i . We define the process $(X_\infty^t)_{t \geq 0}$ with values in $\{0, 1\}^{\mathbb{Z}}$ by

$$X_\infty^t(i) = \begin{cases} 1 & \text{if } X_\infty(i) = 1 \text{ and } \tau_i \leq t, \\ 0 & \text{otherwise.} \end{cases}$$

Here, $\tau_i = \xi_{i-1}$ if the car parked at i is parked at $(i-1, i)$ and $\tau_i = \xi_i$ if this car is at $(i, i+1)$. Then τ_i is indeed the time arrival of the corresponding car, we set $\tau_i = +\infty$ if there is no car at i .

Recall that F is the distribution function of the variables ξ . Note that in the case where $F(t) = 1 - e^{-t}$ then $(X_\infty^t)_t$ defines a homogeneous Markov process.

Theorem 3 (Evolution of the density of cars).

Let τ_i be the arrival time of the car i ,

$$\mathbb{E}[X_\infty^t(i)] = \mathbb{P}(\tau_i \leq t) = 1 - e^{-2F(t)}.$$

Of course we recover $\mathbb{P}(X_\infty = 1) = \mathbb{P}(\tau_i < +\infty) = \lim_{t \rightarrow +\infty} 1 - e^{-2F(t)} = 1 - e^{-2}$. Note that the particular case of Theorem 3 with $F(t) = 1 - e^{-t}$ was proved in ([7], eq. (19)).

Proof. By translation-invariance we assume $i = 0$. Lemma 1 gives that $\tau_i \leq t$ if and only if

- there is an odd rise at 0 and $\xi_{-1} \leq t$,
- or there is an odd descent at 0 and $\xi_0 \leq t$.

These two events being independent we have

$$\mathbb{P}(\tau_i \leq t) = 2f(t) - f(t)^2 = f(t)(2 - f(t))$$

where

$$f(t) = \mathbb{P}(\xi_0 \leq t; \text{ odd descent at } 0).$$

Now,

$$\begin{aligned} f(t) &= \mathbb{P}(t \geq \xi_0 < \xi_1) + \sum_{k \geq 1} \mathbb{P}(t \geq \xi_0 > \xi_1 > \cdots > \xi_{2k} < \xi_{2k+1}) \\ &= \int_0^t (1 - F(r)) dF(r) + \sum_{k \geq 1} \int_0^t \mathbb{P}(t \geq \xi_0 > \xi_1 > \cdots > \xi_{2k-1} > r ; r < \xi_{2k+1}) dF(r), \\ &= \int_0^t (1 - F(r)) dF(r) + \sum_{k \geq 1} \int_0^t \mathbb{P}(t \geq \xi_0 > \xi_1 > \cdots > \xi_{2k-1} > r) \mathbb{P}(r < \xi_{2k+1}) dF(r), \end{aligned}$$

at second line we have conditioned respectively on $\{\xi_0 = r\}$ and on $\{\xi_{2k} = r\}$.

Set $A = \{\xi_0, \xi_1, \dots, \xi_{2k-1} \in (r, t)\}$, then $\mathbb{P}(A) = (F(t) - F(r))^{2k}$ and conditional on A these random variables are ordered as a uniform permutation:

$$\begin{aligned}
f(t) &= \int_{r=0}^t (1 - F(r))dF(r) + \sum_{k \geq 1} \int_0^t \frac{1}{(2k)!} (F(t) - F(r))^{2k} (1 - F(r))dF(r) \\
&= \int_0^t \sum_{k \geq 0} \frac{1}{(2k)!} (F(t) - F(r))^{2k} (1 - F(r))dF(r) \\
&= \int_0^{F(t)} \sum_{k \geq 0} \frac{1}{(2k)!} (F(t) - s)^{2k} (1 - s)ds \\
&= \int_0^{F(t)} \cosh(F(t) - s) (1 - s)ds \\
&= 1 - \exp(-F(t)).
\end{aligned}$$

□

3 Parking on an interval

3.1 The convergence

We obtain the following refinement of Theorem 1:

Theorem 4 (The density of cars on an interval).

$$\left| \frac{\mathbb{E}[M_n]}{n} - (1 - e^{-2}) \right| \leq \frac{13e}{n}. \quad (2)$$

One could improve this estimate into $\mathbb{E}[M_n] = n(1 - e^{-2}) + (1 - 3e^{-2}) + o(n)$ which has been proved by Friedman [5] (see also [4]). The proof we provide here is more probabilist, it is the same as that of Theorem 2 except that we have to take the boundary terms into account.

Proof.

$$\begin{aligned}
\mathbb{P}(X_n(i) = 0) &= \mathbb{P}(\text{even rise at } i) \mathbb{P}(\text{even descent at } i) \\
&= \left(\varepsilon_i + \sum_{1 \leq \ell \leq i/2-1} \mathbb{P}(\xi_{i-2\ell-1} > \xi_{i-2\ell} < \xi_{i-2\ell+1} \cdots < \xi_{i-1}) \right) \\
&\quad \times \left(\varepsilon_{n-i+1} + \sum_{1 \leq \ell \leq (n-i-1)/2} \mathbb{P}(\xi_i > \xi_{i+1} > \cdots > \xi_{i+2\ell-1} < \xi_{i+2\ell}) \right) \\
&= (\varepsilon_i + S_{\lfloor i/2-1 \rfloor}) \times (\varepsilon_{n-i+1} + S_{\lfloor (n-i-1)/2 \rfloor}) \quad (3)
\end{aligned}$$

where $\sum_{\emptyset} = 0$ and

$$\varepsilon_i = \mathbf{1}_{i \text{ is odd}} \mathbb{P}(\text{rise from } 1 \text{ to } i) = \mathbf{1}_{i \text{ is odd}} \mathbb{P}(\xi_1 < \xi_2 < \cdots < \xi_{i-1}) = \mathbf{1}_{i \text{ is odd}} \frac{1}{(i-1)!}.$$

and

$$S_k = \sum_{\ell=1}^k \frac{2\ell}{(2\ell+1)!}.$$

We get, using the estimate $0 \leq e^{-1} - S_k \leq 1/(2k+1)!$,

$$\begin{aligned} |\mathbb{P}(X_n(i) = 0) - e^{-2}| &\leq |S_{\lfloor i/2-1 \rfloor} S_{\lfloor (n-i-1)/2 \rfloor} - e^{-2}| + \varepsilon_i S_{\lfloor (n-i-1)/2 \rfloor} + \varepsilon_{n-i+1} S_{\lfloor i/2-1 \rfloor} + \varepsilon_i \varepsilon_{n-i+1} \\ &\leq 6 \max \left\{ \frac{1}{(i-1)!}, \frac{1}{(n-i-1)!} \right\}. \end{aligned} \quad (4)$$

Now, by symmetry $i \leftrightarrow n-i$

$$|\mathbb{E}[M_n] - n(1 - e^{-2})| \leq 1 + 2 \sum_{i=1}^{\lfloor n/2 \rfloor} |1 - \mathbb{P}(X_n(i) = 0) - 1 + e^{-2}| \leq 1 + 2 \sum_{i=1}^{\lfloor n/2 \rfloor} 6 \frac{1}{(i-1)!} \leq 13e$$

□

3.2 Number of trials

Let T_n be the number of cars that have tried to park before the parking process is over. It is clear that T_n is stochastically smaller than the number of trials needed to pick each number in $\{1, \dots, n-1\}$ at least once, *i.e.* stochastically smaller than a coupon collector with n coupons. Thus, the lim sup (in probability) of $T_n/(n \log n)$ is less than one.

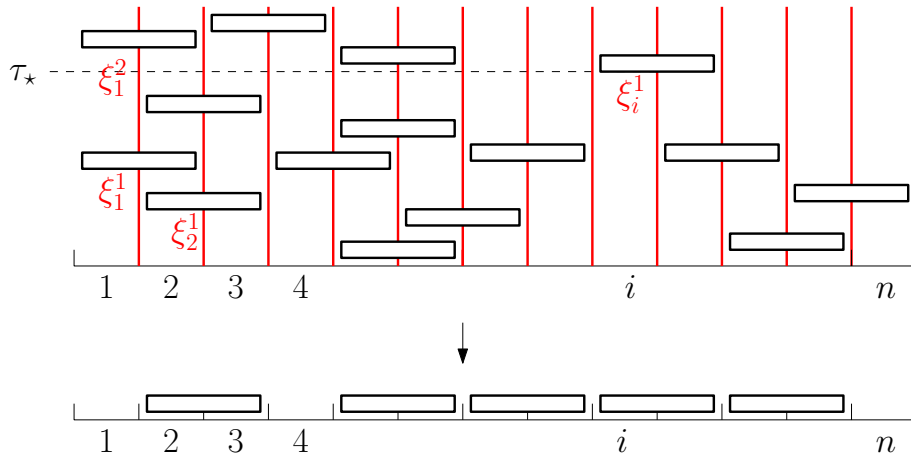
In order to estimate T_n we use another construction of the arrival process, in order to take into account the arrivals of cars that did not succeed to park. We now are given for each $i \in \{1, \dots, n\}$ a sequence of random variables $(\xi_i^j)_{j \geq 1}$, the family $\{\xi_i^j\}_{i,j}$ being i.i.d. exponentially distributed with mean one. At $(i, i+1)$, cars try to park at times

$$\xi_i^1, \xi_i^1 + \xi_i^2, \xi_i^1 + \xi_i^2 + \xi_i^3, \dots$$

Let τ_\star be the arrival of the last car that parks:

$$\tau_\star = \max \{ \tau_i, \tau_i < +\infty \}.$$

Here is a picture that sums up notations (here the last car parks at $(i, i+1)$, $T_n = 11$, note that $\tau_\star \neq \max \xi_i^1$):



By construction and by Markovianity we have

$$T_n \stackrel{(d)}{=} \sum_{i=1}^n \max \{ j; \xi_i^1 + \dots + \xi_j^i \leq \tau_\star \},$$

since $\max \left\{ j; \xi_1^i + \dots + \xi_j^i \leq t \right\}$ is the number of cars that tried to park at $(i, i+1)$ before time t .

Theorem 5 (Number of trials).

$$\frac{T_n}{n \log(n)} \xrightarrow{prob.} 1.$$

Proof. Upper bound. As noted just above, T_n is stochastically smaller than a coupon collector with n coupons. It is classical (see for instance [2] Example 2.2.3) that for each $\varepsilon > 0$ we have

$$\mathbb{P} \left(\frac{T_n}{n \log(n)} \geq 1 + \varepsilon \right) \rightarrow 0$$

Lower bound. The strategy is the following: Theorem 3 suggests that, as long as i is bounded away from 0 and n , we have $\mathbb{P}(u \leq \tau_i < +\infty) \approx e^{-2(1-e^{-u})} - e^{-2} \sim_{u \rightarrow +\infty} 2e^{-2}e^{-u}$. The τ_i 's being weakly dependent, we expect $\tau_\star = \max \tau_i$ to be of order $\log(n)$. To conclude, we will use the fact that $T_n \approx n \times \tau_\star$.

Lemma 2. For every $\delta > 0$ and for n large enough,

$$\mathbb{P}(\tau_\star \geq (1 - \delta) \log(n)) \leq 1 - \exp(-n^{\delta/2}).$$

Proof of Lemma 2. For two integers i, ℓ such that $[i - \ell, i + \ell] \subset [1, n]$, let $A_{i,\ell}(u)$ be the event $E_{i,\ell}(u) \cup F_{i,\ell}(u) \cup G_{i,\ell}(u)$ where

$$E_{i,\ell}(u) = \{ \text{odd rise of length } \leq \ell - 2 \text{ at } i, \xi_{i-1} \geq u, \text{ even descent of length } \leq \ell - 2 \text{ at } i \}$$

$$F_{i,\ell}(u) = \{ \text{even rise of length } \leq \ell - 2 \text{ at } i, \xi_i \geq u, \text{ odd descent of length } \leq \ell - 2 \text{ at } i \}$$

$$G_{i,\ell}(u) = \{ \text{odd rise of length } \leq \ell - 2 \text{ at } i, \xi_{i-1} \geq u, \xi_i \geq u, \text{ odd descent of length } \leq \ell - 2 \text{ at } i \}$$

Event $A_{i,\ell}$ only depends on $\xi_{i-\ell+1}, \dots, \xi_{i+\ell-1}$. Again by Lemma 1 we have

$$\{u \leq \tau_i < +\infty\} \supset A_{i,\ell}(u) \supset \begin{cases} u \leq \tau_i < +\infty, \\ \text{and there is a local minimum among } \xi_{i-\ell+2}, \dots, \xi_{i-1} \\ \text{and there is a local minimum among } \xi_i, \dots, \xi_{i+\ell-2}. \end{cases}$$

Then, for ℓ large enough,

$$\begin{aligned} 0 \leq \mathbb{P}(u \leq \tau_i < +\infty) - \mathbb{P}(A_{i,\ell}(u)) &\leq \mathbb{P} \left(\begin{array}{l} \text{no local min. between } i - \ell + 2 \text{ and } i - 1 \\ \text{or no local min. between } i \text{ and } i + \ell - 2 \end{array} \right) \\ &\leq 2(2/3)^{\lfloor (\ell-2)/3 \rfloor} \leq 2(2/3)^{\ell/4} \end{aligned}$$

for large ℓ . (Here we have used the fact that local minima appear independently at $i+1, i+4, i+7, \dots, i+\ell$ with probability $1/3$.)

Besides,

$$\begin{aligned} \mathbb{P}(u \leq \tau_i < +\infty) &= 1 - \mathbb{P}(\tau_i = +\infty) - \mathbb{P}(\tau_i < u) \\ &= 1 - \mathbb{P}(\tau_i = +\infty) - \mathbb{P} \left(\{ \xi_{i-1} < u; \text{ odd rise at } i \} \cup \{ \xi_i < u; \text{ odd descent at } i \} \right) \end{aligned}$$

Aside from the boundary effects, the argument of the proof of Theorem 3 are still valid and we get for $i \geq 2$

$$\begin{aligned}
\mathbb{P}(\xi_{i-1} < u; \text{ odd rise at } i) &= \sum_{k=0}^{\lfloor i/2 \rfloor - 1} \mathbb{P}(\xi_{i-(2k+2)} > \xi_{i-(2k+1)} < \xi_{i-2k} < \cdots < \xi_{i-1} \leq u) \\
&= \int_0^{F(u)} \sum_{k=0}^{\lfloor i/2 \rfloor - 1} \frac{1}{(2k)!} (s - F(t))^{2k} (1 - s) ds \\
&= \delta_i + \int_0^{F(u)} \cosh(s - F(t)) (1 - s) ds \\
&= 1 - \exp(-F(u)) + \delta_i,
\end{aligned}$$

where δ_i does not depend on n and $|\delta_i| \leq 2/i!$ (δ_i is obtained by bounding the remainder of the Taylor series of \cosh). By symmetry $i \leftrightarrow n - i$ we have the symmetric estimate on $\mathbb{P}(\xi_i \leq u; \text{ odd descent at } i)$.

Combining this with (4) we obtain

$$\mathbb{P}(u \leq \tau_i < +\infty) \geq e^{-2F(u)} - e^{-2} - 20\eta_i \geq 2e^{-2}(1 - F(u)) - 20\eta_i = 2e^{-2}e^{-u} - 20\eta_i.$$

where $|\eta_i| \leq \max\left\{\frac{1}{i!}, \frac{1}{(n-i)!}\right\}$.

Now, events

$$A_{\ell, \ell}(u), A_{3\ell, \ell}(u), A_{5\ell, \ell}(u), \dots, A_{\lfloor n/\ell \rfloor, \ell}(u)$$

are independent and (we skip integer parts in order to lighten notations):

$$\begin{aligned}
\mathbb{P}(\tau_\star \leq u) &\leq \mathbb{P}\left(\text{not } A_{\ell, \ell}(u), \text{not } A_{3\ell, \ell}(u), \text{not } A_{5\ell, \ell}(u), \dots, \text{not } A_{\lfloor n/\ell \rfloor, \ell}(u)\right) \\
&\leq \prod_{j=1}^{n/\ell} \left(1 - 2e^{-2}e^{-u} + 20\eta_j + 2(2/3)^{\ell/4}\right) \\
&\leq \prod_{j=\log(n)}^{n/\ell - \log(n)} \left(1 - 2e^{-2}e^{-u} + 20\eta_j + 2(2/3)^{\ell/4}\right).
\end{aligned}$$

Choose now $\ell = 50 \log(n)$ and take $u = (1 - \delta) \log(n)$, so that for large n the inner is less than $1 - e^{-2}e^{-u}$,

$$\mathbb{P}(\tau_\star \leq (1 - \delta) \log(n)) \leq \left(1 - \frac{e^{-2}}{n^{1-\delta}}\right)^{n/50 \log(n) - 2 \log(n)} \leq \exp(-n^{\delta/2}).$$

□

We now conclude the lower bound:

$$\begin{aligned}
\mathbb{P}(T_n \leq (1 - \varepsilon)n \log(n)) &\leq \mathbb{P}(\tau_\star \leq (1 - \varepsilon) \log(n)) + \mathbb{P}(T_n \leq (1 - \varepsilon)n \log(n); \tau_\star > (1 - \delta) \log(n)) \\
&\leq \mathbb{P}(\tau_\star \leq (1 - \varepsilon) \log(n)) \\
&\quad + \mathbb{P}\left(\sum_{i=1}^n \max\{j; \xi_i^1 + \cdots + \xi_j^i \leq (1 - \delta) \log(n)\} \leq (1 - \varepsilon)n \log(n)\right) \\
&\leq \mathbb{P}(\tau_\star \leq (1 - \varepsilon) \log(n)) \\
&\quad + \mathbb{P}\left(\sum_{i=1}^n \text{Pois}_i((1 - \delta) \log(n)) \leq (1 - \varepsilon)n \log(n)\right),
\end{aligned}$$

where $\text{Poiss}_i(\lambda)$ are i.i.d. Poisson with mean λ . The first term in the right-hand side goes to zero thanks to Lemma 2, so does the second one by taking $\delta = \varepsilon/2$ and using Tchebyshev's inequality. \square

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