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COVERING TECHNIQUES IN AUSLANDER-REITEN THEORY

CLAUDIA CHAIO, PATRICK LE MEUR, AND SONIA TREPODE

Abstract. Given a finite dimensional algebra over a perfect field the text introduces covering functors over the mesh category of any modulated Auslander-Reiten component of the algebra. This is applied to study the composition of irreducible morphisms between indecomposable modules in relation with the powers of the radical of the module category.

Introduction

Let $A$ be a finite-dimensional algebra over a field $k$. The representation theory of $A$ deals with the category $\text{mod} \ A$ of finitely generated (right) $A$-modules. In particular it aims at describing the indecomposable modules up to isomorphism and the morphisms between them. For this purpose the Auslander-Reiten theory gives useful tools such as irreducible morphisms and almost split sequences. These two particular concepts have been applied to study singularities of algebraic varieties and Cohen-Macaulay modules over commutative rings.

Let $\text{ind} \ A$ be the full subcategory of $\text{mod} \ A$ containing one representative of each isomorphism class of indecomposable $A$-modules. Given $X, Y \in \text{ind} \ A$, a morphism $f: X \rightarrow Y$ is called irreducible if it lies in $\text{rad} \setminus \text{rad}^2$. Here $\text{rad}$ denotes the radical of the module category, that is, the ideal in $\text{mod} \ A$ generated by the non-isomorphisms between indecomposable modules. The powers $\text{rad}^\ell$ of the radical are recursively defined by $\text{rad}^{\ell+1} = \text{rad} \cdot \text{rad} = \text{rad} \cdot \text{rad}^\ell$. The Auslander-Reiten theory encodes part of the information of $\text{mod} \ A$ in the Auslander-Reiten quiver $\Gamma(\text{mod} \ A)$. This concentrates much of the combinatorial information on the irreducible morphisms and almost split sequences. However it does not give a complete information on the composition of two (or more) irreducible morphisms. For example the composition of $n$ irreducible morphisms obviously lies in $\text{rad}^n$ but it may lie in $\text{rad}^{n+1}$. It is proved in [IT84b, Thm. 13.3] that if these irreducible morphisms form a sectional path then their composition lies in $\text{rad}^n \setminus \text{rad}^{n+1}$. This result was made more precise for finite-dimensional algebras over algebraically closed fields in a study [CLMT11] of the degrees of irreducible morphisms (in the sense of [Liu92]) and their relationship to the representation type of the algebra. The results in [CLMT11] are based on well-behaved functors introduced first in [Rie80, BG82] for (selfinjective) algebras of finite representation type. This text presents general constructions of well-behaved functors with application to composition of irreducible morphisms.

Let $\Gamma$ be a connected component of the Auslander-Reiten quiver of $A$ (or, anAuslander-Reiten component, for short). Let $\text{ind} \Gamma$ be the full subcategory of $\text{ind} A$...
with set of objects the modules $X \in \text{ind } A$ lying in $\Gamma$. Beyond the combinatorial structure on $\Gamma$, the mesh-category $k(\Gamma)$ is a first approximation of $\text{ind } \Gamma$ taking into account the composition of irreducible morphisms. Actually Igusa and Todorov have shown that $\Gamma$ comes equipped with a $k$-modulation $k\text{-modulation}$, which is called standard here and which includes the division algebra $\kappa_X = \text{End}_A(X)/\rad(X, X)$ and the $\kappa_X - \kappa_Y$-bimodule $\text{irr}(X, Y) = \rad(X, Y)/\rad^2(X, Y)$ for every $X, Y \in \Gamma$. The category $k(\Gamma)$ may be defined by generators and relations (see Section 1 for details). Its objects are the modules $X \in \Gamma$, the generators are the classes of morphisms $u \in \kappa_X$ (as morphisms in $k(\Gamma)(X, X)$) and $u \in \text{irr}(X, Y)$ (as morphisms in $k(\Gamma)(X, Y)$), for every $X, Y \in \Gamma$, and the ideal of relations is the mesh ideal.

When $k$ is a perfect field this text introduces a covering functor of $\text{ind } \Gamma$ in order to get information about the composition of irreducible morphisms in $\Gamma$.

The Auslander-Reiten component is called standard if there exists an isomorphism of categories $k(\Gamma) \cong \text{ind } \Gamma$. Not all Auslander-Reiten components are standard and in many cases there even exist no functor $k(\Gamma) \to \text{ind } \Gamma$. For instance if $\Gamma$ has oriented cycles then such a functor is likely not to exist. This may be bypassed replacing the mesh category $k(\Gamma)$ by that of a suitable translation quiver $\tilde{\Gamma}$ with a $k$-modulation such that there exists a covering $\pi : \tilde{\Gamma} \to \Gamma$. It appears that the composition of irreducible morphisms in $\text{ind } \Gamma$ may be studied using $k(\tilde{\Gamma})$ provided that there exists a so-called well-behaved functor $k(\tilde{\Gamma}) \to \text{ind } \Gamma$. These functors were first considered by Riedtmann [Rie80] in her study of the shapes of the Auslander-Reiten quivers of selfinjective algebras of finite representation type over algebraically closed fields and next by Bongartz and Gabriel [BG82] for algebras of finite representation type over algebraically closed fields. The present article considers well-behaved functors for modulated translation quivers when $k$ is a perfect field. Let $(\kappa_x, M(x, y))_{x,y}$ be the $k$-modulation of $\tilde{\Gamma}$ induced by the standard modulation of $\Gamma$, that is, $\kappa_x = \kappa_{\pi x}$ and $M(x, y) = \text{irr}(\pi x, \pi y)$ for every $x, y \in \tilde{\Gamma}$. Then a functor $F : k(\tilde{\Gamma}) \to \text{ind } \Gamma$ is well-behaved if it induces isomorphisms $\kappa_x \cong \kappa_{\pi x}$ and $M(x, y) \cong \text{irr}(\pi x, \pi y)$, for every $x, y \in \Gamma$. The construction of $F$ relies on three fundamental facts. Firstly, if one tries to construct such an $F$ then it is quite natural to proceed by induction. The translation quiver $\tilde{\Gamma}$ is called with length if any two paths in $\Gamma$ having the same source and the same target have the same length. As mentioned above an inductive construction is likely not to work if $\Gamma$ has oriented cycles and actually simple examples show that this construction fails if $\tilde{\Gamma}$ is not with length. Note that $\Gamma$ is with length when $\tilde{\Gamma}$ is the universal cover of $[\text{GS}2]$. Secondly, if $x \in \tilde{\Gamma}$ then the ring homomorphism $\kappa_x \to k(\tilde{\Gamma})(x, x) \to \text{End}_A(\pi x)$ is a section of the quotient homomorphism $\text{End}_A(X) \to \kappa_x$. In view of the Wedderburn-Malcev theorem this section is most likely to exist in the framework of algebras over perfect fields. Finally, given an irreducible morphism $f : X \to Y$ with $X, Y \in \Gamma$ then there exist $x, y \in \tilde{\Gamma}$ and $u \in k(\tilde{\Gamma})(x, y)$ such that $f - Fu \in \rad^2$. In view of studying the composition of irreducible morphisms in $\text{ind } \Gamma$ one may wish to have an equality $f = Fu$. This would permit to lift the study into $k(\tilde{\Gamma})$ where the composition of morphisms is better understood because of the mesh ideal. Keeping in mind these comments the main result of this text is the following.

**Theorem A.** Let $A$ be a finite-dimensional algebra over a perfect field $k$. Let $\Gamma$ be an Auslander-Reiten component of $A$. Let $\pi : \tilde{\Gamma} \to \Gamma$ be a covering of translation quivers where $\tilde{\Gamma}$ is with length. There exists a well-behaved functor $F : k(\tilde{\Gamma}) \to \text{ind } \Gamma$. 
The study of the composition of irreducible morphisms in $\text{ind}\Gamma$ using such a covering functor $F$ is made possible by the following lifting (or, covering) property of $F$ which is the second main result of the text. No assumption is made on length.

**Theorem B.** Let $A$ be a finite-dimensional algebra over a perfect field $k$. Let $\Gamma$ be an Auslander-Reiten component of $A$. Let $\pi: \bar{\Gamma} \rightarrow \Gamma$ be a covering of translation quivers. Let $F: k(\bar{\Gamma}) \rightarrow \text{ind}\Gamma$ be a well-behaved functor, $x, y \in \bar{\Gamma}$ and let $n \geq 0$.

(a) The two following maps induced by $F$ are bijective

$$\bigoplus_{Fz = Fy} R^n k(\bar{\Gamma})(x, z)/R^{n+1} k(\bar{\Gamma})(x, z) \rightarrow \text{rad}^n(Fx, Fy)/\text{rad}^{n+1}(Fx, Fy)$$

$$\bigoplus_{Fz = Fy} R^n k(\bar{\Gamma})(z, x)/R^{n+1} k(\bar{\Gamma})(z, x) \rightarrow \text{rad}^n(Fy, Fx)/\text{rad}^{n+1}(Fy, Fx).$$

(b) The two following maps induced by $F$ are injective

$$\bigoplus_{Fz = Fy} k(\bar{\Gamma})(x, z) \rightarrow \text{Hom}_A(Fx, Fy) \quad \text{and} \quad \bigoplus_{Fz = Fy} k(\bar{\Gamma})(z, x) \rightarrow \text{Hom}_A(Fy, Fx).$$

(c) $\Gamma$ is generalized standard if and only if $F$ is a covering functor, that is, the two maps of (b) are bijective (see [BG82, 3.1]).

Here $Rk(\bar{\Gamma})$ is the ideal in $k(\bar{\Gamma})$ generated by the morphisms in $M(x, y)$, for every arrow $x \rightarrow y$ in $\bar{\Gamma}$. Call it the *radical* of $k(\bar{\Gamma})$ by abuse of terminology. Define its powers $R^nk(\bar{\Gamma})$ like for the radical of mod $A$. Here is an interpretation of Theorem B. Both $k(\bar{\Gamma})$ and ind $\Gamma$ are filtered by the powers of their respective radicals. The above theorem asserts that $F$ induces a covering functor $\text{gr} k(\bar{\Gamma}) \rightarrow \text{gr} \text{ind} \Gamma$ (in the sense of [BGS2]) between the associated graded categories.

This text is therefore organised as follows. Section 1 is a reminder on Auslander-Reiten theory, modulated translation quivers and their mesh-categories, and coverings of translation quivers; it also introduces strongly irreducible morphisms. Section 2 proves the above theorems. Section 3 gives an application to the composition of irreducible morphisms.

In the sequel $k$ denotes a perfect field. Hence the tensor product over $k$ of two finite-dimensional division algebras is semi-simple. Also if $R$ is a finite-dimensional $k$-algebra and $J \subseteq R$ is a two-sided ideal such that $R/J$ is a division $k$-algebra, then the natural surjection $R \twoheadrightarrow R/J$ admits a section $R/J \rightarrowtail R$ as a $k$-algebra. The composition of mappings $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ is denoted by $fg: X \rightarrow Z$.

1. **Preliminaries**

1.1. **Reminder on Auslander-Reiten theory.** Following [ARS97, V.7, p.178], denote by rad the *radical* of mod $A$. This is the ideal of mod $A$ generated by the non invertible morphisms between indecomposable modules. Its powers are defined recursively by $\text{rad}^0 = \text{mod}A$ and $\text{rad}^{n+1} = \text{rad}\cdot\text{rad}^n$ ($=\text{rad}^n\cdot\text{rad}$) for every integer $n \geq 0$. For $X \in \text{mod}A$ indecomposable, the division algebra $\text{End}_A(X)/\text{rad}(X, X)$ is denoted by $\kappa_X$ and, given $u \in \text{End}_A(X)$, its residue class in $\kappa_X$ is denoted by $\overline{u}$.

A morphism $f$ in mod $A$ is *irreducible* ([ARS97, V.5, p. 166]) if it is neither a section nor a retraction and for all decompositions $f = uv$, then $u$ is a section or else $v$ is a retraction. Given $X, Y \in \text{mod}A$ indecomposable, a morphism $X \rightarrow Y$ is irreducible if and only if it belongs to rad and not to rad$^2$, and the quotient
rad(X,Y)/rad^2(X,Y) is called the space of irreducible morphisms from X to Y and denoted by irr(X,Y). Given f ∈ rad(X,Y), its image in irr(X,Y) is denoted by f. Hence, irr(X,Y) is a \( \kappa_X - \kappa_Y \)-bimodule, or, a left \( \kappa_X \otimes_k \kappa_Y^{\text{op}} \)-module, such that \( \overline{u} \cdot f \cdot \overline{v} = ufv \) for all \( u \in \text{End}_A(X), f \in \text{rad}(X,Y) \) and \( v \in \text{End}_A(Y) \).

A minimal left almost split morphism \([\text{ARS97}, \text{V.1}, 137-138]\) is a morphism \( f: X \to Y \) in mod \( A \) that is not a section, such that any morphism with domain \( X \) and which is not a section factors through \( f \) and such that, for all \( u \in \text{End}_A(X) \), if \( uf = f \) then \( u \) is invertible. Such a morphism features the following properties \([\text{ARS97}, \text{V.1}, \text{VII.1 Prop. 1.3 p. 230}]\): X is indecomposable and, given any direct sum decomposition \( Y = \oplus_{i=1}^r X_i^n \) where \( X_1, \ldots, X_r \) are indecomposable and pairwise non isomorphic, if \( f \) is denoted coordinatewise as \( [f_{i,j} : 1 \leq i \leq r, 1 \leq j \leq n_i] \), then, for all \( i \in \{1, \ldots, r\} \), the family \( \{\overline{f_{i,j}}\}_j \) is a basis of \( \text{irr}(X, X_i) \) over \( \kappa_{X_i} \). These properties characterise minimal left almost split morphisms. Any indecomposable \( A \)-module \( X \) is the domain of such a morphism \( X \to Y \) which happens to be the quotient morphism \( X \to X/\text{soc}(X) \) when \( X \) is injective and, in general, any irreducible morphism \( X \to Z \) is equal to the composite morphism \( X \to Y \to Z \) for some retraction \( Y \to Z \). Minimal right almost split morphisms are defined dually and feature dual properties.

An almost split sequence \([\text{ARS97}, \text{V.1, p.144}]\) is an exact sequence \( 0 \to X \to Y \to Z \to 0 \) in mod \( A \) such that \( X \to Y \) is minimal left almost split and \( Y \to Z \) is minimal right almost split. Any indecomposable and non-projective \( Z \in \text{mod} A \) (or, non-injective \( X \in \text{mod} A \)) is the end term (or, the first term, respectively) of such a (unique up to isomorphism) sequence \([\text{ARS97}, \text{V.1, Thm. 1.15 p. 145}]\), in such a case, \( X \) is called the Auslander-Reiten translate of \( Z \) and denoted by \( \tau_A Z \).

The Auslander-Reiten quiver of mod \( A \) \([\text{ARS97, VII.1 p. 225}]\) is the pair \((\Gamma(\text{mod} A), \tau_A)\) where \( \Gamma(\text{mod} A) \) is the quiver with vertices the objects of \( \text{ind} A \) and such that for all \( X, Y \in \text{ind} A \) there is an arrow \( X \to Y \) if and only if there exists an irreducible morphism \( X \to Y \). Its connected components are called Auslander-Reiten components. These quivers are particular instances of translation quivers \([\text{ARS97, VII.4 p. 248}]\). Those are the pairs \((\Gamma, \tau)\) where \( \Gamma \) is a locally finite quiver with two distinguished sets of vertices called projectives and injectives, respectively, and \( \tau \), the translation, is a bijective mapping from non-projective vertices to non-injective vertices and such that such that for all vertices \( x, y \) with \( x \) non-projective, there is an arrow \( y \to x \) if and only if there is an arrow \( \tau x \to y \).

The valuation \((a, b)\) of an arrow \( X \to Y \) in \( \Gamma(\text{mod} A) \) is defined such that \( a \) is the maximal integer such that there exists a minimal right almost split morphism \( X^a \oplus Z \to Y \) and \( b \) is the maximal integer such that there exists a minimal left almost split morphism \( X \to Y^b \oplus Z \). Equivalently, \((a, b) = (\dim_{\kappa_X} \text{irr}(X,Y), \dim_{\kappa_Y^{\text{op}}} \text{irr}(X,Y))\).

1.2. Strongly irreducible morphisms. The following definition is given over arbitrary fields. However, it is relevant mostly when \( k \) is a perfect field, see below.

**Definition.** Let \( k \) be any field. Let \( A \) be a finite dimensional \( k \)-algebra. Let \( X, X_1, \ldots, X_r \in \text{ind} A \) be such that \( X_1, \ldots, X_r \in \text{ind} A \) are pairwise non isomorphic and let \( n_1, \ldots, n_r \) be positive integers. A strongly irreducible morphism \( X \to \oplus_{i=1}^r X_i^{n_i} \) is an irreducible morphism \([f_{i,j} : 1 \leq i \leq r, 1 \leq j \leq n_i] \) with the following properties in the left \( \kappa_X \otimes_k \kappa_X^{\text{op}} \)-module \( \text{irr}(X, X_i) \), for all \( i \in \{1, \ldots, r\} \).

\[
\begin{align*}
\sum_i \kappa_X \cdot f_{i,j} \cdot \kappa_X & = \oplus_j \kappa_X \cdot f_{i,j} \cdot \kappa_X, \\
\sum_j \kappa_X \cdot f_{i,j} \cdot \kappa_X & \text{ is a direct summand of } \text{irr}(X, X_i) \text{ as a left } \kappa_X \otimes_k \kappa_X^{\text{op}} \text{-module.}
\end{align*}
\]
Note that, $f$ is strongly irreducible if and only if, for all $i$, so is $X \to X_i^{n_i}$. As mentioned previously, this definition is mostly relevant when $k$ is a perfect field because of the following standard result.

**Lemma.** Let $k$ be a perfect field.

1. Let $E, F$ be finite dimensional division $k$-algebras. Then, $E \otimes_k F$ is semi-simple.

2. Let $A$ be a finite dimensional $k$-algebra. Let $X, Y \in \text{ind } A$. Then, $\kappa_X \otimes_k \kappa_Y^{\text{op}}$ is semi-simple.

**Proof.** (1) follows from [Pie82, 10.7 Corollary of p. 188] and [Pie82, 10.7 Corollary b p. 192]). (2) follows from (1). □

Here are some comparison elements between strongly irreducible morphisms and irreducible morphisms, $k$ here need not be perfect. Keep the notation used in the previous definition. First, condition (b) holds automatically when $k$ is perfect. Next, $f$ is irreducible if and only if so is each $f_{i,j}$ and the following assertions are true in the right $\kappa_X$-vector space $\text{irr}(X, X_i)$, for all $i \in \{1, \ldots, r\}$.

(a') $\sum_j f_{i,j} \cdot \kappa_{X_i} = \oplus_j f_{i,j} \cdot \kappa_{X_i}$.

(b') $\sum_j f_{i,j} \cdot \kappa_{X_i}$ is a direct summand of $\text{irr}(X, X_i)$.

Note that condition (b') is independent of $f$ because $\kappa_X$ is a division algebra. Clearly, conditions (a) and (b) in the above definition are stronger than conditions (a') and (b'). In particular, there are irreducible morphisms which are not strongly irreducible. Here is an example (see [ARS97, VII.2, p. 235]). Let $k = \mathbb{R}$ and

$$A = \{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \mid a \in \mathbb{C}, b \in \mathbb{C}, c \in \mathbb{R} \} .$$

Then, $\Gamma(\text{mod } A)$ has the following shape, as a valued quiver,

$$\xymatrix{ & P \ar[r]_{(2,1)} & S' \ar[l]_{(1,2)} \\
S & P & I \ar[l]_{(2,1)} \ar[u]_{(1,2)}
}$$

where $S$, $P$, $I$, $S'$ are simple projective, projective non simple, injective non simple and simple injective, respectively. Consider an almost split sequence

$$0 \to S \xrightarrow{f} P \xrightarrow{g} I \to 0 .$$

Then, there exists an automorphism $u: P \to P$ fitting in an almost split sequence

$$0 \to P \xrightarrow{[g, u g]} I^2 \to S' \to 0 .$$

The valuation of the arrow $P \to I$ of $\Gamma(\text{mod } A)$ is $(1, 2)$. Therefore $\dim_{\kappa_P} \text{irr}(P, I) = 1$. Consequently, $\text{irr}(P, I)$ is generated by a single element as a left $\kappa_P$-vector space, and hence also as a left $\kappa_P \otimes_{\mathbb{R}} \kappa_I^{\text{op}}$-module. Accordingly, the left $\kappa_P \otimes_{\mathbb{R}} \kappa_I^{\text{op}}$-module $\text{irr}(P, I)$ is simple. Thus, it equals both $\kappa_P \cdot \tilde{g} \cdot \kappa_I$ and $\kappa_P \cdot \tilde{u g} \cdot \kappa_I$. In particular, $g: P \to I$ is strongly irreducible; and $[g, u g]$ is not strongly irreducible despite being irreducible, because the sum $\kappa_P \cdot \tilde{g} \cdot \kappa_I + \kappa_P \cdot \tilde{u g} \cdot \kappa_I$ is not direct.

From now on, assume that $k$ is perfect. Here are hints on how to construct strongly irreducible morphisms and how to obtain irreducible morphisms from them. Consider an arrow $X \to Y$ in $\Gamma(\text{mod } A)$. Since $\kappa_X \otimes_k \kappa_Y^{\text{op}}$ is a semi-simple algebra, there is a direct sum decomposition $\text{irr}(X, Y) = \oplus_{j=1}^n S_j$, where $S_1, \ldots, S_n$ are simple left $\kappa_X \otimes_k \kappa_Y^{\text{op}}$-modules. For each $j$, let $f_j \in \text{rad}(X, Y)$ be such that $\overrightarrow{f_j}$
is a generator of the simple module $S_j$. Then, the following morphism is strongly irreducible
\begin{equation}
[f_j; 1 \leq j \leq n]: X \to Y^n.
\end{equation}

Any strongly irreducible morphism $X \to Y^m$ ($m \geq 1$) is equal to the composition of a morphism such as (1), for adequate choices, with a retraction $Y^n \to Y^m$. Now, for every $j \in \{1, \ldots, n\}$, consider $S_j$ as a right $\kappa_Y$-vector space; then, using a basis of $\kappa_X$ over $k$ it follows that there exists a family $\{u_{j,k}\}_{k}$ of automorphisms of $X$ with at most $\dim_k \kappa_X$ terms and such that
\begin{equation}
S_j = \bigoplus_k u_{j,k} f_j \cdot \kappa_Y.
\end{equation}

Accordingly, there is a direct sum decomposition of right $\kappa_Y$-vector spaces
\begin{equation}
\text{irr}(X, Y) = \bigoplus_{j,k} u_{j,k} f_j \cdot \kappa_Y.
\end{equation}

Hence, this sum has $\dim_{\kappa_Y} \text{irr}(X, Y)$ terms and the following morphism is irreducible,
\begin{equation}
[u_{j,k} f_j; j, k]: X \to Y^{\dim_{\kappa_Y} \text{irr}(X, Y)}.
\end{equation}

In particular, if the valuation of the arrow $X \to Y$ of $\Gamma(\text{mod} \ A)$ has the shape $(a, 1)$, then all the direct sums in (1) and (2) have exactly one term. Actually, these considerations show the equivalence of the following assertions.

(i) The irreducible morphism (2) is strongly irreducible.

(ii) $\text{irr}(X, Y)$ is the direct sum of $\dim_{\kappa_Y} \text{irr}(X, Y)$ simple $\kappa_X \otimes_k \kappa_Y$-modules.

(iii) For all $j \in \{1, \ldots, n\}$, the k-vector spaces $\kappa_X \cdot f_j \cdot \kappa_Y$ and $f_j \cdot \kappa_Y$ are equal.

(iv) $\kappa_X \cdot f_j$ is contained in $f_j \cdot \kappa_Y$, for all $j \in \{1, \ldots, n\}$.

Going back to the general situation of an irreducible morphism $f: X \to \bigoplus_{i=1}^r X_i^{n_i}$, it would be worth characterising when it is strongly irreducible. In view of the previous considerations, each of the following conditions is sufficient.

1. $n_1 = \cdots = n_r = 1$.
2. $\kappa_X \cdot f_{i,j}$ is contained in $f_{i,j} \cdot \kappa_X$, for all $i, j$; indeed, in such case, the sum $\sum_{i,j} \kappa_X \cdot f_{i,j} \cdot \kappa_X$ is direct because the morphism $X \to \bigoplus_{i} X_i^{n_i}$ is irreducible and, for every $i, j$, the k-vector spaces $\kappa_X \cdot f_{i,j} \cdot \kappa_X$ and $f_{i,j} \cdot \kappa_X$ are equal.
3. $\kappa_X = k$; indeed, this assertion implies (2).
4. It is algebraically closed; indeed, this assertion implies (3).
5. For each $i \in \{1, \ldots, r\}$, the arrow $X \to X_i$ of $\Gamma(\text{mod} \ A)$ has valuation $(a_i, 1)$; indeed, this entails (1).

Recall ([Liu92, 1.7]) that if an arrow $X \to Y$ of $\Gamma(\text{mod} \ A)$ has finite left degree, then it has valuation $(1, b)$ or $(a, 1)$. In the former case, if $b \geq 2$, then there exists an irreducible morphism $X \to Y^b$ which is necessarily not strongly irreducible.

1.3. Factorisation through minimal almost split morphisms. The reader is referred to [ARS97] for basics on Auslander-Reiten theory. In the sequel the factorisation property of minimal almost split morphisms is used as follows.

Lemma. Let $u: X \to Y$ be a left minimal almost split morphism, $Z \in \text{mod} \ A$ and $v \in \text{rad}^{n+1}(X, Z)$ for some $n \geq 0$. There exists $w \in \text{rad}^n(Y, Z)$ such that $v = uw$.

Proof. Since $v \in \text{rad}^{n+1}$, there exist an integer $N \geq 1$, indecomposable modules $X_1, \ldots, X_N$ and morphisms $v_i' \in \text{rad}(X, X_i)$ and $v_i'' \in \text{rad}^n(X_i, Z)$, for every $i \in \{1, \ldots, N\}$, such that $v = \sum_{i=1}^N v_i' v_i''$. For every $i \in \{1, \ldots, N\}$, there exists $w_i \in$
\begin{align*}
\text{Hom}_A(Y, X_i) \text{ such that } v'_i = uw_i. \text{ Let } w = \sum_{i=1}^{N} w_i v''_i. \text{ Then, } w \in \text{rad}^n(Y, Z) \text{ and } v = uw. \end{align*}

1.4. Modulated translation quivers and their mesh-categories. Let $\Gamma$ be a translation quiver. For a non-projective vertex $x$, the subquiver of $\Gamma$ formed by the arrows starting in $tx$ and the arrows arriving in $x$ is called the mesh starting in $\tau x$.

A $(k)$-modulation on $\Gamma$ is the following data

(i) a division $k$-algebra $\kappa_x$ for every vertex $x \in \Gamma$,
(ii) a non-zero $\kappa_x - \kappa_y$ bimodule $M(x, y)$ for every arrow $x \rightarrow y$ in $\Gamma$,
(iii) a $k$-algebra isomorphism $\tau_x: \kappa_x \rightarrow \kappa_{\tau x}$ for every vertex $x \in \Gamma$,
(iv) a non-degenerate $\kappa_y - \kappa_x$-linear map $\sigma_x: M(y, x) \otimes_{\kappa_x} M(\tau x, y) \rightarrow \kappa_y$ (the left $\kappa_x$-module structure on $M(\tau x, y)$ is defined using its structure of left $\kappa_{\tau x}$-module and $\tau_x: \kappa_x \rightarrow \kappa_{\tau x}$).

With such a structure, $\Gamma$ is called a modulated translation quiver. If $A$ is a finite-dimensional algebra over a field $k$ then the Auslander-Reiten quiver $\Gamma(\text{mod } A)$ has a $k$-modulation as follows (IT84a 2.4, 2.5), in the rest of the text this modulation is called the standard modulation of $\Gamma(\text{mod } A)$; the restriction of the standard modulation of $\Gamma(\text{mod } A)$ to any Auslander-Reiten component is also called the standard modulation of that component. For every non-projective $X \in \text{ind } A$ fix an almost split sequence $0 \rightarrow \tau_A X \rightarrow E \rightarrow X \rightarrow 0$ in $\text{mod } A$. Then

- $\kappa_X = \text{End}_A(X)/\text{rad}(X, X)$ for every $X \in \text{ind } A$,
- $M(X, Y) = \text{irr}(X, Y)$ for every arrow $X \rightarrow Y$ in $\text{mod } A$,
- for every $X \in \text{ind } A$ and every morphism $u: X \rightarrow X$ defining the residue class $\overline{u} \in \kappa_X$, let $\tau_A \overline{u}: \tau_A X \rightarrow \tau_A X$ be the residue class $\overline{u}$ where $\tau_A X \rightarrow \tau_A X$ is a morphism fitting into a commutative diagram

\begin{align*}
0 & \longrightarrow \tau_A X \longrightarrow E \longrightarrow X \longrightarrow 0 \\
0 & \longrightarrow \tau_A X \longrightarrow E \longrightarrow X \longrightarrow 0,
\end{align*}

- let $X, Y \in \text{ind } A$ with $X$ non-projective and assume that there is an arrow $Y \rightarrow X$ in $\text{mod } A$. Let $\overline{u} \in M(\tau_A X, Y)$ and $\overline{v} \in M(Y, X)$ be residue classes of morphisms $u: \tau_A X \rightarrow Y$ and $v: Y \rightarrow X$, respectively. Then define $\sigma_A(\overline{v} \otimes \overline{u})$ as the composition $v'u$ where $u', v'$ are morphisms fitting into a commutative diagram

\begin{align*}
0 & \longrightarrow \tau_A X \longrightarrow E \longrightarrow X \longrightarrow 0 \\
0 & \longrightarrow \tau_A X \longrightarrow E \longrightarrow X \longrightarrow 0
\end{align*}

This construction does not depend on the initial choice of the almost split sequences up to an isomorphism of modulated translation quivers (IT84a 2.5). In the sequel $\Gamma(\text{mod } A)$ is considered as a modulated translation quiver as above.

If $\Gamma$ is a modulated translation quiver, its mesh-category $k(\Gamma)$ is defined as follows (IT84a 1.7). Let $S$ be the semi-simple category whose object set is the set of vertices in $\Gamma$ and such that $S(x, y) = \kappa_x$ if $x = y$ and $S(x, y) = 0$ otherwise. The collection $\{M(x, y)\}_{x \rightarrow y}$ naturally defines an $S - S$-bimodule denoted by $M$. 

\begin{align*}
\text{Hom}_A(Y, X_i) \text{ such that } v'_i = uw_i. \text{ Let } w = \sum_{i=1}^{N} w_i v''_i. \text{ Then, } w \in \text{rad}^n(Y, Z) \text{ and } v = uw. \end{align*}
The path-category is the tensor category $T_S(M)$ also denoted by $k\Gamma$. The mesh-ideal is the ideal in $k\Gamma$ generated by a collection of morphisms $\gamma_x: \tau x \to x$ indexed by the non-projective vertices $x \in \Gamma$. Given a non-projective vertex $x \in \Gamma$, a morphism $\gamma_x: \tau x \to x$ in $k\Gamma$ is defined as follows. For every arrow $y \to x$ ending in $x$, fix a basis $(u_1, \ldots, u_d)$ of the $\kappa_y$-vector space $M(y, x)$. Let $(u_1^*, \ldots, u_d^*)$ be the associated dual basis of the $\kappa_x$-vector space $M(\tau x, y)$ under the pairing $\sigma$ (that is, $\sigma(u_i \otimes u_j^*)$ is 1 if $i = j$ and 0 otherwise). Then $\gamma_x = \sum_{y \to x \in \Gamma} \sum_{i=1}^d u_i^* u_i \in k\Gamma(\tau x, x)$. This morphism does not depend on the choice of the basis $(u_1, \ldots, u_d)$. The mesh-category is then defined as the quotient category of $k\Gamma$ by the mesh-ideal.

Let $\Gamma$ be a component of $\Gamma(\text{mod} A)$ endowed with its standard modulation. As proved in [IT84a Sect. 2], the mesh-category $k(\Gamma)$ does not depend on the choice of the almost split sequences used to define the modulation up to an isomorphism of $k$-linear categories. The following lemma explains how to recover the mesh-relations $\gamma_X$ and the pairing $\sigma$, starting from a different choice of almost split sequences.

Lemma. In the previous setting, let $X \in \Gamma$ be non-projective and let

$$0 \to \tau_A X \xrightarrow{\alpha} \bigoplus_{i=1}^r X_i^{n_i} \xrightarrow{\beta} X \to 0$$

be the almost split sequence ending in $X$ that is used in the definition of the modulation on $\Gamma$, where $X_1, \ldots, X_r \in \Gamma$ are pairwise distinct. Let

$$0 \to \tau_A X \xrightarrow{f} \bigoplus_{i=1}^r X_i^{n_i} \xrightarrow{g} X \to 0$$

be another almost split sequence where the $f_{i,j}: \tau_A X \to X_i$ and the $g_{i,j}: X_i \to X$ are the components of $f$ and $g$, respectively. Then there is a commutative diagram

$$
\begin{array}{ccc}
0 & \xrightarrow{\alpha} & \tau_A X \xrightarrow{\beta} X \\
\downarrow{u} & & \downarrow{v} \\
0 & \xrightarrow{f} & \bigoplus_{i=1}^r X_i^{n_i} \xrightarrow{g} X \\
\end{array}
$$

where $u, v$ are isomorphisms. With this data:

(a) $(\tilde{g}_{i,1}, \ldots, \tilde{g}_{i,n_i})$ is a basis of the left $\kappa_X$,-module $\text{irr}(X_i, X)$ and the corresponding dual basis of the right $\kappa_X^*$-module $\text{irr}(\tau_A X, X_i)$ under the pairing $\sigma$ is $(\tilde{u}_{i,1}, \ldots, \tilde{u}_{i,n_i})$, for each $i \in \{1, \ldots, r\}$;

(b) $\gamma_X = \sum_{i,j} \tilde{f}_{i,j} \otimes \tilde{g}_{j,i}$.  

Proof. (b) follows directly from (a) and from the definition of $\gamma_X$. It therefore suffices to prove (a). The existence of $u$ and $v$ are direct consequences of the basic properties of almost split sequences; also, for every $i$, the given $n_i$-tuples are indeed bases because they arise from a left (or right) minimal almost split morphism (see [1.1]). Let $w: \bigoplus X_i^{n_i} \to \bigoplus X_i^{n_i}$ be $v^{-1}$. For every $i, i' \in \{1, \ldots, r\}$, and every $1 \leq j \leq n_i$, and every $1 \leq j' \leq n_i'$, let $v_{(i,j),(i',j')}: X_i \to X_{i'}$ and $w_{(i,j),(i',j')}: X_i \to X_{i'}$ be the respective components of $v$ and $w$, from the $j$-th component $X_i$ of $X_i^{n_i}$ to the $j'$-th component $X_{i'}$ of $X_{i'}^{n_{i'}}$. Then, the equality
Let \( \Gamma \) be a modulated translation quiver. Recall that the radical \( \mathcal{R}(\Gamma) \) of \( k(\Gamma) \) was defined in the introduction. For every arrow \( x \to y \in \Gamma \) the natural map \( M(x, y) \to k(\Gamma)(x, y) \) is one-to-one. And the \( \kappa_x - \kappa_y \)-bimodule \( \mathcal{R}(\Gamma)(x, y) \) decomposes as \( M(x, y) \oplus \mathcal{R}^2 k(\Gamma)(x, y) \). The description of the ideal \( \mathcal{R}(k(\Gamma)) \) is easier when \( \Gamma \) is with length as shows the following proposition. It is central in this text and used without further reference. The proof is a small variation of [Cha10, 2.1] where the description was first proved in the case \( \kappa_x = k \) for every vertex \( x \in \Gamma \).

**Proposition.** Let \( \Gamma \) be a translation quiver with length and \( x, y \in \Gamma \). If there is a path of length \( \ell \) from \( x \) to \( y \) in \( \Gamma \), then:

(a) \( k(\Gamma)(x, y) = \mathcal{R}(\Gamma)(x, y) = \mathcal{R}^2 k(\Gamma)(x, y) = \cdots = \mathcal{R}^\ell k(\Gamma)(x, y) \).

(b) \( \mathcal{R}^i k(\Gamma)(x, y) = 0 \) if \( i > \ell \).

**1.6. Coverings of translation quivers.** See [BG82, 1.3] for more details. A covering of translation quivers is a quiver morphism \( p : \tilde{\Gamma} \to \Gamma \) such that

(a) \( \tilde{\Gamma} \) are translation quivers, \( \Gamma \) is connected,

(b) a vertex \( x \in \tilde{\Gamma} \) is projective (or injective, respectively) if and only if so is \( px \),

(c) \( p \) commutes with the translations in \( \Gamma \) and \( \tilde{\Gamma} \) (where these are defined),

(d) for every vertex \( x \in \tilde{\Gamma} \) the map \( \alpha \mapsto p(\alpha) \) induces a bijection from the set of arrows in \( \tilde{\Gamma} \) starting in \( x \) (or ending in \( x \)) to the set of arrows in \( \Gamma \) starting in \( px \) (or ending in \( px \), respectively).

Let \( \pi : \tilde{\Gamma} \to \Gamma \) be a covering of translation quivers. If \( \Gamma \) is modulated by division \( k \)-algebras \( \kappa_x \) and bimodules \( M(x, y) \) for every vertex \( x \) and every arrow \( x \to y \), then \( \tilde{\Gamma} \) is modulated by the division \( k \)-algebra \( \kappa_x := \kappa_{\pi x} \) at the vertex \( x \in \tilde{\Gamma} \) and by the bimodule \( M(\pi x, \pi y) \) for every arrow \( x \to y \) in \( \tilde{\Gamma} \). In this text, this modulations on \( \tilde{\Gamma} \) is called induced by the modulations of \( \Gamma \). When \( \tilde{\Gamma} \) is with length, it has a length function, that is, a map \( x \mapsto \ell(x) \) defined on vertices such that \( \ell(y) = \ell(x) + 1 \) for every arrow \( x \to y \). See [BG82, 1.6] for the construction of such
a function in the particular case where \( \tilde{\Gamma} \) is the universal cover of \( \Gamma \) ([BG82, 1.2.1.3]). Let \( \tilde{\Gamma} \) be a covering of \( \Gamma \) such that \( \Gamma \) is connected. This section introduces the notion of well-behaved functors \( F: k(\tilde{\Gamma}) \to \text{ind}\Gamma \) where \( \tilde{\Gamma} \) is equipped with the modulation induced from the standard one restricted to \( \Gamma \) (see 1.4). These functors are proved to exist when \( \tilde{\Gamma} \) is with length. Such objects were first introduced over algebraically closed fields for the standard one restricted to algebraically closed fields (see [BG82, Sect. 2]). The section also proves the lifting properties of well-behaved functors (see [CLMT11, Sect. 2] when \( k \) is algebraically closed).

Until the end of 2.5 the following convention is used implicitly. If \( \tilde{\Gamma} \) be a finite dimensional \( k \)-algebra canonical surjection \( E/\text{rad}E \to E \) of the \( k \)-algebra \( \text{End}_A(X) \) is meant a section \( \kappa \). For short, \( k_X \) is called a section of \( \kappa_X \).

2.1. Sections of residue fields and of spaces of irreducible morphisms. Let \( X \in \Gamma \) be a vertex. The following theorem plays a central role in this article.

**Theorem** (Wedderburn, Malcev (see [Pie82, 11.6 Corollary on p. 211])). Let \( k \) be a perfect field. Let \( E \) be a finite dimensional \( k \)-algebra. Then, there exists a section \( E/\text{rad}E \to E \) of the \( k \)-algebra canonical surjection \( E/\text{rad}E \to E \).

For a given section \( \kappa_X = \text{End}_A(X)/\text{rad}(X,X) \to \text{End}_A(X) \) of the \( k \)-algebra surjection \( \text{End}_A(X) \to \text{End}_A(X)/\text{rad}(X,X) \), the image is denoted by \( k_X \). Then \( k_X \subseteq \text{End}_A(X) \) is a subalgebra such that \( \text{End}_A(X) = k_X \oplus \text{rad}(X,X) \) as a \( k \)-vector space and the surjection \( \text{End}_A(X) \to \kappa_X \) restricts to a \( k \)-algebra isomorphism \( k_X \xrightarrow{\sim} \kappa_X \). For short, \( k_X \) is called a section of \( \kappa_X \).

Let \( X \to Y \) be an arrow in \( \Gamma \). Then \( X \not\cong Y \) and \( \text{Hom}_A(X,Y) = \text{rad}(X,Y) \). Suppose given sections \( k_X \subseteq \text{End}_A(X) \) and \( k_Y \subseteq \text{End}_A(Y) \) of \( \kappa_X \) and \( \kappa_Y \), respectively. Then \( \text{irr}(X,Y) \) is a \( k_X - k_Y \)-bimodule using the isomorphisms \( k_X \xrightarrow{\sim} \kappa_X \) and \( k_Y \xrightarrow{\sim} \kappa_Y \). A by-state \( k_X - k_Y \)-linear section (of \( \text{irr}(X,Y) \)) is meant a section \( \text{irr}(X,Y) \to \text{rad}(X,Y) \) of the canonical surjection \( \text{rad}(X,Y) \to \text{irr}(X,Y) \) in the category of \( k_X - k_Y \)-bimodules. Such a section always exists because the \( k \)-algebra \( k_X \otimes_k k_Y^\text{op} \) is semisimple. Note that the datum of a linear section depends on the choice of the algebra sections \( k_X \) and \( k_Y \).
2.2. Well-behaved functors. The following definition extends to the case of perfect fields the already known definition of well-behaved functors when the base field is algebraically closed ([BGS82, 3.1] and [Rie80, 2.2]). Keep the setting established at the beginning of this section. A \( k \)-linear functor \( F: k(\Gamma) \to \text{ind} \Gamma \) is called well-behaved if

(a) \( Fx = \pi x \) for every vertex \( x \in \Gamma \),

(b) for every vertex \( x \in \Gamma \), the k-algebra map from \( \kappa_{\pi x} = \kappa_x \) to \( \text{End}_A(\pi x) \) defined by \( u \mapsto F(u) \) is a section of the natural surjection \( \text{End}_A(\pi x) \to \kappa_{\pi x} \). Its image is denoted by \( k_x \),

(c) for every arrow \( x \to y \) in \( \Gamma \), the induced map \( Fx \to \pi x \) is well-behaved. Moreover, any well-behaved functor arises in this way.

Note that if \( F \) is as in the definition then distinct vertices \( x, x' \in \Gamma \) such that \( \pi x = \pi x' \) may give rise to different sections \( k_x \) and \( k_{x'} \) in \( \text{End}_A(\pi x) \). The data of a section \( k_x \subseteq \text{End}_A(\pi x) \) of \( \kappa_{\pi x} \), for every vertex \( x \in \Gamma \), and that of a \( k_x - k_y \)-linear section \( M(x, y) \to \text{Hom}_A(\pi x, \pi y) \), for every arrow \( x \to y \) in \( \Gamma \) determine a unique \( k \)-linear functor \( k(\Gamma) \to \text{ind} \Gamma \). It induces a \( k \)-linear functor \( F: k(\Gamma) \to \text{ind} \Gamma \) if and only if it vanishes on \( \gamma_x = \gamma_{\pi x} \) for every non-projective vertex \( x \in \Gamma \). In such a case, \( F \) is well-behaved. Moreover, any well-behaved functor arises in this way.

2.3. Local sections on almost split sequences. The existence of well-behaved functors is based on the following technical lemma. It aims at constructing sections that are compatible with the standard modulation of \( \Gamma \), in some sense.

**Lemma.** Let \( X \in \Gamma \) be a non-injective vertex and let

\[
0 \to X \xrightarrow{f} \bigoplus_{i=1}^{r} X_i^{n_i} \xrightarrow{g} \tau_A^{-1}X \to 0
\]

be an almost split sequence. Let \( k_X \subseteq \text{End}_A(X) \) and \( k_{X_i} \subseteq \text{End}_A(X_i) \) (for every \( i \in \{1, \ldots, r\} \) be sections of \( \kappa_X \) and \( \kappa_{X_i} \) respectively, and let \( \text{irr}(X, X_i) \to \text{rad}(X, X_i) \) be a \( k_X - k_{X_i} \)-linear section which maps \( f_{i,1}, \ldots, f_{i,n_i} \) to \( f_{i,1}, \ldots, f_{i,n_i} \), respectively, for every \( i \in \{1, \ldots, r\} \). There exists a section \( k_{\tau_A^{-1}X} \to \text{End}_A(\tau_A^{-1}X) \) of \( \kappa_{\tau_A^{-1}X} \) and a \( k_X - k_{\tau_A^{-1}X} \)-linear section \( \text{irr}(X_i, \tau_A^{-1}X) \to \text{rad}(X_i, \tau_A^{-1}X) \) (for every \( i \in \{1, \ldots, r\} \) such that

(a) it maps \( g_{i,1}, \ldots, g_{i,n_i} \) to \( g_{i,1}, \ldots, g_{i,n_i} \), respectively, for every \( i \in \{1, \ldots, r\} \),

(b) the induced map \( \bigoplus_{i=1}^{r} \text{irr}(X, X_i) \otimes_{\kappa_{X_i}} \text{irr}(X_i, \tau_A^{-1}X) \to \text{Hom}_A(X, \tau_A^{-1}X) \) vanishes on \( \gamma_{\tau_A^{-1}X} \).

**Proof.** Note that if such sections do exist then \( (g_{i,1}, \ldots, g_{i,n_i}) \) must be a basis over \( k_X \) of the image of \( \text{irr}(X_i, \tau_A^{-1}X) \to \text{rad}(X_i, \tau_A^{-1}X) \) because \( g \) is a right minimal almost split morphism. In particular, the section \( k_{\tau_A^{-1}X} \subseteq \text{End}_A(\tau_A^{-1}X) \) must be such that the left \( k_X \)-submodule of \( \text{rad}(X_i, \tau_A^{-1}X) \) generated by \( g_{i,1}, \ldots, g_{i,n_i} \) is a \( k_X - k_{\tau_A^{-1}X} \)-submodule of \( \text{rad}(X_i, \tau_A^{-1}X) \), for every \( i \in \{1, \ldots, r\} \). The proof therefore proceeds as follows: 1) define a section \( k_{\tau_A^{-1}X} \subseteq \text{End}_A(\tau_A^{-1}X) \) satisfying this last condition, 2) define sections \( \text{irr}(X_i, \tau_A^{-1}X) \to \text{rad}(X_i, \tau_A^{-1}X) \) so that (a) is satisfied, and 3) prove (b).
1) Let \( \varphi \in \text{End}_A(\tau_A^{-1}X) \) and define a new representative \( \varphi_1 \in \text{End}_A(\tau_A^{-1}X) \) of \( \varphi \in \kappa_{\tau_A^{-1}X} \) as follows. Let \( 0 \to X \xrightarrow{\alpha} \oplus_i X_i^{n_i} \xrightarrow{\beta} \tau_A^{-1}X \to 0 \) be the almost split sequence used in the definition of the standard modulation of \( \Gamma \). So there exists an isomorphism of exact sequences

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{\alpha} & \oplus_i X_i^{n_i} & \xrightarrow{\beta} & \tau_A^{-1}X & \longrightarrow & 0 \\
\quad & v & \downarrow & \quad & v & \downarrow & \quad & \Downarrow & \\
0 & \longrightarrow & X & \xrightarrow{f} & \oplus_i X_i^{n_i} & \xrightarrow{g} & \tau_A^{-1}X & \longrightarrow & 0.
\end{array}
\]

There also exists a commutative diagram

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{f} & \oplus_i X_i^{n_i} & \xrightarrow{g} & \tau_A^{-1}X & \longrightarrow & 0 \\
\quad & \psi & \downarrow & \quad & \theta & \downarrow & \quad & \phi & \\
0 & \longrightarrow & X & \xrightarrow{f'} & \oplus_i X_i^{n_i} & \xrightarrow{g'} & \tau_A^{-1}X & \longrightarrow & 0
\end{array}
\]

for some morphisms \( \theta \) and \( \psi \). The pair \((\theta, \psi)\) in the above diagram is not unique; however for any different pair \((\theta', \psi')\) such that the diagram \((\psi')\) still commutes after replacing \( \theta \) and \( \psi \) by \( \theta' \) and \( \psi' \), respectively, then there exists \( h: \oplus_i X_i^{n_i} \to X \) such that \( \theta - \theta' = hf \); given that \( (\psi - \psi')f = f(\theta - \theta') \) and that \( f \) is a monomorphism, it follows that \( \psi - \psi' = fh \); note that none of the \( X_i \) is isomorphic to \( X \) because \( f \) is irreducible; therefore \( h \) lies in \( \text{rad} \), and hence \( \psi - \psi' \) lies in \( \text{rad}^2 \). The diagrams \((4)\) and \((5)\) yield the following commutative diagram

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{\alpha} & \oplus_i X_i^{n_i} & \xrightarrow{\beta} & \tau_A^{-1}X & \longrightarrow & 0 \\
\quad & u\psi u^{-1} & \downarrow & \quad & v\theta v^{-1} & \downarrow & \quad & \phi & \\
0 & \longrightarrow & X & \xrightarrow{\alpha} & \oplus_i X_i^{n_i} & \xrightarrow{\beta} & \tau_A^{-1}X & \longrightarrow & 0
\end{array}
\]

Therefore, \( \tau_*(\varphi) = u\psi u^{-1} \in \kappa_X \). The previous discussion on the uniqueness of the pair \((\theta, \psi)\) shows that \( \overline{\psi} \) is uniquely determined by \( \varphi \) even though \( \psi \) is not. Now let \( \psi_1 \in \kappa_X \) be the representative of \( \psi \), that is, \( \overline{\psi_1} = \overline{\psi} \). Since the section \( \text{irr}(X, X_i) \to \text{rad}(X, X_i) \) is \( k_X \)-linear and maps \( f_{i,1}, \ldots, f_{i,n_i} \) to \( f_{i,1}, \ldots, f_{i,n_i} \), respectively, and since \( (f_{i,1}, \ldots, f_{i,n_i}) \) is a basis of the right \( k_X \)-module \( \text{irr}(X, X_i) \), there is a unique matrix \( \eta_i \in M_{n_i}(k_X) \), considered as an endomorphism of \( X_i^{n_i} \), making the following square commute for \( i \in \{1, \ldots, r\} \)

\[
\begin{array}{ccc}
X & \xrightarrow{f_{i,1}, \ldots, f_{i,n_i}} & X_i^{n_i} \\
\psi_1 & \downarrow & \eta_i \\
X & \xrightarrow{f_{i,1}, \ldots, f_{i,n_i}} & X_i^{n_i}
\end{array}
\]

Therefore there exists a unique \( \varphi_1 \in \text{End}_A(\tau_A^{-1}X) \) making the following diagram commute where \( \eta: \oplus_i X_i^{n_i} \to \oplus_i X_i^{n_i} \) is the morphism defined by \( \{\eta_i: X_i^{n_i} \to X_i^{n_i}\}_{i} \).

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{f} & \oplus_i X_i^{n_i} & \xrightarrow{g} & \tau_A^{-1}X & \longrightarrow & 0 \\
\quad & \psi_1 & \downarrow & \quad & \eta & \downarrow & \quad & \phi_1 & \\
0 & \longrightarrow & X & \xrightarrow{f} & \oplus_i X_i^{n_i} & \xrightarrow{g} & \tau_A^{-1}X & \longrightarrow & 0
\end{array}
\]
Hence the following diagram commutes

\[
\begin{array}{cccccc}
0 & \longrightarrow & X & \overset{\alpha}{\longrightarrow} & \bigoplus_i X_i^n & \overset{\beta}{\longrightarrow} & \tau_A^{-1}X & \longrightarrow & 0 \\
\downarrow u\psi_1u^{-1} & & \downarrow v\psi_1u^{-1} & & \downarrow \varphi_1 & & \downarrow \varphi & & \\
0 & \longrightarrow & X & \overset{\alpha}{\longrightarrow} & \bigoplus_i X_i^n & \overset{\beta}{\longrightarrow} & \tau_A^{-1}X & \longrightarrow & 0 \\
\end{array}
\]

This entails \( \overline{\varphi_1} = \tau_A^{-1}(u\psi_1u^{-1}) \). But \( \overline{\psi_1} = \overline{\psi} \) so that \( u\psi_1u^{-1} = \overline{u\psi_1u^{-1}} \). Thus \( \overline{\varphi_1} = \tau_A^{-1}(\overline{u\psi_1u^{-1}}) = \overline{\varphi} \). This construction therefore yields a well-defined map

\[
s: \quad \text{End}_A(\tau_A^{-1}X) \to \text{End}_A(\tau_A^{-1}X)
\]

\[
\varphi \mapsto \varphi_1 \quad (\text{such that } \overline{\varphi_1} = \overline{\varphi}).
\]

Since \( \tau_\kappa: \kappa_A^{-1}X \to \kappa_X \) is a \( k \)-algebra isomorphism and \( \eta_1, \ldots, \eta_r \) are determined by \( \eta_1 \) (and the fixed \( k \)-algebra sections), the map \( s \) is a \( k \)-algebra homomorphism. Moreover if \( \varphi \in \text{rad}(\tau_A^{-1}X, \tau_A^{-1}X) \) then \( \overline{\varphi} = 0 \) and the representative \( \psi_1 \) in \( k_X \) of \( \tau_\kappa(\overline{\varphi}) = 0 \); therefore, \( \eta_1, \ldots, \eta_r = 0 \) and \( \varphi_1 = 0 \); in other words \( s \) vanishes on \( \text{rad}(\tau_A^{-1}X, \tau_A^{-1}X) \). Hence, \( s \) induces a \( k \)-algebra homomorphism

\[
\overline{s}: \quad \kappa_A^{-1}X \to \text{End}_A(\tau_A^{-1}X)
\]

such that \( \overline{s}(\overline{\varphi}) = \varphi \) for all \( \varphi \in \text{End}_A(\tau_A^{-1}X) \). Denote by \( k_A^{-1}X \) the image of \( \overline{s} \).

Thus, \( k_A^{-1}X \subseteq \text{End}_A(\tau_A^{-1}X) \) is a \( k \)-algebra section of \( \kappa_A^{-1}X \).

2) In order to prove (a) there only remains to define a \( k_X \)-\( k_A^{-1}X \)-linear section \( s_i: \text{irr}(X_i, \tau_A^{-1}X) \to \text{rad}(X_i, \tau_A^{-1}X) \) which maps \( \overline{g_{i,j}} \) to \( g_{i,j} \) for every \( j \in \{1, \ldots, n_i\} \). In view of the assumption on the section \( k_X \subseteq \text{End}_A(X_i) \) of \( \kappa_X \), the construction of \( s_i \) entails that the identity mapping of \( \text{irr}(X_i, \tau_A^{-1}X) \) is equal to the composite morphism

\[
\text{irr}(X_i, \tau_A^{-1}X) \xrightarrow{s_i} \text{rad}(X_i, \tau_A^{-1}X) \to \text{irr}(X_i, \tau_A^{-1}X),
\]

where the right hand-side arrow is the natural surjection. In order to prove that \( s_i \) is a \( k_X \)-\( k_A^{-1}X \)-linear section, there only remains to prove that \( s_i \) is \( k_A^{-1}X \)-linear.

Let \( \varphi \in k_A^{-1}X \). Using the above notation in the construction of \( k_A^{-1}X \), one has \( \varphi = \varphi_1 \). It follows from (6) that

\[
[g_{i,1}, \ldots, g_{i,n_i}]^t \cdot \varphi = [\eta_{i,1}, \ldots, \eta_{i,n_i}]^t.
\]

Writing the endomorphism \( \eta_i: X_i^{n_i} \to X_i^{n_i} \) coordinatewise as \( [\eta_{i,j,j'}; 1 \leq j, j' \leq n_i] \), where \( \eta_{i,j,j'} \in k_X \) for all \( j, j' \), it follows that

\[
(\forall j \in \{1, \ldots, n_i\}) \quad g_{i,j} \varphi = \sum_{j'=1}^{n_i} \eta_{i,j,j'} g_{i,j'}.
\]

Applying \( s_i \) which is \( k_X \)-linear and satisfies \( s_i(\overline{g_{i,j}}) = g_{i,j} \) for all \( j \) entails that

\[
(\forall j \in \{1, \ldots, n_i\}) \quad s_i(\overline{g_{i,j}} \varphi) = g_{i,j} \varphi = s_i(\overline{g_{i,j}}) \varphi.
\]

Accordingly, \( s_i \) is \( k_A^{-1}X \)-linear, which finishes the proof of (a).
3) There only remains to prove (b). According to the lemma in 1.4, the commutative diagram (4) entails that \( \gamma_{\tau_A^{-1}X} = \sum_{i,j=1}^{r} u_i j \otimes g_{i,j} \). Let \( u' \in k_X \) be the representative of \( \pi \in k_X \). The image of \( \gamma_{\tau_A^{-1}X} \) under the map \( \oplus_{i=1}^{r} \text{irr}(X, X_i) \otimes \kappa_{X_i} \) is \( \text{irr}(X_i, \tau_A^{-1}X) \to \text{Hom}_A(X, \tau_A^{-1}X) \) induced by all the considered sections is therefore \( \sum_{i,j} f_{i,j} g_{i,j} = u' \sum_{i,j} f_{i,j} g_{i,j} = u'fg = 0. \)

Dual considerations yield the following dual version of the preceding lemma.

**Lemma.** Let \( X \in \Gamma \) be a non-injective vertex and let

\[
0 \to X \xrightarrow{f} \bigoplus_{i=1}^{r} X_i \xrightarrow{g} \tau_A^{-1} X \to 0
\]

be an almost split sequence. Let \( k_{\tau_A^{-1}X} \subseteq \text{End}_A(X) \) and \( k_X \subseteq \text{End}_A(X_i) \) (for \( i \in \{1, \ldots, r\} \)) be sections of \( \kappa_{\tau_A^{-1}X} \) and \( \kappa_X \), respectively, and let \( \text{irr}(X_i, \tau_A^{-1}X) \hookrightarrow \text{rad}(X_i, \tau_A^{-1}X) \) be a \( k_X \)-linear section under which \( f_{i,1}, \ldots, f_{i,n_i} \) are mapped to \( g_{i,1}, \ldots, g_{i,n_i} \), respectively, for every \( i \in \{1, \ldots, r\} \). There exists a section \( k_X \hookrightarrow \text{End}_A(X) \) of \( k_X \) and a \( k_X \)-linear section \( \text{irr}(X, X_i) \to \text{rad}(X, X_i) \) (for every \( i \in \{1, \ldots, r\} \)).

(a) it maps \( f_{i,1}, \ldots, f_{i,n_i} \) to \( f_{i,1}, \ldots, f_{i,n_i} \), respectively, for every \( i \in \{1, \ldots, r\} \),

(b) the induced map \( \oplus_{i=1}^{r} \text{irr}(X, X_i) \otimes \kappa_{X_i} \text{irr}(X_i, \tau_A^{-1}X) \to \text{Hom}_A(X, \tau_A^{-1}X) \) vanishes on \( \gamma_{\tau_A^{-1}X} \).

2.4. Inductive construction of well-behaved functors. In view of proving the existence of well-behaved functors \( k(\Gamma) \to \text{ind}\Gamma \) it is necessary to consider \( k \)-linear functors \( F : k\chi \to \text{ind}\Gamma \). Here the notation \( k\chi \) stands for the full subcategory of \( k\Gamma \) with object set being the set of vertices in \( \chi \). Following [22] such a functor \( F : k\chi \to \text{ind}\Gamma \) is called well-behaved if:

(a) \( Fx = \pi x \) for every vertex \( x \in \chi \); (b) the \( k \)-algebra homomorphism \( \kappa_x \to k\chi(x, x) \to \text{End}_A(\pi x) \) is a section of the natural surjection \( \text{End}_A(\pi x) \to \kappa_{\pi x} = \kappa_x \), for every vertex \( x \in \chi \) (as above, the image of the section is denoted by \( k_x \)); (c) the \( k \)-linear composite map \( \text{irr}(\pi x, \pi y) = M(x, y) \to k\chi(x, y) \to \text{Hom}_A(\pi x, \pi y) \) is a \( k_x \)-linear section, for every arrow \( x \to y \) in \( \chi \); and (d) it vanishes on \( \gamma_x = \gamma_{\pi x} \) for every non-projective vertex \( x \in \chi \) such that \( \tau x \in \chi \).

**Lemma.** Assume that \( \Gamma \) is with length. Let \( \ell \) be a length function on the vertices in \( \Gamma \). Let \( F : k\chi \to \text{ind}\Gamma \) be a well-behaved functor where \( \chi \subseteq \Gamma \) is a full and convex subquiver distinct from \( \Gamma \) and satisfying the following two conditions

(i) either there is no arrow \( x \to x_1 \) in \( \Gamma \) such that \( x_1 \in \chi \) and \( x \not\in \chi \), or else there is an upper bound on the integers \( \ell(x) \) where \( x \) runs through the vertices in \( \Gamma \setminus \chi \) such that there exists an arrow \( x \to x_1 \) in \( \Gamma \) satisfying \( x_1 \in \chi \),

(ii) either there is no arrow \( x_1 \to x \) in \( \Gamma \) such that \( x_1 \in \chi \) and \( x \not\in \chi \), or else there is a lower bound on the integers \( \ell(x) \) where \( x \) runs through the vertices in \( \Gamma \setminus \chi \) such that there exists an arrow \( x_1 \to x \) in \( \Gamma \) satisfying \( x_1 \in \chi \).

Then there exists at least one arrow such as in (i) or (ii) and for every arrow \( x \to x_1 \) (or, \( x_1 \to x \)) in \( \Gamma \) such that \( x \not\in \chi \) and \( x_1 \in \chi \), with \( \ell(x) \) maximal (or, minimal, respectively) for these properties, there exists a well-behaved functor.
\[ F': k\chi' \rightarrow \text{ind}\Gamma \] which extends \( F \), where \( \chi' \) denotes the full subquiver of \( \Gamma \) with vertices \( x \) and the vertices in \( \chi \).

**Proof.** There exists an arrow \( x \rightarrow x_1 \) or \( x_1 \rightarrow x \) in \( \Gamma \) such that \( x \notin \chi \) and \( x_1 \in \chi \) because \( \chi \subseteq \Gamma \). Assume the former (the latter is dealt with similarly) and choose \( x \) so that \( \ell(x) \) is maximal ((ii)). Let \( \chi' \) be the full subquiver of \( \Gamma \) with vertices \( x \) and those of \( \chi \). Then \( \chi' \) is convex in \( \Gamma \) because \( \ell(x) \) is maximal. Let \( x \rightarrow x_1, \ldots, x \rightarrow x_r \) be the arrows in \( \Gamma \) starting in \( x \) and ending in some vertex in \( \chi \). Note that if \( x \) is non-injective and \( \tau^{-1}x \in \chi \) then these are all the arrows in \( \Gamma \) starting in \( x \) because \( \ell(x) \) is maximal. In order to extend \( F \): \( k\chi \rightarrow \text{ind}\Gamma \) to a functor \( F': k\chi' \rightarrow \text{ind}\Gamma \) distinguish two cases according to whether \( x \) is non-injective and \( \tau^{-1}x \in \chi \), or not.

For every \( i \in \{1, \ldots, r\} \) let \( k_i \subseteq \text{End}_A(x_i) \) be the section \( F(k_{\pi \chi_i}) \) of \( k_{\pi \chi_i} \).

Assume first that either \( x \) is injective, or else \( x \) is non-injective and \( \tau^{-1}x \notin \chi \). Fix any \( k \)-algebra section \( k_x \subseteq \text{End}_A(px) \) of \( k_{\pi \chi} \). Let \( i \in \{1, \ldots, r\} \) be the \( k \)-algebra isomorphisms \( k_{\pi x_i} \rightarrow k_i \) (induced by \( F \)) and \( k_x \rightarrow k_x \) allow one to consider \( M(x, x_i) \) as a \( k_x \)-linear section. For this structure, the quotient map \( \text{Hom}_A(px, px_i) \rightarrow \text{rad}(px, px_i) \) is is \( k_x \)-linear. On the other hand the \( k \)-algebra \( k_x \otimes_k k_i \) is semi-simple. Hence there exists a \( k_x \)-linear section \( M(x, x_i) = \text{irr}(px, px_i) \rightarrow \text{Hom}_A(px, px_i) \) of \( p_i \). The sections \( k_x \subseteq \text{End}_A(px) \) and \( s_i (i \in \{1, \ldots, r\}) \) extend \( F: k\chi \rightarrow \text{ind}\Gamma \) to a \( k \)-linear functor \( F' \): \( k\chi' \rightarrow \text{ind}\Gamma \) satisfying the conditions (a), (b) and (c) in the definition of well-behaved functors. Moreover, condition (d) is satisfied for \( F' \) because it is so for \( F \) and because, if \( y \in \Gamma \) is non-projective and \( y, \tau y \in \chi \) then \( y, \tau y \in \chi \). This proves the lemma when either \( x \) is injective or else \( x \) is non-injective and \( \tau^{-1}x \notin \chi \).

Assume now that \( x \) is non-injective and \( \tau^{-1}x \in \chi \). The mesh in \( \Gamma \) starting in \( x \) has the form

\[
\begin{array}{ccc}
  x & \xrightarrow{\chi} & x_1 \\
  \downarrow & & \downarrow \\
  x_r & \xrightarrow{\tau^{-1}x} & \tau^{-1}x.
\end{array}
\]

For simplicity let \( X = \pi x \) and \( X_i = \pi x_i \) for every \( i \in \{1, \ldots, r\} \) so that \( \pi \tau^{-1}x = \tau^{-1}_AX \). Let \( n_1, \ldots, n_r \geq 1 \) be the integers such that there is an almost split sequence

\[
0 \rightarrow X \rightarrow \bigoplus_{i=1}^r X_i^{n_i} \rightarrow \tau^{-1}_AX \rightarrow 0.
\]

Since \( F: k\chi \rightarrow \text{ind}\Gamma \) is well-behaved, it induces a \( k \)-algebra section \( k_{\tau^{-1}_AX} \subseteq \text{End}_A(\tau^{-1}_AX) \) of \( k_{\tau^{-1}_AX} \). It also induces a \( k \)-algebra section \( k_{\tau^{-1}_AX} \)-linear section \( M(x, \tau^{-1}x) = \text{irr}(X_i, \tau^{-1}AX) \rightarrow \text{rad}(X_i, \tau^{-1}AX) \), for every \( i \in \{1, \ldots, r\} \). Let \( i \in \{1, \ldots, r\} \), let \( \{g_{i,j}: X_i \rightarrow \tau^{-1}_AX \}_{1 \leq j \leq n_i} \) be the image under this \( k_i \otimes_k k_r \)-linear section of a basis of the \( k \)-vector space \( M(x, \tau^{-1}x) \). Thus, the morphism \( \bigoplus M(x, x_i) \otimes_{k_{\pi x_i}} M(x, \tau^{-1}x) \rightarrow \text{Hom}_A(X, \tau^{-1}_AX) \) vanishes on \( \gamma_{\tau^{-1}_AX} \). These new sections clearly extend \( F: k\chi \rightarrow \text{ind}\Gamma \) to a well-behaved functor \( F': k\chi' \rightarrow \text{ind}\Gamma \). Like in the previous case \( F' \): \( k\chi' \rightarrow \text{ind}\Gamma \) is
well-behaved. Finally $\chi'$ satisfies both conditions (i) and (ii) in the statement of the lemma since $\chi' \setminus \chi$ consists of one vertex. □

2.5. **Existence of well-behaved functors.** For a strongly irreducible morphism $f: X \to \oplus_{i=1}^r X_i^{n_i}$, the $\kappa_X - \chi_X$-bimodule $\kappa_X \cdot \tilde{f}_{i,j} \cdot \kappa_{X_i}$ is projective and given by an idempotent of $\kappa_X \otimes \kappa_{X_i}^{\text{op}}$; for all $i,j$, hence, the following are equivalent

- for all $i,j$, the annihilator of $\tilde{f}_{i,j}$ over $\kappa_X \otimes \kappa_{X_i}^{\text{op}}$ is trivial,
- for all $i$, the family $\{\tilde{f}_{i,j}\}_j$ of $\text{irr}(X, X_i)$ is free over $\kappa_X \otimes \kappa_{X_i}^{\text{op}}$.

The following result implies Theorem A.

**Proposition.** Let $A$ be a finite-dimensional algebra over a perfect field $k$. Let $\Gamma$ be an Auslander-Reiten component of $A$. Let $\pi: \tilde{\Gamma} \to \Gamma$ be a covering of translation quivers where $\tilde{\Gamma}$ is connected and with length. Endow $\Gamma$ with its standard $k$-modulation and $\tilde{\Gamma}$ with the induced $k$-modulation.

1. There exists a well-behaved functor $F: k(\tilde{\Gamma}) \to \text{ind } \Gamma$.
2. Let $X \in \Gamma$ and let $f: X \to \oplus_{i=1}^r X_i^{n_i}$ be a strongly irreducible morphism. Assume that the annihilator of $\tilde{f}_{i,j}$ over $\kappa_X \otimes \kappa_{X_i}^{\text{op}}$ is zero for all $i,j$. Let $x \in \pi^{-1}(X)$ and let

$$
\begin{array}{ccc}
& x_1 & \\
x & \longleftrightarrow & \text{ } \\
& x_r & \\
\end{array}
$$

be the full subquiver of $\tilde{\Gamma}$ such that $\pi x_i = X_i$. Then there exists a well-behaved functor $F: k(\tilde{\Gamma}) \to \text{ind } \Gamma$ such that $F$ maps $\tilde{f}_{i,j}$ to $f_{i,j}$ for every $i \in \{1, \ldots, r\}$ and $j \in \{1, \ldots, n_i\}$.

**Proof.** It is possible to assume that $A$ is connected and not semi-simple. Both (1) and (2) are proved with the same inductive construction yet with a specific initialisation step for each. In case (1), let $r = 1$, let $X \to X_1$ be any arrow in $\Gamma$ and let $x \to x_1$ be any arrow in $\tilde{\Gamma}$ with image under $\pi$ equal to $X \to X_1$. In both cases, let $\chi_0$ be the full subquiver of $\tilde{\Gamma}$ with vertices $x, x_1, \ldots, x_r$.

Let $\Sigma$ be the set of pairs $(\chi, F): k\chi \to \text{ind } \Gamma$ where $\chi \subseteq \tilde{\Gamma}$ is a full and convex subquiver containing $\chi_0$ and which satisfies conditions (i) and (ii) in 2.4, and $F: k\chi \to \text{ind } \Gamma$ is a well-behaved functor with the following additional condition in case (2): It maps $\tilde{f}_{i,j}$ to $f_{i,j}$ for all $i,j$. This set is ordered: $(\chi, F) \leq (\chi', F')$ if and only if $\chi \subseteq \chi'$ and $F'$ restricts to $F$.

Let $\ell$ be a length function on $\tilde{\Gamma}$. Denote $\ell(x)$ by $\ell_0$. Denote by $\tilde{\Gamma}_{\geq \ell_0}$ the full subquiver of $\tilde{\Gamma}$ with set of vertices being $\{x' \in \tilde{\Gamma} \mid \ell(x') \geq \ell_0\}$. Consider the subsets $\Sigma_1, \Sigma_2$ and $\Sigma_3$ of $\Sigma$ defined as follows.

- $\Sigma_1$ consists of those $(\chi, F) \in \Sigma$ such that the restriction of $\ell$ to $\chi$ is bounded below by $\ell_0$ and bounded above by $\ell_0 + 1$.
- $\Sigma_2$ consists of those $(\chi, F) \in \Sigma$ such that $\chi \subseteq \tilde{\Gamma}_{\geq \ell_0}$ and $\chi$ is stable under predecessors in $\tilde{\Gamma}_{\geq \ell_0}$, that is, any oriented path in $\tilde{\Gamma}_{\geq \ell_0}$ with endpoint in $\chi$ is contained in $\chi$.
- $\Sigma_3$ consists of the those $(\chi, F) \in \Sigma$ such that $\tilde{\Gamma}_{\geq \ell_0} \subseteq \chi$ and $\chi$ is stable under successors in $\tilde{\Gamma}$.

The set $\Sigma_1$ is not empty. Indeed, let $F: k\chi_0 \to \text{ind } \Gamma$ be as follows. In case (1), let $k\chi$ and $k\chi_1$ be any sections of $\kappa_X$ and $\kappa_X$ in $\text{End}_A(X)$ and $\text{End}_A(X_1)$,
respectively, let \( \text{irr}(X, X_1) \to \text{rad}(X, X_1) \) be any section of the \( k_X \otimes k_{X_1} \)-linear canonical surjection \( \text{rad}(X, X_1) \to \text{irr}(X, X_1) \) and let \( F : k_{\chi_0} \to \text{ind} \Gamma \) be given by the sections \( k_X, k_{X_1} \), and \( \text{irr}(X, X_1) \to \text{rad}(X, X_1) \). In this case, \( (\chi_0, F) \) lies in \( \Sigma_1 \).

In case (2), fix \( k \)-algebra sections \( k_X \subseteq \text{End}_A(X) \) of \( \kappa_X \), and \( k_{X_1} \subseteq \text{End}_A(X_1) \) of \( \kappa_{X_1} \), for every \( i \in \{1, \ldots, r\} \). By assumption on \( f \), there exists a direct sum decomposition \( \text{irr}(X, X_1) = (\bigoplus_i k_X \cdot f_{i,j} \cdot k_{X_1}) \oplus M_1 \) as \( k_X \otimes_k k^{\text{op}}_{X_1} \)-modules. Let \( \text{irr}(X, X_i) \to \text{rad}(X, X_i) \) be the following \( k_X \otimes_k k^{\text{op}}_{X_i} \)-linear section of the natural surjection \( \text{rad}(X, X_i) \to \text{irr}(X, X_i) \) (recall that the algebra \( k_X \otimes_k k^{\text{op}}_{X_i} \) is semi-simple,

- for all \( j \), the composite morphism \( k_X \cdot f_{i,j} \cdot k_{X_1} \to \text{irr}(X, X_i) \) is the \( k_X \otimes_k k^{\text{op}}_{X_1} \)-linear section of the composite morphism \( \text{rad}(X, X_i) \to \text{irr}(X, X_i) \) which maps \( f_{i,j} \) onto \( f_{i,j} \), recall that the annihilator of \( f_{i,j} \) over \( k_X \otimes_k k^{\text{op}}_{X_1} \) is trivial,
- the composite morphism \( M_i \to \text{irr}(X, X_i) \) is any \( k_X \otimes_k k^{\text{op}}_{X_i} \)-linear section of the composite morphism \( \text{rad}(X, X_i) \to \text{irr}(X, X_i) \to M_i \).

These sections define a well-behaved functor \( F : k_{\chi_0} \to \text{ind} \Gamma \). Then \( (\chi_0, F) \in \Sigma_1 \).

Therefore, by definition, \( \Sigma_1 \) is totally inductive. Hence it has at least one maximal element, say, \( (\chi_1, F_1) \), by the Kuratowski-Zorn Lemma. Apply \textbf{2.4} to it: since \( (\chi_1, F_1) \) is maximal in \( \Sigma_1 \), it follows that \( \chi_1 \) is stable under predecessors in \( \Gamma_{\geq \ell_0} \).

Thus, \( \Sigma_2 \) contains \( (\chi_1, F_1) \), and hence it is not empty.

Since \( \Sigma_2 \) is not empty, then it is totally inductive by construction. Consider a maximal element \( (\chi_2, F_2) \) in \( \Sigma_2 \). In this case, \( \chi_2 \) equals \( \Gamma_{\geq \ell_0} \). Indeed, apply \textbf{2.4} to it: since \( (\chi_2, F_2) \) is maximal in \( \Sigma_2 \), it follows that there is no arrow in \( \Gamma \) connecting a vertex in \( \chi_2 \) to a vertex in \( \Gamma_{\geq \ell_0} \setminus \chi_2 \). Hence, should there exist \( x' \in \Gamma_{\geq \ell_0} \setminus \chi_2 \), then \( \chi_2 \cup \{x'\} \) would be a full subquiver of \( \Gamma \) and any \( k \)-algebra section \( k_{\pi x'} \) in \( \text{End}_A(\pi x') \) would define an extension of \( F_2 \) to a well-behaved functor \( \chi_2 \cup \{x'\} \to \text{ind} A \), a contradiction to \( (\chi_2, F_2) \) being maximal in \( \Sigma_2 \). Thus, \( \chi_2 = \Gamma_{\geq \ell_0} \). In particular, \( \Sigma_3 \) is not empty.

Therefore, by definition, \( \Sigma_3 \) is totally inductive. Consider a maximal element \( (\chi_3, F_3) \) in \( \Sigma_3 \). By absurd, assume that the inclusion \( \chi_3 \subseteq \tilde{\Gamma} \) is strict. Then, there exists \( x' \in \tilde{\Gamma} \setminus \chi_3 \) with \( \ell(x') \) maximal because \( \Gamma_{\geq \ell_0} \subseteq \chi_3 \).

There is no arrow \( x_1 \to x' \) such that \( x_1 \in \chi_3 \) because \( \chi_3 \) is stable under successors. Should there exist no arrow \( x' \to x_1 \) in \( \Gamma \) such that \( x_1 \in \chi_3 \), then \( \chi_3 \cup \{x'\} \) would be a full subquiver of \( \tilde{\Gamma} \) containing \( \Gamma_{\geq \ell_0} \) and stable under successors, and any \( k \)-algebra section of \( k_{\pi x'} \) in \( \text{End}_A(\pi x') \) would define an extension of \( F_3 \) to a well-behaved functor \( \chi_3 \cup \{x'\} \to \text{ind} A \), a contradiction to \( (\chi_3, F_3) \) being maximal in \( \Sigma_3 \). There hence exists an arrow \( x' \to x_1 \) such that \( x_1 \in \chi_3 \).

Now, apply \textbf{2.4} to \( (\chi_3, F_3) \): by assumption on \( \ell(x') \), there exists a well-behaved functor \( F_3' : \chi_3 \to \text{ind} A \) which extends \( F_3 \), where \( \chi_3' \) is the full subquiver of \( \tilde{\Gamma} \) with vertices \( x' \) and those of \( \chi_3 \). The assumption on \( \ell(x') \) entails that \( (\chi_3', F_3') \in \Sigma_3 \), a contradiction to \( (\chi_3, F_3) \) being maximal in \( \Sigma_3 \). Thus, \( \chi_3 = \tilde{\Gamma} \).

The authors acknowledge the referee for pointing out the missing hypothesis on the annihilators in a previous version of this text. It would be interesting to determine when such annihilators are indeed trivial. For instance, this is the case when \( \kappa_X = k \) because, then, \( k_X \otimes_k k^{\text{op}}_{X_i} \) is a division algebra for all \( i \).

Proof of Theorem [3] The proof uses a specific left minimal almost split morphism and a specific almost split sequence that arise from $F$ and which are now introduced. Let $X = \pi_x x$. For every arrow in $\Gamma$ with source $x$ (say, with target $x'$), fix one basis over $\kappa_{x,x'}$ of $\text{irr}(\pi_x, \pi_{x'})$. Putting these bases together (for all the arrows in $\Gamma$ with source $x$) yields a sequence of morphisms in $k(\Gamma)$ with domain $x$. Say, the sequence is $(\alpha_i)_{i=1,\ldots,r}$ where the codomain of $\alpha_j$ is denoted by $x_i$ (there may be repetitions in the sequence of codomains). Set $X_i = \pi_x x_i$ and $a_i = F(\alpha_i)$ for every $i$. In particular $a_i : X \to X_i$ is an irreducible morphism and $\overline{a_i} = \alpha_i$ if $\overline{a_i}$ is considered as lying in $k(\Gamma)(x, x_i)$. By construction, $[a_1, \ldots, a_r] : X \to \oplus_{i=1}^r X_i$ is a left minimal almost split morphism. If $x$ is non-injective then $a$ completes into an almost split sequence $0 \to X \xleftarrow{\phi} \oplus_{i=1}^r X_i \xrightarrow{\tau_a^{-1}} X \to 0$ as follows. For every $X' \in \{X_1, \ldots, X_r\}$ the family $\{\tau_{a_i}\}_{i \in I, X' = x_i}$ is a basis of $\text{irr}(X, X')$ over $\kappa_{X'}$; let $\{\beta_j\}_{i \in I, X' = x_i}$ be the corresponding dual basis of $\text{irr}(X', \tau_a^{-1} X)$ over $\kappa_{X'}$ (for the standard k-modulation of $\Gamma$); for every $i \in \{1, \ldots, r\}$ such that $X' = x_i$, set $b_i : X_i \to \tau_a^{-1} X$ to be the image of $\beta_i \in k(\Gamma)(x_i, \tau^{-1} x)$ under $F$ (here, if one considers $b_i$ as lying in $k(\Gamma)(x_i, \tau^{-1} x)$ then $b_i = \beta_i$). By construction $\gamma_{t-1} = \sum_{i \in I} \overline{a_i} \otimes b_i$. Therefore $\sum_{i \in I} a_i u_i = 0$ because $F$ is well-behaved and maps each $\overline{a_i}$ to $a_i$ and $b_i$, respectively. Since moreover $a$ is left minimal almost split, the morphism $b = [b_1, \ldots, b_r]^t$ is right minimal almost split. Thus $(a, b)$ forms the announced almost split sequence.

(a) The two maps are dual to each other so only the first one is taken care of.

The surjectivity for every $x$ is proved by induction on $n \geq 0$. If $n = 0$ it follows from: 

rad $^0(\text{rad}(Fx, Fy)) = \kappa_Fx$ or 0 according to whether $Fx = Fy$ or $Fx \neq Fy$, and $R^0k(\Gamma)(x, z)/Rk(\Gamma)(x, z)$ is $\kappa_{x,z}$ or 0 according to whether $x = z$ or $x \neq z$. If $n \geq 1$, if the surjectivity is already proved for indices smaller that $n$, and if $f \in \text{rad}^n(\text{rad}(Fx, Fy))$ is given, then there exists $(u_i)_{i \in I} \in \oplus_{i=1}^n \text{rad}^{n-1}(Fx_i, Fy_i)$ such that $f = \sum_{i \in I} a_i u_i$. Let $(\phi_z)_{i \in I} \in \oplus_{i=1}^n k(\Gamma)(x, z)$ yields $(\psi_z)_{z \in \oplus_{i=1}^n k(\Gamma)(x, z)}$ together with $(u_i)_{i \in I} \in \oplus_{i=1}^n \text{rad}^{n-1}(Fx_i, Fy_i)$ such that $\sum_{z} F(\phi_z - \psi_z) = \sum_{i} a_i u_i$. On the other hand, $n \geq 1$ and $(a_j)_{j \in \{1, \ldots, r\}}$ contains a basis of $M(x, x_i)$, which is part of the induced modulation of $\Gamma$, over $\kappa_{x_i}$, for every $i \in \{1, \ldots, r\}$. The construction of $k(\Gamma)$ therefore yields $(\theta_{i,z,i})_{i \in \oplus_{i=1}^r k(\Gamma)(x_i, z)}$ such that $a_i - \psi_z = \sum_{i} a_i \theta_{i,z}$, for every $z$. Putting these morphisms together and using that $a_i = F(\alpha_i)$ for every $i$ yields $\sum a_i (\sum z F(\theta_{i,z}) - u_i) = 0$. Now distinguish two cases according to whether $x$ is injective or not. If $x$ is injective then $\sum_z F(\theta_{i,z}) = u_i$, which, following the induction hypothesis, implies that $\theta_{i,z} \in \text{rad}^{n+1} k(\Gamma)$ for every $i$ and every $z$. Thus $\phi_z = \psi_x + \sum a_i \theta_{i,z} \in \text{rad}^{n+1} k(\Gamma)$ for every $z$. If $x$ is not injective there exists $v \in \text{Hom}_A(\tau_a^{-1}Fx, Fy)$ such that $\sum_z F(\theta_{i,z}) - u_i = \sum_z F(\phi_z - 1)$. This proves the surjectivity at index $n$. The injectivity for every $x$ is also proved by induction on $n \geq 0$. If $n = 0$ it follows from: 

rad $^0 Fx \cap \text{rad}(Fx, Fy) = \kappa_Fx$ or 0 according to whether $Fx = Fy$ or $Fx \neq Fy$, and $R^0k(\Gamma)(x, z)/Rk(\Gamma)(x, z)$ is $\kappa_{x,z}$ or 0 according to whether $x = z$ or $x \neq z$. If $n \geq 1$, if the injectivity is already proved for indices smaller that $n$, and if $f \in \text{rad}^n(\text{rad}(Fx, Fy))$ is given, then there exists $(u_i)_{i \in I} \in \oplus_{i=1}^n \text{rad}^{n-1}(Fx_i, Fy_i)$ such that $f = \sum_{i \in I} a_i u_i$. Let $(\phi_z)_{i \in I} \in \oplus_{i=1}^n k(\Gamma)(x, z)$ yields $(\psi_z)_{z \in \oplus_{i=1}^n k(\Gamma)(x, z)}$ together with $(u_i)_{i \in I} \in \oplus_{i=1}^n \text{rad}^{n-1}(Fx_i, Fy_i)$ such that $\sum_{z} F(\phi_z - \psi_z) = \sum_{i} a_i u_i$. On the other hand, $n \geq 1$ and $(a_j)_{j \in \{1, \ldots, r\}}$ contains a basis of $M(x, x_i)$, which is part of the induced modulation of $\Gamma$, over $\kappa_{x_i}$, for every $i \in \{1, \ldots, r\}$. The construction of $k(\Gamma)$ therefore yields $(\theta_{i,z,i})_{i \in \oplus_{i=1}^r k(\Gamma)(x_i, z)}$ such that $a_i - \psi_z = \sum_{i} a_i \theta_{i,z}$, for every $z$. Putting these morphisms together and using that $a_i = F(\alpha_i)$ for every $i$ yields $\sum a_i (\sum z F(\theta_{i,z}) - u_i) = 0$. Now distinguish two cases according to whether $x$ is injective or not. If $x$ is injective then $\sum_z F(\theta_{i,z}) = u_i$, which, following the induction hypothesis, implies that $\theta_{i,z} \in \text{rad}^{n+1} k(\Gamma)$ for every $i$ and every $z$. Thus $\phi_z = \psi_x + \sum a_i \theta_{i,z} \in \text{rad}^{n+1} k(\Gamma)$ for every $z$. If $x$ is not injective there exists $v \in \text{Hom}_A(\tau_a^{-1}Fx, Fy)$ such that $\sum_z F(\theta_{i,z}) - u_i = \sum_z F(\phi_z - 1)$.
Using the surjectivity yields $\chi_z \in \bigoplus_{x} k(\widehat{\Gamma}) (\tau^{-1} x, z)$ such that $v = \sum_i F(\chi_z)$ mod $\text{rad}^{n-1}$. In particular $bv = \sum_i F(\beta_{i} x) \mod \text{rad}^{n}$ for every $i$. Hence $\sum z F(\theta_{i} x - \beta_{i} x) = u_i \mod \text{rad}^{n}$. Therefore $\theta_{i} x - \beta_{i} x \in \mathcal{R}^{n} k(\widehat{\Gamma})$ for every $i$ and every $z$ (by induction and because $u_i \in \text{rad}^{n+1}$). Since moreover $\sum_i \alpha_i \beta_i = 0$, $\psi_z \in \mathcal{R}^{n+1} k(\widehat{\Gamma})$ and $\phi_z = \psi_z + \sum_i \alpha_i \theta_i x$ for every $z$, it follows that $\phi_z \in \mathcal{R}^{n+1} k(\widehat{\Gamma})$.

(b) follows from (a) and from [1.5] (part (b)).

(c) follows from (a), (b) and the fact that $\Gamma$ is generalised standard, that is, $\bigcap_{n \geq 0} \text{rad}^{n}(X, Y) = 0$ for every $X, Y \in \Gamma$. 

3. APPLICATION TO COMPOSITIONS OF IRREDUCIBLE MORPHISMS

Recall the convention set in the introduction: whenever $f: X \to Y$ and $g: Y \to Z$ are two mappings, their composition is denoted by $fg: X \to Z$. The following equivalence was proved in [CLMT11] Prop. 5.1 when $k$ is algebraically closed and under the additional assumption that the valuation of the involved arrows are trivial. This last assumption is dropped here.

**Proposition.** Let $X_1, \ldots, X_{n+1} \in \text{ind} A$. The following conditions are equivalent

(a) there exist irreducible morphisms $X_1 \xrightarrow{h_1} \cdots \xrightarrow{h_n} X_{n+1}$ such that $h_1 \cdots h_n \in \text{rad}^{n+1} \setminus \{0\}$,

(b) there exist irreducible morphisms $f_i: X_i \to X_{i+1}$ and morphisms $\varepsilon_i: X_i \to X_{i+1}$, for every $i$, such that $f_1 \cdots f_n = 0$, such that $\varepsilon_1 \cdots \varepsilon_n \neq 0$ and such that, for every $i$, either $\varepsilon_i \in \text{rad}^{2}$ or else $\varepsilon_i = f_i$.

**Proof.** The implication (b) $\Rightarrow$ (a) was proved in [CT10] Thm. 2.7 (the proof there works for artin algebras and the standard hypothesis made there plays no role for this implication). Assume (a). Let $\Gamma$ be the component of $\Gamma(\text{mod} A)$ containing $X_1, \ldots, X_{n+1}$, let $\pi: \widehat{\Gamma} \to \Gamma$ be the universal covering and $F: k(\widehat{\Gamma}) \to \text{ind} \Gamma$ be a well-behaved functor [2.5]. Let $x_1 \in \pi^{-1}(X_1)$. There is a unique path $\gamma: x_1 \to x_2 \to \cdots \to x_{n+1}$ in $\widehat{\Gamma}$ which image under $\pi$ is $X_1 \to X_2 \to \cdots \to X_{n+1}$. Let $h_i: X_i \to X_{i+1}$ (1 $\leq i \leq n$) be irreducible morphisms such that $h_1 \cdots h_n \in \text{rad}^{n+1} \setminus \{0\}$ and consider $\overline{h_i} \in \text{irr}(X_i, X_{i+1})$ as lying in $k(\widehat{\Gamma})(x_i, x_{i+1})$. Let $h'_i = h_i - F(\overline{h_i})$ for $1 \leq i \leq n$. Then $h'_i \in \text{rad}^{2}$ because $F$ is well-behaved. Therefore $F(\overline{h_1} \cdots \overline{h_n}) \in \text{rad}^{n+1}$. Since $9^{n+1} k(\widehat{\Gamma})(x_i, x_{i+1}) = 0$ (the path $\gamma$ has length $n$), it follows that $\overline{h_1} \cdots \overline{h_n} = 0$ [2.6]. This and $h_1 \cdots h_n \neq 0$ imply that $F(\overline{h_1} \cdots \overline{h_n}) - h_1 \cdots h_n \neq 0$ that is, the sum of the morphisms $F(\overline{h_1}) \cdots F(\overline{h_{i-1}})h'_i \cdots F(\overline{h_{i+1}})h'_i \cdots F(\overline{h_n})$, for $t \in \{1, \ldots, n\}$ and $1 \leq i_1 < \cdots < i_t \leq n$ is non-zero. Hence there exists $t \in \{1, \ldots, n\}$ and $1 \leq i_1 < \cdots < i_t \leq n$ such that the corresponding term in the above sum is non-zero. Define $f_j := F(\overline{h_j})$, and $\varepsilon_j := F(\overline{h_j})$ if $j \notin \{i_1, \ldots, i_t\}$ or $\varepsilon_j := h'_j$ if $j \in \{i_1, \ldots, i_t\}$. Then $\{f_i, \varepsilon_i\}_{i=1, \ldots, n}$ fits the requirements of (b). 

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