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Pinning Model with Heavy Tailed Disorder

Niccolò Torri

ABSTRACT. We study the so-called pinning model, which describes the behavior of a Markov chain interacting with a distinguished state. The interaction depends on an external source of randomness, called disorder, which can attract or repel the Markov chain path, and is tuned by a parameter β . Inspired by [AL11, HM07], we focus on the case when the disorder is heavy-tailed, with exponent $\alpha \in (0, 1)$, while the return times of the Markov chain have a stretched-exponential distribution, with exponent $\gamma \in (0, 1)$. We prove that the set of times at which the Markov chain visits the distinguished state, suitably rescaled, converges in distribution to a limit set, which depends only on the disorder and on the interplay of the parameters α, γ, β . We also show that there exists a random threshold of β below which the limit set is trivial. As a byproduct of our techniques, we improve and complete a result of A. Auffinger and O. Louidor [AL11, Proposition 2.5] on the directed polymer in a random environment with heavy tailed disorder.

1. Set-up and Results

The Pinning Model can be defined as a random perturbation of a random walk or, more generally, of a Markov chain called S . In this model we modify the law of the Markov chain by weighing randomly the probability of a given trajectory up to time N : each time S touches a distinguished state, called 0, before N , say at time n , we give a reward or a penalty to this contact by assigning an exponential weight $\exp(\beta\omega_n - h)$, where $\beta \in \mathbb{R}_+ := (0, \infty)$, $h \in \mathbb{R}$ and $(\omega = (\omega_n)_{n \in \mathbb{N}}, \mathbb{P})$ is an independent random sequence called disorder. The precise definition of the model is given below.

Since we regard the process S only on the set where it takes value 0, it is convenient to work directly on this random set. For this purpose we consider a renewal process $(\tau = (\tau_n)_{n \in \mathbb{N}}, \mathbb{P})$, that is an \mathbb{N}_0 -valued random process such that $\tau_0 = 0$ and $(\tau_j - \tau_{j-1})_{j \in \mathbb{N}}$ is an i.i.d. sequence. This type of random process can be thought of as a random subset of \mathbb{N}_0 , in particular if $S_0 = 0$, then by setting $\tau_0 = 0$ and $\tau_j = \inf\{k > \tau_{j-1} : S_k = 0\}$, for $j > 0$, we recover the set of zeros of the Markov chain S . From this viewpoint the notation $\{n \in \tau\}$ means that there exists $j \in \mathbb{N}$ such that $\tau_j = n$. We refer to [Asm03, Gia07] for more details on the theory of the renewal processes.

Typically in the literature (e.g. [dH07, Gia10, Gia07]) the law of τ_1 , the inter-arrival law of the renewal process, has a polynomial tail and the disorder has finite exponential moments. In our paper we study the case in which the disorder has polynomial tails, in analogy with the articles of A. Auffinger & O. Louidor [AL11] and B. Hambly & J.B. Martin [HM07]. To get interesting results we work with a renewal process where the law of τ_1 is stretched-exponential (cf. Assumptions 1.2). Possible generalization will be discussed in Section 7.

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1.1. The Pinning Model. In this paper we are interested in the behavior of $\tau/N \cap [0, 1] = \{\tau_j/N : \tau_j \leq N\}$, the rescaled renewal process up to time N .

We call P_N the law of $\tau/N \cap [0, 1]$, which turns out to be a probability measure on the space of all subsets of $\{0, 1/N, \dots, 1\}$. On this space for $\beta, h \in \mathbb{R}$ we define the *Pinning Model* $P_{\beta, h, N}^\omega$ as a probability measure defined by the following Radon-Nikodym derivative

$$(1.1) \quad \frac{dP_{\beta, h, N}^\omega}{dP_N}(I) = \frac{1}{Z_{\beta, h, N}^\omega} \exp\left(\sum_{n=1}^{N-1} (\beta\omega_n - h)\mathbb{1}(n/N \in I)\right) \mathbb{1}(1 \in I),$$

where $Z_{\beta, h, N}^\omega$ is a normalization constant, called Partition Function, that makes $P_{\beta, h, N}^\omega$ a probability. Let us stress that the term $\mathbb{1}(1 \in I)$ constrains the last point of $\tau/N \cap [0, 1]$ to be always equal to 1. In such way the Pinning Model is a *random* probability measure on the space X of the all closed subsets of $[0, 1]$ which contain both 0 and 1

$$(1.2) \quad X = \{I \subset [0, 1] : I \text{ is closed and } 0, 1 \in I\}$$

with support given by $X^{(N)}$, the set of the all subset of $\{0, 1/N, \dots, 1\}$ which contain both 0 and 1.

The adjective *random* stresses that $P_{\beta, h, N}^\omega$ is a probability measure indexed by ω , an independent random sequence (called disorder). Therefore in the Pinning Model we have two source of the randomness: the renewal process (τ, P) and the disorder (ω, \mathbb{P}) . To complete the definition we thus need to specify our assumptions on the disorder and on the renewal process.

ASSUMPTION 1.1. *We assume that the disorder ω is an i.i.d. sequence of random variables whose tail is regularly varying with index $\alpha \in (0, 1)$, namely*

$$(1.3) \quad \mathbb{P}(\omega_1 > t) \sim L_0(t)t^{-\alpha}, \quad t \rightarrow \infty,$$

where $\alpha \in (0, 1)$ and $L_0(\cdot)$ is a slowly varying function, cf. [BGT89]. Moreover we assume that the law of ω_1 has no atom and it is supported in $[0, \infty)$, i.e. ω_1 is a positive random variables. The reference example to consider is given by the Pareto Distribution.

ASSUMPTION 1.2. *Given a renewal process, we denote the law of its first point τ_1 by $K(n) := P(\tau_1 = n)$, which characterizes completely the process. Through out this paper we consider a non-terminating renewal process τ , i.e. such that $\sum_{n \in \mathbb{N}} K(n) = 1$, satisfying the following two assumptions*

- (1) *subexponential: $\lim_{n \rightarrow \infty} K(n+k)/K(n) = 1$ for any $k > 0$ and $\lim_{n \rightarrow \infty} K^{*(2)}(n)/K(n) = 2$ (see Appendix A),*
- (2) *stretched-exponential: $\lim_{n \rightarrow \infty} \log K(n)/N^\gamma = -\mathcal{C}$, for a suitable constant $\mathcal{C} > 0$ and $\gamma \in (0, 1)$.*

Roughly speaking, up to local regularity assumptions (subexponentiality), we take $K(n) \cong e^{-\mathcal{C}n^\gamma}$. For instance (cf. Section A) these conditions are satisfied if

$$(1.4) \quad K(n) \sim n^\rho L(n)e^{-\mathcal{C}n^\gamma}, \quad n \rightarrow \infty,$$

with $\rho \in \mathbb{R}$ and $L(\cdot)$ a slowly varying function.

1.2. Main Results. The goal of this paper is to study the behavior of $\tau/N \cap [0, 1]$ under the probability $P_{\beta, h, N}^\omega$ when $N \rightarrow \infty$. To see interesting things in this limit we need to fix $h > 0$ (which is actually equivalent to setting $h = 0$ in (1.1) and to consider a terminating renewal process, cf. Section 4.1) and to rescale β by sending it to 0 with N . If β goes to 0

too slowly (or if it does not go to 0 at all), then $\tau/N \cap [0, 1]$ will always converge to the whole $[0, 1]$, if it goes too fast, it will converge to $\{0, 1\}$. The interesting regime is the following:

$$(1.5) \quad \beta_N \sim \hat{\beta} N^{\gamma - \frac{1}{\alpha}} \ell(N), \quad N \rightarrow \infty,$$

with ℓ a particular slowly varying function depending of L_0 in (1.3). Under this rescaling of β and this choice of $h > 0$ we prove the existence of a random threshold $\hat{\beta}_c$: if $\hat{\beta} < \hat{\beta}_c$ then $\tau/N \cap [0, 1]$ converges to $\{0, 1\}$, while if $\hat{\beta} > \hat{\beta}_c$ then its limit has at least three points.

To prove these facts we proceed by steps. In the first one we show that there exists a random set around which $\tau/N \cap [0, 1]$ is concentrated in the Hausdorff distance: given two non-empty sets $A, B \subset [0, 1]$

$$(1.6) \quad d_H(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right\},$$

where $d(z, C) = \inf_{c \in C} |z - c|$ is the usual distance between a point and a set.

THEOREM 1.3. *Let $(\beta_N)_N$ be as in Equation (1.5). Then for any $N \in \mathbb{N}$, $\hat{\beta} > 0$ there exists a random set $I_{\beta_N, N}$ (i.e. a \mathbf{X} -valued random variable) such that for any $\delta, h > 0$ one has that $\mathbb{P}_{\beta_N, h, N}^\omega(d_H(I, I_{\beta_N, N}) > \delta)$ converges to 0 as $N \rightarrow \infty$ in probability (with respect to the disorder ω). More precisely for any $\epsilon > 0$ there exists $\nu = \nu(\epsilon, \delta)$ and \hat{N} such that for all $N > \hat{N}$*

$$(1.7) \quad \mathbb{P} \left(\mathbb{P}_{\beta_N, h, N}^\omega(d_H(I, I_{\beta_N, N}) > \delta) < e^{-\nu N^\gamma} \right) > 1 - \epsilon.$$

The second step regards the convergence in law of $I_{\beta_N, N}$.

THEOREM 1.4. *Let $(\beta_N)_N$ be as in Equation (1.5). Then for any $\hat{\beta} > 0$ there exists a random closed subset $\hat{I}_{\hat{\beta}, \infty} \in \mathbf{X}$ (i.e. a \mathbf{X} -valued random variable), which depends of a suitable continuum disorder (defined in section 2.1), such that*

$$(1.8) \quad I_{\beta_N, N} \xrightarrow{(d)} \hat{I}_{\hat{\beta}, \infty}$$

on (\mathbf{X}, d_H) .

As a consequence of these Theorems if we look at $\mathbb{P}_{\beta_N, h, N}^\omega$ as a random probability on \mathbf{X} , i.e. as a random variable which takes values in $\mathcal{M}_1(\mathbf{X}, d_H)$, the space of the probability measures on \mathbf{X} , then Theorems 1.3 and 1.4 imply that it converges in law to the δ -measure concentrated on the limit set $\hat{I}_{\hat{\beta}, \infty}$.

THEOREM 1.5. *Let $(\beta_N)_N$ as in Equation (1.5). Then for any $h, \hat{\beta} \in (0, \infty)$,*

$$(1.9) \quad \mathbb{P}_{\beta_N, h, N}^\omega \xrightarrow{(d)} \delta_{\hat{I}_{\hat{\beta}, \infty}}$$

on $\mathcal{M}_1(\mathbf{X}, d_H)$ equipped with the weak topology.

This concludes the results on the convergence of the random set $\tau/N \cap [0, 1]$. It turns out that to say when $\tau/N \cap [0, 1]$ has a trivial limit or not it is sufficient to study when the limit set $\hat{I}_{\hat{\beta}, \infty}$ is given by $\{0, 1\}$ or not. For this purpose we define the random threshold $\hat{\beta}_c$ as

$$(1.10) \quad \hat{\beta}_c = \inf\{\hat{\beta} : \hat{I}_{\hat{\beta}, \infty} \neq \{0, 1\}\}.$$

Denoting by \mathbb{P} the law of the continuum disorder, by a monotonicity argument (cf. Section 5) we have that

- (1) If $\hat{\beta} < \hat{\beta}_c$ then $\hat{I}_{\hat{\beta}, \infty} \equiv \{0, 1\}$, \mathbb{P} -a.s.
- (2) If $\hat{\beta} > \hat{\beta}_c$ then $\hat{I}_{\hat{\beta}, \infty} \neq \{0, 1\}$, \mathbb{P} -a.s.

Moreover the structure of $\hat{\beta}_c$ is described by the following theorem

THEOREM 1.6. *For any choice of $\alpha, \gamma \in (0, 1)$ we have that $\hat{\beta}_c > 0$ \mathbb{P} -a.s., where α is the disorder exponent in Assumption 1.1, while γ is the renewal exponent of Assumption 1.2.*

By using the same technique we complete the result [AL11, Prop. 2.5] about the structure of β_c , the random threshold defined for the directed polymer model in a random environment with heavy tails (we recall the definition in section 6). Precisely

THEOREM 1.7. *Let β_c as in (6.5), then denoting by \mathbb{P}_∞ the law of the continuum environment*

- (1) *For any $\alpha \in (0, \frac{1}{2}), \beta_c > 0$, \mathbb{P}_∞ -a.s.*
- (2) *For any $\alpha \in [\frac{1}{2}, 2), \beta_c = 0$, \mathbb{P}_∞ -a.s.*

Let us stress that in the original paper [AL11] the value of β_c was unknown for $\alpha \in (1/3, 1/2)$.

1.3. Organization of the Paper. In rest of the paper we prove the results of this section. Section 2 contains some preliminary definitions and tools that we use for our proofs. Sections 3 contains the proof of Theorem 1.4 and Section 4 the proof of Theorems 1.3 and 1.5. In Section 5 we prove Theorem 1.6 and then in Section 6 we recall the definition of the Directed Polymer Model, proving Theorem 1.7. Finally in Section 7 we discuss the choice of the parameters α, γ and the future perspectives.

2. Energy & Entropy

In this section we define the random set $I_{\beta, N}$, $\hat{I}_{\hat{\beta}, \infty}$ and we motivate the choice of β_N in (1.5).

To define the random set $I_{\beta, N}$ we compare the *Energy* and the *Entropy* of a given configuration: for a finite set $I = \{x_0 = 0 < x_1 < \dots < x_\ell = 1\}$ we define its *Energy* as

$$(2.1) \quad \sigma_N(I) = \sum_{n=1}^{N-1} \omega_n \mathbb{1}(n/N \in I)$$

and its *Entropy* as

$$(2.2) \quad E(I) = \sum_{k=1}^{\ell} (x_k - x_{k-1})^\gamma.$$

By using these two ingredients we define

$$(2.3) \quad I_{\beta, N} = \operatorname{argmax}_{I \in \mathbf{X}^{(N)}} (\beta \sigma_N(I) - \mathcal{C} N^\gamma E(I)),$$

where γ and \mathcal{C} are the same constants introduced in (2) of Assumption 1.2 and $\mathbf{X}^{(N)}$ is defined as the space of the all possible subsets of $\{0, 1/N, \dots, 1\}$ containing 0 and 1.

By using (2.3) we can find the right rescaling for β : indeed it has to be chosen in such way to make the Energy and the Entropy comparable. To make this, it is convenient to work with a rescaled version of the disorder. For this purpose we consider $(\tilde{M}_i^{(N)})_{i=1}^{N-1}$ the ordered statistics of $(\omega_i)_{i=1}^{N-1}$ — which means that $\tilde{M}_1^{(N)}$ is the biggest value among $\omega_1, \dots, \omega_{N-1}$, $\tilde{M}_2^{(N)}$ is the second biggest one and so on — and $(Y_i^{(N)})_{i=1}^{N-1}$ a random permutation of $\{\frac{1}{N}, \dots, 1 - \frac{1}{N}\}$ independent of the ordered statistics. Therefore the sequence $((\tilde{M}_i^{(N)}, Y_i^{(N)})_{i=1}^{N-1}$ recovers the disorder $(\omega_i)_{i=1}^{N-1}$. The interesting point is that the asymptotic behavior of this sequence is known. Let us recall the main results that we need to find the right rescaling for β .

2.1. The Disorder. Let us start to note that for any fixed k as $N \rightarrow \infty$

$$(2.4) \quad (Y_i^{(N)})_{i=1, \dots, k} \xrightarrow{(d)} (Y_i^{(\infty)})_{n=1, \dots, k},$$

where $(Y_i^{(\infty)})_{i \in \mathbb{N}}$ is an i.i.d. sequence of $\text{Uniform}([0, 1])$.

Moreover, coming to the ordered statistics, from classical extreme value theory (see e.g. [Res87, Section 1.1]) we have that there exists a sequence $(b_N)_N$ such that for any fixed $k > 0$ as $N \rightarrow \infty$

$$(2.5) \quad (M_i^{(N)} := b_N^{-1} \tilde{M}_i^{(N)})_{i=1, \dots, k} \xrightarrow{(d)} (M_i^{(\infty)})_{n=1, \dots, k},$$

where $M_i^{(\infty)} = T_i^{-1/\alpha}$, with T_i be a sum of i independent exponentials of mean 1 and α is the exponent of the disorder introduced in Equation (1.3).

In particular b_N can be written as $b_N \sim N^{\frac{1}{\alpha}} \ell_0(N)$, where $\ell_0(\cdot)$ is a suitable slowly varying function depending of $L_0(\cdot)$ (cf. Equation (1.3)).

We can get a stronger result without much effort, which will be very useful in the sequel. Let us take the sequences $(M_i^{(N)})_{i=1}^{N-1}$ and $(Y_i^{(N)})_{i=1}^{N-1}$ independently and consider

$$(2.6) \quad w_i^{(N)} := \begin{cases} (M_i^{(N)}, Y_i^{(N)})_{i=1}^{N-1}, & i < N, \\ 0, & i \geq N, \end{cases}$$

$$(2.7) \quad w_i^{(\infty)} := (M_i^{(\infty)}, Y_i^{(\infty)})_{i \in \mathbb{N}},$$

We can look at $w^{(N)} = (w_i^{(N)})_{i \in \mathbb{N}}$ and $w^{(\infty)} = (w_i^{(\infty)})_{i \in \mathbb{N}}$ as random variables taking values in $\mathcal{S} := (\mathbb{R}^2)^{\mathbb{N}}$. Let us equip \mathcal{S} with the product topology: a sequence $x^{(N)}$ converges to $x^{(\infty)}$ if and only if for any fixed $i \in \mathbb{N}$ one has $\lim_{N \rightarrow \infty} x_i^{(N)} = x_i^{(\infty)}$. In such way \mathcal{S} is a completely metrizable space and a \mathcal{S} -valued random sequence $(w^{(N)})_N$ converges in law to $w^{(\infty)}$ if and only if for any fixed k , the truncated sequence $(w_1^{(N)}, \dots, w_k^{(N)}, 0, \dots)$ converges in law to $(w_1^{(\infty)}, \dots, w_k^{(\infty)}, 0, \dots)$. Therefore Equations (2.4) and (2.5) imply that

$$(2.8) \quad w^{(N)} \xrightarrow{(d)} w^{(\infty)}$$

in \mathcal{S} . Henceforth we refer to $w^{(N)}$ as the Discrete Disorder of size N , and to $w^{(\infty)}$ as the Continuum Disorder.

2.2. The Energy. Recalling (2.1) we define the rescaled discrete Energy function $\hat{\sigma}_N : X \rightarrow \mathbb{R}_+$ as

$$(2.9) \quad \hat{\sigma}_N(\cdot) = \frac{\sigma_N(I)}{b_N} = \sum_{i=1}^{N-1} M_i^{(N)} \mathbb{1}(Y_i^{(N)} \in \cdot),$$

and Equation (2.3) becomes

$$(2.10) \quad I_{\frac{N\gamma}{b_N} \beta, N} = \arg \max_{I \in X^{(N)}} (\beta \hat{\sigma}_N(\cdot) - \mathcal{E}E(I)),$$

Therefore we choose β_N such that

$$(2.11) \quad \hat{\beta}_N := \frac{b_N}{N\gamma} \beta_N$$

converges to $\hat{\beta} \in (0, \infty)$. This is equivalent to the relation (1.5). Since in the sequel we will study the set $I_{\frac{N\gamma}{b_N} \beta, N}$, it is convenient introduce the notation

$$(2.12) \quad \hat{I}_{\beta, N} = I_{\frac{N\gamma}{b_N} \beta, N}.$$

In particular $\hat{I}_{\hat{\beta}_N, N} = I_{\beta_N, N}$.

REMARK 2.1. Let us stress that the value of \mathcal{C} is quite inessential and it can be included in the parameter $\hat{\beta}$ by a simple rescaling. Therefore from now on we assume $\mathcal{C} = 1$.

It is essential for the sequel to extend the definition of $\hat{I}_{\beta,N}$ to the whole space X equipped with the Hausdorff metric. This generalization leads us to define the same kind of random set introduced in (2.10) in which we use suitable continuum Energy and Entropy.

We define the continuum Energy Function $\hat{\sigma}_\infty : X \rightarrow \mathbb{R}_+$ as

$$(2.13) \quad \hat{\sigma}_\infty(\cdot) = \sum_{i=1}^{\infty} M_i^{(\infty)} \mathbb{1}(Y_i^{(\infty)} \in \cdot),$$

where $(M_i^{(\infty)})_{i \in \mathbb{N}}$ and $(Y_i^{(\infty)})_{i \in \mathbb{N}}$ are the two independent random sequences introduced in (2.4) and (2.5). Since $\alpha \in (0, 1)$ we have that $\hat{\sigma}_\infty(I) < \infty$ for all $I \in X$, because the series $\sum_i M_i^{(\infty)}$ converges a.s.

Before extending the definition of the Entropy to the whole space X , let us conclude this section on the Energy function by proving that $\hat{\sigma}_N$, with $N \in \mathbb{N} \cup \{\infty\}$ is an upper semi-continuous function. The proof of this fact for N finite is simple, while the continuum case requires an argument of approximation. Thus it is useful to define for $k, N \in \mathbb{N} \cup \{\infty\}$ the k -truncated Energy function as

$$(2.14) \quad \hat{\sigma}_N^{(k)}(\cdot) = \sum_{i=1}^{(N-1) \wedge k} M_i^{(N)} \mathbb{1}(Y_i^{(N)} \in \cdot).$$

Let us stress that the support of $\hat{\sigma}_N^{(k)}$ is the space of the all possible subsets of $Y^{(N,k)}$, the set of the positions of the first k -maxima

$$(2.15) \quad Y^{(N,k)} = \{Y_i^{(N)}, i = 1, 2, 3, \dots, (N-1) \wedge k\} \cup \{0, 1\}.$$

Whenever $k \geq N$ we write simply $Y^{(N)}$.

THEOREM 2.2. *For any fixed $k, N \in \mathbb{N} \cup \{\infty\}$ and for a.e. realization of the disorder $w^{(N)}$, the function $\hat{\sigma}_N^{(k)} : X \rightarrow \mathbb{R}_+$ is upper semi-continuous (u.s.c.).*

REMARK 2.3. For sake of clarity let us underline that in the Hausdorff metric (cf. (1.6)) $d_H(A, B) < \epsilon$ if and only if for any $x_1 \in A$ there exists $x_2 \in B$ such that $|x_1 - x_2| < \epsilon$ and vice-versa switching the role of A and B .

PROOF. Let us start to consider the case $N \wedge k < \infty$. For a given $I_0 \in X$, let ι be the set of all points of $Y^{(N,k)}$ which are not in I_0 . Since $Y^{(N,k)}$ has a finite number of points there exists $\eta > 0$ such that $d(z, I_0) > \eta$ for any $z \in \iota$. Then if $I \in X$ is sufficiently close to I_0 , namely $d_H(I, I_0) \leq \eta/2$, then $d(z, I) > \eta/2 > 0$ for any $z \in \iota$. Therefore, among the first k -maxima, I can at most hit only the points hit by I_0 , namely $\hat{\sigma}_N^{(k)}(I) \leq \hat{\sigma}_N^{(k)}(I_0)$ and this concludes the proof of this first part.

For the case $N \wedge k = \infty$ it is enough to observe that the difference between the truncated Energy and the original one

$$(2.16) \quad \sup_{I \in X} \left| \hat{\sigma}_\infty(I) - \hat{\sigma}_\infty^{(k)}(I) \right| = \sup_{I \in X} \left| \sum_{i=1}^{\infty} M_i^{(\infty)} \mathbb{1}(Y_i^{(\infty)} \in I) - \sum_{i=1}^k M_i^{(\infty)} \mathbb{1}(Y_i^{(\infty)} \in I) \right| \leq \sum_{i>k} M_i^{(\infty)},$$

converges to 0 as $k \rightarrow \infty$. Therefore the sequence of u.s.c. functions $\hat{\sigma}_\infty^{(k)}$ converges uniformly to $\hat{\sigma}_\infty$ and this implies the u.s.c. of the limit. \square

2.3. The Entropy.

Let us define

$$(2.17) \quad \mathbf{X}^{(\text{fin})} = \{I \in \mathbf{X} : |I| < \infty\}$$

and stress that it is a countable dense subset of \mathbf{X} with respect to the Hausdorff Metric.

For a given set $I = \{x_0 < x_1 < \dots < x_\ell\} \in \mathbf{X}^{(\text{fin})}$ we define the *Entropy* as

$$(2.18) \quad E(I) = \sum_{k=1}^{\ell} (x_k - x_{k-1})^\gamma.$$

THEOREM 2.4. *The following hold*

- (1) *The Entropy $E(\cdot)$ is strictly increasing with respect to the inclusion of finite sets, namely if $I_1, I_2 \in \mathbf{X}^{(\text{fin})}$ and $I_1 \subsetneq I_2$, then $E(I_2) > E(I_1)$,*
- (2) *The function $E : \mathbf{X}^{(\text{fin})} \rightarrow \mathbb{R}_+$ is lower semi continuous (l.s.c.).*

PROOF. To prove (1) let us note that if $I_2 = \{0, a_1, x, a_2, 1\}$ and $I_1 = \{0, a_1, a_2, 1\}$, with $0 \leq a_1 < x < a_2 \leq 1$ then $E(I_2) - E(I_1) = (x - a_1)^\gamma + (a_2 - x)^\gamma - (a_2 - a_1)^\gamma > 0$ because $\gamma < 1$, thus $a^\gamma + b^\gamma > (a + b)^\gamma$ for any $a, b > 0$. The claim for the general case follows by a simple induction argument.

To prove (2) we fix $I_0 \in \mathbf{X}^{(\text{fin})}$ and we show that if $(I_n)_n$ is a sequence of finite set converging (in the Hausdorff metric) to I_0 , then it must be $\liminf_{n \rightarrow \infty} E(I_n) \geq E(I_0)$ and by the arbitrariness of the sequence the proof will follow.

Let $I_0 \in \mathbf{X}^{(\text{fin})}$ be fixed and let us observe that if we fix $\epsilon > 0$ small, namely smaller than the half of the minimum of the distance between the points of I_0 , then by Remark 2.3 any set I for which $d_H(I, I_0) < \epsilon$ must have at least the same number of points of I_0 , i.e. $|I| \geq |I_0|$. In such way if (I_n) is a sequence of finite sets converging to I_0 , then for any n large enough we can pick out a subset I'_n of I_n with the same number of points of I_0 such that $(I'_n)_n$ still converges to I_0 . Necessary the points of I'_n converge to the ones of I_0 , so that $\lim_{n \rightarrow \infty} E(I'_n) = E(I_0)$. By using Part (1) we have that for any n , $E(I_n) \geq E(I'_n)$, so that $\liminf_{n \rightarrow \infty} E(I_n) \geq E(I_0)$ and the proof follows. \square

Now we are ready to define the Entropy of a generic set $I \in \mathbf{X}$. The goal is to obtain an object for which the properties of the Entropy E on $\mathbf{X}^{(\text{fin})}$ still hold. This extension is not trivial because E is strictly l.s.c., namely given $I \in \mathbf{X}^{(\text{fin})}$ it is always possible to find two sequences $(I_N^{(1)})_N, (I_N^{(2)})_N \in \mathbf{X}^{(\text{fin})}$ converging to I such that $\lim_{N \rightarrow \infty} E(I_N^{(1)}) = E(I)$ and $\lim_{N \rightarrow \infty} E(I_N^{(2)}) = \infty$. For instance let us consider the simplest case, when $I = \{0, 1\}$. Then we may consider $I_N^{(1)} \equiv I$ for any N , so that $E(I_N^{(1)}) \equiv E(\{0, 1\})$, and $I_N^{(2)}$ the set made by $2N$ points such that the first N are equispaced in a neighborhood of 0 of radius $N^{-\epsilon}$ and the others N in a neighborhood of 1 always of radius $N^{-\epsilon}$, with $\epsilon = \epsilon(\gamma)$ small, then $I_N^{(2)} \rightarrow I$ as $N \rightarrow \infty$ and $E(I_N^{(2)}) = 2N \cdot 1/N^{\gamma(1+\epsilon)} + (1 - 2/N^\epsilon)^\gamma = O(N^{1-\gamma(1+\epsilon)}) \rightarrow \infty$ as $N \rightarrow \infty$ if $\epsilon < (1 - \gamma)/\gamma$.

In order to avoid this problem for $I \in \mathbf{X}$ we define

$$(2.19) \quad \bar{E}(I) = \liminf_{J \rightarrow I, J \in \mathbf{X}^{(\text{fin})}} E(J).$$

Let us stress that \bar{E} is nothing but the l.s.c. extension of E to the whole space \mathbf{X} , see e.g. [Bou71, Prop. 5 TG IV.31].

THEOREM 2.5. *The following hold:*

- (1) *The function $\bar{E}(\cdot)$ is increasing with respect to the inclusion of sets, namely if $I_1, I_2 \in \mathbf{X}$ with $I_1 \subset I_2$ then $\bar{E}(I_2) \geq \bar{E}(I_1)$.*
- (2) *The function $\bar{E} : \mathbf{X} \rightarrow \mathbb{R}_+$ is l.s.c. and $\bar{E}|_{\mathbf{X}^{(\text{fin})}} \equiv E$.*

REMARK 2.6. To make more clear the definition we recall that

$$(2.20) \quad \bar{E}(I) = \liminf_{J \rightarrow I, J \in X^{(\text{fin})}} E(J) := \lim_{\delta \rightarrow 0} \left[\inf \left\{ E(J) : J \in B_H(I, \delta) \cap X^{(\text{fin})} \setminus \{I\} \right\} \right],$$

where $B_H(I, \delta)$ denotes the disc of radius δ centered on I in the Hausdorff Metric. If $\bar{E}(I) \in \mathbb{R}$ the definition is equivalent to say

- (a) For any $\epsilon > 0$ and for any $\delta > 0$ there exists $J \in B_H(\delta, I) \cap X^{(\text{fin})} \setminus \{I\}$ such that $\bar{E}(I) + \epsilon > E(J)$.
- (b) For any $\epsilon > 0$ there exists $\delta_0 > 0$ such that for any $J \in B_H(\delta_0, I) \cap X^{(\text{fin})} \setminus \{I\}$, $E(J) > \bar{E}(I) - \epsilon$.

PROOF. We have only to prove (1). Let $I, J \in X$ such that $J \subset I$. If $\bar{E}(I) = \infty$ there is nothing to prove, therefore we can assume that $\bar{E}(I) \in \mathbb{R}$. Then, by property (a), for any $\epsilon, \delta > 0$ there exists $I' \in X^{(\text{fin})}$ such that $\bar{E}(I) + \epsilon \geq E(I')$ and $d_H(I, I') < \delta$. Directly by the definition of the Hausdorff metric, the family of discs of radius δ indexed by $I' - (B(x, \delta))_{x \in I'}$ — covers I and thus also J , therefore if $J' \subset I'$ is the minimal cover of J coming from I' , i.e. $J' := \min\{L \subset I' : J \subset \cup_{x \in L} B(x, \delta)\}$, then it must be $d_H(J, J') < \delta$. Moreover by Theorem 2.4 we have $E(I') \geq E(J')$ and thus $\bar{E}(I) + \epsilon \geq E(J')$. Let now $\delta_0 = \delta_0(\epsilon) > 0$ as prescript in (b), then as soon as $\delta < \delta_0$ it must hold that $E(J') \geq \bar{E}(I) - \epsilon$ and this conclude the proof. \square

From now on in order to simplify the notation we use E instead of \bar{E} to indicate the function E defined on all X .

COROLLARY 2.7. *Let $I \in X$ and let us suppose that there exists $x \notin I$ and $E(I) < \infty$, then $E(I \cup \{x\}) > E(I)$. It follows that the function E is strictly increasing whenever it is finite: if $I \subsetneq J$ and $E(I) < \infty$, then $E(I) < E(J)$.*

PROOF. Let us recall that $x \notin I$ means that there exists $\delta > 0$ such that $I \cap (x - \delta, x + \delta) = \emptyset$ because I is closed. Let $a, b \in I$ the two points of I closest to x respectively at its left and at its right, then the proof will follow by proving that

$$(2.21) \quad E(I \cup \{x\}) - E(I) \geq (x - a)^\gamma + (b - x)^\gamma - (b - a)^\gamma,$$

because the r.h.s. is a quantity strictly bigger than 0. For this purpose we start by observing that for any ϵ small enough if A is a set of a ϵ -neighborhood of $I \cup \{x\}$, then it can be written as union of two disjoint sets D, C where D is in a ϵ -neighborhood of I and C in a ϵ -neighborhood of $\{x\}$. namely $B_H(I \cup \{x\}, \epsilon) \cap X^{(\text{fin})} = \{A \in X^{(\text{fin})} : A = D \cup C, D \in B_H(I, \epsilon) \text{ and } C \in B_H(\{x\}, \epsilon)\}$; moreover we can partition any such D in two disjoint sets $D = D' \cup D''$ where $D' = D \cap [0, x)$ and $D'' = (x, 1]$.

For a fixed set S , let l_S be its smallest point bigger than 0 and r_S its biggest point smaller than 1. By using this notation it follows from the definition of the Entropy for a finite set (2.18) that for any such $A \in B_H(I \cup \{x\}, \epsilon) \cap X^{(\text{fin})}$ we can write

$$(2.22) \quad E(A) = E(D \cup C) = E(D) - (l_{D''} - r_{D'})^\gamma + E(C \cup 0, 1) - l_C^\gamma - (1 - r_C)^\gamma + (l_C - r_{D'})^\gamma + (l_{D''} - r_C)^\gamma.$$

By Theorem 2.4 we have that $E(C \cup 0, 1) \geq l_C^\gamma + (1 - r_C)^\gamma + (r_C - l_C)^\gamma$, from which follows

$$(2.23) \quad E(A) = E(D \cup C) \geq E(D) - (l_{D''} - r_{D'})^\gamma + (l_C - r_{D'})^\gamma + (l_{D''} - r_C)^\gamma + (r_C - l_C)^\gamma \geq E(D) + e(\epsilon),$$

where $e(\epsilon) = \inf\{(l_C - r_{D'})^\gamma + (l_{D''} - r_C)^\gamma + (r_C - l_C)^\gamma - (l_{D''} - r_{D'})^\gamma\}$. This inf is taken among the all possible $D = D' \cup D'' \in B_H(I, \epsilon)$ and $C \in B_H(\{x\}, \epsilon)$. Now the r.h.s. of (2.23) is independent of C , so that $\inf E(A) \geq \inf E(D) + e(\epsilon)$, where the inf is taken among the all possible $A = D \cup C \in B_H(I \cup \{x\}, \epsilon) \cap X^{(\text{fin})} \setminus \{I \cup \{x\}\}$. We conclude the proof by taking the limit for $\epsilon \rightarrow 0$, because $e(\epsilon) \rightarrow (x - a)^\gamma + (b - x)^\gamma - (b - a)^\gamma$ as $\epsilon \rightarrow 0$. \square

PROPOSITION 2.8. *For any $0 \leq a < b \leq 1$ we have that $E([a, b]) = \infty$.*

PROOF. Let us consider the case where $a = 0, b = 1$, the other cases follow in a similar way. By Theorem 2.4 we have that $E([0, 1]) \geq E(\{0, 1/N, \dots, 1\}) = N^{1-\gamma} \uparrow \infty$ as $N \uparrow \infty$ because $\gamma < 1$. \square

2.4. The Energy-Entropy.

DEFINITION 2.9. For any $N, k \in \mathbb{N} \cup \{\infty\}$ and $\beta \in (0, \infty)$ we define, recalling (2.14) and (2.19),

$$(2.24) \quad U_{\beta, N}^{(k)}(I) = \beta \hat{\sigma}_N^{(k)}(I) - E(I).$$

Note that $U_{\beta, N}^{(k)}$ is u.s.c. on (X, d_H) , a compact metric space, then its maximizer

$$(2.25) \quad \hat{u}_{\beta, N}^{(k)} = \max_{I \in X} U_{\beta, N}^{(k)}(I).$$

is well defined.

Whenever $k \geq N$ we will omit the superscript (k) from the notation.

THEOREM 2.10. *For any $N, k \in \mathbb{N} \cup \{\infty\}$, $\beta > 0$ and for a.e. realization of the disorder $w^{(N)}$, the maximum $\hat{u}_{\beta, N}^{(k)}$ is achieved in only one set, i.e. the solution at*

$$(2.26) \quad \hat{I}_{\beta, N}^{(k)} = \operatorname{argmax}_{I \in X} U_{\beta, N}^{(k)}(I)$$

is given by only one set. Moreover for any $N \in \mathbb{N}$ we have that $\hat{I}_{\beta, N}^{(k)} \in X^{(N)}$.

PROOF. We claim that if I is a solution of Equation (2.26), then by using Corollary 2.7

$$(2.27) \quad I \subset Y^{(N, k)}, \quad \text{if } N \wedge k < \infty,$$

$$(2.28) \quad I = \overline{I \cap Y^{(\infty)}} \quad \text{if } N \wedge k = \infty.$$

Indeed if $N \wedge k < \infty$ and (2.27) fails, then there exists $x \in I$ such that $x \notin Y^{(N, k)}$ and this implies $\hat{\sigma}_N^{(k)}(I) = \hat{\sigma}_N^{(k)}(I - \{x\})$, but $E(I - \{x\}) < E(I)$ by Corollary 2.7. Therefore $U_{\beta, N}^{(k)}(I - \{x\}) > U_{\beta, N}^{(k)}(I) = \hat{u}_{\beta, N}^{(k)}$, a contradiction. The case $N \wedge k = \infty$ follows in an analogous way always by using Corollary 2.7 because the set in the r.h.s. of (2.28), which is a subset of I , has the same Energy as I but smaller Entropy. Now we are able to conclude the uniqueness (following the same ideas used in [HM07, Proposition 4.1] or [AL11, Lemma 4.1]): let I^1, I^2 be two subsets achieving the maximum. By using (2.27) and (2.28) if $I^1 \neq I^2$, then there would exist $Y_j^{(N)}$ such that $Y_j^{(N)} \in I_1$ and $Y_j^{(N)} \notin I_2$, so that

$$(2.29) \quad \max_{I: Y_j^{(N)} \in I} U_{\beta, N}^{(k)}(I) = \max_{I: Y_j^{(N)} \notin I} U_{\beta, N}^{(k)}(I)$$

and this leads to

$$(2.30) \quad \beta M_j^{(N)} = \beta \hat{\sigma}_N^{(k)}(Y_j^{(N)}) = \max_{I: Y_j^{(N)} \notin I} U_{\beta, N}^{(k)}(I) - \max_{I: Y_j^{(N)} \in I} \left\{ \beta \sum_{k \neq j: Y_k^{(N)} \in I} M_k^{(N)} - E(I) \right\}.$$

Let us stress that the r.h.s. is independent of $M_j^{(N)}$, which is in the l.h.s. Then, by conditioning on the values of $(M_i^{(N)})_{i \in \mathbb{N}, i \neq j}$ and $(Y_i^{(N)})_{i \in \mathbb{N}}$ we have that the l.h.s. has a continuous distribution, while the r.h.s. is a constant, so that the event in which the r.h.s. is equal to the l.h.s. has zero probability. By countable sub-additivity of the probability we have that a.s. $I_1 = I_2$. \square

3. Convergence

The aim of this section is to discuss the convergence of $\hat{I}_{\hat{\beta}_N, N}$ and $\hat{u}_{\hat{\beta}_N, N}$ under the assumption that $\lim_{N \rightarrow \infty} \hat{\beta}_N = \hat{\beta} \in (0, \infty)$. See respectively (2.26), (2.25) and (2.11).

For technical convenience let us start to build a coupling between the discrete disorder and the continuum one. We recall that by Equation (2.8) $w^{(N)}$ converges in distribution to $w^{(\infty)}$ on \mathcal{S} , a completely metrizable space. Therefore by using Skorokhod's representation theorem (see [Bil99, Theorem 6.7]) we can define $w^{(N)}$ and $w^{(\infty)}$ on a common probability space in order to assume that their convergence holds almost surely.

LEMMA 3.1. *There is a coupling, which with a slight abuse of notation we still call \mathbb{P} , of the continuum model and the discrete one, under which*

$$(3.1) \quad w^{(N)} = (M_i^{(N)}, Y_i^{(N)})_{i \in \mathbb{N}} \xrightarrow[\mathbb{P}\text{-a.s.}]{\mathcal{S}} w^{(\infty)} = (M_i^{(\infty)}, Y_i^{(\infty)})_{i \in \mathbb{N}}, \text{ as } N \rightarrow \infty.$$

In particular for any fixed $\epsilon, \delta > 0$ and $k \in \mathbb{N}$ there exists $\hat{N} < \infty$ such that for all $N > \hat{N}$

$$(3.2) \quad \mathbb{P} \left(\sum_{j=1}^{(N-1) \wedge k} |M_j^{(N)} - M_j| < \epsilon \right) > 1 - \delta,$$

$$(3.3) \quad \mathbb{P} \left(\sum_{j=1}^{(N-1) \wedge k} |Y_j^{(N)} - Y_j| < \epsilon \right) > 1 - \delta,$$

3.1. Convergence Results. Let us rewrite an equivalent, but more handy definition of $\hat{I}_{\hat{\beta}_N, N}^{(k)}$ and $\hat{u}_{\hat{\beta}_N, N}^{(k)}$: for a given $k \in \mathbb{N}$ let

$$(3.4) \quad \mathcal{C}_k = \{A : A \subset \{1, \dots, k\}\}$$

and for any $k \in \mathbb{N}$, $N \in \mathbb{N} \cup \{\infty\}$ and $A \subset \{1, \dots, k\}$ let $Y_A^{(N)} = \{Y_i^{(N)}\}_{i \in A} \cup \{0, 1\}$, which is well defined also for $A = \emptyset$. Therefore by Theorem 2.10 we can write

$$(3.5) \quad \hat{u}_{\hat{\beta}_N, N}^{(k)} = \max_{A \in \mathcal{C}_k} \left[\hat{\beta} \sum_{i \in A} M_i^{(N)} - E(Y_A) \right],$$

$$(3.6) \quad \hat{I}_{\hat{\beta}_N, N}^{(k)} = Y_{A_{\hat{\beta}_N, N}^{(k)}}^{(N)},$$

for a suitable random index set $A_{\hat{\beta}_N, N}^{(k)}$ (which can be empty or not). We have our first convergence result.

PROPOSITION 3.2. *Assume that $\hat{\beta}_N \rightarrow \hat{\beta}$ as $N \rightarrow \infty$. Then for any fixed $\delta > 0$ and $k \in \mathbb{N}$ there exists N_k such that for any $N > N_k$*

$$(3.7) \quad \mathbb{P} \left(A_{\hat{\beta}_N, N}^{(k)} = A_{\hat{\beta}, \infty}^{(k)} \right) > 1 - \delta.$$

PROOF. To prove the claim by using the sub-additivity of the probability, it is enough to prove that for any $r \in \{1, \dots, k\}$

$$(3.8) \quad \mathbb{P} \left(r \notin A_{\hat{\beta}_N, N}^{(k)}, r \in A_{\hat{\beta}, \infty}^{(k)} \right) \rightarrow 0, \quad N \rightarrow \infty,$$

$$(3.9) \quad \mathbb{P} \left(r \in A_{\hat{\beta}_N, N}^{(k)}, r \notin A_{\hat{\beta}, \infty}^{(k)} \right) \rightarrow 0, \quad N \rightarrow \infty.$$

We detail the first one, the second one follows in an analogous way. On the event $\{r \notin A_{\hat{\beta}_N, N}^{(k)}, r \in A_{\hat{\beta}, \infty}^{(k)}\}$ we consider

$$(3.10) \quad \hat{u}_{(r)} := \max_{A \in \mathcal{C}_k, r \notin A} \left[\hat{\beta} \sum_{i \in A} M_i - E(Y_A) \right] < \hat{u}_{\hat{\beta}, \infty}^{(k)}$$

because $r \in A_{\hat{\beta}, \infty}^{(k)}$ by our assumption and by uniqueness of the set that achieves the maximum. Now

$$(3.11) \quad \max_{A \in \mathcal{C}_k, r \notin A} \left[\hat{\beta}_N \sum_{i \in A} M_i^{(N)} - E(Y_A) \right] \\ \leq \max_{A \in \mathcal{C}_k, r \notin A} \left[\hat{\beta} \sum_{i \in A} M_i - E(Y_A) \right] + |\hat{\beta}_N - \hat{\beta}| \sum_{i=1}^k M_i^{(N)} + \hat{\beta} \sum_{i=1}^k |M_i^{(N)} - M_i| \\ = \hat{u}_{(r)} + |\hat{\beta}_N - \hat{\beta}| \sum_{i=1}^k M_i^{(N)} + \hat{\beta} \sum_{i=1}^k |M_i^{(N)} - M_i|$$

and in the same way, always on the event $\{r \notin A_{\hat{\beta}_N, N}^{(k)}, r \in A_{\hat{\beta}, \infty}^{(k)}\}$,

$$(3.12) \quad \max_{A \in \mathcal{C}_k, r \in A} \left[\hat{\beta}_N \sum_{i \in A} M_i^{(N)} - E(Y_A) \right] \geq \hat{u}_{\hat{\beta}, \infty}^{(k)} - |\hat{\beta}_N - \hat{\beta}| \sum_{i=1}^k M_i^{(N)} - \hat{\beta} \sum_{i=1}^k |M_i^{(N)} - M_i|.$$

Therefore by using the assumption that $r \notin A_{\hat{\beta}_N, N}^{(k)}$ we have that the l.h.s. of (3.11) is smaller than that of (3.12), thus $\hat{u}_{\hat{\beta}, \infty}^{(k)} - \hat{u}_{(r)} \leq 2\hat{\beta} \sum_{i=1}^k |M_i^{(N)} - M_i| + 2|\hat{\beta}_N - \hat{\beta}| \sum_{i=1}^k M_i^{(N)}$. Summarizing a simple inclusion of events gives

$$(3.13) \quad \mathbb{P} \left(r \notin A_{\hat{\beta}_N, N}^{(k)}, r \in A_{\hat{\beta}, \infty}^{(k)} \right) \leq \mathbb{P} \left(0 < \hat{u}_{\hat{\beta}, \infty}^{(k)} - \hat{u}_{(r)} \leq 2\hat{\beta} \sum_{i=1}^k |M_i^{(N)} - M_i| + 2|\hat{\beta}_N - \hat{\beta}| \sum_{i=1}^k M_i^{(N)} \right)$$

and the proof follows by observing that the r.h.s. converges to 0 as $N \rightarrow \infty$ by Lemma 3.1. \square

The following Proposition contains mainly the convergence results on the truncated quantities $\hat{u}_{\hat{\beta}_N, N}^{(k)}$ and $\hat{I}_{\hat{\beta}_N, N}^{(k)}$. To make such computation we need to compare the maximum on the all space X with the one achieved out of a neighborhood of radius δ of $\hat{I}_{\hat{\beta}, \infty}^{(k)}$.

DEFINITION 3.3. For any $\delta > 0$, $\beta \in (0, \infty)$ we define

$$(3.14) \quad \hat{u}_{\beta, N}^{(k)}(\delta) = \max_{I \in X: d_H(I, \hat{I}_{\beta, \infty}^{(k)}) \geq \delta} U_{\beta, N}^{(k)}(I).$$

PROPOSITION 3.4. Assume that $\hat{\beta}_N \rightarrow \hat{\beta}$ as $N \rightarrow \infty$. The following hold

- (1) For every fixed $\delta > 0$, $\beta \in (0, \infty)$ $\mathbb{P} \left(\liminf_{k \rightarrow \infty} (\hat{u}_{\beta, \infty}^{(k)} - \hat{u}_{\beta, \infty}^{(k)}(\delta)) > 0 \right) = 1$.
- (2) For any $\epsilon, \delta > 0$ and for any fixed k there exists N_k such that $\mathbb{P} \left(|\hat{u}_{\hat{\beta}_N, N}^{(k)} - \hat{u}_{\hat{\beta}, \infty}^{(k)}| < \epsilon \right) > 1 - \delta$, for any $N > N_k$.
- (3) For any $\epsilon, \delta > 0$ and for any fixed k there exists N_k such that $\mathbb{P} \left(d_H(\hat{I}_{\hat{\beta}_N, N}^{(k)}, \hat{I}_{\hat{\beta}, \infty}^{(k)}) < \epsilon \right) > 1 - \delta$, for any $N > N_k$.
- (4) For any $\epsilon, \delta > 0$, there exist $\eta, K > 0$, $(N_k)_{k > K}$ such that $\mathbb{P} \left(\hat{u}_{\hat{\beta}_N, N}^{(k)}(\epsilon) < \hat{u}_{\hat{\beta}_N, N}^{(k)} - \eta \right) > 1 - \delta$, for any $k > K$ and $N > N_k$.

PROOF. Part (1). By observing that $U_{\beta, N}^{(k)}$ is an u.s.c. function on a compact set, we use the same outline of [AL11, Part (3,4) of Proof of lemma 4.1]. Indeed by contradiction if there exists $\delta > 0$ such that $\liminf_{k \rightarrow \infty} (\hat{u}_{\beta, \infty}^{(k)} - \hat{u}_{\beta, \infty}^{(k)}(\delta)) = 0$, then we may find a sequence I_{k_j} such that $\limsup_{j \rightarrow \infty} U_{\beta, \infty}^{(k_j)}(I_{k_j}) \geq \liminf_{j \rightarrow \infty} U_{\beta, \infty}^{(k_j)}(\hat{I}_{\beta, \infty}^{(k_j)})$ and $d_H(\hat{I}_{\beta, \infty}^{(k_j)}, I_{k_j}) > \delta$. By compactness of the space X we can suppose that there exists $I_0 \in X$ such that $\lim_{j \rightarrow \infty} I_{k_j} = I_0$, therefore by

using the u.s.c. property of $U_{\beta,N}^{(k)}$ we get, recalling that for any fixed $k \in \mathbb{N}$, $U_{\beta,\infty}(I) \geq U_{\beta,\infty}^{(k)}(I)$ and $U_{\beta,\infty}^{(k)}(I) \uparrow U_{\beta,\infty}(I)$ as $k \uparrow \infty$,

$$(3.15) \quad U_{\beta,\infty}(I_0) \geq \limsup_{j \rightarrow \infty} U_{\beta,\infty}(I_{k_j}) \geq \limsup_{j \rightarrow \infty} U_{\beta,\infty}^{(k_j)}(I_{k_j}) \geq \\ \geq \liminf_{j \rightarrow \infty} U_{\beta,\infty}^{(k_j)}(\hat{I}_{\beta,\infty}^{(k_j)}) \geq \liminf_{j \rightarrow \infty} U_{\beta,\infty}^{(k_j)}(\hat{I}_{\beta,\infty}) = U_{\beta,\infty}(\hat{I}_{\beta,\infty}) = \hat{u}_{\beta,\infty},$$

that is $U_{\beta,\infty}(I_0) = \hat{u}_{\beta,\infty}$ and by uniqueness of the maximizer we obtain that $I_0 = \hat{I}_{\beta,\infty}$. Thus if we show that $\lim_{k \rightarrow \infty} \hat{I}_{\beta,\infty}^{(k)} = \hat{I}_{\beta,\infty}$, then we obtain the desired contradiction, because the two sequences $(I_{k_j})_j$ and $(\hat{I}_{\beta,\infty}^{(k_j)})_j$ are at distance at least δ and cannot converge to the same limit. By compactness of X we can assume that $\hat{I}_{\beta,\infty}^{(k)}$ converges to I_1 . Therefore by u.s.c. of $U_{\beta,\infty}$ we get

$$(3.16) \quad U_{\beta,\infty}(I_1) \geq \limsup_{k \rightarrow \infty} U_{\beta,\infty}(\hat{I}_{\beta,\infty}^{(k)}) \geq \limsup_{k \rightarrow \infty} U_{\beta,\infty}^{(k)}(\hat{I}_{\beta,\infty}^{(k)}) \geq \limsup_{k \rightarrow \infty} U_{\beta,\infty}^{(k)}(\hat{I}_{\beta,\infty}) = U_{\beta,\infty}(\hat{I}_{\beta,\infty}).$$

The uniqueness of the maximizer forces to have $\hat{I}_{\beta,\infty} = I_1$ and this concludes the proof. \square

To prove Part (2) we observe that

$$(3.17) \quad \hat{u}_{\hat{\beta},\infty}^{(k)} = \max_{A \in \mathcal{C}_k} \left[\hat{\beta} \sum_{i \in A} M_i - E(Y_A) \right] \leq \hat{u}_{\hat{\beta}_N, N}^{(k)} + |\hat{\beta}_N - \hat{\beta}| \sum_{i=1}^k M_i + \hat{\beta}_N \sum_{i=1}^k |M_i - M_i^{(N)}|,$$

$$(3.18) \quad \hat{u}_{\hat{\beta}_N, N}^{(k)} = \max_{A \in \mathcal{C}_k} \left[\hat{\beta}_N \sum_{i \in A} M_i^{(N)} - E(Y_A) \right] \leq \hat{u}_{\hat{\beta},\infty}^{(k)} + |\hat{\beta}_N - \hat{\beta}| \sum_{i=1}^k M_i^{(N)} + \hat{\beta} \sum_{i=1}^k |M_i - M_i^{(N)}|.$$

and the proof follows by Lemma 3.1 and the assumption on $\hat{\beta}_N$. \square

Part (3) is a consequence of Proposition 3.2 and Lemma 3.1. \square

Part (4) is a direct consequence of the previous Parts (1), (2) and (3). First of all let us prove an intermediate statement: for all $\delta, \epsilon, \eta > 0$ and $k \in \mathbb{N}$ there exists $N_k < \infty$ such that for all $N > N_k$

$$(3.19) \quad \mathbb{P} \left(\hat{u}_{\hat{\beta}_N, N}^{(k)}(\epsilon) < \hat{u}_{\hat{\beta},\infty}^{(k)}(\epsilon/4) + \eta/4 \right) > 1 - \delta/2.$$

For this purpose let us fix $\delta, \epsilon > 0$. By Part (3) for any fixed $k \in \mathbb{N}$ there exists $N_k > 0$ such that for all $N > N_k$, $d_H(\hat{I}_{\hat{\beta}_N, N}^{(k)}, \hat{I}_{\hat{\beta},\infty}^{(k)}) < \frac{\epsilon}{4}$ w.p. $1 - \delta/4$. Let I be a set achieving $\hat{u}_{\hat{\beta}_N, N}^{(k)}(\epsilon)$, so that by definition $d_H(I, \hat{I}_{\hat{\beta}_N, N}^{(k)}) \geq \epsilon$. It is not difficult to see that $I \subset Y^{(N,k)}$ (points outside $Y^{(N,k)}$ does not contribute to the Energy, but increase the Entropy). We claim that for any $\eta > 0$ w.p. at least $1 - \delta/4$ there exists $I' \subset Y^{(\infty,k)} \in X^{(\text{fin})}$ such that $d_H(I', I) < \epsilon/2$ and $U_{\hat{\beta}_N, N}^{(k)}(I) \leq U_{\hat{\beta},\infty}^{(k)}(I') + \eta/4$. If it is true then $\hat{u}_{\hat{\beta}_N, N}^{(k)}(\epsilon) \leq \hat{u}_{\hat{\beta},\infty}^{(k)}(\epsilon/4) + \eta/4$ because $d_H(I', \hat{I}_{\hat{\beta},\infty}^{(k)}) > \epsilon/4$. To prove the existence of I' we observe that $I = \{0, Y_{i_1}^{(N)}, \dots, Y_{i_\ell}^{(N)}, 1\}$, for a suitable choice of indexes $\{i_1, \dots, i_\ell\} \subset \{1, \dots, k\}$ so that by using Lemma 3.1 it is not difficult to show that we can choose $I' = \{0, Y_{i_1}^{(\infty)}, \dots, Y_{i_\ell}^{(\infty)}, 1\}$, possibly enlarging N .

To conclude the proof, by Part (1) for any $\epsilon > 0$ there exists $\eta > 0$ and $K > 0$ such that for all $k > K$, $\hat{u}_{\hat{\beta},\infty}^{(k)}(\epsilon/4) \leq \hat{u}_{\hat{\beta},\infty}^{(k)} - \eta$ w.p. $1 - \delta/4$. Then by using Part (2) together with Equation (3.19) for any $k > K$ we can find N_k such that for all $N > N_k$

$$(3.20) \quad \mathbb{P} \left(\hat{u}_{\hat{\beta}_N, N}^{(k)}(\epsilon) \leq \hat{u}_{\hat{\beta}_N, N}^{(k)} - \eta/4 \right) > 1 - \delta.$$

\square

Let us note that for any fixed $N \in \mathbb{N}$ we have that $\hat{I}_{\hat{\beta}_N, N}^{(k)} \equiv \hat{I}_{\hat{\beta}_N, N}$ as $k > N$. In the following Proposition we show that this convergence holds uniformly on N .

PROPOSITION 3.5. *The following hold*

(1) *For any $N, k \in \mathbb{N} \cup \{\infty\}$ we define*

$$(3.21) \quad \rho_N^{(k)} := \sup_{I \in \bar{X}} \left| \hat{\sigma}_N(I) - \hat{\sigma}_N^{(k)}(I) \right| = \sum_{i > k} M_i^{(N)}.$$

Then for any $\epsilon, \delta > 0$ there exists $K > 0$ such that $\mathbb{P}(\rho_N^{(k)} > \epsilon) < \delta$ for all $k > K$, uniformly on $N \in \mathbb{N}$.

$$(2) \quad \mathbb{P} \left(\lim_{k \rightarrow \infty} \hat{I}_{\hat{\beta}, \infty}^{(k)} = \hat{I}_{\hat{\beta}, \infty} \right) = 1.$$

(3) *For any $\epsilon, \delta > 0$ there exists K such that $\mathbb{P} \left(d_H(\hat{I}_{\hat{\beta}_N, N}^{(k)}, \hat{I}_{\hat{\beta}_N, N}^{(\ell)}) < \epsilon \right) > 1 - \delta$ for all $k > K$ and uniformly on $N > k$.*

PROOF. Part (1) is like [HM07, Proposition 3.3]. \square

Part (2) has been already proven in the proof of Part (1) of Proposition 3.4. \square

To prove Part (3) let $\epsilon, \delta > 0$ be fixed and let us fix $k \in \mathbb{N}$ and $\eta > 0$ to have that Part (4) of Proposition 3.4 holds for any $N > N_k$. Moreover possibly enlarging k we can suppose that $\mathbb{P}(\rho_N^{(k)} < \eta/2) > 1 - \delta$ uniformly on N . Then we claim that $d_H(\hat{I}_{\hat{\beta}_N, N}^{(k)}, \hat{I}_{\hat{\beta}_N, N}^{(\ell)}) < \epsilon$ with high probability for any $\ell > k$ and $N > N_k$. Otherwise if $d_H(\hat{I}_{\hat{\beta}_N, N}^{(k)}, \hat{I}_{\hat{\beta}_N, N}^{(\ell)}) \geq \epsilon$ for some $\ell > k$, then it must hold that $\hat{u}_{\hat{\beta}_N, N}^{(\ell)} \leq \hat{u}_{\hat{\beta}_N, N}^{(k)}(\epsilon) + \hat{\beta}_N \rho_N^{(k)}$. Therefore w.p. greater than $1 - 2\delta$ it must be $\hat{u}_{\hat{\beta}_N, N}^{(\ell)} \leq \hat{u}_{\hat{\beta}_N, N}^{(k)} - \eta/2$ and this is a contradiction because $\ell \mapsto \hat{u}_{\hat{\beta}_N, N}^{(\ell)}$ is non-decreasing and thus $\hat{u}_{\hat{\beta}_N, N}^{(\ell)} \geq \hat{u}_{\hat{\beta}_N, N}^{(k)}$. The restriction on $N > k$ is obtained by observing that we only left a finite number of N , namely all those in the set $\{k+1, \dots, N_k\}$. For any of such N there exists k_N such that $\hat{I}_{\hat{\beta}_N, N}^{(k_N)} = \hat{I}_{\hat{\beta}_N, N}$ (actually we can choose any $k_N > N$). Therefore by setting $K = \max\{k, k_{k+1}, \dots, k_{N_k}\}$ we extend the result at any $k > K$ and $N > k$. \square

3.2. Proof of Theorem 1.4. The proof is a consequence of [Bil99, Theorem 3.2], which can be written as follows

THEOREM 3.6. *Let us suppose that the r.v's $X_N^{(k)}, X^{(k)}, X_N, X$ they take values in a separable metric space $(\mathcal{S}, d_{\mathcal{S}})$ and $X_N^{(k)}, X_N$ are defined on the same probability space. Then if the following diagram holds*

$$\begin{array}{ccc} X_N^{(k)} & \xrightarrow[N \rightarrow \infty]{(d)} & X^{(k)} \\ \text{in probability, uniformly in } N & \downarrow k \rightarrow \infty & \downarrow k \rightarrow \infty \\ & X_N & X \end{array}$$

then $X_N \xrightarrow{(d)} X$. The expression in probability, uniformly in N means

$$(3.22) \quad \lim_{k \rightarrow \infty} \limsup_{N \rightarrow \infty} \mathbb{P} \left(d_{\mathcal{S}}(X_N^{(k)}, X_N) \geq \epsilon \right) = 0,$$

for any fixed $\epsilon > 0$.

In our case we have $X_N^{(k)} = \hat{I}_{\hat{\beta}_N, N}^{(k)}$, $X^{(k)} = \hat{I}_{\hat{\beta}, \infty}^{(k)}$, $X_N = \hat{I}_{\hat{\beta}_N, N}$ and $X = \hat{I}_{\hat{\beta}, \infty}$ and by Propositions 3.4 and 3.5 the diagram above holds.

REMARK 3.7. Let us stress that under the coupling introduced in Lemma 3.1 we have that in Theorem 1.4 the convergence of $\hat{I}_{\hat{\beta}_N, N}$ to $\hat{I}_{\hat{\beta}, \infty}$ holds in probability, namely for any $\epsilon, \delta > 0$ one has $\mathbb{P}\left(d_H(\hat{I}_{\hat{\beta}_N, N}, \hat{I}_{\hat{\beta}, \infty}) < \epsilon\right) > 1 - \delta$ for all N large enough. This follows by Part (3) of Proposition 3.4 and Parts (2), (3) of Proposition 3.5.

4. Concentration

In this section we discuss the concentration of $\tau/N \cap [0, 1]$ around the set $\hat{I}_{\hat{\beta}_N, N}$ (cf. (2.12)), giving a proof of Theorems 1.3, 1.5.

4.1. General Setting of the Section. Let us stress that in the Pinning Model (1.1), the factor $h > 0$ in the exponent of the Radon-Nikodym derivative plays a role to bias the law of the renewal process. W.l.o.g. we may set $h = 0$, provided to replace the original renewal τ with a new one, $\tilde{\tau}$ defined by $\mathbb{P}(\tilde{\tau}_1 = n) = e^{-h} \mathbb{P}(\tau_1 = n)$ and $\mathbb{P}(\tilde{\tau}_1 = \infty) = 1 - e^{-h}$. Note that the renewal process $\tilde{\tau}$ is terminating because $h > 0$. In this case (cf. Appendix A) the renewal function $\tilde{u}(n) := \mathbb{P}(n \in \tilde{\tau})$ satisfies

$$(4.1) \quad \lim_{n \rightarrow \infty} \frac{\log \tilde{u}(n)}{n^\gamma} = -\mathcal{C},$$

with the same γ and \mathcal{C} used in Assumptions 1.2 for the original renewal process τ .

In the sequel we assume $\mathcal{C} = 1$, as already discussed in the Section 3, $h = 0$ and we omit the tilde-sign on the notations, writing simply τ , $u(\cdot)$ instead of $\tilde{\tau}$, $\tilde{u}(\cdot)$.

4.2. Proof of Theorem 1.3. To prove Theorem 1.3 we proceed in two steps. In the first one we consider a truncated version of the Gibbs measure (1.1) in which we regard only the firsts k -maxima of $\omega_1, \dots, \omega_{N-1}$ proving an analogous result for this truncated Pinning Model (cf. Lemma 4.3). In the second step we show how to deduce Theorem 1.3.

Let us thus define the truncated Pinning Model. For technical reasons it is useful write the Energy by using $\hat{\sigma}_N$ defined in (2.9).

DEFINITION 4.1. For $N, k \in \mathbb{N}$ we define the k -truncated Pinning Model measure by the following Radon-Nikodym derivative

$$(4.2) \quad \frac{d\hat{\mathbb{P}}_{\beta, N}^{(k)}(I)}{d\mathbb{P}_N} = \frac{e^{N^\gamma \beta \hat{\sigma}_N^{(k)}(I)} \mathbb{1}(1 \in I)}{\hat{Z}_{\beta, N}^{(k)}},$$

where \mathbb{P}_N is the law of $\tau/N \cap [0, 1]$ used in (1.1).

In the sequel we use the convention that whenever $k \geq N$, the superscript (k) will be omitted.

REMARK 4.2. Note that whenever $\beta = \hat{\beta}_N$ the Equation (4.2) with $k \geq N$ recovers the original Gibbs measure (1.1) with $\beta = \beta_N$.

LEMMA 4.3. Let $(\hat{\beta}_N)_N$ be a sequence converging to $\hat{\beta} \in (0, \infty)$. For any fixed $\epsilon, \delta > 0$ there exist $\nu = \nu(\epsilon, \delta) > 0$, $K = K(\epsilon, \delta)$ and $(N_k)_{k \geq K}$ such that

$$(4.3) \quad \mathbb{P}\left(\hat{\mathbb{P}}_{\hat{\beta}_N, N}^{(k)}\left(d_H(I, \hat{I}_{\hat{\beta}_N, N}^{(k)}) > \delta\right) \leq e^{-N^\gamma \nu}\right) > 1 - \epsilon$$

for all $k > K$ and $N > N_k$.

REMARK 4.4. Roughly speaking the proof of the Lemma 4.3 is based on the estimation of the probability that a given set $\iota = \{\iota_1, \dots, \iota_\ell\}$, with $\iota_j < \iota_{j+1}$, is included in τ/N , namely to compute the probability that $\iota_1, \dots, \iota_\ell \in \tau/N$ when N is large enough.

Precisely we fix $\iota = \{\iota_1, \dots, \iota_\ell\} \subset [0, 1]$ and consider $\iota^{(N)} = \{\iota_1^{(N)}, \dots, \iota_\ell^{(N)}\}$, where $\iota_i^{(N)}$ is the nearest point to ι_i on the lattice $\{0, 1/N, \dots, 1\}$. We define $u_N(\iota) = \prod_{i=1}^{\ell} u(N(\iota_i^{(N)} - \iota_{i-1}^{(N)}))$ (here and henceforth we use the notation $\iota_0 := 0$) and we want to know the behavior of $u_N(\iota)$ as $N \rightarrow \infty$. The answer is given by the following result

PROPOSITION 4.5. *Let $\iota = \{\iota_1, \dots, \iota_\ell\} \subset [0, 1]$ be a fixed and finite set and consider the associated real sequence $(u_N(\iota))_N$. Then $\lim_{N \rightarrow \infty} \frac{1}{N^\gamma} \log u_N(\iota) = -\sum_{i=1}^{\ell} (\iota_i - \iota_{i-1})^\gamma$ and it holds uniformly in the space of the all subsets ι with points spaced at least ξ , for any fixed $\xi > 0$.*

PROOF. The convergence for a fixed set is a consequence of (4.1). To prove the uniformity we note that if $\iota_i - \iota_{i-1} > \xi$, then $\iota_i^{(N)} - \iota_{i-1}^{(N)} > \xi/2$ as soon as $1/N < \xi/2$, which is independent of such ι . This shows the claim for all such ι with two points and it is sufficient to conclude the proof because in general each factor of $u_N(\iota)$ is written on the form of distance between two points and we have at most $\frac{1}{\xi} + 1$ -factors. \square

Another simple, but important, observation is that for a fixed $k \in \mathbb{N}$ w.h. probability the minimal distance between $Y_1^{(N)}, \dots, Y_k^{(N)}$, the positions of the first k -maxima introduced in Section 2, cannot go to 0 as $N \rightarrow \infty$. Precisely, by using Lemma 3.1, we have that for any fixed $\epsilon > 0$ and $k \in \mathbb{N}$ there exists $\xi = \xi(k, \epsilon) > 0$ and N_k such that for any $N > N_k$ the event

$$(4.4) \quad \left\{ \left| Y_i^{(N)} - Y_j^{(N)} \right| > \xi, Y_\ell^{(N)} \in (\xi, 1 - \xi), \forall \ell, i \neq j \in \{1, \dots, k\} \right\}$$

holds w.p. larger than $1 - \epsilon$. By Proposition 4.5 this implies that for any fixed $\zeta > 0$ on the event (4.4) for all N large enough and uniformly on $\iota = \{\iota_0 = 0 < \iota_1 < \dots < \iota_\ell < 1 = \iota_{\ell+1}\} \subset Y^{(N,k)}$, the set of the first k -maxima (2.15), we have that

$$(4.5) \quad e^{-N^\gamma E(\iota) - \zeta N^\gamma} \leq \mathbb{P}(\iota_1, \dots, \iota_\ell \in \tau/N) \leq e^{-N^\gamma E(\iota) + \zeta N^\gamma},$$

where $E(\iota) = \sum_{i=1}^{\ell+1} (\iota_i - \iota_{i-1})^\gamma$ is the entropy (2.18).

PROOF OF LEMMA 4.3. The aim of this proof is to prove that for any given $\delta > 0$ and $k \in \mathbb{N}$ large enough, $\hat{\mathbb{P}}_{\hat{\beta}_{N,N}}^{(k)} \left(d_H(\hat{I}_{\hat{\beta}_{N,N}}^{(k)}, I) > \delta \right) \rightarrow 0$ as $N \rightarrow \infty$, with an explicit rate of convergence. Our strategy is the following: given a set $I \subset \{0, 1/N, \dots, 1\}$, with $0, 1 \in I$ (chosen according to the Pinning Model of Definition 4.2) we regard $I_{(N,k)} := I \cap Y^{(N,k)}$, the intersection of I with the set of the position of the firsts k -maxima: it can have distance bigger or smaller than $\frac{\delta}{2}$ from $\hat{I}_{\hat{\beta}_{N,N}}^{(k)}$. This leads to make a partition of the set of the all possible I 's, therefore by using an inclusion of events and the triangle inequality we conclude

$$(4.6) \quad \left\{ d_H(\hat{I}_{\hat{\beta}_{N,N}}^{(k)}, I) > \delta \right\} \subset \left\{ d_H(\hat{I}_{\hat{\beta}_{N,N}}^{(k)}, I_{(N,k)}) \geq \frac{\delta}{2} \right\} \cup \left\{ d_H(\hat{I}_{\hat{\beta}_{N,N}}^{(k)}, I_{(N,k)}) < \frac{\delta}{2}, d_H(I_{(N,k)}, I) > \frac{\delta}{2} \right\}.$$

We have thus to prove our original statement for

$$(4.7) \quad \hat{\mathbb{P}}_{\hat{\beta}_{N,N}}^{(k)} \left(d_H(\hat{I}_{\hat{\beta}_{N,N}}^{(k)}, I_{(N,k)}) \geq \frac{\delta}{2} \right),$$

$$(4.8) \quad \hat{\mathbb{P}}_{\hat{\beta}_{N,N}}^{(k)} \left(d_H(\hat{I}_{\hat{\beta}_{N,N}}^{(k)}, I_{(N,k)}) < \frac{\delta}{2}, d_H(I, I_{(N,k)}) > \frac{\delta}{2} \right).$$

Let us define the good event to consider for the disorder. Let $\epsilon, \delta, \zeta > 0$ and the associated $k \in \mathbb{N}, \eta, K > 0$ be fixed such that for any $N > N_k$

- (1) the event (4.4) and the relation (4.5) hold w.p. larger than $1 - \epsilon/2$ (the precise value of ζ will be chosen in the sequel),
- (2) the event $\{\hat{u}_{\hat{\beta}_{N,N}}^{(k)}(\delta/2) < \hat{u}_{\hat{\beta}_{N,N}}^{(k)} - \eta\}$ holds w.p. larger than $1 - \epsilon/2$ (cf. Part (4) of Proposition 3.4, Definition 2.9 and 3.3).

In the rest of the proof we work on the intersection of such two events, which has \mathbb{P} -probability larger than $1 - \epsilon$.

We recall that our goal is to find a good upper bound for (4.7) and (4.8). Let us start to consider (4.7). Let

$$(4.9) \quad A = \left\{ \iota \in Y^{(N,k)} : d_H(\iota, \hat{I}_{\hat{\beta}_{N,N}}^{(k)}) \geq \frac{\delta}{2} \text{ and } 0, 1 \in \iota \right\}.$$

Then (4.7) $\leq \sum_{\iota \in A} \hat{\mathbb{P}}_{\hat{\beta}_{N,N}}^{(k)}(I_{(N,k)} = \iota)$ and if N_k is sufficiently large

$$(4.10) \quad \begin{aligned} \hat{\mathbb{P}}_{\hat{\beta}_{N,N}}^{(k)}(I_{(N,k)} = \iota) &= \frac{\mathbb{E}_N \left(e^{N\gamma \hat{\beta}_{N,N} \hat{\sigma}_N^{(k)}(I)} \mathbb{1}(I_{(N,k)} = \iota); 1 \in \tau/N \right)}{\mathbb{E}_N \left(e^{N\gamma \hat{\beta}_{N,N} \hat{\sigma}_N^{(k)}(I)}; 1 \in \tau/N \right)} \leq \frac{e^{N\gamma \hat{\beta}_{N,N} \hat{\sigma}_N^{(k)}(\iota)} \mathbb{P}_N(\iota \subset I)}{e^{N\gamma \hat{\beta}_{N,N} \hat{\sigma}_N^{(k)}(\hat{I}_{\hat{\beta}_{N,N}}^{(k)})} \mathbb{P}_N(\hat{I}_{\hat{\beta}_{N,N}}^{(k)} \subset I)} \stackrel{(4.5)}{\leq} \\ &\leq \exp \left\{ -N\gamma (\hat{u}_{\hat{\beta}_{N,N}}^{(k)} - U_{\hat{\beta}_{N,N}}^{(k)}(\iota)) + 2N\gamma \zeta \right\} \leq \exp \left\{ -N\gamma (\hat{u}_{\hat{\beta}_{N,N}}^{(k)} - \hat{u}_{\hat{\beta}_{N,N}}^{(k)}(\delta/2)) + 2N\gamma \zeta \right\}, \end{aligned}$$

where $U_{\hat{\beta}_{N,N}}^{(k)}$ has been defined in Definition 2.9. By assumption (2) we have that $\hat{u}_{\hat{\beta}_{N,N}}^{(k)} - \hat{u}_{\hat{\beta}_{N,N}}^{(k)}(\delta/2) > \eta$. We conclude that if ζ is chosen small than $\eta/4$, depending only of η , then the l.h.s. of (4.10) $\leq e^{-N\gamma \frac{\eta}{2}}$, uniformly in $\iota \in A$. Summarizing for all such k and N , by observing that A has at most 2^k elements,

$$(4.11) \quad (4.7) \leq \sum_{\iota \in A} \hat{\mathbb{P}}_{\hat{\beta}_{N,N}}^{(k)}(I_{(N,k)} = \iota) \leq |A| e^{-N\gamma \eta/2} \leq 2^k e^{-N\gamma \eta/2}.$$

For (4.8) we introduce

$$(4.12) \quad B = \left\{ \iota \in Y^{(N,k)} : d_H(\iota, \hat{I}_{\hat{\beta}_{N,N}}^{(k)}) < \frac{\delta}{2} \text{ and } 0, 1 \in \iota \right\},$$

Thus (4.8) $\leq \sum_{\iota \in B} \hat{\mathbb{P}}_{\hat{\beta}_{N,N}}^{(k)} \left(d_H(\iota, I) > \frac{\delta}{2}, I_{(N,k)} = \iota \right)$. Let us observe that for a given such ι

$$(4.13) \quad \begin{aligned} \hat{\mathbb{P}}_{\hat{\beta}_{N,N}}^{(k)} \left(d_H(\iota, I) > \frac{\delta}{2}, I_{(N,k)} = \iota \right) &= \frac{\mathbb{E}_N \left(e^{N\gamma \hat{\beta}_{N,N} \hat{\sigma}_N^{(k)}(I)} \mathbb{1}(d_H(\iota, I) > \frac{\delta}{2}, I_{(N,k)} = \iota); 1 \in \tau/N \right)}{\mathbb{E}_N \left(e^{N\gamma \hat{\beta}_{N,N} \hat{\sigma}_N^{(k)}(I)}; 1 \in \tau/N \right)} \\ &\leq \frac{\mathbb{P}_N \left(d_H(\iota, I) > \frac{\delta}{2}, I_{(N,k)} = \iota \right)}{\mathbb{P}_N(\iota \subset I)}. \end{aligned}$$

We have brought back our problem to compute the probability under the renewal distribution \mathbb{P}_N . Note that $\left\{ d_H(\iota, I) > \frac{\delta}{2}, I_{(N,k)} = \iota \right\} \subset \{ \exists x \in I, d_H(x, \iota) > \frac{\delta}{2}, I_{(N,k)} = \iota \}$ and that if $\iota = \{\iota_0 = 0 < \iota_1 < \dots < \iota_\ell = 1\} \in B$ and we define $U_{j,\delta} = [\iota_j + \frac{\delta}{2}, \iota_{j+1} - \frac{\delta}{2}] \cap \frac{\mathbb{N}}{N}$, which is empty if the distance between ι_j and ι_{j+1} is strictly smaller than δ , then $\{ \exists x \in I, d_H(x, \iota) > \frac{\delta}{2}, I_{(N,k)} = \iota \} = \cup_{j=0}^{\ell-1} \cup_{x \in U_{j,\delta}} \{ x \in I, I_{(N,k)} = \iota \}$. Therefore

$$(4.14) \quad \mathbb{P}_N(d_H(\iota, I) > \delta, I_{(N,k)} = \iota) \leq \sum_{j=0}^{\ell-1} \sum_{x \in U_{j,\delta}} \mathbb{P}_N(x \in I, I_{(N,k)} = \iota) \leq \sum_{j=0}^{\ell-1} \sum_{x \in U_{j,\delta}} \mathbb{P}_N(x \in I, \iota \subset I).$$

Take j for which $l_j < x < l_{j+1}$, then by using the i.i.d. structure of the interval-arrival of the renewal process, we also have, recalling that $u(n) = \mathbb{P}(n \in \tau)$,

$$(4.15) \quad \frac{\mathbb{P}_N(\{x\} \cup \iota \subset I)}{\mathbb{P}_N(\iota \subset I)} = \frac{u(N(x-l_j))u(N(l_{j+1}-x))}{u(N(l_{j+1}-l_j))} \stackrel{(4.5)}{\leq} e^{-N^\gamma((x-l_j)^\gamma + (l_{j+1}-x)^\gamma - (l_{j+1}-l_j)^\gamma) + 2\zeta N^\gamma} \leq e^{-N^\gamma(2^{1-\gamma}-1)\delta^\gamma + 2\zeta N^\gamma},$$

uniformly on $N \in \mathbb{N}$ and on all such l_j, l_{j+1} and x . Note that the last inequality follows by observing that for all such l_j, l_{j+1} and x one has $(x-l_j)^\gamma + (l_{j+1}-x)^\gamma - (l_{j+1}-l_j)^\gamma \geq (2^{1-\gamma}-1)\delta^\gamma$.

We conclude that possibly decreasing the value of ζ , there exists a constant $C_\delta > 0$ such that $\frac{\mathbb{P}_N(\{x\} \cup \iota \subset I)}{\mathbb{P}_N(\iota \subset I)} \leq e^{-C_\delta N^\gamma}$ uniformly in $\iota \in B$.

This leads to have that

$$(4.16) \quad (4.8) \leq \sum_{\iota \in B} \hat{\mathbb{P}}_{\hat{\beta}_N, N}^{(k)} \left(d_H \left(\hat{I}_{\hat{\beta}_N, N}^{(k)}, I \right) > \frac{\delta}{2}, I_{(N, k)} = \iota \right) \leq |B| N e^{-C_\delta N^\gamma} \leq 2^k N e^{-C_\delta N^\gamma}.$$

□

PROOF OF THEOREM 1.3. First of all we are going to prove concentration around $\hat{I}_{\hat{\beta}_N, N}^{(k)}$. Let $k > 0$ be fixed (which we will choose in the following). Then, recalling Definition 4.1,

$$(4.17) \quad \hat{\mathbb{P}}_{\hat{\beta}_N, N} \left(d_H \left(\hat{I}_{\hat{\beta}_N, N}^{(k)}, I \right) > \delta \right) \leq \sum_{\hat{\beta}_N, N} \hat{\mathbb{P}}_{\hat{\beta}_N, N}^{(k)} \left(d_H \left(\hat{I}_{\hat{\beta}_N, N}^{(k)}, I \right) > \delta \right) \cdot \sup \left\{ \frac{d\hat{\mathbb{P}}_{\hat{\beta}_N, N}^{(k)}}{d\hat{\mathbb{P}}_{\hat{\beta}_N, N}^{(k)}}(I) : d_H \left(\hat{I}_{\hat{\beta}_N, N}^{(k)}, I \right) > \delta \right\}.$$

By Lemma 4.3 for any $\epsilon, \delta > 0$ there exists $\nu > 0$ independent of k and \hat{N}_k such that for all $N > \hat{N}_k$, $\mathbb{P} \left(\hat{\mathbb{P}}_{\hat{\beta}_N, N}^{(k)} \left(d_H \left(\hat{I}_{\hat{\beta}_N, N}^{(k)}, I \right) > \delta \right) \leq e^{-N^\nu \nu} \right) > 1 - \epsilon$. To control the Radon-Nikodym derivative we may write

$$(4.18) \quad \frac{d\hat{\mathbb{P}}_{\hat{\beta}_N, N}^{(k)}}{d\hat{\mathbb{P}}_{\hat{\beta}_N, N}^{(k)}}(I) = \frac{\hat{Z}_{\hat{\beta}_N, N}^{(k)} e^{N^\gamma \hat{\beta}_N \hat{\sigma}_N(I)}}{\hat{Z}_{\hat{\beta}_N, N} e^{N^\gamma \hat{\beta}_N \hat{\sigma}_N^{(k)}(I)}} \leq e^{\hat{\beta}_N N^\gamma (\hat{\beta}_N \hat{\sigma}_N(I) - \hat{\beta}_N \hat{\sigma}_N^{(k)}(I))} \leq e^{\hat{\beta}_N N^\gamma \rho_N^{(k)}},$$

where $\rho_N^{(k)} = \sum_{i > k} M_i^{(N)}$ is defined in (3.21). By using Part (1) of Proposition 3.5 we choose k large enough such that $\hat{\beta}_N \rho_N^{(k)} < \nu/2$ w.p. $1 - \epsilon$, uniformly in N . This forces to have

$$(4.19) \quad \mathbb{P} \left(\hat{\mathbb{P}}_{\hat{\beta}_N, N} \left(d_H \left(\hat{I}_{\hat{\beta}_N, N}^{(k)}, I \right) > \delta \right) \leq e^{-N^\nu \nu/2} \right) \geq 1 - 2\epsilon.$$

The theorem follows by observing that for all large k , $d_H(\hat{I}_{\hat{\beta}_N, N}^{(k)}, \hat{I}_{\hat{\beta}_N, N}) < \delta/2$ w.p. $1 - \epsilon$, uniformly on N (Proposition 3.5). □

4.3. Proof of Theorem 1.5. In this section we prove Theorem 1.5. The proof is based on the following result

LEMMA 4.6. *Let $(\mathcal{S}, d_{\mathcal{S}})$ be a metric space and let x_N be a sequence converging to \bar{x} . Let $\mu_N \in \mathcal{M}_1(\mathcal{S})$ be such that for any $\epsilon > 0$, $\lim_{N \rightarrow \infty} \mu_N(x : d(x_N, x) > \epsilon) = 0$. Then $\mu_N \rightarrow \delta_{\bar{x}}$.*

PROOF. The proof is a consequence of the Portmanteau's lemma [Bil99, Section 2]. □

PROOF OF THEOREM 1.5. Let $\mu_N = \hat{\mathbb{P}}_{\hat{\beta}_N, N}$ and $\mu_\infty = \delta_{\hat{I}_{\hat{\beta}_N, \infty}}$. Note that μ_N is a random measure on \mathbb{X} depending of the discrete disorder $w^{(N)}$, while μ_∞ depends of the continuum disorder $w^{(\infty)}$. Therefore if we couple together these disorders as in Lemma 3.1 we have that by Theorems 1.3, 1.4 (see Remark 3.7) $\mu_N \left(I \mid d_H(\hat{I}_{\hat{\beta}_N, N}, I) > \delta \right) \xrightarrow{\mathbb{P}} 0$ and $\hat{I}_{\hat{\beta}_N, N} \xrightarrow{\mathbb{P}} \hat{I}_{\hat{\beta}_N, \infty}$.

To conclude the proof let us observe that the law of μ_N is a probability measure on $\mathcal{M}_1(X)$, the space of the probability measures on X , which is a compact space because X is. Therefore we can assume that μ_N has a limit in distribution. We have thus to show that this limit is the law of μ_∞ . For this purpose it is enough to show that there exists a subsequence N_k such that $\mu_{N_k} \xrightarrow{(d)} \mu_\infty$. It is standard to check that we can find a subsequence N_k such that $\mu_{N_k} \left(I \mid d_H(\hat{I}_{\hat{\beta}_{N_k}, N_k}, I) > \delta \right) \xrightarrow{\mathbb{P}\text{-a.s.}} 0$ and $\hat{I}_{\hat{\beta}_{N_k}, N_k} \xrightarrow{\mathbb{P}\text{-a.s.}} \hat{I}_{\hat{\beta}, \infty}$, therefore by Lemma 4.6 we conclude that $\mu_{N_k} \xrightarrow{\mathbb{P}\text{-a.s.}} \mu_\infty$ and this concludes the proof. \square

5. Proof of Theorem 1.6

The goal of this section is to give a proof of Theorem 1.6.

As preliminary fact let us show that if $\beta < \hat{\beta}_c$ then $\hat{I}_{\beta, \infty} \equiv \{0, 1\}$, while if $\beta > \hat{\beta}_c$ then $\hat{I}_{\beta, \infty} \neq \{0, 1\}$.

To make this let us consider the maximum of the difference between the Continuum Energy (2.13) and the Entropy (2.19), $\hat{u}_{\beta, \infty} = \hat{\sigma}_\infty(\hat{I}_{\beta, \infty}) - E(\hat{I}_{\beta, \infty})$, defined in (2.25). Then whenever $\hat{u}_{\beta, \infty} \leq -1$, we have that $-1 = -E(\{0, 1\}) \leq \hat{u}_{\beta, \infty} \leq -1$ and this implies that $\hat{I}_{\beta, \infty} \equiv \{0, 1\}$ by uniqueness of the maximizer. On the other hand, if $\hat{u}_{\beta, \infty} > -1$, then there exists $I \neq \{0, 1\}$ such that $U_{\beta, \infty}(I) > -1$ because $U_{\beta, \infty}(\{0, 1\}) = -1$, so that $\{0, 1\} \subsetneq \hat{I}_{\beta, \infty}$. In particular, since $\beta \mapsto \hat{u}_{\beta, \infty}$ is non-decreasing, we have that $\hat{I}_{\beta, \infty} \equiv \{0, 1\}$ if $\beta < \hat{\beta}_c$ and $\hat{I}_{\beta, \infty} \neq \{0, 1\}$ if $\beta > \hat{\beta}_c$.

To prove the theorem we proceed in two steps: in the first one we show that a.s. for any $\epsilon > 0$ there exists $\beta_0 = \beta_0(\epsilon) > 0$ random for which $\hat{I}_{\beta, \infty} \subset [0, \epsilon] \cup [1 - \epsilon, 1]$ for all $\beta < \beta_0$. In the second one we show that if ϵ is small enough, then the quantity of Energy that we can gain is always too small to hope to compensate the Entropy. To improve this strategy we use some results on the Poisson Point Process that we are going to recall.

Let us start to note that the process $(Y_i^{(\infty)}, M_i^{(\infty)})_{i \in \mathbb{N}} \subset [0, 1] \times \mathbb{R}_+$ is a realization of a Poisson Point Process Π with intensity

$$(5.1) \quad \mu(dxdz) = \mathbf{1}_{[0,1]}(x) \frac{\alpha}{z^{1+\alpha}} \mathbf{1}_{[0,\infty)}(z) dx dz.$$

In such way, as proved in [Kin93], the process

$$(5.2) \quad X_t = \sum_{(x,z) \in \Pi} z \mathbb{1}(x \in [0, t] \cup [1-t, 1]), \quad t \in \left[0, \frac{1}{2}\right]$$

is a α -stable subordinator. The behavior of a α -stable subordinator in a neighborhood of 0 is described by [Ber96, Thm 10 Ch. 3], precisely if $(X_t)_t$ is such subordinator with $\alpha \in (0, 1)$ and $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is an increasing function, then $\limsup_{t \rightarrow 0^+} X_t/h(t) = \infty$ or 0 a.s. depending on whether the integral $\int_0^1 h(t)^{-\alpha} dt$ diverges or converges. In particular by taking $q > 1$ and $h(t) = t^{1/\alpha} \log^{q/\alpha}(1/t)$ in a neighborhood of 0, we have the following result

PROPOSITION 5.1. *Let $(X_t)_t$ be a α -stable subordinator, with $\alpha \in (0, 1)$, then for every $q > 1$ a.s. there exists a random constant $C > 0$ such that*

$$(5.3) \quad X_t \leq C t^{\frac{1}{\alpha}} \log^{\frac{q}{\alpha}} \left(\frac{1}{t} \right)$$

in a neighborhood of 0.

REMARK 5.2. Let us stress that the process X_t is the value of the sum of the all charges in the set $[0, t] \cup [1-t, 1]$. Therefore it gives an upper bound on the energy that we can gain by visiting this set.

5.0.1. *Step One.* Let us show that a.s. for any $\epsilon > 0$ there exists $\beta_0 = \beta_0(\epsilon) > 0$ for which $\hat{I}_{\beta, \infty} \subset [0, \epsilon] \cup [1 - \epsilon, 1]$ for all $\beta < \beta_0$. Otherwise there should exist $\epsilon > 0$ and a sequence $\beta_k > 0$, $\beta_k \rightarrow 0$ as $k \rightarrow \infty$ such that $\hat{I}_{\beta_k, \infty} \cap (\epsilon, 1 - \epsilon) \neq \emptyset$. Let x be one of such points, then, by Theorem 2.5 we have that $E(\hat{I}_{\beta_k, \infty}) \geq E(\{0, x, 1\}) \geq \epsilon^\gamma + (1 - \epsilon)^\gamma$. Let $S = \sum_{i \in \mathbb{N}} M_i^{(\infty)}$, which is a.s. finite. Therefore by observing that $\hat{u}_{\beta_k, \infty} = \beta_k \hat{\sigma}_\infty(\hat{I}_{\beta_k, \infty}) - E(\hat{I}_{\beta_k, \infty}) \geq -1$ we get

$$(5.4) \quad \beta_k S \geq \beta_k \hat{\sigma}_\infty(\hat{I}_{\beta_k, \infty}) \geq E(\hat{I}_{\beta_k, \infty}) - 1 \geq \epsilon^\gamma + (1 - \epsilon)^\gamma - 1.$$

There is a contradiction because the l.h.s. goes to 0 as $\beta_k \rightarrow 0$, while the r.h.s. is a strictly positive number.

REMARK 5.3. Let us note that if we set $\beta_0 = \beta_0(\epsilon) = (\epsilon^\gamma + (1 - \epsilon)^\gamma - 1)/S$ then for all $\beta < \beta_0$ it must be that $\hat{I}_{\beta, \infty} \subset [0, \epsilon] \cup [1 - \epsilon, 1]$. Moreover let us stress that which such choice $\beta_0 \downarrow 0$ as $\epsilon \downarrow 0$.

5.0.2. *Step Two.* Now let us fix $\epsilon > 0$ small and $\beta_0 = \beta_0(\epsilon) \leq 1$ as in Remark 5.3. Let

$$(5.5) \quad \epsilon_1 = \sup \hat{I}_{\beta, \infty} \cap [0, \epsilon],$$

$$(5.6) \quad \epsilon_2 = \inf \hat{I}_{\beta, \infty} \cap [1 - \epsilon, 1].$$

Let $\hat{\epsilon} = \max\{\epsilon_1, 1 - \epsilon_2\}$. If $\hat{\epsilon} = 0$ we have finished. Then we may assume that $\hat{\epsilon} > 0$ and we choose $q > 1$, $C > 0$ for which Proposition 5.1 holds for any $t < \epsilon$, namely

$$(5.7) \quad \beta \hat{\sigma}_\infty(\hat{I}_{\beta, \infty}) \leq \beta_0 X_{\hat{\epsilon}} \leq C \hat{\epsilon}^{\frac{1}{\alpha}} \log^{\frac{q}{\alpha}} \left(\frac{1}{\hat{\epsilon}} \right),$$

Moreover, by Theorem 2.5 we have that

$$(5.8) \quad E(\hat{I}_{\beta, \infty}) \geq \hat{\epsilon}^\gamma + (1 - \hat{\epsilon})^\gamma.$$

In particular if ϵ is small enough, we have that

$$(5.9) \quad E(\hat{I}_{\beta, \infty}) - 1 > \frac{\hat{\epsilon}^\gamma}{2}.$$

Therefore with, if necessary, a further restrictions on ϵ and β_0 by recalling that $\alpha, \gamma \in (0, 1)$ we conclude that for all $\beta < \beta_0$

$$(5.10) \quad E(\hat{I}_{\beta, \infty}) - 1 \leq \beta \hat{\sigma}_\infty(\hat{I}_{\beta, \infty}) \leq C \hat{\epsilon}^{\frac{1}{\alpha}} \log^{\frac{q}{\alpha}} \left(\frac{1}{\hat{\epsilon}} \right) \leq \frac{\hat{\epsilon}^\gamma}{2} < E(\hat{I}_{\beta, \infty}) - 1,$$

which is a contradiction. Therefore $\hat{\epsilon}$ must be 0 and this implies that $\hat{I}_{\beta, \infty} \equiv \{0, 1\}$ for each $\beta < \beta_0$.

6. The Directed Polymer in Heavy Tailed Random Environment

Originally introduced by [HH85], the directed polymer is a model to describe an interaction between a polymer chain and a medium with microscopic impurities. From a mathematical point of view we consider the set of all paths of a $1 + 1$ - dimensional simple random walk starting from 0 and constrained to come back to 0 after N -steps. The impurities — and so the medium-polymer interactions — are an i.i.d. sequence $(\{\omega_{i,j}\}_{i,j \in \mathbb{N}}, \mathbb{P})$ placed on all integers that can be touched by the walk (note that for any N we have about N^2 points which can be touched). More precisely for a given path s we define the Gibbs measure

$$(6.1) \quad \mu_{\beta, N}(s) = \frac{e^{\beta \sigma_N(s)}}{Q_{\beta, N}},$$

where $\sigma_N(\cdot) = \sum_{i,j} \omega_{i,j} \mathbb{1}(s_i = j)$ is the Energy and $Q_{\beta, N}$ is a normalization constant.

In [AL11] is studied the case in which the impurities have heavy tails, namely the distribution of $\omega_{1,1}$ is regularly varying with index $\alpha \in (0, 2)$. In this case we have to consider

an ordered statistics built on a sample of i.i.d. random variables of size about N^2 , therefore in (2.5) we have to consider b_{N^2} instead of b_N as good rescaling for the Energy, cf. [AL11, (2.4),(2.5)]. This leads to have that to see interesting behavior in the limit of $N \rightarrow \infty$ we have to send β to 0 with N by choosing $\beta = \beta_N \sim \hat{\beta} N^{1-2/\alpha} L(N)$, with L a slowly varying function.

Under this scaling of β by [AL11, Theorem 2.1] one has that the trajectories of the polymer are concentrated in the uniform topology around a favorable curve $\hat{\gamma}_{\beta_N, N}$. In [AL11, Theorem 2.2] one shows that there exists a limit in distribution for the sequence of curves $\hat{\gamma}_{\beta_N, N}$, denoted by $\hat{\gamma}_\beta$. By working on this limit curve one shows that the existence of a random threshold β_c , cf. [AL11, Proposition 2.5]. A problem still open concerns the complete description of its structure. In our work we answer this problem, see Theorem 1.7. The aim of this section is to give its proof.

DEFINITION 6.1 (Entropy). Let us consider $\mathcal{L}^0 = \{s : [0, 1] \rightarrow \mathbb{R} : s \text{ is } 1\text{-Lipschitz}, s(0) = s(1) = 0\}$ equipped with L^∞ -norm, denoted by $\|\cdot\|_\infty$.

For a curve $\gamma \in \mathcal{L}^0$ we define its entropy as

$$(6.2) \quad E(\gamma) = \int_0^1 e\left(\frac{d}{dx}\gamma(x)\right) dx,$$

where $e(x) = \frac{1}{2}((1+x)\log(1+x) + (1-x)\log(1-x))$.

Let us observe that such $E(\cdot)$ is the rate function in the large deviations principle for the sequence of uniform measures on \mathcal{L}_N^0 , the set of linearly interpolated $\frac{1}{N}$ -scaled trajectories of a simple random walk.

DEFINITION 6.2. We introduce the continuous environment π_∞ as

$$(6.3) \quad \pi_\infty(\gamma) = \sum_i T_i^{-\frac{1}{\alpha}} \delta_{Z_i}(\text{graph}(\gamma)), \gamma \in \mathcal{L}^0.$$

Here $\text{graph}(\gamma) = \{(x, \gamma(x)) : x \in [0, 1]\} \subset \mathcal{D} = \{(x, y) \in \mathbb{R}^2 : |y| \leq x \wedge (1-x)\}$ is the graph of γ , $\alpha \in (0, 2)$ is the parameter related to the disorder, T_i is a sum of i -independent exponentials of mean 1 and $(Z_i)_{i \in \mathbb{N}}$ is an i.i.d.-sequence of Uniform(\mathcal{D}) r.v.'s. These two sequences are assumed to be independent with joint law denoted by \mathbb{P}_∞ .

For $\beta < \infty$ we introduce

$$(6.4) \quad \hat{\gamma}_\beta = \operatorname{argmax}_{\gamma \in \mathcal{L}^0} \{\beta \pi_\infty(\gamma) - E(\gamma)\}$$

and we set $u_\beta = \beta \pi_\infty(\hat{\gamma}_\beta) - E(\hat{\gamma}_\beta)$. Since $\beta \pi_\infty(\gamma \equiv 0) - E(\gamma \equiv 0) = 0$ a.s. we have that $u_\beta \geq 0$ a.s., consequently we define the Random Phase Transition as

$$(6.5) \quad \beta_c = \inf\{\beta > 0 : u_\beta > 0\} = \inf\{\beta > 0 : \hat{\gamma}_\beta \neq 0\}.$$

6.1. The Structure of β_c . The random set $(Z_i, T_i^{-\frac{1}{\alpha}})_{i \in \mathbb{N}} \subset \mathcal{D} \times \mathbb{R}_+$ is a realization of a PPP(μ^*) called Π^* with density given by

$$(6.6) \quad \mu^*(dxdydz) = \frac{\mathbf{1}_{\mathcal{D}}(x, y)}{|\mathcal{D}|} \frac{\alpha}{z^{1+\alpha}} \mathbf{1}_{[0, \infty)}(z) dxdydz.$$

Let us introduce the process

$$(6.7) \quad U_t = \sum_{(x, y, z) \in \Pi^*} z \mathbb{1}((x, y) \in A(t)), \quad t \in \left[0, \frac{1}{2}\right]$$

with $A(t) = \{(x, y) \in \mathcal{D} : x \in [0, 1], |y| \leq t\}$. The problem is that $(U_t)_{t \in [0, \frac{1}{2}]}$ is not a Lévy Process, but, at the same time, it is not too far from it. Indeed it has càdlàg trajectories and

independent but not homogeneous increments because the area of $A(t)$ does not grow linearly. Therefore by introducing a suitable function $\phi(t) > t$ we can replace $A(t)$ by $A(\phi(t))$ in order to obtain a process with homogeneous increment. In particular we take $\phi(t) = 1/2(1 - \sqrt{1-4t})$ in such a way that $\text{Leb}(A(\phi(t))) = t$ for all $t \in [0, 1/4]$. Then the process $W_t = U_{\phi(t)}$ is a subordinator and $W_t \geq U_t$ for any $t \in [0, 1/4]$.

To prove Theorem 1.7 we start showing a general fact of this model: for any $\alpha \in (0, 2)$ and for any $\epsilon > 0$ we have that $\|\hat{\gamma}_\beta\|_\infty < \epsilon$ (that is $\text{graph}(\hat{\gamma}_\beta) \subset A(\epsilon)$) if β small enough. To do this we need some preliminary results

PROPOSITION 6.3. *Let E be the entropy of Definition 6.1. Then for all $\gamma \in \mathcal{L}^0$ if $z = (x, y) \in \text{graph}(\hat{\gamma}_\beta)$ we have that*

$$(6.8) \quad E(\gamma) \geq E(\gamma_z),$$

where γ_z is the curve obtained by linear interpolation of $\{(0, 0), z, (1, 0)\}$.

PROOF. [AL11, Proposition 3.1] □

Moreover as shown in [AL11, Proof of Proposition 2.5], there exist two constants $C_1, C_2 > 0$ such that for all $z = (x, y) \in \mathcal{D}$ we have $C_1 \left(\frac{y^2}{x} + \frac{y^2}{1-x} \right) \leq E(\gamma_z) \leq C_2 \left(\frac{y^2}{x} + \frac{y^2}{1-x} \right)$. This leads to have that there exists $C_0 > 0$ for which

$$(6.9) \quad E(\gamma_z) \geq C_0 y^2,$$

uniformly on $z \in \mathcal{D}$.

PROPOSITION 6.4. *For any fixed $\alpha \in (0, 2)$, \mathbb{P}_∞ -a.s. for any $\epsilon > 0$ there exists $\beta_0 = \beta_0(\epsilon) > 0$ such that $\|\hat{\gamma}_\beta\|_\infty < \epsilon$ for all $\beta < \beta_0$.*

PROOF. By contradiction let us suppose that there exists $\epsilon > 0$ such that for a sequence $\beta_k \rightarrow 0$ as $k \rightarrow \infty$ we have $\|\hat{\gamma}_{\beta_k}\|_\infty \geq \epsilon$. By continuity of $\hat{\gamma}_{\beta_k}$ there exists a point $x \in [\epsilon, 1 - \epsilon]$ such that $\hat{\gamma}_{\beta_k}(x) = \epsilon$.

By [HM07, Proposition 4.1] with probability 1 there exists a random set $A \subset \mathbb{N}$ such that $S = \sum_{i \in A} T_i^{-\frac{1}{\alpha}} < \infty$ and for any $\gamma \in \mathcal{L}^0$ it holds that $S \geq \pi_\infty(\gamma)$. For instance if $\alpha \in (0, 1)$, then $A \equiv \mathbb{N}$, while if $\alpha > 1$ then $A \subsetneq \mathbb{N}$. Since $u_{\beta_k} = \beta_k \pi_\infty(\hat{\gamma}_{\beta_k}) - E(\hat{\gamma}_{\beta_k}) \geq 0$ we obtain that a.s.

$$(6.10) \quad \beta_k S \geq \beta_k \pi_\infty(\hat{\gamma}_{\beta_k}) \geq E(\hat{\gamma}_{\beta_k}) \geq E(\gamma_{z=(x, \epsilon)}) \geq C_0 \epsilon^2.$$

Sending $\beta_k \rightarrow 0$ we obtain a contradiction because the l.h.s. converges to 0. □

PROOF OF PROPOSITION 1.7. We have to prove only the point (1), the other one has been already proven in [AL11]. Let $\epsilon > 0$ be fixed and $\beta_0 = \beta_0(\epsilon)$ such that $\|\hat{\gamma}_\beta\|_\infty < \epsilon$ for all $\beta < \beta_0 \leq 1$. Let moreover $\hat{\epsilon} = \max |\hat{\gamma}_\beta(x)|$. Our intention is to estimate the energy gained by $\hat{\gamma}_\beta$ by computing the total charges contained in the region $A_{\hat{\epsilon}}$. For this purpose let, $q > 1$ and $C > 0$ for which Proposition 5.1 holds for the process W_t and then for U_t . Therefore

$$(6.11) \quad \pi_\infty(\hat{\gamma}_\beta) \leq \sum_{i \in \mathbb{N}} T_i^{-\frac{1}{\alpha}} \mathbf{1}_{(Z_i \in A_{\hat{\epsilon}})} \leq U_{\hat{\epsilon}} \leq C \hat{\epsilon}^{\frac{1}{\alpha}} \log^{\frac{q}{\alpha}} \left(\frac{1}{\hat{\epsilon}} \right).$$

For the Entropy we use directly Equation (6.9):

$$(6.12) \quad E(\hat{\gamma}_\beta) \geq C_0 \hat{\epsilon}^2.$$

Conclusion: if ϵ is small enough we get

$$(6.13) \quad \beta \pi_\infty(\hat{\gamma}_\beta) \leq C \hat{\epsilon}^{\frac{1}{\alpha}} \log^{\frac{q}{\alpha}} \left(\frac{1}{\hat{\epsilon}} \right) \leq C_0 \hat{\epsilon}^2 \leq E(\hat{\gamma}_\beta),$$

because $\alpha < \frac{1}{2}$ and this forces to have $u_\beta = 0$ for all $\beta < \beta_0$. □

7. Possible Generalizations and Perspectives

Let us discuss what happens with different choices of the parameters γ and h .

($\gamma \geq 1$) If $\gamma \geq 1$, then for any $(x_i)_{i=1}^\ell \subset (0, 1)$ one has that $(\sum_{k=1}^\ell x_k)^\gamma \geq \sum_{k=1}^\ell x_k^\gamma$ (where the equality holds only if $\gamma = 1$). This implies (by a simple induction argument) that the Entropy function $E(I)$ is non-increasing (strictly non-increasing if $\gamma > 1$) on $\mathbf{X}^{(N)}$, therefore for any fixed $\beta > 0$ and $N \in \mathbb{N}$, the solution of Equation (2.3) is $I_{\beta, N} = \{0, 1/N, \dots, 1\}$. Thus whenever $N \rightarrow \infty$ the limit set is the interval $[0, 1]$, independently of our choice of β_N . Unfortunately with this choice of γ , the function $K(\cdot)$ is not sub-exponential and thus we are not able to prove concentration of τ/N around $I_{\beta_N, N}$. Anyway we conjecture that τ/N converges always to the whole segment $[0, 1]$.

($\gamma = 0$) To give sense to this limiting case it is convenient to consider Equation (1.4), that is $K(n) := P(\tau_1 = n) \sim L(n)n^\rho$, with $\rho < -1$. This particular case is very interesting, since the power-polynomial case is the most studied case in this type of model, due to the physical interpretation and its mathematical richness, see for instance [Gia07, Gia10, dH07]. If we work with such renewal process, we conjecture that the correct rescaling will be given by $\beta = \beta_N \sim N^{-1/\alpha} \log N$ and the limit measure for the the sequence $P_{\beta_N, h, N}^\omega(\cdot)$ will be give by a more complicated structure than the δ -measure of a single set. This would mean that we do not have concentration around a single favorable set.

The conjecture on β_N appears by analyzing the specific case in which $\rho \in (-2, -1)$. In that case our strategy is to write the Gibbs measure in a Polynomial Chaos expansion, as done in [CSZ14] and consider the first term of the sum. Then we prove that if we take $\beta = \beta_N$ as above, the Laplace transform converges, therefore if a good rescaling exists it should be β_N . We thus conjecture that the same rescaling works for all possible choice of $\rho < -1$.

($h \leq 0$) According to the first part of Section 4, in the Pinning Model (1.1) we can always assume $h = 0$ by changing the underlying renewal process. If $h > 0$ this new renewal process turns out to be terminating. In this regime, however, the behavior of its renewal function $u(n) := P(n \in \tau)$ cannot be stretch-exponential, as proven in [BL08, Theorem 2.1], thus our proof about the concentration cannot work. We conjecture that in this case τ/N converges always to $[0, 1]$.

Appendix A. Asymptotic Behavior for Terminating Renewal Processes

In this section we consider a terminating renewal process (τ, P) and $K(n) = P(\tau_1 = n)$, with $K(\infty) > 0$. The aim is to study the asymptotic behavior of the renewal function $u(N) = P(N \in \tau) = \sum_m K^{*(m)}(N)$, where $K^{*(m)}$ is the m^{th} -convolution of K with itself under the assumption that $K(\cdot)$ is subexponential. We refer to [FKZ09] for the general theory on the subexponential distribution and for the notations that we follow.

DEFINITION A.1 (Long-Tailed Distribution). We say that a discrete probability density q on \mathbb{N}_0 is long-tailed if $q(n) > 0$ for any n large enough and if

$$(A.1) \quad \lim_{n \rightarrow \infty} \frac{q(n+k)}{q(n)} = 1$$

for any fixed $k > 0$.

DEFINITION A.2 (Sub-exponential Distribution). We say that a discrete probability density q on \mathbb{N} is subexponential if it is long-tailed and

$$(A.2) \quad q^{*(2)}(n) \sim 2q(n), \quad n \rightarrow \infty.$$

The result in which we are interested in is the following

THEOREM A.3. *Let $K(\cdot)$ be a discrete probability density on $\mathbb{N} \cup \{\infty\}$ such that $K(\infty) > 0$ and let $\delta = 1 - K(\infty) < 1$. Let $q(\cdot)$ defined as $q(n) = \delta^{-1}K(n)$. If q is subexponential, then*

$$(A.3) \quad \lim_{n \rightarrow \infty} \frac{u(n)}{K(n)} = \frac{1}{K(\infty)^2}.$$

Its proof is a simple consequence of the Dominated Convergence Theorem by using the following results

LEMMA A.4. *Let q be a subexponential discrete probability density on \mathbb{N} , then for any $m \geq 1$*

$$(A.4) \quad q^{*(m)}(n) \stackrel{n \rightarrow \infty}{\sim} mq(n).$$

PROOF. [FKZ09, Corollary 4.13]. □

THEOREM A.5. *Let q be a subexponential discrete probability density on \mathbb{N} . Then we have that for any $\epsilon > 0$ there exists $N_0 = N_0(\epsilon)$ and $c = c(\epsilon)$ such that for any $n > N_0$ and $m \geq 1$*

$$(A.5) \quad q^{*(m)}(n) \leq c(1 + \epsilon)^m q(n).$$

PROOF. [FKZ09, Theorem 4.14]. □

A.1. The case of $K(n) \cong e^{-\mathcal{C}n^\gamma}$. The goal of this section is to prove that if we take a non-terminating renewal process such that $K(n) \sim n^\rho L(n)e^{-\mathcal{C}n^\gamma}$, where $\gamma \in (0, 1)$, $\mathcal{C} > 0$, $\rho \in \mathbb{R}$ and $L(n)$ is a slowly varying function, then it is subexponential.

By [FKZ09, Theorem 4.11] w.l.o.g. we can assume $K(n) = n^\rho \tilde{L}(n)e^{-\mathcal{C}n^\gamma}$, with \tilde{L} be another slowly varying function. Since $\gamma \in (0, 1)$ it is not difficult to see that such $K(\cdot)$ is long-tailed. We have to show that $\lim_{n \rightarrow \infty} K^{*(2)}(n)/K(n) = 2$. By [FKZ09, Theorem 4.7] it is enough to prove that for any choice of $h = h(n) \rightarrow \infty$ as $n \rightarrow \infty$, with $h(n) < n/2$, we have that $\sum_{m=h(n)}^{n-h(n)} K(n-m)K(m) = o(K(n))$, as $n \rightarrow \infty$.

Let us consider $R(y) = y^\gamma$, with $\gamma \in (0, 1)$. R is a concave increasing function and $R'(y) = \gamma y^{\gamma-1}$ is strictly decreasing, so that given two integer points n, m such that $n - m > m$ we have

$$(A.6) \quad R(n) - R(n - m) \leq mR'(n - m) \leq mR'(m) = \gamma m^\gamma = \gamma R(m),$$

By Karamata's representation for slowly varying functions [BGT89, Theorem 1.2.1] there exists $c_1 \geq 1$ for which $\tilde{L}(xr) \leq c_1 \tilde{L}(r)$ for any $x \in [\frac{1}{2}, 1]$ and $r \geq 1$. This implies also that for any $\rho \in \mathbb{R}$ there exists $c = c(\rho)$ such that $(xr)^\rho \tilde{L}(xr) \leq cr^\rho \tilde{L}(r)$ for any $x \in [\frac{1}{2}, 1]$ and $r \geq 1$. Therefore in our case, whenever $n - m \geq n/2$ we have that $K(n - m) \leq n^\rho \tilde{L}(n)e^{-\mathcal{C}(n-m)^\gamma} = K(n)e^{R(n) - R(n-m)}$.

Summarizing, by using all these observations we conclude that

$$(A.7) \quad \sum_{m=h(n)}^{\frac{n}{2}} \frac{K(n-m)K(m)}{K(n)} \leq c \sum_{m=h(n)}^{\infty} m^\rho \tilde{L}(m)e^{-\mathcal{C}(1-\gamma)R(m)},$$

which goes to 0 as $h(n) \rightarrow \infty$ and the proof follows by observing that

$$(A.8) \quad \sum_{m=h(n)}^{n-h(n)} \frac{K(n-m)K(m)}{K(n)} = 2 \sum_{m=h(n)}^{\frac{n}{2}} \frac{K(n-m)K(m)}{K(n)}.$$

□

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References

- [AL11] A. Auffinger and O. Louidor. Directed polymers in random environment with heavy tails. *Comm. on Pure and Applied Math.*, 64:183–204, 2011.
- [Asm03] S. Asmussen. *Applied Probability and Queues*, volume 51 of *Stochastic Modelling and Applied Probability*. Springer, 2003.
- [Ber96] J. Bertoin. *Lévy Processes*. Cambridge University Press, 1996.
- [BGT89] N. H. Bingham, C. M. Goldie, and J. L. Teugels. *Regular Variation*. Cambridge University Press, 1989.
- [Bil99] P. Billingsley. *Convergence of Probability Measures*. John Wiley and Sons, Inc., 2nd edition, 1999.
- [BL08] V.S. Barbu and N. Limnios. *Semi-Markov Chains and Hidden Semi-Markov Models toward Applications*, volume 191 of *Lecture Notes in Statistics*. Springer, 2008.
- [Bou71] N. Bourbaki. *Topologie Général, Chapitres 1 à 4*. Diffusion C.C.L.S., Paris, 1971.
- [CSZ14] F. Caravenna, R. Sun, and N. Zygouras. Polynomial chaos and scaling limits of disordered systems. *arXiv:1312.3357*, 2014.
- [dH07] F. den Hollander. *Random Polymers*, volume 1974 of *Lecture Notes in Mathematics, Ecole d'Eté de Probabilités de Saint-Flour XXXVII*. Springer, 2007.
- [FKZ09] S. Foss, D. Korshunov, and S. Zachary. *An introduction to Heavy-Tailed and Subexponential Distribution*. Springer, 2009.
- [Gia07] G. Giacomin. *Random Polymer Models*. Imperial College Press, World Scientific, 2007.
- [Gia10] G. Giacomin. *Disorder and Critical Phenomena Through Basic Probability Models*, volume 2025 of *Lecture Notes in Mathematics, Ecole d'Eté de Probabilités de Saint-Flour XL*. Springer, 2010.
- [HH85] D.A. Huse and C.L. Henveley. Pinning and roughening of domain walls in ising systems due to random impurities. *Phys. Rev. Lett.*, 54:2708–2711, 1985.
- [HM07] B. Hambly and J. B. Martin. Heavy tails in last-passage percolation. *Probability Theory and Related Fields*, 137:227–275, 2007.
- [Kin93] J. F. C. Kingman. *Poisson Processes*. Oxford Studies in Probability, 1993.
- [Res87] R. Resnick. *Extreme values, regular variation and point process*. Series in Operations Research and Financial Engineering. New York, Springer, 1987.

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