# Multiplicities of equivariant Spin$^c$ Dirac operators

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## Contents



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# 1 Introduction

Let $M$ be a compact connected manifold. We assume that $M$ is even dimensional and oriented. We consider a spin$^c$ structure on $M$, and denote by $\mathcal{S}$ the corresponding Spin$^c$ bundle. Let $K$ be a compact connected Lie group acting on $M$ and $\mathcal{S}$ and we denote by $D : \Gamma(M, \mathcal{S}^+) \to \Gamma(M, \mathcal{S}^-)$ the corresponding $K$-equivariant Spin$^c$ Dirac operator. The equivariant index of $D$, denoted $\mathcal{Q}_K(M, \mathcal{S})$, belongs to the Grothendieck group of representations of $K$ :

$$\mathcal{Q}_K(M, \mathcal{S}) = \sum_{\pi \in \widehat{K}} \mathrm{m}(\pi)\ \pi.$$

An important example is when $M$ is a compact complex manifold, $K$ a compact group of holomorphic transformations of $M$, and $\mathcal{L}$ any holomorphic $K$-equivariant line bundle on $M$, not necessarily ample. Then the Dolbeaut operator twisted by $\mathcal{L}$ can be realized as a Spin$^c$ Dirac operator $D$ acting on sections of a Spin$^c$-bundle $\mathcal{S}$. In this case $\mathcal{Q}_K(M, \mathcal{S}) = \sum_q (-1)^q H^{0,q}(M, \mathcal{L})$.

Another example is when $M$ is a compact even dimensional oriented manifold with a $K$-invariant spin structure. Let $\mathcal{S}_{\mathrm{spin}}$ be the corresponding canonical spin bundle, $L$ be **any** $K$-equivariant line bundle, and take the Spin$^c$ bundle $\mathcal{S}_{\mathrm{spin}} \otimes L$. Then $\mathcal{Q}_K(M, \mathcal{S}_{\mathrm{spin}} \otimes L)$ is the index of the Dirac operator associated to the spin structure twisted by the line bundle $L$.

The aim of this article is to give a geometric description of the multiplicity $\mathrm{m}(\pi)$ in the spirit of the Guillemin-Sternberg phenomenon $[Q, R] = 0$ [10, 17, 18, 28, 20]. After the remarkable results of Meinrenken-Sjamaar [18], it was tempting to find in what way we can extend these results to other situations. Consider the determinant line bundle $\mathbb{L} = \det(\mathcal{S})$ of the spin$^c$ structure. This is a $K$-equivariant complex line bundle on $M$. The choice of a $K$-invariant hermitian metric and of a $K$-invariant hermitian connection $\nabla$ on $\mathbb{L}$ determines a moment map

$$\Phi_{\mathcal{S}} : M \to \mathfrak{k}^*$$

by the relation $\mathcal{L}(X) - \nabla_{X_M} = 2i\langle \Phi_{\mathcal{S}}, X\rangle$, for all $X \in \mathfrak{k}$. If $M$ is spin and $\mathcal{S} = \mathcal{S}_{\mathrm{spin}} \otimes L$, then $\Phi_{\mathcal{S}}$ is the "moment map" associated to a connection on $L$.

We compute $\mathrm{m}(\pi)$ in term of the reduced "manifolds" $\Phi_{\mathcal{S}}^{-1}(K\xi)/K$. This formula extends the result of [21]. However, in this article, we do not assume any hypothesis on the line bundle $\mathbb{L}$, in particular we do not assume that the curvature of the connection $\nabla$ is a symplectic form. In this pre-symplectic setting, a partial answer to this question has been obtained by [12, 8, 9, 5] when $K$ is a torus.

In a recent preprint [14], Hochs and Mathai use our result to obtain a $[Q, R] = 0$ theorem in the case of an action of a connected Lie group $G$ on a Spin$^c$ manifold $M$. In their work, $G$ or $M$ are not necessarily compact but the $G$-action on $M$ is proper and co-compact : in this context they are able to come back to the compact setting by Abel's slice theorem.

Results obtained here have been announced in [23].

### 1.1 The result

We start to explain our result in the torus case. The general case reduces (in spirit) to this case, using an appropriate slice for the $K$-action on $M$.

Let $T$ be a torus acting effectively on $M$, and let $\mathcal{S} \to M$ be a $T$-equivariant Spin$^c$-bundle (with connection) on $M$. In contrast to the symplectic case, the image $\Phi_{\mathcal{S}}(M)$ might not be convex and depends of the choice of the connection. Let $\Lambda \subset \mathfrak{t}^*$ be the lattice of weights. If $\mu \in \Lambda$, we denote by $\mathbb{C}_\mu$ the corresponding one dimensional representation of $T$. The topological space $M_\mu = \Phi_{\mathcal{S}}^{-1}(\mu)/T$, which may not be connected, is an orbifold provided with a Spin$^c$-structure when $\mu$ in $\mathfrak{t}^*$ is a regular value of $\Phi_{\mathcal{S}}$. In this case we define the integer $\mathrm{Q}^{\mathrm{spin}}(M_\mu)$ as the index of the corresponding Spin$^c$ Dirac operator on the orbifold $M_\mu$. We can define $\mathrm{Q}^{\mathrm{spin}}(M_\mu)$ even if $\mu$ is a singular value. Postponing this definition, our result states that

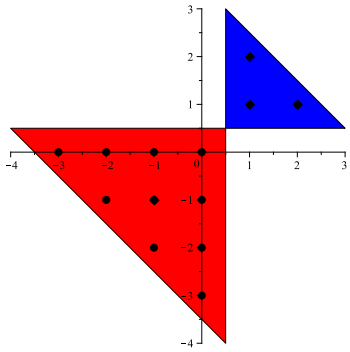


Figure 1: $T$-multiplicities for non ample bundle on Hirzebruch surface

$$\mathcal{Q}_T(M,\mathcal{S}) = \sum_{\mu\in\Lambda\cap\Phi_{\mathcal{S}}(M)} \mathrm{Q}^{\mathrm{spin}}(M_\mu)\,\mathbb{C}_\mu.$$

Here is the definition of $\mathrm{Q}^{\mathrm{spin}}(M_\mu)$ (see Section 5.1). We approach $\mu$ by a regular value $\mu+\epsilon$, and we define $\mathrm{Q}^{\mathrm{spin}}(M_\mu)$ as the index of a Spin$^c$ Dirac operator on the orbifold $M_{\mu+\epsilon}$, and this is independent of the choice of $\epsilon$ sufficiently close. Remark here that $\mu$ has to be an interior point of $\Phi_{\mathcal{S}}(M)$ in order for $\mathrm{Q}^{\mathrm{spin}}(M_\mu)$ to be non zero, as otherwise we can take $\mu+\epsilon$ not in the image. In a forthcoming article, we will give a more detailed description of the function $\mu \to \mathrm{Q}^{\mathrm{spin}}(M_\mu)$ in terms of locally quasi-polynomial functions on $\mathfrak{t}^*$.

When $M$ is a toric manifold, this result was obtained by Karshon-Tolman. In Figure 1, we draw the picture of the function $\mu \mapsto \mathrm{Q}^{\mathrm{spin}}(M_\mu)$ for the Hirzebruch surface, and a non ample line bundle on it (we give the details of this example in the last section). The image of $\Phi_{\mathcal{S}}$ is the union of the two large triangles in red and blue. The multiplicities are 1 on the integral points of the interior of the red triangle, and $-1$ on the integral points of the interior of the blue triangle.

Now consider the case of a compact connected Lie group $K$ acting on $M$ and $\mathcal{S}$. Before describing precisely the multiplicities of $\mathcal{Q}_K(M,\mathcal{S})$, we first

give a vanishing result.

Let $\mathcal{H}_{\mathfrak{k}}$ be the set of conjugacy classes of the reductive algebras $\mathfrak{k}_\xi, \xi \in \mathfrak{k}^*$. We group the coadjoint orbits according to the conjugacy class $(\mathfrak{h}) \in \mathcal{H}_{\mathfrak{k}}$ of the stabilizer, and we consider the Dixmier sheet $\mathfrak{k}^*_{(\mathfrak{h})}$ of orbits $K\xi$ with $\mathfrak{k}_\xi$ conjugated to $\mathfrak{h}$. We denote by $H$ the connected subgroup of $K$ with Lie algebra $\mathfrak{h}$. If $\mathfrak{z}$ is the center of $\mathfrak{h}$, let $\mathfrak{z}^*_0$ be the set of $\xi \in \mathfrak{z}^*$, such that $\mathfrak{k}_\xi = \mathfrak{h}$. We see then that the Dixmier sheet $\mathfrak{k}^*_{(\mathfrak{h})}$ is equal to $K\mathfrak{z}^*_0$.

Let $(\mathfrak{k}_M)$ be the generic infinitesimal stabilizer of the $K$-action on $M$. We prove the following vanishing result in Sections 4.5.1 and 4.5.2.

**Theorem 1.1** *If $\mathcal{Q}_K(M, \mathcal{S})$ is non zero, then there exists a unique $(\mathfrak{h}) \in \mathcal{H}_{\mathfrak{k}}$ such that :*

- $([\mathfrak{k}_M, \mathfrak{k}_M]) = ([\mathfrak{h}, \mathfrak{h}])$,
- *the pullback $\Phi_{\mathcal{S}}^{-1}(\mathfrak{k}^*_{(\mathfrak{h})})$ is open and dense in $M$.*

A typical example of a couple $(M, \mathcal{S})$ satisfying the conditions of Theorem 1.1 if when $M$ is equal to $K \times_H Y$ with $Y$ a compact $H/[H, H]$-manifold (see Subsection 5.4). The Spin$^c$-bundle on $M$ determines a Spin$^c$-bundle $\mathcal{S}_Y$ on $Y$ such that the moment map $\Phi_{\mathcal{S}_Y}$ takes value in $\mathfrak{z}^*$ ($\mathfrak{z}$ is the Lie algebra of $H/[H, H]$). In this case, it is easy to compute $\mathcal{Q}_K(M, \mathcal{S})$ in terms of $\mathcal{Q}_H(Y, \mathcal{S}_Y)$ via an induction formula.

In spirit, we are in this situation. Indeed we can define the non-compact "slice" $\mathcal{Y} = \Phi_{\mathcal{S}}^{-1}(\mathfrak{z}^*_0)$ which is a $H/[H, H]$ submanifold of $M$ such that $K\mathcal{Y}$ is a dense open subset of $M$.

In order to study the $K$-multiplicities of $\mathcal{Q}_K(M, \mathcal{S})$, we need a geometric parametrization of the dual $\widehat{K}$.

We say that a coadjoint orbit $\mathcal{P} \subset \mathfrak{k}^*$ is admissible if $\mathcal{P}$ carries a Spin$^c$-bundle $\mathcal{S}_{\mathcal{P}}$ such that the corresponding moment map $\Phi_{\mathcal{S}}$ is the inclusion $\mathcal{P} \hookrightarrow \mathfrak{k}^*$. We denote simply by $\mathrm{Q}_K^{\mathrm{spin}}(\mathcal{P})$ the element $\mathcal{Q}_K(\mathcal{P}, \mathcal{S}_{\mathcal{P}}) \in R(K)$. It is either 0 or an irreducible representation of $K$, and the map

$$\mathcal{O} \mapsto \pi_{\mathcal{O}} := \mathrm{Q}_K^{\mathrm{spin}}(\mathcal{O})$$

defines a bijection between the regular admissible orbits and the dual $\widehat{K}$.

Denote by $\mathcal{A}((\mathfrak{h}))$ the set of admissible orbits contained in the Dixmier sheet $\mathfrak{k}^*_{(\mathfrak{h})}$. When $\mathcal{O}$ is a regular admissible orbit, a coadjoint orbit $\mathcal{P} \in \mathcal{A}((\mathfrak{h}))$ is called a $(\mathfrak{h})$-ancestor of $\mathcal{O}$ if $\mathrm{Q}_K^{\mathrm{spin}}(\mathcal{P}) = \pi_{\mathcal{O}}$.

When $(M, \mathcal{S})$ satisfy the conditions of Theorem 1.1, we can define the Spin$^c$ index $\mathrm{Q}^{\mathrm{spin}}(M_{\mathcal{P}}) \in \mathbb{Z}$ of the reduced space $M_{\mathcal{P}} = \Phi_{\mathcal{S}}^{-1}(\mathcal{P})/K$, for any

$\mathcal{P} \in \mathcal{A}((\mathfrak{h}))$. We use the slice $\mathcal{Y}$ and the deformation procedure, as explained in the abelian case.

We obtain the following $[Q, R] = 0$ theorem which is the main result of the paper.

**Theorem 1.2** *Assume that* $([\mathfrak{k}_M, \mathfrak{k}_M]) = ([\mathfrak{h}, \mathfrak{h}])$ *with* $(\mathfrak{h}) \in \mathcal{H}_{\mathfrak{k}}$.

- *The multiplicity of the representation* $\pi_{\mathcal{O}}$ *in* $\mathcal{Q}_K(M, \mathcal{S})$ *is equal to*

$$\sum_{\mathcal{P}} \mathrm{Q}^{\mathrm{spin}}(M_{\mathcal{P}})$$

*where the sum runs over the* $(\mathfrak{h})$*-ancestor of* $\mathcal{O}$*. In other words*

$$\mathcal{Q}_K(M, \mathcal{S}) = \sum_{\mathcal{P} \in \mathcal{A}((\mathfrak{h}))} \mathrm{Q}^{\mathrm{spin}}(M_{\mathcal{P}}) \mathrm{Q}_K^{\mathrm{spin}}(\mathcal{P}).$$

- *Furthermore, each term* $\mathrm{Q}^{\mathrm{spin}}(M_{\mathcal{P}}) \in \mathbb{Z}$ *does not depend on the choice of the connection on the determinant line bundle* $\det(\mathcal{S})$.

It may be useful to rephrase this theorem by describing the parametrization of admissible orbits by parameters belonging to the closed Weyl chamber $\mathfrak{t}^*_{\geqslant 0}$. Let $\Lambda_{\geqslant 0} := \Lambda \cap \mathfrak{t}^*_{\geqslant 0}$ be the set of dominant weights, and let $\rho$ be the half sum of the positive roots.

The set of regular admissible orbits is indexed by the set $\Lambda_{\geqslant 0} + \rho$: if $\lambda \in \Lambda_{\geqslant 0} + \rho$, the coadjoint orbit $K\lambda$ is regular admissible and $\pi_{K\lambda}$ is the representation with highest weight $\lambda - \rho$.

Denote by $\mathcal{F}$ the set of the relative interiors of the faces of $\mathfrak{t}^*_{\geqslant 0}$. Thus $\mathfrak{t}^*_{\geqslant 0} = \coprod_{\sigma \in \mathcal{F}} \sigma$. The face $\mathfrak{t}^*_{>0}$ is the open face in $\mathcal{F}$.

Let $\sigma \in \mathcal{F}$. The stabilizer $K_\xi$ of a point $\xi \in \sigma$ depends only of $\sigma$. We denote it by $K_\sigma$, and by $\mathfrak{k}_\sigma$ its Lie algebra. We choose on $\mathfrak{k}_\sigma$ the system of positive roots compatible with $\mathfrak{t}^*_{\geqslant 0}$, and let $\rho^{K_\sigma}$ be the corresponding $\rho$. When $\mu \in \sigma$, the coadjoint orbit $K\mu$ is admissible if and only if $\lambda = \mu - \rho + \rho^{K_\sigma} \in \Lambda$.

The map $\mathcal{F} \longrightarrow \mathcal{H}_{\mathfrak{k}}$, $\sigma \mapsto (\mathfrak{k}_\sigma)$, is surjective but not injective. We denote by $\mathcal{F}((\mathfrak{h}))$ the set of faces of $\mathfrak{t}^*_{\geqslant 0}$ such that $(\mathfrak{k}_\sigma) = (\mathfrak{h})$.

Using the above parameters, we may rephrase Theorem 1.2 as follows.

**Theorem 1.3** *Assume that* $([\mathfrak{k}_M, \mathfrak{k}_M]) = ([\mathfrak{h}, \mathfrak{h}])$ *with* $(\mathfrak{h}) \in \mathcal{H}_{\mathfrak{k}}$. *Let* $\lambda \in \Lambda_{\geqslant 0} + \rho$ *and let* $\mathrm{m}_\lambda \in \mathbb{Z}$ *be the multiplicity of the representation* $\pi_{K\lambda}$ *in* $\mathcal{Q}_K(M, \mathcal{S})$. *We have*

$$\mathrm{m}_\lambda = \sum_{\substack{\sigma \in \mathcal{F}((\mathfrak{h})) \\ \lambda - \rho^{K_\sigma} \in \sigma}} \mathrm{Q}^{\mathrm{spin}}(M_{K(\lambda - \rho^{K_\sigma})}). \tag{1.1}$$

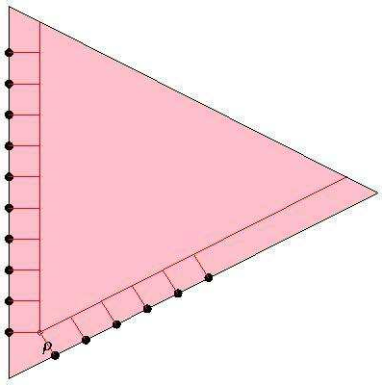


Figure 2: $K$-multiplicities and ancestors

More explicitly, the sum (1.1) is taken over the faces $\sigma$ of the Weyl chamber such that

$$([\mathfrak{k}_M, \mathfrak{k}_M]) = ([\mathfrak{k}_\sigma, \mathfrak{k}_\sigma]), \quad \Phi_{\mathcal{S}}(M) \cap \sigma \neq \varnothing, \quad \lambda \in \{\sigma + \rho^{K_\sigma}\}. \tag{1.2}$$

In Section 6.3, we give an example of a $SU(3)$-manifold $M$ with generic stabilizer $SU(2)$, and a Spin$^c$ bundle $\mathcal{S}$ where several $\sigma$ contribute to the multiplicity of a representation $\pi_{K\lambda}$ in $\mathcal{Q}_K(M, \mathcal{S})$. On Figure 2, the picture of the decomposition of $\mathcal{Q}_K(M, \mathcal{S})$ is given in terms of the representations $\mathrm{Q}_K^{\mathrm{spin}}(\mathcal{P})$ associated to the $SU(2)$-ancestors $\mathcal{P}$. All reduced spaces are points, but the multiplicity $\mathrm{Q}^{\mathrm{spin}}(M_{\mathcal{P}})$ are equal to $-1$, following from the orientation rule. On the picture, the links between admissible regular orbits $\mathcal{O}$ and their ancestors $\mathcal{P}$ are indicated by segments. We see that the orbit $\mathcal{O}(\rho)$ of $\rho$ has two ancestors $\mathcal{P}_1$ and $\mathcal{P}_2$, so that the multiplicity of the trivial representation is equal to

$$\mathrm{Q}^{\mathrm{spin}}(M_{\mathcal{P}_1}) + \mathrm{Q}^{\mathrm{spin}}(M_{\mathcal{P}_2}) = -2$$

and comes from two different faces of the Weyl chamber.

If the generic stabilizer of the action of $K$ in $M$ is abelian, expression (1.1) simplifies as follows. Consider the slice $Y = \Phi_{\mathcal{S}}^{-1}(\mathfrak{t}^*_{>0})$ which is a $T$-invariant submanifold. Let $\Phi_Y$ be the restriction of $\Phi_{\mathcal{S}}$ to $Y$. If $\mathcal{Q}_K(M, \mathcal{S})$ is non zero, then $KY$ is a dense open subset of $M$, and we have simply

$$\mathrm{m}_\lambda = \mathrm{Q}^{\mathrm{spin}}(Y_\lambda) \tag{1.3}$$

where $Y_\lambda = \Phi_Y^{-1}(\lambda)/T$. In other words

$$\mathcal{Q}_K(M,\mathcal{S}) = \sum_{\lambda\in\Lambda_{\geqslant 0}+\rho} \mathrm{Q}^{\mathrm{spin}}(Y_\lambda)\,\pi_{K\lambda}.$$

In particular, if the group $K$ is the circle group, and $\lambda$ is a regular value of the moment map $\Phi$, Identity (1.3) was obtained in [5].

### 1.2 Techniques of the proof

Consider the Kirwan vector field $\kappa_{\mathcal{S}}$ on $M$: at $m \in M$, $\kappa_{\mathcal{S}}$ is the tangent vector obtained by the infinitesimal action of $-\Phi_{\mathcal{S}}(m)$ at $m \in M$ (we have identified $\mathfrak{k}$ and $\mathfrak{k}^*$). We use a topological deformation $\sigma_\kappa$ of the symbol $\sigma$ of the Dirac operator $D$ by pushing the zero section of $\mathrm{T}^*M$ inside $\mathrm{T}^*M$ using the Kirwan vector field $\kappa_{\mathcal{S}}$. We call this deformation the Witten deformation, as it was used by Witten (in the symplectic setting) to show that the computation of integrals of equivariant cohomology classes on $M$ reduces to the study of contributions coming from a neighborhood of $Z_{\mathcal{S}}$, the set of zeroes of $\kappa_{\mathcal{S}}$, leading to the so called non abelian localization formula.

Here we apply the same technique to compute the index $\mathcal{Q}_K(M,\mathcal{S})$ as a sum of equivariant indices of transversally elliptic operators associated to connected components $Z$ of $Z_{\mathcal{S}}$. We are able to identify them to some basic transversally elliptic symbols whose indices were computed by Atiyah-Singer (see [1]). Although these indices are infinite dimensional representations, they are easier to understand than the original finite dimensional representation $\mathcal{Q}_K(M,\mathcal{S})$ (an analogue, strongly related via the theory of toric manifolds, is the Brianchon-Gram decomposition of the characteristic function of a compact convex polytope $P$ as an alternate sum of characteristic functions of cones). We give an example of the decomposition of the representation $\mathcal{Q}_K(M,\mathcal{S})$ in Subsection 4.2.

All properties of the **K**-theory version of Witten deformation that we use here were previously proved in [20]. However, we have written in [24] a hopefully more readable description of the functorial properties of this non abelian localization formula in **K**-theory.

To compute the multiplicity of $\pi_{\mathcal{O}}$ in $\mathcal{Q}_K(M,\mathcal{S})$, we use the shifting trick and compute the $K$-invariant part of the equivariant index $\mathcal{Q}_K(P,\mathcal{S}_P)$ where $P$ is the product manifold $M \times \mathcal{O}^*$. Let $Z_P$ be the zero set of the corresponding Kirwan vector field $\kappa_P$ and $\sigma_\kappa$ the deformed symbol. The computation of the equivariant index is thus reduced to the study of the deformed symbol $\sigma_\kappa$ in a neighborhood of $Z_P$. We have to single out the components $Z$ such that the trivial representation of $K$ occurs with non

zero multiplicity. Here is where we discover that, for $\mathcal{Q}_K(M,\mathcal{S})$ to be non zero, it is necessary that the semi-simple part of the generic stabilizer $\mathfrak{k}_M$ of the action of $K$ on $M$ is equal to the semi-simple part of a Levi subalgebra $\mathfrak{h}$ of $\mathfrak{k}$. It follows that such a component $Z$ is described rather simply as an induced manifold $K \times_H (Y \times o(\mathfrak{h}))$, with $Y$ a $H/[H,H]$ manifold, and $o(\mathfrak{h})$ the $[H,H]$-orbit of the corresponding $\rho^{[H,H]}$ element. Then we use the fact that the quantization of the orbit of $\rho$ is the trivial representation. In fact, to determine the contributing components $Z$, we study a function $d_P : Z_P \to \mathbb{R}$ relating the representation of $K_m$ on $\mathrm{T}_m M$ and the norm of $\Phi_{\mathcal{S}}(m)$. Here $K_m$ is the stabilizer of $m \in M$. It relies on the "magical inequality" (Corollary 3.15) on distance of regular weights to faces of the Weyl chamber. This step differs from the crucial step in the proof of $[Q,R]=0$ theorem in the symplectic case. Both theorems are somewhat both magical, but each one on its own. It maybe useful for the reader to read first [24], where we recall the first author proof of $[Q,R]=0$ in the Hamiltonian case, where the strategy is straightforward. This strategy is also explained in more combinatorial terms in Szenes-Vergne [26].

### 1.3 Outline of the article

Let us explain the contents of the different sections of the article, and their main use in the final proof.

• In Section 2, we give the definition of the index of a Spin$^c$-bundle.

• In Section 3, we describe the canonical Spin$^c$-bundle on admissible coadjoint orbits (see (3.12)). For a $K$-admissible coadjoint orbit $\mathcal{P}$, we determine the regular admissible orbit $\mathcal{O}$ such that if $\mathrm{Q}_K^{\mathrm{spin}}(\mathcal{P})$ is not zero, then $\mathrm{Q}_K^{\mathrm{spin}}(\mathcal{P}) = \pi_{\mathcal{O}}$ (Proposition 3.8).

We prove the magical inequality (Corollary 3.15) on distance of the shifted Weyl chamber $\rho^K + \mathfrak{t}^*_{\geqslant 0}$ to admissible $\mu \in \mathfrak{t}^*$ (that is $K\mu$ is an admissible orbit). This inequality on Weyl chambers will be used over and over again in this article.

• In Section 4, we define the Witten deformation and recall some of its properties (proved in [20, 24]). It allows us to reduce the computation of $\mathcal{Q}_K(M,\mathcal{S})$ to indices $q_Z$ of simpler transversally elliptic operators defined in neighborhoods of connected components of $Z_{\mathcal{S}} = \{\kappa_{\mathcal{S}} = 0\}$.

We introduce a function $d_{\mathcal{S}} : Z_{\mathcal{S}} \to \mathbb{R}$. If $d_{\mathcal{S}}$ takes strictly positive values on some component $Z$ of $Z_{\mathcal{S}}$, then the $K$-invariant part of the (virtual) representation $q_Z$ is equal to 0 (Proposition 4.17). This is a very important technical proposition.

If $\mathcal{O}$ is an admissible regular coadjoint orbit, the shifting trick leads us

to study the manifold $M \times \mathcal{O}^*$ with Spin$^c$-bundle $\mathcal{S} \otimes \mathcal{S}_{\mathcal{O}^*}$. We want to select the component $Z$ of $Z_{\mathcal{S}\otimes\mathcal{S}_{\mathcal{O}^*}}$ so that $[q_Z]^K$ is not zero. Theorem 4.22 summarizes the geometric structure enjoyed by $M$ and $\mathcal{S}$ when there exists such a component. Although this theorem is natural (as we tried to explain it in the introduction), we are able to obtain it only using Witten deformation on $M \times \mathcal{O}^*$ (for all regular admissible orbits $\mathcal{O}$) and a careful study of the function $d_{\mathcal{S}\otimes\mathcal{S}_{\mathcal{O}^*}}$.

We show that the components $Z$ for which $[q_Z]^K \neq 0$ are contained in the subsets $\Phi_{\mathcal{S}}^{-1}(\mathcal{P}) \times \mathcal{O}^*$ of $M \times \mathcal{O}^*$ where $\mathcal{P}$ is a $(\mathfrak{h})$-ancestor to $\mathcal{O}$ (Proposition 4.24).

We then obtain that the multiplicity $\mathrm{m}_{\mathcal{O}}$ of $\pi_{\mathcal{O}}$ in $\mathcal{Q}_K(M, \mathcal{S})$ is the sum $\sum_{\mathcal{P}} \mathrm{m}_{\mathcal{O}}^{\mathcal{P}}$ parametrized by the $(\mathfrak{h})$-ancestors of $\mathcal{O}$. In Proposition 4.25, we prove that each term $\mathrm{m}_{\mathcal{O}}^{\mathcal{P}}$ is independent of the choice of the connection.

• In Section 5, we prove that $\mathrm{m}_{\mathcal{O}}^{\mathcal{P}}$ is equal to $\mathrm{Q}^{\mathrm{spin}}(M_{\mathcal{P}})$. Here we explain how to define indices on singular reduced spaces. The main theorem is their invariance under small deformation.

We then have done all the work needed to be able to prove the main theorem.

We finally verify that (fortunately) the statement $[Q, R] = 0$ in the Spin$^c$ case is compatible with Spin$^c$ induction.

• The last section is dedicated to some examples.

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### Notations

Throughout the paper :

- $K$ denotes a compact connected Lie group with Lie algebra $\mathfrak{k}$.
- $T$ is a maximal torus in $K$ with Lie algebra $\mathfrak{t}$.
- $\Lambda \subset \mathfrak{t}^*$ is the weight lattice of $T$ : every $\mu \in \Lambda$ defines a 1-dimensional $T$-representation, denoted $\mathbb{C}_\mu$, where $t = \exp(X)$ acts by $t^\mu := e^{i\langle \mu, X\rangle}$.
- We fix a $K$-invariant inner product $(\cdot, \cdot)$ on $\mathfrak{k}$. This allows us to identify $\mathfrak{k}$ and $\mathfrak{k}^*$ when needed.

  We denote by $\langle \cdot, \cdot \rangle$ the natural duality between $\mathfrak{k}$ and $\mathfrak{k}^*$.

- We denote by $R(K)$ the representation ring of $K$ : an element $E \in R(K)$ can be represented as finite sum $E = \sum_{\mu \in \widehat{K}} \mathrm{m}_\mu \pi_\mu$, with $\mathrm{m}_\mu \in \mathbb{Z}$. The multiplicity of the trivial representation is denoted $[E]^K$.
- We denote by $\hat{R}(K)$ the space of $\mathbb{Z}$-valued functions on $\hat{K}$. An element $E \in \hat{R}(K)$ can be represented as an infinite sum $E = \sum_{\mu \in \widehat{K}} \mathrm{m}(\mu) \pi_\mu$, with $\mathrm{m}(\mu) \in \mathbb{Z}$.
- If $H$ is a closed subgroup of $K$, the induction map $\mathrm{Ind}_H^K : \hat{R}(H) \to \hat{R}(K)$ is the dual of the restriction morphism $R(K) \to R(H)$. In particular $[\mathrm{Ind}_H^K(E)]^K = [E]^H$.
- When $K$ acts on a set $X$, the stabilizer subgroup of $x \in X$ is denoted $K_x := \{k \in K \mid k \cdot x = x\}$. The Lie algebra of $K_x$ is denoted $\mathfrak{k}_x$.
- An element $\xi \in \mathfrak{k}^*$ is called regular if $K_\xi$ is a maximal torus of $K$.
- When $K$ acts on a manifold $M$, we denote $X_M(m) := \frac{d}{dt}|_{t=0} e^{-tX} \cdot m$ the vector field generated by $-X \in \mathfrak{k}$. Sometimes we will also use the notation $X_M(m) = -X \cdot m$. The set of zeroes of the vector field $X_M$ is denoted $M^X$.
- If $V$ is a complex (ungraded) vector space, then the exterior space $\bigwedge V = \bigwedge^+ V \oplus \bigwedge^- V$ will be $\mathbb{Z}/2\mathbb{Z}$ graded in even and odd elements.
- If $E_1 = E_1^+ \oplus E_1^-$ and $E_2 = E_2^+ \oplus E_2^-$ are two $\mathbb{Z}/2\mathbb{Z}$ graded vector spaces (or vector bundles), the tensor product $E_1 \otimes E_2$ is $\mathbb{Z}/2\mathbb{Z}$-graded with $(E_1 \otimes E_2)^+ = E_1^+ \otimes E_2^+ \oplus E_1^- \otimes E_2^-$ and $(E_1 \otimes E_2)^- = E_1^- \otimes E_2^+ \oplus E_1^+ \otimes E_2^-$. Similarly the spaces $\mathrm{End}(E_i)$ are $\mathbb{Z}/2\mathbb{Z}$ graded. The action of $\mathrm{End}(E_1) \otimes \mathrm{End}(E_2)$ on $E_1 \otimes E_2$ obeys the usual sign rules: for example, if $f \in \mathrm{End}(E_2)^-$, $v_1 \in E_1^-$ and $v_2 \in E_2$, then $f(v_1 \otimes v_2) = -v_1 \otimes f v_2$.
- If $E$ is a vector space and $M$ a manifold, we denote by $[E]$ the trivial vector bundle on $M$ with fiber $E$.

# 2 Spin$^c$ equivariant index

## 2.1 Spin$^c$ modules

Let $V$ be an oriented Euclidean space of even dimension $n = 2\ell$. We denote by $\mathrm{Cl}(V)$ its Clifford algebra. If $e_1, \ldots, e_n$ is an oriented orthonormal frame of $V$, we define the element

$$\epsilon := (i)^\ell e_1 \cdots e_n \in \mathrm{Cl}(V)$$

that depends only of the orientation. We have $\epsilon^2 = 1$ and $\epsilon v = -v\epsilon$ for any $v \in V$.

If $E$ is a $\mathrm{Cl}(V)$-module, the Clifford map is denoted $\mathbf{c}_E : \mathrm{Cl}(V) \to \mathrm{End}(E)$. We see then that the element of order two $\epsilon_E := \mathbf{c}_E(\epsilon)$ defines a $\mathbb{Z}/2\mathbb{Z}$-graduation on $E$ by defining $E^{\pm} := \ker(\mathrm{Id}_E \mp \epsilon_E)$. Moreover the maps $\mathbf{c}_E(v) : E \to E$ for $v \in V$ interchange the subspaces $E^+$ and $E^-$. This graduation will be called the canonical graduation of the Clifford module $E$.

We follow the conventions of [3]. Recall the following fundamental fact.

**Proposition 2.1** *Let $V$ be an even dimensional Euclidean space.*

- *There exists a complex $\mathrm{Cl}(V)$-module $S$ such that the Clifford morphism $\mathbf{c}_S : \mathrm{Cl}(V) \to \mathrm{End}(S)$ induces an isomorphism of complex algebra $\mathrm{Cl}(V) \otimes \mathbb{C} \simeq \mathrm{End}(S)$.*

- *The Clifford module $S$ is unique up to isomorphism. We call it the spinor $\mathrm{Cl}(V)$-module.*

- *Any complex $\mathrm{Cl}(V)$-module $E$ has the following decomposition*

$$E \simeq S \otimes \hom_{\mathrm{Cl}(V)}(S, E) \tag{2.4}$$

  *where $\hom_{\mathrm{Cl}(V)}(S, E)$ is the vector space spanned by the $\mathrm{Cl}(V)$-complex linear maps from $S$ to $E$. If $V$ is oriented and the Clifford modules $S$ and $E$ carry their canonical grading, then (2.4) is an isomorphism of graded Clifford $\mathrm{CL}(V)$-modules.*

Let $V = V_1 \oplus V_2$ be an orthogonal decomposition of even dimensional Euclidean spaces. We choose an orientation $o(V_1)$ on $V_1$. There is a one-to-one correspondence between the graded $\mathrm{Cl}(V_2)$-modules and the graded $\mathrm{Cl}(V)$-modules defined as follows. Let $S_1$ be the spinor module for $\mathrm{Cl}(V_1)$. If $W$ is a $\mathrm{Cl}(V_2)$-module, the vector space $E := S_1 \otimes W$ is a $\mathrm{Cl}(V)$-module with the Clifford map defined by

$$\mathbf{c}_E(v_1 \oplus v_2) := \mathbf{c}_{S_1}(v_1) \otimes \mathrm{Id}_W + \epsilon_{S_1} \otimes \mathbf{c}_W(v_2).$$

Here $v_i \in V_i$ and $\epsilon_{S_1} \in \mathrm{End}(S_1)$ defines the canonical graduation of $S_1$. Conversely, if $E$ is a graded $\mathrm{Cl}(V)$-module, the vector space $W := \hom_{\mathrm{Cl}(V_1)}(S_1, E)$ formed by the complex linear maps $f : S_1 \to E$ commuting with the action of $\mathrm{Cl}(V_1)$ has a natural structure of $\mathrm{Cl}(V_2)$ graded module and $E \simeq S_1 \otimes W$.

If we fix an orientation $o(V)$ on $V$, it fixes an orientation $o(V_2)$ on $V_2$ by the relation $o(V) = o(V_1)o(V_2)$. Then the Clifford modules $E$ and $W$ carries

their canonical $\mathbb{Z}/2\mathbb{Z}$ graduation, and $E \simeq S_1 \otimes W$ becomes an identity of graded Clifford modules.

**Example 2.2** *Let $H$ be an Euclidean vector space equipped with a complex structure $J \in O(H)$: we denote by $\bigwedge_J H$ the exterior product of the space $H$ considered as a complex vector space with complex structure $J$. Denote by $m(v)$ the exterior multiplication by $v$. The action $\mathbf{c}$ of $H$ on $\bigwedge_J H$ given by $\mathbf{c}(v) = m(v) - m(v)^*$ satisfies $c(v)^2 = -\|v\|^2 \mathrm{Id}$. Thus, $\bigwedge_J H$, equipped with the action $\mathbf{c}$, is a realization of the spinor module for $H$. Note that the group $\mathrm{U}(J)$ of unitary transformations of $H$ acts naturally on $\bigwedge_J H$. If one choose the orientation on $H$ induced by the complex structure, one sees that the canonical grading is $(\bigwedge_J H)^{\pm} = \bigwedge_J^{\pm} H$.*

*Consider another complex structure $J' \in O(H)$ : the vector space $\bigwedge_{J'} H$ is another spinor module for $H$. We denote by $\epsilon_J^{J'}$ the ratio between the orientations defined by $J$ and $J'$. One can check that*

$$\bigwedge_{J'} H \simeq \epsilon_J^{J'} \, \mathbb{C}_\chi \otimes \bigwedge_J H, \tag{2.5}$$

*as a graded $\mathrm{Cl}(H)$-module and also as a graded $\mathrm{U}(J') \cap \mathrm{U}(J)$-module. Here $\mathbb{C}_\chi$ is the 1-dimensional representation of $\mathrm{U}(J') \cap \mathrm{U}(J)$ associated to the unique character $\chi$ defined by the relation $\chi(g)^2 = \det_{J'}(g) \det_J(g)^{-1}$, $\forall g \in \mathrm{U}(J') \cap \mathrm{U}(J)$.*

**Example 2.3** *When $V = Q \oplus Q$ with $Q$ an Euclidean space, we can identify $V$ with $Q_{\mathbb{C}}$ by $(x, y) \to x \oplus iy$. Thus $S_Q := \bigwedge Q_{\mathbb{C}}$ is a realization of the spinor module for $V$. It carries a natural action of the orthogonal group $\mathrm{O}(Q)$ acting diagonally. If $Q$ carries a complex structure $J \in O(Q)$, we can consider the spin modules $\bigwedge_J Q$ and $\bigwedge_{-J} Q$ for $Q$. We have then the isomorphism $S_Q \simeq \bigwedge_J Q \otimes \bigwedge_{-J} Q$ of graded $\mathrm{Cl}(V)$-modules (it is also an isomorphism of $\mathrm{U}(J)$-modules).*

## 2.2 Spin$^c$ structures

Consider now the case of an Euclidean vector bundle $\mathcal{V} \to M$ of *even rank*. Let $\mathrm{Cl}(\mathcal{V}) \to M$ be the associated Clifford algebra bundle. A complex vector bundle $\mathcal{E} \to M$ is a $\mathrm{Cl}(\mathcal{V})$-module if there is a bundle algebra morphism $\mathbf{c}_{\mathcal{E}} : \mathrm{Cl}(\mathcal{V}) \longrightarrow \mathrm{End}(\mathcal{E})$.

**Definition 2.4** *Let $\mathcal{S} \to M$ be a $\mathrm{Cl}(\mathcal{V})$-module such that the map $\mathbf{c}_{\mathcal{S}}$ induces an isomorphism $\mathrm{Cl}(\mathcal{V}) \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow \mathrm{End}(\mathcal{S})$. Then we say that $\mathcal{S}$ is a $\mathrm{Spin}^c$-bundle for $\mathcal{V}$.*

As in the linear case, an orientation on the vector bundle $\mathcal{V}$ determines a $\mathbb{Z}/2\mathbb{Z}$ grading of the vector bundle $\mathcal{S}$ (called the canonical graduation) such that for any $v \in \mathcal{V}_m$, the linear map[1] $\mathbf{c}_{\mathcal{S}}(m, v) : \mathcal{S}_m \to \mathcal{S}_m$ is odd.

**Example 2.5** *When $\mathcal{H} \to M$ is a Hermitian vector bundle, the complex vector bundle $\bigwedge \mathcal{H}$ is a $\mathrm{Spin}^c$ bundle for $\mathcal{H}$. If one choose the orientation of the vector bundle $\mathcal{H}$ induced by the complex structure, one sees that the canonical grading is $(\bigwedge \mathcal{H})^{\pm} = \bigwedge^{\pm} \mathcal{H}$.*

We assume that the vector bundle $\mathcal{V}$ is oriented, and we consider two $\mathrm{Spin}^c$-bundles $\mathcal{S}, \mathcal{S}'$ for $\mathcal{V}$, both with their canonical grading. We have the following identity of graded $\mathrm{Spin}^c$-bundles : $\mathcal{S}' \simeq \mathcal{S} \otimes \mathbb{L}_{\mathcal{S},\mathcal{S}'}$ where $\mathbb{L}_{\mathcal{S},\mathcal{S}'}$ is a complex line bundle on $M$ defined by the relation

$$\mathbb{L}_{\mathcal{S},\mathcal{S}'} := \hom_{\mathrm{Cl}(\mathcal{V})}(\mathcal{S}, \mathcal{S}'). \tag{2.6}$$

**Definition 2.6** *Let $\mathcal{V} \to M$ be an Euclidean vector bundle of even rank. The determinant line bundle of a $\mathrm{Spin}^c$-bundle $\mathcal{S}$ on $\mathcal{V}$ is the line bundle $\mathbb{L}_{\mathcal{S}} \to M$ defined by the relation*

$$\mathbb{L}_{\mathcal{S}} := \hom_{\mathrm{Cl}(\mathcal{V})}(\overline{\mathcal{S}}, \mathcal{S})$$

*where $\overline{\mathcal{S}}$ is the $\mathrm{Cl}(\mathcal{V})$-module with opposite complex structure. Sometimes $\mathbb{L}_{\mathcal{S}}$ is also denoted $\det(\mathcal{S})$.*

**Example 2.7** *When $\mathcal{H} \to M$ is a Hermitian vector bundle, the determinant line bundle of the $\mathrm{Spin}^c$-bundle $\bigwedge \mathcal{H}$ is $\det(\mathcal{H}) := \bigwedge^{\max} \mathcal{H}$.*

If $\mathcal{S}$ and $\mathcal{S}'$ are two $\mathrm{Spin}^c$-bundles for $\mathcal{V}$, we see that

$$\mathbb{L}_{\mathcal{S}'} = \mathbb{L}_{\mathcal{S}} \otimes (\mathbb{L}_{\mathcal{S},\mathcal{S}'})^{\otimes 2}.$$

Assume that $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2$ is an orthogonal sum of Euclidean vector bundles of even rank. We assume that $\mathcal{V}_1$ is oriented, and let $\mathcal{S}_1$ be a $\mathrm{Spin}^c$-bundle for $\mathcal{V}_1$ that we equip with its canonical grading. If $\mathcal{E}$ is a Clifford bundle for $\mathcal{V}$, then we have the following isomorphism[2]

$$\mathcal{E} \simeq \mathcal{S}_1 \otimes \mathcal{W} \tag{2.7}$$

[1] The map $\mathbf{c}_{\mathcal{S}}(m, -) : \mathcal{V}_m \to \mathrm{End}(\mathcal{S}_m)$ will also be denoted by $\mathbf{c}_{\mathcal{S}_m}$.

[2] The proof is identical to the linear case explained earlier.

where $\mathcal{W} := \hom_{\mathrm{Cl}(\mathcal{V}_1)}(\mathcal{S}_1, \mathcal{E})$ is a Clifford bundle for $\mathcal{V}_2$. If $\mathcal{V}$ is oriented, it fixes an orientation $o(\mathcal{V}_2)$ on $\mathcal{V}_2$ by the relation $o(\mathcal{V}) = o(\mathcal{V}_1)o(\mathcal{V}_2)$. Then the Clifford modules $\mathcal{E}$ and $\mathcal{W}$ carries their canonical $\mathbb{Z}/2\mathbb{Z}$ grading, and (2.7) becomes an identity of graded Clifford modules.

In the particular situation where $\mathcal{S}$ is a Spin$^c$-bundle for $\mathcal{V}$, then $\mathcal{S} \simeq \mathcal{S}_1 \otimes \mathcal{S}_2$ where $\mathcal{S}_2 := \hom_{\mathrm{Cl}(\mathcal{V}_1)}(\mathcal{S}_1, \mathcal{S})$ is a Spin$^c$-bundle for $\mathcal{V}_2$. At the level of determinant line bundles we obtain $\mathbb{L}_{\mathcal{S}} = \mathbb{L}_{\mathcal{S}_1} \otimes \mathbb{L}_{\mathcal{S}_2}$.

Let us end this section by recalling the notion of Spin-structure and Spin$^c$-structure. Let $\mathcal{V} \to M$ be an oriented Euclidean vector bundle of rank $n$, and let $\mathrm{P}_{SO}(\mathcal{V})$ be its orthogonal frame bundle : it is a principal $\mathrm{SO}_n$ bundle over $M$.

Let us consider the spinor group $\mathrm{Spin}_n$ which is the double cover of the group $\mathrm{SO}_n$. The group $\mathrm{Spin}_n$ is a subgroup of the group $\mathrm{Spin}_n^c$ which covers $\mathrm{SO}_n$ with fiber $U(1)$.

A Spin structure on $\mathcal{V}$ is a $\mathrm{Spin}_n$-principal bundle $\mathrm{P}_{Spin}(\mathcal{V})$ over $M$ together with a $\mathrm{Spin}_n$- equivariant map $\mathrm{P}_{Spin}(\mathcal{V}) \to \mathrm{P}_{SO}(\mathcal{V})$.

We assume now that $\mathcal{V}$ is of even rank $n = 2\ell$. Let $\mathrm{S}_n$ be the irreducible complex spin representation of $\mathrm{Spin}_n$. Recall that $\mathrm{S}_n = \mathrm{S}_n^+ \oplus \mathrm{S}_n^-$ inherits a canonical Clifford action $c : \mathbb{R}^n \to \mathrm{End}(\mathrm{S}_n)$ which is $\mathrm{Spin}_n$-equivariant, and which interchanges the graduation: $c(v) : \mathrm{S}_n^{\pm} \to \mathrm{S}_n^{\mp}$. The spinor bundle attached to the Spin-structure $\mathrm{P}_{Spin}(\mathcal{V})$ is

$$\mathcal{S} := \mathrm{P}_{Spin}(\mathcal{V}) \times_{\mathrm{Spin}_n} \mathrm{S}_n.$$

A Spin$^c$-bundle for $\mathcal{V}$ determines a Spin$^c$ structure, that is a principal bundle over $M$ with structure group $\mathrm{Spin}_n^c$. When $\mathcal{V}$ admits a Spin-structure, any Spin$^c$-bundle for $\mathcal{V}$ is of the form $\mathcal{S}_L = \mathcal{S}_{\mathrm{spin}} \otimes L$ where $\mathcal{S}_{\mathrm{spin}}$ is the spinor bundle attached to the Spin-structure and $L$ is a line bundle on $M$. Then the determinant line bundle for $\mathcal{S}_L$ is $L^{\otimes 2}$.

## 2.3 Moment maps and Kirwan vector field

In this section, we consider the case of a Riemannian manifold $M$ acted on by a compact Lie group $K$. Let $\mathcal{S} \to M$ be a Spin$^c$-bundle on $M$. If the $K$-action lifts to the Spin$^c$-bundle $\mathcal{S}$ in such a way that the bundle map $\mathrm{c}_{\mathcal{S}} : \mathrm{Cl}(\mathrm{T}M) \to \mathrm{End}(\mathcal{S})$ commutes with the $K$-action, we say that $\mathcal{S}$ defines a $K$-equivariant Spin$^c$-bundle on $M$. In this case, the $K$-action lifts also to the determinant line bundle $\mathbb{L}_{\mathcal{S}}$. The choice of an invariant Hermitian connection $\nabla$ on $\mathbb{L}_{\mathcal{S}}$ determines an equivariant map $\Phi_{\mathcal{S}} : M \to \mathfrak{k}^*$ and a

2-form $\Omega_{\mathcal{S}}$ on $M$ by means of the Kostant relations

$$\mathcal{L}(X) - \nabla_{X_M} = 2i\langle \Phi_{\mathcal{S}}, X\rangle \quad \text{and} \quad \nabla^2 = -2i\Omega_{\mathcal{S}} \tag{2.8}$$

for every $X \in \mathfrak{k}$. Here $\mathcal{L}(X)$ denotes the infinitesimal action of $X \in \mathfrak{k}$ on the sections of $\mathbb{L}_{\mathcal{S}}$. We will say that $\Phi_{\mathcal{S}}$ is the moment map for $\mathcal{S}$ (it depends however of the choice of a connection).

Via the equivariant Bianchi formula, Relations (2.8) induce the relations

$$\iota(X_M)\Omega_{\mathcal{S}} = -d\langle \Phi_{\mathcal{S}}, X\rangle \quad \text{and} \quad d\Omega_{\mathcal{S}} = 0 \tag{2.9}$$

for every $X \in \mathfrak{k}$. It follows that $\Phi_{\mathcal{S}}$ is a moment map, as defined in [24].

In particular the function $m \to \langle \Phi_{\mathcal{S}}(m), X\rangle$ is locally constant on $M^X$.

**Remark 2.8** *Let $b \in \mathfrak{k}$ and $m \in M^b$, the set of zeroes of $b_M$. We consider the linear actions $\mathcal{L}(b)|_{\mathcal{S}_m}$ and $\mathcal{L}(b)|_{\mathbb{L}_{\mathcal{S}m}}$ on the fibers at $m$ of the* Spin$^c$*-bundle $\mathcal{S}$ and the line bundle $\mathbb{L}_{\mathcal{S}}$. Kostant relations imply $\mathcal{L}(b)|_{\mathbb{L}_{\mathcal{S}m}} = 2i\langle \Phi_{\mathcal{S}}(m), b\rangle$. The irreducibility of $\mathcal{S}$ implies that*

$$\mathcal{L}(b)|_{\mathcal{S}_m} = i\,\langle \Phi_{\mathcal{S}}(m), b\rangle\, \mathrm{Id}_{\mathcal{S}_m}.$$

*Furthermore the function $m \to \langle \Phi_{\mathcal{S}}(m), b\rangle$ is locally constant on $M^b$.*

Note that the closed 2-form $\Omega_{\mathcal{S}}$, which is *half* of the curvature of $\mathbb{L}_{\mathcal{S}}$, is not (in general) a symplectic form. Furthermore, if we take any (real valued) invariant 1-form $A$ on $M$, $\nabla + iA$ is another connection on $\mathbb{L}_{\mathcal{S}}$. The corresponding curvature and moment map will be modified in $\Omega_{\mathcal{S}} - \frac{1}{2}dA$ and $\Phi_{\mathcal{S}} - \frac{1}{2}\Phi_A$ where $\Phi_A : M \to \mathfrak{k}^*$ is defined by the relation $\langle \Phi_A, X\rangle = -\iota(X_M)A$.

Let $\Phi : M \to \mathfrak{k}$ be a $K$-equivariant map. We define the $K$-invariant vector field $\kappa_\Phi$ on $M$ by

$$\kappa_\Phi(m) := -\Phi(m) \cdot m, \tag{2.10}$$

and we call it the **Kirwan vector field** associated to $\Phi$. The set where $\kappa_\Phi$ vanishes is a K-invariant subset that we denote by $Z_\Phi \subset M$.

We identify $\mathfrak{k}^*$ to $\mathfrak{k}$ by our choice of $K$-invariant scalar product and we will have a particular interest in the equivariant map $\Phi_{\mathcal{S}} : M \to \mathfrak{k}^* \simeq \mathfrak{k}$ associated to the Spin$^c$-bundle $\mathcal{S}$. In this case we may denote the $K$-invariant vector field $\kappa_{\Phi_{\mathcal{S}}}$ simply by $\kappa_{\mathcal{S}}$ (even if it depends of the choice of a connection):

$$\kappa_{\mathcal{S}}(m) := -\Phi_{\mathcal{S}}(m) \cdot m.$$

and we denote $Z_\Phi$ by $Z_{\mathcal{S}}$.

As $\Phi_{\mathcal{S}}$ is a moment map, we have the following basic description (see [20, 24]).

**Lemma 2.9** *If the manifold $M$ is compact, the set $\Phi_{\mathcal{S}}(Z_{\mathcal{S}})$ is a* finite *collection of coadjoint orbits. For any coadjoint orbit $\mathcal{O} = K\beta$, we have*

$$Z_{\mathcal{S}} \cap \Phi_{\mathcal{S}}^{-1}(\mathcal{O}) = K(M^{\beta} \cap \Phi_{\mathcal{S}}^{-1}(\beta)).$$

*Here we have identified $\beta \in \mathfrak{k}^*$ to an element in $\mathfrak{k}$ still denoted by $\beta$. Furthermore, any $\beta$ in the image $\Phi_{\mathcal{S}}(Z_{\mathcal{S}})$ is such that $\|\beta\|^2$ is a critical value of the map $\|\Phi_{\mathcal{S}}\|^2$.*

**Remark 2.10** *Although the map $\Phi_{\mathcal{S}}$ as well as the set $Z_{\mathcal{S}}$ vary when we vary the connection, we see that the image $\Phi_{\mathcal{S}}(Z_{\mathcal{S}})$ is contained in a finite set of coadjoint orbits that does not depend of the connection (see [24]).*

Figure 3 describes the set $\Phi_{\mathcal{S}}(Z_{\mathcal{S}})$ for the action of the diagonal torus of $K = \mathrm{SU}(3)$ on the orbit $K\rho$ equipped with its canonical Spin$^c$-bundle.

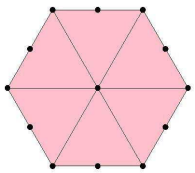

Figure 3: The set $\Phi_{\mathcal{S}}(Z_{\mathcal{S}})$

## 2.4 Equivariant index

Assume in this section that the Riemannian $K$-manifold $M$ is compact, even dimensional, oriented, and equipped with a $K$-equivariant Spin$^c$-bundle $\mathcal{S} \to M$. The orientation induces a decomposition $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$, and the corresponding Spin$^c$ Dirac operator is a first order elliptic operator $\mathcal{D}_{\mathcal{S}} : \Gamma(M, \mathcal{S}^+) \to \Gamma(M, \mathcal{S}^-)$ [3, 7]. Its principal symbol is the bundle map $\sigma(M, \mathcal{S}) \in \Gamma(\mathrm{T}^*M, \hom(p^*\mathcal{S}^+, p^*\mathcal{S}^-))$ defined by the relation

$$\sigma(M, \mathcal{S})(m, \nu) = \mathbf{c}_{\mathcal{S}_m}(\tilde{\nu}) : \mathcal{S}_m^+ \longrightarrow \mathcal{S}_m^-.$$

Here $\nu \in \mathrm{T}^*M \to \tilde{\nu} \in \mathrm{T}M$ is the identification defined by the Riemannian structure.

If $\mathcal{W} \to M$ is a complex $K$-vector bundle, we can define similarly the twisted Dirac operator $\mathcal{D}_{\mathcal{S}}^{\mathcal{W}} : \Gamma(M, \mathcal{S}^+ \otimes \mathcal{W}) \to \Gamma(M, \mathcal{S}^- \otimes \mathcal{W})$.

**Definition 2.11** *Let $\mathcal{S} \to M$ be an equivariant* Spin$^c$*-bundle. We denote :*

- $\mathcal{Q}_K(M, \mathcal{S}) \in R(K)$ *the equivariant index of the operator* $\mathcal{D}_{\mathcal{S}}$*,*
- $\mathcal{Q}_K(M, \mathcal{S} \otimes \mathcal{W}) \in R(K)$ *the equivariant index of the operator* $\mathcal{D}_{\mathcal{S}}^{\mathcal{W}}$*.*

Let $\widehat{A}(M)(X)$ be the equivariant Â-genus class of $M$: it is an equivariant analytic function from a neighborhood of $0 \in \mathfrak{k}$ with value in the algebra of differential forms on $M$. Berline-Vergne equivariant index formula [3][Theorem 8.2] asserts that

$$\mathcal{Q}_K(M, \mathcal{S})(e^X) = (\frac{i}{2\pi})^{\frac{\dim M}{2}} \int_M e^{i(\Omega_{\mathcal{S}} + \langle \Phi_{\mathcal{S}}, X \rangle)} \widehat{A}(M)(X) \tag{2.11}$$

for $X \in \mathfrak{k}$ small enough. Here we write $\mathcal{Q}_K(M, \mathcal{S})(e^X)$ for the trace of the element $e^X \in K$ in the virtual representation $\mathrm{Q}_K(M, \mathcal{S})$ of $K$. It shows in particular that $\mathrm{Q}_K(M, \mathcal{S}) \in R(K)$ is a topological invariant : it only depends of the class of the equivariant form $\Omega_{\mathcal{S}} + \langle \Phi_{\mathcal{S}}, X \rangle$, which represents *half* of the first equivariant Chern class of the line bundle $\mathbb{L}_{\mathcal{S}}$.

**Example 2.12** *We consider the simplest case of the theory. Let $M := \mathbb{P}^1(\mathbb{C})$ be the projective space of (complex) dimension one. We write an element of $M$ as $[z_1, z_2]$ in homogeneous coordinates. Consider the (ample) line bundle $\mathcal{L} \to \mathbb{P}^1$, dual of the tautological bundle. Let $\mathcal{S}(n)$ be the* Spin$^c$*-bundle $\bigwedge_{\mathbb{C}} \mathrm{T}M \otimes \mathcal{L}^{\otimes n}$. The virtual representation $\mathcal{Q}_T(M, \mathcal{S}(n))$ is equal to $H^0(\mathbb{P}^1, \mathcal{O}(n)) - H^1(\mathbb{P}^1, \mathcal{O}(n))$. Then for $n \geqslant 0$,*

$$\mathcal{Q}_T(M, \mathcal{S}(n)) = \sum_{k=0}^{n} t^k.$$

*Here $T = \{t \in \mathbb{C}; |t| = 1\}$ acts on $[z_1, z_2]$ via $t \cdot [z_1, z_2] = [t^{-1} z_1, z_2]$.*

# 3 Coadjoint orbits and the magical inequality

In this section, we describe Spin$^c$-bundles on admissible coadjoint orbits of $K$ and the equivariant indices of the associated Dirac operators.

## 3.1 Conjugacy classes of centralizers

For any $\xi \in \mathfrak{k}^*$, the stabilizer $K_\xi$ is a connected subgroup of $K$ with same rank. We denote by $\mathfrak{k}_\xi$ its Lie algebra.

Let $\mathcal{H}_{\mathfrak{k}}$ be the set of conjugacy classes of the reductive algebras $\mathfrak{k}_\xi, \xi \in \mathfrak{k}^*$. The set $\mathcal{H}_{\mathfrak{k}}$ contains the conjugacy class formed by the Cartan sub-algebras.

It contains also $\mathfrak{k}$ (stabilizer of 0). A coadjoint orbit $\mathcal{O}$ belongs to the Dixmier sheet $\mathfrak{k}^*_{(\mathfrak{h})}$, for $(\mathfrak{h}) \in \mathcal{H}_\mathfrak{k}$ if $(\mathfrak{k}_\xi) = (\mathfrak{h})$ for (any) $\xi \in \mathcal{O}$.

**Remark 3.1** *If $\mathfrak{h} = \mathfrak{k}_\xi$, then $\mathfrak{h}_\mathbb{C}$ is the Levi subalgebra of the parabolic subalgebra determined by $\xi$. Parabolics are classified by subsets of simple roots. However, different conjugacy classes of parabolics might give rise to the same conjugacy class of Levi subalgebras (as seen immediately for type $A_n$).*

We denote by $\mathcal{S}_\mathfrak{k}$ the set of conjugacy classes of the semi-simple parts $[\mathfrak{h}, \mathfrak{h}]$ of the elements $(\mathfrak{h}) \in \mathcal{H}_\mathfrak{k}$.

**Lemma 3.2** *The map $(\mathfrak{h}) \to ([\mathfrak{h}, \mathfrak{h}])$ induces a bijection between $\mathcal{H}_\mathfrak{k}$ and $\mathcal{S}_\mathfrak{k}$*

**Proof.** Assume that $[\mathfrak{h}, \mathfrak{h}] = [\mathfrak{h}', \mathfrak{h}'] = \mathfrak{s}$. Consider $\mathfrak{n}$ the normalizer of $\mathfrak{s}$. Then $\mathfrak{h}$ and $\mathfrak{h}'$ are both contained in $\mathfrak{n}$. Let $\mathfrak{t}, \mathfrak{t}'$ be Cartan subalgebras of $\mathfrak{h}, \mathfrak{h}'$. Then $\mathfrak{t}$ and $\mathfrak{t}'$ are conjugated inside the normalizer of $\mathfrak{s}$. As $\mathfrak{h} = \mathfrak{s} + \mathfrak{t}$, we see that $\mathfrak{h}$ is conjugated to $\mathfrak{h}'$.

The connected Lie subgroup with Lie algebra $\mathfrak{h}$ is denoted $H$, that is if $\mathfrak{h} = \mathfrak{k}_\xi$, then $H = K_\xi$. We write $\mathfrak{h} = \mathfrak{z} \oplus [\mathfrak{h}, \mathfrak{h}]$ where $\mathfrak{z}$ is the center and $[\mathfrak{h}, \mathfrak{h}]$ is the semi-simple part of $\mathfrak{h}$. Thus $\mathfrak{h}^* = \mathfrak{z}^* \oplus [\mathfrak{h}, \mathfrak{h}]^*$ and $\mathfrak{z}^*$ is the set of elements in $\mathfrak{h}^*$ vanishing on the semi-simple part of $\mathfrak{h}$. We write $\mathfrak{k} = \mathfrak{h} \oplus [\mathfrak{z}, \mathfrak{k}]$, so we embed $\mathfrak{h}^*$ in $\mathfrak{k}^*$ as a $H$-invariant subspace, that is we consider an element $\xi \in \mathfrak{h}^*$ also as an element of $\mathfrak{k}^*$ vanishing on $[\mathfrak{z}, \mathfrak{k}]$.

### 3.2 Statement of results on admissible coadjoint orbits

We first define the $\rho$-orbit. Let $T$ be a Cartan subgroup of $K$. Then $\mathfrak{t}^*$ is imbedded in $\mathfrak{k}^*$ as the subspace of $T$-invariant elements. Choose a system of positive roots $\Delta^+ \subset \mathfrak{t}^*$, and let $\rho^K = \frac{1}{2}\sum_{\alpha>0} \alpha$. The definition of $\rho^K$ requires the choice of a Cartan subgroup $T$ and of a positive root system. However a different choice leads to a conjugate element. Thus we can make the following definition.

**Definition 3.3** *We denote by $o(\mathfrak{k})$ the coadjoint orbit of $\rho^K \in \mathfrak{k}^*$. We call $o(\mathfrak{k})$ the $\rho$-orbit.*

If $K$ is abelian, then $o(\mathfrak{k})$ is $\{0\}$.

The notion of admissible coadjoint orbit is defined in [6] for any real Lie group $G$. When $K$ is a compact connected Lie group, we adopt the following equivalent definition: a coadjoint orbit $\mathcal{O} \subset \mathfrak{k}^*$ is admissible if $\mathcal{O}$

carries a $K$-equivariant Spin$^c$-bundle $\mathcal{S}_{\mathcal{O}}$, such that the associated moment map is the injection $\mathcal{O} \hookrightarrow \mathfrak{k}^*$. If $K\xi$ is an admissible orbit, we also say that the element $\xi$ is admissible. An admissible coadjoint orbit $\mathcal{O}$ is oriented by its symplectic structure, and we denote by $Q_K^{\rm spin}(\mathcal{O}) := \mathcal{Q}_K(\mathcal{O}, \mathcal{S}_{\mathcal{O}})$ the corresponding equivariant spin$^c$ index.

We have $\langle \xi, [\mathfrak{k}_\xi, \mathfrak{k}_\xi]\rangle = 0$. The quotient space $\mathfrak{q} = \mathfrak{k}/\mathfrak{k}_\xi$ is equipped with the symplectic form $\Omega_\xi(\bar{X}, \bar{Y}) := \langle \xi, [X, Y]\rangle$, and with a unique $K_\xi$-invariant complex structure $J_\xi$ such that $\Omega_\xi(-, J_\xi -)$ is a scalar product. We denote by $\mathfrak{q}^\xi$ the space $\mathfrak{k}/\mathfrak{k}_\xi$ **considered as a complex vector space** via the complex structure $J_\xi$. Any element $X \in \mathfrak{k}_\xi$ defines a complex linear map $\mathrm{ad}(X) : \mathfrak{q}^\xi \to \mathfrak{q}^\xi$.

**Definition 3.4** • *For any $\xi \in \mathfrak{k}^*$, we denote by $\rho(\xi)$ the element of $\mathfrak{k}_\xi^*$ such that*

$$\langle \rho(\xi), X\rangle = \frac{1}{2i}\mathbf{Tr}_{\mathfrak{q}^\xi}\mathrm{ad}(X), \quad X \in \mathfrak{k}_\xi.$$

*We extend $\rho(\xi)$ to an element of $\mathfrak{k}^*$, that we still denote by $\rho(\xi)$.*

Thus, given a $\xi \in \mathfrak{t}^*$, and $H = K_\xi$, we have written $\rho^K$ as sum of $\rho^{K_\xi} + \rho(\xi)$, according to the decomposition $\mathfrak{k} = \mathfrak{k}_\xi \oplus \mathfrak{q}$.

If $i\theta : \mathfrak{k}_\xi \to i\mathbb{R}$ is the differential of a character of $K_\xi$, we denote by $\mathbb{C}_\theta$ the corresponding 1-dimensional representation of $K_\xi$, and by $[\mathbb{C}_\theta] = K \times_{K_\xi} \mathbb{C}_\theta$ the corresponding line bundle over the coadjoint orbit $K\xi \subset \mathfrak{k}^*$. The condition that $K\xi$ is admissible means that there exists a Spin$^c$-bundle $\mathcal{S}$ on $K\xi$ such that $\det(\mathcal{S}) = [\mathbb{C}_{2\xi}]$ ($2i\xi$ needs to be the differential of a character of $K_\xi$).

**Lemma 3.5** *1. $\langle \rho(\xi), [\mathfrak{k}_\xi, \mathfrak{k}_\xi]\rangle = 0$.*

*2. The coadjoint orbit $K\xi$ is admissible if and only if $i(\xi - \rho(\xi))$ is the differential of a 1-dimensional representation of $K_\xi$.*

**Proof.** Consider the character $k \mapsto \det_{\mathfrak{q}^\xi}(k)$ of $K_\xi$. Its differential is $2i\rho(\xi)$. Thus $\langle \rho(\xi), [\mathfrak{k}_\xi, \mathfrak{k}_\xi]\rangle = 0$.

We can equip $K\xi \simeq K/K_\xi$ with the Spin$^c$-bundle

$$\mathcal{S}_\xi := K \times_{K_\xi} \bigwedge \mathfrak{q}^\xi$$

with determinant line bundle $\det(\mathcal{S}_\xi) = [\mathbb{C}_{2\rho(\xi)}]$. Any other $K$-equivariant Spin$^c$-bundle on $K\xi$ is of the form $\mathcal{S}_\xi \otimes [\mathbb{C}_\theta]$ where $i\theta$ is the differential of

a character of $K_\xi$. Then $\det(\mathcal{S}_\xi \otimes [\mathbb{C}_\theta]) = [\mathbb{C}_{2\xi}]$ if and only if $\xi - \rho(\xi) = \theta$. The lemma then follows.

In particular the orbit $o(\mathfrak{k})$ is admissible. Indeed if $\xi = \rho^K$, then $\xi - \rho(\xi) = 0$.

An admissible coadjoint orbit $\mathcal{O} = K\xi$ is then equipped with the Spin$^c$-bundle

$$\mathcal{S}^{\pm}_{\mathcal{O}} := K \times_{K_\xi} \left( \bigwedge\nolimits^{\pm} \mathfrak{q}^\xi \otimes \mathbb{C}_{\xi - \rho(\xi)} \right). \tag{3.12}$$

Its Spin$^c$ equivariant index is

$$\mathrm{Q}^{\mathrm{spin}}_K(\mathcal{O}) = \mathrm{Ind}^K_{K_\xi} \left( \bigwedge \mathfrak{q}^\xi \otimes \mathbb{C}_{\xi - \rho(\xi)} \right). \tag{3.13}$$

See [24].

The following proposition is well known. We will recall its proof in Lemma 3.11 in the next subsection.

**Proposition 3.6** • *The map $\mathcal{O} \mapsto \pi_{\mathcal{O}} := \mathrm{Q}^{\mathrm{spin}}_K(\mathcal{O})$ defines a bijection between the set of regular admissible orbits and $\widehat{K}$.*

• *$\mathrm{Q}^{\mathrm{spin}}_K(o(\mathfrak{k}))$ is the trivial representation of $K$.*

We now describe the representation $\mathrm{Q}^{\mathrm{spin}}_K(\mathcal{O})$ attached to any admissible orbit in terms of regular admissible orbits.

**Definition 3.7** *To any coadjoint orbit $\mathcal{O} \subset \mathfrak{k}^*$, we associate the coadjoint orbit $s(\mathcal{O}) \subset \mathfrak{k}^*$ which is defined as follows : if $\mathcal{O} = K\mu$, take $s(\mathcal{O}) = K\xi$ with $\xi \in \mu + o(\mathfrak{k}_\mu)$. We call $s(\mathcal{O})$ the shift of the orbit $\mathcal{O}$.*

If $\mathcal{O}$ is regular, $s(\mathcal{O}) = \mathcal{O}$. If $\mathcal{O} = \{0\}$, then $s(\mathcal{O}) = o(\mathfrak{k})$.

The following proposition will be proved in the next subsection.

**Proposition 3.8** *Let $\mathcal{P}$ be an admissible orbit.*

• *$\mathcal{P}^* := -\mathcal{P}$ is also admissible and $\mathrm{Q}^{\mathrm{spin}}_K(\mathcal{P}^*) = \mathrm{Q}^{\mathrm{spin}}_K(\mathcal{P})^*$.*

• *If $s(\mathcal{P})$ is regular, then $s(\mathcal{P})$ is also admissible.*

• *Conversely, if $\mathcal{O}$ is regular and admissible, and $\mathcal{P}$ is such that $s(\mathcal{P}) = \mathcal{O}$, then $\mathcal{P}$ is admissible.*

- – *If $s(\mathcal{P})$ is not regular, then $\mathrm{Q}_K^{\mathrm{spin}}(\mathcal{P}) = 0$.*
  – *If $s(\mathcal{P})$ is regular, then $\mathrm{Q}_K^{\mathrm{spin}}(\mathcal{P}) = \mathrm{Q}_K^{\mathrm{spin}}(s(\mathcal{P})) = \pi_{s(\mathcal{P})}$.*

It is important to understand what are the admissible orbits $\mathcal{P}$ such that $s(\mathcal{P})$ is equal to a fixed regular admissible orbit $\mathcal{O}$.

For the remaining part of this subsection, we fix a conjugacy class $(\mathfrak{h})$. We denote by $\mathcal{A}((\mathfrak{h}))$ the set of admissible orbits belonging to the Dixmier sheet $\mathfrak{k}^*_{(\mathfrak{h})}$.

**Definition 3.9** *Let $\mathcal{O} \subset \mathfrak{k}^*$ be a $K$-orbit. A $K$-orbit $\mathcal{P}$ is called a $(\mathfrak{h})$ ancestor of $\mathcal{O}$ is $\mathcal{P} \subset \mathfrak{k}^*_{(\mathfrak{h})}$ and $s(\mathcal{P}) = \mathcal{O}$.*

We make the choice of a connected Lie subgroup $H$ with Lie algebra $\mathfrak{h}$ and write $\mathfrak{h} = \mathfrak{z} \oplus [\mathfrak{h}, \mathfrak{h}]$. We denote by $\mathfrak{z}_0^*$ the set of elements $\xi \in \mathfrak{z}^*$ such that $K_\xi = H$. The orbit $o(\mathfrak{h})$ (the $\rho$-orbit for $H$) is contained in $[\mathfrak{h}, \mathfrak{h}]^*$. The orbit $\mathcal{P}$ is a $(\mathfrak{h})$-ancestor to $\mathcal{O}$, if and only if there exists $\mu \in \mathfrak{z}_0^*$ such that $K\mu = \mathcal{P}$ and $\rho^H \in o(\mathfrak{h})$ such that $\mathcal{O} = K(\mu + \rho^H)$. If $\mathcal{O}$ is admissible then $\mathcal{P}$ is admissible (see Lemma 3.16).

Given a regular admissible orbit $\mathcal{O}$, there might be several $(\mathfrak{h})$-ancestors to $\mathcal{O}$.

**Example 3.10** *Consider the group $K = SU(3)$ and let $(\mathfrak{h})$ be the centralizer class of a subregular element $f \in \mathfrak{k}^*$ with centralizer $H = S(U(2) \times U(1))$.*

*We consider the Cartan subalgebra of diagonal matrices and choose a Weyl chamber. Let $\omega_1, \omega_2$ be the two fundamental weights. Let $\sigma_1, \sigma_2$ be the half lines $\mathbb{R}_{>0}\omega_1$, $\mathbb{R}_{>0}\omega_2$. The set $\mathcal{A}((\mathfrak{h}))$ is equal to the collection of orbits $K \cdot (\frac{1+2n}{2}\omega_1), n \in \mathbb{Z}$ (see Figure 4).*

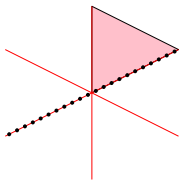

Figure 4: $H$-admissible orbits

*As $-\omega_1$ is conjugated to $\omega_2$, we see that the set $\mathcal{A}((\mathfrak{h}))$ is equal to the collection of orbits $K\cdot(\frac{1+2n}{2}\omega_i), n\in\mathbb{Z}_{\geqslant 0}, i=1,2$. Here we have chosen the representatives in the chosen closed Weyl chamber.*

*One has $s(K\cdot(\frac{1+2n}{2}\omega_i))=K(\rho^K+(n-1)\omega_i)$. Thus the shifted orbit is a regular orbit if and only if $n>0$. For $n=1$, both admissible orbits $K\cdot\frac{3}{2}\omega_1$ and $K\cdot(\frac{-3}{2}\omega_1)=K\cdot\frac{3}{2}\omega_2$ are $(\mathfrak{h})$-ancestors to the orbit $K\rho^K=o(\mathfrak{k})$.*

*Both admissible orbits $\mathcal{P}_1=K\cdot\frac{1}{2}\omega_1$ and $\mathcal{P}_2=K\cdot\frac{1}{2}\omega_2$ are such that $\mathrm{Q}_K^{\mathrm{spin}}(\mathcal{P}_i)=0$*

*In Figure 5, we draw the link between $H$-admissible orbits and their respective shifts.*

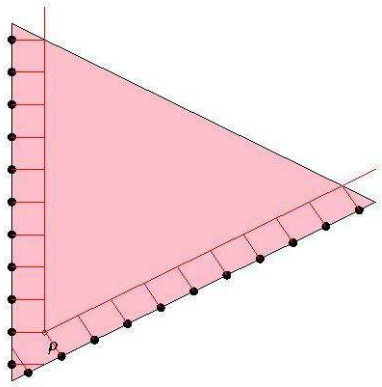


Figure 5: $H$-admissible orbits and their shifts

There might also be several classes of conjugacy $(\mathfrak{h})$ such that $\mathcal{O}$ admits a $(\mathfrak{h})$-ancestor $\mathcal{P}$. For example, let $\mathcal{O}=o(\mathfrak{k})$. Then, for any $\mathfrak{h}\in\mathcal{H}_{\mathfrak{k}}$, the orbit $K(\rho^K-\rho^H)$ is a $(\mathfrak{h})$-ancestor to $\mathcal{O}$. Here we have chosen a Cartan subgroup $T$ contained in $H$, $H=K_\xi$ and a positive root system such that $\xi$ is dominant to define $\rho^K$ and $\rho^H$.

### 3.3 Admissible coadjoint orbits and Weyl chamber

In order to parameterize coadjoint orbits, we choose a Cartan subgroup $T$ of $K$ with Lie algebra $\mathfrak{t}$. Let $\Lambda\subset\mathfrak{t}^*$ be the lattice of weights of $T$. Let $W$ be the Weyl group. Choose a system of positive roots $\Delta^+\subset\mathfrak{t}^*$, and let

$$\rho^K=\frac{1}{2}\sum_{\alpha>0}\alpha.$$

If $\alpha \in \mathfrak{t}^*$ is a root, we denote by $H_\alpha \in \mathfrak{t}$ the corresponding coroot (so $\langle \alpha, H_\alpha \rangle = 2$). Then $\langle \rho^K, H_\alpha \rangle = 1$ if and only if $\alpha$ is a simple root.

Define the positive closed Weyl chamber by

$$\mathfrak{t}^*_{\geqslant 0} = \{\xi \in \mathfrak{t}^*; \langle \xi, H_\alpha \rangle \geqslant 0 \text{ for all } \alpha > 0\},$$

and we denote by $\Lambda_{\geqslant 0} := \Lambda \cap \mathfrak{t}^*_{\geqslant 0}$ the set of dominant weights. Any coadjoint orbit $\mathcal{O}$ of $K$ is of the form $\mathcal{O} = K\xi$ with $\{\xi\} = \mathcal{O} \cap \mathfrak{t}^*_{\geqslant 0}$.

We index the set $\widehat{K}$ of classes of finite dimensional irreducible representations of $K$ by the set $\rho^K + \Lambda_{\geqslant 0}$. The irreducible representation $\pi_\lambda$ corresponding to $\lambda \in \rho^K + \Lambda_{\geqslant 0}$ is the irreducible representation with infinitesimal character $\lambda$. Its highest weight is $\lambda - \rho^K$. The representation $\pi_{\rho^K}$ is the trivial representation of $K$. The Weyl character formula for the representation $\pi_\lambda$ is, for $X \in \mathfrak{t}$,

$$\mathbf{Tr}\, \pi_\lambda(e^X) = \frac{\sum_{w \in W} \epsilon(w) e^{i\langle w\lambda, X\rangle}}{\prod_{\alpha > 0} e^{i\langle \alpha, X\rangle/2} - e^{-i\langle \alpha, X\rangle/2}}.$$

For any $\mu \in \mathfrak{t}^*$, we consider its element $\rho(\mu) \in \mathfrak{k}^*$ (Definition 3.4).

**Lemma 3.11** *Let $\lambda \in \mathfrak{t}^*_{\geqslant 0}$ be a regular admissible element of $\mathfrak{k}^*$. Then*

1. $\lambda \in \rho^K + \Lambda_{\geqslant 0}$.
2. $\mathrm{Q}^{\mathrm{spin}}_K(K\lambda) = \pi_\lambda$.

**Proof.** Let $\lambda \in \mathfrak{t}^*_{\geqslant 0}$ be regular and admissible, then $\rho(\lambda) = \rho^K$, so $\lambda \in \{\rho^K + \Lambda\} \cap \mathfrak{t}^*_{>0}$. If $\alpha$ is a simple root, then **the integer** $\langle \lambda - \rho^K, H_\alpha \rangle = \langle \mu, H_\alpha \rangle - 1$ is non negative, as $\langle \lambda, H_\alpha \rangle > 0$. So $\lambda - \rho^K$ is a dominant weight.

Atiyah-Bott fixed point for the trace of the representation $\mathrm{Q}^{\mathrm{spin}}_K(K\lambda)$ is Weyl character formula.

Thus we obtain Lemma 3.11 and Proposition 3.6.

If $\mathfrak{h} \in \mathcal{H}_\mathfrak{k}$, we denote by $\|\rho^H\|$ the norm of any element in the coadjoint orbit $o(\mathfrak{h}) \subset \mathfrak{h}^*$ for $H$.

The positive Weyl chamber is the simplicial cone determined by the equations $\langle \lambda, H_\alpha \rangle \geqslant 0$ for the simple roots $\alpha \geqslant 0$. We denote by $\mathcal{F}_\mathfrak{k}$ the set of the relative interiors of the faces of $\mathfrak{t}^*_{\geqslant 0}$. Thus $\mathfrak{t}^*_{\geqslant 0} = \coprod_{\sigma \in \mathcal{F}_\mathfrak{k}} \sigma$, and we denote by $\mathfrak{t}^*_{>0} \in \mathcal{F}_\mathfrak{k}$ the interior of $\mathfrak{t}^*_{\geqslant 0}$.

Let $\sigma \in \mathcal{F}_\mathfrak{k}$. Thus $\mathbb{R}\sigma$, the linear span of $\sigma$, is the subspace determined by $\langle \lambda, H_\alpha \rangle = 0$ where the $\alpha$ varies over a subset of the simple roots.

The stabilizer $K_\xi$ does not depend of the choice of the point $\xi \in \sigma$ : we denote it by $K_\sigma$. The map $\sigma \to \mathfrak{k}_\sigma$ induces a surjective map from $\mathcal{F}_\mathfrak{k}$ to $\mathcal{H}_\mathfrak{k}$.

For $\sigma \in \mathcal{F}_\mathfrak{k}$, we have the decomposition $\mathfrak{k}_\sigma = [\mathfrak{k}_\sigma, \mathfrak{k}_\sigma] \oplus \mathfrak{z}(\mathfrak{k}_\sigma)$ with dual decomposition $\mathfrak{k}_\sigma^* = [\mathfrak{k}_\sigma, \mathfrak{k}_\sigma]^* \oplus \mathbb{R}\sigma$. Let

$$\rho^{K_\sigma} := \frac{1}{2} \sum_{\substack{\alpha>0 \\ (\alpha,\sigma)=0}} \alpha$$

be the $\rho$-element of the group $K_\sigma$ associated to the positive root system $\{\alpha > 0, (\alpha, \sigma) = 0\}$ for $K_\sigma$. Then

$$\rho^K - \rho^{K_\sigma} = \frac{1}{2} \sum_{\substack{\alpha>0 \\ (\alpha,\sigma)>0}} \alpha,$$

and for any $\mu \in \sigma$, the element $\rho(\mu) \in \mathfrak{k}^*$ is equal to $\rho^K - \rho^{K_\sigma}$. In particular, $\rho^K - \rho^{K_\sigma}$ vanishes on $[\mathfrak{k}_\sigma, \mathfrak{k}_\sigma]$, so $\rho^K - \rho^{K_\sigma} \in \mathbb{R}\sigma$, while $\rho^{K_\sigma} \in [\mathfrak{k}_\sigma, \mathfrak{k}_\sigma]^*$. The decomposition $\rho^K = (\rho^K - \rho^{K_\sigma}) + \rho^{K_\sigma}$ is an orthogonal decomposition.

Figure 6 shows this orthogonal decomposition of $\rho$ for the case $SU(3)$.

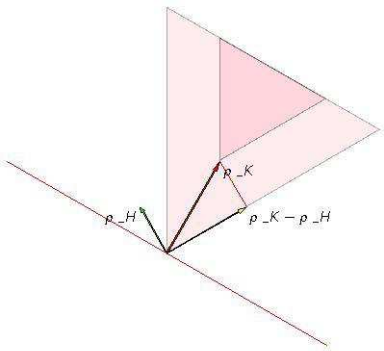


Figure 6: Orthogonal decomposition of $\rho_K$

We start by proving some geometric properties of the Weyl chamber. The subset $\rho_K + \mathfrak{t}^*_{\geqslant 0}$ of the positive Weyl chamber will be called the shifted Weyl chamber. It is determined by the inequalities $\langle \lambda, H_\alpha \rangle \geqslant 1$ for any simple root $\alpha \geqslant 0$, and thus $\langle \lambda, H_\alpha \rangle \geqslant 1$ for any positive root. The following proposition is illustrated in Figure 7 in the case $SU(3)$.

**Proposition 3.12** *1. If $\lambda \in \rho^K + \mathfrak{t}^*_{\geqslant 0}$, then $(\lambda, \lambda) \geqslant (\lambda, \rho^K) \geqslant (\rho^K, \rho^K)$. The equality $(\lambda, \lambda) = (\lambda, \rho^K)$ holds only if $\lambda = \rho^K$.*

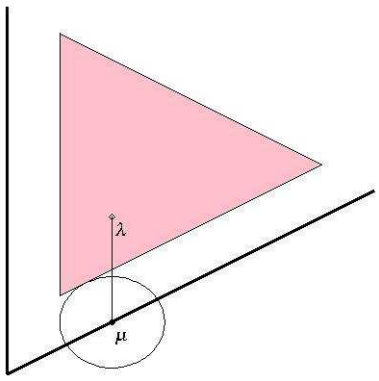


Figure 7: Distance of a singular element $\mu$ to a strongly regular element $\lambda$

2. *Let $\sigma \in \mathcal{F}_{\mathfrak{k}}$.*
   - *The orthogonal projection of $\xi \in \mathfrak{t}^*_{>0}$ onto $\mathbb{R}\sigma$ belongs to $\sigma$.*
   - *We have $\rho^K - \rho^{K_\sigma} \in \sigma$ for any $\sigma \in \mathcal{F}_{\mathfrak{k}}$.*
3. *For any $(\mathfrak{h}) \in \mathcal{H}_{\mathfrak{k}}$, $\|\rho^K\| \geqslant \|\rho^H\|$, and $\|\rho^K\| = \|\rho^H\|$ only if $H = K$.*
4. *If $\lambda \in \rho^K + \mathfrak{t}^*_{\geqslant 0}$ and $\mu \in \mathfrak{t}^*$, then:*

$$\|\lambda - \mu\|^2 \geqslant \frac{1}{2} \sum_{\substack{\alpha>0 \\ (\alpha,\mu)=0}} (\lambda, \alpha) \geqslant \|\rho^{K_\mu}\|^2. \tag{3.14}$$

*The equality*

$$\|\lambda - \mu\|^2 = \frac{1}{2} \sum_{\substack{\alpha>0 \\ (\alpha,\mu)=0}} (\lambda, \alpha)$$

*holds if and only if $\mu$ belongs to $\mathfrak{t}^*_{\geqslant 0}$, and if $\mu$ is the projection of $\lambda$ on the face $\sigma$ of $\mathfrak{t}^*_{\geqslant 0}$ containing $\mu$. In particular $\lambda - \rho(\lambda) = \mu - \rho(\mu)$.*

**Proof.** If $\lambda = \rho^K + c$, with $c \in \mathfrak{t}^*_{\geqslant 0}$, inequalities $(\lambda, \lambda) \geqslant (\lambda, \rho^K) \geqslant (\rho^K, \rho^K)$ follows from the fact that $(\lambda, c)$ and $(\rho^K, c)$ are non negative, as the scalar product of two elements of $\mathfrak{t}^*_{\geqslant 0}$ is non negative.

The second point follows from the fact that the dual cone to $\mathfrak{t}^*_{\geqslant 0}$ is generated by the simple roots $\alpha_i$, and $(\alpha_i, \alpha_j) \leqslant 0$, if $i \neq j$.

We have the orthogonal decomposition $\rho^K = \rho^{K_\sigma} + (\rho^K - \rho^{K_\sigma})$: hence $\rho^K - \rho^{K_\sigma}$, which is the orthogonal projection of $\rho^K \in \mathfrak{t}^*_{>0}$ on $\mathbb{R}\sigma$, belongs to $\sigma$.

For the third point, we might choose $H$ conjugated to $K_\sigma$, so $\|\rho^K\|^2 = \|\rho^{K_\sigma}\|^2 + \|\rho^K - \rho^{K_\sigma}\|^2$.

We now prove the last point.

Let $\mathfrak{k}_\mu$ be the centralizer of $\mu$ and let $\mathfrak{z}$ be the center of $\mathfrak{k}_\mu$. Consider the orthogonal decomposition $\mathfrak{t}^* = \mathfrak{z}^* \oplus \mathfrak{a}^*$ where $\mathfrak{a}$ is a Cartan subalgebra for $[\mathfrak{k}_\mu, \mathfrak{k}_\mu]$, that is $\mathfrak{a} = \sum_{(\alpha,\mu)=0} \mathbb{R}H_\alpha$. Let $\rho^{K_\mu} \in \mathfrak{a}^*$ be the $\rho$ element for the system $\Delta^1_+ = \{\alpha > 0, (\alpha, \mu) = 0\}$ of $[\mathfrak{k}_\mu, \mathfrak{k}_\mu]$.

Let us write $\lambda = \rho^K + c$, with $c$ dominant, and decompose $\rho^K = p_0 + p_1$, $c = c_0 + c_1$, with $p_0, c_0 \in \mathfrak{z}^*$, $p_1, c_1 \in \mathfrak{a}^*$. Thus $\lambda = \lambda_0 + \lambda_1$, with $\lambda_0 \in \mathfrak{z}^*$ and $\lambda_1 = p_1 + c_1$. Now $p_1$ belongs to the shifted Weyl chamber in $\mathfrak{a}^*$. Indeed, for any $\alpha > 0$ such that $(\alpha, \mu) = 0$, we have $\langle p_1, H_\alpha\rangle = \langle \rho^K, H_\alpha\rangle \geqslant 1$. Similarly $c_1$ is dominant for the system $\Delta^1_+$.

As $\mu \in \mathfrak{z}^*$, we have $\|\lambda - \mu\|^2 = \|\lambda_0 - \mu\|^2 + \|p_1 + c_1\|^2$. Using the first point of 3.12, we obtain

$$\|\lambda - \mu\|^2 = \|\lambda_0 - \mu\|^2 + \|p_1 + c_1\|^2 \geqslant (p_1 + c_1, \rho^{K_\mu}) \geqslant \|\rho^{K_\mu}\|^2.$$

As

$$(p_1 + c_1, \rho^{K_\mu}) = \frac{1}{2} \sum_{\substack{\alpha>0 \\ (\alpha,\mu)=0}} (\lambda, \alpha)$$

we obtain Inequalities (3.14).

If the inequality $\|\lambda - \mu\|^2 \geqslant (p_1 + c_1, \rho^{K_\mu})$ is an equality, then $c_1 = 0$, $p_1 = \rho^{K_\mu}$, and $\lambda_0 = \mu$. Thus for roots $\alpha \in \Delta^1_+$, $\langle \rho^{K_\mu}, H_\alpha\rangle = \langle \rho^K, H_\alpha\rangle$. As $\rho^{K_\mu}$ takes value 1 on simple roots for $K_\mu$, it follows that the set $S_1$ of simple roots for the system $\Delta^1_+$ is contained in the set of simple roots for $\Delta^+$. As $\mathfrak{a} = \oplus_{\alpha\in S_1} \mathbb{R}H_\alpha$, the orthogonal $\mathfrak{z}$ of $\mathfrak{a}$ is $\mathbb{R}\sigma$ for the face $\sigma$ of $\mathfrak{t}^*$ orthogonal to the subset $S_1$ of simple roots. We then have $K_\mu = K_\sigma$. Furthermore, $\lambda = \mu + \rho^{K_\sigma}$. Thus $\mu$ is the projection of $\lambda$ on $\mathbb{R}\sigma$, so $\mu \in \sigma \subset \mathfrak{t}^*_{\geqslant 0}$. As $\rho(\lambda) = \rho^K$, and $\rho(\mu) = \rho^K - \rho^{K_\sigma}$, we obtain $\lambda - \rho(\lambda) = \mu - \rho(\mu)$. So all assertions are proved.

**Corollary 3.13** *Let $\sigma \in \mathcal{F}_\mathfrak{k}$. The distance between the shifted Weyl chamber $\rho^K + \mathfrak{t}^*_{\geqslant 0}$ and the vector space $\mathbb{R}\sigma$ is equal to $\|\rho^{K_\sigma}\|$. Furthermore, if $\rho^K + \lambda'$, with $\lambda' \in \mathfrak{t}^*_{\geqslant 0}$ and $\lambda \in \mathbb{R}\sigma$ are at distance $\|\rho^{K_\sigma}\|$, then $\rho^K + \lambda' = \rho^{K_\sigma} + \lambda$.*

**Proof.** Indeed, if $\mu \in \mathbb{R}\sigma$, and $\lambda \in \rho^K + \mathfrak{t}^*_{\geqslant 0}$, then Inequality (3.14) implies that $\|\lambda - \mu\| \geqslant \|\rho^{K_\mu}\|$. As $K_\sigma \subset K_\mu$, $\|\rho^{K_\mu}\| \geqslant \|\rho^{K_\sigma}\|$.

Let us reformulate Inequalities (3.14) above independently of the choice of a positive root system.

**Definition 3.14** *A regular element $\lambda \in \mathfrak{k}^*$ determines a closed positive Weyl chamber $C_\lambda \subset \mathfrak{k}^*_\lambda$. We say that $\lambda$ is very regular if $\lambda \in \rho(\lambda) + C_\lambda$.*

Regular admissible elements are very regular.

Here is the magical inequality that we will use over and over again to get vanishing results.

**Corollary 3.15 (The magical inequality)** *Let $\lambda, \mu$ be two elements of $\mathfrak{t}^*$. Assume that $\lambda$ is very regular, then*

$$\|\lambda - \mu\|^2 \geqslant \frac{1}{2} \sum_{\substack{(\alpha,\lambda)>0 \\ (\alpha,\mu)=0}} (\lambda, \alpha) \geqslant \|\rho^{K_\mu}\|^2.$$

*If the equality*

$$\|\lambda - \mu\|^2 = \frac{1}{2} \sum_{\substack{(\alpha,\lambda)>0 \\ (\alpha,\mu)=0}} (\lambda, \alpha)$$

*holds, then $\mu \in C_\lambda$ and $\lambda - \rho(\lambda) = \mu - \rho(\mu)$.*

Let us now study the admissible coadjoint orbits and their shifts. The following lemma just restate properties which follow directly from the preceding discussions.

**Lemma 3.16** *For any $\mu \in \sigma$,*

- $\rho(\mu) = \rho^K - \rho^{K_\sigma}$ *and* $\rho^K - \rho^{K_\sigma} \in \sigma$,
- $o(\mathfrak{k}_\mu) = K_\sigma \rho^{K_\sigma}$,
- $K\mu$ *is admissible if and only if* $\mu + \rho^{K_\sigma} \in \rho^K + \Lambda$,
- $s(K\mu) = K(\mu + \rho^{K_\sigma})$.

Proposition 3.17 below says that the shifts of **admissible elements** stay in the closure of the Weyl chamber. Figure 8 illustrate this fact in the case $SU(3)$.

**Proposition 3.17** *Let $\sigma$ be a relative interior of a face of $\mathfrak{t}^*_{\geqslant 0}$ and let $\mu$ be* **an admissible element** *of $\mathfrak{t}^*_{\geqslant 0}$.*

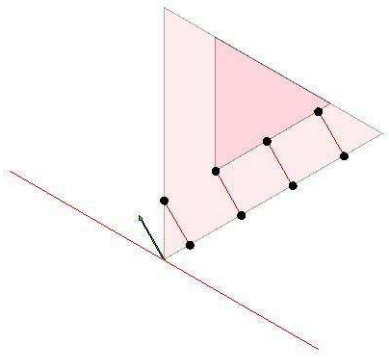

Figure 8: Shifts of admissible orbits

1. *If $\mu$ is regular and $\mu - \rho^K \in \overline{\sigma}$, then $\mu - \rho^{K_\sigma} \in \sigma$.*
2. *If $\mu \in \sigma$ and $\mu + \rho^{K_\sigma}$ is regular, then $\mu + \rho^{K_\sigma} \in \rho^K + (\Lambda_{\geqslant 0} \cap \overline{\sigma})$.*
3. *If $\mu \in \sigma$, we have*

$$\mathrm{Q}_K^{\mathrm{spin}}(K\mu) = \begin{cases} 0 & \text{if } \mu + \rho^{K_\sigma} \text{ is singular,} \\ \pi_{\mu+\rho^{K_\sigma}} & \text{if } \mu + \rho^{K_\sigma} \text{ is regular.} \end{cases}$$

**Proof.** The first point follows from the fact that $\rho^K - \rho^{K_\sigma} \in \sigma$.

We prove the second point. Let $\mu \in \sigma$ such that $\lambda = \mu + \rho^{K_\sigma}$ is regular. Thus $\|\lambda - \mu\|^2 = \|\rho^{K_\mu}\|^2$. Then $\lambda$ being regular and admissible, $\lambda$ is very regular. We use Corollary 3.15. The equality $\|\lambda - \mu\|^2 = \|\rho^{K_\mu}\|^2$ implies $\lambda - \rho(\lambda) = \mu - \rho(\mu) = \mu - (\rho^K - \rho^{K_\sigma})$. Thus $\rho(\lambda) = \rho^K$, so $\lambda \in \mathfrak{t}^*_{\geqslant 0}$. The element $\lambda - \rho^K = \mu - (\rho^K - \rho^{K_\sigma})$ is in $\mathbb{R}\sigma$. As it is dominant, it is in $\overline{\sigma}$.

Let us prove the last point. Let $\mathfrak{q}^\mu$ be the complex space $\mathfrak{k}/\mathfrak{k}_\mu$ equipped with the complex structure $J_\mu$. The equivariant index $\Theta$ of the Dirac operator associated to the Spin$^c$-bundle $\mathcal{S}_{K\mu} = K \times_{K_\mu} (\bigwedge \mathfrak{q}^\mu \otimes \mathbb{C}_{\mu-\rho^K+\rho^{K_\sigma}})$ is given by Atiyah-Bott fixed point formula: for $X \in \mathfrak{t}$, $\Theta(e^X) = \sum_{w \in W/W_\mu} w \cdot \frac{e^{i\langle \mu, X\rangle}}{\prod_{\langle \alpha,\mu\rangle>0} e^{i\langle\alpha,X\rangle/2} - e^{-i\langle\alpha,X\rangle/2}}$. Here $W_\mu$, the stabilizer of $\mu$ in $W$, is equal to the Weyl group of the group $K_\sigma$. Using $\sum_{w\in W_\sigma} \epsilon(w) e^{w\rho^{K_\sigma}} = \prod_{\alpha>0,\langle\alpha,\sigma\rangle=0}(e^{\alpha/2} - e^{-\alpha/2})$, we obtain

$$\Theta(e^X) = \frac{\sum_{w\in W} \epsilon(w) e^{i\langle w(\mu+\rho^{K_\sigma}), X\rangle}}{\prod_{\alpha>0} e^{i\langle\alpha,X\rangle/2} - e^{-i\langle\alpha,X\rangle/2}}. \tag{3.15}$$

If $\mu + \rho^{K_\sigma}$ is singular, $\Theta$ is equal to zero. If $\mu + \rho^{K_\sigma}$ is regular, thanks to the second point, $\mu + \rho^{K_\sigma}$ is in $\rho^K + \Lambda_{\geqslant 0}$, so $\Theta = \pi_{\mu+\rho^{K_\sigma}}$.

Remark that $\rho^{K_\sigma}$ itself is not dominant, so it is not true that any element $\mu + \rho^{K_\sigma}$, with $\mu \in \sigma$, is dominant. Thus the integrality conditions on $\mu$ are needed to obtain Proposition 3.17.

**Let us prove Proposition 3.8.**

We choose a Cartan subgroup $T$ and a positive root system, and let $\mathcal{P} = K\mu$ be an admissible orbit, with $\mu \in \mathfrak{t}^*_{\geqslant 0}$. Let $\sigma$ be the face (interior) of $\mathfrak{t}^*_{\geqslant 0}$ where $\mu$ belongs. By Lemma 3.16, $s(\mathcal{P}) = K(\mu + \rho^{K_\sigma})$. Thus the two first points of Proposition 3.8 as well as the last point are consequence of Proposition 3.17. From the Atiyah-Bott fixed point formula, we obtain $\mathbf{Tr}(\mathrm{Q}^{\mathrm{spin}}_K(\mathcal{P}^*))(g) = \mathbf{Tr}(\mathrm{Q}^{\mathrm{spin}}_K(\mathcal{P}))(g^{-1})$, so $\mathrm{Q}^{\mathrm{spin}}_K(\mathcal{P}^*) = \mathrm{Q}^{\mathrm{spin}}_K(\mathcal{P})^*$.

## 3.4 Complex structures

We often will use complex structures and normalized traces on real vector spaces defined by the following procedure.

**Definition 3.18** *Let $N$ be a real vector space and $b : N \to N$ a linear transformation, such that $-b^2$ is diagonalizable with non negative eigenvalues. Define*

- *the diagonalizable transformation $|b|$ of $N$ by $|b| = \sqrt{-b^2}$,*
- *the complex structure $J_b = b|b|^{-1}$ on $N/\ker(b)$*
- *we denote by $\mathbf{nTr}_N|b| = \frac{1}{2}\mathbf{Tr}_N|b|$, that is half of the trace of the action of $|b|$ in the real vector space $N$. We call $\mathbf{nTr}_N|b|$ the normalized trace of $b$.*

If $N$ has a Hermitian structure invariant by $b$, $\frac{1}{2}\mathbf{Tr}_N|b|$ is the trace of $|b|$ considered as a Hermitian matrix. The interest of our notation is that we do not need complex structures to define $\mathbf{nTr}_N|b|$.

If $N$ is an Euclidean space and $b$ a skew-symmetric transformation of $N$, then $-b^2$ is diagonalizable with non negative eigenvalues. By definition of $J_b$, the transformation $b$ of $N$ determines a complex diagonalizable transformation of $N/\ker(b)$, and the list of its complex eigenvalues is $[ia_1, \ldots, ia_\ell]$ where the $a_k$ are strictly positive real numbers. We have $\mathbf{nTr}_N|b| = \sum_{k=1}^{\ell} a_k \geqslant 0$.

Recall our identification $\mathfrak{k} = \mathfrak{k}^*$ with the help of a scalar product. When $\beta \in \mathfrak{k}^*$, denote by $b$ the corresponding element of $\mathfrak{k}$. We have defined a complex structure $J_\beta$ on $\mathfrak{k}/\mathfrak{k}_\beta$. On the other hand, $b$ defines an invertible transformation of $\mathfrak{k}/\mathfrak{k}_\beta$. It can be checked that $J_\beta = J_b$. If we choose a Cartan subalgebra containing $b$, then $\mathbf{nTr}_{\mathfrak{k}}|b| = \sum_{\alpha>0} |\langle \alpha, b\rangle|$.

For further use, we include a lemma. Let us consider $\mathfrak{k}_{\mathbb{C}}$, the complexified space of $\mathfrak{k}$. Consider the complex space $\bigwedge \mathfrak{k}_{\mathbb{C}}$.

**Lemma 3.19** *Let $b \in \mathfrak{k}$. Let $x \in \mathbb{R}$ be an eigenvalue for the action of $\frac{b}{i}$ in $\bigwedge \mathfrak{k}_{\mathbb{C}}$. Then $x \geqslant -\mathbf{nTr}_{\mathfrak{k}}|b|$*

**Proof.** Indeed, consider a Cartan subalgebra $\mathfrak{t}$ containing $b$, the system of roots $\Delta$ and an order such that $\langle \alpha, b \rangle \geqslant 0$ for all $\alpha > 0$. An eigenvalue $x$ on $\bigwedge \mathfrak{k}_{\mathbb{C}}$ is thus of the form $\sum_{\alpha \in I \subset \Delta} \langle \alpha, b \rangle$. Thus we see that the lowest eigenvalue is $-\sum_{\alpha>0} \langle \alpha, b \rangle = -\mathbf{nTr}_{\mathfrak{k}}|b|$.

Assume now that $\mathcal{N} \to M$ is a real vector bundle equipped with an action of a compact Lie group $K$. For any $b \in \mathfrak{k}$, and any $m \in M$ such that $b_M(m) = 0$, we may consider the linear action $\mathcal{L}(b)|_{\mathcal{N}_m}$ which is induced by $b$ on the fibers $\mathcal{N}_m$. It is easy to check that $(\mathcal{L}(b)|_{\mathcal{N}_m})^2$ is diagonalizable with eigenvalues which are negative or equal to zero. We denote by $|\mathcal{L}_m(b)| = \sqrt{-(\mathcal{L}(b)|_{\mathcal{N}_m})^2}$.

**Definition 3.20** *We denote by $\mathbf{nTr}_{\mathcal{N}_m}|b| = \frac{1}{2}\mathbf{Tr}|\mathcal{L}_m(b)|$ that is half of the trace of the real endomorphism $|\mathcal{L}_m(b)|$ on $\mathcal{N}_m$. We call $\mathbf{nTr}_{\mathcal{N}_m}|b|$ the normalized trace of the action of $b$ on $\mathcal{N}_m$.*

For further use, we rewrite Corollary 3.15 as an inequality on normalized traces.

For any $b \in \mathfrak{k}$ and $\mu \in \mathfrak{k}^*$ fixed by $b$, we may consider the action $\mathrm{ad}(b) : \mathfrak{k}_\mu \to \mathfrak{k}_\mu$ and the corresponding normalized trace $\mathbf{nTr}_{\mathfrak{k}_\mu}|\mathrm{ad}(b)|$ denoted simply by $\mathbf{nTr}_{\mathfrak{k}_\mu}|b|$.

**Proposition 3.21** *Let $b \in \mathfrak{k}$ and denote by $\beta$ the corresponding element in $\mathfrak{k}^*$. Let $\lambda, \mu$ be elements of $\mathfrak{k}^*$ fixed by $b$. Assume that $\lambda$ is very regular and that $\mu - \lambda = \beta$. Then*

$$\|\beta\|^2 \geqslant \frac{1}{2}\mathbf{nTr}_{\mathfrak{k}_\mu}|b|.$$

*If the equality holds, then $\mu$ belongs to the positive Weyl chamber $C_\lambda$ and*

1. *$\lambda - \rho(\lambda) = \mu - \rho(\mu)$, hence $\lambda$ is admissible if and only if $\mu$ is admissible,*
2. *$s(K\mu) = K\lambda$.*

**Proof.** Indeed, as $\lambda$ is fixed by $b$, we see that $\beta$ belong to $\mathfrak{k}^*_\lambda$. We may assume that $\mathfrak{k}^*_\lambda = \mathfrak{t}^*$. Thus $\beta, \lambda$ and $\mu = \lambda - \beta$ belong to $\mathfrak{t}^*$. The element $\lambda$ is a very regular element of $\mathfrak{t}^*$. Proposition is thus a restatement of Corollary 3.15.

### 3.5 Induced Spin$^c$ bundles

Let $H \subset K$ be the stabilizer subgroup of some element in $\mathfrak{k}^*$. We denote by $\mathfrak{h}$ the Lie algebra of $H$ and we consider the open subset $\mathfrak{h}_0^* := \{\xi \in \mathfrak{h}^* \mid K_\xi \subset H\}$. Equivalently, the element $\xi$, identified to an element of $\mathfrak{h}$, is such that the transformation $\mathrm{ad}(\xi)$ is invertible on $\mathfrak{k}/\mathfrak{h}$, so it determines a complex structure on $\mathfrak{k}/\mathfrak{h}$ denoted $J_\xi$. The complex structure $J_\xi$ on $\mathfrak{k}/\mathfrak{h}$ determined by $\xi \in \mathfrak{h}_0^*$ depends only of the connected component $C$ of $\mathfrak{h}_0^*$ containing $\xi$. Thus we denote by $J_C$ the corresponding complex structure on $\mathfrak{q} = \mathfrak{k}/\mathfrak{h}$, by $\rho_C \in \mathfrak{z}^*$ the element $\rho^K(\xi) - \rho^H(\xi)$ for any $\xi \in C$ and by $\mathfrak{q}^C$ the complex vector space $\mathfrak{q}$ equipped with $J_C$. If $C$ and $C'$ are two connected components, we denote by $\epsilon_C^{C'}$ the ratio of the orientation $o(J_C)$ and $o(J_{C'})$ on $\mathfrak{q}$.

Consider a compact $H$-manifold $Y$ and the manifold $M = K \times_H Y$. Assume $M$ is oriented and equipped with a $K$-equivariant Spin$^c$-bundle $\mathcal{S}$. At the level of tangent spaces we have $\mathrm{T}M|_Y \simeq [\mathfrak{q}] \oplus \mathrm{T}Y$ where $[\mathfrak{q}] = Y \times \mathfrak{q}$. We orient the manifold $Y$ through the relation $o(M) = o(J_C)o(Y)$. We consider the Spin$^c$-bundle $\mathcal{S}_Y$ defined by

$$\mathcal{S}|_Y = \left[\bigwedge \mathfrak{q}^C\right] \otimes \mathcal{S}_Y. \tag{3.16}$$

Here $\left[\bigwedge \mathfrak{q}^C\right] = Y \times \bigwedge \mathfrak{q}^C$ is a Spin$^c$-bundle for the trivial bundle $[\mathfrak{q}]$.

This gives a bijection (depending of $C$) between the $K$-equivariant Spin$^c$-bundles $\mathcal{S}$ on $M$ and the $H$-equivariant Spin$^c$-bundles $\mathcal{S}_Y$ on $Y$. If the relation (3.16) holds, we say that $\mathcal{S}$ is the Spin$^c$-bundle induced by $\mathcal{S}_Y$. In this "induced setting", we have

$$\mathcal{Q}_K(M, \mathcal{S}) = \mathrm{Ind}_H^K \left( \bigwedge \mathfrak{q}^C \otimes \mathcal{Q}_H(Y, \mathcal{S}_Y) \right). \tag{3.17}$$

See [24].

We end this section by considering the particular case of an induced manifold $M := K \times_H H\mu$ where $H\mu$ is an admissible $H$-coadjoint orbit. Here $H\mu$ is equipped with its canonical Spin$^c$-bundle $\mathcal{S}_{H\mu}$, and the representation $\mathcal{Q}_H(H\mu, \mathcal{S}_{H\mu})$ is simply denoted by $\mathrm{Q}_H^{\mathrm{spin}}(H\mu)$.

The Spin$^c$ index on the manifold $M = K \times_H H\mu$ is equal to the character

$$I_\mu^C := \mathrm{Ind}_H^K \left( \bigwedge \mathfrak{q}^C \otimes \mathrm{Q}_H^{\mathrm{spin}}(H\mu) \right).$$

The following result will be used in Section 5.4.

**Proposition 3.22** • *If $\mu + \rho_C \notin \mathfrak{h}_0^*$, then $I_\mu^C = 0$.*

- *If $\mu+\rho_C \in \mathfrak{h}_0^*$, then $\mu+\rho_C$ is $K$-admissible, and*

$$I_\mu^C = \epsilon_{C'}^C \, \mathrm{Q}_K^{\mathrm{spin}}(K(\mu+\rho_C))$$

*where $C'$ is the connected component of $\mathfrak{h}_0^*$ containing $\mu+\rho_C$.*

**Proof.** By definition $\mathrm{Q}_H^{\mathrm{spin}}(H\mu) = \mathrm{Ind}_{H_\mu}^H\Big(\bigwedge_{J_\mu} \mathfrak{h}/\mathfrak{h}_\mu \otimes \mathbb{C}_{\mu-\rho^H(\mu)}\Big)$. We assume first that $\mu' := \mu+\rho_C \in \mathfrak{h}_0^*$ : let $C'$ be the connected component of $\mathfrak{h}_0^*$ containing $\mu+\rho_C$. As $K_{\mu'} = H_{\mu'} = H_\mu$, we have

$$I_\mu^C = \mathrm{Ind}_{K_{\mu'}}^K\Big(\bigwedge \mathfrak{q}^C \otimes \bigwedge_{J_{\mu'}} \mathfrak{h}/\mathfrak{h}_{\mu'} \otimes \mathbb{C}_{\mu'-\rho_C-\rho^H(\mu')}\Big).$$

Now we use the fact that the graded $K_{\mu'}$-module $\bigwedge \mathfrak{q}^C$ is equal to $\epsilon_{C'}^C \bigwedge \mathfrak{q}^{C'} \otimes \mathbb{C}_{\rho_C-\rho'_C}$ (see Example 2.2). It gives that

$$\begin{aligned} I_\mu^C &= \epsilon_{C'}^C \, \mathrm{Ind}_{K_{\mu'}}^K\Big(\bigwedge \mathfrak{q}^{C'} \otimes \bigwedge_{J_{\mu'}} \mathfrak{h}/\mathfrak{h}_{\mu'} \otimes \mathbb{C}_{\mu'-\rho_{C'}-\rho^H(\mu')}\Big) \\ &= \epsilon_{C'}^C \, \mathrm{Ind}_{K_{\mu'}}^K\Big(\bigwedge_{J_{\mu'}} \mathfrak{k}/\mathfrak{k}_{\mu'} \otimes \mathbb{C}_{\mu'-\rho(\mu')}\Big) \\ &= \epsilon_{C'}^C \, \mathrm{Q}_K^{\mathrm{spin}}(K\mu'). \end{aligned}$$

Assume now that $I_\mu^C \neq 0$. The equivariant index $\mathrm{Q}_H^{\mathrm{spin}}(H\mu)$ must be non zero. Hence we have $\mathrm{Q}_H^{\mathrm{spin}}(H\mu) = \mathrm{Q}_H^{\mathrm{spin}}(H\tilde{\mu})$ where $\tilde{\mu} \in \mu + o(\mathfrak{h}_\mu)$ is an $H$-admissible and $H$-regular element.

Consider the maximal torus $T := H_{\tilde{\mu}}$, and the Weyl chamber $\mathcal{C} = \mathfrak{t}^*_{\geqslant 0}$ for $K$ containing $\tilde{\mu}$. Let $J_{\mathcal{C}}$ be the corresponding complex structure on $\mathfrak{k}/\mathfrak{t}$. Let $\rho^K$ be the $\rho$ element associated to the choice of Weyl chamber. Let $C'$ be the connected component of $\mathfrak{h}_0^*$ that contains the open face $\mathfrak{t}^*_{>0}$. We check that $\rho^K = \rho_{C'} + \rho^H(\tilde{\mu})$.

Like before one has

$$\begin{aligned} I_\mu^C &= \mathrm{Ind}_H^K\Big(\bigwedge \mathfrak{q}^C \otimes \mathrm{Q}_H^{\mathrm{spin}}(H\tilde{\mu})\Big) \\ &= \mathrm{Ind}_T^K\Big(\bigwedge \mathfrak{q}^C \otimes \bigwedge_{J_{\tilde{\mu}}} \mathfrak{h}/\mathfrak{t} \otimes \mathbb{C}_{\tilde{\mu}-\rho^H(\tilde{\mu})}\Big) \\ &= \epsilon_{C'}^C \, \mathrm{Ind}_T^K\Big(\bigwedge_{J_{\mathcal{C}}} \mathfrak{k}/\mathfrak{t} \otimes \mathbb{C}_{\tilde{\mu}+\rho_C-\rho^K}\Big). \end{aligned}$$

We see then that $I_\mu^C \neq 0$ only if $\lambda := \tilde{\mu}+\rho_C = \mu' + \rho^{H_{\mu'}}$ is a $K$-regular element.

Here we have $\|\rho^{H_{\mu'}}\| = \|\lambda - \mu'\|$, and one the other hand by the magical inequality we must have $\|\lambda - \mu'\| \geqslant \|\rho^{K_{\mu'}}\|$ since $\lambda$ is $K$-regular and admissible. It forces $\|\rho^{K_{\mu'}}\|$ to be equal to $\|\rho^{H_{\mu'}}\|$, and then $K_{\mu'} = H_{\mu'}$ : the element $\mu' = \mu + \rho_C$ belongs to $\mathfrak{h}_0^*$.

The proof is completed.

**Remark 3.23** *This proposition is a particular case of the vanishing theorem that we will prove later on in Section 4.5.1. Indeed the generic stabilizer of the action of $K$ on $M \simeq K/H_\mu$ is $H_\mu$, and the moment map associated to the induced bundle is $\overline{k} \mapsto k \cdot \mu'$. Our vanishing Theorem 4.19 says then that for $\mathcal{Q}_K(M, \mathcal{S})$ to be non zero, the subalgebras $\mathfrak{h}_\mu$ and $\mathfrak{k}_{\mu'}$ have to be equal.*

## 3.6 Slices

We assume here that $M$ is a $K$-manifold and that $\Phi : M \to \mathfrak{k}^*$ is a $K$-equivariant map. If $\mathcal{O}$ is a coadjoint orbit, a neighborhood of $\Phi^{-1}(\mathcal{O})$ in $M$ can be identified with an induced manifold, and the restriction of Spin$^c$-bundles to a neighborhood of $\Phi^{-1}(\mathcal{O})$ can be identified to an induced bundle. To this aim, let us recall the notion of slice [16].

**Definition 3.24** *Let $M$ be a $K$-manifold and $m \in M$ with stabilizer subgroup $K_m$. A submanifold $Y \subset M$ containing $m$ is a slice at $m$ if $Y$ is $K_m$-invariant, $KY$ is a neighborhood of $m$, and the map*

$$K \times_{K_m} Y \longrightarrow M, \ [k, y] \mapsto ky$$

*is an isomorphism on $KY$.*

Consider the coadjoint action of $K$ on $\mathfrak{k}^*$. Define $U_\xi$ to be the connected component of the open subset $(\mathfrak{k}_\xi^*)_0 := \{\zeta \in \mathfrak{k}_\xi^* \mid \mathfrak{k}_\zeta \subset \mathfrak{k}_\xi\}$ of $\mathfrak{k}_\xi^*$ containing $\xi$. Then $K \times_{K_\xi} U_\xi \to KU_\xi$ is a diffeomorphism. We call $U_\xi$ the maximal slice at $\xi$.

The following construction was used as a fundamental tool in the symplectic setting [11].

**Proposition 3.25** *Let $\Phi : M \to \mathfrak{k}^*$ be a $K$-invariant map. Let $\xi \in \mathfrak{k}^*$, and let $U_\xi$ be the maximal slice at $\xi$.*

- *$Y = \Phi^{-1}(U_\xi)$ is a $K_\xi$-invariant submanifold of $M$ (perhaps empty).*
- *$KY$ is an open neighborhood of $\Phi^{-1}(K\xi)$ diffeomorphic to $K \times_{K_\xi} Y$.*

The manifold $Y$, when is not empty, is called the slice (of $M$) at $\xi \in \mathfrak{k}^*$. Note that $Y$ can be disconnected.

**Proof.** Let us consider the $K_\xi$-invariant decompositions $\mathfrak{k} = \mathfrak{k}_\xi \oplus \mathfrak{q}$, $\mathfrak{k}^* = \mathfrak{k}_\xi^* \oplus \mathfrak{q}^*$: we denote $\xi \to [\xi]_{\mathfrak{q}^*}$ the corresponding projection to $\mathfrak{q}^*$.

A point $\zeta$ is in $(\mathfrak{k}_\xi^*)_o$ if and only if the map $Y \in \mathfrak{q} \to Y\zeta$ is an isomorphism from $\mathfrak{q}$ to $\mathfrak{q}^*$. Thus for any $y \in Y$, the linear map $\Pi_y := [-]_{\mathfrak{q}^*} \circ \mathrm{T}_y\Phi : \mathrm{T}_yM \to \mathfrak{q}^*$ is onto. Indeed, the tangent space to $Ky$ projects onto the tangent space to $K\Phi(y)$, which contains $[\mathfrak{q}, \Phi(y)] = \mathfrak{q}^*$. Thus we obtain that $Y$ is a submanifold with tangent space $\ker(\Pi_y)$ and furthermore $\mathrm{T}_yM = \mathrm{T}_yY \oplus \mathfrak{q}\cdot y$.

The rest of the assertions follow from the fact that $U_\xi$ is a slice at $\xi$ for the coadjoint action.

# 4 Computing the multiplicities

## 4.1 Transversally elliptic operators

In this subsection, we recall the basic definitions from the theory of transversally elliptic symbols (or operators) defined by Atiyah and Singer in [1]. We refer to [4, 22] for more details.

Let $M$ be a compact $K$-manifold with cotangent bundle $\mathrm{T}^*M$. Let $p : \mathrm{T}^*M \to M$ be the projection. If $\mathcal{E}$ is a vector bundle on $M$, we may denote still by $\mathcal{E}$ the vector bundle $p^*\mathcal{E}$ on the cotangent bundle $\mathrm{T}^*M$. If $\mathcal{E}^+, \mathcal{E}^-$ are $K$-equivariant complex vector bundles over $M$, a $K$-equivariant morphism $\sigma \in \Gamma(\mathrm{T}^*M, \hom(\mathcal{E}^+, \mathcal{E}^-))$ is called a *symbol* on $M$. For $x \in M$, and $\nu \in \mathrm{T}_x^*M$, thus $\sigma(x,\nu) : \mathcal{E}_x^+ \to \mathcal{E}_x^-$ is a linear map from $\mathcal{E}_x^+$ to $\mathcal{E}_x^-$. The subset of all $(x,\nu) \in \mathrm{T}^*M$ where the map $\sigma(x,\nu)$ is not invertible is called the *characteristic set* of $\sigma$, and is denoted by $\mathrm{Char}(\sigma)$. A symbol is elliptic if its characteristic set is compact. An elliptic symbol $\sigma$ on $M$ defines an element $[\sigma]$ in the equivariant $\mathbf{K}$-theory of $\mathrm{T}^*M$ with compact support, which is denoted by $\mathbf{K}_K^0(\mathrm{T}^*M)$. The index of $\sigma$ is a virtual finite dimensional representation of $K$, that we denote by $\mathrm{Index}_K^M(\sigma) \in R(K)$.

Recall the notion of *transversally elliptic symbol*. Let $\mathrm{T}_K^*M$ be the following $K$-invariant closed subset of $\mathrm{T}^*M$

$$\mathrm{T}_K^*M \ = \{(x,\nu) \in \mathrm{T}^*M,\ \langle \nu, X \cdot x\rangle = 0 \quad \text{for all } X \in \mathfrak{k}\}\,.$$

Its fiber over a point $x \in M$ is formed by all the cotangent vectors $\nu \in \mathrm{T}_x^*M$ which vanish on the tangent space to the orbit of $x$ under $K$, in the point $x$. A symbol $\sigma$ is *$K$-transversally elliptic* if the restriction of $\sigma$ to $\mathrm{T}_K^*M$

is invertible outside a compact subset of $\mathrm{T}^*_K M$ (i.e. $\mathrm{Char}(\sigma) \cap \mathrm{T}^*_K M$ is compact).

A *$K$-transversally elliptic* symbol $\sigma$ defines an element of $\mathbf{K}^0_K(\mathrm{T}^*_K M)$, and the index of $\sigma$ defines an element $\mathrm{Index}^M_K(\sigma)$ of $\hat{R}(K)$ defined in [1].

We will use the following obvious remark. Let $\sigma \in \Gamma(\mathrm{T}^*M, \hom(\mathcal{E}^+, \mathcal{E}^-))$ be a transversally elliptic symbol on $M$.

**Lemma 4.1** *Assume an element $b \in K$ acts trivially on $M$, and that $\mathcal{E}^\pm$ are $K$-equivariant vector bundles on $M$ such that the subbundles $[\mathcal{E}^\pm]^b$ fixed by $b$ are equal to $\{0\}$. Then $[\mathrm{Index}^M_K(\sigma)]^K = 0$*

**Proof.** The space $[\mathrm{Index}^M_K(\sigma)]^K$ is constructed as the (virtual) subspace of invariant $C^\infty$-sections of the bundle $\mathcal{E}^\pm$ which are solutions of a $K$-invariant pseudo-differential operator on $M$ with symbol $\sigma$. But, as the action of $b$ is trivial on the basis, and $[\mathcal{E}^\pm]^b = \{0\}$, the space of $b$-invariant $C^\infty$-sections of the bundle $\mathcal{E}^\pm$ is reduced to 0.

Any elliptic symbol is $K$-transversally elliptic, hence we have a restriction map $\mathbf{K}^0_K(\mathrm{T}^*M) \to \mathbf{K}^0_K(\mathrm{T}^*_K M)$, and a commutative diagram

$$\begin{array}{ccc} \mathbf{K}^0_K(\mathrm{T}^*M) & \longrightarrow & \mathbf{K}^0_K(\mathrm{T}^*_K M) \\ \Big\downarrow{\scriptstyle \mathrm{Index}^M_K} & & \Big\downarrow{\scriptstyle \mathrm{Index}^M_K} \\ R(K) & \longrightarrow & \hat{R}(K)\ . \end{array} \tag{4.18}$$

Using the *excision property*, one can easily show that the index map $\mathrm{Index}_K : \mathbf{K}^0_K(\mathrm{T}^*_K \mathcal{U}) \to \hat{R}(K)$ is still defined when $\mathcal{U}$ is a $K$-invariant relatively compact open subset of a $K$-manifold (see [20][section 3.1]).

In the rest of this article, $M$ will be a Riemannian manifold, and we denote $\nu \in \mathrm{T}^*M \to \tilde{\nu} \in \mathrm{T}M$ the corresponding identification.

## 4.2 The Witten deformation

In this section $M$ is an oriented $K$-manifold of even dimension (not necessarily compact). Let $\Phi : M \to \mathfrak{k}^*$ be a $K$-equivariant map. Let $\kappa_\Phi$ be the *Kirwan vector field* associated to $\Phi$ (see (2.10)). We denote by $Z_\Phi$ the set of zeroes of $\kappa_\Phi$ (clearly $Z_\Phi$ contains the set of fixed points of the action of $K$ on $M$ as well as $\Phi^{-1}(0)$).

**Definition 4.2** *Let $\sigma(M, \mathcal{S})(m, \nu) = \mathbf{c}_{\mathcal{S}_m}(\tilde{\nu}) : \mathcal{S}^+_m \to \mathcal{S}^-_m$ be the symbol of the Dirac operator attached to the* $\mathrm{Spin}^c$*-bundle $\mathcal{S}$, and let $\Phi : M \to \mathfrak{k}^*$ be*

*an equivariant map. The symbol $\sigma(M,\mathcal{S},\Phi)$ pushed by the vector field $\kappa_\Phi$ is the symbol defined by*

$$\sigma(M,\mathcal{S},\Phi)(m,\nu) = \mathbf{c}_{\mathcal{S}_m}(\tilde{\nu} - \kappa_\Phi(m)) : \mathcal{S}_m^+ \longrightarrow \mathcal{S}_m^-$$

*for any $(m,\nu) \in \mathrm{T}M$.*

*Similarly if $\mathcal{W} \to M$ is a $K$-equivariant vector bundle, we define*

$$\sigma(M,\mathcal{S}\otimes\mathcal{W},\Phi)(m,\nu) = \sigma(M,\mathcal{S},\Phi)(m,\nu) \otimes \mathrm{Id}_{\mathcal{W}_m}.$$

Note that $\sigma(M,\mathcal{S},\Phi)(m,\nu)$ is invertible except if $\tilde{\nu} = \kappa_\Phi(m)$. If furthermore $(m,\nu)$ belongs to the subset $\mathrm{T}^*_K M$ of cotangent vectors orthogonal to the $K$-orbits, then $\nu = 0$ and $\kappa_\Phi(m) = 0$. Indeed $\kappa_\Phi(m)$ is tangent to $K\cdot m$ while $\tilde{\nu}$ is orthogonal. So we note that $(m,\nu) \in \mathrm{Char}(\sigma(M,\mathcal{S},\Phi_\mathcal{S})) \cap \mathrm{T}^*_K M$ if and only if $\nu = 0$ and $\kappa_\Phi(m) = 0$.

For any $K$-invariant open subset $\mathcal{U} \subset M$ such that $\mathcal{U} \cap Z_\Phi$ is compact in $M$, we see that the restriction $\sigma(M,\mathcal{S},\Phi)|_\mathcal{U}$ is a transversally elliptic symbol on $\mathcal{U}$, and so its equivariant index is a well defined element in $\hat{R}(K)$.

Thus we can define the following localized equivariant indices.

**Definition 4.3**

- *A closed invariant subset $Z \subset Z_\Phi$ is called a* component *if it is a union of connected components of $Z_\Phi$.*
- *If $Z \subset Z_\Phi$ is a* compact component*, and $\mathcal{W}$ is a $K$-equivariant vector bundle over $M$, we denote by*

  $$\mathcal{Q}_K(M,\mathcal{S}\otimes\mathcal{W},Z,\Phi) \ \in \ \hat{R}(K)$$

  *the equivariant index of $\sigma(M,\mathcal{S}\otimes\mathcal{W},\Phi)|_\mathcal{U}$ where $\mathcal{U}$ is an invariant neighborhood of $Z$ so that $\mathcal{U}\cap Z_\Phi = Z$.*
- *If we make the Witten deformation with the map $\Phi = \Phi_\mathcal{S}$, the term $\mathcal{Q}_K(M,\mathcal{S}\otimes\mathcal{W},Z,\Phi_\mathcal{S})$ is denoted simply by $\mathcal{Q}_K(M,\mathcal{S}\otimes\mathcal{W},Z)$.*

By definition, $Z = \varnothing$ is a component and $\mathcal{Q}_K(M,\mathcal{S}\otimes\mathcal{W},\varnothing,\Phi) = 0$.

When $M$ is compact it is clear that the classes of the symbols $\sigma(M,\mathcal{S},\Phi)$ and $\sigma(M,\mathcal{S})$ are equal in $\mathbf{K}^0_K(\mathrm{T}^*_K M)$, thus we get the first form of the localization theorem.

**Theorem 4.4** *Assume that $M$ is compact. If $Z_\Phi = Z_1 \coprod \ldots \coprod Z_p$ is a decomposition into disjoint (compact) components, we have the following equality in $\hat{R}(K)$ :*

$$\mathcal{Q}_K(M,\mathcal{S}) = \sum_{i=1}^{p} \mathcal{Q}_K(M,\mathcal{S},Z_i,\Phi)$$

**Remark 4.5** *Write $\Phi_{\mathcal{S}}(Z_{\mathcal{S}}) = \coprod_j \mathcal{O}_j$ as a disjoint union of a finite set of coadjoint orbits. Then we obtain the decomposition*

$$\mathcal{Q}_K(M, \mathcal{S}) = \sum_j \mathcal{Q}_{\mathcal{O}_j}$$

*with $\mathcal{Q}_{\mathcal{O}} = \mathcal{Q}_K(M, \mathcal{S}, \Phi_{\mathcal{S}}^{-1}(\mathcal{O}) \cap Z_{\mathcal{S}})$. As in [20], this decomposition is the main tool of our study. However, in this work, we will need to introduce a further refinement of this decomposition.*

**Example 4.6** *We return to our basic example (Example 2.12). Let $p_+ = [1, 0]$ and $p_- = [0, 1]$ be the fixed points of the $T$-action on $M = \mathbb{P}^1(\mathbb{C})$. The determinant line bundle of $\mathcal{S}(n)$ is $\mathbb{L}_n = [\mathbb{C}_{-1}] \otimes \mathcal{L}^{\otimes 2n+2}$ where $[\mathbb{C}_{-1}]$ is the trivial line bundle equipped with the representation $t^{-1}$ on $\mathbb{C}$. We choose the moment map $\Phi_n$ associated to a connection on the determinant bundle (see more details in Section 6):*

$$\Phi_n([z_1, z_2]) = (n+1)\frac{|z_1|^2}{|z_1|^2 + |z_2|^2} - \frac{1}{2}.$$

*Then, for $n \geqslant 0$, $\mathcal{Z} = \{p_+\} \cup \{p_-\} \cup \Phi_n^{-1}(0)$, thus $\Phi_n(Z_{\mathcal{S}}) = \{-\frac{1}{2}\} \cup \{0\} \cup \{n + \frac{1}{2}\}$. Remark that $Z_{\mathcal{S}}$ is smooth: it has 3 connected components, the two fixed points, and $\Phi_n^{-1}(0)$ a circle with free action of $T$. Then we obtain the associated decomposition $\mathcal{Q}_T(M, \mathcal{S}(n)) = Q_{-\frac{1}{2}} + Q_0 + Q_{\frac{1}{2}}$ with*

$$Q_{-\frac{1}{2}} = -\sum_{k=-1}^{-\infty} t^k, \quad Q_0 = \sum_{k=-\infty}^{-\infty} t^k, \quad Q_{\frac{1}{2}} = -\sum_{k=n+1}^{\infty} t^k.$$

**Example 4.7** *Take the product $N = \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$, with $\mathrm{Spin}^c$ bundle $\mathcal{S} = \mathcal{S}(0) \otimes \mathcal{S}(0)$, moment map $\Phi_0$ and we consider the diagonal action of $T$ with moment map $\Phi(m_1, m_2) = \Phi_0(m_1) + \Phi_0(m_2)$. As $\mathcal{Q}_T(\mathbb{P}^1(\mathbb{C}), \mathcal{S}(0))$ is the trivial representation of $T$, $\mathcal{Q}_T(N, \mathcal{S})$ is still the trivial representation of $T$.*

*We have $\Phi(Z_{\mathcal{S}}) = \{-1\} \cup \{0\} \cup \{1\}$. In this case $\Phi^{-1}(\pm 1) = \{(p_\pm, p_\pm)\}$, and $\Phi^{-1}(0)$ is not smooth.*

*Consider the associated decomposition of $\mathcal{Q}_T(N, \mathcal{S}) = Q_{-1} + Q_0 + Q_1$. We have*

$$Q_{-1} = \sum_{k=-\infty}^{-2} (-k-1)t^k, \quad Q_0 = \sum_{k=-\infty}^{-\infty} (|k|-1)t^k, \quad Q_1 = \sum_{k=2}^{\infty} (k-1)t^k.$$

*We see that indeed $Q_{-1} + Q_0 + Q_1 = t^0$. Figure 9 shows the corresponding multiplicity functions.*

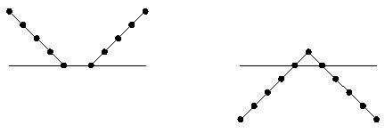

Figure 9: The graph of $Q_{-1} + Q_1$ and the graph of $Q_0$

## 4.3 Some properties of the localized index

In this subsection, we recall the properties of the localized index $\mathcal{Q}_K(M, \mathcal{S}, Z, \Phi)$ that we will use in this article.

### 4.3.1 Fixed point submanifolds and $\mathrm{Spin}^c$-bundles

Let $\mathcal{S}$ be a $K$-equivariant $\mathrm{Spin}^c$-bundle over the tangent bundle $\mathrm{T}M$ of a $K$-manifold $M$ (equipped with an invariant Riemannian metric). The manifold $M$ is oriented and the Clifford bundle $\mathcal{S}$ is equipped with its canonical $\mathbb{Z}/2\mathbb{Z}$-grading. Let $b \in \mathfrak{k}$ be a non-zero $K$-invariant element, and consider the submanifold $M^b$ where the vector field $b_M$ vanishes. We have an orthogonal decomposition

$$\mathrm{T}M|_{M^b} = \mathcal{N} \oplus \mathrm{T}M^b.$$

The normal bundle $\mathcal{N}$ inherits a fibrewise linear endomorphism $\mathcal{L}(b)$ which is anti-symmetric relatively to the metric.

**Definition 4.8** • *We denote by $\mathcal{N}_b$ the vector bundle $\mathcal{N}$ over $M^b$ equipped with the complex structure $J_b := \mathcal{L}(b)|\mathcal{L}(b)|^{-1}$.*

• *We take on $\mathcal{N}$ the orientation $o(\mathcal{N})$ induced by the complex structure $-J_b$. On $M^b$ we take the orientation $o(M^b)$ defined by $o(\mathcal{N})o(M^b) = o(M)$.*

Note that the endomorphism $\mathcal{L}(b) : \mathcal{N}_b \to \mathcal{N}_b$ is $\mathbb{C}$-linear, diagonalizable, with eigenvalues $i\theta^1_{\mathcal{X}}, \ldots, i\theta^p_{\mathcal{X}}$ that depends of the connected component $\mathcal{X}$ of $M^b$. For further use, we note the following positivity result which follows directly from the definition of $J_b$.

**Lemma 4.9** *The eigenvalues of the action of $\frac{1}{i}\mathcal{L}(b)$ on $\mathcal{N}_b$ are positive.*

If we consider the complex line bundle $\det(\mathcal{N}_b) \to M^b$, we see that $\frac{1}{i}\mathcal{L}(b)$ acts on the fibers of $\det(\mathcal{N}_b)|_{\mathcal{X}}$ by multiplication by the positive number

$$\mathbf{nTr}_{\mathcal{N}_b|_{\mathcal{X}}}|b| = \sum_{j=1}^{p} \theta^j_{\mathcal{X}}.$$

**Proposition 4.10** *Let $\mathbb{L}_{\mathcal{S}}$ be the determinant line bundle of the* $\mathrm{Spin}^c$ *bundle $\mathcal{S}$. There exists a equivariant* $\mathrm{Spin}^c$*-bundle $\mathbb{d}_b(\mathcal{S})$ on the tangent bundle $\mathrm{T}M^b$ with determinant line bundle equal to*

$$\mathbb{L}_{\mathbb{d}_b(\mathcal{S})} := \mathbb{L}_{\mathcal{S}}|_{M^b} \otimes \det(\mathcal{N}_b). \tag{4.19}$$

**Proof.** The restriction $\mathcal{S}|_{M^b}$ is a $\mathrm{Spin}^c$-bundle over the tangent bundle $\mathrm{T}M|_{M^b} = \mathcal{N} \oplus \mathrm{T}M^b$. We denote $\overline{\mathcal{N}_b}$ the vector bundle $\mathcal{N}$ with the complex structure $-J_b$. Let $\bigwedge \overline{\mathcal{N}_b}$ be the $\mathrm{Spin}^c$ bundle on $\mathcal{N}$ with its canonical grading : since $o(\mathcal{N}) = o(-J_b)$ we have $(\bigwedge \overline{\mathcal{N}_b})^{\pm} = \bigwedge^{\pm} \overline{\mathcal{N}_b}$.

Since $\bigwedge \overline{\mathcal{N}_b}$ is a graded $\mathrm{Spin}^c$-bundle over $\mathcal{N}$, we know that there exists an equivariant $\mathrm{Spin}^c$ bundle $\mathbb{d}_b(\mathcal{S})$ over the tangent bundle $\mathrm{T}M^b$ (with its canonical grading) such that

$$\mathcal{S}|_{M^b} = \bigwedge \overline{\mathcal{N}_b} \otimes \mathbb{d}_b(\mathcal{S}). \tag{4.20}$$

is an isomorphism of graded Clifford modules. At the level of determinant line bundle, we get $\det(\mathcal{S})|_{M^b} = \det(\overline{\mathcal{N}_b}) \otimes \det(\mathbb{d}_b(\mathcal{S}))$. Identity (4.19) follows.

Consider the linear action $\mathcal{L}(b)|_{\mathbb{d}_b(\mathcal{S})}$ of $b$ on the fibers of the $\mathrm{Spin}^c$-bundle $\mathbb{d}_b(\mathcal{S}) \to M^b$.

**Lemma 4.11** *We have $\frac{1}{i}\mathcal{L}(b)|_{\mathbb{d}_b(\mathcal{S})} = a\,\mathrm{Id}_{\mathbb{d}_b(\mathcal{S})}$ where*

$$a(m) = \langle \Phi_{\mathcal{S}}(m), b \rangle + \frac{1}{2}\mathbf{n}\mathbf{Tr}_{\mathrm{T}_m M}|b|$$

*is a locally constant function on $M^b$.*

**Proof.** Thanks to Remark 2.8, we know that $a(m)$ is equal to $\langle \Phi_{\mathbb{d}_b(\mathcal{S})}(m), b \rangle$ where $\Phi_{\mathbb{d}_b(\mathcal{S})}$ is a moment map attached to the line bundle $\mathbb{L}_{\mathbb{d}_b(\mathcal{S})}$. Thanks to (4.19) we see that $\langle \Phi_{\mathbb{d}_b(\mathcal{S})}(m), b \rangle = \langle \Phi_{\mathcal{S}}(m), b \rangle + \frac{1}{2}\mathbf{Tr}_{\mathcal{N}_b}|b|$. But $\mathbf{n}\mathbf{Tr}_{\mathrm{T}M}|b| = \mathbf{Tr}_{\mathcal{N}_b}|b|$ as well as and $\langle \Phi_{\mathcal{S}}(m), b \rangle$ are locally constant on $M^b$.

The localization formula of Atiyah-Segal can be expressed in the following way (see [24]):

**Theorem 4.12** *Let $b \in \mathfrak{k}$ be a non-zero $K$-invariant element and assume that $M$ is compact. For any complex $K$-vector bundle $\mathcal{W} \to M$, we have the following equalities in $\hat{R}(K)$ :*

$$\mathcal{Q}_K(M, \mathcal{S} \otimes \mathcal{W}) = \mathcal{Q}_K\left(M^b, \mathbb{d}_b(\mathcal{S}) \otimes \mathcal{W}|_{M^b} \otimes \mathrm{Sym}(\mathcal{N}_b)\right).$$

*Here* $\mathrm{Sym}(\mathcal{N}_b)$ *is the symmetric algebra of the complex vector bundle $\mathcal{N}_b$.*

### 4.3.2 The localization formula over a coadjoint orbit

Let $\Phi : M \to \mathfrak{k}^*$ be an equivariant map. Let $\beta \in \mathfrak{k}^*$. We also consider $\beta$ as an element of $\mathfrak{k}$ that we denote by the same symbol. In this section we assume that $Z_\beta = K(M^\beta \cap \Phi^{-1}(\beta))$ is a compact component of $Z_\Phi \subset M$. The study of $\mathcal{Q}_K(M, \mathcal{S} \otimes \mathcal{W}, Z_\beta, \Phi) \in \hat{R}(K)$ is thus localized in a neighborhood of $\Phi^{-1}(K\beta)$, an induced manifold. Let us recall the corresponding induction formula.

The restriction of $\Phi$ to $M^\beta$ is a $K_\beta$-equivariant map taking value in $\mathfrak{k}_\beta^*$. The subset $Z'_\beta = M^\beta \cap \Phi^{-1}(\beta)$ is a compact component of $Z_{\Phi|_{M^\beta}} = Z_\Phi \cap M^\beta$. We may then define the localized index

$$\mathcal{Q}_{K_\beta}(M^\beta, \mathbb{d}_\beta(\mathcal{S}) \otimes \mathcal{W}|_{M^\beta}, Z'_\beta, \Phi|_{M^\beta}) \in \hat{R}(K_\beta)$$

where $\mathbb{d}_\beta(\mathcal{S})$ is the graded Spin$^c$-bundle on $M^\beta$ defined in Proposition 4.10.

We consider the normal bundle $\mathcal{N} \to M^\beta$ of $M^\beta$ in $M$. Recall that $\mathcal{N}_\beta$ denotes the vector bundle $\mathcal{N}$ equipped with the complex $J_\beta$. The following formula is proved in [20, 24]:

$$\begin{aligned}&\mathcal{Q}_K(M, \mathcal{S} \otimes \mathcal{W}, Z_\beta, \Phi)\\ &= \operatorname{Ind}_{K_\beta}^K \left( \mathcal{Q}_{K_\beta}(M^\beta, \mathbb{d}_\beta(\mathcal{S}) \otimes \mathcal{W}|_{M^\beta} \otimes \operatorname{Sym}(\mathcal{N}_\beta), Z'_\beta, \Phi|_{M^\beta}) \otimes \bigwedge (\mathfrak{k}/\mathfrak{k}_\beta)_{\mathbb{C}} \right).\end{aligned}$$

**Remark 4.13** *When $K$ is abelian, this gives*

$$\begin{aligned}&\mathcal{Q}_K(M, \mathcal{S} \otimes \mathcal{W}, \Phi^{-1}(\beta) \cap M^\beta, \Phi)\\ &= \mathcal{Q}_K(M^\beta, \mathbb{d}_\beta(\mathcal{S}) \otimes \mathcal{W}|_{M^\beta} \otimes \operatorname{Sym}(\mathcal{N}_\beta), \Phi^{-1}(\beta) \cap M^\beta, \Phi|_{M^\beta})\end{aligned}$$

*which shows that the Atiyah-Segal localization formula (4.12) still holds for the Witten deformation.*

Thus we obtain the following proposition.

**Proposition 4.14** *Let $\mathcal{S}$ be a $K$-equivariant* Spin$^c$*-bundle over $M$, with its canonical grading. Let $\Phi : M \to \mathfrak{k}^*$ be an equivariant map. Let $\mathcal{W} \to M$ be an equivariant complex vector bundle. Assume that $Z_\beta = K(M^\beta \cap \Phi^{-1}(\beta))$ is a compact component of $Z_\Phi \subset M$. Then*

$$\begin{aligned}&[\mathcal{Q}_K(M, \mathcal{S} \otimes \mathcal{W}, Z_\beta, \Phi)]^K =\\ &\left[ \mathcal{Q}_{K_\beta}(M^\beta, \mathbb{d}_\beta(\mathcal{S}) \otimes \mathcal{W}|_{M^\beta} \otimes \operatorname{Sym}(\mathcal{N}_\beta), Z'_\beta, \Phi|_{M^\beta}) \otimes \bigwedge (\mathfrak{k}/\mathfrak{k}_\beta)_{\mathbb{C}} \right]^{K_\beta}. \qquad (4.21)\end{aligned}$$

This proposition will be used to obtain vanishing results, by studying the infinitesimal action of $\beta$ on the vector bundle $\mathbb{d}_\beta(\mathcal{S})\otimes\mathcal{W}|_{M^\beta}\otimes\mathrm{Sym}(\mathcal{N}_\beta)$.

The formula (4.21) can be specialized to each connected component of $M^\beta$. For a connected component $\mathcal{X}\subset M^\beta$ intersecting $\Phi^{-1}(\beta)$, we define the compact subset

$$Z_\beta(\mathcal{X}) = K\left(\mathcal{X}\cap\Phi^{-1}(\beta)\right)\subset Z_\beta.$$

First we note that $\mathcal{Q}_K(M,\mathcal{S}\otimes\mathcal{W},Z_\beta,\Phi)$ is equal to the sum $\sum_{\mathcal{X}}\mathcal{Q}_K(M,\mathcal{S}\otimes\mathcal{W},Z_\beta(\mathcal{X}),\Phi)$ parameterized by the connected component of $M^\beta$ intersecting $\Phi^{-1}(\beta)$ (their are finite in number).

We have a localization formula for each term $\mathcal{Q}_K(M,\mathcal{S}\otimes\mathcal{W},Z_\beta(\mathcal{X}),\Phi)$ separately (see [20, 24]) :

$$[\mathcal{Q}_K(M,\mathcal{S}\otimes\mathcal{W},Z_\beta(\mathcal{X}),\Phi)]^K = \tag{4.22}$$
$$\left[\mathcal{Q}_{K_\beta}(\mathcal{X},\mathbb{d}_\beta(\mathcal{S})|_{\mathcal{X}}\otimes\mathcal{W}|_{\mathcal{X}}\otimes\mathrm{Sym}(\mathcal{N}_\beta)|_{\mathcal{X}},Z'_\beta(\mathcal{X}),\Phi|_{\mathcal{X}})\otimes\bigwedge(\mathfrak{k}/\mathfrak{k}_\beta)_{\mathbb{C}}\right]^{K_\beta}$$

where $Z'_\beta(\mathcal{X}) = \mathcal{X}\cap\Phi^{-1}(\beta)\subset Z'_\beta$.

### 4.3.3 Induction formula

For the Witten deformation, we proved in [24] the following variation on the invariance of the index under direct images.

Let $H$ be a closed subgroup of $K$, and consider a $H$-invariant decomposition

$$\mathfrak{k} = \mathfrak{h}\oplus\mathfrak{q}.$$

Let $B_\mathfrak{q}$ be an open ball in $\mathfrak{q}$, centered at 0 and $H$-invariant. Let $N'$ be a $H$-manifold, and consider $N = K\times_H(B_\mathfrak{q}\times N')$. Then $N'$ is a submanifold of $M$, and the normal bundle of $N'$ in $N$ is isomorphic to the trivial bundle with fiber $\mathfrak{q}\oplus\mathfrak{q}$. Let $S_\mathfrak{q}$ be the Spin$^c$ module for $\mathfrak{q}\oplus\mathfrak{q}$ (we can take $\bigwedge\mathfrak{q}_{\mathbb{C}}$ as realization of $S_\mathfrak{q}$). Thus if $\mathcal{E}$ is a $K$-equivariant graded Clifford bundle on $N$, there exists a $H$-equivariant graded Clifford bundle $\mathcal{E}'$ on $N'$ such that

$$\mathcal{E}|_{N'} = S_\mathfrak{q}\otimes\mathcal{E}'.$$

Let $\Phi' : N'\to\mathfrak{h}^*$ be a $H$-equivariant map, and let $\Phi : N\to\mathfrak{k}^*$ be a $K$-equivariant map. We assume that these maps are linked by the following relations :

$$\begin{cases}\Phi|_{N'} = \Phi',\\ \Phi([1;X,n'])\in\mathfrak{h}^*\Longleftrightarrow X=0,\\ (\Phi([1;X,n']),X)\geqslant 0,\end{cases}\tag{4.23}$$

for $(X, n') \in B_{\mathfrak{q}} \times N'$.

Under these conditions, we see that the critical sets $Z_\Phi \subset N$ and $Z_{\Phi'} \subset N'$ are related by : $Z_\Phi = K \times_H (\{0\} \times Z_{\Phi'})$.

**Proposition 4.15 ([24])** *Let $Z$ be a compact component of $Z_\Phi$ and $Z'$ its intersection with $N'$. Then $Z'$ is a compact component of $Z_{\Phi'}$ and*

$$\mathcal{Q}_K(N, \mathcal{E}, Z, \Phi) = \mathrm{Ind}_H^K \left( \mathcal{Q}_H(N', \mathcal{E}', Z', \Phi') \right).$$

*This leads to the relation $[\mathcal{Q}_K(N, \mathcal{E}, Z, \Phi)]^K = [\mathcal{Q}_H(N', \mathcal{E}', Z', \Phi')]^H$.*

## 4.4 The function $d_{\mathcal{S}}$

Let $M$ be a *compact* oriented even dimensional $K$-manifold, equipped with a $K$-equivariant Spin$^c$ bundle $\mathcal{S}$. Let $\Phi_{\mathcal{S}}$ be the associated moment map on $M$, and $\kappa_{\mathcal{S}}$ be the Kirwan vector field. Let $Z_{\mathcal{S}}$ be the vanishing set of $\kappa_{\mathcal{S}}$ :

$$Z_{\mathcal{S}} = \{m \in M \mid \Phi_{\mathcal{S}}(m) \cdot m = 0\} = \bigcup_\theta M^\theta \cap \Phi_{\mathcal{S}}^{-1}(\theta).$$

We now introduce a function $d_{\mathcal{S}} : Z_{\mathcal{S}} \longrightarrow \mathbb{R}$ which will localize our study of $[\mathcal{Q}_K(M, \mathcal{S}, Z_{\mathcal{S}})]^K$ to special components $Z$ of $Z_{\mathcal{S}}$.

Define $d_{\mathcal{S}} : Z_{\mathcal{S}} \longrightarrow \mathbb{R}$ by the following relation

$$d_{\mathcal{S}}(m) = \|\theta\|^2 + \frac{1}{2}\mathbf{nTr}_{\mathrm{T}_m M}|\theta| - \mathbf{nTr}_{\mathfrak{k}}|\theta|, \quad \text{with} \quad \theta = \Phi_{\mathcal{S}}(m). \tag{4.24}$$

**Lemma 4.16**

- *The function $d_{\mathcal{S}}$ is a $K$-invariant locally constant function on $Z_{\mathcal{S}}$ that takes a finite number of values.*
- *The subsets $Z_{\mathcal{S}}^{>0} = \{d_{\mathcal{S}} > 0\}$, $Z_{\mathcal{S}}^{=0} = \{d_{\mathcal{S}} = 0\}$, $Z_{\mathcal{S}}^{<0} = \{d_{\mathcal{S}} < 0\}$ are components of $Z_{\mathcal{S}}$.*

**Proof.** The $K$-invariance of $d_{\mathcal{S}}$ is immediate.

The image $\Phi_{\mathcal{S}}(Z_{\mathcal{S}})$ is equal to a finite union $\bigcup_j \mathcal{O}_j$ of coadjoint orbits. For each coadjoint orbit $\mathcal{O} = K\beta$, the set $Z_{\mathcal{S}} \cap \Phi_{\mathcal{S}}^{-1}(\mathcal{O})$ is equal to a finite disjoint union $\bigcup_j K(\mathcal{X}^j \cap \Phi_{\mathcal{S}}^{-1}(\beta))$ where $(\mathcal{X}^j)$ are the connected components of $M^\beta$ intersecting $\Phi_{\mathcal{S}}^{-1}(\beta)$. Since $m \mapsto \mathbf{nTr}_{\mathrm{T}_m M}|\theta|$ is well defined and locally constant on $M^\theta$, the map $d_{\mathcal{S}}$ is constant on each component $K(\mathcal{X}^j \cap \Phi_{\mathcal{S}}^{-1}(\beta))$. This proves that $d_{\mathcal{S}}$ is locally constant function that takes a finite number of values.

The second point is a direct consequence of the first.

We now prove the following fundamental fact.

**Proposition 4.17** *Let $Z_{\mathcal{S}}^{\geqslant 0}$ be the component of $Z_{\mathcal{S}}$ where $d_{\mathcal{S}}$ takes strictly positive values. We have $\left[\mathcal{Q}_K(M,\mathcal{S},Z_{\mathcal{S}}^{\geqslant 0})\right]^K = 0$.*

Since $\mathcal{Q}_K(M,\mathcal{S}) = \mathcal{Q}_K(M,\mathcal{S},Z_{\mathcal{S}}^{\leqslant 0}) + \mathcal{Q}_K(M,\mathcal{S},Z_{\mathcal{S}}^{=0}) + \mathcal{Q}_K(M,\mathcal{S},Z_{\mathcal{S}}^{\geqslant 0})$ by Theorem 4.4, note first the following immediate corollary.

**Corollary 4.18** *If $d_{\mathcal{S}}$ takes non negative values on $Z_{\mathcal{S}}$, we have*

$$[\mathcal{Q}_K(M,\mathcal{S})]^K = [\mathcal{Q}_K(M,\mathcal{S},Z_{\mathcal{S}}^{=0})]^K.$$

We now prove Proposition 4.17.

**Proof.** Consider a coadjoint orbit $K\beta$ contained in $\Phi_{\mathcal{S}}(Z_{\mathcal{S}})$. Let $\mathcal{X}$ be the connected component of $M^\beta$ and let $Z'_\beta(\mathcal{X}) := \mathcal{X} \cap \Phi^{-1}(\beta)$. Let $Z_\beta(\mathcal{X}) = KZ'_\beta(\mathcal{X})$. Let us show that $[\mathcal{Q}_K(M,\mathcal{S},Z_\beta(\mathcal{X}))]^K = 0$ if $d_{\mathcal{S}}$ is strictly positive on $Z_\beta(\mathcal{X})$.

As $[\mathcal{Q}_K(M,\mathcal{S},Z_\beta(\mathcal{X}))]^K$ is equal to

$$\left[\mathcal{Q}_{K_\beta}(\mathcal{X},\mathfrak{d}_\beta(\mathcal{S})|_{\mathcal{X}} \otimes \mathrm{Sym}(\mathcal{N}_\beta)|_{\mathcal{X}}, Z'_\beta(\mathcal{X}), \Phi_{\mathcal{S}}|_{\mathcal{X}}) \otimes \bigwedge(\mathfrak{k}/\mathfrak{k}_\beta)_{\mathbb{C}}\right]^{K_\beta} \tag{4.25}$$

by the localization formula (4.22), it is sufficient to prove that the infinitesimal action $\mathcal{L}(\beta)$ on the fibers of the vector bundles $\mathfrak{d}_\beta(\mathcal{S})|_{\mathcal{X}} \otimes \mathrm{Sym}^j(\mathcal{N}_\beta)|_{\mathcal{X}} \otimes \bigwedge(\mathfrak{k}/\mathfrak{k}_\beta)_{\mathbb{C}}$ have only strictly positive eigenvalues. We establish this by minorizing the possible eigenvalues : they are sums of eigenvalues on each factor of the tensor product.

We have

$$\frac{1}{i}\mathcal{L}(\beta) = \begin{cases} \|\beta\|^2 + \frac{1}{2}\mathbf{nTr}_{\mathrm{T}M|_{\mathcal{X}}}|\beta| & \text{on } \mathfrak{d}_\beta(\mathcal{S})|_{\mathcal{X}}, \\ \geqslant 0 & \text{on } \mathrm{Sym}^j(\mathcal{N}_\beta)|_{\mathcal{X}}, \\ \geqslant -\mathbf{nTr}_{\mathfrak{k}}|\beta| & \text{on } \bigwedge(\mathfrak{k}/\mathfrak{k}_\beta)_{\mathbb{C}}. \end{cases}$$

In the first equality, we have used Lemma 4.11: the function $m \mapsto \langle \Phi_{\mathcal{S}}(m), \beta\rangle$ is constant on $\mathcal{X}$, and as $\mathcal{X}$ contains a point projecting on $\beta$, $\frac{1}{i}\mathcal{L}(\beta)|_{\mathfrak{d}_\beta(\mathcal{S})|_{\mathcal{X}}} = (\|\beta\|^2 + \frac{1}{2}\mathbf{nTr}_{\mathrm{T}M|_{\mathcal{X}}}|\beta|)\,\mathrm{Id}_{\mathfrak{d}_\beta(\mathcal{S})|_{\mathcal{X}}}$.

In the second inequality, we used Lemma 4.9, so that the action of $\frac{1}{i}\mathcal{L}(\beta)$ on the graded piece $\mathrm{Sym}^j(\mathcal{N}_\beta)$ is strictly positive for $j > 0$ or equal to 0 for $j = 0$.

In the last inequality, we have used Lemma 3.19.

If $d_{\mathcal{S}}$ takes a strictly positive value on $Z_\beta(\mathcal{X})$, we see that $\frac{1}{i}\mathcal{L}(\beta) > 0$ on $\mathfrak{d}_\beta(\mathcal{S})|_{\mathcal{X}} \otimes \mathrm{Sym}^j(\mathcal{N}_\beta)|_{\mathcal{X}} \otimes \bigwedge(\mathfrak{k}/\mathfrak{k}_\beta)_{\mathbb{C}}$ : this forces (4.25) to be equal to zero.

## 4.5 The Witten deformation on the product $M \times \mathcal{O}^*$

In this section, $M$ is a *compact* oriented even dimensional $K$-manifold, equipped with a $K$-equivariant Spin$^c$ bundle $\mathcal{S}$. Let $\Phi_{\mathcal{S}}$ be the associated moment map on $M$. Our aim is to compute geometrically the multiplicities of the equivariant index $\mathcal{Q}_K(M, \mathcal{S})$.

### 4.5.1 Vanishing theorems

Let $\mathcal{H}_{\mathfrak{k}}$ be the set of conjugacy classes of the reductive algebras $\mathfrak{k}_\xi, \xi \in \mathfrak{k}^*$. We denote by $\mathcal{S}_{\mathfrak{k}}$ the set of conjugacy classes of the semi-simple parts $[\mathfrak{h}, \mathfrak{h}]$ of the elements $(\mathfrak{h}) \in \mathcal{H}_{\mathfrak{k}}$.

Recall that an orbit $\mathcal{P}$ is a $(\mathfrak{h})$-ancestor of $\mathcal{O}$ if $\mathcal{P}$ belongs to the Dixmier sheet $\mathfrak{k}^*_{(\mathfrak{h})}$ and $s(\mathcal{P}) = \mathcal{O}$. Here $s(\mathcal{P})$ is defined as follows : if $\mathcal{P} = K\mu$ with $\mathfrak{k}_\mu = \mathfrak{h}$, then $s(\mathcal{P}) = K(\mu + o(\mathfrak{h}))$ (see Definition 3.7).

Recall that the map $\mathcal{O} \mapsto \pi_{\mathcal{O}} := \mathrm{Q}_K^{\mathrm{spin}}(\mathcal{O})$ is a bijection between the regular admissible orbits and $\widehat{K}$. If $\mathcal{O}$ is a regular admissible orbit, then $\mathcal{O}^* := -\mathcal{O}$ is also admissible and $\pi_{\mathcal{O}^*} = (\pi_{\mathcal{O}})^*$. If we apply the shifting trick, we see that the multiplicity of $\pi_{\mathcal{O}}$ in $\mathcal{Q}_K(M, \mathcal{S})$ is equal to

$$\begin{aligned} \mathrm{m}_{\mathcal{O}} &= [\mathcal{Q}_K(M, \mathcal{S}) \otimes (\pi_{\mathcal{O}})^*]^K \\ &= [\mathcal{Q}_K(M \times \mathcal{O}^*, \mathcal{S} \otimes \mathcal{S}_{\mathcal{O}^*})]^K . \end{aligned} \tag{4.26}$$

Let $(\mathfrak{k}_M)$ be the generic infinitesimal stabilizer of the $K$-action on $M$. In this section, we prove the following vanishing results.

**Theorem 4.19**

- *If* $([\mathfrak{k}_M, \mathfrak{k}_M]) \neq ([\mathfrak{h}, \mathfrak{h}])$ *for any* $(\mathfrak{h}) \in \mathcal{H}_{\mathfrak{k}}$*, then*

$$\mathcal{Q}_K(M, \mathcal{S}) = 0$$

  *for any* $K$*-equivariant* Spin$^c$*-bundle* $\mathcal{S}$ *on* $M$*.*

- *Assume that* $([\mathfrak{k}_M, \mathfrak{k}_M]) = ([\mathfrak{h}, \mathfrak{h}])$ *for* $(\mathfrak{h}) \in \mathcal{H}_{\mathfrak{k}}$*. Then*

$$\mathrm{m}_{\mathcal{O}} = 0$$

  *if there is no* $(\mathfrak{h})$*-ancestor* $\mathcal{P}$ *to* $\mathcal{O}$ *contained in* $\Phi_{\mathcal{S}}(M)$*.*

We consider the product $M \times \mathcal{O}^*$ equipped with the Spin$^c$-bundle $\mathcal{S} \otimes \mathcal{S}_{\mathcal{O}^*}$. The corresponding moment map is $\Phi_{\mathcal{S}\otimes\mathcal{S}_{\mathcal{O}^*}}(m, \xi) = \Phi_{\mathcal{S}}(m) + \xi$. We use the simplified notation $\Phi_{\mathcal{O}}$ for $\Phi_{\mathcal{S}\otimes\mathcal{S}_{\mathcal{O}^*}}$, $\kappa_{\mathcal{O}}$ for the corresponding Kirwan vector field on $M \times \mathcal{O}^*$, $Z_{\mathcal{O}} := \{\kappa_{\mathcal{O}} = 0\}$, and $d_{\mathcal{O}}$ for the function $d_{\mathcal{S}\otimes\mathcal{S}_{\mathcal{O}^*}}$

on $Z_{\mathcal{O}}$. Theorem 4.19 will result from a careful analysis of the function $d_{\mathcal{O}} : Z_{\mathcal{O}} \to \mathbb{R}$ that was introduced in Section 4.4. Thanks to Proposition 4.17 and Corollary 4.18, Theorem 4.19 is a direct consequence of the following theorem.

**Theorem 4.20** *Let $\mathcal{O}$ be a regular admissible orbit.*

- *The function $d_{\mathcal{O}}$ is non negative on $Z_{\mathcal{O}}$.*
- *If the function $d_{\mathcal{O}}$ is not strictly positive, then there exists a unique $(\mathfrak{h}) \in \mathcal{H}_{\mathfrak{k}}$ such that the following conditions are satisfied:*
  1. *$([\mathfrak{k}_M, \mathfrak{k}_M]) = ([\mathfrak{h}, \mathfrak{h}])$.*
  2. *the orbit $\mathcal{O}$ has an $(\mathfrak{h})$-ancestor $\mathcal{P}$ contained in $\Phi_{\mathcal{S}}(M)$.*

**Proof.** Let $P = M \times \mathcal{O}^*$ and let us compute the function $d_{\mathcal{O}}$ on $Z_{\mathcal{O}}$. Let $m \in M$ and $\lambda \in \mathcal{O}$. The point $p = (m, -\lambda) \in Z_{\mathcal{O}} \subset P$ if and only $\Phi_{\mathcal{O}}(p) \cdot p = 0$. Let $\beta = \Phi_{\mathcal{O}}(p)$. This means that $\beta$ stabilizes $m$ and $\lambda$, and if $\mu = \Phi_{\mathcal{S}}(m) \in \mathfrak{k}^*$, then $\beta = \mu - \lambda$.

We write $\mathrm{T}_{(m,-\lambda)}P = \mathrm{T}_m M \oplus \mathrm{T}_{-\lambda}\mathcal{O}^*$ and, since $\mathcal{O}^*$ is a regular orbit, we have $\mathbf{nTr}_{\mathrm{T}_{-\lambda}\mathcal{O}^*}|\beta| = \mathbf{nTr}_{\mathfrak{k}}|\beta|$.

We consider a $K_m$-invariant decomposition $\mathrm{T}_m M = \mathfrak{k} \cdot m \oplus E_m$ where $\mathfrak{k} \cdot m \simeq \mathfrak{k}/\mathfrak{k}_m$, we obtain $\mathbf{nTr}_{\mathrm{T}_m M}|\beta| = \mathbf{nTr}_{E_m}|\beta| + \mathbf{nTr}_{\mathfrak{k}}|\beta| - \mathbf{nTr}_{\mathfrak{k}_m}|\beta|$. Thus,

$$\begin{aligned} d_{\mathcal{O}}(p) &= \|\beta\|^2 + \frac{1}{2}\mathbf{nTr}_{\mathrm{T}_{(m,-\lambda)}P}|\beta| - \mathbf{nTr}_{\mathfrak{k}}|\beta| \\ &= \|\beta\|^2 + \frac{1}{2}\mathbf{nTr}_{\mathrm{T}_m M}|\beta| - \frac{1}{2}\mathbf{nTr}_{\mathfrak{k}}|\beta| \\ &= \|\beta\|^2 + \frac{1}{2}\mathbf{nTr}_{E_m}|\beta| - \frac{1}{2}\mathbf{nTr}_{\mathfrak{k}_m}|\beta| \\ &\geqslant \|\beta\|^2 + \frac{1}{2}\mathbf{nTr}_{E_m}|\beta| - \frac{1}{2}\mathbf{nTr}_{\mathfrak{k}_\mu}|\beta|. \end{aligned} \tag{4.27}$$

In the last inequality, we used $\mathfrak{k}_m \subset \mathfrak{k}_\mu$ as $\mu = \Phi_{\mathcal{S}}(m)$. By Proposition 3.21, $\|\beta\|^2 - \frac{1}{2}\mathbf{nTr}_{\mathfrak{k}_\mu}|\beta| \geqslant 0$ when $\beta = \mu - \lambda$, as $\lambda$ is very regular (being regular and admissible), and $\beta \in \mathfrak{k}_\mu \cap \mathfrak{k}_\lambda$. Then the first point follows.

Assume now that there exists a point $p = (m, -\lambda) \in Z_{\mathcal{O}}$ such that $d_{\mathcal{O}}(p) = 0$. It implies then that $\|\beta\|^2 = \frac{1}{2}\mathbf{nTr}_{\mathfrak{k}_\mu}|\beta|$ and $\mathbf{nTr}_{E_m}|\beta| = 0$. The first equality implies, thanks to Proposition 3.21, that $K\mu$ is an admissible orbit such that $s(K\mu) = \mathcal{O}$. Let us denote $H = K_\mu$ : the relation $s(K\mu) = \mathcal{O}$ implies that $-\beta \in o(\mathfrak{h}) \subset [\mathfrak{h}, \mathfrak{h}]^*$. We write $-\beta = \rho^H$. Now we have to

explain why the condition $\mathbf{nTr}_{E_m}|\rho^H| = 0$ implies $([\mathfrak{k}_M, \mathfrak{k}_M]) = ([\mathfrak{h}, \mathfrak{h}])$. Since $\Phi_{\mathcal{S}}(m) = \mu$, we have

$$(\mathfrak{k}_M) \subset (\mathfrak{k}_m) \subset (\mathfrak{h}). \tag{4.28}$$

Consider $Y = \Phi_{\mathcal{S}}^{-1}(U_\mu)$ the $H$-invariant slice constructed in Proposition 3.25. The product $KY$ is an invariant neighborhood of $m$ isomorphic to $K \times_H Y$. The subspace $E_m$ can be taken as the subspace $\mathrm{T}_m Y \subset \mathrm{T}_m M$. Now the condition $\mathbf{nTr}_{E_m}|\rho^H| = 0$ implies that $\rho^H$ acts trivially on the connected component $Y_m$ of $Y$ containing $m$. Elements $X \in [\mathfrak{h}, \mathfrak{h}]$ such that $X_{Y_m} = 0$ form an ideal in $[\mathfrak{h}, \mathfrak{h}]$. Since the ideal generated by $\rho^H$ in $[\mathfrak{h}, \mathfrak{h}]$ is equal to $[\mathfrak{h}, \mathfrak{h}]$, we have proved that $[\mathfrak{h}, \mathfrak{h}]$ acts trivially on $Y_m$. Since $KY_m$ is an open subset of $M$, we get

$$([\mathfrak{h}, \mathfrak{h}]) \subset (\mathfrak{k}_M). \tag{4.29}$$

With (4.28) and (4.29) we get $([\mathfrak{k}_M, \mathfrak{k}_M]) = ([\mathfrak{h}, \mathfrak{h}])$. Finally we have proven that if $d_{\mathcal{O}}$ vanishes at some point $p$, then $([\mathfrak{k}_M, \mathfrak{k}_M]) = ([\mathfrak{h}, \mathfrak{h}])$ for some $(\mathfrak{h}) \in \mathcal{H}_{\mathfrak{k}}$, and there exists an admissible orbit $K\mu \subset \mathfrak{k}^*_{(\mathfrak{h})} \cap \Phi_{\mathcal{S}}(M)$ such that $s(K\mu) = \mathcal{O}$.

### 4.5.2 Geometric properties

We summarize here some of the geometric properties enjoyed by $(M,\ \Phi = \Phi_{\mathcal{S}})$, when $\mathcal{Q}_K(M, \mathcal{S})$ is not zero.

Let $(\mathfrak{h}) \in \mathcal{H}_{\mathfrak{k}}$. We choose a representative $\mathfrak{h}$. Let $H$ be the corresponding group and $N_K(H)$ be the normalizer of $H$ in $K$. Consider the decomposition $\mathfrak{h} = [\mathfrak{h}, \mathfrak{h}] \oplus \mathfrak{z}$ where $\mathfrak{z}$ is the center of $\mathfrak{h}$. Thus $\mathfrak{z}^* \subset \mathfrak{h}^*$. Consider the open set

$$\mathfrak{h}_0^* = \{\xi \in \mathfrak{h}^* \mid \mathfrak{k}_\xi \subset \mathfrak{h}\}$$

of $\mathfrak{h}^*$. Let $\mathfrak{z}_0^* = \mathfrak{h}_0^* \cap \mathfrak{z}^*$ be the corresponding open subset of $\mathfrak{z}_0^*$.

We first note the following basic proposition.

**Proposition 4.21** *Let $M$ be a $K$-manifold such that $([\mathfrak{k}_M, \mathfrak{k}_M]) = ([\mathfrak{h}, \mathfrak{h}])$ and let $\Phi : M \to \mathfrak{k}^*$ be an equivariant map. Then*

- $\Phi(M) \subset K\mathfrak{z}^*$.
- *Assume $\mathcal{Y} := \Phi^{-1}(\mathfrak{h}_0^*)$ non empty, then*
  - *a) $\mathcal{Y}$ is a submanifold of $M$ invariant by the action of $N_K(H)$, and $[H, H]$ acts trivially on $\mathcal{Y}$.*

*b) The image $\Phi(\mathcal{Y})$ is contained in $\mathfrak{z}_0^*$.*

*c) The open subset $K\mathcal{Y}$ is diffeomorphic to $K \times_{N_K(H)} \mathcal{Y}$.*

**Proof.** Let us prove the first item. Using our $K$-invariant inner product, we consider $\Phi$ as a map $\Phi : M \to \mathfrak{k}$. The condition on the infinitesimal stabilizer $(\mathfrak{k}_M)$ gives that $M = KM^{[H,H]}$. If $m \in M^{[H,H]}$, the term $\Phi(m)$ belongs to the Lie algebra $\mathfrak{g}$ of the centralizer subgroup $G := Z_K([H,H])$. But one can easily prove that $\mathfrak{z}$ is a Cartan subalgebra of $\mathfrak{g}$: hence $\Phi(m)$ is conjugated to an element of $\mathfrak{z}$. This proves the first item.

If $\mathcal{Y}$ is non empty, the proof that it is a submanifold follows the same line than the proof of Proposition 3.25. The set $K\mathcal{Y}$ is a non empty open set in $M$ : so on $\mathcal{Y}$ we have $(\mathfrak{k}_M) = (\mathfrak{k}_y)$ on a dense open subset $\mathcal{Y}'$. The condition $([\mathfrak{k}_M, \mathfrak{k}_M]) = ([\mathfrak{h}, \mathfrak{h}])$ implies that $\dim[\mathfrak{h}, \mathfrak{h}] = \dim[\mathfrak{k}_y, \mathfrak{k}_y]$ on $\mathcal{Y}'$, but since $\mathfrak{k}_y \subset \mathfrak{k}_{\Phi(y)} \subset \mathfrak{h}$, we conclude that $[\mathfrak{h}, \mathfrak{h}] = [\mathfrak{k}_y, \mathfrak{k}_y] \subset \mathfrak{k}_y$ on $\mathcal{Y}'$ : in other words $[H,H]$ acts trivially on $\mathcal{Y}$, and $[\mathfrak{h}, \mathfrak{h}] = [\mathfrak{k}_y, \mathfrak{k}_y]$ for any $y \in \mathcal{Y}$. Furthermore, if $\xi = \Phi(y)$, then $[\mathfrak{h}, \mathfrak{h}]$ acts trivially on $\xi$. So $\xi$ is in the center of $\mathfrak{h}$.

Let us prove that $\pi : K \times_{N_K(H)} \mathcal{Y} \to K\mathcal{Y}$ is one to one. If $y_1 = ky_2$, we have $\xi_1 = k\xi_2$ with $\xi_i = \Phi(y_i)$. As $\Phi(Y) \subset \mathfrak{z}_0^*$, the stabilizers of $\xi_1, \xi_2$ are both equal to $H$. It follows that $k$ belongs to the normalizer of $H$.

The following theorem results directly from Theorem 4.20 and Lemma 4.21. Indeed, in the case where $\mathcal{Q}_K(M, \mathcal{S}) \neq \{0\}$, then $([\mathfrak{k}_M, \mathfrak{k}_M]) = ([\mathfrak{h}, \mathfrak{h}])$ for some $(\mathfrak{h}) \in \mathcal{H}_{\mathfrak{k}}$. Furthermore, there exists at least a regular admissible orbit $\mathcal{O}$ such that $m_{\mathcal{O}}$ is non zero, and consequently there exists orbit $\mathcal{P} \subset \mathfrak{k}^*_{(\mathfrak{h})} \cap \Phi_{\mathcal{S}}(M)$.

**Theorem 4.22** *Let $M$ be a $K$-manifold and let $\mathcal{S}$ be an equivariant* Spin$^c$*-bundle on $M$ with moment map $\Phi_{\mathcal{S}}$. Assume $\mathcal{Q}_K(M, \mathcal{S}) \neq \{0\}$. Then*

- *There exists $(\mathfrak{h}) \in \mathcal{H}_{\mathfrak{k}}$ such that $([\mathfrak{k}_M, \mathfrak{k}_M]) = ([\mathfrak{h}, \mathfrak{h}])$.*
- *If $\mathfrak{z}$ is the center of $\mathfrak{h}$, then $\Phi_{\mathcal{S}}(M) \subset K\mathfrak{z}^*$ and the open set $\Phi_{\mathcal{S}}^{-1}(K\mathfrak{z}_0^*)$ is non empty.*
- *The group $[H,H]$ acts trivially on the submanifold $\mathcal{Y} = \Phi_{\mathcal{S}}^{-1}(\mathfrak{z}_0^*)$.*

In the next sections, we consider a connected component $C$ of $\mathfrak{h}_0^*$. We consider the $H$-invariant submanifold $\mathcal{Y}_C := \Phi_{\mathcal{S}}^{-1}(C)$ of $Y$: here the open subset $K\mathcal{Y}_C$ is diffeomorphic to $K \times_H \mathcal{Y}_C$.

We follow here the notations of Section 3.5. We denote $\mathfrak{q}^C$ the vector space $\mathfrak{k}/\mathfrak{h}$ equipped with the complex structure $J_C$. There exists a unique $H$-equivariant Spin$^c$-bundle $\mathcal{S}_{\mathcal{Y}_C}$ on $\mathcal{Y}_C$ such that

$$\mathcal{S}|_{\mathcal{Y}_C} \simeq \bigwedge \mathfrak{q}^C \otimes \mathcal{S}_{\mathcal{Y}_C}. \tag{4.30}$$

At the level of determinant line bundles we have $\det(\mathcal{S}_{\mathcal{Y}_C}) = \det(\mathcal{S})|_{\mathcal{Y}_C} \otimes \mathbb{C}_{-2\rho_C}$, and the corresponding moment map satisfy the relation $\Phi_{\mathcal{Y}_C} = \Phi_{\mathcal{S}}|_{\mathcal{Y}_C} - \rho_C$.

We know already that the subgroup $[H, H]$ acts trivially on the submanifold $\mathcal{Y}_C$ (see Theorem 4.22). It acts also trivially on the bundle $\mathcal{S}_{\mathcal{Y}_C}$ since the moment map $\Phi_{\mathcal{Y}_C}$ takes value in $\mathfrak{z}^*$ (see Remark 2.8).

### 4.5.3 Localization on $Z_{\mathcal{O}}^{=0}$

Let $\mathcal{O}$ be a regular admissible orbit. By Theorem 4.20 and Corollary 4.18, we know that our object of study

$$m_{\mathcal{O}} = [\mathcal{Q}_K(M \times \mathcal{O}^*, \mathcal{S} \otimes \mathcal{S}_{\mathcal{O}^*})]^K$$

is equal to $\left[\mathcal{Q}_K(M \times \mathcal{O}^*, \mathcal{S} \otimes \mathcal{S}_{\mathcal{O}^*}, Z_{\mathcal{O}}^{=0}, \Phi_{\mathcal{O}})\right]^K$.

Let us give a description of the subset $Z_{\mathcal{O}}^{=0}$ of $Z_{\mathcal{O}} \subset M \times \mathcal{O}^*$ where $d_{\mathcal{O}}$ vanishes. We denote by $q : M \times \mathcal{O}^* \to \mathfrak{k}^* \oplus \mathfrak{k}^*$ the map given by $q(m, \xi) = (\Phi_{\mathcal{S}}(m), -\xi)$. If $\mu$ belongs to a coadjoint orbit $\mathcal{P}$, and $\xi \in \mu + o(\mathfrak{k}_\mu)$, then $\mathcal{P}$ is an ancestor to the orbit $\mathcal{O}$ of $\xi$.

**Definition 4.23** *Let $\mathcal{P}$ be a coadjoint orbit.*

- *Define the following subset of $\mathfrak{k}^* \oplus \mathfrak{k}^*$:*

$$R(\mathcal{P}) = \{(\mu, \xi); \mu \in \mathcal{P}; \xi \in \mu + o(\mathfrak{k}_\mu)\}.$$

- *Define $Z_{\mathcal{O}}^{\mathcal{P}} = q^{-1}(R(\mathcal{P})) \subset M \times \mathcal{O}^*$.*

**Proposition 4.24** *Assume $M$ is a $K$-manifold with $([\mathfrak{k}_M, \mathfrak{k}_M]) = ([\mathfrak{h}, \mathfrak{h}])$. Let $\mathcal{S}$ be a $K$-equivariant* Spin$^c$*-bundle over $M$ with moment map $\Phi_{\mathcal{S}}$. Let $\mathcal{O}$ be a regular admissible coadjoint. Then*

$$Z_{\mathcal{O}}^{=0} = \bigsqcup_{\mathcal{P}} Z_{\mathcal{O}}^{\mathcal{P}}$$

*where the disjoint union is over the set of $(\mathfrak{h})$-ancestors to $\mathcal{O}$. Furthermore, for $\mathcal{P}$ a $(\mathfrak{h})$-ancestor to $\mathcal{O}$, the set $Z_{\mathcal{O}}^{\mathcal{P}}$ is equal to $(\Phi_{\mathcal{S}}^{-1}(\mathcal{P}) \times \mathcal{O}^*) \cap Z_{\mathcal{O}}^{=0}$.*

**Proof.** In the proof of Proposition 4.20, we have seen that, if $d_{\mathcal{O}}(m,-\lambda)=0$, then the element $\mu=\Phi_{\mathcal{S}}(m)$ is such that $(\mathfrak{k}_\mu)=(\mathfrak{h})$ and $\lambda=\beta+\mu$ with $\beta\in o(\mathfrak{k}_\mu)$. So $K\mu$ is a $(\mathfrak{h})$ ancestor of $\mathcal{O}$ and $q(m,-\lambda)\in\bigsqcup_{\mathcal{P}} Z_{\mathcal{O}}^{\mathcal{P}}$. This proves the first assertion.

Conversely take now $(m,-\xi)\in Z_{\mathcal{O}}^{\mathcal{P}}$ , define $\mu=\Phi_{\mathcal{S}}(m)$. So $K\mu$ is a $(\mathfrak{h})$ ancestor of $\mathcal{O}$ and $\xi=\mu+\beta$ with $\beta\in o(\mathfrak{k}_\mu)$. By $K$-invariance, we may assume $\mu\in\mathfrak{z}_0^*$, so $m\in\mathcal{Y}$. We have $T_mM=\mathfrak{k}/\mathfrak{k}_m\oplus T_m\mathcal{Y}$. So

$$d_{\mathcal{O}}(m,-\xi)=\|\beta\|^2-\frac{1}{2}\mathbf{nTr}_{\mathfrak{k}_m}|\beta|+\frac{1}{2}\mathbf{nTr}_{\mathrm{T}_m\mathcal{Y}}|\beta|.$$

As $\beta\in o(\mathfrak{h})\subset[\mathfrak{h},\mathfrak{h}]$ acts trivially on $\mathcal{Y}$ by Lemma 4.21, we have $d_{\mathcal{O}}(m,-\xi)=\|\rho^H\|^2-\frac{1}{2}\mathbf{nTr}_{\mathfrak{k}_m}|\rho^H|$. But since $[\mathfrak{h},\mathfrak{h}]\subset\mathfrak{k}_m\subset\mathfrak{h}$, and then $\frac{1}{2}\mathbf{nTr}_{\mathfrak{k}_m}|\rho^H|=\frac{1}{2}\mathbf{nTr}_{\mathfrak{h}}|\rho^H|=\|\rho^H\|^2$ : finally $d_{\mathcal{O}}(m,-\xi)=0$.

At this stage we have proved that

$$\mathrm{m}_{\mathcal{O}}=\sum_{\mathcal{P}}\mathrm{m}_{\mathcal{O}}^{\mathcal{P}} \tag{4.31}$$

where the sum runs over the $(\mathfrak{h})$-ancestor of $\mathcal{O}$ and

$$\mathrm{m}_{\mathcal{O}}^{\mathcal{P}}=\left[\mathcal{Q}_K(M\times\mathcal{O}^*,\mathcal{S}\otimes\mathcal{S}_{\mathcal{O}^*},Z_{\mathcal{O}}^{\mathcal{P}},\Phi_{\mathcal{O}})\right]^K.$$

In the next section we will go into the computation of the terms $\mathrm{m}_{\mathcal{O}}^{\mathcal{P}}$. We end up this section with the following important fact.

**Proposition 4.25** *Each individual term $\mathrm{m}_{\mathcal{O}}^{\mathcal{P}}$ is independent of the choice of the moment map $\Phi_{\mathcal{S}}$.*

**Proof.** Let $\Phi_{\mathcal{S}}^t, t\in[0,1]$ be a family of moment maps for $\mathcal{S}$. This gives a family $\Phi_{\mathcal{O}}^t(m,\xi):=\Phi_{\mathcal{S}}^t(m)+\xi$ for $\mathcal{S}\otimes\mathcal{S}_{\mathcal{O}^*}$.

Let $\kappa_{\mathcal{O}}^t$ be the Kirwan vector field associated to $\Phi_{\mathcal{O}}^t$, and let $Z_{\mathcal{O}}(t):=\{\kappa_{\mathcal{O}}^t=0\}$. We denote simply by $\sigma^t$ the symbol $\sigma(M\times\mathcal{O}^*,\mathcal{S}\otimes\mathcal{S}_{\mathcal{O}^*},\Phi_{\mathcal{O}}^t)$. For any $t\in[0,1]$, we consider the quantity $Q_{\mathcal{O}}^{\mathcal{P}}(t)\in\hat{R}(K)$ which is the equivariant index of $\sigma^t|_{U_t}$, where $U_t$ is a (small) neighborhood of

$$Z_{\mathcal{O}}^{\mathcal{P}}(t)\subset Z_{\mathcal{O}}(t)$$

such that $U_t\cap Z_{\mathcal{O}}(t)=Z_{\mathcal{O}}^{\mathcal{P}}(t)$.

Let us prove that the multiplicity $\mathrm{m}_{\mathcal{O}}^{\mathcal{P}}(t)=[Q_{\mathcal{O}}^{\mathcal{P}}(t)]^K$ is independent of $t$. It is sufficient to prove that $t\to[Q_{\mathcal{O}}^{\mathcal{P}}(t)]^K$ is locally constant : let us show that it is constant in a neighborhood of 0. We follow the same line of

proof that the proof of the independence of the connection of the local piece $\mathcal{Q}_K(M,\mathcal{S},\Phi_{\mathcal{S}}^{-1}(\mathcal{O})\cap Z_{\mathcal{S}})$ of $\mathcal{Q}_K(M,\mathcal{S})$ in [24].

Let $U_0$ be a neighborhood of $Z_{\mathcal{O}}^{\mathcal{P}}(0)$ such that

$$\overline{U_0}\cap Z_{\mathcal{O}}(0)=Z_{\mathcal{O}}^{\mathcal{P}}(0). \tag{4.32}$$

The vector field $\kappa_{\mathcal{O}}^0$ does not vanish on $\partial U_0$ : there exist $\epsilon>0$ so that $\kappa_{\mathcal{O}}^t$ does not vanish on $\partial U_0$ for $t\in[0,\epsilon]$. The family $\sigma^t|_{U_0}$, $t\in[0,\epsilon]$ is then an homotopy of transversally elliptic symbols : hence they have the same equivariant index.

**Lemma 4.26** *For small $t$ we have*

$$U_0\cap Z_{\overline{\mathcal{O}}}^{=0}(t)=Z_{\mathcal{O}}^{\mathcal{P}}(t).$$

Indeed, by Proposition 4.24, $Z_{\overline{\mathcal{O}}}^{=0}(t)$ projects by the first projection $\Phi_{\mathcal{S}}^t$ : $M\times\mathcal{O}^*\to M\to\mathfrak{k}^*$ to a finite union of coadjoint orbits (the $(\mathfrak{h})$-ancestors to $\mathcal{O}$) and $Z_{\mathcal{O}}(0)$ projects on $\mathcal{P}$. So, for $t$ small, $U_0\cap Z_{\overline{\mathcal{O}}}^{=0}(t)$ is the subset $Z_{\mathcal{O}}^{\mathcal{P}}(t)$ of $Z_{\overline{\mathcal{O}}}^{=0}(t)$ projecting on $\mathcal{P}$.

So, for small $t$, we have the decomposition $U_0\cap Z_{\mathcal{O}}(t)=Z_{\mathcal{O}}^{\mathcal{P}}(t)\cup Z_t$, where $Z_t$ is a component contained in $Z_{\overline{\mathcal{O}}}^{>0}(t)$. Finally, for small $t$, we have

$$\begin{aligned}Q_{\mathcal{O}}^{\mathcal{P}}(0) &= \mathrm{Index}_K(\sigma^0|_{U_0})\\ &= \mathrm{Index}_K(\sigma^t|_{U_0})\\ &= Q_{\mathcal{O}}^{\mathcal{P}}(t)+\mathcal{Q}_K(M\times\mathcal{O}^*,\mathcal{S}\otimes\mathcal{S}_{\mathcal{O}^*},Z_t,\Phi_{\mathcal{O}}^t).\end{aligned}$$

Since $[\mathcal{Q}_K(M\times\mathcal{O}^*,\mathcal{S}\otimes\mathcal{S}_{\mathcal{O}^*},Z_t,\Phi_{\mathcal{O}}^t)]^K=0$ by Proposition 4.17 the proof of Proposition 4.25 is completed.

### 4.5.4 Computation of $\mathrm{m}_{\mathcal{O}}^{\mathcal{P}}$

In this section we compute

$$\mathrm{m}_{\mathcal{O}}^{\mathcal{P}}:=\left[\mathcal{Q}_K(M\times\mathcal{O}^*,\mathcal{S}\otimes\mathcal{S}_{\mathcal{O}^*},Z_{\mathcal{O}}^{\mathcal{P}},\Phi_{\mathcal{O}})\right]^K.$$

Let $C$ be a connected component of $\mathfrak{h}_0^*$ that intersects the orbit $\mathcal{P}$. With the help of Proposition 4.15, we will reduce the computation of $\mathrm{m}_{\mathcal{O}}^{\mathcal{P}}$ to a similar computation where the group $K$ acting on $M$ is replaced with the torus $A_H=H/[H,H]$ acting on the slice $\mathcal{Y}_C$.

Let $\mu\in\mathcal{P}\cap C$ : $\mathfrak{k}_\mu=\mathfrak{h}$ and $\mu-\rho(\mu)$ defines a character of $H$. Then $Z_{\mathcal{O}}^{\mathcal{P}}$ is equal to $K(\Phi_{\mathcal{S}}^{-1}(\mu)\times(-\mu+o(\mathfrak{h})^*)$. Here $o(\mathfrak{h})$ is the $\rho$ orbit for $H$, so $o(\mathfrak{h})=o(\mathfrak{h})^*$ and $\mathrm{Q}_H^{\mathrm{spin}}(o(\mathfrak{h})^*)$ is the trivial representation of $H$.

Let $\mathcal{Y}_C = \Phi_{\mathcal{S}}^{-1}(C)$ be the slice relative to the connected component $C$ (see Section 4.5.2). Thus $K\mathcal{Y}_C$ is an open neighborhood of $\Phi_{\mathcal{S}}^{-1}(\mathcal{P})$ in $M$ diffeomorphic with $K \times_H \mathcal{Y}_C$. We see that

$$Z_{\mathcal{O}}^{\mathcal{P}} \subset (K \times_H \mathcal{Y}_C) \times \mathcal{O}^* \simeq K \times_H (\mathcal{Y}_C \times \mathcal{O}^*).$$

We consider the $H$-manifold $N' := \mathcal{Y}_C \times o(\mathfrak{h})^*$ and the $K$-manifold

$$N = K \times_H (B_{\mathfrak{q}} \times N') = K \times_H (B_{\mathfrak{q}} \times \mathcal{Y}_C \times o(\mathfrak{h})^*),$$

where $B_{\mathfrak{q}}$ is a small open ball in $\mathfrak{q}$, centered at 0 and $H$-invariant.

When $B_{\mathfrak{q}}$ is small enough, the map $(X, \xi) \mapsto \exp(X)(-\mu + \xi)$, from $B_{\mathfrak{q}} \times o(\mathfrak{h})^*$ into $\mathcal{O}^*$, defines a diffeomorphism into a $H$-invariant neighborhood of the $H$-orbit $-\mu + o(\mathfrak{h})^*$ in $\mathcal{O}^*$. Hence a $K$-invariant neighborhood of $Z_{\mathcal{O}}^{\mathcal{P}}$ in $M \times \mathcal{O}^*$ is diffeomorphic to $N$. Under this isomorphism, the equivariant map $\Phi_{\mathcal{O}} = \Phi_{\mathcal{S}} + i_{\mathcal{O}^*}$ defines a map $\Phi$ on $N$. For $k \in K, X \in B_{\mathfrak{q}}, y \in \mathcal{Y}_C, \xi \in o(\mathfrak{h})^*$, we have

$$\Phi([k; X, y, \xi]) := k\,(\Phi_{\mathcal{S}}(y) + \exp(X)(-\mu + \xi)).$$

It restricts to $N'$ as the $H$-equivariant map $\Phi'(y, \xi) = \Phi_{\mathcal{S}}(y) - \mu + \xi$ with value in $\mathfrak{h}^*$. Furthermore, if $B_{\mathfrak{q}}$ is small enough, $\Phi([1; X, y, \xi])$ belongs to $\mathfrak{h}^*$ if and only $X = 0$. As $X \in \mathfrak{q}$, we see also that $(\Phi([1; X, y, \xi]), X) = (\Phi_{\mathcal{S}}(y), X) + (\exp(X)(-\mu + \xi), X) = (\Phi_{\mathcal{S}}(y) - \mu + \xi, X) = 0$ for all $(X, y, \xi) \in B_{\mathfrak{q}} \times \mathcal{Y}_C \times o(\mathfrak{h})^*$. Conditions (4.23) are satisfied. Proposition 4.15 tells us that

$$\mathrm{m}_{\mathcal{O}}^{\mathcal{P}} := \left[\mathcal{Q}_H(N', \mathcal{S}', Z', \Phi')\right]^H$$

where $Z' := \Phi_{\mathcal{S}}^{-1}(\mu) \times o(\mathfrak{h})^*$.

Now we have to explain the nature of the Spin$^c$ bundle $\mathcal{S}'$ over $N' = \mathcal{Y}_C \times o(\mathfrak{h})^*$. Let $\mathcal{S}_{o(\mathfrak{h})^*}$ be the canonical Spin$^c$-bundle of the orbit $o(\mathfrak{h})^*$. Let $\mathcal{S}_{\mathcal{Y}_C}$ be the Spin$^c$-bundle on $\mathcal{Y}_C$ defined by (4.30).

**Proposition 4.27** *We have $\mathcal{S}' = \mathcal{S}_{\mathcal{Y}_C}^{\mathcal{P}} \otimes \mathcal{S}_{o(\mathfrak{h})^*}$ where $\mathcal{S}_{\mathcal{Y}_C}^{\mathcal{P}} = \mathcal{S}_{\mathcal{Y}_C} \otimes \mathbb{C}_{-\mu+\rho(\mu)}$ is a Spin$^c$-bundle on $\mathcal{Y}_C$. The determinant line bundle of $\mathcal{S}_{\mathcal{Y}_C}^{\mathcal{P}}$ is equal to $\det(\mathcal{S})|_{\mathcal{Y}_C} \otimes \mathbb{C}_{-2\mu}$, and the corresponding moment map is $\Phi_{\mathcal{Y}_C}^{\mathcal{P}} := \Phi_{\mathcal{S}}|_{\mathcal{Y}_C} - \mu$.*

*The subgroup $[H, H]$ acts trivially on $(\mathcal{Y}_C, \mathcal{S}_{\mathcal{Y}_C}^{\mathcal{P}})$.*

**Proof.** Let $\lambda$ be an element of the $H$-orbit $\mathcal{O}_{\mathcal{P}} := \mu + o(\mathfrak{h})$. The Spin$^c$ bundle $\mathcal{S}_{\mathcal{O}^*}$ on $\mathcal{O}^* = (K\lambda)^*$ induces a Spin$^c$ bundle $\mathcal{S}_1$ over $\mathcal{O}_{\mathcal{P}}^*$ through the relation $\mathcal{S}_{\mathcal{O}^*}|_{\mathcal{O}_{\mathcal{P}}^*} \simeq \bigwedge \overline{\mathfrak{q}^C} \otimes \mathcal{S}_1$.

We can check that $\mathcal{S}_1$ is the $H$-Spin$^c$ bundle on $\mathcal{O}^*_{\mathcal{P}} = (H\lambda)^* \simeq o(\mathfrak{h})^*$ equal to

$$\begin{aligned} H \times_{H_\lambda} \Big( \bigwedge_{-J_\lambda} \mathfrak{h}/\mathfrak{h}_\lambda \otimes \mathbb{C}_{-\lambda+\rho(\lambda)} \Big) &\simeq \Big( H \times_{H_\lambda} \bigwedge_{-J_\lambda} \mathfrak{h}/\mathfrak{h}_\lambda \Big) \otimes \mathbb{C}_{-\lambda+\rho(\lambda)} \\ &\simeq \mathcal{S}_{o(\mathfrak{h})^*} \otimes \mathbb{C}_{-\mu+\rho(\mu)} \end{aligned}$$

since $\lambda - \rho(\lambda) = \mu - \rho(\mu) \in \mathfrak{z}^*$.

As the Spin$^c$ bundle $\mathcal{S}_{\mathfrak{q}}$ is equal to the product $\bigwedge \overline{\mathfrak{q}^C} \otimes \bigwedge \mathfrak{q}^C$ (see Example 2.3), we know then that $\mathcal{S}' \simeq \mathcal{S}_{\mathcal{Y}_C} \otimes \mathcal{S}_1 \simeq \mathcal{S}_{\mathcal{Y}_C} \otimes \mathcal{S}_{o(\mathfrak{h})^*} \otimes \mathbb{C}_{-\mu+\rho(\mu)}$.

The relation $\det(\mathcal{S}^{\mathcal{P}}_{\mathcal{Y}_C}) = \det(\mathcal{S})|_{\mathcal{Y}_C} \otimes \mathbb{C}_{-2\mu}$ comes from the fact that $\det(\mathcal{S}_{\mathcal{Y}_C}) = \det(\mathcal{S})|_{\mathcal{Y}_C} \otimes \mathbb{C}_{-2\rho(\mu)}$ since $\rho_C = \rho(\mu)$.

We consider now the $H$-manifold $\mathcal{Y}_C$ equipped with the Spin$^c$-bundle $\mathcal{S}^{\mathcal{P}}_{\mathcal{Y}_C}$. Let

$$\mathcal{Q}_H(\mathcal{Y}_C, \mathcal{S}^{\mathcal{P}}_{\mathcal{Y}_C}, \{0\}) \in \hat{R}(H) \tag{4.33}$$

be the equivariant index localized on the compact component $\{\Phi^{\mathcal{P}}_{\mathcal{Y}_C} = 0\} = \{\Phi_{\mathcal{S}} = \mu\} \subset \mathcal{Y}_C$. Let $A_H$ be the torus $H/[H,H]$. Since $[H,H]$ acts trivially on $(\mathcal{Y}_C, \mathcal{S}^{\mathcal{P}}_{\mathcal{Y}_C})$ we may also define the localized index $\mathcal{Q}_{A_H}(\mathcal{Y}_C, \mathcal{S}^{\mathcal{P}}_{\mathcal{Y}_C}, \{0\}) \in \hat{R}(A_H)$.

We can now prove the main result of this section.

**Theorem 4.28** *The multiplicity* $\mathrm{m}^{\mathcal{P}}_{\mathcal{O}}$ *is equal to*

$$\Big[\mathcal{Q}_H(\mathcal{Y}_C, \mathcal{S}^{\mathcal{P}}_{\mathcal{Y}_C}, \{0\})\Big]^H = \Big[\mathcal{Q}_{A_H}(\mathcal{Y}_C, \mathcal{S}^{\mathcal{P}}_{\mathcal{Y}_C}, \{0\})\Big]^{A_H}.$$

**Proof.** Let $Z' := \Phi_{\mathcal{S}}^{-1}(\mu) \times o(\mathfrak{h})^*$. The character $\mathcal{Q}_H(N', \mathcal{S}', Z', \Phi') \in \hat{R}(H)$ is equal to the equivariant index of $\sigma(N', \mathcal{S}', \Phi')|_{\mathcal{U}}$ where $\mathcal{U} \subset N'$ is an invariant open subset such that $\mathcal{U} \cap Z_{\Phi'} = Z'$. For $(y,\xi) \in N' = \mathcal{Y}_C \times o(\mathfrak{h})^*$ and $(v,\eta) \in \mathrm{T}_{(y,\xi)}N'$, the endomorphism $\sigma(N', \mathcal{S}_{N'}, \Phi')|_{(y,\xi)}(v,\eta)$ is equal to

$$\mathbf{c}_1(v + (\Phi_{\mathcal{S}}(y) - \mu + \xi) \cdot y) \otimes \mathrm{Id}_{\mathcal{S}_{o(\mathfrak{h})^*|_\xi}} + \epsilon_1 \otimes \mathbf{c}_2(\eta + (\Phi_{\mathcal{S}}(y) - \mu + \xi) \cdot \xi).$$

Here $\mathbf{c}_1$ acts on $\mathcal{S}^{\mathcal{P}}_{\mathcal{Y}_C}|_y$, $\mathbf{c}_2$ acts on $\mathcal{S}_{o(\mathfrak{h})^*}|_\xi$ and $\epsilon_1$ is the canonical grading operator on $\mathcal{S}^{\mathcal{P}}_{\mathcal{Y}_C}|_y$.

Since $o(\mathfrak{h})^*$ is compact, we can replace the term $\mathbf{c}_2(\eta + (\Phi_{\mathcal{S}}(y) - \mu + \xi) \cdot \xi)$ simply by $\mathbf{c}_2(\eta)$. Since $[H,H]$ acts trivially on $\mathcal{Y}_C$, and $\xi \in [\mathfrak{h},\mathfrak{h}]$, the vector field $y \mapsto (\Phi_{\mathcal{S}}(y) - \mu + \xi) \cdot y$ is equal to $y \mapsto (\Phi_{\mathcal{S}}(y) - \mu) \cdot y$. Thus our symbol is homotopic to the symbol

$$\mathbf{c}_1(v + (\Phi_{\mathcal{S}}(y) - \mu) \cdot y) \otimes \mathrm{Id}_{\mathcal{S}_{o(\mathfrak{h})|_\xi}} + \epsilon_1 \otimes \mathbf{c}_2(\eta).$$

This last expression is the product symbol of the $H$-transversally elliptic symbol $\mathbf{c}_1(v + (\Phi_{\mathcal{S}}(y) - \mu) \cdot y)$ on $\mathcal{Y}_C$ and of the elliptic symbol $\mathbf{c}_2(\eta)$ on $o(\mathfrak{h})^*$. The equivariant indices multiply under the product (as one is elliptic) ([1],[22]).

Now the $H$-equivariant index of $c_2(\eta)$ acting on $\mathcal{S}_{o(\mathfrak{h})^*}$ is the trivial representation of $H$. Thus we obtain our theorem. We have also to remark that the identity $\left[\mathcal{Q}_H(\mathcal{Y}_C, \mathcal{S}^{\mathcal{P}}_{\mathcal{Y}_C}, \{0\})\right]^H = \left[\mathcal{Q}_{A_H}(\mathcal{Y}_C, \mathcal{S}^{\mathcal{P}}_{\mathcal{Y}_C}, \{0\})\right]^{A_H}$ follows from the fact that $[H, H]$ acts trivially on $(\mathcal{Y}_C, \mathcal{S}^{\mathcal{P}}_{\mathcal{Y}_C})$.

# 5 Multiplicities and reduced spaces

In this section, we interpret the multiplicity as an equivariant index on a reduced space.

Let $\mathcal{O} \subset \mathfrak{k}^*$ be a regular admissible orbit, and $(\mathfrak{h}) \in \mathcal{H}_{\mathfrak{k}}$ so that $([\mathfrak{h}, \mathfrak{h}]) = ([\mathfrak{k}_M, \mathfrak{k}_M])$. In the previous section, we have proved that the multiplicity of $\pi_{\mathcal{O}}$ in $\mathcal{Q}_K(M, \mathcal{S})$ is equal to

$$\mathrm{m}_{\mathcal{O}} = \sum_{\mathcal{P}} \mathrm{m}^{\mathcal{P}}_{\mathcal{O}}$$

where the sum runs over the $K$-orbits $\mathcal{P}$ which are $(\mathfrak{h})$-ancestors of $\mathcal{O}$. Furthermore, we have proved that $\mathrm{m}^{\mathcal{P}}_{\mathcal{O}} = \left[\mathcal{Q}_{A_H}(\mathcal{Y}_C, \mathcal{S}^{\mathcal{P}}_{\mathcal{Y}_C}, \{0\})\right]^{A_H}$.

The aim of this section is to prove the following theorem.

**Theorem 5.1** *The multiplicity $\mathrm{m}^{\mathcal{P}}_{\mathcal{O}}$ is equal to the spin$^c$ index of the (possibly singular) reduced space $M_{\mathcal{P}} := \Phi_{\mathcal{S}}^{-1}(\mathcal{P})/K$.*

However, our first task is to give a meaning to a $\mathrm{Q}^{\mathrm{spin}}(M_{\mathcal{P}}) \in \mathbb{Z}$ even if $M_{\mathcal{P}}$ is singular.

## 5.1 Spin$^c$ index on singular reduced spaces

We consider a connected oriented manifold $N$, equipped with a Spin$^c$-bundle $\mathcal{S}$. We assume that a torus $G$ acts on the data $(N, \mathcal{S})$. An invariant connexion on the determinant line bundle $\mathbb{L} = \det(\mathcal{S})$ defines a moment map $\Phi : N \to \mathfrak{g}^*$. We do not assume that $N$ is compact, but we assume that the map $\Phi$ is proper[3]. For any $\xi \in \mathfrak{g}^*$, the reduced space $N_\xi := \Phi^{-1}(\xi)/G$ is compact.

[3]We will use sometimes a slightly different hypothesis : $\Phi$ is proper as a map from $N$ to an open subset of $\mathfrak{g}^*$.

The purpose of this subsection is to explain how we can define the spin$^c$-index, $\mathrm{Q}^{\mathrm{spin}}(N_\mu) \in \mathbb{Z}$, for any $\mu$ in the weight lattice $\Lambda$ of the torus $G$.

Let $\mathfrak{g}_N$ be the generic infinitesimal stabilizer of the $G$-action on $N$ : the image of $N$ under the map $\Phi$ leaves in an affine space $I(N)$ parallel to $\mathfrak{g}_N^\perp$. If $\xi \in I(N)$ is a regular value of $\Phi : N \to I(N)$, the reduced space $N_\xi$ is a compact orbifold (as proved in [24]). We can define Spin$^c$-bundles on orbifolds, as well as Spin$^c$-indices.

We start with the following basic fact.

**Lemma 5.2** *For any regular value $\xi \in I(N)$ of $\Phi : N \to I(N)$, the orbifold $N_\xi$ is oriented and equipped with a family of* Spin$^c$*-bundles $\mathcal{S}_\xi^\mu$ parameterized by $\mu \in \Lambda \cap I(N)$.*

**Proof.** Let $G_N$ be the subtorus with Lie algebra $\mathfrak{g}_N$. Let $G' = G/G_N$. The dual of the Lie algebra $\mathfrak{g}'$ of $G'$ is canonically identified with $\mathfrak{g}_N^\perp$.

We assume that $\xi$ is a regular value of $\Phi : N \to I(N)$ : the fiber $Z = \Phi^{-1}(\xi)$ is a submanifold equipped with a locally free action of $G'$. Let $N_\xi := Z/G'$ be the corresponding "reduced" space, and let $\pi : Z \to N_\xi$ be the projection map. We can define the tangent (orbi)-bundle $TN_\xi$ to $N_\xi$.

On $Z$, we obtain an exact sequence $0 \longrightarrow \mathrm{T}Z \longrightarrow \mathrm{T}N|_Z \xrightarrow{d\Phi_\epsilon} [(\mathfrak{g}')^*] \to 0$, and an orthogonal decomposition $\mathrm{T}Z = \mathrm{T}_{G'}Z \oplus [\mathfrak{g}']$ where $[\mathfrak{g}']$ is the trivial bundle on $Z$ corresponding to the subspace of $\mathrm{T}Z$ formed by the vector fields generated by the infinitesimal action of $\mathfrak{g}'$. So $\mathrm{T}N|_Z$ admits the decomposition $\mathrm{T}N|_Z \simeq \mathrm{T}_{G'}Z \oplus [\mathfrak{g}'] \oplus [(\mathfrak{g}')^*]$. We rewrite this as

$$\mathrm{T}N|_Z \simeq \mathrm{T}_{G'}Z \oplus [\mathfrak{g}'_{\mathbb{C}}] \tag{5.34}$$

with the convention $\mathfrak{g}' \simeq \mathfrak{g}' \otimes i\mathbb{R}$ and $(\mathfrak{g}')^* \simeq \mathfrak{g}' \otimes \mathbb{R}$. Note that the bundle $\mathrm{T}_{G'}Z$ is naturally identified with $\pi^*(\mathrm{T}N_\xi)$.

If we take on $\mathfrak{g}'_{\mathbb{C}}$ the orientation $o(i)$ given by the complex structure, there exists a unique orientation $o(N_\xi)$ on $N_\xi$ such that $o(N) = o(N_\xi)o(i)$.

**Definition 5.3** *Let $\widetilde{\mathcal{S}}_\xi$ be the* Spin$^c$ *bundle on the vector bundle $\mathrm{T}_{G'}Z \to Z$ such that*

$$\mathcal{S}|_Z \simeq \widetilde{\mathcal{S}}_\xi \otimes [\bigwedge \mathfrak{g}'_{\mathbb{C}}].$$

*Here $[\bigwedge \mathfrak{g}'_{\mathbb{C}}] = Z \times \bigwedge \mathfrak{g}'_{\mathbb{C}}$ is a* Spin$^c$*-bundle on the bundle $[\mathfrak{g}'_{\mathbb{C}}] = Z \times \mathfrak{g}'_{\mathbb{C}}$*

The Kostant relation shows that for any $X \in \mathfrak{g}_N$, the element $e^X$ acts on the fibers of $\widetilde{\mathcal{S}}_\xi$ as a multiplication by $e^{i\langle \nu, X\rangle}$ where $\nu$ is any element of $I(N)$.

Hence, for any $\mu \in \Lambda \cap I(N)$, the action of $G_N$ on the tensor $\widetilde{\mathcal{S}}_\xi \otimes [\mathbb{C}_{-\mu}]$ is trivial. We can then define a Spin$^c$-bundle $\mathcal{S}^\mu_\xi$ on $\mathrm{T}N_\xi$ by the relation

$$\widetilde{\mathcal{S}}_\xi \otimes [\mathbb{C}_{-\mu}] = \pi^* \left( \mathcal{S}^\mu_\xi \right).$$

The proof of the following theorem is given in the next subsection.

**Theorem 5.4** *For any $\mu \in I(N) \cap \Lambda$, consider the compact oriented orbifold $N_{\mu+\epsilon}$ associated to a generic[4] element $\epsilon \in \mathfrak{g}_N^\perp$. Then the index*

$$\mathcal{Q}(N_{\mu+\epsilon}, \mathcal{S}^\mu_{\mu+\epsilon})$$

*is independent of the choice of a generic and small enough $\epsilon$.*

Thanks to the previous Theorem, one defines the spin$^c$ index of singular reduced spaces as follows.

**Definition 5.5** *If $\mu \in \Lambda$, the number $\mathrm{Q}^{\mathrm{spin}}(N_\mu)$ is defined by the following dichotomy*

$$\mathrm{Q}^{\mathrm{spin}}(N_\mu) = \begin{cases} 0 & \text{if } \mu \notin I(N), \\ \mathcal{Q}(N_{\mu+\epsilon}, \mathcal{S}^\mu_{\mu+\epsilon}) & \text{if } \mu \in I(N) \text{ and } \epsilon \in \mathfrak{g}_N^\perp \text{ is generic} \\ & \text{and small enough.} \end{cases}$$

The invariant $\mathrm{Q}^{\mathrm{spin}}(N_\mu) \in \mathbb{Z}$ vanishes if $\mu$ does not belongs to the relative interior of $\Phi(N)$ in the affine space $I(N)$. It is due to the fact that we can then approach $\mu$ by elements $\mu + \epsilon$ that are not in the image $\Phi(N)$.

Let us consider the particular case where $\mu \in I(N) \cap \Lambda$ is a regular value of $\Phi : N \to I(N)$ such that the reduced space $N_\mu$ is reduced to a point. Let $m_o \in \Phi^{-1}(\mu)$, and let $\Gamma \subset G'$ be the stabilizer subgroup of $m_o$ ($\Gamma$ is finite). In this case (5.34) becomes $\mathrm{T}_{m_o} N \simeq \mathfrak{g}'_{\mathbb{C}}$, and $o(N_\mu)$ is the quotient between the orientation of $N$ and those of $\mathfrak{g}'_{\mathbb{C}}$. At the level of graded Spin$^c$-bundles we have

$$\mathcal{S}_{m_o} \simeq o(N_\mu) \bigwedge \mathfrak{g}'_{\mathbb{C}} \otimes \mathbb{L}^{1/2}_{m_o}$$

where $\mathbb{L}^{1/2}_{m_o}$ is a one dimensional representation of $\Gamma$ such that $(\mathbb{L}^{1/2}_{m_o})^{\otimes 2} = \mathbb{L}_{m_o}$. In this case Definition 5.5 gives that

$$\mathrm{Q}^{\mathrm{spin}}(N_\mu) = o(N_\mu) \dim \left[ \mathbb{L}^{1/2}_{m_o} \otimes \mathbb{C}_{-\mu} \right]^\Gamma \in \{-1, 0, 1\}. \tag{5.35}$$

[4] So that $\mu + \epsilon$ is a regular value of $\Phi : N \to I(N)$.

## 5.2 Proof of Theorem 5.4

In this subsection we consider a fixed $\mu \in I(N) \cap \Lambda$. For any $\epsilon \in \mathfrak{g}(N)^{\perp}$, we consider the moment map $\Phi_\epsilon = \Phi - \mu - \epsilon$.

We start with the fundamental Lemma

**Lemma 5.6** *The map $\epsilon \mapsto [\mathcal{Q}_G(N, \mathcal{S}, \Phi_\epsilon^{-1}(0), \Phi_\epsilon) \otimes \mathbb{C}_{-\mu}]^G$ is constant in a neighborhood of 0.*

**Proof.** Changing $\mathcal{S}$ to $\mathcal{S} \otimes [\mathbb{C}_{-\mu}]$, we might as well take $\mu = 0$.

Let $r > 0$ be smallest non-zero critical value of $\|\Phi\|^2$, and let $\mathcal{U} := \Phi^{-1}(\{\xi \mid \|\xi\| < r/2\})$. Using Lemma 2.9, we have $\mathcal{U} \cap \{\kappa_0 = 0\} = \Phi^{-1}(0)$.

We describe now $\{\kappa_\epsilon = 0\} \cap \mathcal{U}$ using a parametrization similar to those introduced in [19][Section 6].

Let $\mathfrak{g}_i, i \in I$ be the finite collection of infinitesimal stabilizers for the $G$-action on the compact set $\overline{\mathcal{U}}$. Let $\mathcal{D}$ be the subset of the collection of subspaces $\mathfrak{g}_i^{\perp}$ of $\mathfrak{g}^*$ such that $\Phi^{-1}(0) \cap N^{\mathfrak{g}_i} \neq \varnothing$.

Note that $\mathcal{D}$ is reduced to $I(N)$ if 0 is regular value of $\Phi : N \to I(N)$. If $\Delta = \mathfrak{g}_i^{\perp}$ belongs to $\mathcal{D}$, and $\epsilon \in I(N)$, write the orthogonal decomposition $\epsilon = \epsilon_\Delta + \beta_\Delta$ with $\epsilon_\Delta \in \Delta$, and $\beta_\Delta \in \mathfrak{g}_i$. Let

$$\mathcal{B}_\epsilon = \{\beta_\Delta = \epsilon - \epsilon_\Delta, \Delta \in \mathcal{D}\}$$

the set of $\beta$ so obtained.

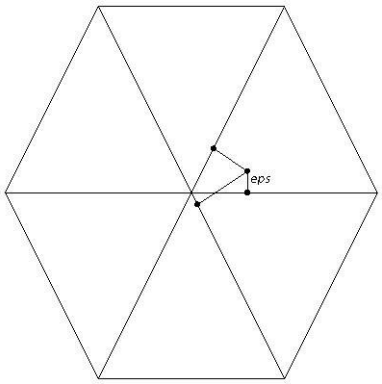


Figure 10: The point $\epsilon$ and its projections $\epsilon_\Delta$

We denote by $Z_\epsilon$ the zero set of the vector field $\kappa_\epsilon$ associated to $\Phi_\epsilon$. Thus, if $\epsilon$ is sufficiently small ($\|\epsilon\| < r/2$),

$$Z_\epsilon \cap \mathcal{U} = \bigcup_{\beta \in \mathcal{D}_\epsilon} N^\beta \cap \Phi_\epsilon^{-1}(\beta). \tag{5.36}$$

With (5.36) in hands, we see easily that $t \in [0,1] \mapsto \sigma(N, \mathcal{S}, \Phi_{t\epsilon})|_{\mathcal{U}}$ is an homotopy of transversally elliptic symbols on $\mathcal{U}$. Hence they have the same index

$$\begin{aligned} \mathcal{Q}_G(\mathcal{U}, \mathcal{S}, \Phi^{-1}(0), \Phi) &= \mathcal{Q}_G(\mathcal{U}, \mathcal{S}, Z_\epsilon \cap \mathcal{U}, \Phi_\epsilon) \\ &= \sum_{\beta \in \mathcal{B}_\epsilon} \mathcal{Q}_G(N, \mathcal{S}, \Phi_\epsilon^{-1}(\beta) \cap N^\beta, \Phi_\epsilon). \end{aligned}$$

The lemma will be proved if we check that $[\mathcal{Q}_G(N, \mathcal{S}, \Phi_\epsilon^{-1}(\beta) \cap N^\beta, \Phi_\epsilon)]^G = 0$ for any non-zero $\beta \in \mathcal{B}_\epsilon$.

If $\beta_\Delta \in \mathcal{B}_\epsilon$ and $n \in \Phi_\epsilon^{-1}(\beta_\Delta) \cap N^{\beta_\Delta}$, $\Phi(n) = \beta_\Delta + \epsilon = \epsilon_\Delta$. So $\langle \Phi(n), \beta_\Delta \rangle = \langle \epsilon_\Delta, \beta_\Delta \rangle = 0$. So the infinitesimal action, $\mathcal{L}(\beta)$, on the fiber of the vector bundle $\mathcal{S}_n$ is equal to 0.

The Atiyah-Segal localization formula for the Witten deformation (Remark 4.13) gives

$$\begin{aligned} \mathcal{Q}_G(N, \mathcal{S}, \Phi_\epsilon^{-1}(\beta) \cap N^\beta, \Phi_\epsilon) &= \mathcal{Q}_G(N^\beta, \mathbb{d}_\beta(\mathcal{S}) \otimes \mathrm{Sym}(\mathcal{V}_\beta), \Phi_\epsilon^{-1}(\beta), \Phi_\epsilon) \\ &= \sum_{\mathcal{X} \subset N^\beta} \mathcal{Q}_G(\mathcal{X}, \mathbb{d}_\beta(\mathcal{S})|_{\mathcal{X}} \otimes \mathrm{Sym}(\mathcal{V}_\beta)|_{\mathcal{X}}, \Phi_\epsilon^{-1}(\beta), \Phi_\epsilon) \end{aligned}$$

where $\mathcal{V}_\beta \to N^\beta$ is the normal bundle of $N^\beta$ in $N$ and the sum runs over the connected components $\mathcal{X}$ of $N^\beta$ that intersects $\Phi_\epsilon^{-1}(\beta)$.

Let us look to the infinitesimal action of $\beta$, denoted $\mathcal{L}(\beta)$, on the fibers of the vector bundle $\mathbb{d}_\beta(\mathcal{S})|_{\mathcal{X}} \otimes \mathrm{Sym}(\mathcal{N}_\beta)|_{\mathcal{X}}$. This action can be checked at a point $n \in \Phi_\epsilon^{-1}(\beta) \cap N^\beta$.As the action of $\beta$ on the fiber of the vector bundle $\mathcal{S}_n$ is equal to 0, we obtain

$$\frac{1}{i}\mathcal{L}(\beta) = \begin{cases} \frac{1}{2}\mathbf{Tr}_{\mathrm{T}N|_{\mathcal{X}}}(|\beta|) & \text{on } \mathbb{d}_\beta(\mathcal{S})|_{\mathcal{X}}, \\ \geqslant 0 & \text{on } \mathrm{Sym}(\mathcal{N}_\beta)|_{\mathcal{X}}. \end{cases}$$

So we have checked that $\frac{1}{i}\mathcal{L}(\beta) \geqslant \frac{1}{2}\mathbf{Tr}_{\mathrm{T}N|_{\mathcal{X}}}(|\beta|)$ on $\mathbb{d}_\beta(\mathcal{S})|_{\mathcal{X}} \otimes \mathrm{Sym}(\mathcal{N}_\beta)|_{\mathcal{X}}$.

Now we remark that $\beta$ does not acts trivially on $N$, since $\beta$ belongs to the direction of the subspace $I(N) = \mathfrak{g}_N^\perp$: this forces $\frac{1}{2}\mathbf{Tr}_{\mathrm{T}N|_{\mathcal{X}}}(|\beta|)$ to be strictly positive. Finally we see that $\frac{1}{i}\mathcal{L}(\beta) > 0$ on $\mathbb{d}_\beta(\mathcal{S})|_{\mathcal{X}} \otimes \mathrm{Sym}(\mathcal{N}_\beta)|_{\mathcal{X}}$, and then

$$\left[\mathcal{Q}_G(\mathcal{X}, \mathbb{d}_\beta(\mathcal{S})|_{\mathcal{X}} \otimes \mathrm{Sym}(\mathcal{V}_\beta)|_{\mathcal{X}}, \Phi_\epsilon^{-1}(\beta), \Phi_\epsilon)\right]^G = 0.$$

if $\beta \neq 0$. The Lemma 5.6 is proved.

The proof of Theorem 5.4 will be completed with the following

**Lemma 5.7** *If $\mu+\epsilon$ is a regular value of $\Phi : N \to I(N)$, the invariant $[\mathcal{Q}_G(N, \mathcal{S}_N, \Phi_\epsilon^{-1}(0), \Phi_\epsilon) \otimes \mathbb{C}_{-\mu}]^G$ is equal to the index $\mathcal{Q}(N_{\mu+\epsilon}, \mathcal{S}^\mu_{\mu+\epsilon})$.*

We assume that $\mu+\epsilon$ is a regular value of $\Phi : N \to I(N)$ : the fiber $Z = \Phi^{-1}(\mu+\epsilon)$ is a submanifold equipped with a locally free action of $G' = G/G_N$. Let $N_{\mu+\epsilon} := Z/G'$ be the corresponding "reduced" space, and let $\pi : Z \to N_{\mu+\epsilon}$ be the projection map. We have the decomposition

$$\mathrm{T}N|_Z \simeq \pi^*(\mathrm{T}N_{\mu+\epsilon}) \oplus [\mathfrak{g}'_{\mathbb{C}}]. \tag{5.37}$$

For any $\nu \in \Lambda \cap I(N)$, $\mathcal{S}^\nu_{\mu+\epsilon}$ is a the Spin$^c$ bundle on $N_{\mu+\epsilon}$ defined by the relation

$$\mathcal{S}_N|_Z \otimes \mathbb{C}_{-\nu} \simeq \pi^*\left(\mathcal{S}^\nu_{\mu+\epsilon}\right) \otimes [\bigwedge \mathfrak{g}'_{\mathbb{C}}].$$

The following result is proved in [24].

**Proposition 5.8** *We have the following equality in $\hat{R}(G)$*

$$\mathcal{Q}_G(N, \mathcal{S}_N, \Phi_\epsilon^{-1}(0), \Phi_\epsilon) = \sum_{\nu \in \Lambda \cap I(N)} \mathcal{Q}(N_{\mu+\epsilon}, \mathcal{S}^\nu_{\mu+\epsilon})\ \mathbb{C}_\nu.$$

*In particular $[\mathcal{Q}_G(N, \mathcal{S}_N, \Phi_\epsilon^{-1}(0), \Phi_\epsilon) \otimes \mathbb{C}_{-\mu}]^G$ is equal to $\mathcal{Q}(N_{\mu+\epsilon}, \mathcal{S}^\mu_{\mu+\epsilon})$.*

## 5.3 $[Q, R] = 0$

We come back to the setting of a compact $K$-manifold $M$, oriented and of even dimension, that is equipped with a $K$-Spin$^c$ bundle $\mathcal{S}$. Let $\mathbb{L}_\mathcal{S}$ be its determinant bundle, and let $\Phi_\mathcal{S} \to \mathfrak{k}^*$ be the moment map that is attached to an invariant connection on $\mathbb{L}_\mathcal{S}$. We assume that there exists $(\mathfrak{h}) \in \mathcal{H}_\mathfrak{k}$ such that $([\mathfrak{k}_M, \mathfrak{k}_M]) = ([\mathfrak{h}, \mathfrak{h}])$. Let $\mathfrak{z}$ be the center of $\mathfrak{h}$.

We consider an admissible element $\mu \in \mathfrak{z}^*$ such that $K_\mu = H$ : the coadjoint orbit $\mathcal{P} := K\mu$ is admissible and contained in the Dixmier sheet $\mathfrak{k}^*_{(\mathfrak{h})}$. Let

$$M_\mathcal{P} := \Phi_\mathcal{S}^{-1}(\mathcal{P})/K$$

In order to define $\mathrm{Q}^{\mathrm{spin}}(M_\mathcal{P}) \in \mathbb{Z}$ we proceed as follows.

Let $\mathfrak{h}_0^* := \{\xi \in \mathfrak{h}^* \ | K_\xi \subset H\}$ and let $\mathcal{Y} := \Phi_\mathcal{S}^{-1}(\mathfrak{h}_0^*)$. We recall that the map $\xi \mapsto \rho(\xi)$ is locally constant on $\mathfrak{h}_0^*$. Let us fix a connected component $C$ of $\mathfrak{h}_0^*$ : we denote $\rho_C = \rho(\xi)$ for any $\xi \in C$. We consider $\mathcal{Y}_C = \Phi_\mathcal{S}^{-1}(C)$ that is a $H$-submanifold of $M$ equipped with a $H$-Spin$^c$ bundle $\mathcal{S}_{\mathcal{Y}_C}$: the associated moment map is $\Phi_{\mathcal{Y}_C} := \Phi_\mathcal{S}|_{\mathcal{Y}_C} - \rho_C$.

For any admissible element $\mu \in C \cap \mathfrak{z}^*$ the element

$$\tilde{\mu} := \mu - \rho(\mu) = \mu - \rho_C$$

belongs to the weight lattice $\Lambda$ of the torus $A_H := H/[H,H]$, and the reduced space $M_{K\mu}$ is equal to

$$(\mathcal{Y}_C)_{\tilde{\mu}} := \{\Phi_{\mathcal{Y}_C} = \tilde{\mu}\}/A_H.$$

By definition, we take $\mathrm{Q}^{\mathrm{spin}}(M_{K\mu}) := \mathrm{Q}^{\mathrm{spin}}((\mathcal{Y}_C)_{\tilde{\mu}})$ where the last term is computed as explained in the previous section. More precisely, let us decompose $\mathcal{Y}_C$ into its connected components $\mathcal{Y}_1, \ldots, \mathcal{Y}_r$. For each $j$, let $\mathfrak{z}_j \subset \mathfrak{z}$ be the generic infinitesimal stabilizer relative to the $A_H$-action on $\mathcal{Y}_j$. Then we take

$$\mathrm{Q}^{\mathrm{spin}}(M_{\mathcal{P}}) = \mathrm{Q}^{\mathrm{spin}}(M_{K\mu}) := \sum_j \mathrm{Q}^{\mathrm{spin}}\left((\mathcal{Y}_j)_{\tilde{\mu}+\epsilon_j}\right)$$

where $\epsilon_j \in \mathfrak{z}_j^{\perp}$ are generic and small enough.

With this definition of quantization of reduced spaces $\mathrm{Q}^{\mathrm{spin}}(M_{\mathcal{P}})$, we obtain the main theorem of this article, inspired by the $[Q,R]=0$ theorem of Meinrenken-Sjamaar.

Let $M$ be a $K$-manifold and $\mathcal{S}$ be a $K$-equivariant Spin$^c$-bundle over $M$. Let $(\mathfrak{h}) \in \mathcal{H}_{\mathfrak{k}}$ such that $([\mathfrak{k}_M, \mathfrak{k}_M]) = ([\mathfrak{h},\mathfrak{h}])$, and consider the set $\mathcal{A}((\mathfrak{h}))$ of admissible orbits contained in the Dixmier sheet $\mathfrak{k}^*_{(\mathfrak{h})}$.

**Theorem 5.9**

$$\mathcal{Q}_K(M,\mathcal{S}) = \sum_{\mathcal{P}\in\mathcal{A}((\mathfrak{h}))} \mathrm{Q}^{\mathrm{spin}}(M_{\mathcal{P}})\mathrm{Q}_K^{\mathrm{spin}}(\mathcal{P}). \tag{5.38}$$

We end this section by giving yet another criterium for the vanishing of $\mathrm{Q}_K^{\mathrm{spin}}(M,\mathcal{S})$.

Consider the map $\Phi_{\mathcal{S}} : M \to \mathfrak{k}^*$. At each point $m \in M$, the differential $d_m\Phi_{\mathcal{S}}$ gives a map $T_mM \to \mathfrak{k}^*$. Let $\mathfrak{k}_m^{\perp} \subset \mathfrak{k}^*$. From the Kostant relations, we see that $d_m\Phi_{\mathcal{S}}$ take value in $\mathfrak{k}_m^{\perp}$.

**Proposition 5.10** *If $\mathcal{Q}_K(M,\mathcal{S}) \neq 0$, then there exists $m \in M\backslash M^K$ such that* $\mathrm{Image}(d_m\Phi_{\mathcal{S}}) = \mathfrak{k}_m^{\perp}$.

**Proof.** If we consider the decomposition of the slice $\mathcal{Y}_C = \bigcup \mathcal{Y}_j$ in connected components, for $\mathcal{Q}_K(M,\mathcal{S}) \neq 0$, then for some $j$, $\Phi(Y_j)$ has non empty interior in $\mathfrak{z}_j^\perp$. Here $\mathfrak{z}_j$ is the infinitesimal stabilizer of the action of $H/[H,H]$ on $\mathcal{Y}_j$. Thus $\mathfrak{z}_j$ is equal to $\mathfrak{k}_m \subset \mathfrak{h}$ for generic $m \in \mathcal{Y}_j$. So there exists a point $m \in \mathcal{Y}_j$ such that the differential of $\Phi_{\mathcal{S}}|_{\mathcal{Y}}$ is surjective on $\mathfrak{z}_j^\perp \subset \mathfrak{h}^*$. Now if we consider $K\mathcal{Y}_j \subset M$, then $\mathrm{Image}(d_m\Phi_{\mathcal{S}}) = \mathfrak{h}^* \oplus \mathfrak{z}_j^\perp$. This is exactly $\mathfrak{k}_m^\perp$.

When the action of $K$ is abelian, we can always reduce ourselves to an effective action with $\mathfrak{k}_M = \{0\}$. Then the support of decomposition of $\mathcal{Q}_K(M,\mathcal{S})$ is contained in the interior of $\Phi_{\mathcal{S}}(M) \cap \Lambda$. If this set has no interior point, then $\mathcal{Q}_K(M,\mathcal{S}) = 0$. This small remark implies the well-known Atiyah-Hirzebruch vanishing theorem in the spin case [2], as well as the variant of Hattori [13].

We also note another corollary.

**Corollary 5.11** *If the two form $\Omega_{\mathcal{S}}$ is exact, and the $K$-action on $M$ is non-trivial then $\mathcal{Q}_K(M,\mathcal{S}) = 0$.*

It is due to the fact that if $\Omega_{\mathcal{S}} = d\alpha$, by modifying the connection on $\mathbb{L}_{\mathcal{S}}$ by $\alpha$, our moment map is constant. So if the action is non trivial, $\mathcal{Q}_K(M,\mathcal{S}) = 0$.

## 5.4 $[Q,R] = 0$ on induced manifolds

Let $H \subset K$ be the stabilizer subgroup of some element in $\mathfrak{k}^*$. We adopt the notations of Section 3.5. Let $C$ be a choice of a connected component of $\mathfrak{h}_0^*$.

Assume that $Y$ is a compact $H$-manifold, and consider the manifold $M = K \times_H Y$. Assume that $M$ is oriented and equipped with a $K$-equivariant Spin$^c$-bundle $\mathcal{S}$. We consider the Spin$^c$-bundle $\mathcal{S}_Y$ on $Y$ such that $\mathcal{S}|_Y = \bigwedge \mathfrak{q}^C \otimes \mathcal{S}_Y$. The equivariant index $\mathcal{Q}_K(M,\mathcal{S})$ verifies the equation

$$\mathcal{Q}_K(M,\mathcal{S}) = \mathrm{Ind}_H^K\Big(\bigwedge \mathfrak{q}^C \otimes \mathcal{Q}_H(Y,\mathcal{S}_Y)\Big). \tag{5.39}$$

The aim of this section is to explain how our $[Q,R] = 0$ theorem matches with the induction formula (5.39) when we apply it to both indices $\mathcal{Q}_K(M,\mathcal{S})$ and $\mathcal{Q}_H(Y,\mathcal{S}_Y)$.

Let $\mathbb{L}_{\mathcal{S}}$ be the determinant line bundle of the Spin$^c$-bundle $\mathcal{S}$. As $\mathbb{L}_{\mathcal{S}} \simeq K \times_H \mathbb{L}_{\mathcal{S}}|_Y$ we can choose an equivariant connection on $\mathbb{L}_{\mathcal{S}}$ such that the corresponding moment map $\Phi_{\mathcal{S}} : M \to \mathfrak{k}^*$, when restricted to $Y$, takes value in $\mathfrak{h}^*$. The determinant line bundle $\mathbb{L}_Y$ of the Spin$^c$-bundle $\mathcal{S}_Y$ is equal to $\mathbb{L}_{\mathcal{S}}|_Y \otimes \mathbb{C}_{-2\rho_C}$, and for the moment map $\Phi_Y$, we have $\Phi_Y = \Phi_{\mathcal{S}}|_Y - \rho_C$.

We can assume that there exists a Levi subalgebra $\mathfrak{l} \subset \mathfrak{h}$ such that the conjugacy class of generic stabilizer $([\mathfrak{h}_Y, \mathfrak{h}_Y]) = ([\mathfrak{k}_M, \mathfrak{k}_M])$ is equal to $([\mathfrak{l}, \mathfrak{l}])$, otherwise $\mathcal{Q}_K(M, \mathcal{S}) = \mathcal{Q}_H(Y, \mathcal{S}_Y) = 0$. We note $\mathcal{A}_{\mathfrak{k}}((\mathfrak{l}))$ (resp. $\mathcal{A}_{\mathfrak{h}}((\mathfrak{l}))$) the set of $K$-admissible (resp. $H$-admissible) orbits belonging to the Dixmier sheet $\mathfrak{k}^*_{((\mathfrak{l}))}$ (resp. $\mathfrak{h}^*_{((\mathfrak{l}))}$).

For any orbit $H\xi \subset \mathfrak{h}^*$, we define the $K$-orbit $\mathbf{t}_C(H\xi) := K(H\xi + \rho_C) = K(\xi + \rho_C)$. Let $\mathcal{A}_{\mathfrak{h}}((\mathfrak{l}))_C$ be the subset of $\mathcal{A}_{\mathfrak{h}}((\mathfrak{l}))$ formed by the $H$-coadjoint orbit $\mathcal{P}'$ such that $\mathcal{P}' + \rho_C$ is contained in $\mathfrak{h}^*_0$. Let $\mathfrak{h}^*_{(\mathfrak{l})} \subset \mathfrak{h}^*$ be the Dixmier sheet of coadjoint orbits $H\xi$ with $H_\xi$ conjugate to $L$.

We have the following basic fact.

**Lemma 5.12** • *If $\mathcal{P}' \in \mathcal{A}_{\mathfrak{h}}((\mathfrak{l}))_C$, then $\mathbf{t}_C(\mathcal{P}')$ belongs to $\mathcal{A}_{\mathfrak{k}}((\mathfrak{l}))$.*

• *For any $\mathcal{P} \in \mathcal{A}_{\mathfrak{k}}((\mathfrak{l}))$ we have*

$$\mathcal{P} \cap \overline{\mathfrak{h}^*_{(\mathfrak{l})}} = \mathcal{P} \cap \mathfrak{h}^*_{(\mathfrak{l})} = \coprod_{\mathcal{P}'} (\mathcal{P}' + \rho_C) \tag{5.40}$$

*where the finite union runs over the orbits $\mathcal{P}' \in \mathcal{A}_{\mathfrak{h}}((\mathfrak{l}))_C$ such that $\mathbf{t}_C(\mathcal{P}') = \mathcal{P}$.*

**Proof.** Let $\mathcal{P}' \in \mathcal{A}_{\mathfrak{h}}((\mathfrak{l}))_C$. Then $P' = H\mu$ with a $H$-admissible element $\mu \in \mathfrak{h}^*$ such that $K_\mu = L$ and $\mu + \rho_C \in \mathfrak{h}^*_0$. We have $K_{\mu+\rho_C} = H_{\mu+\rho_C} = H_\mu = L$ and

$$\mu + \rho_C - \rho^K(\mu + \rho_C) = \mu - \rho^H(\mu) + \rho_C - \rho_{C'} \tag{5.41}$$

where $C'$ is the connected compoent of $\mathfrak{h}^*_0$ containing $a + \rho_C$. As $\rho_C - \rho_{C'}$ belongs to the weight lattice we see that $\mu + \rho_C$ is $K$-admissible. The first point is proved.

The inclusions $\coprod_{\mathbf{t}_C(\mathcal{P}')=\mathcal{P}} (\mathcal{P}' + \rho_C) \subset \mathcal{P} \cap \mathfrak{h}^*_{(\mathfrak{l})} \subset \mathcal{P} \cap \overline{\mathfrak{h}^*_{(\mathfrak{l})}}$ are obvious. Consider now a $H$-orbit $\mathcal{T}$ contained in $\mathcal{P} \cap \overline{\mathfrak{h}^*_{(\mathfrak{l})}}$. We have $\mathcal{T} = H\lambda$ where $\lambda$ is $K$-admissible. As $\lambda \in \overline{\mathfrak{h}^*_{(\mathfrak{l})}}$ the stabilizer $H_\lambda$ is $H$-conjugated to a subgroup containing $L$. In the other hand, the stabilizer sugroup $K_\lambda$ is $K$-conjugate to $L$. If we compare the dimension of the connectd subgroups $H_\lambda$ and $K_\lambda$ we see that $K_\lambda = H_\lambda$ and then $H\lambda \in \mathcal{P} \cap \mathfrak{h}^*_{(\mathfrak{l})}$ : the element $\lambda$ can be choosen so that $K_\lambda = H_\lambda = L$.

We consider $\mu = \lambda - \rho_C$ so that $\mathbf{t}_C(H\mu) = \mathcal{P}$. We see first that $H_\mu = H_\lambda = L$ and (5.41) shows that $\mu$ is $H$-admissible. We have checked that $H\mu \in \mathcal{A}_{\mathfrak{h}}((\mathfrak{l}))$ and $\mathcal{T} = H\mu + \rho_C \subset \mathcal{P} \cap \mathfrak{h}^*_{(\mathfrak{l})}$. The second point is also proved.

For the $H$-manifold $Y$, our $[Q,R]=0$ Theorem says that $\mathcal{Q}_H(Y,\mathcal{S}_Y)$ is equal to $\sum_{\mathcal{P}'\in\mathcal{A}_{\mathfrak{h}}((\mathfrak{l}))}\mathrm{Q}^{\mathrm{spin}}(Y_{\mathcal{P}'})\mathrm{Q}_H^{\mathrm{spin}}(\mathcal{P}')$. If we apply the induction formula proved in Proposition 3.22, we get that

$$\begin{aligned}\mathcal{Q}_K(M,\mathcal{S}) &= \sum_{\mathcal{P}'\in\mathcal{A}_{\mathfrak{h}}((\mathfrak{l}))}\mathrm{Q}^{\mathrm{spin}}(Y_{\mathcal{P}'})\mathrm{Ind}_H^K\Big(\bigwedge\mathfrak{q}^C\otimes\mathrm{Q}_H^{\mathrm{spin}}(\mathcal{P}')\Big)\\ &= \sum_{\mathcal{P}'\in\mathcal{A}_{\mathfrak{h}}((\mathfrak{l}))_C}\epsilon_C^{\mathcal{P}'}\,\mathrm{Q}^{\mathrm{spin}}(Y_{\mathcal{P}'})\mathrm{Q}_K^{\mathrm{spin}}(\mathfrak{t}_C(\mathcal{P}'))\\ &= \sum_{\mathcal{P}\in\mathcal{A}_{\mathfrak{k}}((\mathfrak{l}))}\mathrm{m}^{\mathcal{P}}\mathrm{Q}_K^{\mathrm{spin}}(\mathcal{P})\end{aligned}$$

with $\mathrm{m}^{\mathcal{P}}=\sum_{\mathfrak{t}_C(\mathcal{P}')=\mathcal{P}}\epsilon_C^{\mathcal{P}'}\,\mathrm{Q}^{\mathrm{spin}}(Y_{\mathcal{P}'})$. Here $\epsilon_C^{\mathcal{P}'}$ is the sign $\epsilon_C^{C'}$ where $C'$ is the connected component of $\mathfrak{h}_0^*$ that contains $\mathcal{P}'+\rho_C$ (see Section 3.5).

Finally, we recover the $[Q,R]=0$ Theorem for the $K$-manifold $M$ with the help of the following

**Proposition 5.13** *For any $\mathcal{P}\in\mathcal{A}_{\mathfrak{k}}((\mathfrak{l}))$, the term $\mathrm{m}^{\mathcal{P}}$ is equal to $\mathrm{Q}^{\mathrm{spin}}(M_{\mathcal{P}})$.*

**Proof.** Identity (5.40) and the fact that the image of $\Phi_Y$ is contained in $\overline{\mathfrak{h}^*_{(\mathfrak{l})}}$ gives automatically that

$$\Phi_{\mathcal{S}}^{-1}(\mathcal{P})=\coprod_{\mathfrak{t}_C(\mathcal{P}')=\mathcal{P}}K\times_H\Phi_Y^{-1}(\mathcal{P}').$$

Hence the reduced space $M_{\mathcal{P}}:=\Phi_{\mathcal{S}}^{-1}(\mathcal{P})/K$ decomposes as a disjoint sum $\coprod_{\mathfrak{t}_C(\mathcal{P}')=\mathcal{P}}M_{\mathcal{P}}^{\mathcal{P}'}$ where $M_{\mathcal{P}}^{\mathcal{P}'}=(K\times_H\Phi_Y^{-1}(\mathcal{P}'))/K$ is equal (as a set) to $Y_{\mathcal{P}'}=\Phi_Y^{-1}(\mathcal{P}')/H$.

Let $\mathcal{P}'\in\mathcal{A}_{\mathfrak{k}}((\mathfrak{l}))$ such that $\mathfrak{t}_C(\mathcal{P}')=\mathcal{P}$. The proposition will be proved if we show that $\mathrm{Q}^{\mathrm{spin}}(M_{\mathcal{P}}^{\mathcal{P}'})=\epsilon_C^{\mathcal{P}'}\mathrm{Q}^{\mathrm{spin}}(Y_{\mathcal{P}'})$.

Consider $\mu$ such that $\mathcal{P}'=H\mu$ and $H_\mu=L$. Take $\mu'=\mu+\rho_C$ : we have $\mathcal{P}=K\mu'$ and $K_{\mu'}=L$. Let $B\subset\mathfrak{l}^*$ be a small ball centered at $\mu$, and consider the slice $\mathcal{Y}:=\Phi_Y^{-1}(B)$ : the set $H\mathcal{Y}\subset Y$ is a $H$-invariant open neighborhood of $\Phi_Y^{-1}(\mathcal{P}')$ diffeomorphic to $H\times_L\mathcal{Y}$. Consider the $K$-invariant open subset

$$M^{\mathcal{P}'}:=K\times_H\Big(H\mathcal{Y}\Big)\subset M$$

We note that $M^{\mathcal{P}'}\simeq K\times_L\mathcal{Y}$, and the reduction of $M^{\mathcal{P}'}$, equipped with the moment map $\Phi_{\mathcal{S}}|_{M^{\mathcal{P}'}}$, relatively to $\mathcal{P}$ is equal to $M_{\mathcal{P}}^{\mathcal{P}'}$.

By definition, the quantity $\mathrm{Q}^{\mathrm{spin}}(M_{\mathcal{P}}^{\mathcal{P}'})$ is equal to

$$[\mathcal{Q}_L(\mathcal{Y},\mathcal{S}_{\mathcal{Y}}\otimes\mathbb{C}_{-\mu'+\rho^K(\mu')},\{0\})]^L$$

where $\mathcal{S}_{\mathcal{Y}}$ is the Spin$^c$-bundle defined by relation $\mathcal{S}|_{\mathcal{Y}} = \bigwedge_{J_{\mu'}} \mathfrak{k}/\mathfrak{l} \otimes \mathcal{S}_{\mathcal{Y}}$.

On the other hand, the quantity $\mathrm{Q}^{\mathrm{spin}}(Y_{\mathcal{P}'})$ is equal to

$$[\mathcal{Q}_L(\mathcal{Y}, \mathcal{S}'_{\mathcal{Y}} \otimes \mathbb{C}_{-\mu+\rho^H(\mu)}, \{0\})]^L$$

where $\mathcal{S}'_{\mathcal{Y}}$ is the Spin$^c$-bundle defined by relation $\mathcal{S}_Y|_{\mathcal{Y}} = \bigwedge_{J_\mu} \mathfrak{l}/\mathfrak{h} \otimes \mathcal{S}'_{\mathcal{Y}}$. Now if we use the fact that $\mathcal{S}|_Y = \bigwedge_{J_C} \mathfrak{k}/\mathfrak{h} \otimes \mathcal{S}_Y$, we can check that

$$\mathcal{S}_{\mathcal{Y}} \otimes \mathbb{C}_{-\mu'+\rho^K(\mu')} \simeq \epsilon_C^{C'} \, \mathcal{S}'_{\mathcal{Y}} \otimes \mathbb{C}_{-\mu+\rho^H(\mu)}$$

at the level of $L$-equivariant graded Spin$^c$-bundles. The proof of the relation $\mathrm{Q}^{\mathrm{spin}}(M_{\mathcal{P}}^{\mathcal{P}'}) = \epsilon_C^{\mathcal{P}'} \mathrm{Q}^{\mathrm{spin}}(Y_{\mathcal{P}'})$ then follows.

# 6 Examples

## 6.1 $\mathbb{P}^1(\mathbb{C})$

We consider the simplest case of the theory. Let $\mathbb{P}^1 := \mathbb{P}^1(\mathbb{C})$ be the projective space of (complex) dimension one. Consider the (ample) line bundle $\mathcal{L} \to \mathbb{P}^1$, dual of the tautological bundle. It is obtained as quotient of the trivial line bundle $\mathbb{C}^2\backslash\{(0,0)\} \times \mathbb{C}$ on $\mathbb{C}^2\backslash\{(0,0)\}$ by the action $u \cdot (z_1, z_2, z) = (uz_1, uz_2, uz)$ of $\mathbb{C}^*$. We consider the action of $T = S^1$ on $\mathcal{L} \to \mathbb{P}^1$ defined by $t \cdot [z_1, z_2, z] = [t^{-1}z_1, z_2, z]$.

Let $\mathcal{S}(n)$ be the Spin$^c$-bundle $\bigwedge_{\mathbb{C}} \mathrm{T}\mathbb{P}^1 \otimes \mathcal{L}^{\otimes n}$. The character $\mathrm{Q}_T^{\mathrm{spin}}(M, \mathcal{S}(n))$ is equal to $H^0(\mathbb{P}^1, \mathcal{O}(n)) - H^1(\mathbb{P}^1, \mathcal{O}(n))$ where $\mathcal{O}(n)$ is the sheaf of holomorphic sections of $\mathcal{L}^{\otimes n}$. Note that the holomorphic line bundle $\mathcal{L}^{\otimes n}$ is not ample if $n \leqslant 0$. We have

- $\mathrm{Q}_T^{\mathrm{spin}}(M, \mathcal{S}(n)) = -\sum_{k=n+1}^{-1} t^k$ when $n \leqslant -2$,
- $\mathrm{Q}_T^{\mathrm{spin}}(M, \mathcal{S}(-1)) = 0$,
- $\mathrm{Q}_T^{\mathrm{spin}}(M, \mathcal{S}(n)) = \sum_{k=0}^{n} t^k$ when $n \geqslant 0$.

The determinant line bundle of $\mathcal{S}(n)$ is $\mathbb{L}_n = [\mathbb{C}_{-1}] \otimes \mathcal{L}^{\otimes 2n+2}$ where $[\mathbb{C}_{-1}]$ is the trivial line bundle equipped with the representation $t^{-1}$ on $\mathbb{C}$.

Remark that $\mathbb{P}^1$ is homogeneous under $U(2)$, so there exists a unique $U(2)$-invariant connection on $\mathbb{L}_n$. The corresponding moment map $\Phi_{\mathcal{S}(n)}$ is such that

$$\Phi_{\mathcal{S}(n)}([z_1, z_2]) = (n+1)\frac{|z_1|^2}{|z_1|^2 + |z_2|^2} - \frac{1}{2}. \tag{6.42}$$

The image $I_n = \Phi_{\mathcal{S}(n)}(M)$ is

- the interval $[-\frac{1}{2}, n+\frac{1}{2}]$ when $n \geqslant 0$,
- a point $\{-\frac{1}{2}\}$ when $n = -1$,
- the interval $[n+\frac{1}{2}, -\frac{1}{2}]$ when $n \leqslant -2$,

It is in agreement with our theorem. Indeed all characters occurring in $Q_T^{\text{spin}}(M, \mathcal{S}(n))$ are the integral points in the relative interior of $I_n$, and all reduced spaces are points.

If we consider simply the action of $T$ on $\mathbb{P}^1$, the choice of connection may vary. In fact, given any smooth function $f$ on $\mathbb{R}$, we can modify the connection such that $\Phi_{\mathcal{S}(n)}([z_1, z_2]) = -\frac{1}{2} + (n+1)\frac{|z_1|^2}{|z_1|^2+|z_2|^2} + f(\frac{|z_1|^2}{|z_1|^2+|z_2|^2})\frac{|z_1|^2}{|z_1|^2+|z_2|^2}(1 - \frac{|z_1|^2}{|z_1|^2+|z_2|^2})$. Let $\Omega_{\mathbb{L}}$ be the curvature of $\mathbb{L}$, then the Duistermaat-Heckman measure $(\Phi_{\mathcal{S}(n)})_*\Omega_{\mathbb{L}}$ is independent of the choice of the connection and is equal to the characteristic function of $I_n$.

Take for example

$$\Phi_{\mathcal{S}(n)}([z_1, z_2]) = -\frac{1}{2} + (n+1)\frac{|z_1|^2}{|z_1|^2 + |z_2|^2} - 15\frac{|z_1|^2}{|z_1|^2 + |z_2|^2}(1 - \frac{|z_1|^2}{|z_1|^2 + |z_2|^2}).$$

Figure 11 is the graph on $\Phi_{\mathcal{S}(n)}$ for $n = 4$ in terms of $x = \frac{|z_1|^2}{|z_1|^2+|z_2|^2}$ varying between 0 and 1. We see that the image of $\Phi_{\mathcal{S}(4)}$ is the interval $[-\frac{13}{6}, \frac{9}{2},]$. But the image of the signed measure is still $[-\frac{1}{2}, \frac{9}{2}]$. Above the integral points in $[-\frac{13}{6}, -\frac{1}{2}]$, the reduced space is not connected, it consists of two points giving opposite contributions to the index. So our theorem holds.

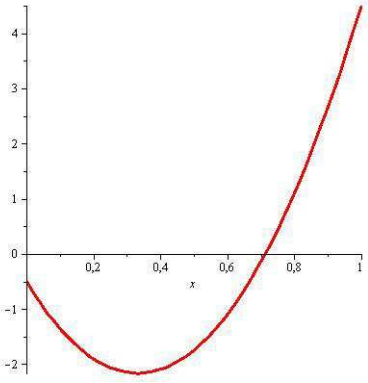


Figure 11: The graph of $\Phi_{\mathbb{L}}$

## 6.2 The Hirzebruch surface

We consider $M$ to be the Hirzebruch surface. Represent $M$ as the quotient of $\mathcal{U} = \mathbb{C}^2 - \{(0,0)\} \times \mathbb{C}^2 - \{(0,0)\}$ by the free action of $\mathbb{C}^* \times \mathbb{C}^*$ acting by

$$(u,v) \cdot (z_1, z_2, z_3, z_4) = (uz_1, uz_2, uvz_3, vz_4)$$

and we denote by $[z_1, z_2, z_3, z_4] \in M$ the equivalence class of $(z_1, z_2, z_3, z_4)$. The map $\pi : [z_1, z_2, z_3, z_4] \to [z_1, z_2]$ is a fibration of $M$ on $P_1(\mathbb{C})$ with fiber $P_1(\mathbb{C})$.

Consider the line bundle $L(n_1, n_2)$ obtained as quotient of the trivial line bundle $\mathcal{U} \times \mathbb{C}$ on $\mathcal{U}$ by the action

$$(u,v) \cdot (z_1, z_2, z_3, z_4, z) = (uz_1, uz_2, uvz_3, vz_4, u^{n_1} v^{n_2} z)$$

for $(u,v) \in \mathbb{C}^* \times \mathbb{C}^*$. The line bundle $L(n_1, n_2)$ is ample if and only if $n_1 > n_2 > 0$.

We have a canonical action of the group $K := U(2)$ on $M$ : $g \cdot [Z_1, Z_2] = [gZ_1, Z_2]$ for $Z_1, Z_2 \in \mathbb{C}^2 - \{(0,0)\}$ and the line bundle $L(n_1, n_2)$ with action $g \cdot [Z_1, Z_2, z] = [gZ_1, Z_2, z]$ is $K$-equivariant.

We are interested in the (virtual) $K$-module

$$H^0(M, \mathcal{O}(n_1, n_2)) - H^1(M, \mathcal{O}(n_1, n_2)) + H^2(M, \mathcal{O}(n_1, n_2))$$

where $\mathcal{O}(n_1, n_2)$ be the sheaf of holomorphic sections of $L(n_1, n_2)$.

In this case, it is in fact possible to compute directly individual cohomology groups $H^i(M, \mathcal{O}(n_1, n_2))$. However, we will describe here only results on the alternate sum and relate them to the moment map.

Let $T = U(1) \times U(1)$ be the maximal torus of $K$. The set $Y := \{[z_1, z_2, z_3, z_4] \in M \,|\, z_1 = 0\}$ is a $T$-invariant complex submanifold of $M$ (with trivial action of $(t_1, 1)$). The map

$$Y \to \mathbb{P}^1(\mathbb{C}), \quad [0, z_2, z_3, z_4] \mapsto [(z_2)^{-1} z_3, z_4]$$

is a $T$-equivariant isomorphism and the map $(g, y) \in K \times Y \mapsto g \cdot y \in M$ factorizes through an isomorphism $K \times_T Y \simeq M$. Thus $M$ is an induced manifold.

For any $(a, b) \in \mathbb{Z}^2$, we denote $\mathbb{C}_{a,b}$ the 1-dimensional representation of $T$ associated to the character $(t_1, t_2) \mapsto t_1^a t_2^b$. We denote by $e_1^*, e_2^*$ the canonical bases of $\mathfrak{t}^* \simeq \mathbb{R}^2$. The Weyl chamber is $\mathfrak{t}^*_{\geqslant 0} = \{x e_1^* + y e_2^*, x \geqslant y\}$. The elements $e_1^*, e_2^*$ are conjugated by the Weyl group.

The line bundle $L(n_1, n_2)$, when restricted to $Y \simeq \mathbb{P}^1(\mathbb{C})$, is isomorphic to $\mathcal{L}^{\otimes n_2} \otimes [\mathbb{C}_{0,-n_1}]$.

We consider $L_\kappa = L(3,2)$ the line bundle obtained from the reduction of the trivial line bundle $\bigwedge^4 \mathbb{C}^4$ with natural action of $\mathbb{C}^* \times \mathbb{C}^*$. We denote $\mathcal{S}_M := \bigwedge_{\mathbb{C}} \mathrm{T}M$ (resp. $\mathcal{S}_Y := \bigwedge_{\mathbb{C}} \mathrm{T}Y$) the Spin$^c$-bundle associated to the complex structure on $M$ (resp. $Y$).

We denote by $\varphi : Y \to [0,1]$ the map defined by $\varphi(y) = \frac{|a_1|^2}{|a_1|^2+|a_2|^2}$ if $y \simeq [a_1, a_2]$.

**Proposition 6.1** • *Let $\mathcal{S}(n_1,n_2)$ be the spin bundle $\mathcal{S}_M \otimes L(n_1,n_2)$ on $M$. Its determinant line bundle is*

$$\mathbb{L}_{n_1,n_2} = [\mathbb{C}_{\det}] \otimes L_\kappa \otimes L(2n_1, 2n_2)$$

*where $[\mathbb{C}_{\det}] \to M$ is the trivial $U(2)$-equivariant line bundle associated to the character* $\det : U(2) \to \mathbb{C}^*$.

• *There exists a connection on $\mathbb{L}_{n_1,n_2}$ such that the corresponding moment map $\Phi_{n_1,n_2} : K \times_T Y \to \mathfrak{k}^*$ is defined by*

$$\Phi_{n_1,n_2}([k,y]) = \Big( -(n_1 + \frac{3}{2}) + (n_2+1)\varphi(y)\Big) k \cdot e_2^* + \frac{1}{2}(e_1^* + e_2^*).$$

**Proof.** For the second point, we construct a $U(2)$-invariant connection on $\mathbb{L}_{n_1,n_2}$ by choosing the $T$-invariant connection on $(\mathbb{L}_{n_1,n_2})|_Y$ having moment map $\big(-(n_1+\frac{3}{2}) + (n_2+1)\varphi(y)\big)\, e_2^* + \frac{1}{2}(e_1^* + e_2^*)$ under the $T$-action (see Equation (6.42)).

From Proposition 6.1, it is not difficult to describe the "Kirwan set" $\Delta(n_1,n_2) = \mathrm{Image}(\Phi_{n_1,n_2}) \cap \mathfrak{t}^*_{\geqslant 0}$ for all cases of $n_1, n_2$. It depends of the signs of $n_1 + \frac{3}{2}, n_2 + 1, n_1 - n_2 + \frac{1}{2}$, that is, as we are working with integers, the signs of $n_1+1$, $n_2+1$ and $n_1 - n_2$. We concentrate in the case where $n_1 + 1 \geqslant 0, n_2 + 1 \geqslant 0$ (other cases are similarly treated). Then, we have two cases:

• If $n_1 \geqslant n_2$, then the Kirwan set $\Delta(n_1,n_2)$ is the interval

$$[(n_1 - n_2) + \frac{1}{2}, n_1 + \frac{3}{2}](-e_2^*) + \frac{1}{2}(e_1^* + e_2^*).$$

• If $n_2 > n_1$, then the Kirwan set $\Delta(n_1,n_2)$ is the union of the intervals

$$[0, n_2 - n_1 - \frac{1}{2}]e_1^* + \frac{1}{2}(e_1^* + e_2^*)$$

and

$$[0, n_1 + \frac{3}{2}](-e_2^*) + \frac{1}{2}(e_1^* + e_2^*).$$

If $n_1 \geqslant n_2 \geqslant 0$ the curvature of the corresponding connection on $\mathbb{L}_{n_1,n_2} = L(2n_1+3, 2n_2+2)$ (which is an ample line bundle) is non degenerate, thus the image is a convex subset of $\mathfrak{t}^*_{\geqslant 0}$ (in agreement with Kirwan convexity theorem) while for $n_2 > n_1$ the image set is not convex.

The character $\mathcal{Q}_K(n_1,n_2) := \mathcal{Q}_K(M, \mathcal{S}(n_1,n_2))$ is equal to the (virtual) $K$-module $H^0(M,\mathcal{O}(n_1,n_2)) - H^1(M,\mathcal{O}(n_1,n_2)) + H^2(M,\mathcal{O}(n_1,n_2))$ where $\mathcal{O}(n_1,n_2)$ is the sheaf of holomorphic sections of $L(n_1,n_2)$.

Let $\Lambda_{\geqslant 0} = \{(\lambda_1,\lambda_2); \lambda_1 \geqslant \lambda_2\}$ be the set of dominant weights for $U(2)$. We index the representations of $U(2)$ by $\rho + \Lambda_{\geqslant 0}$. Here $\rho = (\frac{1}{2}, \frac{-1}{2})$ and $\lambda_1, \lambda_2$ are integers. We then have

$$\pi_{(\frac{1}{2},-k-\frac{1}{2})} = S^k$$

the space of complex polynomials on $\mathbb{C}^2$ homogeneous of degree $k$.

If $n_2 \geqslant 0$, we know that $\mathcal{Q}_T(Y, \mathcal{S}_Y \otimes \mathcal{L}^{\otimes n_2}) = \sum_{k=0}^{n_2} t_2^k$. From the induction formula (3.17) (or direct computation via Cech cohomology !!) we obtain

• If $n_1 \geqslant n_2$, then

$$\mathcal{Q}_K(n_1,n_2) = \sum_{k=n_1-n_2}^{n_1} \pi_{(\frac{1}{2},-k-\frac{1}{2})}.$$

• If $n_2 > n_1$, then

$$\mathcal{Q}_K(n_1,n_2) = \sum_{k=0}^{n_1} \pi_{(\frac{1}{2},-k-\frac{1}{2})} - \sum_{k=0}^{n_2-n_1-2} \pi_{(k+\frac{3}{2},\frac{1}{2})}.$$

Let us checked how our theorem works in these cases. First, we notice that we are in a multiplicity free case : all the non-empty reduced spaces are points.

• Consider the case where $n_1 \geqslant n_2$. We see that the parameter $(\frac{1}{2}, -k-\frac{1}{2})$ belongs to the relative interior of the interval $\Delta(n_1,n_2)$. In particular for $b = (0,0)$, the unique point in the relative interior of the interval $\Delta(0,0)$ is $\rho$. This is in agreement to the fact that the representation $\mathcal{Q}_K(0,0)$ is the trivial representation of $K$.

• Consider the case where $n_2 > n_1$. We see that the parameter $(\frac{1}{2}, -k-\frac{1}{2})$ belongs to the relative interior of $[-n_1-\frac{3}{2}, 0]e_2^* + \frac{1}{2}(e_1^*+e_2^*)$ if and only if $k \leqslant n_1$. Similarly, the parameter $(k+\frac{3}{2}, \frac{1}{2})$ belongs to the relative interior of $[0, n_2-n_1-\frac{1}{2}]e_1^* + \frac{1}{2}(e_1^*+e_2^*)$ if and only if $k \leqslant n_2-n_1-2$.

In Figures 6.2, 6.2, 13, we draw the Kirwan subsets of $\mathfrak{t}^*_{\geqslant 0}$ corresponding to the values $a = [8,5], c = [3,6]$. The circle points on the red line represents the admissible points occurring with multiplicity 1 in $\mathcal{Q}_K(n_1,n_2)$. The

diamond points on the blue line represents the admissible points occurring with multiplicity $-1$ in $\mathcal{Q}_K(n_1, n_2)$.

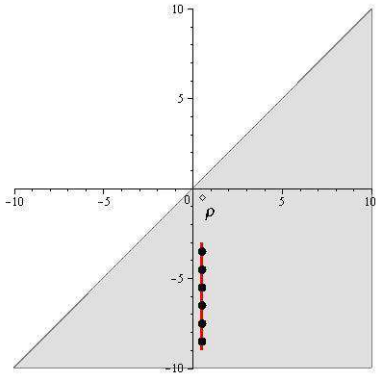


Figure 12: $K$-Multiplicities for $\mathcal{Q}_K(8, 5)$

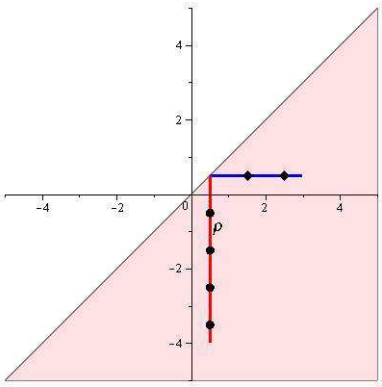


Figure 13: $K$-Multiplicities for $\mathcal{Q}_K(3, 6)$

Consider now $M$ as a $T$-manifold. Let $\Phi_{\mathbb{L}} : M \to \mathfrak{t}^*$ be the moment map relative to the action of $T$ which is the composite of $\Phi_{\mathbb{L}} : M \to \mathfrak{k}^*$ with the projection $\mathfrak{k}^* \to \mathfrak{t}^*$. Thus, the image is the convex hull of $\Delta(n_1, n_2)$ and‘its symmetric image with respect to the diagonal.

Consider first the case where $n_1 = n_2 = 0$. Thus our determinant bundle $\mathbb{L}_{0,0} = L(3, 2)$ is ample. The image of the moment map $\Phi^T_{0,0} : M \to \mathfrak{t}^*$ is equal to the convex polytope $\Delta$ with vertices $(0, \frac{1}{2}), (\frac{1}{2}, 0), (\frac{1}{2}, -1), (-1, \frac{1}{2})$,

the images of the 4 fixed points $[1,0,1,0]$, $[1,0,0,1]$,$[0,1,1,0]$,$[0,1,0,1]$. The only integral point in the interior of the polytope is $(0,0)$ and the reduced space $(\Phi^T_{0,0})^{-1}((0,0))/T$ is a point. The representation $\mathrm{Q}^{\mathrm{spin}}_T(M,\mathcal{S}(0,0))$ is indeed the trivial representation of $T$.

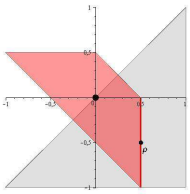

Figure 14: $T$-multiplicities for $\mathcal{Q}_T(0,0)$

We now concentrate on the case $(n_1,n_2)=(3,6)$. The line bundle $\mathbb{L}:=\mathbb{L}_{3,6}$ is not an ample bundle, so that its curvature $\Omega_{\mathbb{L}}$ is degenerate, and the Liouville form $\beta_{\mathbb{L}}=\Omega_{\mathbb{L}}\wedge\Omega_{\mathbb{L}}$ is a signed measure on $M$. Let us draw the Duistermaat measure $(\Phi_{\mathbb{L}})_*\beta_{\mathbb{L}}$, a signed measure on $\mathfrak{t}^*$. In red the measure is with value 1, in blue the measure is with value $-1$.

We also verify that our theorem is true. Indeed the representation $\mathcal{Q}_T(M,\mathcal{S}(3,6))=\mathcal{Q}_K(M,\mathcal{S}(3,6))|_T$ is

$$1+t_1^{-1}+t_2^{-1}+t_1^{-2}+t_1^{-1}t_2^{-1}+t_2^{-2}+t_1^{-3}+t_1^{-2}t_2^{-1}+t_1^{-1}t_1^{-2}+t_2^{-3}-t_1t_2-t_1t_2^2-t_1^2t_2.$$

The $\lambda\in\mathbb{Z}^2$ such that $t^\lambda$ occurs in $\mathcal{Q}_T(M,\mathcal{S}(3,6))$ are the integral points in the interior of the image of $\Phi_{\mathbb{L}}(M)$ : they have multiplicity $\pm1$, and the reduced space are points.

### 6.3 A $SU(3)$ manifold

Consider $\mathbb{C}^4$ with its canonical basis $\{e_1,\dots,e_4\}$. Let $K\simeq SU(3)$ be the subgroup of $SU(4)$ that fixes $e_4$.

Let $T=S(U(1)\times U(1)\times U(1))$ be the maximal torus of $K$ with Lie algebra $\mathfrak{t}=\{(x_1,x_2,x_3),\sum_i x_i=0\}$, and Weyl chamber $\mathfrak{t}^*_{\geqslant 0}:=\{\xi_1\geqslant\xi_2\geqslant\xi_3,\sum_i\xi_i=0\}$. We choose the fundamental roots $\omega_1,\omega_2$ so that $K_{\omega_1}=S(U(2)\times U(1))$ and $K_{\omega_2}=S(U(1)\times U(2))$. Recall that $\omega_1,\omega_2$ generates the weight lattice $\Lambda\subset\mathfrak{t}^*$ so that $\Lambda_{\geqslant 0}=\mathbb{N}\omega_1+\mathbb{N}\omega_2$. Note also that $\rho=\omega_1+\omega_2$. For any $\lambda\in\Lambda_{\geqslant 0}+\rho$, we denote $\pi_\lambda$ the irreducible representation of $K$ with highest weight $\lambda-\rho$.

Let $X=\{0\subset L_1\subset L_2\subset\mathbb{C}^4,\ \dim L_i=i\}$ be the homogeneous partial flag manifold under the action of $SU(4)$. We have two lines bundles over $X$: $\mathcal{L}_1(x)=L_1$ and $\mathcal{L}_2(x)=L_2/L_1$ for $x=(L_1,L_2)$.

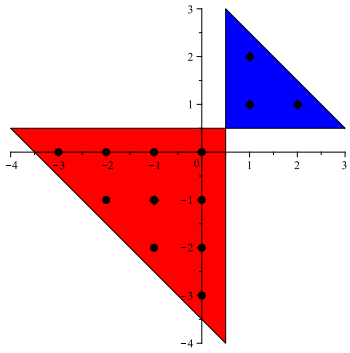

Figure 15: $T$-multiplicities for non ample bundle on Hirzebruch surface

Our object of study is the complex submanifold

$$M = \{(L_1, L_2) \in X \mid \mathbb{C}e_4 \subset L_2\}.$$

The group $K$ acts on $M$, and the generic stabilizer of the action is $[K_{\omega_1}, K_{\omega_1}] \simeq SU(2)$. We consider the family of lines bundles

$$\mathcal{L}(a,b) = \mathcal{L}_1^{\otimes a}|_M \otimes \mathcal{L}_2^{\otimes -b}|_M, \quad (a,b) \in \mathbb{N}^2.$$

Let $\mathcal{S}_M := \bigwedge_{\mathbb{C}} \mathrm{T}M$ be the Spin$^c$-bundle associated to the complex structure on $M$. We compute the characters

$$\mathcal{Q}_K(a,b) := \mathcal{Q}_K(M, \mathcal{S}_M \otimes \mathcal{L}(a,b)) \in R(K).$$

Again

$$\mathcal{Q}_K(a,b) = \sum_{i=0}^{\dim M} (-1)^i H^i(M, \mathcal{O}(\mathcal{L}(a,b))).$$

We notice that $K_{\omega_1}$ corresponds to the subgroup of $K$ that fixes the line $\mathbb{C}e_3$. The set $Y := \{(L_1, L_2) \in X \,|\, L_2 = \mathbb{C}e_3 \oplus \mathbb{C}e_4\}$ is a $K_{\omega_1}$-invariant complex submanifold of $M$ such that the map $(k, y) \in K \times Y \mapsto ky \in M$ factorizes through an isomorphism $K \times_{K_{\omega_1}} Y \simeq M$. Notice that $[K_{\omega_1}, K_{\omega_1}]$

acts trivially on $Y$. Thus we are in the "ideal"situation studied in Section 5.4.

If we take $a \geqslant 4$ and $b \geqslant 1$ we get that

$$\mathcal{Q}_K(a,b) = -\sum_{k=0}^{b-1} \pi_{k\omega_1+\rho} - \sum_{j=0}^{a-4} \pi_{j\omega_2+\rho}. \tag{6.43}$$

In particular the multiplicity of $\pi_\rho$ (the trivial representation) in $\mathcal{Q}_K(a,b)$ is equal to $-2$.

We now verify the formula (5.38) in our case. The Spin$^c$-bundle $\mathcal{S}_M$ is equal to $\mathcal{S}_{K\omega_1} \otimes K \times_{K_{\omega_1}} \mathcal{S}_Y$. The corresponding determinant line bundle $\det(\mathcal{S}_M)$ satisfies

$$\begin{aligned} \det(\mathcal{S}_M) &= K \times_{K_{\omega_1}} \mathbb{C}_{3\omega_1} \otimes K \times_{K_{\omega_1}} \det(\mathcal{S}_Y) \\ &= K \times_{K_{\omega_1}} \mathbb{C}_{2\omega_1} \otimes \mathcal{L}_1^{\otimes -2}. \end{aligned}$$

Hence for the Spin$^c$-bundle $\mathcal{S}_M \otimes \mathcal{L}(a,b)$ we have

$$\begin{aligned} \det(\mathcal{S}_M \otimes \mathcal{L}(a,b)) &= \det(\mathcal{S}_M) \otimes \mathcal{L}(a,b)^{\otimes 2} \\ &= K \times_{K_{\omega_1}} \mathbb{C}_{(2b+2)\omega_1} \otimes \mathcal{L}_1^{\otimes 2(a+b-1)}. \end{aligned}$$

The line bundle $\det(\mathcal{S}_M \otimes \mathcal{L}(a,b))$ is equipped with a natural holomorphic and hermitian connection $\nabla$. To compute the corresponding moment map $\Phi_{a,b} : M \to \mathfrak{k}^*$, we notice that $\mathcal{L}_1 = K \times_{K_{\omega_1}} \mathcal{L}^{-1}$ where $\mathcal{L} \to \mathbb{P}^1$ is the prequantum line bundle over $\mathbb{P}_1$ (equipped with the Fubini-Study symplectic form). If we denote $\varphi : Y \simeq \mathbb{P}^1 \to [0,1]$ the function defined by $\varphi([z_1,z_2]) = \frac{|z_1|^2}{|z_1|^2+|z_2|^2}$, we see that

$$\Phi_{a,b}([k,y]) = k\left[((b+1) - (a+b-1)\varphi(y))\,\omega_1\right].$$

for $[k,y] \in M$. In this case, the Kirwan set $\Phi_{a,b}(M) \cap \mathfrak{t}^*_{\geqslant 0}$ is the non convex set $[0,b+1]\omega_1 \cup [0,a-2]\omega_2$.

We know (see Exemple 3.10) that the set $\mathcal{A}((\mathfrak{k}_{\omega_1}))$ is equal to the collection of orbits $K(\frac{1+2n}{2}\omega_i), n \in \mathbb{N}, i = 1,2$, and we have $\mathcal{Q}_K(K(\frac{1}{2}\omega_i)) = 0$ and $\mathcal{Q}_K(K(\frac{3+2k}{2}\omega_i)) = \pi_{k\omega_i+\rho}$ when $k \geqslant 0$.

If we apply (5.38), we see that $\pi_{k\omega_1+\rho}$ occurs in $\mathcal{Q}_K(a,b)$ only if $\frac{3+2k}{2} < b+1$ : so $k \in \{0,\dots,b-1\}$. Similarly $\pi_{j\omega_2+\rho}$ occurs in $\mathcal{Q}_K(a,b)$ only if $\frac{3+2j}{2} < a-2$ : so $j \in \{0,\dots,a-4\}$. For all this cases the corresponding reduced spaces are points and one could check that the corresponding quantizations are all equal to $-1$ (see (5.35)).

Finally we have checked that (5.38) allows us to recover (6.43).

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