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# Approximation result for a Mumford-Shah functional adapted to a segmentation problem

David Vicente

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## Abstract

We consider a segmentation problem of image close to a binary one. After introducing a Mumford Shah energy like, we perform an approximation of this model. We prove a  $\Gamma$ -convergence result and show why this approximation is suitable for numerics.

## 1 Introduction

This paper is motivated by a 3D-image segmentation process with thin tubular structures. We have introduced a model which is derived from Mumford and Shah's one in [7]. Mathematical analysis of such model is difficult. So, in order to make numerical experiments, we perform an approximation in the sense of  $\Gamma$ -convergence. More precisely, an image is modeled as an application  $g$  defined on  $\Omega$  with values on  $[0, 1]$  where  $\Omega$  is an open, bounded and regular subset of  $\mathbb{R}^n$ . We assume that  $g$  is close to a binary function. More precisely, its histogram has two distincts modes that we assume to be 0 and 1. We prove that the model

$$(\mathcal{P}): \quad \text{Min} \{E(p): p \in SBV(\Omega), p(x) \in \{0; 1\} \text{ a.e. } x \in \Omega\}, \quad (1.1)$$

where  $E(p) = \frac{1}{2} \int_{\Omega} (p - g)^2 dx + \beta \mathcal{H}^{n-1}(S_p)$ .

where  $S_p$  is the jump set of the binary function  $p$ , may be approximated by

$$(\mathcal{P}_{\varepsilon}): \quad \text{Min} \{E_{\varepsilon}(p): p \in W^{1,2}(\Omega), 0 \leq p(x) \leq 1 \text{ a.e. } x \in \Omega\}$$

where  $E_{\varepsilon}(p) = \frac{1}{2} \int_{\Omega} (p - g)^2 dx + \beta \int_{\Omega} \left( 9\varepsilon |\nabla p|^2 + \frac{p^2(1-p)^2}{\varepsilon} \right) dx$

(1.2)

in the sense of the  $\Gamma$ -convergence. In the application which motivated this paper, we set  $n = 3$  but we prove this result for any  $n \geq 1$ .

In a companion paper [4] we perform a tuning for the parameters of this approximated model in order to detect thin structures of tubes in 3D images and we give there a complete bibliography of the subject. In the present paper we introduce the theoretical framework (section 2) then we prove the  $\Gamma$ -convergence of the model (section 3) and give additional properties related to numerical aspects (section 4).

## 2 Functional framework and $\Gamma$ -convergence

### 2.1 Functional spaces

We work in this paper with spaces of functions with bounded variation that we recall thereafter, for more details see [5], [1].

**Definition 2.1.** *Let  $\Omega$  be an open subset  $\mathbb{R}^n$ , and  $u \in L^1(\Omega)$ . We define  $BV(\Omega)$ , the space of functions with bounded variations:*

$$BV(\Omega) = \{u \in L^1(\Omega) : TV(u) < +\infty\},$$

where

$$TV(u) = \sup \left\{ \int_{\Omega} u(x) \operatorname{div}(\xi)(x) dx : \xi \in C_c^1(\Omega), \|\xi\|_{\infty} \leq 1 \right\}.$$

The space  $BV(\Omega)$ , with the norm  $\|u\|_{BV(\Omega)} = \|u\|_{L^1} + TV(u)$ , is a Banach space. The derivative, in a distributional sense, of an element  $u \in BV(\Omega)$  is a Radon measure, denoted  $Du$ , and  $TV(u) = \int_{\Omega} |Du|$  is the total variation of  $u$ . We introduce the following notation for a measure  $\mu$  and a set  $B$ ,  $\mu \llcorner B$  is the measure such that:

$$\mu \llcorner B(A) = \mu(A \cap B).$$

We recall useful properties on the structure of the gradient of  $BV$  functions in next theorem.

**Theorem 2.1.** *Let  $u \in BV(\Omega)$  then we have the following decomposition:*

$$Du = \nabla u \cdot \mathcal{L}_n + (u^+ - u^-) \nu_u \mathcal{H}^{n-1} \llcorner S_u + C_u,$$

where

- $\nabla u = \frac{Du}{\mathcal{L}_n}$ ,
- $S_u$  is a countable union of hypersurfaces,
- $\nu_u : S_u \rightarrow \mathbf{S}^{n-1}$  is a measurable normal unitary vector of  $S_u$ ,
- $u^-, u^+$  are the lower and upper approximated limits of  $u$ ,
- $C_u \perp \mathcal{H}^{n-1}$ .

**Definition 2.2.** *For any  $u \in BV(\Omega)$ , we set:*

- $\nabla u \cdot \mathcal{L}_n$  as the regular part of  $Du$ ,
- $S_u$  as the jump set of  $u$ ,
- $(u^+ - u^-) \nu_u \mathcal{H}^{n-1} \llcorner S_u$  as the jump part of  $Du$ ,

- $C_u$  as the Cantor part of  $Du$ .

**Definition 2.3.** The set of BV-functions whose derivative have a Cantor part equal to zero is called the special set of functions with bounded variation, denoted  $SBV(\Omega)$ .

**Theorem 2.2.** Let  $u = \mathbf{1}_A$  an indicator function such that  $u \in SBV(\Omega)$  then we have the following decomposition:

$$Du = \nu_u \mathcal{H}^{n-1} \llcorner S_u.$$

The space  $SBV$  has a useful slicing property.

**Definition 2.4.** Let  $\nu \in \mathbf{S}^{n-1}$  (unit sphere of  $\mathbb{R}^n$ ) and  $\Omega$  be an open subset of  $\mathbb{R}^n$ . Let  $\Pi_\nu$  be the orthogonal plane to  $\nu$ :

$$\Pi_\nu = \{x \in \mathbb{R}^n : \langle \nu, x \rangle = 0\},$$

where  $\langle \cdot, \cdot \rangle$  stands for the scalar  $\mathbb{R}^n$ -product. For any  $x \in \Pi_\nu$  we define

$$\Omega_x = \{t \in \mathbb{R} : x + t\nu \in \Omega\}$$

and

$$\Omega_\nu = \{x \in \Pi_\nu : \Omega_x \neq \emptyset\},$$

see figure 2.1.

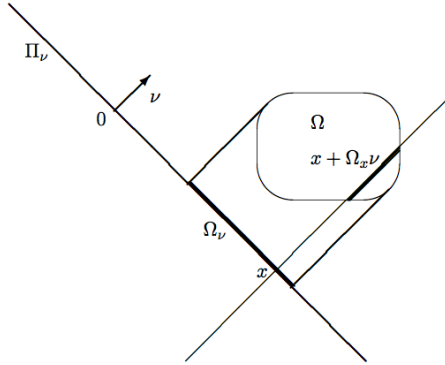


Figure 2.1: Slicing the open  $\Omega$

As in [2], we introduce a projection on the open set  $\Omega_x$  (see figure 2.1). For any function  $u$  defined on  $\Omega \subset \mathbb{R}^n$  and any  $x \in \Omega_\nu$ , we set

$$\begin{aligned} u_x: \Omega_x &\rightarrow \mathbb{R} \\ t &\rightarrow u(x + t\nu) \end{aligned}$$

**Theorem 2.3.** Let  $u \in L^\infty(\Omega)$  be a function such that, for all  $\nu \in \mathbf{S}^{n-1}$ ,

i)  $u_x \in SBV(\Omega_x)$  for  $\mathcal{H}^{n-1}$  a.e.  $x \in \Omega_\nu$ ,

ii)  $\int_{\Omega_\nu} \left\{ \int_{\Omega_x} |\nabla u_x| dt + \mathcal{H}^0(S_{u_x}) \right\} d\mathcal{H}^{n-1}(x) < +\infty$  where  $S_v$  is the jump set of the function  $v$  (see 2.2);

then,  $u \in SBV(\Omega)$  and  $\mathcal{H}^{n-1}(S_u) < +\infty$ .

Conversely, let  $u \in SBV(\Omega) \cap L^\infty$  be such that  $\mathcal{H}^{n-1}(S_u) < +\infty$ . Then i) and ii) are satisfied. Moreover, we have

iii)  $\langle \nabla u(x + t\nu), \nu \rangle = \nabla u_x(t)$ , for a.e.  $t \in \Omega_x$  and  $\mathcal{H}^{n-1}$ -a.e.  $x \in \Omega_\nu$ ,

iv) there exists a measurable function  $\nu_u : S_u \rightarrow \mathbf{S}^{n-1}$  depending on  $S_u$ , such that

$$\int_{S_u} \langle \nu_u, \nu \rangle d\mathcal{H}^{n-1}(x) = \int_{\Omega_\nu} \mathcal{H}^0(S_{u_x}) d\mathcal{H}^{n-1}(x).$$

## 2.2 $\Gamma$ -convergence theory

Usually, a standard compactness argument is sufficient to prove the existence of minimizing solutions. In our case, we use a specific notion of convergence which will be specified here, since compactness is not ensured for usual topologies. As before, for more details see [3].

**Definition 2.5.** Let  $(X, d)$  a metric space,  $(F_n)_{n \in \mathbb{N}}$  a sequence of functions defined on  $X$  with values in  $\mathbb{R} \cup \{+\infty\}$  and  $F : X \rightarrow \mathbb{R} \cup \{+\infty\}$ . The sequence  $(F_n)_{n \in \mathbb{N}}$   $\Gamma$ -converges to  $F$  on  $x \in X$  if the following assertions are true:

i) for all  $(x_n)_{n \in \mathbb{N}}$  which converges to  $x$ :

$$F(x) \leq \liminf_{n \rightarrow \infty} F_n(x_n),$$

ii) there exists  $(y_n)_{n \in \mathbb{N}}$  which converges to  $x$  such that:

$$F(x) \geq \limsup_{n \rightarrow \infty} F_n(y_n),$$

This concept is the convergence of the approximate minimum. More precisely, if for any  $n$ ,  $p_n$  is a minimum of  $F_n$  and  $(p_n)_n$  converges to  $p$ , then according to this concept,  $p$  is a minimum of  $F$ .

We shall use the following result.

**Proposition 2.1.** Let  $X$  a topological space,  $(f_n)_{n \in \mathbb{N}}$  a sequence of functions defined on  $X$  with values in  $\mathbb{R} \cup \{+\infty\}$  which  $\Gamma$ -converges to  $f$  and  $h$  a continuous function defined on  $X$  with values in  $\mathbb{R} \cup \{+\infty\}$ . Then the sequence  $(f_n + h)_{n \in \mathbb{N}}$   $\Gamma$ -converges to  $f + h$ .

### 3 A $\Gamma$ -convergence result

We now prove that the family of functionals  $(E_\varepsilon)_{\varepsilon>0}$  defined by (1.2)  $\Gamma$ -converges to  $E$  defined by (1.1). For more simplicity in the proof we assume that  $\beta = 1$  in this section, its value doesn't change the arguments of the proof. We denote  $W_b^{1,2}(\Omega)$  the following space

$$W_b^{1,2}(\Omega) = \left\{ p \in W^{1,2}(\Omega) : 0 \leq p(x) \leq 1 \text{ a.e. } x \in \Omega \right\}.$$

Let  $\mathbb{B}(\Omega; [0, 1])$  be the space of measurable functions defined on  $\Omega$  which take their values in  $[0, 1]$ . This space is endowed with the almost everywhere convergence topology. It's a Fréchet space (see [2]).

Let  $F$  be the penalization term of the functional  $E$ :

$$F: \mathbb{B}(\Omega; [0, 1]) \rightarrow \mathbb{R} \cup \{+\infty\} \quad (3.1)$$

$$p \mapsto \begin{cases} \mathcal{H}^{n-1}(S_p) & \text{if } p \in SBV(\Omega), \\ +\infty & \text{otherwise} \end{cases} \quad (3.2)$$

and  $F_\varepsilon$  be its approximation:

$$F_\varepsilon: \mathbb{B}(\Omega; [0, 1]) \rightarrow \mathbb{R} \cup \{+\infty\}$$

$$p \mapsto \begin{cases} \int_\Omega \left( 9\varepsilon |\nabla p|^2 + \frac{p^2(1-p)^2}{\varepsilon} \right) dx, & \text{if } p \in W_b^{1,2}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Let  $H$  be the fitting data term:

$$H: \mathbb{B}(\Omega; [0, 1]) \rightarrow \mathbb{R} \cup \{+\infty\}$$

$$p \rightarrow \frac{1}{2} \int_\Omega (p - g)^2 dx.$$

Let  $(p_k)_{k \in \mathbb{N}} \subset \mathbb{B}(\Omega; [0, 1])$  be a a.e. converging sequence to some  $p$ . According Lebesgue theorem and the fact that  $\|p_k\|_\infty \leq 1$ , the sequence  $(H(p_k))_{k \in \mathbb{N}}$  converges to  $H(p)$  and  $H$  is continuous for the almost everywhere convergence topology. So, it is sufficient to prove the  $\Gamma$ -convergence of  $F_\varepsilon$  to  $F$ . Indeed, proposition 2.1 with  $X = \mathbb{B}(\Omega; [0, 1])$ , implies the  $\Gamma$ -convergence of  $E_\varepsilon = F_\varepsilon + H$  to  $E = F + H$ .

We introduce the following notation:

$$F_\varepsilon(p, U) = \int_U \left( 9\varepsilon |\nabla p|^2 + \frac{p^2(1-p)^2}{\varepsilon} \right) dx, \quad F(p, U) = \mathcal{H}^{n-1}(S_p \cap U)$$

and

$$F_-(p, U) = \inf \left\{ \liminf_{k \rightarrow \infty} F_{\varepsilon_k}(p_k, U) \mid p_k \xrightarrow{a.e.} p, (p_k)_{k \in \mathbb{N}} \subset W_b^{1,2}(U) \right\}. \quad (3.3)$$

The functional  $F_-$  is also called the *lower- $\Gamma$  limit* of  $F$ .

We shall prove the following theorem.

**Theorem 3.1.** *Let  $(\varepsilon_k)_{k \in \mathbb{N}}$  be a sequence which converges to  $0^+$  and  $p \in SBV(\Omega)$  be a function taking its values in  $\{0, 1\}$ . Then*

$$i) \quad F(p, \Omega) \leq F_-(p, \Omega) \quad (3.4)$$

*holds;*

*ii) there exists a sequence  $(p_k)_{k \in \mathbb{N}} \subset W_b^{1,2}(\Omega)$  which converges a.e. to  $p$  such that:*

$$F(p, \Omega) \geq \limsup_{k \rightarrow \infty} F_{\varepsilon_k}(p_k, \Omega). \quad (3.5)$$

According to this theorem, we deduce that  $F(p, \Omega) = F_-(p, \Omega)$  for  $p \in SBV(\Omega)$  taking its values in  $\{0, 1\}$ . The proof is detailed thereafter.

### 3.1 The inequality for the *lower* $\Gamma$ -limit (3.4)

We now prove the first inequality (3.4). We shall need the following properties of this functional:

a)  $F_-(p, \cdot)$  is superadditive on open sets, that is

$$A \cap B = \emptyset \quad \Rightarrow \quad F_-(p, A \cup B) \geq F_-(p, A) + F_-(p, B),$$

b)  $F_-(p, \cdot)$  is non decreasing:

$$A \subset B \quad \Rightarrow \quad F_-(p, A) \leq F_-(p, B).$$

By a diagonal extraction, there exists a sequence  $(p_k)_{k \in \mathbb{N}} \subset W_b^{1,2}(\Omega)$  such that:

$$\begin{cases} p_k \xrightarrow{a.e.} p, \\ F_{\varepsilon_k}(p_k, \Omega) \rightarrow F_-(p, \Omega) \end{cases} \quad (3.6)$$

We assume that  $F_-(p, \Omega) < +\infty$  otherwise (3.4) is ensured. We have to prove:

$$F(p, \Omega) \leq \liminf_{k \rightarrow \infty} F_{\varepsilon_k}(p_k, \Omega)$$

As in [2], we perform the proof in two steps: the first step deals with dimension 1. The second generalizes it for dimension  $n \geq 2$ .

#### 3.1.1 The one-dimensional case

Assume  $\Omega = I$  is an open interval. To avoid confusion, when  $n = 1$ , we denote the approximating functionals by  $F_\varepsilon^1$  et their lower  $\Gamma$ -limit, defined in (3.3), by  $F_-^1$ .

**Lemma 3.1.** *Let  $B_\eta(x) \subset \mathbb{R}$  be an open ball and  $q \in \mathbb{B}(B_\eta(x))$  be a function taking its values in  $\{0; 1\}$ . Assuming*

$$\forall \rho \in ]0, \eta[, \quad q \notin W^{1,2}(B_\rho(x)),$$

*then we have:*

$$\forall \rho \in ]0, \eta[, \quad F_-^1(q, B_\rho(x)) \geq 1.$$

*In particular, if  $I \subset \mathbb{R}$  is an open and bounded interval such that  $q \in \mathbb{B}(I)$  takes its values in  $\{0; 1\}$  and  $F_-^1(q, I) < +\infty$  then  $q \in SBV(I)$ . Moreover, there exists  $J \subset I$  with cardinal less than  $F_-^1(q, I)$  such that*

*i)  $p$  is constant on each connected component of  $I \setminus J$ ,*

*ii)  $\mathcal{H}^0(S_q \cap I) \leq F_-^1(q, I)$ .*

*Proof.* We can assume that  $F_-^1(q, B_\rho(x)) < +\infty$  for any  $\rho \in ]0; \eta[$ . As before, there exists a sequence  $(q_k)_{k \in \mathbb{N}} \subset W_b^{1,2}(B_\rho(x))$  such that:

$$\begin{cases} q_k \xrightarrow{\text{a.e.}} q, \\ F_{\varepsilon_k}^1(q_k, B_\rho(x)) \rightarrow F_-^1(q, B_\rho(x)) \end{cases} \quad (3.7)$$

Since  $q$  takes its values a.e. in  $\{0, 1\}$  and  $q \notin W^{1,2}(B_{\rho'}(x))$  for any  $\rho' \in ]0, \rho[$ , there exists two sequences  $(y_k^1)_{k \in \mathbb{N}}$  and  $(y_k^2)_{k \in \mathbb{N}}$  such that:

$$y_k^1 \rightarrow x, \quad q_k(y_k^1) \rightarrow 1 \quad \text{and} \quad y_k^2 \rightarrow x, \quad q_k(y_k^2) \rightarrow 0.$$

However, we have:

$$F_{\varepsilon_k}^1(q_k, B_\rho(x)) \geq \int_{x-\rho}^{x+\rho} \left( 9\varepsilon_k |\nabla q_k(y)|^2 + \frac{q_k^2(1 - q_k(y))^2}{\varepsilon_k} \right) dy.$$

We introduce:

$$A := 3\sqrt{\varepsilon_k} |\nabla q_k(y)|, \quad B := \frac{q_k(1 - q_k(y))}{\sqrt{\varepsilon_k}}.$$

With the inequality  $A^2 + B^2 \geq 2AB$ , we get:

$$F_{\varepsilon_k}^1(q_k, B_\rho(x)) \geq 6 \int_{x-\rho}^{x+\rho} |\nabla q_k(y)| q_k(y) (1 - q_k(y)) dy.$$

As  $[y_k^1, y_k^2] \subset B_\rho(x)$ , we obtain:

$$F_{\varepsilon_k}^1(q_k, B_\rho(x)) \geq 6 \int_{y_k^1}^{y_k^2} |\nabla q_k(y)| q_k(y) (1 - q_k(y)) dy.$$



Since  $q_k \in W^{1,2}(I)$ , we may use the change of variable  $t = q_k(s)$ . This yields:

$$\begin{aligned} F_{\varepsilon_k}^1(q_k, B_\rho(x)) &\geq 6 \int_{q_k(y_k^1)}^{q_k(y_k^2)} t(1-t) dt, \\ F_{\varepsilon_k}^1(q_k, B_\rho(x)) &\geq 6 \left| \frac{q_k^2(y_k^1) - q_k^2(y_k^2)}{2} - \frac{q_k^3(y_k^1) - q_k^3(y_k^2)}{3} \right| \end{aligned}$$

We know that  $q_k(y_k^1) \rightarrow 1$  and  $q_k(y_k^2) \rightarrow 0$ , so that we deduce:

$$\frac{q_k^2(y_k^1) - q_k^2(y_k^2)}{2} - \frac{q_k^3(y_k^1) - q_k^3(y_k^2)}{3} \rightarrow \frac{1}{6}.$$

It comes,

$$\liminf F_{\varepsilon_k}^1(q_k, B_\rho(x)) \geq 1.$$

and we can conclude

$$F_-^1(q, B_\rho(x)) \geq 1. \quad (3.8)$$

We set

$$J = \left\{ x \in I : \forall \rho > 0, q \notin W^{1,2}(B_\rho(x)) \right\}.$$

The functional  $F_-^1(q, \cdot)$  is super-additive and non-decreasing (see (3.1)). With inequality (3.8) we get

$$F_-^1(q, I) \geq \mathcal{H}^0(J).$$

According to the fact that  $q$  takes its values a.e. in  $\{0; 1\}$ , then  $q$  is constant on the connected components of  $I \setminus J$  and  $S_q \cap I = J$ . Therefore, we may conclude

$$F_-^1(q, I) \geq \mathcal{H}^0(S_q \cap I).$$

□

### 3.1.2 Generalization to dimensions $n \geq 2$

We now extend the proof to any higher dimension  $n$  by a slicing argument.

Recall that we want to prove  $F_-(p, \Omega) \geq F(p, \Omega)$ .

Let  $(p_k)_{k \in \mathbb{N}} \subset W_b^{1,2}(\Omega)$  as in (3.6). Let  $A$  be an arbitrary open subset of  $\Omega$ . Assume that  $F_-(p, A) < +\infty$  and  $F_{\varepsilon_k}(p_k, A) \rightarrow F_-(p, A)$ . Let  $\nu \in \mathbf{S}^{n-1}$  be an unit vector, we have:

$$\begin{aligned} F_{\varepsilon_k}(p_k, A) &= \int_A \left( 9\varepsilon_k |\nabla p_k(x)|^2 + \frac{p_k^2(x)(1-p_k(x))^2}{\varepsilon_k} \right) dx \\ &\geq \int_A \left( 9\varepsilon_k |\langle \nabla p_k(x), \nu \rangle|^2 + \frac{p_k^2(x)(1-p_k(x))^2}{\varepsilon_k} \right) dx \end{aligned}$$

The Lebesgue measure of  $\mathbb{R}^n$  projected on  $\Pi_\nu$  is the Hausdorff measure with dimension  $n - 1$  (see for example [5]). Applying Fubini theorem, we get:

$$\begin{aligned} F_{\varepsilon_k}(p_k, A) &\geq \int_{A_\nu} \int_{A_x} \left( 9\varepsilon_k |\langle \nabla p_k(x + t\nu), \nu \rangle|^2 + \frac{p_k(x + t\nu)^2 (1 - p_k(x + t\nu))^2}{\varepsilon_k} \right) dt \, d\mathcal{H}^{n-1}(x) \\ &\geq \int_{A_\nu} F_{\varepsilon_k}^1((p_k)_x, A_x) \, d\mathcal{H}^{n-1}(x). \end{aligned}$$

Fatou lemma yields:

$$F_-(p, A) \geq \int_{A_\nu} \liminf F_{\varepsilon_k}^1((p_k)_x, A_x) \, d\mathcal{H}^{n-1}(x).$$

As  $(p_k)_x \in SBV(A_x)$  a.e.  $x \in A_\nu$ , with lemma 3.1, we have:

$$F_-(p, A) \geq \int_{A_\nu} \mathcal{H}^0(S_{(p)_x} \cap A_x) \, d\mathcal{H}^{n-1}(x).$$

Using theorem 2.3 gives:

$$\int_{A_\nu} \mathcal{H}^0(S_{(p)_x} \cap A_x) \, d\mathcal{H}^{n-1}(x) = \int_{S_p \cap A} |\langle \nu_p, \nu \rangle| \, d\mathcal{H}^{n-1}(x),$$

so that

$$F_-(p, A) \geq \int_{S_p \cap A} |\langle \nu_p, \nu \rangle| \, d\mathcal{H}^{n-1}(x). \quad (3.9)$$

As  $p$  belongs to  $SBV(\Omega)$ , according to theorem 2.1, the set  $S_p$  is a countable union of  $\mathcal{C}^1$ -hypersurfaces. Then, we obtain:

$$\mathcal{H}^{n-1}(S_p) = \sup_{\mathcal{A}, \mathcal{V}} \left\{ \sum_{i=1}^{\infty} \int_{S_p \cap A_i} |\langle \nu_p, \nu_i \rangle| \, d\mathcal{H}^{n-1}(x) \right\}$$

where the supremum is taken over all the families  $\mathcal{A} = (A_i)_{i=1 \dots \infty}$  of open and pairwise disjoint subsets of  $\Omega$  and  $\mathcal{V} = (\nu_i)_{i=1 \dots \infty} \subset \mathbf{S}^{n-1}$ . Applying the inequality (3.9) to each term, we have:

$$\mathcal{H}^{n-1}(S_p) \leq \sup_{\mathcal{A}} \left\{ \sum_{i=1}^{\infty} F_-(p, A_i) \right\}.$$

where the supremum is only taken over all the families  $\mathcal{A}$ .

Since  $F_-(p, \cdot)$  is super-additive and non decreasing (see (3.1)), then we may conclude that

$$\mathcal{H}^{n-1}(S_p) \leq F_-(p, \Omega).$$

□

### 3.2 The inequality for the *higher* $\Gamma$ -limit (3.5)

To prove the second part (3.5), we construct a sequence of functions  $(p_k)_{k \in \mathbb{N}}$  converging to  $p$ , so that the higher bound of the energy is lower than  $F(p, \Omega)$ . We will use the following lemma due to Modica [6].

**Lemma 3.2.** *Let  $A$  be an open and bounded subset of  $\Omega$  with a non empty Lipschitz boundary  $\partial A$ . Let*

$$h(x) = \begin{cases} +\text{dist}(x, \partial A) & \text{if } x \in A, \\ -\text{dist}(x, \partial A) & \text{otherwise} \end{cases}$$

and for all  $t > 0$ ,  $S_t = \{x \in A: h(x) = t\}$ .

Then  $h$  is a Lipschitz function, and we have:

$$|Dh(x)| = 1 \text{ a.e. } x \in \Omega \text{ and } \lim_{t \rightarrow 0} \mathcal{H}^{n-1}(S_t \cap \Omega) = \mathcal{H}^{n-1}(\partial A \cap \Omega).$$

We may now prove the second part (3.5) of theorem 3.

*Proof.* Set:

$$h(x) = \begin{cases} +\text{dist}(x, \{p = 0\}) & \text{if } p(x) = 1, \\ -\text{dist}(x, \{p = 1\}) & \text{if } p(x) = 0, \end{cases}$$

$$\chi_0(t) = \begin{cases} 1 & \text{si } t \geq 0, \\ 0 & \text{si } t < 0. \end{cases}$$

It is clear that  $p = \chi_0 \circ h$ . We construct  $p_k$  as  $p_k = \chi_k \circ h$ . In this case, we have:

$$F_{\varepsilon_k}(p_k) = \int_{\Omega} \left( 9\varepsilon_k |\nabla \chi_k(h(x))|^2 |\nabla h(x)|^2 + \frac{\chi_k^2(h(x))(1 - \chi_k(h(x)))^2}{\varepsilon_k} \right) dx.$$

The function  $p$  belongs to  $SBV(\Omega)$ . According to theorem 2.1 and lemma 3.2, we have  $|\nabla h(x)| = 1$  for almost every  $x \in \Omega$ . Then, we may apply coarea formula (see for example [5]) to obtain:

$$F_{\varepsilon_k}(p_k) = \int_{t=-\infty}^{+\infty} \varphi_{\varepsilon,k}(t) \mathcal{H}^{n-1}(\{h = t\}) dt,$$

where 
$$\varphi_{\varepsilon,k}(t) = 9\varepsilon_k |\nabla \chi_k(t)|^2 + \frac{\chi_k^2(t)(1 - \chi_k(t))^2}{\varepsilon_k}.$$

The minimization of this energy is a 1D-problem. The Euler-lagrange equation associated to this problem is:

$$(9\varepsilon_k (\chi_k')^2)' = \left( \frac{\chi_k^2(1 - \chi_k)^2}{\varepsilon_k} \right)'$$

Thus, there exists  $c_k > 0$  such that:

$$9\varepsilon_k(\chi'_k)^2 = c_k + \frac{\chi_k^2(1-\chi_k)^2}{\varepsilon_k} \quad (3.10)$$

Let  $\overline{\chi}_k$  a solution of the equation (3.10) such that:

$$\begin{cases} \overline{\chi}_k(t) = 0 & \text{if } t \leq 0, \\ 3\sqrt{\varepsilon_k}\overline{\chi}'_k = \sqrt{c_k + \frac{\overline{\chi}_k^2(1-\overline{\chi}_k)^2}{\varepsilon_k}} & \text{if } t \in [0, \eta_k], \\ \overline{\chi}_k(t) = 1 & \text{if } t \geq \eta_k. \end{cases}$$

As  $\overline{\chi}'_k \geq \sqrt{\frac{c_k}{9\varepsilon_k}}$ , we have  $\eta_k \leq \sqrt{\frac{c_k}{9\varepsilon_k}}$ . If  $\frac{c_k}{\varepsilon_k} \rightarrow 0$  then  $\overline{\chi}_k \circ h$  converges to  $p$  almost everywhere.

Set  $A = 3\sqrt{\varepsilon_k}\overline{\chi}'_k$  and  $B = \frac{\overline{\chi}_k(1-\overline{\chi}_k)}{\sqrt{\varepsilon_k}}$ . We have  $A^2 = c_k + B^2$  and

$$\begin{aligned} A^2 + B^2 &= 2AB + c_k + 2B^2 - 2AB, \\ &= 2AB + c_k + 2B(B - A), \\ &= 2AB + c_k + 2B(B - \sqrt{c_k + B^2}), \\ &= 2AB + c_k - 2c_k \frac{B}{B + \sqrt{c_k + B^2}}. \end{aligned}$$

Replacing it in  $F_{\varepsilon_k}(p_k)$  leads to:

$$\begin{aligned} F_{\varepsilon_k}(p_k) &= \int_{t=0}^{\eta_k} (A^2 + B^2) \mathcal{H}^{n-1}(\{h = t\}) \, dt \\ &= \int_{t=0}^{\eta_k} (2AB + c_k - 2c_k \frac{B}{B + \sqrt{c_k + B^2}}) \mathcal{H}^{n-1}(\{h = t\}) \, dt \\ &\leq \int_{t=0}^{\eta_k} (2AB + c_k) \mathcal{H}^{n-1}(\{h = t\}) \, dt. \end{aligned}$$

Replacing  $A$  and  $B$  by its values, we get:

$$F_{\varepsilon_k}(p_k) \leq \int_{t=0}^{\eta_k} 6\overline{\chi}'_k \overline{\chi}_k (1 - \overline{\chi}_k) \mathcal{H}^{n-1}(\{h = t\}) \, dt + \int_{t=0}^{\eta_k} c_k \mathcal{H}^{n-1}(\{h = t\}) \, dt.$$

As  $p \in SBV$ . So,  $t \rightarrow \mathcal{H}^{n-1}(\{h = t\})$  is continuous at  $t = 0$  and converges to  $\mathcal{H}^{n-1}(S_p)$  with theorem 2.1. Thus, the second term converges to 0. Applying Fatou lemma gives

$$\begin{aligned} \limsup F_{\varepsilon_k}(p_k) &\leq \mathcal{H}^{n-1}(S_p) \limsup \int_{t=0}^{\eta_k} \frac{3}{4} \overline{\chi}'_k (1 - \overline{\chi}_k^2) \, dt \\ &\leq \mathcal{H}^{n-1}(S_p) \int_0^1 6s(1-s) \, ds \\ &\leq \mathcal{H}^{n-1}(S_p). \end{aligned}$$

□

## 4 Convergence of solutions

In this section we prove an existence result for the minimizing problem of  $E_\varepsilon$ , with  $\varepsilon > 0$  fixed; we set  $p_\varepsilon$  such solution. Then, we show the convergence a.e. of an extracted subsequence  $(p_{\varepsilon_k})_{\varepsilon_k \in \mathbb{N}}$ , with  $\varepsilon_k \rightarrow 0^+$ , to a binary function  $p \in SBV(\Omega)$ . To demonstrate that, we use the  $\Gamma$ -convergence result proved in section 3.

### 4.1 Lower semi-continuity of $E_\varepsilon$

For  $\varepsilon > 0$  fixed, we can find a minimizer of  $E_\varepsilon$  on  $W_b^{1,2}(\Omega)$ . Contrary to  $E$ , the functional  $E_\varepsilon$  is lower semi-continuous for the usual topology.

**Theorem 4.1.** *Let  $\varepsilon > 0$ , the problem  $(\mathcal{P}_\varepsilon)$  (1.2) admits solutions. More precisely, if  $(p_k)_{k \in \mathbb{N}} \subset W_b^{1,2}(\Omega)$  is a minimizing sequence of  $E_\varepsilon$ , then there exists a sub-sequence  $(p_{k_l})_{l \in \mathbb{N}}$  converging a.e. to  $p \in W^{1,2}(\Omega)$  such that*

$$E_\varepsilon(p) = \lim_{l \rightarrow \infty} E_\varepsilon(p_{k_l}) = \min \left\{ E_\varepsilon(q) : q \in W_b^{1,2}(\Omega) \right\}.$$

*Proof.* Let  $(p_k)_{k \in \mathbb{N}} \subset W_b^{1,2}(\Omega)$  be a minimizing sequence of  $E_\varepsilon$ . In particular, the sequence  $(E_\varepsilon(p_k))_{k \in \mathbb{N}}$  is bounded. Since

$$9\varepsilon\beta \|\nabla p_k\|_{L^2(\Omega)}^2 \leq E_\varepsilon(p_k)$$

and

$$\|p_k\|_{L^2(\Omega)}^2 \leq \|p_k\|_{L^\infty(\Omega)} \mathcal{L}^n(\Omega) \leq \mathcal{L}^n(\Omega),$$

then this sequence is bounded in  $W^{1,2}(\Omega)$ . So, there exists a subsequence  $(p_{k_l})_{l \in \mathbb{N}}$  which weakly converges to  $p \in W^{1,2}(\Omega)$ . The functional  $q \rightarrow \int_\Omega |\nabla q|^2$  is lower semi-continuous for the weak topology of  $W^{1,2}(\Omega)$ . According to Fatou lemma, the functionals

- $q \rightarrow \int_\Omega (q - g)^2$
- $q \rightarrow \int_\Omega q^2(1 - q)^2$

are lower semi-continuous for the topology of the a.e. convergence. Then, we may conclude

$$E_\varepsilon(p) \leq \liminf_{l \rightarrow \infty} E_\varepsilon(p_{k_l}).$$

Since  $(p_k)_{k \in \mathbb{N}}$  is a minimizing sequence then

$$E_\varepsilon(p) = \lim_{l \rightarrow \infty} E_\varepsilon(p_{k_l}) = \min \left\{ E_\varepsilon(q) : q \in W_b^{1,2}(\Omega) \right\}.$$

□

## 4.2 Compactness result of $(\mathcal{P}_\varepsilon)_{\varepsilon>0}$

According to 4.1, there exists  $p_\varepsilon$  solution of  $(\mathcal{P}_\varepsilon)$ . For numerical applications, we need the convergence of the solutions  $(p_{\varepsilon_k})_{k \in \mathbb{N}}$  for any  $\varepsilon_k \rightarrow 0^+$  to a solution of  $(\mathcal{P})$ . In this section, we prove that this family is compact in the following sense.

**Theorem 4.2.** *Let  $(\varepsilon_k)_{k \in \mathbb{N}}$  be a sequence converging to  $0^+$  and let  $(p_k)_{k \in \mathbb{N}} \subset W_b^{1,2}(\Omega)$  be such that for any  $k \in \mathbb{N}$ ,  $p_k$  is a minimizer of  $E_{\varepsilon_k}$ . Then, there exists a subsequence  $(p_{k_l})_{l \in \mathbb{N}}$  which converges a.e. to  $p \in \mathbb{B}(\Omega)$ . Moreover, we have:*

- i)  $p(x) \in \{0; 1\}$  a.e.  $x \in \Omega$ ,
- ii)  $p \in SBV(\Omega)$  and  $\mathcal{H}^{n-1}(S_p) < +\infty$ ,
- iii)  $p$  is a solution of  $(\mathcal{P})$ .

This result also prove that the space  $SBV(\Omega; \{0; 1\})$  is "optimal" for the previous  $\Gamma$ -convergence result: it is the largest domain for which this result is satisfied.

*Proof.* Let  $q \in SBV(\Omega)$  an arbitrary function such that  $q$  takes its values in  $\{0; 1\}$  and  $\mathcal{H}^{n-1}(S_q) < \infty$ . According theorem 3, there exists a sequence  $(q_k)_{k \in \mathbb{N}}$  such that  $(E_{\varepsilon_k}(q_k))_{k \in \mathbb{N}}$  converges to  $E(q)$ . In particular, the sequence  $(E_{\varepsilon_k}(q_k))_{k \in \mathbb{N}}$  is bounded, we denote  $M$  its higher bound. Since  $p_k$  is a minimizer of  $E_{\varepsilon_k}$  for any  $k$  then  $(E_{\varepsilon_k}(p_k))_{k \in \mathbb{N}}$  is bounded by  $M$  too. We set  $c_k = 3p_k^2 - 2p_k^3$ . Note that  $c_k \in W^{1,2}(\Omega)$  and that the chain rule gives  $\nabla c_k = 6\nabla p_k p_k(1 - p_k)$ , so we have

$$\begin{aligned} \int_{\Omega} |\nabla c_k| &= 6 \int_{\Omega} |\nabla p_k| p_k(1 - p_k) dx, \\ &\leq \int_{\Omega} 9\varepsilon_k |\nabla p_k|^2 dx + \int_{\Omega} \frac{p_k^2(1 - p_k)^2}{\varepsilon_k} dx, \\ &\leq \frac{M}{\beta}. \end{aligned}$$

Thus  $(c_k)_{k \in \mathbb{N}}$  is bounded in  $W^{1,1}(\Omega)$ . So, there exists a subsequence  $(c_{k_l})_{l \in \mathbb{N}}$  which converges a.e. to  $c \in \mathbb{B}(\Omega)$ . According that  $x \rightarrow 3x^2 - 2x^3$  is a bicontinuous isomorphism from  $[0, 1]$  to itself, there exists  $p \in \mathbb{B}(\Omega)$  such that  $(p_{k_l})_{l \in \mathbb{N}}$  converges a.e. to  $p$ . We have

$$\forall k \in \mathbb{N}, \quad \int_{\Omega} \frac{p_k^2(1 - p_k)^2}{\varepsilon_k} dx \leq \frac{M}{\beta}.$$

As  $(\varepsilon_k)_{k \in \mathbb{N}}$  converges to  $0^+$ , Fatou lemma yields

$$\int_{\Omega} p^2(1 - p)^2 dx = 0,$$

so that  $p(x) \in \{0; 1\}$  a.e.  $x \in \Omega$ .

Let us prove **ii**. As

$$\forall k \in \mathbb{N}, \quad E_{\varepsilon_k}(p_k) \leq M.$$

then  $F_-(p, \Omega) \leq M < +\infty$ .

Let  $\nu \in \mathbb{S}^{n-1}$  be a fixed vector, and  $(r_k)_{k \in \mathbb{N}} \subset W_b^{1,2}(\Omega)$  such that

$$\begin{cases} r_k \xrightarrow{\text{a.e.}} p, \\ F_{\varepsilon_k}(r_k, \Omega) \rightarrow F_-(p, \Omega). \end{cases}$$

Since  $r_k \in W^{1,2}(\Omega)$ , then  $(r_k)_x$  (defined by [2.1](#)) belongs to  $W^{1,2}(\Omega_x)$  and we have:

$$\langle \nabla r_k(x + t\nu), \nu \rangle = \nabla(r_k)_x.$$

As

$$\int_{\Omega} \left( 9\varepsilon_k |\langle \nabla r_k, \nu \rangle|^2 + \frac{r_k^2(1-r_k)^2}{\varepsilon_k} \right) dx \leq F_{\varepsilon_k}(r_k, \Omega).$$

Fubini theorem gives

$$\int_{\Omega_\nu} \int_{\Omega_x} \left[ 9\varepsilon_k |\langle \nabla r_k(x + t\nu), \nu \rangle|^2 + \frac{r_k^2(1-r_k)^2}{\varepsilon_k} \right] dt \, d\mathcal{H}^{n-1}(x) \leq F_{\varepsilon_k}(r_k, \Omega).$$

and

$$\int_{\Omega_\nu} F_{\varepsilon_k}^1((r_k)_x, \Omega_x) d\mathcal{H}^{n-1}(x) \leq F_{\varepsilon_k}(r_k, \Omega).$$

We apply Fatou lemma to get

$$\begin{aligned} \int_{\Omega_\nu} \liminf F_{\varepsilon_k}^1((r_k)_x, \Omega_x) d\mathcal{H}^{n-1}(x) &\leq F_-(p, \Omega), \\ \int_{\Omega_\nu} F_-^1((p_x), \Omega_x) d\mathcal{H}^{n-1}(x) &\leq F_-(p, \Omega). \end{aligned}$$

We deduce that  $F_-^1((p_x), \Omega_x) < +\infty$  is finite  $\mathcal{H}^{n-1}$ -a.e.  $x \in \Omega_\nu$ . Otherwise,  $((r_k)_x)_{k \in \mathbb{N}}$  converges to  $p_x$  a.e. on  $\Omega_x$  and  $p_x$  takes its values on  $\{0; 1\}$   $\mathcal{H}^{n-1}$ -a.e.  $x \in \Omega_\nu$ . According to lemma [3.1](#), we have

$$p_x \in SBV(\Omega_x)$$

for  $\mathcal{H}^{n-1}$ -a.e.  $x \in \Omega_\nu$ . Moreover, we have proved that  $\mathcal{H}^0(S_{p_x} \cap \Omega_x) \leq F_-^1(p_x, \Omega_x)$  and  $\nabla p_x = 0$ , so we have

$$\int_{\Omega_\nu} \left\{ \int_{\Omega_x} |\nabla p_x| dt + \mathcal{H}^0(S_{p_x}) \right\} d\mathcal{H}^{n-1}(x) < +\infty.$$

Now, the conditions of theorem [2.3](#) are satisfied. We can conclude that  $p \in SBV(\Omega)$  and  $\mathcal{H}^{n-1}(S_p) < +\infty$ .

Let us prove **iii**. □

## 5 Conclusion

We have shown in section 3 that  $(E_\varepsilon)_\varepsilon$   $\Gamma$ -converges to  $E$  (theorem 3.1) in  $SBV$  and this is the best we can have (theorem 4.2). Furthermore, we have shown that this approximation is suitable for numerical experiments since the solutions of  $(\mathcal{P}_\varepsilon)$  are approximations of  $(\mathcal{P})$  (theorem 4.1).

Practically, we replace the problem

$$(\mathcal{P}): \text{Min} \{E(p): p \in SBV(\Omega), \text{ and } p \text{ takes its values in } \{0, 1\}\}$$

by the problem

$$(\mathcal{P}_\varepsilon): \text{Min} \left\{ E_\varepsilon(p): p \in W_b^{1,2}(\Omega) \right\}$$

with  $\varepsilon$  fixed and small. In the companion paper [4], we study this approximate problem with the specific constraints for thin tubes.

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