A study of the assumptions used in statistical energy analysis

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ABSTRACT

This paper is a discussion on the coupling strength, the frequency bandwidth and indirect coupling loss factor in statistical energy analysis. The example of coupled oscillators is taken. First, it is shown that the coupling power proportionality is always valid for two coupled oscillators for any strength of coupling as long as the resonant modes are within the frequency band. Then, the case of three coupled oscillators indicated that the apparition of indirect coupling loss factors depends on the coupling strength. Finally, it is shown that modes outside the frequency bandwidth of excitation can lead to invalid the reciprocity relation. Some test cases are presented to illustrate these conclusions.

Keywords: Statistical energy analysis, coupling power proportionality, indirect coupling, coupling loss factors.

1. INTRODUCTION

Statistical energy analysis (SEA) is a well-known statistical theory of vibroacoustics which evaluate energy transfers between subsystems subjected to random forces. The main result is the so-called *coupling power proportionality* (CPP) which states that the power transmitted between two subsystems is proportional to the difference of their modal energies. The proof of this relation is based on the modal approach of SEA where the modes are viewed as mechanical oscillators. The basic model is composed of two coupled oscillators conservatively coupled and submitted to random and uncorrelated external forces having the power spectral density of a white noise (1, 2). A generalization to an arbitrary number of oscillators has been achieved by Newland (3) but with the additional hypothesis that the coupling is weak. A review of the hypotheses useful in SEA is presented in reference (4).

Mace (5) considers two types of systems: the one which fully respect SEA hypotheses and where the CPP applies (*proper-SEA*) and systems which approach CPP introducing indirect coupling loss factors (*quasi-SEA*). These indirect coupling loss factors appear, for some cases, in the coupling loss factor matrix even for not physically connected subsystems.

This paper is in the continuation of what has been previously done in (4) and has two objectives. First, studying the coupling power proportionality when the hypothesis of weak coupling is violated. Then, showing what could happen when taking non-resonant modes in the frequency bandwidth of integration. For that a reference calculation which makes a link between the power injected and the total energy is done on the example of three coupled oscillators. Then it is compared to the coupling power proportionality given by SEA.

2. REFERENCE CALCULATION

In this section, the total energies of each oscillator and the power transmitted between them are calculated. These expressions are useful to be compared with SEA coupling power proportionality.

2.1 Motion of three oscillators

Let us consider three oscillators coupled by elastic couplings (Fig 1).

In the Fourier space, the equations of motion of three oscillators coupled by elastic couplings are **DX=F** where **X** is the displacement vector, **F** the force vector and **D** the dynamic stiffness matrix. The frequency response function H_i^j is the response of oscillator i when a harmonic force is applied to oscillator j. This is

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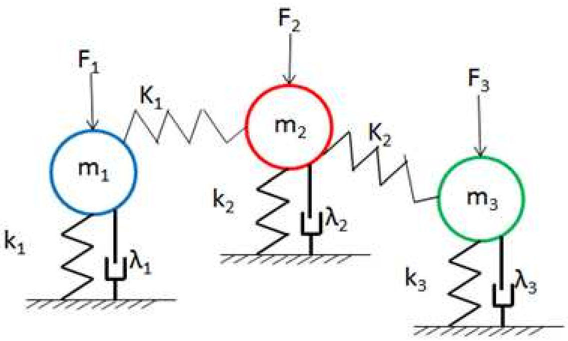

Figure 1 – Three oscillators excited by uncorrelated random forces F_i coupled by elastic couplings K_i.

the input of row i and column j of the matrix $\mathrm{H} = \mathrm{D}^{-1}$. For any harmonic force vector **F** the displacement vector is **X=HF**. For the mechanical oscillators of Fig 1, **H** is

$$H = \begin{bmatrix} m_1(\omega_1^2 - \omega^2 + j\omega\Delta_1) & -K_1 & 0 \\ -K_1 & m_2(\omega_2^2 - \omega^2 + j\omega\Delta_2) & -K_2 \\ 0 & -K_2 & m_3(\omega_3^2 - \omega^2 + j\omega\Delta_3) \end{bmatrix}^{-1} \tag{1}$$

where $\omega_i = ((k_i + K_i)/m_i)^{1/2}$ is the blocked natural angular frequency of oscillator i.

2.2 Total energies of oscillators

2.3 Vibrational energies

When the forces are random, stationary and uncorrelated and denoting S_j the power spectral density of the random process F_j, the expectation of the square of velocity of oscillator i is

$$< \dot{X}_i^2 >= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \omega^2 \sum_j |H_i^j|^2 S_j d\omega \tag{2}$$

where $< . >$ denotes a probability expectation. Let us consider that the forces have a power spectral density constant ($S_j = 1$) in the bandwidth $[\omega_{min}, \omega_{max}]$ and zero elsewhere. The vibrational energy defined as twice the kinetic energy is

$$< E_i >= \frac{1}{\pi} \int_{\omega_{min}}^{\omega_{max}} m_i \omega^2 \left(\sum_j |H_i^j|^2 S_j \right) d\omega \tag{3}$$

This expression is valid for any combination of random, stationary and uncorrelated forces of power spectral density S_j.

2.4 Transmitted power

The net power transmitted from oscillator i to oscillator j when the force is applied to oscillator k is

$$< P_{ij} >= \frac{1}{2} K_{ij} (< X_i \dot{X}_j > - < X_j \dot{X}_i >) = \frac{1}{2\pi} \int_{\omega_{min}}^{\omega_{max}} \Re[i K_{ij} \omega \left(\sum_k H_i^{k*} H_j^k S_k \right)] d\omega \tag{4}$$

where $*$ denotes the complex conjugate and $\Re$ the real part. Since oscillators 1 and 3 are not directly connected $< P_{13} >= 0$.

2.5 Injected power

The power injected into oscillator i is,

$$< P_{inj}^i >=< F_i \dot{X}_i >= \frac{1}{2\pi} \Re \int_{\omega_{min}}^{\omega_{max}} i\omega H_i^i S_i d\omega \tag{5}$$

A simpler expression of the injected power is obtained by Gersch (6). For a white noise excitation ($S_i = 1$ over an infinite frequency band), the injected power i is inversely proportional to the mass m_i.

$$< P_{inj}^i >= \frac{S_i}{2m_i} \tag{6}$$

3. SEA CALCULATION

In this section, the expression of the coupling power proportionality and notably the expression of beta factor are shown.

3.1 Direct SEA approach: the coupling power proportionality

Assuming that the coupling between two oscillators is conservative and that the external forces are uncorrelated white noises it is proved by Lyon and Maidanik (1), Lyon and Sharton (7) and Lotz and Crandall (8) that the power transmitted is proportional to the difference of the total energies. The coupling power proportionality between oscillators i and j can then be written

$$P_{ij} = \eta_{ij}\omega_c(E_i - E_j) \tag{7}$$

Where the coefficient $\eta_{ij}\omega_c$ is, for an elastic coupling given by

$$\eta_{ij}\omega_c = \frac{K_i^2(\Delta_1+\Delta_2)}{m_i m_j[(\omega_i^2-\omega_j^2)^2+(\Delta_i+\Delta_j)(\Delta_i\omega_j^2-\Delta_j\omega_i^2)]} \tag{8}$$

where $\Delta_i = \lambda_i/m_i$ is the half power bandwidth, λ_i the viscous damping coefficient, m_i the mass, K_i the coupling stiffness and ω_i the natural frequency of oscillator i. The coupling loss factors verify the reciprocity relationship,

$$\eta_{ij}\omega_c = \eta_{ji}\omega_c \tag{9}$$

and will be referred to as consistency condition. In the case of three coupled oscillators the SEA system traducing the CPP is

$$\begin{bmatrix} P_{inj}^1 \\ P_{inj}^2 \\ P_{inj}^3 \end{bmatrix} = \omega_c \begin{bmatrix} \eta_1+\eta_{12} & -\eta_{21} & 0 \\ -\eta_{12} & \eta_2+\eta_{21}+\eta_{23} & -\eta_{32} \\ 0 & -\eta_{23} & \eta_3+\eta_{32} \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} \tag{10}$$

Such a system is called proper-SEA system. The conditions of conservation of energy and consistency are verified and the indirect coupling loss factors (η_{13} and η_{31}) are null.

4. ENERGY INFLUENCE COEFFICIENT

The energy influence coefficient method introduced by Guyader (9, 10) is used to compute the indirect coupling loss factors. The oscillators are excited one after another by uncorrelated forces. Three systems are obtained for each excitation giving the total energies of each oscillator. Let us call E_i^j the vibrational energy of oscillator i when the excitation concerns oscillator j. Superposing each equations allows to build a matrix system which has the form $F = E\eta$ where F is the power injected vector, E the matrix composed of vibrational energies E_i^j and η the vector which contains the damping loss factors, the coupling loss factors and the indirect coupling loss factors. For three oscillators the matrix system writes,

$$\begin{bmatrix} P_{inj}^1 \\ 0 \\ 0 \\ 0 \\ P_{inj}^2 \\ 0 \\ 0 \\ 0 \\ P_{inj}^3 \end{bmatrix} = \omega_c \begin{bmatrix} E_1^1 & E_1^1 & E_1^1 & -E_2^1 & 0 & 0 & -E_3^1 & 0 & 0 \\ 0 & -E_1^1 & 0 & E_2^1 & E_2^1 & E_2^1 & 0 & -E_3^1 & 0 \\ 0 & 0 & -E_1^1 & 0 & 0 & -E_2^1 & E_3^1 & E_3^1 & E_3^1 \\ E_1^2 & E_1^2 & E_1^2 & -E_2^2 & 0 & 0 & -E_3^2 & 0 & 0 \\ 0 & -E_1^2 & 0 & E_2^2 & E_2^2 & E_2^2 & 0 & -E_3^2 & 0 \\ 0 & 0 & -E_1^2 & 0 & 0 & -E_2^2 & E_3^2 & E_3^2 & E_3^2 \\ E_1^3 & E_1^3 & E_1^3 & -E_2^3 & 0 & 0 & -E_3^3 & 0 & 0 \\ 0 & -E_1^3 & 0 & E_2^3 & E_2^3 & E_2^3 & 0 & -E_3^3 & 0 \\ 0 & 0 & -E_1^3 & 0 & 0 & -E_2^3 & E_3^3 & E_3^3 & E_3^3 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_{12} \\ \eta_{13} \\ \eta_{21} \\ \eta_2 \\ \eta_{23} \\ \eta_{31} \\ \eta_{32} \\ \eta_3 \end{bmatrix} \tag{11}$$

5. RESULTS AND DISCUSSION

5.1 Influence of the coupling strength

Let us consider the oscillators having characteristics as shown in Table 1. The computation of the total energies follows Eq (3). The coupling stiffness denoted $K_1 = K_2$ vary from $98E-5$ N/m to $98E+5$ N/m.

Type	Oscillator 1	Oscillator 2	Oscillator 3
Mass m_i (kg)	0.0039	0.0040	0.0036
Stiffness k_i (N/m)	9e3	9e3	9e3
Damping λ_i (S.I)	0.001	0.001	0.001

Table 1 – Oscillator parameters

The forces have a power spectral density constant ($S_i = 1$) in the time-frequency bandwidth [ω_{min} , ω_{max}] and zero elsewhere. All modes are resonant (natural frequency within [ω_{min} , ω_{max}]).

The parameters presented in Table 1 are alike giving close natural frequencies (f_1=241.77Hz; f_2=238.75Hz and f_3=251.64Hz when $K_i << k_i$). For high coupling values, the natural frequency of each oscillator is affected and can increase. Then, in order to detect the resonant mode for each values of coupling the frequency bandwidth of integration is taken sufficiently large [0; 20 kHz].

5.2 Case of two coupled oscillators

Let us consider the case of two oscillators taking the coupling stiffness $K_2 = 0$ between oscillators 2 and 3.

The expression of the proportionality constant $\eta_{12}\omega_c$ given by SEA (Eq (8)) is called β_{SEA} and is directly compared to the ratio (called β_{REF}) of the power transmitted given by Eq (4) and the difference of the total energies (Eq (3)): $\beta_{REF} = P_{12}/(E_1 - E_2)$. Figure 2 shows the evolution of beta estimated by SEA and beta estimated by the reference calculation versus the coupling strength.

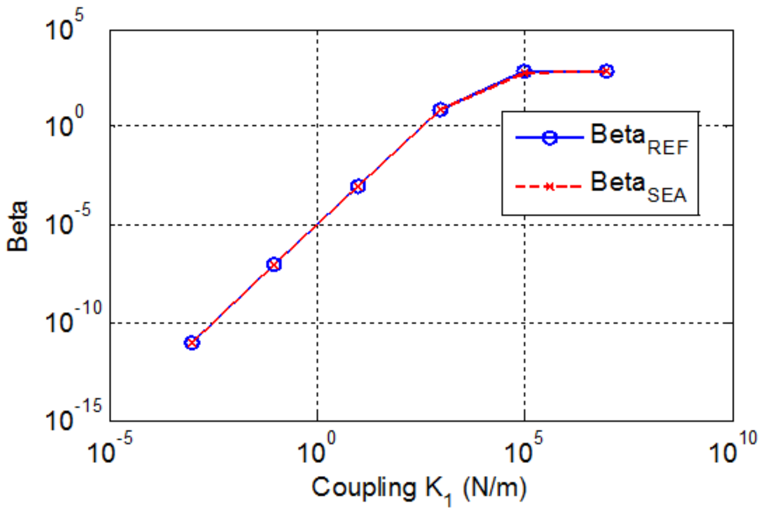

Figure 2 – Evolution of β_{REF} and β_{SEA} for two oscillators when oscillator 1 is excited versus the coupling strength K_1.

A perfect agreement is observed for any strength of coupling illustrating that the coupling power proportionality is always valid for two oscillators.

5.3 Case of three coupled oscillators

Let us consider the case of three oscillators ($K_2 \neq 0$). As before $\beta_{ij,REF}$, computed from Eq (3) and (4) is compared with the SEA beta factor (Eq (8)) while the couplings $K_1 = K_2$ vary. Figure 3 presents the evolution of the beta factors. The gap between $\beta_{ij,REF}$ and $\beta_{ij,SEA}$ do not have the same rough size compare to their evolutions. That's why results are presented in two scales: linear (Fig. 3 (a)) and logarithmic (Fig. 3 (b)).

For weak couplings ($K_1 << k_1$ and $K_2 << k_2$), $\beta_{ij,REF}$ are equal to $\beta_{ij,SEA}$ in particular until K_1=K_2=$98E-1$ N/m. For stronger couplings ($K_1 \geq 98E+1$ N/m) it is shown (Fig. 3 a) that $\beta_{ij,REF}$ and $\beta_{ij,SEA}$ do not fit traducing that the coupling power proportionality is no longer valid.

In Figure 4, energy influence coefficient method has been used to compute the damping loss factor (DLF), the coupling loss factors (CLF) and the possible indirect coupling loss factors (CLFi) when the couplings K_1 and K_2 grows. Table 2 gives each element of vector η (Eq (11)).

The first observation is that reciprocity relations holds whatever the couplings values are. Indeed, η_{12}

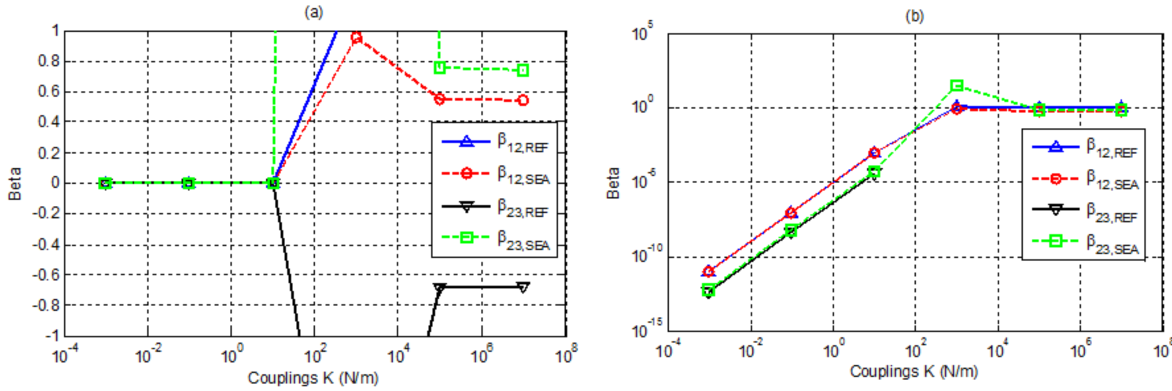

Figure 3 – Evolution of $\beta_{ij,REF}$ and $\beta_{ij,SEA}$ for three oscillators when oscillator 1 is excited versus the coupling strength $K_1 = K_2$.

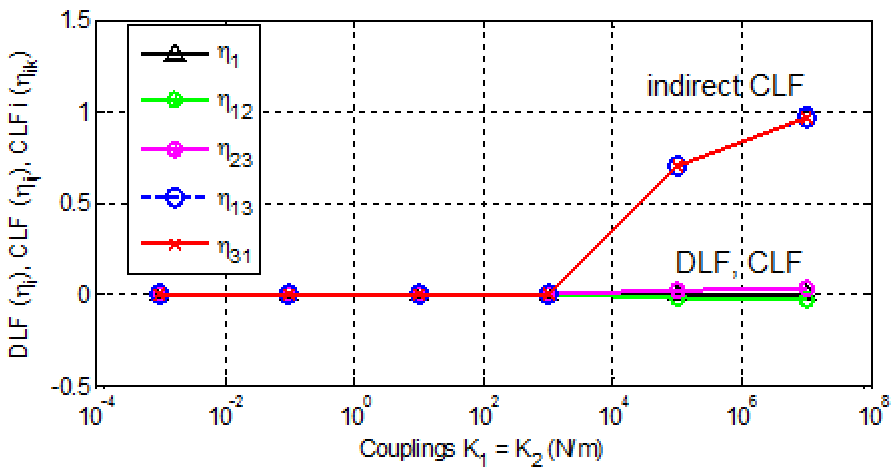

Figure 4 – Evolution of the CLF, DLF and CLFi versus the couplings stiffness K_1 =K_2 with large bandwidth.

K	$98E-5$	$98E-3$	$98E-1$	$98E+1$	$98E+3$	$98E+5$
η_1	0.001	0.001	0.001	0.001	0.001	0.001
η_2	0.001	0.001	0.001	0.001	0.001	0.001
η_3	0.001	0.001	0.001	0.001	0.001	0.001
η_{12}	$3.7E-14$	$3.7E-10$	$4.1E-6$	0.0018	-0.0194	-0.0288
η_{21}	$3.7E-14$	$3.7E-10$	$4.1E-6$	0.0018	-0.0194	-0.0288
η_{23}	$2.2E-15$	$2.2E-11$	$2.3E-7$	0.0021	0.0215	0.0313
η_{32}	$2.2E-15$	$2.2E-11$	$2.3E-7$	0.0021	0.0215	0.0313
η_{13}	$-2E-26$	$-2E-18$	$-2E-10$	0.0045	0.7060	0.9692
η_{31}	$-2E-26$	$-2E-18$	$-2E-10$	0.0045	0.7060	0.9692

Table 2 – Damping loss factor, coupling loss factor, indirect coupling loss factor values while the couplings $K_1 = K_2$ vary.

= η_{21}; η_{23} = η_{32}. The second observation is that the damping loss factors are well estimated compared to the prescribed values (see Table 1). The third observation is that, for weak couplings, indirect coupling loss factors values are very low compared to the coupling loss factor values ($\eta_{ik} << \eta_{ij} << \eta_i$) and thus can be neglected. Conversely, they appear when the couplings are strong (above K_1=K_2=98$E-1$ N/m) and cannot be neglected anymore.

5.4 Study of the influence of a non resonant mode

In this section, the influence of a non resonant mode is viewed. For that, oscillator's parameters are fixed in such a way that the natural frequency of oscillator 2 is outside the frequency band. Then, the mass of oscillator 2 is adjusted to get closer to the bandwidth of integration. The margin between the natural frequency of oscillator 2 and the bandwidth is evaluated by a simple difference: $f_2 - f_{min}$. Three cases are observed :

- f_2 is outside the bandwidth with $f_2 < f_{min}$. The mode of oscillator 2 is denoted *non-resonant low frequency*.
- f_2 is within the bandwidth with $f_{min} < f_2 < f_{max}$. The mode of oscillator 2 is *resonant*.
- f_2 is outside the bandwidth with $f_{max} < f_2$. The mode of oscillator 2 is denoted *non-resonant high frequency*.

Figure 5 illustrates by a drawing the simulation protocol.

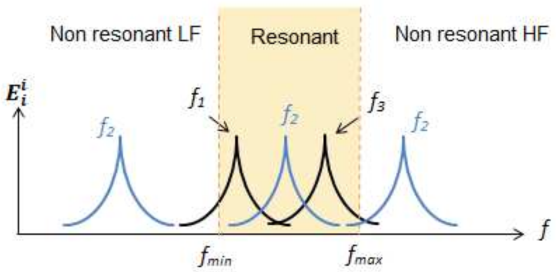

Figure 5 – Sketch of the natural frequencies of each oscillator. The resonance frequency of oscillator 2 varies.

The damping loss factors are prescribed equal for the three oscillators (see Table 3). The couplings are weak ($K_1 = K_2 = 98E-3$ N/m) the bandwidth of integration is an octave defined by the central frequency $f_c = 250$ Hz. In this way, the resonance pic of oscillator 2 is within the frequency band from the value $f_2 - f_{min} = 0$ to $f_2 - f_{min} = 177$ Hz.

Type	Oscillator 1	Oscillator 2	Oscillator 3
Mass m_i (kg)	0.0039	variable	0.0036
Stiffness k_i (N/m)	9e3	9e3	9e3
Damping λ_i (S.I)	0.001	0.001	0.001

Table 3 – Oscillator parameters

Energy influence coefficient method is again used to estimate the DLF, the CLF and the CLFi. Figure 6 presents the DLF, the CLF and the CLFi while the natural frequency of oscillator 2 varies.

First, it is observed that the DLF η_1 and η_3 are correctly estimated. η_2 grows with the shifting of the natural frequency f_2 but correct values are estimated when the resonance pic is within the bandwidth of integration. The CLF are no more symmetric ($\eta_{ij} \neq \eta_{ji}$) when $f_2 - f_{min}$ is below zero and greater than 177 Hz (the resonance frequency is before or after the bandwidth of integration). Nevertheless, the equality becomes valid ($\eta_{ij} = \eta_{ji}$) when f_2 is within the bandwidth.

The indirect coupling loss factors are symmetric ($\eta_{13} = \eta_{31}$) and can have negative values. Their evolutions are small compared to the evolution of the CLF and their values are 10^5 times lower than the CLF, as consequence, they can be neglected.

6. CONCLUSIONS

It has been shown that the main relation in SEA, the coupling power proportionality, is valid whatever the coupling strength is, provided that the system is limited to two subsystems and the frequency bandwidth of excitation contains the natural frequencies (only resonant modes). For more than two subsystems, the CPP can be verified as long as the couplings are weak ($K_i << k_i$ is a fair condition) and the modes are resonant.

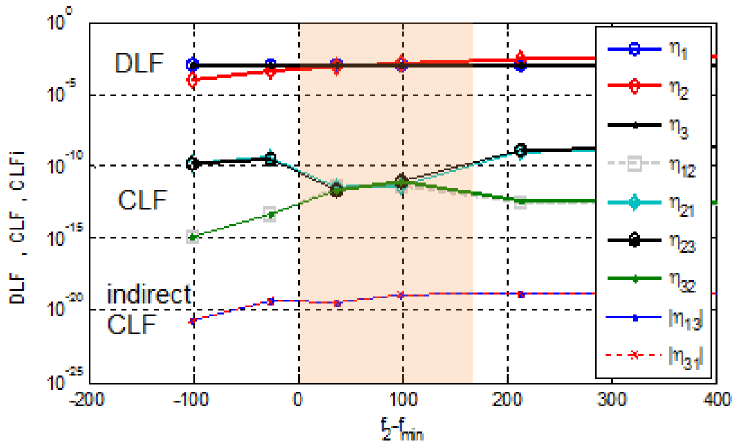

Figure 6 – Evolution of the DLF and the CLF using inverse SEA. Bandwidth: [176; 353 Hz], weak couplings ($K = 98E-3$ N/m).

For more than two oscillators, the indirect coupling loss factors appear when the coupling between subsystems is strong but the reciprocity relation always applies.

The frequency bandwidth of excitation is also important. Non-resonant modes (modes outside the frequency bandwidth of excitation) lead to the invalidity of the reciprocity relation and thus pull down the coupling power proportionality relation.

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