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# On isometries of Product of normed linear spaces.

Mohammed Bachir

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*Laboratoire SAMM 4543, Université Paris 1 Panthéon-Sorbonne, Centre P.M.F. 90 rue Tolbiac 75634 Paris cedex 13*

*Email : Mohammed.Bachir@univ-paris1.fr,*

**Abstract.** We give a condition on norms under which two vector normed spaces  $X$  and  $Y$  are isometrically isomorphic if and only if  $X \times \mathbb{R}$  and  $Y \times \mathbb{R}$  are isometrically isomorphic. We also prove that this result fail for arbitrary norms even if  $X = Y = \mathbb{R}^2$  by building a generic counterexamples.

**Keyword, phrase:** Normed vector space and isometries.

## 1 Introduction

We are interested in this paper in the following question. Let  $X$  and  $Y$  be vector spaces and let  $N_X$  and  $N_Y$  be two norms on  $(X \times \mathbb{R}, N_X)$  and  $(Y \times \mathbb{R}, N_Y)$  respectively. The norm  $N_X$  on  $X$  (and in a similar way  $N_Y$ ) denotes  $N_X(x, 0)$  for all  $x \in X$ .

**Problem.** It is true that  $(X \times \mathbb{R}, N_X)$  and  $(Y \times \mathbb{R}, N_Y)$  are isometrically isomorphic if and only if  $(X, N_X)$  and  $(Y, N_Y)$  are isometrically isomorphic?

We begin by showing that in the general case the answer to this question is no for arbitrary norms  $N_X$  and  $N_Y$ , even when  $X$  and  $Y$  are two dimensional vector spaces, by constructing a generic counterexamples (See Theorem 1). We prove then in Theorem 2 that the result is true for all norms  $(N_X, N_Y)$  satisfying the following property  $(P)$ .

**Definition 1** *Let  $X$  and  $Y$  be two vector spaces. Let  $N_X$  and  $N_Y$  be two norms on  $X \times \mathbb{R}$  and  $Y \times \mathbb{R}$  respectively. We say the the pair  $(N_X, N_Y)$  satisfy the property  $(P)$  if for all  $x \in X$  and all  $y \in Y$ :*

$$N_X(x, 0) = N_Y(y, 0) \Rightarrow N_X(x, \lambda) = N_Y(y, \lambda), \forall \lambda \in \mathbb{R}.$$

*In all the article we identify  $X$  with  $X \times \{0\}$  and the norm  $N_X$  on  $X$  denotes  $N_X(x, 0)$  for all  $x \in X$ .*

**Exemples 1** *Let  $(X, \|\cdot\|_X)$  and  $(Y, \|\cdot\|_Y)$  be two normed vector spaces. Let  $p \in [1, +\infty[$  and*

$$N_{X,p}(x, t) := (\|x\|_X^p + |t|^p)^{\frac{1}{p}},$$

$$N_{X,\infty}(x, t) := \max(\|x\|_X, |t|),$$

*for all  $(x, t) \in X \times \mathbb{R}$ . In a similar way we define  $N_{Y,p}$  and  $N_{Y,\infty}$ . Then the pairs  $(N_{X,p}, N_{Y,p})$  and  $(N_{X,\infty}, N_{Y,\infty})$  satisfies the property  $(P)$ .*

We give in the following proposition a more general examples.

**Proposition 1** *Let  $N_{\mathbb{R}^2}$  be any norm on  $\mathbb{R}^2$  such that  $N_X(x, t) := N_{\mathbb{R}^2}(\|x\|_X, |t|)$  for all  $(x, t) \in X \times \mathbb{R}$  defined a norm on  $X \times \mathbb{R}$  (Similarly we define  $N_Y$  on  $Y \times \mathbb{R}$ ). Then  $(N_X, N_Y)$  satisfy the property (P).*

*Proof.* Let  $x \in X$  and  $y \in Y$  be such that  $N_X(x, 0) = N_Y(y, 0)$ . Then  $N_{\mathbb{R}^2}(\|x\|_X, 0) = N_{\mathbb{R}^2}(\|y\|_Y, 0)$  and so  $\|x\|_X N_{\mathbb{R}^2}(1, 0) = \|y\|_Y N_{\mathbb{R}^2}(1, 0)$ , which implies that  $\|x\|_X = \|y\|_Y$ . It follows that  $N_{\mathbb{R}^2}(\|x\|_X, |\lambda|) = N_{\mathbb{R}^2}(\|y\|_Y, |\lambda|)$  for all  $\lambda \in \mathbb{R}$ . In other words  $N_X(x, \lambda) = N_Y(y, \lambda)$  for all  $\lambda \in \mathbb{R}$ . ■

The problem mentioned above was motivated at the first time in [1] by questions connected to the Banach-Stone theorem, and solved positively only for the particular norms  $N_{X,p}$  and  $N_{Y,p}$  when  $p \in [1, +\infty[ \setminus \{2\}$ . The technique used in [1] did not include the case  $p=2$ . The property (P) here is more general and allowed to include varied norms. We give in section 4 other simple examples of applications of Theorem 2.

## 2 A generic counterexample.

**Theorem 1** *Let  $X = Y = \mathbb{R}^2$ . For each norm  $\|\cdot\|_X$  on  $X$  there exists a norm  $\|\cdot\|_Y$  on  $Y$ , a norm  $N_X$  on  $X \times \mathbb{R}$  and a norm  $N_Y$  on  $Y \times \mathbb{R}$  such that :*

- (1)  $(X, \|\cdot\|_X)$  is not isometrically isomorphic to  $(Y, \|\cdot\|_Y)$ .
- (2)  $(X \times \mathbb{R}, N_X)$  is isometrically isomorphic to  $(Y \times \mathbb{R}, N_Y)$ .
- (3) the restriction of  $N_X$  to  $X$  coincide with  $\|\cdot\|_X$  and the restriction of  $N_Y$  to  $Y$  coincide with  $\|\cdot\|_Y$ .

*Proof.* Let  $p \in [1, +\infty[$ . Let us define  $N_X$  and  $N_Y$  as follow :

$$N_X(x_1, x_2, t) := (\|(x_1, x_2)\|_X^p + |t|^p)^{\frac{1}{p}}, \quad \forall (x_1, x_2, t) \in X \times \mathbb{R}$$

and

$$N_Y(y_1, y_2, s) := (|y_2|^p + \frac{\|(y_1, s)\|_X^p}{a^p})^{\frac{1}{p}}, \quad \forall (y_1, y_2, s) \in Y \times \mathbb{R}.$$

Where  $a = \|(1, 0)\|_X$ . Let us define the norm  $\|\cdot\|_{Y,p}$  on  $Y$  as follows  $\|(y_1, y_2)\|_{Y,p} := (|y_1|^p + |y_2|^p)^{\frac{1}{p}}$  for all  $(y_1, y_2) \in Y$ . Clearly,

$$N_X(x_1, x_2, 0) = \|(x_1, x_2)\|_X, \quad \forall (x_1, x_2) \in X$$

and

$$N_Y(y_1, y_2, 0) = (|y_1|^p + |y_2|^p)^{\frac{1}{p}} := \|(y_1, y_2)\|_{Y,p}, \quad \forall (y_1, y_2) \in Y.$$

( Since  $\frac{\|(y_1, 0)\|_X^p}{a^p} = |y_1| \frac{\|(1, 0)\|_X^p}{a^p} = |y_1|$ ). On the other hand, the following map is an isometric isomorphism:

$$\begin{aligned} \Theta : (X \times \mathbb{R}, N_X) &\rightarrow (Y \times \mathbb{R}, N_Y) \\ (x_1, x_2, t) &\mapsto (ax_1, t, ax_2). \end{aligned}$$

Now, there exist two cases:

*Case 1 :* If every point of the sphere  $S_X$  of  $X$  is an extreme point, we choose  $p = 1$  and so  $S_Y$  has a non extreme point since in this case  $\|(y_1, y_2)\|_{Y,1} = |y_1| + |y_2|$  (For example  $(\frac{1}{2}, \frac{1}{2})$  is not extreme for  $\|\cdot\|_{Y,1}$ ). Consequently  $X$  and  $Y$  cannot be isometrically isomorphic.

*Case 2 :* If there exists some point of the sphere  $S_X$  which is not extreme point then we choose  $p = 2$  and so every points of  $S_Y$  is an extreme point since  $\|(y_1, y_2)\|_{Y,2} = (|y_1|^2 + |y_2|^2)^{\frac{1}{2}}$  is the euclidean norm. Also  $X$  and  $Y$  cannot be isometrically isomorphic. ■

### 3 Isometries between product spaces.

**Theorem 2** *Let  $X$  and  $Y$  be a vector spaces. Suppose that  $(N_X, N_Y)$  satisfy the property (P). Then  $(X \times \mathbb{R}, N_X)$  and  $(Y \times \mathbb{R}, N_Y)$  are isometrically isomorphic if and only if  $(X, N_X)$  and  $(Y, N_Y)$  are isometrically isomorphic.*

The proof of the above theorem is given in section 3.2 after some lemmas.

#### 3.1 Notations and lemmas.

We need some notations and lemmas. Let  $\Theta : (X \times \mathbb{R}, N_X) \rightarrow (Y \times \mathbb{R}, N_Y)$  be an isomorphism isometric. We set  $(a, u) = \Theta^{-1}(0, 1)$  and  $(b, v) = \Theta(0, 1)$ . Let us define the linear continuous map  $\chi_X$  as follow :

$$\begin{aligned} \chi_X : X \times \mathbb{R} &\longrightarrow \mathbb{R} \\ (x, t) &\longmapsto t \end{aligned}$$

We define analogously the map  $\chi_Y$  by

$$\begin{aligned} \chi_Y : Y \times \mathbb{R} &\longrightarrow \mathbb{R} \\ (y, t) &\longmapsto t \end{aligned}$$

We obtain the following linear map on  $Y \times \{0\}$ :

$$\chi_X \circ \Theta^{-1} : Y \times \{0\} \longrightarrow \mathbb{R}$$

Analogously we have also the linear map on  $X \times \{0\}$ :

$$\chi_Y \circ \Theta : X \times \{0\} \longrightarrow \mathbb{R}$$

Let us set  $X_0 := \text{Ker}(\chi_Y \circ \Theta)$  and  $Y_0 := \text{Ker}(\chi_X \circ \Theta^{-1})$ .

**Remark 1** *The linear spaces  $X_0$  and  $Y_0$  are not necessarily closed since  $\chi_X$  and  $\chi_Y$  are not necessarily continuous.*

**Lemma 1**  *$X_0$  and  $Y_0$  are isometrically isomorphic. More precisely, the map*

$$\begin{aligned} \Theta : (X_0, N_X) &\rightarrow (Y_0, N_Y) \\ (z, 0) &\mapsto \Theta(z, 0) \end{aligned} \tag{1}$$

*is an isomorphism isometric.*

*Proof.* Since  $\Theta$  is an isomorphism isometric, it suffices to show that the restriction of  $\Theta$  to  $X_0$  is onto. Indeed, let  $(y, 0) \in Y_0$ . Clearly,  $(z, 0) := \Theta^{-1}(y, 0) \in X_0$  since  $\chi_Y \circ \Theta(\Theta^{-1}(y, 0)) = \chi_Y(y, 0) = 0$  and we have  $(y, 0) = \Theta(z, 0)$ . ■

**Lemma 2** *We have only two cases.*

**Case1:**  $u \neq 0$ . *In this case, we have  $X \times \{0\} = X_0$ .*

**Case2:**  $u = 0$ . *In this case we have  $\Theta^{-1}(0, 1) = (a, 0)$  and  $X \times \{0\} = X_0 \oplus \mathbb{R}(a, 0)$ . Similarly we have,*

**Case1:**  $v \neq 0$ . *In this case, we have  $Y \times \{0\} = Y_0$ .*

**Case2:**  $v = 0$ . *In this case we have  $\Theta(0, 1) = (b, 0)$  and  $Y \times \{0\} = Y_0 \oplus \mathbb{R}(b, 0)$ .*

*Proof.* For all  $x \in X$  there exists  $(y_x, \lambda_x) \in Y \times \mathbb{R}$  such that

$$\begin{aligned}(x, 0) = \Theta^{-1}(y_x, \lambda_x) &= \Theta^{-1}(y_x, 0) + \lambda \Theta^{-1}(0, 1) \\ &= \Theta^{-1}(y_x, 0) + \lambda_x(a, u) \\ &= \Theta^{-1}(y_x, 0) + (\lambda_x a, \lambda_x u)\end{aligned}\tag{2}$$

Since  $\Theta^{-1}(y_x, 0) \in X_0 \subset X \times \{0\}$  and also  $(x, 0) \in X \times \{0\}$ , then from the above equation we obtain that  $(\lambda_x a, \lambda_x u) \in X \times \{0\}$  which implies that  $\lambda_x u = 0$ . So we have :

**Case1:**  $u \neq 0$ . In this case,  $X \times \{0\} = X_0$ . Indeed, if  $u \neq 0$  then  $\lambda_x = 0$  and so  $(x, 0) = \Theta^{-1}(y_x, 0) \in X_0$ , for all  $x \in X$  i.e  $X \times \{0\} \subset X_0$ . On the other hand we know that  $X_0 \subset X \times \{0\}$ .

**Case2:**  $u = 0$ . In this case we have  $\Theta^{-1}(0, 1) = (a, 0)$  and so  $X = X_0 \oplus \mathbb{R}(a, 0)$ . Indeed, We have  $X_0 \cap \mathbb{R}(a, 0) = (0, 0)$ , since if  $\alpha$  is a real number such that  $\alpha(a, 0) \in X_0$  then  $0 = \chi_Y \circ \Theta(\alpha(a, 0)) = \alpha \chi_Y(0, 1) = \alpha$ . In other words from (2), for all  $x \in X$ , there exist  $(y_x, \lambda_x) \in Y \times \mathbb{R}$  such that

$$(x, 0) = \Theta^{-1}(y_x, 0) + \lambda_x(a, 0).$$

whith  $\Theta^{-1}(y_x, 0) \in X_0$ . Thus  $X \times \{0\} \subset X_0 \oplus \mathbb{R}(a, 0) \subset X \times \{0\}$  and so  $X \times \{0\} = X_0 \oplus \mathbb{R}(a, 0)$ .

In a similar way we obtain the second part of the lemma. ■

**Lemma 3** *We have,  $u = 0$  if and only if  $v = 0$ .*

*Proof.* Suppose that  $v = 0$ . Then for all  $(x, t) \in X \times \mathbb{R}$ , we have  $\Theta(x, t) = \Theta(x, 0) + \Theta(0, t) = \Theta(x, 0) + t\Theta(0, 1) = \Theta(x, 0) + t(b, v) = \Theta(x, 0) + (tb, 0)$ . Now, we are going to prove that  $u = 0$ . Suppose that the contrary hold, that is  $u \neq 0$ . Then  $X \times \{0\} = X_0$  (See the case 1. in Lemma 2). So  $\Theta(x, 0) \in \Theta(X \times \{0\}) = \Theta(X_0) = Y_0$ , since  $\Theta$  is an isomorphism isometric from  $X_0$  onto  $Y_0$  (See the formula (1)). Now since  $Y_0 \subset Y \times \{0\}$ , then  $\Theta(x, 0) + t(b, 0) \in Y \times \{0\}$ . In other words,  $\Theta(x, t) \in Y \times \{0\}$  for all  $(x, t) \in X \times \mathbb{R}$ . So  $\Theta(X \times \mathbb{R}) \subset Y \times \{0\}$ . But  $\Theta$  is an isomorphism between  $X \times \mathbb{R}$  and  $Y \times \mathbb{R}$ . This implies that  $Y \times \{0\} = Y \times \mathbb{R}$  which is impossible. Thus  $u = 0$ . In a similar way we obtain the converse. ■

### 3.2 Proof of Theorem 2 and some corollaries.

We give now the proof of the main result.

*Proof of Theorem 2.* For the “if” part, let  $T : (X, N_X) \rightarrow (Y, N_Y)$  be an isomorphism isometric. Let us define  $\Theta : (X \times \mathbb{R}, N_X) \rightarrow (Y \times \mathbb{R}, N_Y)$  by  $\Theta(x, \lambda) = (T(x), \lambda)$ . Then, clearly  $\Theta$  is an isomorphism and by the property (P) it is also isometric. We prove now the “only if part”. By combining Lemma 2 and Lemma 3 we have that:

*Case1.* If  $u \neq 0$  and  $v \neq 0$ , then  $X \times \{0\} = X_0$  and  $Y \times \{0\} = Y_0$ . So by Lemma 1 we conclude that  $X \times \{0\}$  and  $Y \times \{0\}$  are isometrically isomorphic for the norms  $N_X$  and  $N_Y$ . So  $(X, N_X)$  and  $(Y, N_Y)$  are isometrically isomorphic.

*Case2.* If  $u = 0$  and  $v = 0$ , using Lemma 2 we have that  $\Theta^{-1}(0, 1) = (a, 0)$  and  $X \times \{0\} = X_0 \oplus \mathbb{R}(a, 0)$  and  $\Theta(0, 1) = (b, 0)$  and  $Y \times \{0\} = Y_0 \oplus \mathbb{R}(b, 0)$ . Now we prove that the map

$$\begin{aligned}\psi : X \times \{0\} = X_0 \oplus \mathbb{R}(a, 0) &\rightarrow Y \times \{0\} = Y_0 \oplus \mathbb{R}(b, 0) \\ (z, 0) + \lambda(a, 0) &\mapsto \Theta(z, 0) + \lambda(b, 0)\end{aligned}$$

is an isomorphism isometric. Indeed, the fact that  $\psi$  is linear and onto map is clear by using Lemma 1. Let us prove that  $\psi$  is isometric for the norms  $N_X$  and  $N_Y$ . Since  $(z, 0) \in X_0$ , by (1) there exist  $(y, 0) \in Y_0$  such that  $\Theta(z, 0) = (y, 0)$ . Since

$$\begin{aligned}(z, 0) + \lambda(a, 0) &= \Theta^{-1}(\Theta(z, 0)) + \lambda \Theta^{-1}(0, 1) \\ &= \Theta^{-1}(\Theta(z, 0) + (0, \lambda)) \\ &= \Theta^{-1}(y, \lambda)\end{aligned}$$

then, using the fact that  $\Theta^{-1}$  is isometric we have

$$\begin{aligned} N_X((z, 0) + \lambda(a, 0)) &= N_X(\Theta^{-1}(y, \lambda)) \\ &= N_Y(y, \lambda). \end{aligned} \quad (3)$$

On the other hand we know that  $(b, 0) = \Theta(0, 1)$  so  $\Theta(z, 0) + \lambda(b, 0) = \Theta(z, 0) + \lambda\Theta(0, 1) = \Theta(z, \lambda)$ . Thus, using the fact that  $\Theta$  is isometric we have,

$$\begin{aligned} N_Y(\psi((z, 0) + \lambda(a, 0))) &= N_Y(\Theta(z, 0) + \lambda(b, 0)) \\ &= N_Y(\Theta(z, \lambda)) \\ &= N_X(z, \lambda). \end{aligned} \quad (4)$$

But  $N_X(z, 0) = N_Y(y, 0)$  since  $\Theta(z, 0) = (y, 0)$  and  $\Theta$  is isometric. Since  $(N_X, N_Y)$  satisfy the property (P) then  $N_X(z, \lambda) = N_Y(y, \lambda)$ . Thus, using the formulas (3) and (4) we obtain that  $\psi$  is isometric. ■

**Remark 2** *By induction, we can easily extend the above theorem to  $X \times \mathbb{R}^n$  ( $n \in \mathbb{N}^*$ ) if we assume that  $(N_X, N_Y)$  is a pair of norms satisfying the following property ( $P^n$ ): for all  $x \in X$  all  $y \in Y$ , all  $i \in \{1, 2, \dots, n\}$  and all  $(s_1, s_2, \dots, s_i); (s'_1, s'_2, \dots, s'_i) \in \mathbb{R}^i$  : if  $N_X(x, s_1, s_2, \dots, s_i, 0, 0, \dots, 0) = N_Y(y, s'_1, s'_2, \dots, s'_i, 0, 0, \dots, 0)$  then  $N_X(x, s_1, s_2, \dots, s_i, \lambda, 0, \dots, 0) = N_Y(y, s'_1, s'_2, \dots, s'_i, \lambda, 0, \dots, 0), \forall \lambda \in \mathbb{R}$ .*

**Examples 2** *Let  $p \in [1, +\infty[$ , and*

$$N_{X,p}(x, s_1, \dots, s_n) = (\|x\|_X^p + \sum_{k=1}^n |s_k|^p)^{\frac{1}{p}},$$

$$N_{X,\infty}(x, s_1, \dots, s_n) = \max(\|x\|_X, |s_1|, \dots, |s_n|)$$

for all  $(x, s_1, \dots, s_n) \in X \times \mathbb{R}^n$ . In a similar way we define  $N_{Y,p}$  and  $N_{Y,\infty}$ . Then the pairs  $(N_{X,p}, N_{Y,p})$  and  $(N_{X,\infty}, N_{Y,\infty})$  satisfies the property ( $P^n$ ).

**Corollary 1** *Let  $X$  and  $Y$  be a vector spaces. Let  $n \in \mathbb{N}^*$  and suppose that  $(N_X, N_Y)$  satisfy ( $P^n$ ). Then  $(X \times \mathbb{R}^n, N_X)$  and  $(Y \times \mathbb{R}^n, N_Y)$  are isometrically isomorphic if and only if  $(X, N_X)$  and  $(Y, N_Y)$  are isometrically isomorphic.*

As a remark we have the following corollary for inner product spaces. Note that a non complete inner product space has no orthonormal basis in general (See [5]). The symbol  $\cong$  means “isometrically isomorphic”.

**Corollary 2** *Let  $(H, \|\cdot\|_H)$  and  $(L, \|\cdot\|_L)$  be two inner product space (not necessary complete). Then  $(H, \|\cdot\|_H) \cong (L, \|\cdot\|_L)$  if and only if for all finite dimensional subspaces  $E \subset H$  and  $F \subset L$  such that  $\dim(E) = \dim(F)$  we have that  $(E^\perp, \|\cdot\|_H) \cong (F^\perp, \|\cdot\|_L)$ . Where  $E^\perp$  and  $F^\perp$  denotes the orthogonal of  $E$  and  $F$  respectively.*

*Proof.* Let  $E \subset H$  and  $F \subset L$  such that  $\dim(E) = \dim(F) = n$  for  $n \in \mathbb{N}$ . By the classical projection theorem on a complete vector subspace of an inner product space, we have  $H = E^\perp \oplus E$  and  $L = F^\perp \oplus F$ . On the other hand it is clear that  $(H, \|\cdot\|_H) \cong (E^\perp \times \mathbb{R}^n, N_{E^\perp, 2})$  and  $(L, \|\cdot\|_L) \cong (F^\perp \times \mathbb{R}^n, N_{F^\perp, 2})$ , where  $N_{E^\perp, 2}$  and  $N_{F^\perp, 2}$  are defined as in the Example 2 with  $p = 2$ . Since  $(N_{E^\perp, 2}, N_{F^\perp, 2})$  satisfy ( $P^n$ ) then from Corollary 1 we obtain  $(E^\perp, \|\cdot\|_H) \cong (F^\perp, \|\cdot\|_L)$ . The converse is clear. ■

## 4 Applications.

We give in this section two applications of Theorem 2. We denote by  $(C^1[0, 1], N_{C^1[0,1]})$  the space of continuously differentiable functions on  $[0, 1]$  endowed with the norm  $N_{C^1[0,1]}(f) := N_{\mathbb{R}^2}(\|f'\|_\infty, |f(0)|)$ , where  $N_{\mathbb{R}^2}$  denotes any norm satisfying Proposition 1. Let  $(X, \|\cdot\|_X)$  be a Banach space. We denote by  $N_X$  the norm defined on  $X \times \mathbb{R}$  by  $N_X(x, t) := N_{\mathbb{R}^2}(\|x\|_X, |t|)$  for all  $(x, t) \in X \times \mathbb{R}$ . Finally, we denote by  $(C[0, 1], \|\cdot\|_\infty)$  the space of continuous functions on  $[0, 1]$  endowed with the supremum norm.

**Proposition 2** *We have  $(X \times \mathbb{R}, N_X) \cong (C^1[0, 1], N_{C^1[0,1]})$ , if and only if  $(X, \|\cdot\|_X) \cong (C[0, 1], \|\cdot\|_\infty)$ .*

*Proof.* Let us define the norm  $N_{C[0,1]}$  on  $C[0, 1] \times \mathbb{R}$  by  $N_{C[0,1]}(g, t) := N_{\mathbb{R}^2}(\|g\|_\infty, |t|)$  for all  $(g, t) \in C[0, 1] \times \mathbb{R}$ . Let us consider the map

$$\begin{aligned} \chi : (C^1[0, 1], N_{C^1[0,1]}) &\rightarrow (C[0, 1] \times \mathbb{R}, N_{C[0,1]}) \\ f &\mapsto (f', f(0)) \end{aligned}$$

Clearly,  $\chi$  is an isomorphism isometric. So we have  $(X \times \mathbb{R}, N_X) \cong (C[0, 1] \times \mathbb{R}, N_{C[0,1]})$ . Since  $(N_X, N_{C[0,1]})$  satisfy the property (P) by Proposition 1 then using Theorem 2 we obtain that  $(X, \|\cdot\|_X) \cong (C[0, 1], \|\cdot\|_\infty)$ , since  $N_X(\|x\|_X, 0) = \|x\|_X N_{\mathbb{R}^2}(1, 0)$  and  $N_{C[0,1]}(g, 0) = \|g\|_\infty N_{\mathbb{R}^2}(1, 0)$ . ■

Let us recall some notions. Let  $K$  and  $C$  be convex subsets of vector spaces. A function  $T : K \rightarrow C$  is said to be affine if for all  $x, y \in K$  and  $0 \leq \lambda \leq 1$ ,  $T(\lambda x + (1 - \lambda)y) = \lambda T(x) + (1 - \lambda)T(y)$ . The set of all continuous real-valued affine functions on a convex subset  $K$  of a topological vector space will be denoted by  $Aff(K)$ . Clearly, all translates of continuous linear functionals are elements of  $Aff(K)$ , but the converse is not true in general (see [4] page 22.). However, we do have the following relationship.

**Proposition 3** ([4], Proposition 4.5) *Assume that  $K$  is a compact convex subset of a separated locally convex space  $X$  then*

$$\left\{ a \in Aff(K) : a = r + x_K^* \text{ for some } x^* \in X^* \text{ and some } r \in \mathbb{R} \right\}$$

*is dense in  $(Aff(K), \|\cdot\|_\infty)$ , where  $\|\cdot\|_\infty$  denotes the norm of uniform convergence.*

But in the particular case when  $X$  is a Banach space and  $K = (B_{X^*}, w^*)$  is the unit ball of the dual space  $X^*$  endowed with the weak star topology, the well known result due to Banach and Dieudonné states that:

**Theorem 3 (Banach-Dieudonné).** *The space  $(Aff_0(B_{X^*}), \|\cdot\|_\infty)$  is isometrically identified to  $(X, \|\cdot\|)$ . In other words,  $Aff_0(B_{X^*}) = \{\hat{z}|_{B_{X^*}} : z \in X\}$ . Where  $Aff_0(B_{X^*})$  denotes the space of all affine weak star continuous functions that vanish at 0 and  $\hat{z} : p \mapsto p(z)$  for all  $p \in X^*$  and  $\hat{z}|_{B_{X^*}}$  denotes the restriction of  $\hat{z}$  to  $B_{X^*}$ .*

Now, let  $X$  and  $Y$  be two Banach spaces and let us endowed the space  $Aff(B_{X^*})$  (and in a similar way the space  $Aff(B_{Y^*})$ ) with the norm  $N(f) := N_{\mathbb{R}^2}(\|f - f(0)\|_\infty, |f(0)|)$  for all  $f \in Aff(B_{X^*})$ , where  $N_{\mathbb{R}^2}$  denotes any norm on  $\mathbb{R}^2$  satisfying Proposition 1. We obtain the following version of the Banach-Stone theorem for affine functions (For more information about the Banach-Stone theorem see [2] and [3]).

**Proposition 4** *Let  $X$  and  $Y$  be two Banach spaces. Then the following assertions are equivalent.*

- (1)  $(Aff(B_{X^*}), N)$  and  $(Aff(B_{Y^*}), N)$  are isometrically isomorphic.
- (2)  $(X, \|\cdot\|_X)$  and  $(Y, \|\cdot\|_Y)$  are isometrically isomorphic.

*Proof.* Let  $\tilde{N}$  be the norm on  $Aff_0(B_{X^*}) \times \mathbb{R}$  defined by  $\tilde{N}(f_0, t) := N_{\mathbb{R}^2}(\|f_0\|_\infty, |t|)$  for all  $(f_0, t) \in Aff_0(B_{X^*}) \times \mathbb{R}$ . Let us consider the map,

$$\begin{aligned} \chi : (Aff(B_{X^*}), N) &\rightarrow (Aff_0(B_{X^*}) \times \mathbb{R}, \tilde{N}) \\ f &\mapsto (f - f(0), f(0)) \end{aligned}$$

Clearly,  $\chi$  is an isometric isomorphism. Thus using Theorem 1 we have that  $(Aff(B_{X^*}), N)$  and  $(Aff_0(B_{X^*}), N)$  are isometrically isomorphic if and only if  $(Aff_0(B_{X^*}), \|\cdot\|_\infty)$  and  $(Aff_0(B_{Y^*}), \|\cdot\|_\infty)$  are isometrically isomorphic, which is equivalent by Theorem 3 to the fact that  $(X, \|\cdot\|_X)$  and  $(Y, \|\cdot\|_Y)$  are isometrically isomorphic. ■

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