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MULTIVARIABLE BOUNDARY PI CONTROL AND REGULATION OF A FLUID FLOW SYSTEM

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Abstract. The paper is concerned with the control of a fluid flow system governed by nonlinear hyperbolic partial differential equations. The control and the output observation are located on the boundary. We study local stability of spatially heterogeneous equilibrium states by using Lyapunov approach. We prove that the linearized system is exponentially stable around each subcritical equilibrium state. A systematic design of proportional and integral controllers is proposed for the system based on the linearized model. Robust stabilization of the closed-loop system is proved by using a spectrum method.

1. Introduction.

In the paper we study a fluid flow system governed by the Saint Venant equation

$$\left\{\begin{array}{l} \dfrac{\partial S(x,t)}{\partial t}+\dfrac{\partial Q(x,t)}{\partial x}=0, \quad (x,t)\in(0,l)\times(0,\infty), \\ \dfrac{\partial Q(x,t)}{\partial t}+\dfrac{\partial}{\partial x}\left[\dfrac{Q^2(x,t)}{S(x,t)}\right]+gS(x,t)\dfrac{\partial Z(x,t)}{\partial x}+g\eta^2R^{-4/3}\left[\dfrac{Q^2(x,t)}{S(x,t)}\right]=0, \\ Q(0,t)=Q_{in}(t), \quad Z(l,t)=Z_{out}(t), \\ y(t)=(Z(0,t),Q(l,t)) \end{array}\right. \tag{1}$$

The partial differential equation (PDE) (1) models dynamic behaviour of the unsteady flow in a one-reach canal for shallow water, where x denotes the spatial domain (m), t the time (s), $S(x,t)$ the flow cross-section (m^2), $Z(x,t)$ the water level (m), $Q(x,t)$ the flow discharge (m^2/s), g the Newton acceleration constant (m/s^2), η the Manning coefficient for friction slope, R the hydraulic radius (sectional area/wetted perimeter), $Q_{in}(t)$ the upstream flow rate and $Z_{out}(t)$ the downstream level in the canal. The PDE (1) has three unknown functions $S(x,t)$, $Q(x,t)$

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This work is dedicated to Professor Gauthier Sallet for his 65th birthday.

and $Z(x,t)$. There is a relation between $S(x,t)$ and $Z(x,t)$ once the cross-section geometry is fixed :

$$S(x,t) = \mathcal{S}(Z(x,t)). \tag{2}$$

Therefore we choose $Z(x,t)$ and $Q(x,t)$ as state, and $Q_{in}(t)$ and $Z_{out}(t)$ as control. The upstream level and the downstream flow rate are chosen as the output measurement :

$$y(t) = (Z(0,t), Q(l,t)). \tag{3}$$

The objective is to control and regulate the output $y(t)$ by using proportional and integral controllers (PI controllers) on the boundary. The one-reach canal system described by (1), (2) and (3) is a nonlinear multi-inputs and multi-outputs infinite-dimensional system. As a first approach to tackling the control problem, we study the linearized model of the system around equilibrium states. Because of friction slope $\eta > 0$ each equilibrium state is spatially heterogeneous or function of space variable x. As a consequence the linearized model has for the coefficients functions of space variable x. Therefore studying local stability of spatially heterogeneous equilibrium states becomes more difficult than that of homogeneous one (see [1, 2, 25]) as the same as synthesis of stabilizing controllers [25].

Our work is motivated by the problem of controlling an open channel governed by the Saint Venant equation. The hyperbolic PDE describes the dynamics of open channel hydraulic systems, such as rivers, irrigation or drainage canals, sewers etc, assuming one dimensional flow. Description of recent developments in these aspects can be found in the published works [2, 6, 5].

Davison's robust PI controllers for finite dimensional systems [4] has been extended first by Pohjolainen [14] to a class of analytic semigroup systems and then to strongly continuous semigroup systems in [15, 24] where either the control operator or the observation operator is bounded. The extended synthesis has found applications for systems such as counter flow heat exchanger system [24]. Synthesis of robust PI controllers has been exploited in [12] for a rather general class of infinite-dimensional systems called regular systems where both the control operator and the observation operator may be unbounded. In particular, if the input-output transfer matrix $G(s)$ is invertible at $s = 0$, then it is possible to construct an integral controller which stabilizes the system and guarantees the regulation of the set point. Guided by the result we consider the linearized model of the Saint Venant system (1)-(3) where both the control and the observation are located on the boundary, and so represented by unbounded control operator and unbounded observation operator. We propose a systematic method of designing boundary PI controllers for the linearized system.

Irrigation systems have received considerable attention since ten years and numerous interesting results on the feedback stabilization have been obtained by the Lyapunov approach [3] and by the transfer function approach [11]. However there are still interesting and open questions as suggested in [1]. Specifically, when friction slope is taken into account in the Saint Venant model, i.e. $\eta > 0$ in (1), feedback stabilizability of heterogeneous equilibrium states has been established by solving some nonlinear ordinary differential equation. Indeed it is not easy to say if the ordinary differential equation has a bounded solution (see [1]). The present paper is a further endeavour towards understanding the nonlinear fluid flow system described by the Saint Venant equation. We have for the first objective to find easily checkable conditions for local stability of heterogeneous equilibrium states.

We assume the presence of friction slope in the model and study local stability of heterogeneous equilibrium states. Under general conditions exponential stability of the linearized system is proved by using a Lyapunov direct approach. Roughly speaking, if the Manning coefficient of friction slope η is small, then the linearized system is exponentially stable. Then stabilizing PI controllers are designed based on the linearized system. Exponential stability of the closed-loop system is shown by a spectral analysis method so that the output regulation is guaranteed. The contribution of the paper is twofold: proof of exponential stability of the linearized Saint Venant system and synthesis of multivariable boundary PI controllers for the hyperbolic system.

Although our study is carried out directly on the PDE describing the system, concepts such as operator semigroups, admissibility of observation and control operators, and well-posedness and regularity of linear systems are both helpful and important for the clarity of reasoning throughout the paper. We shall give a brief presentation of these notions in the next section. The interested reader is referred to Pazy [13], Weiss [22] and Tucsnak and Weiss [21] for more details.

The paper is organized as follows : Section 2 is devoted to show that integral stabilization implies regulation for regular systems; Section 3 is devoted to computing equilibrium states which are spatially heterogeneous; In Section 4 is proposed some necessary and sufficient condition for exponential stability of two coupled hyperbolic equations; The open-loop exponential stability is proved in Section 5 by taking into account friction slope; The closed-loop stability of the linearized system by the designed PI controllers is proved in Section 6 with a spectral analysis method; Section 7 contains our conclusions and the Appendix is added for proofs of technical results.

2. Integral stabilization implies regulation.

Let X be a Hilbert space and let $A : \mathcal{D}(A) \to X$ is the generator of an exponentially stable C_0 semigroup e^{tA} on X. The Hilbert space X_1 is $\mathcal{D}(A)$ with the norm $\|z\|_1 = \|Az\|$, where $0 \in \rho(A)$, the resolvent set of A. The Hilbert space X_{-1} is the completion of X with respect to the norm $\|z\|_{-1} = \|A^{-1}z\|$. This space is isomorphic to $\mathcal{D}(A^*)'$, the dual space of $\mathcal{D}(A^*)$ and

$$X_1 \subset X \subset X_{-1}\,,$$

densely and with continuous embeddings. The semigroup e^{tA} extends to a semigroup on X_{-1}, denoted by the same symbol. The generator of the extended semigroup is an extension of A, whose domain is X, so that $A : X \to X_{-1}$. Without loss of generality for our applications it is assumed that the input space is the same as the output space and equal to a finite dimensional real Euclidian space $\mathbb{R}^m$. We consider a well-posed linear system Σ with input space $U = \mathbb{R}^m$, state space X, output space $Y = \mathbb{R}^m$, semigroup generator A, control operator B, observation operator C and transfer function G. Thus, $B \in \mathcal{L}(\mathbb{R}^m, X_{-1})$ is an admissible control operator for e^{tA} and $C \in \mathcal{L}(X_1, \mathbb{R}^m)$ is an admissible observation operator for e^{tA}. The control operator B is called *bounded* if $B \in \mathcal{L}(U, X)$, and C is called *bounded* if it can be extended such that $C \in \mathcal{L}(X, Y)$.

An important subclass of the well-posed linear systems are the regular linear systems [22]. The system Σ is called *regular* if for each $v \in \mathbb{R}^m$, the following limit exists (in $\mathbb{R}^m$) :

$$Dv = \lim_{s\to+\infty} G(s)v. \tag{4}$$

The operator $D \in \mathcal{L}(U, Y)$ is then called the *feedthrough operator* of Σ. We define the following extension of C by

$$C_\Lambda x = \lim_{\lambda \to +\infty} C\lambda(\lambda I - A)^{-1}x \ \ \forall\, x \in \mathcal{D}(C_\Lambda), \tag{5}$$

where $\mathcal{D}(C_\Lambda)$ is the space of those $x \in X$ for which the above limit exists. The system Σ is regular if and only if $(sI - A)^{-1}BU \subset \mathcal{D}(C_\Lambda)$ for some (hence, for every) $s \in \rho(A)$ and, if this is the case, then $G(s) = C_\Lambda(sI - A)^{-1}B + D$ for all $s \in \rho(A)$. Moreover, for a regular system, $y(t) = C_\Lambda x(t) + Du(t)$ holds for almost every $t \geq 0$ (for every initial state $x(0) \in X$ and every input signal $u \in L^2_{loc}(\mathbb{R}^+, \mathbb{R}^m)$).

On the state space X the system is written under the form of evolution equations as follows

$$\begin{cases} \dot{\phi}(t) = A\phi(t) + Bu(t) \\ y(t) = C_\Lambda \phi(t) + Du(t) \\ \phi(0) = \phi_0. \end{cases} \tag{6}$$

Definition 2.1. The system (6) is called well-posed if the linear mapping $(\phi_0, u) \mapsto (\phi(T), y)$ is continuous from $X \times L^2((0,T), \mathbb{R}^m)$ to $X \times L^2((0,T), \mathbb{R}^m)$ for every $T > 0$.

The following result can be found in the literature (cf. [22] or [12]).

Proposition 1. *Assume that the system (6) is regular and that the semigroup e^{tA} is exponentially stable on X. Then the following properties hold true :*
(i) *The observation mapping $\phi_0 \mapsto y(t) = C_\Lambda e^{tA}\phi_0$ is continuous from X to $L^2(\mathbb{R}^+, \mathbb{R}^m)$;*
(ii) *The control-state mapping $u \mapsto \phi(t) = \int_0^t e^{(t-\tau)A}Bu(\tau)d\tau$ is continuous from $L^2(\mathbb{R}^+, \mathbb{R}^m)$ to $C([0,\infty), X)$ and from $L^2(\mathbb{R}^+, \mathbb{R}^m)$ to $L^2(\mathbb{R}^+, X)$, and $\lim_{t\to\infty}\phi(t) = 0\ \forall\, u \in L^2(\mathbb{R}^+, \mathbb{R}^m)$;*
(iii) *The control-output mapping $u \mapsto y(t) = C_\Lambda \int_0^t e^{(t-\tau)A}Bu(\tau)d\tau + Du(t)$ is continuous from $L^2(\mathbb{R}^+, \mathbb{R}^m)$ to $L^2(\mathbb{R}^+, \mathbb{R}^m)$.*

We consider the closed-loop system (6) controlled by the following integral controller : $\dot{\xi}(t) = y(t) - y^r$ and $u(t) = K_I\xi$ where $K_I \in \mathbb{R}^{m\times m}$ is the integral controller gain and $y^r \in \mathbb{R}^m$ is the set point. It is easy to see that the closed-loop system is governed by the following evolution equation

$$\begin{cases} \dot{\phi}(t) = A\phi(t) + BK_I\xi + Bw_c \\ \dot{\xi} = C_\Lambda\phi + DK_I\xi + w_o - y^r \\ y(t) = C_\Lambda\phi + DK_I\xi + w_o \\ \phi(0) = \phi_0, \ \ \xi(0) = \xi_0. \end{cases} \tag{7}$$

where w_c and w_o added here represent control and output disturbances, respectively. It is well known that the closed-loop system (7) is also regular (see [12]). Stabilization of the closed-loop system by the integral controller implies the regulation guaranteed, independently of unknown constant disturbances w_c, $w_o \in \mathbb{R}^m$. Here we prove a little more on the regulation result. We say that the disturbances $w_c(t)$ and $w_o(t)$ are quadratically close to constant if there are constants $\tilde{w}_c$, $\tilde{w}_o \in \mathbb{R}^m$ such that $\epsilon_c(t) = (w_c(t) - \tilde{w}_c)$ and $\epsilon_o(t) = (w_o(t) - \tilde{w}_o)$ belong to $L^2(\mathbb{R}^+, \mathbb{R}^m)$. If the disturbances $w_c(t)$ and $w_o(t)$, instead of being constants, are quadratically close to constant, then the regulation result is still true if stabilization is fulfilled. However the regulation must be understood in a broad sense as follows.

Definition 2.2. The closed-loop system (7) is said regulated if we have

$$\lim_{t\to\infty}\int_t^\infty \|y(\tau)-y^r\|^2_{\mathbb{R}^m}d\tau = 0.$$

The following general result holds true for regular linear systems having possibly unbounded control and observation operators. It means that regulation is automatically deduced from stabilization. For the reader's convenience a simple proof of the following proposition is given in the Appendix.

Proposition 2. *Assume that the matrix $G_0 = (-C_\Lambda A^{-1}B + D) \in \mathbb{R}^{m\times m}$ is invertible. Stabilization implies regulation independently of all control disturbance $w_c(t)$ or output disturbance $w_o(t)$ quadratically close to constant.*

Remark 1. If the control and output disturbances are null, i.e., $w_c(t) \equiv 0$ and $w_o(t) \equiv 0$, then we recover the regulation in the classical sense : $\lim_{t\to\infty}\|y(t) - y^r\|_{\mathbb{R}^m} = 0 \ \forall\, \phi_0 \in \mathcal{D}(A)$.

3. Equilibrium solutions.

In the paper the synthesis of boundary PI-controllers will be carried out based on the linearized Saint Venant model. For the sake of simplicity it is assumed that the canal has a rectangular cross section. As a consequence we have the relations : $S(x,t) = BZ(x,t)$ and $R = BZ(x,t)/(B+2Z(x,t))$ where B is the base width.

Let $\tilde{Q}_{in}$ and $\tilde{Z}_{out}$ be positive constants. The equilibrium solution of the Saint Venant equation (1) is given by

$$\begin{cases} \dfrac{dQ(x)}{dx} = 0, \qquad x\in(0,l) \\ \dfrac{d}{dx}\left[\dfrac{Q^2(x)}{BZ(x)}\right] + gBZ(x)\dfrac{dZ(x)}{dx} + g\eta^2R^{-4/3}\left[\dfrac{Q^2(x,t)}{BZ(x)}\right] = 0 \\ Q(0) = \tilde{Q}_{in}, \quad Z(l) = \tilde{Z}_{out}. \end{cases} \tag{8}$$

Definition 3.1. We call subcritical equilibrium state for the Saint Venant system (1) each solution $(\tilde{Q}(x), \tilde{Z}(x))$ of (8) satisfying the subcritical hydraulic condition :

$$gB^2\tilde{Z}^3_{out} - \tilde{Q}^2_{in} > 0.$$

As the reader will see each subcritical equilibrium solution satisfies the inequality on the interval

$$gB^2\tilde{Z}^3(x) - \tilde{Q}^2(x) > 0, \quad \forall\, x\in[0,l]. \tag{9}$$

Lemma 3.2. *Assume that the upstream flow rate $\tilde{Q}_{in}$ and the downstream level $\tilde{Z}_{out}$ satisfy the subcritical condition (9). Then the system (8) has a unique equilibrium solution $(\tilde{Q}(x), \tilde{Z}(x))$ of class C^1. Further $\tilde{Q}(x) = \tilde{Q}_{in}$ is constant and $\tilde{Z}(x)$ is a decreasing function of x obtained by solving the following differential equation :*

$$\begin{cases} \tilde{Z}_x = \dfrac{g\eta^2R^{-4/3}\tilde{Q}^2\tilde{Z}}{\tilde{Q}^2 - gB^2\tilde{Z}^3} \\ \tilde{Z}(l) = \tilde{Z}_{out} \end{cases} \tag{10}$$

where (and hereafter) $\tilde{Z}_x$ denotes the partial derivative of $\tilde{Z}(x)$ w.r.t. x.

Remark 2. The right member of the equation (10) is a C^1 function of $\tilde{Z}$. It has a unique local solution. The subcritical boundary condition being satisfied the unique solution $\tilde{Z}(x)$ is bounded on the interval $[0,l]$ whatever $l > 0$.

We set

$$a_1 = \frac{1}{B}, \quad a_2 = \frac{gB^2\tilde{Z}^3 - \tilde{Q}^2}{B\tilde{Z}^2}, \quad a_3 = \frac{2\tilde{Q}}{B\tilde{Z}}, \tag{11}$$

$$a_4 = \frac{g\eta^2 R^{-4/3}\tilde{Q}^4}{B\tilde{Z}^2(gB^2\tilde{Z}^3 - \tilde{Q}^2)} + \frac{4g\eta^2 R^{-4/3} B\tilde{Q}^2}{3\tilde{Z}^2(B+2\tilde{Z})}, \quad a_5 = \frac{2g^2\eta^2 R^{-4/3} B\tilde{Q}\tilde{Z}^2}{gB^2\tilde{Z}^3 - \tilde{Q}^2}. \tag{12}$$

where $R = B\tilde{Z}(x)/(B + 2\tilde{Z}(x))$.

Remark 3. If the Manning coefficient $\eta = 0$, then the equilibrium state is given by two positive constants $(\tilde{Q}, \tilde{Z})$ independent of x. In this case each term a_k, $k = 1, ..., 5$, is constant with $a_4 = a_5 = 0$. If $\eta > 0$, then we have $a_i(x) > 0$, $\forall\, i = 1$ and $\forall\, x \in [0, l]$.

The following lemmas are easy to prove by direct computation.

Lemma 3.3. *The linearized system around the equilibrium solution is governed by*

$$\left\{\begin{array}{l} \dfrac{\partial}{\partial t}\begin{bmatrix} z(x,t) \\ q(x,t) \end{bmatrix} = \begin{bmatrix} 0 & -a_1 \\ -a_2 & -a_3 \end{bmatrix} \dfrac{\partial}{\partial x}\begin{bmatrix} z(x,t) \\ q(x,t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ a_4 & -a_5 \end{bmatrix}\begin{bmatrix} z(x,t) \\ q(x,t) \end{bmatrix} \\ q(0,t) = u_1(t) \\ z(l,t) = u_2(t) \\ y(t) = [z(0,t), q(l,t)]^\tau \end{array}\right. \tag{13}$$

Let the matrices $A(x)$ and $B(x)$ be defined by

$$A(x) = \begin{bmatrix} 0 & -a_1 \\ -a_2(x) & -a_3(x) \end{bmatrix}, \quad B(x) = \begin{bmatrix} 0 & 0 \\ a_4(x) & -a_5(x) \end{bmatrix}. \tag{14}$$

Lemma 3.4. *The matrix $A(x)$ defined in (14) has two different eigenvalues under the subcritical condition: the one is negative and the other is positive, given by*

$$\lambda_1(x) = -\frac{1}{2}\left(a_3 + \sqrt{a_3^2 + 4a_1a_2}\right), \quad \lambda_2(x) = \frac{1}{2}\left(\sqrt{a_3^2 + 4a_1a_2} - a_3\right). \tag{15}$$

By substituting the expressions $a_k(x)$ into the eigenvalues (15) we get the following relation :

$$\lambda_1(x) = -\left(\sqrt{g\tilde{Z}} + \frac{\tilde{Q}}{B\tilde{Z}}\right), \quad \lambda_2(x) = \sqrt{g\tilde{Z}} - \frac{\tilde{Q}}{B\tilde{Z}}. \tag{16}$$

In the next section we present a result of stability concerned with two hyperbolic PDE. To the best of our knowledge the result is new in the sense that linear PDE of spatially heterogeneous coefficients are dealt with.

4. Exponential stability of coupled hyperbolic PDE.

Consider the linear hyperbolic system governed by the following PDE

$$\left\{\begin{array}{l} \dfrac{\partial}{\partial t}\begin{bmatrix} R_1(x,t) \\ R_2(x,t) \end{bmatrix} = \begin{bmatrix} \lambda_1(x) & 0 \\ 0 & \lambda_2(x) \end{bmatrix} \dfrac{\partial}{\partial x}\begin{bmatrix} R_1(x,t) \\ R_2(x,t) \end{bmatrix}, \quad (x,t) \in (0,l) \times \mathbb{R}^+ \\ R_1(0,t) = \alpha_1 R_2(0,t), \\ R_2(l,t) = \alpha_2 R_1(l,t), \end{array}\right. \tag{17}$$

where $\lambda_i(x)$ is C^1 and α_i is a real constant $\forall\, i = 1, 2$. Assume that $\lambda_1(x) < 0$ and $\lambda_2(x) > 0\; \forall\, x \in [0, l]$. Consider for the state space the Hilbert space $X = (L^2(0,1))^2$ equipped with the usual quadratic norm.

Theorem 4.1. *The null solution $R(\cdot,t) \equiv 0$ is exponentially stable for the system (17) if and only if $\alpha_1^2\alpha_2^2 < 1$.*

Remark 4. Greenberg and Li [7] have proved that, in the quasilinear case and under the form of two Riemann invariants, the above condition was a sufficient condition for the null equilibrium solution to be exponentially stable. Our condition is necessary and sufficient for exponential stability of spatially heterogeneous linear PDE. It is concerned with two different situations.

Proof of Theorem 4.1. We prove the sufficient condition first. Consider the candidate of Lyapunov functional $V_W : X \to \mathbb{R}^+$ such that

$$V_W(R) = \int_0^l R(x)^* W(x) R(x) dx \tag{18}$$

where the weight matrix W is a diagonal matrix defined by

$$W(x) = \operatorname{diag}\left(\left(1 + e^{-\theta(1+x)}\right)\tilde{W}_1(x), \left(1 - e^{-\theta(1+x)}\right)\tilde{W}_2(x)\right) \tag{19}$$

and $\tilde{W}_i(x)$, $i = 1, 2$, by

$$\tilde{W}_1(x) = \frac{\lambda_1(0)}{\lambda_1(x)}, \quad \tilde{W}_2(x) = \left(\alpha_1^2 + \varepsilon\right)\left(\frac{-\lambda_1(0)}{\lambda_2(x)}\right)\left(\frac{1 + e^{-\theta}}{1 - e^{-\theta}}\right). \tag{20}$$

Notice that the constant ε is fixed, small and positive such that $\left(\alpha_1^2 + \varepsilon\right)\alpha_2^2 < 1$.

Computing the time derivative of the Lyapunov functional $V_W(R(\cdot,t))$ along the smooth trajectories of the system (17) gives us the following

$$\dot{V}_W(R(\cdot,t)) = \int_0^l 2R^*(x,t)W(x)R_t(x,t)dx = \int_0^l 2R^*(x,t)W(x)\Lambda(x)R_x(x,t)dx.$$

Hence we have

$$\dot{V}_W(R(\cdot,t)) = \int_0^l \left(R^* W(x)\Lambda(x)R\right)_x dx - \int_0^l R^* \left(W(x)\Lambda(x)\right)_x R dx, \tag{21}$$

where $\Lambda(x) = \operatorname{diag}\left(\lambda_1(x), \lambda_2(x)\right)$ and

$$\left(W(x)\Lambda(x)\right)_x = -\theta\lambda_1(0)e^{-\theta(1+x)} \begin{bmatrix} 1 & 0 \\ 0 & \dfrac{\left(\alpha_1^2 + \varepsilon\right)\left(1 + e^{-\theta}\right)}{1 - e^{-\theta}} \end{bmatrix}.$$

Using the boundary condition in (17) we get

$$\int_0^l \left(R^* W(x)\Lambda(x)R\right)_x dx \le \lambda_1(0)\left[1 + e^{-\theta(1+l)}\right] R_1^2(l,t)\left[1 - (\alpha_1^2 + \varepsilon)\alpha_2^2 F(\theta)\right], \tag{22}$$

where

$$F(\theta) = \left(\frac{1 + e^{-\theta}}{1 - e^{-\theta}}\right)\left(\frac{1 - e^{-\theta(1+l)}}{1 + e^{-\theta(1+l)}}\right).$$

From the fact that $F(\theta)$ is a decreasing function of θ having limit equal to 1 as $\theta \to \infty$ and by the hypothesis $(\alpha_1^2 + \varepsilon)\alpha_2^2 < 1$, there exists a positive constant $\theta_1 > 0$ such that, for every $\theta \ge \theta_1$,

$$\int_0^l \left(R^* W(x)\Lambda(x)R\right)_x dx \le 0. \tag{23}$$

Let $\tilde{W}$ be the diagonal matrix defined by

$$\tilde{W}(x) = \mathrm{diag}\left(-\lambda_1(0), \frac{-\lambda_1(0)\left(\alpha_1^2+\varepsilon\right)\left(1+e^{-\theta_1}\right)}{1-e^{-\theta_1}}\right) \tag{24}$$

and let $V_{\tilde{W}}$ be the functional defined as in (18) with the weight $\tilde{W}$. It is easy to find positive constants K_1, K_2, $\tilde{K}_1$ and $\tilde{K}_2$ such that

$$K_1\|R\|_X^2 \le V_W(R) \le K_2\|R\|_X^2, \quad \tilde{K}_1\|R\|_X^2 \le V_{\tilde{W}}(R) \le \tilde{K}_2\|R\|_X^2, \quad \forall\, R \in X. \tag{25}$$

For every smooth initial condition $R^0 \in X$ and by substituting (23) and (25) into (21), the following inequality is obtained :

$$\dot{V}_W(R(\cdot,t) \le -\theta_1 \tilde{K}_1 K_2^{-1} e^{-\theta_1(1+l)} V_W(R(\cdot,t),$$

or, with $K_3 = \theta_1 \tilde{K}_1 K_2^{-1} e^{-\theta_1(1+l)}$,

$$V_W(R(\cdot,t) \le e^{-tK_3} V_W(R^0).$$

So we have proved exponential stability of the system (18).

To prove the necessary part, consider the Lyapunov candidate $V_{\hat{W}}(R)$ as above with the weight $\hat{W}$:

$$\hat{W}(x) = \mathrm{diag}(\hat{W}_1(x), \hat{W}_2(x))$$

where $\hat{W}_i$, $i = 1,2$, is the same as in (19)-(20) except with $\varepsilon = 0$. Suppose that $\alpha_1^2\alpha_2^2 \ge 1$. Computing the time derivative of $V_{\hat{W}}(R(\cdot,t))$ along the smooth solutions of (18) gives us $\dot{V}_{\hat{W}}(R(\cdot,t)) \ge 0$, or $V_{\hat{W}}(R(\cdot,t)) \ge V_{\hat{W}}(R^0)$ $\forall\, t \ge 0$. Hence the system is not asymptotically stable, which contradicts exponential stability of the system. The necessary part is proved. The proof of Theorem 4.1 is complete. $\square$

5. **Open-loop stability of the Saint Venant system.** We investigate first local stability of the Saint Venant system around each equilibrium solution. More precisely we are interested to know if the linearized system (13) is exponentially stable. If the viscous friction slope is null, our Theorem 4.1 is applied to prove exponential stability of the linearized system. If the friction slope is small, exponential stability still holds true for the linearized system.

The following useful identities may be proved by direct computation.

Lemma 5.1. *Let a_i, $i = 1, ..., 5$, be defined in (10)-(12) and let λ_j, $j = 1,2$, be defined in (15). Then we have*

$$a_1a_2 - \lambda_k a_3 = \lambda_k^2, \quad k = 1,2.$$

Corollary 1. *The linearized system (13) around each subcritical equilibrium state is exponentially stable if the viscous friction slope is equal to zero, i.e. $\eta = 0$.*

Proof of Corollary 1. If the viscous friction slope is null, each a_i is constant and positive $\forall\, i = 1,2,3$ and $a_4 = a_5 = 0$. Thus the matrix $A(x)$ is equal to constant matrix A

$$A = \begin{bmatrix} 0 & -a_1 \\ -a_2 & -a_3 \end{bmatrix}$$

which has two real eigenvalues of opposite sign λ_1 and λ_2. Consider the invertible linear transformation $T_1\ : L^2(0,l) \times L^2(0,l) \to L^2(0,l) \times L^2(0,l)$ such that

$$\begin{pmatrix} z \\ q \end{pmatrix} = T_1 \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -\frac{\lambda_1}{a_1} & -\frac{\lambda_2}{a_1} \end{pmatrix} \begin{pmatrix} R_1 \\ R_2 \end{pmatrix}. \tag{26}$$

The transformed open-loop system from (13) is written as follows

$$\left\{\begin{array}{l}\dfrac{\partial}{\partial t}\left(\begin{array}{c}R_1(x,t)\\R_2(x,t)\end{array}\right)=\left(\begin{array}{cc}\lambda_1 & 0\\0 & \lambda_2\end{array}\right)\dfrac{\partial}{\partial x}\left(\begin{array}{c}R_1(x,t)\\R_2(x,t)\end{array}\right)\\[2ex] R_1(0,t)=-\left(\dfrac{\lambda_2(0)}{\lambda_1(0)}\right)R_2(0,t)-\left(\dfrac{a_1}{\lambda_1(0)}\right)u_1(t)\\[2ex] R_2(l,t)=-R_1(l,t)+u_2(t)\\[2ex] y(t)=\left(R_1(0,t)+R_2(0,t),-\dfrac{\lambda_1(l)}{a_1}R_1(l,t)-\dfrac{\lambda_2(l)}{a_1}R_2(l,t)\right)^\tau.\end{array}\right. \tag{27}$$

Putting $\alpha_1=-(\lambda_2(0)/\lambda_1(0))$ and $\alpha_2=-1$, we check easily that $\alpha_1^2\alpha_2^2<1$ for the subcritical equilibrium state. Direct application of Theorem 4.1 proves exponential stability of the linearized system for $u_1=u_2=0$. □

Remark 5. It is not difficult to show that the open-loop system (27) is well-posed and regular (cf. [26, 20, 22]).

If the viscous friction slope is positive, we have the following result.

Theorem 5.2. *The null solution $R(\cdot,t)\equiv 0$ is exponentially stable for the linearized system (13) around each subcritical equilibrium solution, provided that the viscous friction slope is small. More precisely there exists some positive constant $\eta^*>0$ such that the linearized system (13) is exponentially stable $\forall\ \eta\in[0,\eta^*]$.*

Proof of Theorem 5.2. Consider the same transformation T_1 as the previous one in (26). Without ambiguity we set

$$T_1(x)=\left[\begin{array}{cc}1 & 1\\ -\dfrac{\lambda_1(x)}{a_1} & -\dfrac{\lambda_2(x)}{a_1}\end{array}\right]. \tag{28}$$

Note that $T_1(x)$ is a smooth functions of x because λ_1 and λ_2 are. The same transformation carried out on the linearized system (13) gives us

$$\left\{\begin{array}{l}\dfrac{\partial R}{\partial t}=\Lambda(x)\dfrac{\partial R}{\partial x}+\tilde{B}(x)R\\[2ex] R_1(0,t)=-\left(\dfrac{\lambda_2(0)}{\lambda_1(0)}\right)R_2(0,t)-\left(\dfrac{a_1}{\lambda_1(0)}\right)u_1(t)\\[2ex] R_2(l,t)=-R_1(l,t)+u_2(t)\end{array}\right. \tag{29}$$

where

$$\begin{array}{l}\Lambda(x)=\mathrm{diag}(\lambda_1(x),\lambda_2(x))\\ \tilde{B}(x)=T_1^{-1}(x)\left[A(x)T_1'(x)+B(x)T_1(x)\right].\end{array} \tag{30}$$

Direct computation gives us $\tilde{B}(x)$

$$\tilde{B}(x)=\frac{a_1}{\lambda_1-\lambda_2}\left[\begin{array}{cc}\dfrac{\lambda_1}{a_1}(\lambda_1'-a_5)-a_4 & \dfrac{1}{a_1}(\lambda_1\lambda_2'-a_5\lambda_2)-a_4\\[2ex] \dfrac{1}{a_1}(a_5\lambda_1-\lambda_2\lambda_1')+a_4 & \dfrac{\lambda_2}{a_1}(a_5-\lambda_2')+a_4\end{array}\right].$$

(λ_1' denotes the derivative of $\lambda_1(x)$ w.r.t. x.) Moreover $\tilde{B}(x)$ can be written as follows

$$\tilde{B}(x)=\tilde{Z}_x(x)\left[\begin{array}{cc}b_{11}(x) & b_{12}(x)\\ b_{21}(x) & b_{22}(x)\end{array}\right] \tag{31}$$

where $b_{ij}(x)$ can be computed explicitly from the equilibrium state. From the equilibrium equation (10) there exists some positive constant $\beta > 0$ such that

$$\tilde{Z}_{out} \leq \tilde{Z}(x) \leq e^{\eta^2 \beta l} \tilde{Z}_{out}, \quad \forall\, x \in [0, l]. \tag{32}$$

In (29), we set $\alpha_1 = -\lambda_2(0)/\lambda_1(0)$ and $\alpha_2 = -1$. By applying Theorem 4.1 exponential stability of the open-loop system is proved when $\tilde{B}(x) \equiv 0$. From Lemma 3.2 et by (32), there exists a positive constant $K_4 > 0$ such that

$$\|\tilde{B}(x)\|_{\mathcal{L}(\mathbb{R}^2)} \leq \eta^2 K_4, \quad \forall\, x \in [0, l]. \tag{33}$$

Let $V_W(R)$ be the same Lyapunov functional as in the proof of Theorem 4.1 but with $\varepsilon = 0$. By computing the time derivative of the Lyapunov functional $V_W(R)$ along the smooth solutions of (29), we get the following

$$\dot{V}_W(R(\cdot, t)) \leq -\theta_1 \tilde{K}_1 K_2^{-1} e^{-\theta_1(1+l)} V_W(R(\cdot, t)) + \int_0^l 2R^* W(x) \tilde{B}(x) R dx, \tag{34}$$

where the constants $\tilde{K}_1$, K_2 and θ_1 are chosen as in the proof of Theorem 4.1. We have used the fact that

$$|\alpha_1 \alpha_2| = \frac{B\sqrt{g}\tilde{Z}^{3/2}(0) - \tilde{Q}_{in}}{B\sqrt{g}\tilde{Z}^{3/2}(0) + \tilde{Q}_{in}} \leq \frac{B\sqrt{g}\tilde{Z}_{out}^{3/2} e^{3\eta^2 \beta L/2} - \tilde{Q}_{in}}{B\sqrt{g}\tilde{Z}_{out}^{3/2} e^{3\eta^2 \beta L/2} + \tilde{Q}_{in}} < 1. \tag{35}$$

By (25) and (33) we have some positive constant K_5 such that the following inequality holds

$$\left| \int_0^l 2R^* W(x) \tilde{B}(x) R dx \right| \leq 2\eta^2 K_1^{-1} K_4 K_5 V_W(R(\cdot, t)). \tag{36}$$

Substituting (36) into (34) allows to find some constant $\eta^* > 0$ such that for every $\eta \in [0, \eta^*]$,

$$\dot{V}_W(R(\cdot, t)) \leq -\frac{1}{2} \theta_1 \tilde{K}_1 K_2^{-1} e^{-\theta_1(1+l)} V_W(R(\cdot, t)). \tag{37}$$

With the same notation as in the proof of Theorem 4.1, we get

$$V_W(R(\cdot, t) \leq e^{-tK_3/2} V_W(R^0), \quad \forall\, \eta \in [0, \eta^*].$$

So is proved exponential stability of the linearized system (13) for $\eta > 0$. □

6. **PI-controllers and closed-loop stability.** We have shown that the linearized model (13) can be transformed into the form (29). In this section our Theorem 5.2 is applied to prove exponential stability of the linearized model controlled by the proportional output feedback law. Moreover the closed-loop system is stabilized and regulated by PI-controllers of the form :

$$\begin{cases} \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = K_p(y(t) - y^r) + k_I K_I \xi \\ \dot{\xi} = y(t) - y^r, \end{cases} \tag{38}$$

where K_p, $K_I \in \mathbb{R}^{2\times 2}$, $k_I \in \mathbb{R}$ and $y^r \in \mathbb{R}^2$ is the control setpoint. In other words we give a systematic design method for tuning the controller matrices K_P and K_I such that the following properties hold true : (a) the closed-loop system (13) by the PI output feedback law (38) is well-posed and the associated semigroup is exponentially stable; (b) the output regulation is achieved in the sense of Definition 2.2 :

$$\lim_{t\to+\infty} \|y(t) - y^r\|_{\mathbb{R}^2} = 0 \tag{39}$$

independently of known or unknown disturbances quadratically close to constant.

Recall that the matrix $\begin{pmatrix} 0 & -a_1 \\ -a_2 & -a_3 \end{pmatrix}$ in (13) has two real eigenvalues of opposite sign. Denote by λ_1 the negative one and by λ_2 the positive one

$$\begin{cases} \lambda_1 = -\frac{1}{2}\left(a_3 + \sqrt{a_3^2 + 4a_1a_2}\right), \\ \lambda_2 = \frac{1}{2}\left(-a_3 + \sqrt{a_3^2 + 4a_1a_2}\right). \end{cases} \tag{40}$$

Let us set

$$K_P = \operatorname{diag}(K_{P,1}, K_{P,2}) = \begin{bmatrix} \dfrac{-\lambda_2(0)}{a_1} & 0 \\ 0 & \dfrac{-a_1}{\lambda_1(l)} \end{bmatrix} \tag{41}$$

and

$$K_I = \begin{bmatrix} K_{P,1} & -1 \\ -\exp\left(\int_0^l \tilde{a}_4(\xi)d\xi\right) & K_{P,2} + \displaystyle\int_0^l \exp\left(\int_\eta^l \tilde{a}_4(\xi)d\xi\right)\tilde{a}_5(\eta)d\eta \end{bmatrix} \tag{42}$$

where the constants $K_{P,1}$ and $K_{P,2}$ are defined in (41) and

$$\tilde{a}_4(x) = \frac{a_4(x)}{a_2(x)}, \quad \tilde{a}_5(x) = \frac{a_5(x)}{a_2(x)}. \tag{43}$$

Notice that the matrix K_I is invertible.

Remark 6. The integral controller matrix K_I in (42) is obtained by computing $K_I = -G_0^{-1}$ where G_0 is the mapping $v \mapsto y$ defined by the following ordinary differential equation :

$$\begin{aligned} &0 = A(x)\frac{\partial}{\partial x}\begin{bmatrix} z \\ q \end{bmatrix} + B(x)\begin{bmatrix} z \\ q \end{bmatrix} \\ &[q(0), \ \ z(l)]^\tau = K_P(y + v) \\ &y = [z(0), \ \ q(l)]^\tau, \end{aligned} \tag{44}$$

where $A(x)$ and $B(x)$ are defined in (14). Notice that G_0 is just the transfer matrix $G(s)$ of the system (13) with $u = K_P y + v$ evaluated at $s = 0$. If $K_P = 0$, we recover a pure integral controller which also works for the considered system.

Our main result is summarized in the coming theorem.

Theorem 6.1. (a) *There exists some positive $\eta^* > 0$ such that, for every Manning number $\eta \in [0, \eta^*]$, the closed-loop system (13) by the proportional controller (38) with $k_I = 0$ is exponentially stable.*
(b) *There exists a constant $k_I^* > 0$ such that the PI-controller (38) and (41)-(42) stabilizes exponentially the linearized Saint Venant system (13) with the output regulation guaranteed, $\forall\ 0 < k_I < k_I^*$ and $\forall\ 0 \le \eta \le \eta^*$.*

In the following, we show how the linearized Saint Venant system (13) controlled by the PI controller (38) is transformed into the form (29) and prove Theorem 6.1 by a spectral method. The natural state space for the closed-loop system is the Hilbert space $H = L^2(0, l) \times L^2(0, l) \times \mathbb{R}^2$.

6.1. **P-controller design.** Synthesis of the proportional controller is to keep satisfied the following basic requirements : 1) On each boundary point the outgoing information should be determined by the incoming information for existence and uniqueness of the PDE solutions (see [18]); 2) The dissipation conditions are made satisfied "at the very most" ; 3) The exponential decay rate is made as large as possible for the underlying semigroup.

After the proportional output feedback $u = K_P(y - y^r) + v$ and by the transformation $(z, q)^\tau = T_1 R$ in (26), the closed-loop system (13) or (29) is governed by the following PDE

$$\left\{\begin{array}{l} \dfrac{\partial R}{\partial t} = \Lambda(x)\dfrac{\partial R}{\partial x} + \tilde{B}(x)R \\ \left[\begin{array}{c} R_1(0,t) \\ R_2(l,t) \end{array}\right] = \left(\begin{array}{c} \dfrac{-\lambda_2(0)}{\lambda_1(0)}R_2(0,t) \\ -R_1(l,t) \end{array}\right) + \left(\begin{array}{cc} \dfrac{-a_1}{\lambda_1(0)} & 0 \\ 0 & 1 \end{array}\right)\Big[K_P(y(t) - y^r) + v(t)\Big] \\ y(t) = \Big[R_1(0,t) + R_2(0,t),\ \ \dfrac{-1}{a_1}\left(\lambda_1(l)R_1(l,t) + \lambda_2(l)R_2(l,t)\right)\Big]^\tau \end{array}\right. \tag{45}$$

where the matrices $\tilde{B}(x)$ and K_P are defined in (31) and (41), respectively.

Lemma 6.2. *Let $y^r = 0$. The P-controlled system in (45) is well-posed and exponentially stable $\forall\, \eta \in [0, \eta^*]$ for some $\eta^* > 0$.*

Proof of Lemma 6.2 . Substituting y into the boundary condition from (45) leads to the following PDE

$$\left\{\begin{array}{l} \dfrac{\partial}{\partial t}\left(\begin{array}{c} R_1(x,t) \\ R_2(x,t) \end{array}\right) = \Lambda(x)\dfrac{\partial}{\partial x}\left(\begin{array}{c} R_1(x,t) \\ R_2(x,t) \end{array}\right) + \tilde{B}(x)\left(\begin{array}{c} R_1(x,t) \\ R_2(x,t) \end{array}\right) \\ \\ \left(\begin{array}{c} R_1(0,t) \\ R_2(l,t) \end{array}\right) = C_3 v + \left(\begin{array}{cc} \dfrac{\lambda_2(0)}{\lambda_2(0) - \lambda_1(0)} & 0 \\ 0 & \dfrac{a_1}{\lambda_1(l) - \lambda_2(l)} \end{array}\right) y^r \\ \\ y(t) = \left(R_1(0,t) + R_2(0,t), -\dfrac{\lambda_1(l)}{a_1}R_1(l,t) - \dfrac{\lambda_2(l)}{a_1}R_2(l,t)\right)^\tau, \end{array}\right. \tag{46}$$

where C_3 is a constant matrix given by (56). The well-posedness of the system has been proved in [26]. Let us prove just exponential stability of the system.

Let $v = 0$ and $y^r = 0$. Define the unbounded linear operator $\tilde{A}\ : \mathcal{D}(\tilde{A}) \to X$ as follows :

$$\mathcal{D}(\tilde{A}) = \{(f_1, f_2) \in (H^1(0,l))^2 \mid f_1(0) = f_2(l) = 0\} \tag{47}$$

and for every $f \in \mathcal{D}(\tilde{A})$,

$$\tilde{A}f(x) = \Lambda(x)f_x(x) + \tilde{B}(x)f(x). \tag{48}$$

It is easy to see that $\tilde{A}$ is the generator of a C_0 semigroup on X. By using (31) and the same argument as in the proof of Theorem 5.2 we prove exponential stability of the semigroup $e^{t\tilde{A}}$ and hence exponential stability of the P-controlled system. $\square$

6.2. **I-controller design.** By adding the integral controller the closed-loop system (46) is described by the following PDE

$$\frac{\partial}{\partial t}\begin{pmatrix} R_1(x,t) \\ R_2(x,t) \end{pmatrix} = \Lambda(x)\frac{\partial}{\partial x}\begin{pmatrix} R_1(x,t) \\ R_2(x,t) \end{pmatrix} + \tilde{B}(x)\begin{pmatrix} R_1(x,t) \\ R_2(x,t) \end{pmatrix} \tag{49}$$

$$\dot{\xi} = y(t) - y^r$$

$$\begin{pmatrix} R_1(0,t) \\ R_2(l,t) \end{pmatrix} = \begin{pmatrix} \frac{\lambda_2(0)}{\lambda_2(0)-\lambda_1(0)} & 0 \\ 0 & \frac{a_1}{\lambda_1(l)-\lambda_2(l)} \end{pmatrix} y^r + C_3(k_I K_I \xi + v) \tag{50}$$

$$y(t) = \left(R_1(0,t) + R_2(0,t), -\frac{\lambda_1(l)}{a_1}R_1(l,t) - \frac{\lambda_2(l)}{a_1}R_2(l,t)\right)^\tau. \tag{51}$$

Consider the homogeneous PDE corresponding to the above closed-loop system (49)-(50):

$$\begin{aligned} R_t &= (\Lambda(x)\partial_x + \tilde{B})R \\ \xi_t &= y \end{aligned} \tag{52}$$

$$\begin{bmatrix} R_1(0,t) \\ R_2(l,t) \end{bmatrix} = C_3 k_I K_I \xi \tag{53}$$

$$y = C_1 R(\cdot,t) + C_2 k_I K_I \xi, \tag{54}$$

where the linear operators $C_1 \ : (H^1(0,1))^2 \to \mathbb{R}^2$ and $C_2,\ C_3 \ : \mathbb{R}^2 \to \mathbb{R}^2$, defined by

$$\begin{aligned} C_1 R &= \begin{pmatrix} 1 & 0 \\ 0 & \frac{-\lambda_1(l)}{a_1} \end{pmatrix}\begin{pmatrix} R_2(0) \\ R_1(l) \end{pmatrix} \\ C_2 \xi &= \begin{pmatrix} \frac{a_1}{\lambda_2(0)-\lambda_1(0)} & 0 \\ 0 & \frac{-\lambda_1(l)\lambda_2(l)}{a_1(\lambda_1(l)-\lambda_2(l))} \end{pmatrix}\xi \end{aligned} \tag{55}$$

and

$$C_3 \xi = \begin{pmatrix} \frac{a_1}{\lambda_2(0)-\lambda_1(0)} & 0 \\ 0 & \frac{\lambda_1(l)}{\lambda_1(l)-\lambda_2(l)} \end{pmatrix}\xi. \tag{56}$$

The first objective is to prove exponential stability of the system governed by the PDE (52)-(53). For each $k_I > 0$ we define the unbounded operator $A_1 \ : \mathcal{D}(A_1) \to H$ by

$$\mathcal{D}(A_1) = \left\{ \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} \in (H^1(0,l))^2 \times \mathbb{R}^2 \ \middle| \ \begin{pmatrix} f_1(0) \\ f_2(l) \end{pmatrix} = C_3 k_I K_I f_3 \right\} \tag{57}$$

and for every $f \in \mathcal{D}(A_1)$,

$$A_1 f(x) = \begin{bmatrix} \begin{pmatrix} \lambda_1(x) & 0 \\ 0 & \lambda_2(x) \end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} + \tilde{B}(x)\begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} \\ \\ C_1\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} + k_I C_2 K_I f_3 \end{bmatrix} \tag{58}$$

which is exactly the right hand member of (52).

Then we have the following result whose proof will be presented at the end of the section.

Theorem 6.3. (i) *There exists some constant $k_I^* > 0$ such that A_1 is the generator of an exponentially stable C_0 semigroup on H whatever $0 < k_I < k_I^*$.*
(ii) *The output regulation is guaranteed $\forall\ 0 < k_I < k_I^*$: For all control and output disturbances quadratically close to constant the output converges to the set point in the sense that*

$$\lim_{T\to+\infty}\int_T^{\infty}\|y(t)-y^r\|_{\mathbb{R}^2}^2dt=0.$$

Since the integral part is applied on the boundary and the output observation is also on the boundary, the existing theory in [14, 15] and [24] is not directly useful to prove exponential stability of the semigroup e^{tA_1}. The reason is that its domain $\mathcal{D}(A_1)$ depends on K_I and k_I. It makes incongruous to assign the spectrum of A_1 by using the perturbation theory [9]. Our idea is to look for a transformation such that the domain of the new generator is independent of K_I and k_I. Then the technical methods in [14], [24] and [9] can be used for our transformed system. Since the transformation is continuous and invertible, the transformed system is exponentially stable if and only if the original one is.

For the purpose we consider the invertible transformation $T_2\ : H\to H$ such that

$$\begin{pmatrix} R_1(x)\\ R_2(x)\\ \xi\end{pmatrix}=T_2\begin{pmatrix} \tilde R_1(x)\\ \tilde R_2(x)\\ \tilde\xi\end{pmatrix}=\begin{bmatrix}\begin{pmatrix}\tilde R_1(x)\\ \tilde R_2(x)\end{pmatrix}+\begin{pmatrix}\frac{l-x}{l} & 0\\ 0 & \frac{x}{l}\end{pmatrix}C_3k_IK_I\tilde\xi\\ \\ \tilde\xi\end{bmatrix} \tag{59}$$

where the matrix $C_3\ :\mathbb{R}^2\to\mathbb{R}^2$ is defined in (56). Applying the transformation $(R,\xi)^\tau=T_2(\tilde R,\tilde\xi)^\tau$ on the homogeneous equation (52)-(54) leads us to the following PDE :

$$\begin{aligned}
&\tilde R_t=(\Lambda(x)\partial_x+\tilde B)\tilde R+B_1(x)C_3k_IK_I\tilde\xi-C_4(x)C_3k_IK_I(C_1\tilde R+C_2k_IK_I\tilde\xi)\\
&\tilde\xi_t=C_1\tilde R+C_2k_IK_I\tilde\xi\\
&\tilde R_1(0,t)=0,\quad \tilde R_2(l,t)=0\\
&y=C_1\tilde R+C_2k_IK_I\tilde\xi
\end{aligned} \tag{60}$$

where

$$C_4(x)=\begin{pmatrix}\frac{l-x}{l} & 0\\ 0 & \frac{x}{l}\end{pmatrix},\quad B_1(x)=\Lambda(x)\begin{pmatrix}\frac{-1}{l} & 0\\ 0 & \frac{1}{l}\end{pmatrix}+\tilde B(x)C_4(x). \tag{61}$$

Now we define the unbounded operator $A_2\ :\mathcal{D}(A_2)\to H$ by

$$\mathcal{D}(A_2)=\left\{\begin{pmatrix}f_1\\ f_2\\ f_3\end{pmatrix}\in(H^1(0,l))^2\times\mathbb{R}^2\ \middle|\ \begin{pmatrix}f_1(0)\\ f_2(l)\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\right\} \tag{62}$$

and for every $f \in \mathcal{D}(A_2)$,

$$A_2 f = \left[\begin{array}{l} [\tilde{A} - C_4 C_3 k_I K_I C_1] \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} + [B_1 C_3 - C_4 C_3 k_I K_I C_2] k_I K_I f_3 \\ C_1 \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} + C_2 k_I K_I f_3 \end{array} \right] \tag{63}$$

which is exactly the right-hand member of (60) and where $\tilde{A}$ has been defined in (47)-(48). Obviously A_1 is the generator of an exponentially stable semigroup on H if and only if A_2 is. If the resolvent $R(\lambda, A_2) = (\lambda I - A_2)^{-1}$ is bounded in some right half plane $\Re e(\lambda) \geq -\alpha$ for some $\alpha > 0$, it follows from the result of Huang [8] or Prüss [16] that the semigroup e^{tA_2} is exponentially stable.

Remark 7. The operator A_2 is the generator of a C_0 semigroup on H and its domain $\mathcal{D}(A_2)$ is independent of K_I and k_I. However they appear in the perturbation term.

Let $\alpha > 0$ and let $\mathbb{C}^+_{-\alpha}$ denote the closed right half plane : $\mathbb{C}^+_{-\alpha} = \{\lambda \in \mathbb{C} \mid \Re e(\lambda) \geq -\alpha\}$. The complementary set of $\mathbb{C}^+_{-\alpha}$ is the open left half plane noted as $\mathbb{C}^-_{-\alpha} = \{\lambda \in \mathbb{C} \mid \Re e(\lambda) < -\alpha\}$. We consider also the closed set $\Omega_{-\alpha}$ defined in $\mathbb{C}$ by (see Figure 1) :

$$\Omega_{-\alpha} = \{\lambda \mid \Re e(\lambda) \geq -\alpha, |\lambda| \geq \alpha\}. \tag{64}$$

Define the unbounded operator $A_3 \ : \mathcal{D}(A_2) \to H$ by

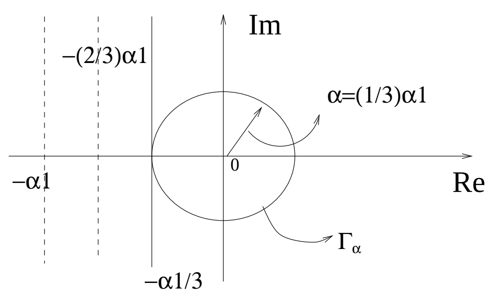

FIGURE 1. Separation of the spectrums

$$A_3 f(x) = \begin{bmatrix} \tilde{A} & 0 \\ C_1 & 0 \end{bmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}, \quad \forall\ f \in \mathcal{D}(A_2). \tag{65}$$

Then the operator A_2 can be written as follows

$$A_2 = A_3 + P \tag{66}$$

where the perturbation term P is given by

$$P = \begin{bmatrix} -C_4 C_3 k_I K_I C_1 & [B_1 C_3 - C_4 C_3 k_I K_I C_2] k_I K_I \\ 0 & C_2 k_I K_I \end{bmatrix}. \tag{67}$$

The resolvent $R(\lambda, A_2)$ is written by

$$R(\lambda, A_2) = R(\lambda, A_3) \sum_{n=0}^{\infty} (P R(\lambda, A_3))^n \tag{68}$$

where $R(\lambda, A_3)$ is given by

$$R(\lambda, A_3) = \left[\begin{array}{cc} R(\lambda, \tilde{A}) & 0 \\ \lambda^{-1} C_1 R(\lambda, \tilde{A}) & \lambda^{-1} \end{array} \right]. \tag{69}$$

Recall that $\tilde{A}$ and C_1 are defined in (47)-(48) and (55), respectively. By referring to Russell and Weiss [19] and our Theorem 5.2 the following result can be proved by direct computation.

Lemma 6.4. (i) *The semigroup $e^{t\tilde{A}}$ is exponentially stable on X.*
(ii) *$(C_1, \tilde{A})$ is admissible and there exist some positive constants α and M such that*

$$\|C_1 R(\lambda, \tilde{A})\| \le \frac{M}{\sqrt{\Re e(\lambda) + \alpha}}, \quad \forall\, \Re e(\lambda) + \alpha > 0.$$

(iii) *There are some positive constants α and k_I^* such that $\|R(\lambda, A_2)\|$ is bounded on the closed set $\Omega_{-\alpha}$ uniformly w.r.t. $k_I \in (0, k_I^*)$.*

In the following we keep using the same notation as in Lemma 6.4 and take the constant k_I^* smaller if necessary. Let D_α denote the open disc $D_\alpha = \{\lambda \in \mathbb{C} \mid |\lambda| < \alpha\}$. The following lemma may be proved similarly as in [14, 15] or [24]. To be complete we give a simpler proof in the Appendix.

Lemma 6.5. (i) *With the notation of Lemma 6.4, the spectrum $\sigma(A_2) \subset \mathbb{C}^-_{-\alpha} \cup D_\alpha$ $\forall\, k_I \in (0, k_I^*)$. Moreover the subset $\sigma(A_2) \cap D_\alpha$ is equal to two eigenvalues of A_2.*
(ii) *The two eigenvalues of A_2 in the disc D_α are located in the left half part, i.e. $\sigma(A_2) \cap D_\alpha \subset \mathbb{C}^-_{-k_I/2}$ $\forall\, k_I \in (0, k_I^*)$.*

The state space being an Hilbert space, our Theorem 6.3 is easily proved by using Lemmas 6.4-6.5 and a result of Huang [8].

Proof of Theorem 6.3 . Since $\|R(\lambda, A_2)\|$ is continuous on $\rho(A_2)$, by Lemma 6.5-(ii) it is bounded on the right half disc $D_\alpha \cap \mathbb{C}^+_{-k_I/2}$. This fact combined with Lemma 6.4-(iii) implies that $\|R(\lambda, A_2)\|$ is bounded on the right half plane $\mathbb{C}^+_{-k_I/2}$. The theorem of Huang [8] (see also Prüss [16]) tells us that the semigroup e^{tA_2} is exponentially stable.

Since the system is regular, stabilization implies regulation by Proposition 2. □

Remark 8. We have introduced y^r to illustrate the regulation effect. However, for the designed PI controllers to be useful for the nonlinear system (1), it is more realistic to take $y^r = 0$.

7. Conclusions.

In the paper we have studied dynamical behavior of a fluid flow system governed by the Saint Venant equation and local stability of its subcritical equilibrium states. As the friction slope is taken into account, the considered equilibrium states are spatially heterogeneous. We have proposed different conditions allowing to guarantee exponential stability of the linearized models according to friction slope. We have shown that the linearized Saint Venant system is exponentially stable at least for small friction slope. What is more interesting to note is that exponential stability might be proved by our approach even for big friction slope facing to concrete models. Systematic synthesis of stabilizing PI controllers has been worked out based on the linearized models. Since both the control and the observation are boundary, a spectral analysis approach has been used to prove exponential stability of the closed-loop system resulted from the designed PI controllers. It is our future work to study the closed-loop stability by the PI controllers from the nonlinear models.

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Appendix A. Proof of Proposition 2. The closed-loop system (7) is still regular with $(\mathbb{A}, \mathbb{B}, \mathbb{C}, \mathbb{D})$ as follows

$$\mathbb{A} = \begin{bmatrix} A & BK_I \\ C_\Lambda & DK_I \end{bmatrix}, \ \mathbb{B} = \begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix}, \ \mathbb{C} = \begin{bmatrix} C_\Lambda & DK_I \end{bmatrix}, \ \mathbb{D} = \begin{bmatrix} 0 & I \end{bmatrix}.$$

Let $\tilde{w}_c$, $\tilde{w}_o \in \mathbb{R}^m$ such that $\epsilon_c(t) = (w_c(t) - \tilde{w}_c)$ and $\epsilon_o(t) = (w_o(t) - \tilde{w}_o)$ belong to $L^2(\mathbb{R}^+, \mathbb{R}^m)$. The unique solution of the system (7) is written as

$$\begin{bmatrix} \phi(t) \\ \xi(t) \end{bmatrix} = e^{t\mathbb{A}} \left(\begin{bmatrix} \phi_0 \\ \xi_0 \end{bmatrix} + \mathbb{A}^{-1} \begin{bmatrix} B\tilde{w}_c \\ \tilde{w}_0 - y^r \end{bmatrix} \right) + \int_0^t e^{(t-\tau)\mathbb{A}} \mathbb{B} \begin{bmatrix} \epsilon_c(\tau) \\ \epsilon_o(\tau) \end{bmatrix} d\tau + s_\infty$$

where

$$s_\infty = -\mathbb{A}^{-1} \begin{bmatrix} B\tilde{w}_c \\ \tilde{w}_0 - y^r \end{bmatrix}.$$

As $\mathbb{A}$ is the generator of an exponentially stable semigroup on $H = X \times \mathbb{R}^m$, by Proposition 1-(ii), the state $(\phi(t), \xi(t))^\tau$ converges to s_∞ as time goes to infinity. By Proposition 1-(i) and (iii), the output $y(t)$ is quadratically close to constant, that is

$$\lim_{t\to\infty} \int_t^\infty \|y(\tau) - (\mathbb{C}s_\infty + \tilde{w}_0)\|^2 d\tau = 0.$$

We have only to prove that $y^r = \mathbb{C}s_\infty + \tilde{w}_0$.

Indeed, set

$$x = -\mathbb{A}^{-1} \begin{bmatrix} B\tilde{w}_c \\ \tilde{w}_0 - y^r \end{bmatrix} = s_\infty$$

where $x = (x_1, x_2)$ with $x_1 \in X$ and $x_2 \in \mathbb{R}^m$. Direct algebraic computation gives us

$$x_1 = -A^{-1}BG_0^{-1}(y^r - \tilde{w}_o + D\tilde{w}_c), \quad K_I x_2 = G_0^{-1}(C_\Lambda A^{-1}B\tilde{w}_c + y^r - \tilde{w}_0).$$

One may check easily that $C_\Lambda x_1 + DK_I x_2 + \tilde{w}_o = y^r$. $\square$

Appendix B. Proof of Lemma 6.5. (i) By Lemma 6.4-(iii), $\Omega_{-\alpha} \subset \rho(A_2)$, which implies that $\sigma(A_2) \subset \mathbb{C}^-_{-\alpha} \cup D_\alpha$. The resolvent $R(\lambda, A_3)$ being compact, the bounded set $\sigma(A_3) \cap D_\alpha$ is constituted of eigenvalues only and $\sigma(A_3) = \sigma(\tilde{A}) \cup \{0\}$. Denote by Γ_α the circle centered at zero of radius α (see Figure 1). The algebraic multiplicity of $\lambda = 0$ is equal to the dimension of the projector P_0

$$P_0 = \frac{1}{2\pi i} \oint_{\Gamma_\alpha} R(\lambda, A_3) d\lambda = \begin{pmatrix} 0 & 0 \\ -C_1\tilde{A}^{-1} & I \end{pmatrix}. \tag{70}$$

Obviously Dim(Range(P_0))=2 and so the algebraic multiplicity of $\lambda = 0$ is equal to 2.

It is easy to find that $PR(\lambda, A_3)$ is given by

$$PR(\lambda, A_3) = k_I \begin{pmatrix} [-C_4C_3 + \lambda^{-1}C_5]K_IC_1R(\lambda, \tilde{A}) & \lambda^{-1}C_5K_I \\ \lambda^{-1}C_2K_IC_1R(\lambda, \tilde{A}) & \lambda^{-1}C_2K_I \end{pmatrix}, \tag{71}$$

where $C_5 = B_1C_3 - C_4C_3k_IK_IC_2$ and B_1, C_3, K_I and C_2 are defined in (61), (56), (42) and (55), respectively. Denote by P_λ the operator defined by the matrix in (71). By Lemma 6.4-(ii), we have

$$\sup_{\lambda\in\Omega_{-\alpha}} \|k_IP_\lambda\| < 1, \quad \forall\, 0 < k_I < k_I^*$$

Thus the following sum

$$R(\lambda, A_2) = R(\lambda, A_3)\sum_{n=0}^{+\infty} k_I^n P_\lambda^n \tag{72}$$

converges normally on $\Omega_{-\alpha}$. Similarly $\sigma(A_2) \cap D_\alpha$ contains only eigenvalues of A_2. More precisely it contains two eigenvalues only. Indeed the dimension of the eigen-space encircled by Γ_α is still equal to two. Consider the projector P_1

$$P_1 = \frac{1}{2\pi i}\oint_{\Gamma_\alpha} R(\lambda, A_2)d\lambda = P_0 + o(k_I), \tag{73}$$

From (70) and (73), we have $\|P_0 - P_1\|_{\mathcal{L}(H)} < 1$ for every k_I small enough, which implies that the two projectors have the same dimension (cf. [9, p.34]). The assertion (i) is proved.

It is sufficient to assign by means of K_I the two eigenvalues λ_1 and λ_2 in D_α to the left half part. From (72),

$$A_2P_1 = \frac{1}{2\pi i}\oint_{\Gamma_\alpha} \lambda R(\lambda, A_2)d\lambda = k_I\begin{pmatrix} 0 & 0 \\ -C_6K_IC_1\tilde{A}^{-1} & C_6K_I \end{pmatrix} + o(k_I^2),$$

where $C_6 = (C_2 - C_1\tilde{A}^{-1}B_1C_3)$. Let $M' = P_1H$ and let $A_2^{M'}$ denote the part of A_2 in M'. From Theorem 6.17 in [9, p.178], we have

$$\sigma(A_2^{M'}) = \{\lambda_1, \lambda_2\}.$$

For small k_I a basis of M' is formed by $\left\{P_1\begin{bmatrix}0\\1\\0\end{bmatrix}, P_1\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$. With respect to the basis is the matrix representation of $A_2^{M'}$ given by

$$A_2^{M'} = k_IC_6K_I + O(k_I^2). \tag{74}$$

We claim that the following identity holds true

$$K_I = -C_6^{-1}. \tag{75}$$

By the claim and (74), one finds some $k_I^* > 0$ such that

$$\max\{\Re e(\lambda_1), \Re e(\lambda_2)\} = -k_I + O(k_I^2) < 0, \quad \forall\, k_I \in (0, k_I^*).$$

The rest of the spectrum $\sigma(A_2)$ satisfies $\Re e(\lambda) \le -\alpha$ for $\lambda \ne \lambda_1$ or λ_2. So we prove that

$$\sup\{\Re e(\lambda) \mid \lambda \in \sigma(A_2)\} < -k_I/2.$$

To finish we prove the claim. Notice that the matrix C_6 is the mapping $v \mapsto y$ defined by the following ordinary differential equation

$$\begin{aligned} &0 = \tilde{A}\tilde{R} + B_1C_3v \\ &\tilde{R}_1(0) = \tilde{R}_2(l) = 0 \\ &y = C_1\tilde{R} + C_2v. \end{aligned}$$

By the successive transformations $R = T_2^{-1}\tilde{R}$ and $(z,q)^\tau = T_1 R$, we recover the differential equation (44). The mapping being invariant with respect to the transformations we have $C_6 = G_0$. □

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