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| Christophe Cornut. Filtrations and Buildings. 2014. hal-01085000v2

HAL Id: hal-01085000

<https://hal.science/hal-01085000v2>

Preprint submitted on 9 Mar 2015 (v2), last revised 22 Oct 2019 (v4)

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FILTRATIONS AND BUILDINGS

CHRISTOPHE CORNUT

En hommage à Alexander Grothendieck

ABSTRACT. We construct and study a scheme theoretical version of the Tits vectorial building, relate it to filtrations on fiber functors, and use them to clarify various constructions pertaining to affine Bruhat-Tits buildings, for which we also provide a Tannakian description.

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Date: November 13, 2014, v1.1.

2000 Mathematics Subject Classification. 14L15, 18D10, 20E42, 20G05, 51E24.

Key words and phrases. Filtrations, Buildings and the Tannakian formalism.

1. INTRODUCTION

The *combinatorial* Tits building of a reductive group G over a field K reflects the incidence relations between the parabolic subgroups of G . Its geometric realization, the *spherical* Tits building, is obtained by gluing spheres along common sectors. It has an action of $G(K)$ and can be equipped with a non-canonical $G(K)$ -invariant metric, which turns it into a $CAT(1)$ -space. When K is a local field, the spherical building can also be realized as the visual boundary of an affine building attached to G , namely its symmetric space or Bruhat-Tits building, depending upon whether K is archimedean or not (in the non-archimedean case, the spherical buildings of various reductive groups over the residue field also show up in the local description of the Bruhat-Tits building). In both cases, the affine building itself has a $G(K)$ -action and a non-canonical $G(K)$ -invariant metric for which it is a $CAT(0)$ -space, and the cone of its visual boundary acts transitively on the affine building by non-expanding maps [10]. Looking at things the other way around, the choice of a base point in the affine building realizes it as a quotient of the *vectorial* Tits building, the latter being the cone of the spherical Tits building.

This vectorial Tits building is the unifying theme of our somewhat eclectic paper, whose initial intention was to clarify and canonify the above constructions. It is yet another affine building with an action of $G(K)$ (which can now be defined over any field K) and it is equipped with a $CAT(0)$ -metric canonically attached to any choice of a faithful representation $\tau : G \hookrightarrow GL(V)$, see section 4.2.3.

In section 2, we actually start with a reductive group G over an arbitrary base scheme S . For a totally ordered commutative group $\Gamma = (\Gamma, +, \leq)$, we define our fundamental G -equivariant cartesian diagram of S -schemes

$$\begin{array}{ccccc} \mathbb{G}^\Gamma(G) & \xrightarrow{\text{Fil}} & \mathbb{F}^\Gamma(G) & \xrightarrow{t} & \mathbb{C}^\Gamma(G) \\ F \downarrow & & F \downarrow & & F \downarrow \\ \mathbb{O}PP(G) & \xrightarrow{p_1} & \mathbb{P}(G) & \xrightarrow{t} & \mathbb{O}(G) \end{array}$$

where $\mathbb{P}(G)$ and $\mathbb{O}PP(G)$ are respectively the S -schemes of parabolic subgroups P of G and pairs of opposed parabolic subgroups (P, P') of G , $\mathbb{O}(G)$ is the S -scheme of G -orbits in $\mathbb{P}(G)$ or $\mathbb{O}PP(G)$, and $\mathbb{G}^\Gamma(G) = \underline{\text{Hom}}(\mathbb{D}_S(\Gamma), G)$ where $\mathbb{D}_S(\Gamma)$ is the (diagonalizable) multiplicative group over S with character group Γ . For $\Gamma = \mathbb{Z}$, $\mathbb{D}_S(\Gamma) = \mathbb{G}_{m,S}$ and $\mathbb{G}^\Gamma(G)$ is merely the scheme of cocharacters of G . However, we do not require Γ to be finitely generated over \mathbb{Z} , and we are in fact particularly interested by the cases where $\Gamma = \mathbb{Q}$ or \mathbb{R} . In the above diagram, the *facet* morphisms F are surjective and locally constant in the étale topology on their base. The p_1 and Fil morphisms are affine smooth surjective with geometrically connected fibers and the *type* morphisms t are projective smooth surjective with geometrically connected fibers. Since $\mathbb{O}(G)$ is finite étale over S , all of the above schemes are smooth, separated and surjective over S . We equip $\mathbb{C}^\Gamma(G)$ and $\mathbb{O}(G)$ with S -monoid structures, and the facet map $F : \mathbb{C}^\Gamma(G) \rightarrow \mathbb{O}(G)$ is compatible with them. For $\Gamma = \mathbb{R}$, $\mathbb{F}^\Gamma(G)$ is a scheme theoretical version of the Tits vectorial building and $\mathbb{C}^\Gamma(G)$ is a scheme theoretical version of a closed Weyl chamber. For $\Gamma = \mathbb{Z}$, the S -scheme $\mathbb{C}^\Gamma(G)$ classifies the G -orbits of cocharacters of G .

In section 3, we show that $\mathbb{G}^\Gamma(G)$ and $\mathbb{F}^\Gamma(G)$ represent functors respectively related to Γ -graduations and Γ -filtrations on a variety of fiber functors. The main

difficulty here is to show that the Γ -filtrations split fpqc-locally on the base scheme. For $\Gamma = \mathbb{Z}$, this was essentially established in the thesis of Saavedra Rivano [30], at least when S is the spectrum of a field. We strictly follow Saavedra's proof (which he attributes to Deligne), adding a considerable amount of details and some patch when needed. We advise our reader to read both texts side by side, only switching to ours when he feels uncomfortable with (the necessary generalizations of) Saavedra's arguments.

For $\Gamma = \mathbb{Z}$, Ziegler recently established the fpqc-splitting of \mathbb{Z} -filtrations on fiber functors on arbitrary Tannakian categories [38], thereby proving a conjecture which was left open after Saavedra's thesis. In particular, the \mathbb{Z} -filtrations we consider have fpqc-splittings even when G is not reductive, but defined over a field. In the reductive case, the final arguments in Ziegler's proof simplify those of Saavedra's, but rely more on the Saavedra-Deligne theorem that all fiber functors on Tannakian categories are fpqc-locally isomorphic [11]. According to D. Schäppi, it follows from his own work [31, 32] and Lurie's note on Tannaka duality that the same result holds for any \otimes -functor $\mathrm{Rep}^{fp}(G)(S) \rightarrow \mathrm{QCoh}(T)$ where: S is affine, T is an S -scheme, G is affine flat over S , $\mathrm{Rep}^{fp}(G)(S)$ is the \otimes -category of algebraic representations of G on finitely presented \mathcal{O}_S -modules, and G has the resolution property: any finitely presented algebraic representation of G is covered by another one which is locally free. It then seems likely that Ziegler's proof could yield a common generalization of his result ($\Gamma = \mathbb{Z}$, G affine over a field) and ours (Γ and S arbitrary, but G reductive) on the existence of fpqc-splittings of Γ -filtrations, using a hefty dose of the stack formalism. We have chosen to stick to the constructive, down-to-earth original proof of Saavedra/Deligne – and to reductive groups as well.

In section 4, we study the sections of our schemes over a local ring \mathcal{O} . We first equip $\mathbf{F}^\Gamma(G) = \mathbb{F}^\Gamma(G)(\mathcal{O})$ with a collection of *apartments* $\mathbf{F}^\Gamma(S)$ indexed by the maximal split subtorus S of G and with the collection of *facets* $F^{-1}(P)$ indexed by the parabolic subgroups P of G . The key properties of the resulting combinatorial structure are well-known when \mathcal{O} is a field and $\Gamma = \mathbb{R}$, in which case $\mathbf{F}^\Gamma(G)$ is the Tits vectorial building, but most of them carry over to this more general situation, thanks to the wonderful last chapter of SGA3. We describe the behavior of these auxiliary structures under specialization (when \mathcal{O} is Henselian) or generalization (when \mathcal{O} is a valuation ring). When Γ is a subring of \mathbb{R} , we also attach to every finite free faithful representation τ of G a partially defined *scalar product* on $\mathbf{F}^\Gamma(G)$ and the corresponding *distance* and *angle* functions, and we study their basic properties. When \mathcal{O} is a field, a theorem of Borel and Tits [7] implies that these functions are defined everywhere, and one thus retrieves the aforementioned non-canonical distances on the vectorial Tits building $\mathbf{F}(G) = \mathbf{F}^{\mathbb{R}}(G)$.

Over a field K and with $\Gamma = \mathbb{R}$, we next define a notion of *affine $\mathbf{F}(G)$ -spaces*, which interact with the vectorial Tits building $\mathbf{F}(G)$ as affine spaces do with their underlying vector space. Strongly influenced by the formalism set up by Rousseau in [29] and Parreau in [26], we introduce various axioms that these spaces may satisfy, leading to the more restricted class of *affine $\mathbf{F}(G)$ -buildings*. Most of the abstract definitions of buildings that have already been proposed involve a covering atlas of charts, which are bijections from a given fixed affine space onto subsets of the building (its apartments) subject to various conditions. Our definition also involves a covering by apartments, but their affine structure is inherited from a globally defined $G(K)$ -equivariant transitive operation $x \mapsto x + \mathcal{F}$ of $\mathbf{F}(G)$ on the

building. It is therefore essentially a boundary-based formalism for buildings, as opposed to the more usual apartment-based formalism.

Of course $\mathbf{F}(G)$ is itself an affine $\mathbf{F}(G)$ -building, with a distinguished point. When K is equipped with a non-trivial, non-archimedean absolute value, we show that the (extended) affine building $\mathbf{B}^e(G)$ constructed by Bruhat and Tits [7, 8] is canonically equipped with a structure of affine $\mathbf{F}(G)$ -building in our sense. This is our precise formalization of the previous all too vague assertion that the visual boundary of the Bruhat-Tits building is a geometric realization of the combinatorial Tits building. This being done, we may now fix a base point \circ in the Bruhat-Tits building $\mathbf{B}^e(G)$, and try to describe $\mathbf{B}^e(G)$ as a quotient of $\mathbf{F}(G)$ using the surjective map $\mathbf{F}(G) \ni \mathcal{F} \mapsto \circ + \mathcal{F} \in \mathbf{B}^e(G)$. We do just this in the last subsection, assuming that our base point \circ is hyperspecial, i.e. corresponds to a reductive group G over the valuation ring \mathcal{O} of K , which we also assume to be Henselian.

More precisely, we first define a space of K -norms on the fiber functor

$$\omega_G^\circ : \text{Rep}^\circ(G)(\mathcal{O}) \rightarrow \text{Vect}(K)$$

where $\text{Rep}^\circ(G)(\mathcal{O})$ is the category of algebraic representations of G on finite free \mathcal{O} -modules. This space is equipped with a $G(K)$ -action, an explicit $G(K)$ -equivariant operation of $\mathbf{F}(G_K)$ and a base point α_G fixed by $G(\mathcal{O})$. We show that the map $\circ + \mathcal{F} \mapsto \alpha_G + \mathcal{F}$ is well-defined, injective, $G(K)$ -equivariant and compatible with the operations of $\mathbf{F}(G_K)$. It thus defines an isomorphism α of affine $\mathbf{F}(G_K)$ -buildings from $\mathbf{B}^e(G_K)$ to a set $\mathbf{B}(\omega_G^\circ, K) = \alpha_G + \mathbf{F}(G_K)$ of K -norms on ω_G° .

This tannakian description of the extended Bruhat-Tits building immediately implies that the assignment $G \mapsto \mathbf{B}^e(G_K)$ is functorial in the reductive group G over \mathcal{O} . Such a functoriality was already established by Landvogt [21], with fewer assumptions on G_K but more assumptions on K . It also suggests a possible definition of Bruhat-Tits buildings for reductive groups over valuation rings of height greater than 1, as well as a similar tannakian description of symmetric spaces (in the archimedean case). It is related to previous constructions as follows.

Our canonical isomorphism $\alpha : \mathbf{B}^e(G_K) \rightarrow \mathbf{B}(\omega_G^\circ, K)$ assigns to a point x in $\mathbf{B}^e(G_K)$ and an algebraic representation τ of G on a flat \mathcal{O} -module $V(\tau)$ a K -norm $\alpha(x)(\tau)$ on $V_K(\tau) = V(\tau) \otimes K$. For the adjoint representation τ_{ad} of G on $\mathfrak{g} = \text{Lie}(G)$, the adjoint-regular and regular representations ρ_{adj} and ρ_{reg} of G on $\mathcal{A}(G) = \Gamma(G, \mathcal{O}_G)$, we obtain respectively: a K -norm $\alpha_{\text{ad}}(x)$ on $\mathfrak{g}_K = \text{Lie}(G_K)$ whose closed balls give the Moy-Prasad filtration of x on \mathfrak{g}_K [24], the K -norm $\alpha_{\text{adj}}(x)$ in G_K^{an} constructed in [27], and an embedding $x \mapsto \alpha_{\text{reg}}(x)$ of the extended Bruhat-Tits building in the analytic Berkovich space G_K^{an} attached to G_K .

A different tannakian formalism for Bruhat-Tits buildings had already been proposed by Haines and Wilson [37], with alcoves and their parahorics playing the leading role. It is closely related to ours: their Moy-Prasad filtrations are the lattice chains of closed balls of our norms. We owe them the essential shape of our formalism, if not the very idea that such a formalism was indeed possible: we were first naively looking for a base-point free description of the Bruhat-Tits buildings.

This paper grew out of a question by J-F. Dat and numerous discussions with D. Mauger on buildings and cocharacters. I am very grateful to G. Rousseau and A. Parreau, who always had answers to my questions. Apart from the emphasis on the boundary, most of the definitions and results of section 4.3 are either taken from his survey [29] or from her preprint [26]. P. Deligne kindly provided the patch

at the very end of the proof of the splitting theorem, dealing with groups of type G_2 in characteristic 2.

2. THE GROUP THEORETICAL FORMALISM

2.1. Γ -graduations on smooth affine groups.

Theorem 1. *Let H and G be group schemes over a base scheme S , with H of multiplicative type and quasi-isotrivial, G smooth and affine. Then the functor*

$$\underline{\mathrm{Hom}}_{S-\mathrm{Gr}}(H, G) : (\mathrm{Sch}/S)^\circ \rightarrow \mathrm{Set}, \quad T \mapsto \mathrm{Hom}_{T-\mathrm{Gr}}(H_T, G_T)$$

is representable by a smooth and separated scheme over S .

Remark 2. When H is of finite type, it is quasi-isotrivial by [1, X 4.5]. The theorem is then due to Grothendieck, see [1, XI 4.2]. The proof given there relies on the density theorem of [1, IX 4.7], definitely a special feature of finite type multiplicative groups. When H is trivial, we may still reduce the proof of the above theorem to the finite type case, see remark 12 below. For the general case, we have to find another road through SGA3, passing through [1, X 5.6] which has no finite type assumption on H but requires H and G to be of multiplicative type and quasi-isotrivial:

Proposition 3. *Let H and G be group schemes of multiplicative type over S , with H quasi-isotrivial and G of finite type. Then $\underline{\mathrm{Hom}}_{S-\mathrm{Gr}}(H, G)$ is representable by a quasi-isotrivial twisted constant group scheme X over S .*

Proof. This is [1, X 5.6], since G is also quasi-isotrivial by [1, X 4.5]. \square

Lemma 4. *Let X be a quasi-isotrivial twisted constant scheme over S . Then $X \rightarrow S$ is separated étale and satisfies the valuative criterion of properness. If moreover S is irreducible and geometrically unibranch with generic point η , then*

$$X = \coprod_{\lambda \in X_\eta} X(\lambda) \quad \text{with} \quad X(\lambda) = \overline{\{\lambda\}} \text{ open and closed in } X,$$

each $X(\lambda)$ is a connected finite étale cover of S and $\Gamma(X/S) = \Gamma(X_\eta/\eta)$.

Proof. The morphism $X \rightarrow S$ is separated by [17, 2.7.1] and étale by [19, 17.7.3]. Since valuation rings are normal integral domains, it remains to establish the last claims. Suppose therefore that S is irreducible and geometrically unibranch with generic point η . Then by [19, 18.10.7] applied to $X \rightarrow S$,

$$X = \coprod_{\lambda \in X_\eta} X(\lambda) \quad \text{with} \quad X(\lambda) = \overline{\{\lambda\}} \text{ open and closed in } X,$$

thus $X(\lambda)$ is already étale over S . Fix an étale covering $\{S_i \rightarrow S\}$ trivializing X , so that $X \times_S S_i = Q_{i,S_i}$ for some set Q_i . Using [19, 18.10.7] again, we may assume that each S_i is connected, in which case we obtain decompositions

$$Q_i = \coprod_{\lambda \in X_\eta} Q_i(\lambda) \quad \text{with} \quad X(\lambda) \times_S S_i = Q_i(\lambda)_{S_i}.$$

Since the generic fiber $\lambda \rightarrow \eta$ of $X(\lambda) \rightarrow S$ is finite of degree $n(\lambda) = [k(\lambda) : k(\eta)]$, each $Q_i(\lambda)$ is a finite subset of Q_i of order $n(\lambda)$, therefore $X(\lambda) \times_S S_i$ is finite over S_i and $X(\lambda)$ is finite over S by [17, 2.7.1]. Being finite and étale over the connected S , $X(\lambda)$ is a finite étale cover of S . Being irreducible, it is also connected. By [19, 17.4.9], the map which sends a section g of $X \rightarrow S$ to its image $g(S)$ identifies $\Gamma(X/S)$ with the set of connected components $X(\lambda)$ of X for which $X(\lambda) \rightarrow S$ is an isomorphism, i.e. such that $n(\lambda) = 1$. Therefore $\Gamma(X/S) = \Gamma(X_\eta/\eta)$. \square

Lemma 5. *Let $f : H \rightarrow G$ be a morphism of group schemes over S , with H of multiplicative type and G separated of finite presentation. Then there is a unique closed multiplicative subgroup Q of G such that f factors through a faithfully flat morphism $f' : H \rightarrow Q$. Moreover f' is also uniquely determined by f .*

Proof. Everything being local for the fpqc topology, we may assume that S is affine and $H = \mathbb{D}_S(M)$ for some abstract commutative group M . Then $M = \varinjlim M'$ where M' runs through the filtered set $\mathcal{F}(M)$ of finitely generated subgroups of M , thus also $\mathbb{D}_S(M) = \varprojlim \mathbb{D}_S(M')$. Since $\mathbb{D}_S(M')$ is affine for all M' and $G \rightarrow S$ is locally of finite presentation, it follows from [18, 8.13.1] that f factors through $f_1 : \mathbb{D}_S(M') \rightarrow G$ for some $M' \in \mathcal{F}(M)$. Applying [1, IX 6.8] to f_1 yields a closed multiplicative subgroup Q of G such that f_1 factors through a faithfully flat (and affine) morphism $f'_1 : \mathbb{D}_S(M') \rightarrow Q$, whose composite with the faithfully flat (and affine) morphism $\mathbb{D}_S(M) \rightarrow \mathbb{D}_S(M')$ is the desired factorization. Since Q is then also the image of f in the category of fpqc sheaves on Sch/S , it is already unique as a subsheaf of G . Since $Q \rightarrow G$ is a monomorphism, also f' is unique. \square

Definition 6. We call Q the image of f and denote it by $Q = \text{im}(f)$.

Lemma 7. *Let $f : H \rightarrow G$ be a morphism of group schemes over S , with H of multiplicative type and G smooth and affine. Then the centralizer of f is equal to the centralizer of its image, and is representable by a closed smooth subgroup of G .*

Proof. Let $f = \iota \circ f'$ be the factorization of the previous lemma. Since f' is faithfully flat (and quasi-compact, being a morphism between affine S -schemes, therefore even affine), it is an epimorphism in the category of schemes. It then follows from the definitions in [13, VIB §6] that the centralizers of f , ι and $\text{im}(f)$ are equal. By [1, XI 5.3], the centralizer of ι is a closed smooth subgroup of G . \square

Lemma 8. *Let $f : H \rightarrow Q$ be a morphism of group schemes of multiplicative type over S , with Q of finite type. Define $U = \{s \in S : f_s \text{ is faithfully flat}\}$. Then U is open and closed and $f_U : H_U \rightarrow Q_U$ is faithfully flat.*

Proof. Let I be the image of f . Then U is the set of points $s \in S$ where $I_s = Q_s$. Now apply [1, IX 2.9] to $I \hookrightarrow Q$. \square

We may now prove Theorem 1. Define presheaves A, B, C on Sch/S by

$$\begin{aligned} C(S') &= \{\text{multiplicative subgroups } Q \text{ of } G_{S'}\}, \\ B(S') &= \{(Q, f') : Q \in C(S') \text{ and } f' : H_{S'} \rightarrow Q \text{ is a morphism}\}, \\ A(S') &= \{(Q, f') \in B(S') \text{ with } f' \text{ faithfully flat}\}. \end{aligned}$$

Then C is representable, smooth and separated by [1, XI 4.1], $B \rightarrow C$ is relatively representable by étale and separated morphisms by Proposition 3 and Lemma 4, $A \rightarrow B$ is relatively representable by open and closed immersions by Lemma 8 and finally A is isomorphic to $\underline{\text{Hom}}_{S-\text{Gr}}(H, G)$ by Lemma 5. Therefore $\underline{\text{Hom}}_{S-\text{Gr}}(H, G)$ is indeed representable by a smooth and separated scheme over S .

Definition 9. For an abstract commutative group $\Gamma = (\Gamma, +)$ and a smooth and affine group scheme G over S , we set $\mathbb{G}^\Gamma(G) = \underline{\text{Hom}}_{S-\text{Gr}}(\mathbb{D}_S(\Gamma), G)$. Therefore

$$\mathbb{G}^\Gamma(G) : (\text{Sch}/S)^\circ \rightarrow \text{Set}$$

is representable by a smooth and separated scheme over S .

Proposition 10. *Let $f : \mathbb{D}_S(\Gamma) \rightarrow G$ be a morphism of group schemes over S , with G separated and of finite presentation. Then for each s in S ,*

$$\Gamma(s) = \{\gamma \in \Gamma : \gamma \equiv 1 \text{ on } \ker(f_s)\}$$

belongs to the set $\mathcal{F}(\Gamma)$ of finitely generated subgroups of Γ . For each $\Lambda \in \mathcal{F}(\Gamma)$,

$$S(\Lambda) = \{s \in S : \Gamma(s) = \Lambda\}$$

is open and closed in S , and finally

$$\ker(f)_{S(\Lambda)} = \mathbb{D}_{S(\Lambda)}(\Gamma/\Lambda) \quad \text{and} \quad \text{im}(f)_{S(\Lambda)} = \mathbb{D}_{S(\Lambda)}(\Lambda).$$

Proof. We may assume that S is affine and G is of multiplicative type (using Lemma 5 for the latter). Since $\mathbb{D}_S(\Gamma) = \varprojlim \mathbb{D}_S(\Lambda)$ it follows again from [18, 8.13.1] that there is some Λ in $\mathcal{F}(\Gamma)$ such that f factors through $g : \mathbb{D}_S(\Lambda) \rightarrow G$, i.e. $\mathbb{D}_S(\Gamma/\Lambda) \subset \ker(f)$. But then $\Gamma(s) \subset \Lambda$ for every $s \in S$, which proves the first claim. Applying now [1, IX 2.11 (i)] to g gives a finite partition of S into open and closed subset S_i , together with a collection of distinct subgroups Λ_i of Λ such that $\ker(g)_{S_i} = \mathbb{D}_{S_i}(\Lambda/\Lambda_i)$ and $\text{im}(g)_{S_i} \simeq \mathbb{D}_{S_i}(\Lambda_i)$. But then $\ker(f)_{S_i} = \mathbb{D}_{S_i}(\Gamma/\Lambda_i)$, $\text{im}(f)_{S_i} \simeq \mathbb{D}_{S_i}(\Lambda_i)$ and $S_i = S(\Lambda_i)$, which proves the remaining claims. \square

Corollary 11. *If Γ is torsion free, then $\text{im}(f)$ is a locally trivial subtorus of G .*

Remark 12. The proposition suggests another proof of Theorem 1 for $H = \mathbb{D}_S(\Gamma)$. It shows indeed that the Zariski sheaf $\mathbb{G}^\Gamma(G)$ is the disjoint union of relatively open and closed subsheaves $\mathbb{G}^\Gamma(G)(\Lambda)$, indexed by $\Lambda \in \mathcal{F}(\Gamma)$. Moreover, $\mathbb{G}^\Gamma(G)(\Lambda)$ is isomorphic to the subsheaf $\mathbb{G}^\Lambda(G)(\Lambda)$ of $\mathbb{G}^\Lambda(G)$, which is representable by a smooth and separated scheme over S by [1, XI 4.2].

2.2. Γ -graduations and Γ -filtrations on reductive groups. Let S be a scheme, G a reductive group over S , $\mathfrak{g} = \text{Lie}(G)$ its Lie algebra. Let $\Gamma = (\Gamma, +, \leq)$ be a non-trivial totally ordered commutative group.

2.2.1. Recall from [12, XXVI 3.5] that the sheaf $\mathbb{P}(G) : (\text{Sch}/S)^\circ \rightarrow \text{Set}$ given by

$$\mathbb{P}(G)(T) = \{\text{parabolic subgroups } P \text{ of } G_T\}$$

is representable, smooth and projective over S , with Stein factorization given by

$$\mathbb{P}(G) \xrightarrow{t} \mathbb{O}(G) \rightarrow S$$

where $\mathbb{O}(G)$ is the S -scheme of open and closed subschemes of the Dynkin S -scheme $\mathbb{DYN}(G)$ of the reductive group G/S , see [12, XXIV 3.3]. Both $\mathbb{DYN}(G)$ and $\mathbb{O}(G)$ are twisted constant finite schemes over S , thus finite étale over S by [17, 2.7.1.xv] and [19, 17.7.3], and $\mathbb{O}(G)$ is actually a finite étale cover of S . The morphism t is smooth, projective, with non-empty geometrically connected fibers; it classifies the parabolic subgroups of G in the following sense: two parabolic subgroups P_1 and P_2 of G are conjugated locally in the fpqc topology on S if and only if $t(P_1) = t(P_2)$.

2.2.2. For a parabolic subgroup P of G with unipotent radical U , we denote by $\overline{R}(P)$ the radical of P/U [12, XXII 4.3.6]. For the universal parabolic subgroup P_u of $G_{\mathbb{P}(G)}$, we obtain a $\mathbb{P}(G)$ -torus $R_{\mathbb{P}(G)} = \overline{R}(P_u)$. We claim that it descends canonically to an $\mathbb{O}(G)$ -torus $R_{\mathbb{O}(G)}$ over $\mathbb{O}(G)$. Since t is faithfully flat and quasi-compact, it is a morphism of effective descent for affine group schemes by [2, VIII 2.1], thus also for tori by definition [1, IX 1.3]. Our claim now follows from:

Lemma 13. *There exists a canonical descent datum on $R_{\mathbb{P}(G)}$ with respect to t .*

Proof. We have to show that for any $T \rightarrow S$ and any pair of parabolic subgroups P_1 and P_2 of G_T such that $t(P_1) = t(P_2)$, there exists a canonical isomorphism $\overline{R}(P_1) \simeq \overline{R}(P_2)$. Let $M_i = P_i/U_i$ be the maximal reductive quotient of P_i , so that $R_i = \overline{R}(P_i)$ is the radical of M_i . We may assume that $T = S$ and, by a descent argument, that $P_2 = \text{Int}(g)(P_1)$ for some $g \in G(S)$. Then $\text{Int}(g)$ induces isomorphisms $P_1 \rightarrow P_2$, $M_1 \rightarrow M_2$ and $R_1 \rightarrow R_2$. Since g is well-defined up to right multiplication by an element of $P_1(S)$ thanks to [12, XXVI 1.2], $M_1 \rightarrow M_2$ is well-defined up to an inner automorphism of M_1 and $R_1 \rightarrow R_2$ does not depend upon any choice: this is our canonical isomorphism. \square

2.2.3. By [12, XXVI 4.3.4 and 4.3.5], the formula

$$\mathbb{O}\text{PP}(G)(T) = \{(P_1, P_2) \text{ pair of opposed parabolic subgroups of } G_T\}$$

defines an open subscheme $\mathbb{O}\text{PP}(G)$ of $\mathbb{P}(G)^2$ and the two projections

$$p_1, p_2 : \mathbb{O}\text{PP}(G) \rightarrow \mathbb{P}(G)$$

are isomorphic U_u -torsors, thus affine smooth surjective morphisms with geometrically connected fibers. Here U_u is the unipotent radical of the universal parabolic subgroup P_u of $G_{\mathbb{P}(G)}$, it acts by conjugation on the fibers, and the isomorphism is the involution $\iota(P_1, P_2) = (P_2, P_1)$ of the S -scheme $\mathbb{O}\text{PP}(G)$. We denote by $(P_u^1, P_u^2) = (p_1^*P_u, p_2^*P_u)$ the universal pair of opposed parabolic subgroups of $G_{\mathbb{O}\text{PP}(G)}$, by $U_u^i = p_i^*U_u$ the unipotent radical of P_u^i , and by $R_{\mathbb{O}\text{PP}(G)}$ the radical of the corresponding universal Levi subgroup $L_u = P_u^1 \cap P_u^2$ of $G_{\mathbb{O}\text{PP}(G)}$. Thus

$$L_u \simeq P_u^i/U_u^i \simeq p_i^*(P_u/U_u) \quad \text{and} \quad R_{\mathbb{O}\text{PP}(G)} \simeq p_i^*R_{\mathbb{P}(G)}.$$

2.2.4. The S -scheme $\mathbb{G}^\Gamma(R_{\mathbb{O}\text{PP}(G)})$ represents the functor which maps $T \rightarrow S$ to the set of all triples (P_1, P_2, f) where (P_1, P_2) is a pair of opposed parabolic subgroups of G_T with Levi subgroup $L = P_1 \cap P_2$, and $f : \mathbb{D}_T(\Gamma) \rightarrow L$ is a central morphism. The next proposition uses the total ordering on $\Gamma = (\Gamma, +, \leq)$ to define a section

$$\mathbb{G}^\Gamma(G) \rightarrow \mathbb{G}^\Gamma(R_{\mathbb{O}\text{PP}(G)}), \quad f \mapsto (P_f, P_{\iota f}, f)$$

of the obvious forgetful morphism of S -schemes

$$\mathbb{G}^\Gamma(R_{\mathbb{O}\text{PP}(G)}) \rightarrow \mathbb{G}^\Gamma(G), \quad (P_1, P_2, f) \mapsto f.$$

Proposition 14. *Let $f : \mathbb{D}_S(\Gamma) \rightarrow G$ be a morphism and write $\mathfrak{g} = \bigoplus_\gamma \mathfrak{g}_\gamma$ for the corresponding weight decomposition of $\text{ad} \circ f : \mathbb{D}_S(\Gamma) \rightarrow GL_S(\mathfrak{g})$. There exists a unique parabolic subgroup P_f of G containing the centralizer L_f of f such that*

$$\text{Lie}(P_f) = \bigoplus_{\gamma \geq 0} \mathfrak{g}_\gamma.$$

Moreover L_f is a Levi subgroup of P_f , thus $P_f = U_f \rtimes L_f$ where U_f is the unipotent radical of P_f . For $\iota f = f^{-1}$, $P_{\iota f}$ is opposed to P_f , $L_f = P_f \cap P_{\iota f}$ and

$$\begin{aligned} \text{Lie}(P_f) &= \bigoplus_{\gamma \geq 0} \mathfrak{g}_\gamma & \text{Lie}(P_{\iota f}) &= \bigoplus_{\gamma \leq 0} \mathfrak{g}_\gamma & \text{and} & \text{Lie}(L_f) = \mathfrak{g}_0. \\ \text{Lie}(U_f) &= \bigoplus_{\gamma > 0} \mathfrak{g}_\gamma & \text{Lie}(U_{\iota f}) &= \bigoplus_{\gamma < 0} \mathfrak{g}_\gamma \end{aligned}$$

Proof. Let Q be the image of f . Then L_f is the centralizer of Q by Lemma 7 and Q is a locally trivial subtorus of G by Proposition 10 (since Γ is torsion free). We may assume that Q is trivial, i.e. $Q \simeq \mathbb{D}_S(\Lambda)$ for some finitely generated subgroup Λ of Γ . The proposition then follows from [12, XXVI 6.1]. \square

Proposition 15. *The S -morphism $\mathbb{G}^\Gamma(G) \rightarrow \mathbb{G}^\Gamma(R_{\mathbb{O}\text{PP}(G)})$ is an open and closed immersion, and $\mathbb{G}^\Gamma(G) \rightarrow \mathbb{O}\text{PP}(G)$ is a quasi-isotrivial twisted constant morphism.*

Proof. The second assertion follows from the first one together with Grothendieck's proposition 3. Given a section (P_1, P_2, f) of $\mathbb{G}^\Gamma(R_{\mathbb{O}\mathbb{P}\mathbb{P}(G)})$ over some S -scheme T , we have to show that the condition $(P_1, P_2) = (P_f, P_{t_f})$ is representable by an open and closed subscheme of T . It is plainly representable by the inverse image of the diagonal of $\mathbb{O}\mathbb{P}\mathbb{P}(G)$ under the S -morphism $T \rightarrow \mathbb{O}\mathbb{P}\mathbb{P}(G)^2$ defined by our two pairs (P_1, P_2) and (P_f, P_{t_f}) , which is a closed subscheme of T since $\mathbb{O}\mathbb{P}\mathbb{P}(G)$ is separated over S . On the other hand, since the Levi subgroup $L = P_1 \cap P_2$ of G is contained in $L_f = P_f \cap P_{t_f}$, our condition $(P_1, P_2) = (P_f, P_{t_f})$ is equivalent to $(\mathfrak{p}_1, \mathfrak{p}_2) = (\oplus_{\gamma \geq 0} \mathfrak{g}_\gamma, \oplus_{\gamma \leq 0} \mathfrak{g}_\gamma)$ where $\mathfrak{p}_i = \text{Lie}(P_i)$: this last claim is local in the fpqc topology on T , we may thus assume that L contains a maximal torus of G and then apply [12, XXII 5.3.5]. Now write $\mathfrak{u}_i = \oplus \mathfrak{u}_{i,\gamma}$ for the weight decomposition of the Lie algebra of the unipotent radical of P_i , and set $\mathfrak{u}_i^\pm = \oplus_{\pm\gamma \geq 0} \mathfrak{u}_{i,\gamma}$. Then our Lie algebra condition is equivalent to the vanishing of the locally free sheaf $\mathfrak{u}_1^- \oplus \mathfrak{u}_2^+$, and it is therefore representable by the open complement of its support. \square

Remark 16. This gives yet another proof of Theorem 1 (using Grothendieck's proposition 3), when G is reductive and Γ torsion free (using [22] to construct \leq on Γ).

2.2.5. The following cartesian diagram (in the fibered category of tori over schemes):

$$\begin{array}{ccccc} R_{\mathbb{O}\mathbb{P}\mathbb{P}(G)} & \xrightarrow{p_1} & R_{\mathbb{P}(G)} & \xrightarrow{t} & R_{\mathbb{O}(G)} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{O}\mathbb{P}\mathbb{P}(G) & \xrightarrow{p_1} & \mathbb{P}(G) & \xrightarrow{t} & \mathbb{O}(G) \end{array}$$

induces an analogous cartesian diagram (in the fibered category of quasi-isotrivial twisted constant group schemes over schemes):

$$\begin{array}{ccccc} \mathbb{G}^\Gamma(R_{\mathbb{O}\mathbb{P}\mathbb{P}(G)}) & \xrightarrow{\text{Fil}} & \mathbb{G}^\Gamma(R_{\mathbb{P}(G)}) & \xrightarrow{t} & \mathbb{G}^\Gamma(R_{\mathbb{O}(G)}) \\ F \downarrow & & F \downarrow & & F \downarrow \\ \mathbb{O}\mathbb{P}\mathbb{P}(G) & \xrightarrow{p_1} & \mathbb{P}(G) & \xrightarrow{t} & \mathbb{O}(G) \end{array}$$

which is given on T -valued points by the following formulas:

$$\begin{array}{ccccc} (P_1, P_2, f) & \xrightarrow{\text{Fil}} & (P_1, \bar{f}) & \xrightarrow{t} & (t(P_1), \bar{f}) \\ F \downarrow & & F \downarrow & & F \downarrow \\ (P_1, P_2) & \xrightarrow{p_1} & P_1 & \xrightarrow{t} & t(P_1) \end{array}$$

Here $\bar{f} : \mathbb{D}_T(\Gamma) \rightarrow \overline{R}(P_1)$ is defined by the diagram

$$\begin{array}{ccccc} \mathbb{D}_T(\Gamma) & \xrightarrow{f} & R(L) & \hookrightarrow & L \\ & \searrow \bar{f} & \downarrow \simeq & & \downarrow \simeq \\ & & \overline{R}(P_1) & \hookrightarrow & P_1/U_1 \end{array}$$

where $L = P_1 \cap P_2$ and U_1 is the unipotent radical of P_1 .

Lemma 17. *The open and closed subscheme $\mathbb{G}^\Gamma(G)$ of $\mathbb{G}^\Gamma(R_{\mathbb{O}\mathbb{P}\mathbb{P}(G)})$ is saturated with respect to $\mathbb{G}^\Gamma(R_{\mathbb{O}\mathbb{P}\mathbb{P}(G)}) \rightarrow \mathbb{G}^\Gamma(R_{\mathbb{P}(G)})$ and $\mathbb{G}^\Gamma(R_{\mathbb{O}\mathbb{P}\mathbb{P}(G)}) \rightarrow \mathbb{G}^\Gamma(R_{\mathbb{O}(G)})$.*

Proof. It is sufficient to establish that it is saturated with respect to the second map. We have to show that for an S -scheme T , a morphism $f : \mathbb{D}_T(\Gamma) \rightarrow G_T$, a pair of opposed parabolic subgroups (P_1, P_2) of G_T with Levi $L = P_1 \cap P_2$, and a central morphism $h : \mathbb{D}_T(\Gamma) \rightarrow L$, if $(P_f, P_{\iota f}, f)$ and (P_1, P_2, h) have the same image in $\mathbb{G}^\Gamma(R_{\mathbb{O}(G)})(T)$, then $(P_1, P_2) = (P_h, P_{\iota h})$. This is local in the fpqc topology on T . Since $t(P_f) = t(P_1)$ by assumption, we may assume that there is a $g \in G(T)$ such that $\text{Int}(g)(P_f, P_{\iota f}) = (P_1, P_2)$ by [12, 4.3.4 iii]. But then also $\text{Int}(g) \circ f = h$ (by assumption), thus $(P_1, P_2) = \text{Int}(g)(P_f, P_{\iota f}) = (P_h, P_{\iota h})$. \square

2.2.6. By an elementary case of fpqc descent (along p_1 and t), we thus obtain a cartesian diagram of open and closed embeddings of smooth S -schemes

$$\begin{array}{ccccc} \mathbb{G}^\Gamma(G) & \xrightarrow{\text{Fil}} & \mathbb{F}^\Gamma(G) & \xrightarrow{t} & \mathbb{C}^\Gamma(G) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{G}^\Gamma(R_{\mathbb{O}^{\text{pp}}(G)}) & \xrightarrow{p_1} & \mathbb{G}^\Gamma(R_{\mathbb{P}(G)}) & \xrightarrow{t} & \mathbb{G}^\Gamma(R_{\mathbb{O}(G)}) \end{array}$$

which in turns gives our fundamental cartesian diagram of smooth S -schemes

$$\begin{array}{ccccc} \mathbb{G}^\Gamma(G) & \xrightarrow{\text{Fil}} & \mathbb{F}^\Gamma(G) & \xrightarrow{t} & \mathbb{C}^\Gamma(G) \\ \downarrow F & & \downarrow F & & \downarrow F \\ \mathbb{O}^{\text{pp}}(G) & \xrightarrow{p_1} & \mathbb{P}(G) & \xrightarrow{t} & \mathbb{O}(G) \end{array}$$

The morphism $\text{Fil} : \mathbb{G}^\Gamma(G) \rightarrow \mathbb{F}^\Gamma(G)$ is a $U_{\mathbb{F}^\Gamma(G)}$ -torsor, where $U_{\mathbb{F}^\Gamma(G)}$ is the unipotent radical of the pull-back $P_{\mathbb{F}^\Gamma(G)}$ of the universal parabolic subgroup P_u of $G_{\mathbb{P}(G)}$. In particular, it is affine smooth surjective with geometrically connected fibers. The morphism $t : \mathbb{F}^\Gamma(G) \rightarrow \mathbb{C}^\Gamma(G)$ is projective smooth surjective with geometrically connected fibers. The three *facet* morphisms F are quasi-isotrivial twisted constant (i.e. locally constant in the étale topology on their base), in particular they are separated and étale by Lemma 4. We will see in due time that they are also surjective. Since $\mathbb{O}(G)$ is a finite étale cover of S , everyone is smooth and separated over S .

If S is irreducible and geometrically unibranch, then so are the connected components of $\mathbb{O}(G)$. Over each of them, the facet map $F : \mathbb{C}^\Gamma(G) \rightarrow \mathbb{O}(G)$ is then merely an infinite disjoint union of finite étale covers, see Lemma 4. The same decomposition then also holds for its pull-backs over $\mathbb{P}(G)$ or $\mathbb{O}^{\text{pp}}(G)$.

2.2.7. For an S -scheme T and morphisms $x, y : \mathbb{D}_T(\Gamma) \rightarrow G_T$, we have

$$\begin{aligned} \text{Fil}(x) = \text{Fil}(y) \quad \text{in} \quad \mathbb{F}^\Gamma(G)(T) & \iff \exists p \in P_x(T) : \text{Int}(p)(x) = y \\ & \iff \exists u \in U_x(T) : \text{Int}(u)(x) = y \end{aligned}$$

and then such a u is unique. This equivalence relation is known as the Par-equivalence and denoted by $x \sim_{\text{Par}} y$. If T is an (absolutely) affine scheme, then

$$\mathbb{F}^\Gamma(G)(T) = \mathbb{G}^\Gamma(G)(T) / \sim_{\text{Par}}$$

by [12, XXVI 2.2]. On the other hand,

$$\begin{aligned} t \circ \text{Fil}(x) = t \circ \text{Fil}(y) \quad \text{in} \quad \mathbb{C}^\Gamma(G)(T) \\ \iff \text{fpqc locally on } T, \exists g \in G(T) : \text{Int}(g)(x) = y. \end{aligned}$$

If T is semi-local, then by [12, XXVI 5.2],

$$\begin{aligned} & t \circ \text{Fil}(x) = t \circ \text{Fil}(y) \quad \text{in } \mathbb{C}^\Gamma(G)(T) \\ \iff & \exists g \in G(T) : \text{Int}(g)(x) = y. \end{aligned}$$

2.2.8. $\mathbb{C}^\Gamma(G)$ is a monoid. There is natural structure of commutative monoid on the S -scheme $\mathbb{O}(G)$, given by the intersection morphism

$$\cap : \mathbb{O}(G) \times_S \mathbb{O}(G) \rightarrow \mathbb{O}(G) \quad (a, b) \mapsto a \cap b$$

Let $\mathbb{O}'(G)$ be the open and closed subscheme of $\mathbb{O}(G) \times_S \mathbb{O}(G)$ on which $a \cap b = a$, i.e. $a \subset b$. Let p_1 and $p_2 : \mathbb{O}'(G) \rightarrow \mathbb{O}(G)$ be the two projections. We claim:

Lemma 18. *There exists a canonical morphism $p_2^* R_{\mathbb{O}(G)} \rightarrow p_1^* R_{\mathbb{O}(G)}$.*

Proof. Let $\mathbb{P}'(G)$ be the inverse image of $\mathbb{O}'(G)$ in $\mathbb{P}(G) \times_S \mathbb{P}(G)$, and denote by q_1 and $q_2 : \mathbb{P}'(G) \rightarrow \mathbb{P}(G)$ the two projections. Then $q_i^*(R_{\mathbb{P}(G)}) = (p_i^* R_{\mathbb{O}(G)})_{\mathbb{P}'(G)}$ for $i \in \{1, 2\}$. We have to show that there is a canonical morphism $q_2^* R_{\mathbb{P}(G)} \rightarrow q_1^* R_{\mathbb{P}(G)}$, compatible with the descent data on both sides. This boils down to: for any $S' \rightarrow S$ and $(P_1, P_2) \in \mathbb{P}'(G)(S')$, there exists a canonical morphism $\overline{R}(P_2) \rightarrow \overline{R}(P_1)$. We may assume that $S' = S$. Since $t(P_1) \subset t(P_2)$, there exists by [12, XXVI 3.8] a unique parabolic subgroup P_2' of G , containing P_1 , such that $t(P_2) = t(P_2')$. Using the canonical isomorphism $\overline{R}(P_2') \simeq \overline{R}(P_2)$, we may thus assume that $P_2' = P_2$, i.e. $P_1 \subset P_2$. Let U_i be the unipotent radical of P_i , so that $U_2 \subset U_1$ is a normal subgroup of P_1 . Then P_1/U_2 is a parabolic subgroup of P_2/U_2 with maximal reductive quotient P_1/U_1 , which reduces us further to the case where $G = P_2$. Then P_1 contains the radical $\overline{R}(G) = R(G)$ of G , and $P_1 \rightarrow P_1/U_1$ maps $R(G)$ to the radical $\overline{R}(P_1)$ of P_1/U_1 . This yields our canonical morphism $\overline{R}(P_2) \rightarrow \overline{R}(P_1)$. \square

Pulling back the above morphism through

$$\begin{aligned} \mathbb{O}(G) \times_S \mathbb{O}(G) & \rightarrow \mathbb{O}'(G) \\ (a, b) & \mapsto (a \cap b, b) \end{aligned}$$

we obtain a morphism $p_2^* R_{\mathbb{O}(G)} \rightarrow (\cap)^* R_{\mathbb{O}(G)}$ of tori over $\mathbb{O}(G) \times_S \mathbb{O}(G)$. By symmetry, there is also a morphism $p_1^* R_{\mathbb{O}(G)} \rightarrow (\cap)^* R_{\mathbb{O}(G)}$. The product of these two yields a morphism in the fibered category of tori over Sch/S ,

$$\begin{array}{ccc} R_{\mathbb{O}(G)} \times_S R_{\mathbb{O}(G)} & \longrightarrow & R_{\mathbb{O}(G)} \times_{\mathbb{O}(G)} R_{\mathbb{O}(G)} \\ \downarrow & & \downarrow \\ \mathbb{O}(G) \times_S \mathbb{O}(G) & \xrightarrow{\cap} & \mathbb{O}(G) \end{array}$$

Composing it with the multiplication map on the $\mathbb{O}(G)$ -torus $R_{\mathbb{O}(G)}$, we obtain yet another such morphism, namely

$$\begin{array}{ccc} R_{\mathbb{O}(G)} \times_S R_{\mathbb{O}(G)} & \longrightarrow & R_{\mathbb{O}(G)} \\ \downarrow & & \downarrow \\ \mathbb{O}(G) \times_S \mathbb{O}(G) & \xrightarrow{\cap} & \mathbb{O}(G) \end{array}$$

Applying now the $\mathbb{G}^\Gamma(-)$ construction to the latter diagram yields a morphism

$$\begin{array}{ccc} \mathbb{G}^\Gamma(R_{\mathbb{O}(G)}) \times_S \mathbb{G}^\Gamma(R_{\mathbb{O}(G)}) & \longrightarrow & \mathbb{G}^\Gamma(R_{\mathbb{O}(G)}) \\ \downarrow & & \downarrow \\ \mathbb{O}(G) \times_S \mathbb{O}(G) & \xrightarrow{\cap} & \mathbb{O}(G) \end{array}$$

in the fibered category of commutative group schemes over Sch/S . The top map of this diagram defines a commutative monoid structure on the S -scheme $\mathbb{G}^\Gamma(R_{\mathbb{O}(G)})$. By construction, the structural morphism $\mathbb{G}^\Gamma(R_{\mathbb{O}(G)}) \rightarrow \mathbb{O}(G)$ is compatible with the monoid structures on both sides.

Lemma 19. *The S -scheme $\mathbb{C}^\Gamma(G)$ is a submonoid of $\mathbb{G}^\Gamma(R_{\mathbb{O}(G)})$.*

Proof. Using additive notations, we have to show that for $S' \rightarrow S$ and c_1, c_2 in $\mathbb{C}^\Gamma(G)(S')$, there exists an fpqc cover $S'' \rightarrow S'$ and an element $f \in \mathbb{G}^\Gamma(G)(S'')$ such that $c_1 + c_2 = t \circ \text{Fil}(f)$ in $\mathbb{G}^\Gamma(R_{\mathbb{O}(G)})$. We may assume that $S' = S$ and $c_i = t \circ \text{Fil}(f_i)$ for some $f_i : \mathbb{D}_S(\Gamma) \rightarrow G$. Using [12, XXVI 1.14 and XXIV 1.5], we may also assume that there is an épinglage (G, T, Δ, \dots) which is adapted to $P_1 = P_{f_1}$ and $P_2 = P_{f_2}$. Then by [12, XXVI 1.6 and 1.8], we may assume that $L_1 = L_{f_1}$ and $L_2 = L_{f_2}$ both contain the maximal torus T of G , so that both f_1 and f_2 factor through T . Let $f = f_1 + f_2 : \mathbb{D}_S(\Gamma) \rightarrow T \hookrightarrow G$ and $P = P_f$. We claim that $c_1 + c_2 = t \circ \text{Fil}(f)$. By [12, XXVI 3.2], $t(P_i) = \Delta(P_i)_S$ where $\Delta(P_i) \subset \Delta \subset \text{Hom}(T, \mathbb{G}_{m,S})$ is the set of simple roots occurring in $\text{Lie}(L_i)$, i.e. $\Delta(P_i) = \{\alpha \in \Delta : \alpha \circ f_i = 0 \in \Gamma\}$. By construction, $\alpha \circ f_i \geq 0$ in Γ for every $\alpha \in \Delta$, thus also $\alpha \circ f = \alpha \circ f_1 + \alpha \circ f_2 \geq 0$ in Γ for every $\alpha \in \Delta$ with $\alpha \circ f = 0$ if and only if $\alpha \circ f_1 = 0 = \alpha \circ f_2$. It follows that our épinglage is also adapted to P , with $\Delta(P) = \Delta(P_1) \cap \Delta(P_2)$, i.e. $t(P) = t(P_1) \cap t(P_2)$ in $\mathbb{O}(G)(S)$. The inclusion $P \subset P_i$ induces the canonical morphism $\text{can}_i : \overline{R}(P_i) \rightarrow \overline{R}(P)$ and one checks easily that $\overline{f} = \text{can}_1 \circ \overline{f}_1 + \text{can}_2 \circ \overline{f}_2$. Thus by definition,

$$c_1 + c_2 = t(P, \text{can}_1 \circ \overline{f}_1 + \text{can}_2 \circ \overline{f}_2) = t(P, \overline{f}) = t \circ \text{Fil}(f)$$

as was to be shown. \square

2.2.9. *Isogenies.* Suppose that Γ is uniquely divisible, i.e. is a \mathbb{Q} -vector space.

Proposition 20. *The cartesian diagram below is invariant under central isogenies:*

$$\begin{array}{ccccc} \mathbb{G}^\Gamma(G) & \xrightarrow{\text{Fil}} & \mathbb{F}^\Gamma(G) & \xrightarrow{t} & \mathbb{C}^\Gamma(G) \\ \downarrow F & & \downarrow F & & \downarrow F \\ \mathbb{O}\text{PP}(G) & \xrightarrow{p_1} & \mathbb{P}(G) & \xrightarrow{t} & \mathbb{O}(G) \end{array}$$

Proof. The bottom line only depends upon the adjoint group $G^{\text{ad}} = G/Z(G)$: this is true for $\mathbb{O}(G)$ because $\mathbb{D}\text{YN}(G) = \mathbb{D}\text{YN}(G^{\text{ad}})$ by definition of the Dynkin S -scheme [12, XXIV 3.3] in view of [12, XXII 4.3.7], and the maps $P \mapsto P/Z(G)$ and $P^{\text{ad}} \mapsto \text{ad}^{-1}(P^{\text{ad}})$ (where $\text{ad} : G \rightarrow G^{\text{ad}}$ is the quotient map) induce mutually inverse bijections between parabolic subgroups of G and parabolic subgroups of G^{ad} , which are compatible with the type maps and with opposition. For the top line, let $f : G_1 \rightarrow G_2$ be a central isogeny [12, XXII 4.2.9]. We first claim that composition with f yields an isomorphism $\mathbb{G}^\Gamma(G_1) \rightarrow \mathbb{G}^\Gamma(G_2)$: for split tori, this immediately follows from [13, VIII 1.5] and our assumption on Γ ; for tori, our

claim is local in the fpqc topology on S by [17, 2.7.1], which reduces us to the previous case; for arbitrary reductive groups, use Lemma 5 and [1, XVII 7.1.1]. If now P_1 is a parabolic subgroup of G_1 with image P_2 in G_2 , then f induces an isogeny $\overline{R}(P_1) \rightarrow \overline{R}(P_2)$. Thus f yields an isogeny $R_f : R_{\mathbb{P}(G_1)} \rightarrow R_{\mathbb{P}(G_2)}$ of tori over $\mathbb{P}(G_1) \simeq \mathbb{P}(G_2)$. The induced isomorphism $\mathbb{G}^\Gamma(R_{\mathbb{P}(G_1)}) \simeq \mathbb{G}^\Gamma(R_{\mathbb{P}(G_2)})$ is compatible with the morphisms $\mathbb{G}^\Gamma(G_i) \rightarrow \mathbb{G}^\Gamma(R_{\mathbb{P}(G_i)})$, therefore also $\mathbb{F}^\Gamma(G_1) \simeq \mathbb{F}^\Gamma(G_2)$. Since R_f is also compatible with the canonical descent data of Lemma 13, it descends to an isogeny $R_f : R_{\mathbb{O}(G_1)} \rightarrow R_{\mathbb{O}(G_2)}$ of tori over $\mathbb{O}(G_1) \simeq \mathbb{O}(G_2)$. The induced isomorphism $\mathbb{G}^\Gamma(R_{\mathbb{O}(G_1)}) \simeq \mathbb{G}^\Gamma(R_{\mathbb{O}(G_2)})$ is again compatible with the morphisms $\mathbb{F}^\Gamma(G_i) \rightarrow \mathbb{G}^\Gamma(R_{\mathbb{O}(G_i)})$, therefore also $\mathbb{C}^\Gamma(G_1) \simeq \mathbb{C}^\Gamma(G_2)$. \square

Plainly, the above diagrams are also compatible with products. Considering the canonical diagram of central isogenies [12, XXII 4.3 & 6.2]

$$R(G) \times G^{\text{der}} \rightarrow G \rightarrow G^{\text{ab}} \times G^{\text{ss}} \rightarrow G^{\text{ab}} \times G^{\text{ad}}$$

where $R(G)$ is the radical of G , G^{der} its derived group, $G^{\text{ab}} = G/G^{\text{der}}$ its coradical, $G^{\text{ss}} = G/R(G)$ its semi-simplification and $G^{\text{ad}} = G/Z(G)$ its adjoint group, we obtain compatible canonical decompositions

$$\begin{aligned} \mathbb{G}^\Gamma(G) &= \mathbb{G}^\Gamma(G)^r \times \mathbb{G}^\Gamma(G)^c \\ \mathbb{F}^\Gamma(G) &= \mathbb{F}^\Gamma(G)^r \times \mathbb{F}^\Gamma(G)^c \\ \mathbb{C}^\Gamma(G) &= \mathbb{C}^\Gamma(G)^r \times \mathbb{C}^\Gamma(G)^c \end{aligned}$$

with $\mathbb{G}^\Gamma(G)^c = \mathbb{F}^\Gamma(G)^c = \mathbb{C}^\Gamma(G)^c = \mathbb{G}^\Gamma(R(G)) = \mathbb{G}^\Gamma(G^{\text{ab}}) = \mathbb{G}^\Gamma(Z(G))$ and

$$\begin{aligned} \mathbb{G}^\Gamma(G)^r &= \mathbb{G}^\Gamma(G^{\text{der}}) = \mathbb{G}^\Gamma(G^{\text{ss}}) = \mathbb{G}^\Gamma(G^{\text{ad}}), \\ \mathbb{F}^\Gamma(G)^r &= \mathbb{F}^\Gamma(G^{\text{der}}) = \mathbb{F}^\Gamma(G^{\text{ss}}) = \mathbb{F}^\Gamma(G^{\text{ad}}), \\ \mathbb{C}^\Gamma(G)^r &= \mathbb{C}^\Gamma(G^{\text{der}}) = \mathbb{C}^\Gamma(G^{\text{ss}}) = \mathbb{C}^\Gamma(G^{\text{ad}}). \end{aligned}$$

The decomposition of $\mathbb{C}^\Gamma(G)$ is compatible with its monoid structure.

3. THE TANNAKIAN FORMALISM

Let G be an affine and flat group scheme over S and let $\Gamma = (\Gamma, +, \leq)$ be a non-trivial, totally ordered commutative group. We will define below an equivariant diagram of fpqc sheaves $(\text{Sch}/S)^\circ \rightarrow \text{Group}$ or $(\text{Sch}/S)^\circ \rightarrow \text{Set}$:

$$\begin{array}{ccc} \begin{array}{c} G \\ \downarrow \iota \\ \text{Aut}^\otimes(V) \\ \downarrow \\ \text{Aut}^\otimes(V^\circ) \text{ or } \text{Aut}^\otimes(\omega) \\ \downarrow \\ \text{Aut}^\otimes(\omega^\circ) \end{array} & \text{acting on} & \begin{array}{ccc} \mathbb{G}^\Gamma(G) & \xrightarrow{\text{Fil}} & \mathbb{F}^\Gamma(G) \\ \downarrow \iota & & \downarrow \iota \\ \mathbb{G}^\Gamma(V) & \xrightarrow{\text{Fil}} & \mathbb{F}^\Gamma(V) \\ \downarrow & & \downarrow \\ \mathbb{G}^\Gamma(V^\circ) \text{ or } \mathbb{G}^\Gamma(\omega) & \xrightarrow{\text{Fil}} & \mathbb{F}^\Gamma(V^\circ) \text{ or } \mathbb{F}^\Gamma(\omega) \\ \downarrow & & \downarrow \\ \mathbb{G}^\Gamma(\omega^\circ) & \xrightarrow{\text{Fil}} & \mathbb{F}^\Gamma(\omega^\circ) \end{array} \\ \dots & & \dots \\ \dots & & \dots \end{array}$$

Our main result for this section will then be the following Theorem:

Theorem 21. *If G is a reductive group over S , then*

$$\begin{aligned} G &= \mathrm{Aut}^{\otimes}(V) = \mathrm{Aut}^{\otimes}(V^{\circ}) = \mathrm{Aut}^{\otimes}(\omega) \\ \mathbb{G}^{\Gamma}(G) &= \mathbb{G}^{\Gamma}(V) = \mathbb{G}^{\Gamma}(V^{\circ}) = \mathbb{G}^{\Gamma}(\omega) \\ \mathbb{F}^{\Gamma}(G) &= \mathbb{F}^{\Gamma}(V) = \mathbb{F}^{\Gamma}(V^{\circ}) \subset \mathbb{F}^{\Gamma}(\omega) \end{aligned}$$

If moreover G is isotrivial and S quasi-compact, then also

$$G = \mathrm{Aut}^{\otimes}(\omega^{\circ}), \quad \mathbb{G}^{\Gamma}(G) = \mathbb{G}^{\Gamma}(\omega^{\circ}) \quad \text{and} \quad \mathbb{F}^{\Gamma}(G) = \mathbb{F}^{\Gamma}(\omega) = \mathbb{F}^{\Gamma}(\omega^{\circ}).$$

More precisely, we will first show that for any affine flat group scheme G over S ,

$$\begin{aligned} G &= \mathrm{Aut}^{\otimes}(V) = \mathrm{Aut}^{\otimes}(\omega) \\ \mathbb{G}^{\Gamma}(G) &= \mathbb{G}^{\Gamma}(V) = \mathbb{G}^{\Gamma}(\omega) \\ \mathbb{F}^{\Gamma}(G) &\subset \mathbb{F}^{\Gamma}(V) \subset \mathbb{F}^{\Gamma}(\omega) \end{aligned}$$

Then, under technical assumptions which are satisfied by all reductive groups (resp. all isotrivial reductive groups over quasi-compact bases), we will also establish that

$$\begin{array}{l} \mathrm{Aut}^{\otimes}(V) = \mathrm{Aut}^{\otimes}(V^{\circ}) \\ \mathbb{G}^{\Gamma}(V) = \mathbb{G}^{\Gamma}(V^{\circ}) \\ \mathbb{F}^{\Gamma}(V) \subset \mathbb{F}^{\Gamma}(V^{\circ}) \end{array} \left(\begin{array}{l} \text{resp.} \\ \mathbb{F}^{\Gamma}(\omega), \mathbb{F}^{\Gamma}(V^{\circ}) \subset \mathbb{F}^{\Gamma}(\omega^{\circ}) \end{array} \right).$$

We will finally show that for G reductive and isotrivial over a quasi-compact S , the morphism $\mathbb{G}^{\Gamma}(G) \rightarrow \mathbb{F}^{\Gamma}(\omega^{\circ})$ is an epimorphism of fpqc sheaves on S . Thus

$$\mathbb{F}^{\Gamma}(G) = \mathbb{F}^{\Gamma}(V) = \mathbb{F}^{\Gamma}(V^{\circ}) = \mathbb{F}^{\Gamma}(\omega) = \mathbb{F}^{\Gamma}(\omega^{\circ})$$

in this case, and the remaining statement, namely

$$\mathbb{F}^{\Gamma}(G) = \mathbb{F}^{\Gamma}(V) = \mathbb{F}^{\Gamma}(V^{\circ})$$

for a reductive group G over an arbitrary S easily follows.

3.1. Γ -graduations and Γ -filtrations on quasi-coherent sheaves.

3.1.1. Let \mathcal{M} be a quasi-coherent sheaf on a scheme X . A Γ -graduation on \mathcal{M} is a collection $\mathcal{G} = (\mathcal{G}_{\gamma})_{\gamma \in \Gamma}$ of quasi-coherent subsheaves of \mathcal{M} such that $\mathcal{M} = \bigoplus_{\gamma \in \Gamma} \mathcal{G}_{\gamma}$. A Γ -filtration on \mathcal{M} is a collection $\mathcal{F} = (\mathcal{F}^{\gamma})_{\gamma \in \Gamma}$ of quasi-coherent subsheaves of \mathcal{M} such that, locally on X for the fpqc topology, there exists a Γ -graduation $\mathcal{G} = (\mathcal{G}_{\gamma})_{\gamma \in \Gamma}$ on \mathcal{M} for which $\mathcal{F}^{\gamma} = \bigoplus_{\eta \geq \gamma} \mathcal{G}_{\eta}$. We call any such \mathcal{G} a splitting of \mathcal{F} and write $\mathcal{F} = \mathrm{Fil}(\mathcal{G})$. We set $\mathcal{F}_+^{\gamma} = \bigcup_{\eta > \gamma} \mathcal{F}^{\eta}$ and $\mathrm{Gr}_{\mathcal{F}}^{\gamma} \mathcal{M} = \mathcal{F}^{\gamma} / \mathcal{F}_+^{\gamma}$.

Lemma 22. *Let \mathcal{F} be a Γ -filtration on \mathcal{M} . Then $\gamma \mapsto \mathcal{F}^{\gamma}$ is non-increasing, exhaustive ($\bigcup \mathcal{F}^{\gamma} = \mathcal{M}$), separated ($\bigcap \mathcal{F}^{\gamma} = 0$), and for every $\gamma \in \Gamma$,*

$$0 \rightarrow \mathcal{F}^{\gamma} \rightarrow \mathcal{M} \rightarrow \mathcal{M} / \mathcal{F}^{\gamma} \rightarrow 0 \quad \text{and} \quad 0 \rightarrow \mathcal{F}_+^{\gamma} \rightarrow \mathcal{F}^{\gamma} \rightarrow \mathrm{Gr}_{\mathcal{F}}^{\gamma}(\mathcal{M}) \rightarrow 0$$

are pure exact sequences of quasi-coherent sheaves (see 3.12).

Proof. Everything is local in the fpqc topology on X , trivial if \mathcal{F} has a splitting. \square

3.1.2. These definitions give rise to a diagram of fpqc stacks over Sch

$$\text{Gr}^\Gamma \text{QCoh} \begin{array}{c} \xrightarrow{\text{Fil}} \\ \xleftarrow{\text{Gr}} \end{array} \text{Fil}^\Gamma \text{QCoh} \xrightarrow{\text{forg}} \text{QCoh}$$

whose fiber over a scheme X is the diagram of exact \otimes -functors

$$\text{Gr}^\Gamma \text{QCoh}(X) \begin{array}{c} \xrightarrow{\text{Fil}} \\ \xleftarrow{\text{Gr}} \end{array} \text{Fil}^\Gamma \text{QCoh}(X) \xrightarrow{\text{forg}} \text{QCoh}(X)$$

where $\text{QCoh}(X)$ is the abelian \otimes -category of quasi-coherent sheaves \mathcal{M} on X , $\text{Gr}^\Gamma \text{QCoh}(X)$ is the abelian \otimes -category of Γ -graded quasi-coherent sheaves $(\mathcal{M}, \mathcal{G})$ on X , and $\text{Fil}^\Gamma \text{QCoh}(X)$ is the exact (in Quillen's sense) \otimes -category of Γ -filtered quasi-coherent sheaves $(\mathcal{M}, \mathcal{F})$ on X . The morphisms in these last two categories are the morphisms of the underlying quasi-coherent sheaves which preserve the given collections of subsheaves, and the \otimes -products are given by the usual formulas

$$\begin{aligned} (\mathcal{M}_1, \mathcal{G}_1) \otimes (\mathcal{M}_2, \mathcal{G}_2) &= (\mathcal{M}_1 \otimes \mathcal{M}_2, \mathcal{G}) \quad \text{with} \quad \mathcal{G}_\gamma = \bigoplus_{\gamma_1 + \gamma_2 = \gamma} \mathcal{G}_{1, \gamma_1} \otimes \mathcal{G}_{2, \gamma_2}, \\ (\mathcal{M}_1, \mathcal{F}_1) \otimes (\mathcal{M}_2, \mathcal{F}_2) &= (\mathcal{M}_1 \otimes \mathcal{M}_2, \mathcal{F}) \quad \text{with} \quad \mathcal{F}^\gamma = \sum_{\gamma_1 + \gamma_2 = \gamma} \mathcal{F}_1^{\gamma_1} \otimes \mathcal{F}_2^{\gamma_2}. \end{aligned}$$

The second formula makes sense by the purity mentioned above, and indeed defines a Γ -filtration on $\mathcal{M}_1 \otimes \mathcal{M}_2$: if \mathcal{G}_i splits \mathcal{F}_i for $i \in \{1, 2\}$, then \mathcal{G} splits \mathcal{F} . We have

$$\begin{aligned} \mathcal{F}_+^\gamma &= \sum_{\gamma_1 + \gamma_2 > \gamma} \mathcal{F}_1^{\gamma_1} \otimes \mathcal{F}_2^{\gamma_2} \\ \text{and } \text{Gr}_{\mathcal{F}}^\gamma(\mathcal{M}_1 \otimes \mathcal{M}_2) &\simeq \bigoplus_{\gamma_1 + \gamma_2 = \gamma} \text{Gr}_{\mathcal{F}_1}^{\gamma_1}(\mathcal{M}_1) \otimes \text{Gr}_{\mathcal{F}_2}^{\gamma_2}(\mathcal{M}_2). \end{aligned}$$

The first formula is trivial and gives the morphism (from right to left) in the second formula, which is easily seen to be an isomorphism by localization to an fpqc cover of X over which \mathcal{F}_1 and \mathcal{F}_2 both acquire a splitting. The neutral object for \otimes are

$$1_X = (\mathcal{O}_X, \mathcal{G} \text{ of } \mathcal{F}) \quad \text{with } \mathcal{G}_\gamma = \begin{cases} \mathcal{O}_X & \text{for } \gamma = 0, \\ 0 & \text{otherwise} \end{cases} \quad \text{and } \mathcal{F}^\gamma = \begin{cases} \mathcal{O}_X & \text{for } \gamma \leq 0, \\ 0 & \text{otherwise.} \end{cases}$$

A morphism $(\mathcal{M}_1, \mathcal{F}_1) \rightarrow (\mathcal{M}_2, \mathcal{F}_2)$ is strict if $\text{Im}(\mathcal{F}_1^\gamma) = \mathcal{F}_2^\gamma \cap \text{Im}(\mathcal{M}_1)$ in \mathcal{M}_2 for every $\gamma \in \Gamma$. The short exact sequences of $\text{Fil}^\Gamma \text{QCoh}(X)$ are those made of strict arrows whose underlying sequence of sheaves is short exact. The formulas

$$\text{Fil}(\mathcal{M}, \mathcal{G}) = (\mathcal{M}, \text{Fil}(\mathcal{G})), \quad \text{Gr}(\mathcal{M}, \mathcal{F}) = \bigoplus_\gamma \text{Gr}_{\mathcal{F}}^\gamma \mathcal{M} \quad \text{and} \quad \text{forg}(\mathcal{M}, -) = \mathcal{M}$$

define the exact \otimes -functors between our three categories. Finally the ‘‘base change functors’’ defining the fibered category structures on $\text{Gr}^\Gamma \text{QCoh}$ and $\text{Fil}^\Gamma \text{QCoh}$ are induced by the base change functors on QCoh (thanks to the purity of the subsheaves). It is well-known that QCoh is an fpqc stack over Sch (see for instance [35, Theorem 4.23]) and it follows rather formally from their definitions that the other two fibered categories are also fpqc stacks over Sch . We denote by

$$\text{Gr}^\Gamma \text{QCoh}/S \begin{array}{c} \xrightarrow{\text{Fil}} \\ \xleftarrow{\text{Gr}} \end{array} \text{Fil}^\Gamma \text{QCoh}/S \xrightarrow{\text{forg}} \text{QCoh}/S$$

the corresponding stacks over Sch/S where S is any base scheme.

3.2. Γ -graduations and Γ -filtrations on fiber functors.

3.2.1. Let $s : G \rightarrow S$ be an affine and flat group scheme. We denote by $\text{Rep}(G)$ the fpqc stack over Sch/S whose fiber over $T \rightarrow S$ is the abelian \otimes -category $\text{Rep}(G)(T)$ of quasi-coherent G_T - \mathcal{O}_T -modules as defined in [13, I 4.7.1]. Set $\mathcal{A}(G) = s_*\mathcal{O}_G$. Then $\mathcal{A}(G)$ is a quasi-coherent Hopf algebra over S and $\text{Rep}(G)(T)$ is \otimes -equivalent to the category of quasi-coherent $\mathcal{A}(G_T)$ -comodules where $\mathcal{A}(G_T) = \mathcal{A}(G)_T$. Let

$$V : \text{Rep}(G) \rightarrow \text{QCoh}/S$$

be the forgetful functor. For any S -scheme $q : T \rightarrow S$, we denote by

$$V_T : \text{Rep}(G_T) \rightarrow \text{QCoh}/T \quad \text{and} \quad \omega_T : \text{Rep}(G)(S) \rightarrow \text{QCoh}(T)$$

the induced morphism of fpqc stack over Sch/T and fiber functor. Note that ω_T is a right exact \otimes -functor. It also commutes with arbitrary colimits and preserves pure monomorphisms and pure short exact sequences, where purity in $\text{Rep}(G)(S)$ refers to purity of the underlying objects in $\text{QCoh}(S)$.

3.2.2. A Γ -graduation \mathcal{G} on $V_T : \text{Rep}(G_T) \rightarrow \text{QCoh}/T$ is a factorization

$$\text{Rep}(G_T) \xrightarrow{\mathcal{G}} \text{Gr}^\Gamma \text{QCoh}/T \xrightarrow{\text{forg}} \text{QCoh}/T$$

of V_T such that if $\mathcal{G}_\gamma : \text{Rep}(G_T) \rightarrow \text{QCoh}/T$ is the γ -component of \mathcal{G} ,

(G0) For every T -morphism $f : X \rightarrow Y$, $\rho \in \text{Rep}(G)(Y)$ and $\gamma \in \Gamma$,

$$f^*(\mathcal{G}_\gamma(\rho)) = \mathcal{G}_\gamma(f^*\rho).$$

(G1) For every T -scheme $X \rightarrow T$, $\rho_1, \rho_2 \in \text{Rep}(G)(X)$ and $\gamma \in \Gamma$,

$$\mathcal{G}_\gamma(\rho_1 \otimes \rho_2) = \bigoplus_{\gamma_1 + \gamma_2 = \gamma} \mathcal{G}_{\gamma_1}(\rho_1) \otimes \mathcal{G}_{\gamma_2}(\rho_2).$$

Thus (G0) says that each \mathcal{G}_γ is a morphism of fibered categories over Sch/T . Then (G1) implies that $\mathcal{G}_0(\rho) = \mathcal{M}$ and $\mathcal{G}_\gamma(\rho) = 0$ for $\gamma \neq 0$ when ρ is the trivial representation of G_X on $\mathcal{M} \in \text{QCoh}(X)$ (one proves it first for $\mathcal{M} = \mathcal{O}_X$).

3.2.3. A Γ -graduation \mathcal{G} on $\omega_T : \text{Rep}(G)(S) \rightarrow \text{QCoh}(T)$ is a factorization

$$\text{Rep}(G)(S) \xrightarrow{\mathcal{G}} \text{Gr}^\Gamma \text{QCoh}(T) \xrightarrow{\text{forg}} \text{QCoh}(T)$$

of ω_T such that if $\mathcal{G}_\gamma : \text{Rep}(G)(S) \rightarrow \text{QCoh}(T)$ is the γ -component of \mathcal{G} ,

(G1) For every $\rho_1, \rho_2 \in \text{Rep}(G)(S)$ and $\gamma \in \Gamma$,

$$\mathcal{G}_\gamma(\rho_1 \otimes \rho_2) = \bigoplus_{\gamma_1 + \gamma_2 = \gamma} \mathcal{G}_{\gamma_1}(\rho_1) \otimes \mathcal{G}_{\gamma_2}(\rho_2).$$

(G2) For the trivial representation ρ of G on $\mathcal{M} \in \text{QCoh}(S)$,

$$\mathcal{G}_0(\rho) = \mathcal{M} \quad \text{and} \quad \mathcal{G}_\gamma(\rho) = 0 \quad \text{if } \gamma \neq 0.$$

Note that each \mathcal{G}_γ is right exact, commutes with arbitrary colimits and preserves pure monomorphisms and pure short exact sequences.

3.2.4. A Γ -filtration \mathcal{F} on $V_T : \text{Rep}(G_T) \rightarrow \text{QCoh}/T$ is a factorization

$$\text{Rep}(G_T) \xrightarrow{\mathcal{F}} \text{Fil}^\Gamma \text{QCoh}/T \xrightarrow{\text{forg}} \text{QCoh}/T$$

of V_T such that if $\mathcal{F}^\gamma : \text{Rep}(G_T) \rightarrow \text{QCoh}/T$ is the γ -component of \mathcal{G} ,

(F0) For every T -morphism $f : X \rightarrow Y$, $\rho \in \text{Rep}(G)(Y)$ and $\gamma \in \Gamma$,

$$f^*(\mathcal{F}^\gamma(\rho)) = \mathcal{F}^\gamma(f^*\rho).$$

(F1) For every $X \rightarrow T$, $\rho_1, \rho_2 \in \text{Rep}(G)(X)$ and $\gamma \in \Gamma$,

$$\mathcal{F}^\gamma(\rho_1 \otimes \rho_2) = \sum_{\gamma_1 + \gamma_2 = \gamma} \mathcal{F}^{\gamma_1}(\rho_1) \otimes \mathcal{F}^{\gamma_2}(\rho_2).$$

(F3) For every $X \rightarrow T$ and $\gamma \in \Gamma$, $\mathcal{F}^\gamma : \text{Rep}(G)(X) \rightarrow \text{QCoh}(X)$ is exact.

Thus (F0) says that each \mathcal{F}^γ is a morphism of fibered categories over Sch/T . Then again (F1) and (F3) imply that $\mathcal{F}^\gamma(\rho) = \mathcal{M}$ for $\gamma \leq 0$ and $\mathcal{F}^\gamma(\rho) = 0$ for $\gamma > 0$ when ρ is the trivial representation of G on $\mathcal{M} \in \text{QCoh}(X)$.

3.2.5. A Γ -filtration \mathcal{F} on $\omega_T : \text{Rep}(G)(S) \rightarrow \text{QCoh}(T)$ is a factorization

$$\text{Rep}(G)(S) \xrightarrow{\mathcal{F}} \text{Fil}^\Gamma \text{QCoh}(T) \xrightarrow{\text{forg}} \text{QCoh}(T)$$

of ω_T such that if $\mathcal{F}^\gamma : \text{Rep}(G)(S) \rightarrow \text{QCoh}(T)$ is the γ -component of \mathcal{G} ,

(F1) For every $\rho_1, \rho_2 \in \text{Rep}(G)(S)$ and $\gamma \in \Gamma$,

$$\mathcal{F}^\gamma(\rho_1 \otimes \rho_2) = \sum_{\gamma_1 + \gamma_2 = \gamma} \mathcal{F}^{\gamma_1}(\rho_1) \otimes \mathcal{F}^{\gamma_2}(\rho_2).$$

(F2) For the trivial representation ρ of G on $\mathcal{M} \in \text{QCoh}(S)$,

$$\mathcal{F}^\gamma(\rho) = \mathcal{M} \text{ if } \gamma \leq 0 \quad \text{and} \quad \mathcal{F}^\gamma(\rho) = 0 \text{ if } \gamma > 0.$$

(F3) For every $\gamma \in \Gamma$, $\mathcal{F}^\gamma : \text{Rep}(G)(S) \rightarrow \text{QCoh}(T)$ is right exact.

Since \mathcal{F}^γ preserves arbitrary direct sums (as a subfunctor of ω_T which does), this last axiom implies that \mathcal{F}^γ commutes with arbitrary colimits. It also preserves pure monomorphisms and pure short exact sequences.

3.2.6. We may now introduce a diagram of fpqc sheaves $(\text{Sch}/S)^\circ \rightarrow \text{Set}$,

$$\begin{array}{ccc} \mathbb{G}^\Gamma(V) & \xrightarrow{\text{res}} & \mathbb{G}^\Gamma(\omega) \\ \text{Fil} \downarrow & & \text{Fil} \downarrow \\ \mathbb{F}^\Gamma(V) & \xrightarrow{\text{res}} & \mathbb{F}^\Gamma(\omega) \end{array}$$

The four presheaves map an S -scheme T to the corresponding set of Γ -graduations or Γ -filtrations on V_T or ω_T , the Fil -morphisms are given by post-composition with the eponymous functors, and the res morphisms map \mathcal{G} or \mathcal{F} on V_T to

$$\text{Rep}(G)(S) \rightarrow \text{Rep}(G)(T) \xrightarrow{\mathcal{G}_T} \text{Gr}^\Gamma \text{QCoh}(T) \quad \text{or} \quad \dots \xrightarrow{\mathcal{F}_T} \text{Fil}^\Gamma \text{QCoh}(T).$$

The fact that all four presheaves are actually fpqc sheaves on S is essentially a formal consequence of the fact that the corresponding fibered categories of Γ -graded and Γ -filtered quasi-coherent sheaves are fpqc stacks over Sch/S .

3.2.7. The above diagram is equivariant with respect to a morphism

$$\mathrm{Aut}^{\otimes}(V) \xrightarrow{\mathrm{res}} \mathrm{Aut}^{\otimes}(\omega)$$

of fpqc sheaves of groups on Sch/S , with $\mathrm{Aut}^{\otimes}(\star)$ acting on $\mathbb{G}^{\Gamma}(\star)$ and $\mathbb{F}^{\Gamma}(\star)$ and mapping an S -scheme T to a group $\mathrm{Aut}^{\otimes}(\star_T)$ defined as follows: $\mathrm{Aut}^{\otimes}(V_T)$ is the group of all automorphisms $\eta : V_T \rightarrow V_T$ such that:

(A0) For every T -morphism $f : X \rightarrow Y$ and $\rho \in \mathrm{Rep}(G)(Y)$,

$$\eta_{f^*(\rho)} = f^*(\eta_{\rho}).$$

(A1) For every T -scheme $X \rightarrow T$ and $\rho_1, \rho_2 \in \mathrm{Rep}(G)(X)$,

$$\eta_{\rho_1 \otimes \rho_2} = \eta_{\rho_1} \otimes \eta_{\rho_2}.$$

These conditions imply as above that $\eta_{\rho} = \mathrm{Id}_{\mathcal{M}}$ when ρ is the trivial representation of G_X on a quasi-coherent \mathcal{O}_X -module \mathcal{M} . Similarly, $\mathrm{Aut}^{\otimes}(\omega_T)$ is the group of all automorphisms $\eta : \omega_T \rightarrow \omega_T$ such that:

(A1) For every $\rho_1, \rho_2 \in \mathrm{Rep}(G)(S)$,

$$\eta_{\rho_1 \otimes \rho_2} = \eta_{\rho_1} \otimes \eta_{\rho_2}.$$

(A2) For the trivial representation ρ of G on $\mathcal{M} \in \mathrm{QCoh}(S)$,

$$\eta_{\rho} = \mathrm{Id}_{\mathcal{M}}.$$

The fact that these two presheaves are actually fpqc sheaves on S is essentially a formal consequence of the fact that QCoh/S is a stack over Sch/S . The morphism between them sends $\eta \in \mathrm{Aut}^{\otimes}(V_T)$ to the automorphism of ω_T which maps ρ in $\mathrm{Rep}(G)(S)$ to the automorphism η_{ρ_T} of $V(\rho_T) = \omega_T(\rho)$, the actions mentioned above are the obvious ones, and the claimed equivariance is equally straightforward.

3.2.8. For $\star \in \{V, \omega\}$ and $\mathcal{X} \in \mathbb{G}^{\Gamma}(\star)(T)$ or $\mathbb{F}^{\Gamma}(\star)(T)$, we denote by

$$\mathrm{Aut}^{\otimes}(\mathcal{X}) : (\mathrm{Sch}/T)^{\circ} \rightarrow \mathrm{Group}$$

the stabilizer of \mathcal{X} in the restriction $\mathrm{Aut}^{\otimes}(\star)|_T$ of $\mathrm{Aut}^{\otimes}(\star)$ to Sch/T . It is an fpqc subsheaf of $\mathrm{Aut}^{\otimes}(\star)|_T$. For $\mathcal{X} = \mathcal{F}$ in $\mathbb{F}^{\Gamma}(\star)(T)$, there is also a morphism

$$\mathrm{Gr}^{\bullet} : \mathrm{Aut}^{\otimes}(\mathcal{F}) \rightarrow \mathrm{Aut}^{\otimes}(\mathrm{Gr}_{\mathcal{F}}^{\bullet}).$$

Here $\mathrm{Aut}^{\otimes}(\mathrm{Gr}_{\mathcal{F}}^{\bullet})$ is an fpqc sheaf of groups on Sch/T which maps $X \rightarrow T$ to a group of automorphisms of $\mathrm{Gr}_{\mathcal{F}_X}^{\bullet} = \mathrm{Gr}^{\bullet} \circ \mathcal{F}_X$ subject to conditions whose precise formulation will be left to the reader. The kernel of this morphism is an fpqc sheaf

$$\mathrm{Aut}^{\otimes!}(\mathcal{F}) : (\mathrm{Sch}/T)^{\circ} \rightarrow \mathrm{Group}.$$

If \mathcal{G} is a splitting of \mathcal{F} , then $\mathrm{Gr}_{\mathcal{F}}^{\bullet} \simeq \mathcal{G}$, thus $\mathrm{Aut}^{\otimes}(\mathrm{Gr}_{\mathcal{F}}^{\bullet}) \simeq \mathrm{Aut}^{\otimes}(\mathcal{G})$ and

$$\mathrm{Aut}^{\otimes}(\mathcal{F}) \simeq \mathrm{Aut}^{\otimes!}(\mathcal{F}) \rtimes \mathrm{Aut}^{\otimes}(\mathcal{G}).$$

3.2.9. There is finally another equivariant diagram of fpqc sheaves on S ,

$$\begin{array}{ccc} G & & \mathbb{G}^\Gamma(G) \xrightarrow{\text{Fil}} \mathbb{F}^\Gamma(G) \\ \downarrow \iota & \text{acting on} & \downarrow \iota \\ \text{Aut}^\otimes(V) & & \mathbb{G}^\Gamma(V) \xrightarrow{\text{Fil}} \mathbb{F}^\Gamma(V) \end{array}$$

The morphism $\iota : G \rightarrow \text{Aut}^\otimes(V)$ sends $g \in G(T)$ to the automorphism $\iota(g)$ of V_T which maps $\rho \in \text{Rep}(G)(X)$ to the automorphism $\rho(g_X)$ of $V(\rho)$ – for an S -scheme T and a T -scheme X . The morphism $\iota : \mathbb{F}^\Gamma(G) \hookrightarrow \mathbb{F}^\Gamma(V)$ is the image of

$$\mathbb{G}^\Gamma(G) \xrightarrow{\iota} \mathbb{G}^\Gamma(V) \xrightarrow{\text{Fil}} \mathbb{F}^\Gamma(V)$$

where $\iota : \mathbb{G}^\Gamma(G) \rightarrow \mathbb{G}^\Gamma(V)$ is defined as follows. Recall from [13, I 4.7.3] that the fpqc stacks $\text{Gr}^\Gamma \text{QCoh}$ and $\text{Rep} \mathbb{D}(\Gamma)$ over Sch are \otimes -equivalent: A Γ -graded quasi-coherent sheaf $\mathcal{M} = \bigoplus_{\gamma \in \Gamma} \mathcal{G}_\gamma$ on a scheme X is mapped to the unique representation ρ of $\mathbb{D}_X(\Gamma)$ on \mathcal{M} such that for every $f : Y \rightarrow X$ and $\alpha : \Gamma \rightarrow \Gamma(Y, \mathcal{O}_Y^*)$ in $\mathbb{D}_X(\Gamma)(Y)$, $\rho(\alpha)(x)$ equals $\alpha(\gamma) \cdot x$ for every $\gamma \in \Gamma$ and $x \in \Gamma(Y, f^* \mathcal{G}_\gamma)$. Conversely, a representation ρ of $\mathbb{D}_X(\Gamma)$ on a quasi-coherent \mathcal{O}_X -module \mathcal{M} is sent to the Γ -grading on \mathcal{M} defined by the eigenspace decomposition of ρ . Then ι maps a morphism $\chi : \mathbb{D}_T(\Gamma) \rightarrow G_T$ in $\mathbb{G}^\Gamma(G)(T)$ to the Γ -graduation on V_T defined by

$$\text{Rep}(G_T) \xrightarrow{-\circ \chi} \text{Rep}(\mathbb{D}_T(\Gamma)) \simeq \text{Gr}^\Gamma \text{QCoh}/T \xrightarrow{\text{forg}} \text{QCoh}/T.$$

Remark 23. We will show in corollary 40 that for a *reductive* group G , the definition of the fpqc sheaf $\mathbb{F}^\Gamma(G)$ on Sch/S given here (image of $\mathbb{G}^\Gamma(G) \rightarrow \mathbb{F}^\Gamma(V)$) coincides with the definition of section 2.2 (image of $\mathbb{G}^\Gamma(G) \rightarrow \mathbb{G}^\Gamma(R_{\mathbb{P}(G)})$).

3.3. The subcategories of rigid objects. We briefly discuss the $-^\circ$ variants of the above definitions, mostly mentioning the new features.

3.3.1. *Finite locally free sheaves.* Let $\text{LF} \rightarrow \text{Sch}$ be the fibered category whose fiber over X is the full subcategory $\text{LF}(X)$ of $\text{QCoh}(X)$ whose objects are the finite locally free sheaves on X . Then LF is a substack of QCoh by [17, 2.5.2]. Pulling back everything through $\text{LF} \hookrightarrow \text{QCoh}$, we obtain a diagram of fpqc stacks over Sch ,

$$\text{Gr}^\Gamma \text{LF} \begin{array}{c} \xrightarrow{\text{Fil}} \\ \xleftarrow{\text{Gr}} \end{array} \text{Fil}^\Gamma \text{LF} \xrightarrow{\text{forg}} \text{LF}$$

whose fiber over a scheme X is a diagram of exact (in Quillen's sense) \otimes -functors

$$\text{Gr}^\Gamma \text{LF}(X) \begin{array}{c} \xrightarrow{\text{Fil}} \\ \xleftarrow{\text{Gr}} \end{array} \text{Fil}^\Gamma \text{LF}(X) \xrightarrow{\text{forg}} \text{LF}(X).$$

An alternative and useful description of the objects of $\text{Fil}^\Gamma \text{LF}(X)$ is provided by Proposition 24 below, which also implies that the Gr functor is indeed well-defined. Over a base scheme S , there is the corresponding diagram of fpqc stacks:

$$\text{Gr}^\Gamma \text{LF}/S \begin{array}{c} \xrightarrow{\text{Fil}} \\ \xleftarrow{\text{Gr}} \end{array} \text{Fil}^\Gamma \text{LF}/S \xrightarrow{\text{forg}} \text{LF}/S$$

3.3.2. These categories have compatible inner Hom's and duals given by

$$\underline{\mathrm{Hom}}(x, y) = x^\vee \otimes y \quad \text{with} \quad (\mathcal{M}, \mathcal{G})^\vee = (\mathcal{M}^\vee, \mathcal{G}^\vee) \quad \text{and} \quad (\mathcal{M}, \mathcal{F})^\vee = (\mathcal{M}^\vee, \mathcal{F}^\vee)$$

where \mathcal{M}^\vee is the dual of \mathcal{M} , $(\mathcal{G}^\vee)_\gamma = (\mathcal{G}_{-\gamma})^\vee$ and $(\mathcal{F}^\vee)^\gamma = (\mathcal{F}_+^{-\gamma})^\perp = (\mathcal{M}/\mathcal{F}_+^{-\gamma})^\vee$. Thus if \mathcal{G} is a splitting of \mathcal{F} , then \mathcal{G}^\vee is a splitting of \mathcal{F}^\vee . Moreover, we have

$$(\mathcal{F}^\vee)_+^\gamma = (\mathcal{F}^{-\gamma})^\perp \simeq (\mathcal{M}/\mathcal{F}^{-\gamma})^\vee \quad \text{and} \quad \mathrm{Gr}_{\mathcal{F}^\vee}^\gamma(\mathcal{M}^\vee) \simeq \mathrm{Gr}_{\mathcal{F}}^{-\gamma}(\mathcal{M})^\vee.$$

For the inner Homs, we obtain the following formula:

$$\mathrm{Gr}_{\mathcal{F}}^\gamma(\underline{\mathrm{Hom}}(\mathcal{M}_1, \mathcal{M}_2)) \simeq \bigoplus_{\gamma_2 - \gamma_1 = \gamma} \underline{\mathrm{Hom}}(\mathrm{Gr}_{\mathcal{F}_1}^{\gamma_1}(\mathcal{M}_1), \mathrm{Gr}_{\mathcal{F}_2}^{\gamma_2}(\mathcal{M}_2)).$$

3.3.3. Γ -filtrations on finite locally free sheaves.

Proposition 24. *Let \mathcal{M} be a finite locally free sheaf on X . Let $(\mathcal{F}^\gamma)_{\gamma \in \Gamma}$ be a non-increasing collection of quasi-coherent subsheaves of \mathcal{M} . Then the following conditions are equivalent:*

- (1) *For every affine open subset U of X , there is a Γ -graduation $\mathcal{M}_U = \bigoplus_{\gamma \in \Gamma} \mathcal{G}_\gamma$ such that $\mathcal{F}_U^\gamma = \bigoplus_{\eta \geq \gamma} \mathcal{G}_\eta$ for every $\gamma \in \Gamma$.*
- (2) *Locally on X for the Zariski topology, there is a Γ -graduation $\mathcal{M} = \bigoplus_{\gamma \in \Gamma} \mathcal{G}_\gamma$ such that $\mathcal{F}^\gamma = \bigoplus_{\eta \geq \gamma} \mathcal{G}_\eta$ for every $\gamma \in \Gamma$.*
- (3) *Locally on X for the fpqc topology, there exists a Γ -graduation $\mathcal{M} = \bigoplus_{\gamma \in \Gamma} \mathcal{G}_\gamma$ such that $\mathcal{F}^\gamma = \bigoplus_{\eta \geq \gamma} \mathcal{G}_\eta$ for every $\gamma \in \Gamma$, i.e. \mathcal{F} is a Γ -filtration on \mathcal{M} .*
- (4) *For every $\gamma \in \Gamma$, $\mathrm{Gr}_{\mathcal{F}}^\gamma(\mathcal{M})$ is finite locally free and for every $x \in X$,*

$$\dim_{k(x)} \mathcal{M}(x) = \sum_\gamma \dim_{k(x)} \mathrm{Gr}_{\mathcal{F}(x)}^\gamma(\mathcal{M}(x)).$$

In (4), $\mathcal{F}(x)$ is the image of \mathcal{F} in $\mathcal{M}(x) = \mathcal{M} \otimes k(x)$ and $\mathrm{Gr}_{\mathcal{F}}^\gamma(\mathcal{M})$, $\mathrm{Gr}_{\mathcal{F}(x)}^\gamma(\mathcal{M}(x))$ are defined as usual. Under the above equivalent conditions, for all $\gamma \in \Gamma$: \mathcal{F}^γ , \mathcal{F}_+^γ and $\mathrm{Gr}_{\mathcal{F}}^\gamma(\mathcal{M})$ are finite locally free sheaves on X and for every $x \in X$,

$$\mathcal{F}^\gamma(x) \simeq \mathcal{F}^\gamma \otimes k(x), \quad \mathcal{F}_+^\gamma(x) \simeq \mathcal{F}_+^\gamma \otimes k(x), \quad \mathrm{Gr}_{\mathcal{F}(x)}^\gamma(\mathcal{M}(x)) \simeq \mathrm{Gr}_{\mathcal{F}}^\gamma(\mathcal{M}) \otimes k(x).$$

Proof. Plainly (1) \Rightarrow (2) \Rightarrow (3). Moreover (3) \Rightarrow (4) is easy (using [17, 2.5.2.iii]) and the last assertions follow from (1). To prove that (4) \Rightarrow (1), we may assume that $X = U$ is affine. Since $\mathrm{Gr}_{\mathcal{F}}^\gamma(\mathcal{M})$ is finite locally free by assumption, it is then projective in $\mathrm{QCoh}(X)$ by [23, Corollary of 7.12]. Therefore, there exists a quasi-coherent subsheaf \mathcal{G}_γ of \mathcal{F}_γ such that $\mathcal{F}^\gamma = \mathcal{G}_\gamma \oplus \mathcal{F}_+^\gamma$. We will show that

$$\mathcal{M} = \bigoplus_{\gamma \in \Gamma} \mathcal{G}_\gamma \quad \text{and} \quad \forall \gamma : \mathcal{F}^\gamma = \bigoplus_{\eta \geq \gamma} \mathcal{G}_\eta.$$

This being now a local question in the Zariski topology of X , we may assume that the rank of \mathcal{M} is constant on X , and also nonzero. Fix $x \in X$ and define

$$\Gamma(x) = \{\gamma : \mathrm{Gr}_{\mathcal{F}(x)}^\gamma(\mathcal{M}(x)) \neq 0\} = \{\gamma_1 < \dots < \gamma_r\}.$$

Define $U_0 = \mathrm{Supp}(\mathcal{M}/\mathcal{F}^{\gamma_1})^c$, $U_i = \mathrm{Supp}(\mathcal{F}_+^{\gamma_i}/\mathcal{F}^{\gamma_{i+1}})^c \cap U_{i-1}$ for $0 < i < r$ and $U_r = \mathrm{Supp}(\mathcal{F}_+^{\gamma_r})^c \cap U_{r-1}$. Since \mathcal{M} is finite locally free, $\mathcal{M}/\mathcal{F}^{\gamma_1}$ is finitely generated and U_0 is open in X . Since $\mathcal{M} = \mathcal{F}^{\gamma_1}$ over U_0 and $\mathcal{F}^{\gamma_1} = \mathcal{F}_+^{\gamma_1} \oplus \mathcal{G}_{\gamma_1}$ over X , $\mathcal{M} = \mathcal{F}_+^{\gamma_1} \oplus \mathcal{G}_{\gamma_1}$ over U_0 . Therefore $\mathcal{F}_+^{\gamma_1}$ is finite locally free over U_0 . Repeating this argument successively with (\mathcal{M}, X) replaced by $(\mathcal{F}_+^{\gamma_1}, U_0)$, $(\mathcal{F}_+^{\gamma_2}, U_1)$ etc... we obtain: U_r is open in X , $\mathcal{M} = \bigoplus_i \mathcal{G}_{\gamma_i}$ and $\mathcal{F}^\gamma = \bigoplus_{i: \gamma_i \geq \gamma} \mathcal{G}_{\gamma_i}$ over U_r for every $\gamma \in \Gamma$,

with everyone finite locally free over U_r . All we have to do now is to show that the formula of (4) implies that x belongs to U_r . The formula is equivalent to:

$$\mathcal{F}^\gamma(x) = \begin{cases} \mathcal{M}(x) & \text{if } \gamma \leq \gamma_1, \\ \mathcal{F}^{\gamma_{i+1}}(x) & \text{if } \gamma \in]\gamma_i, \gamma_{i+1}], \\ 0 & \text{if } \gamma > \gamma_r. \end{cases}$$

Since \mathcal{M} is finitely generated over X , $\mathcal{F}^{\gamma_1}(x) = \mathcal{M}(x)$ implies $\mathcal{F}_x^{\gamma_1} = \mathcal{M}_x$ by Nakayama's Lemma, thus x belongs to U_0 . Since $\mathcal{M} = \mathcal{F}^{\gamma_1} = \mathcal{F}_+^{\gamma_1} \oplus \mathcal{G}_{\gamma_1}$ over U_0 , $\mathcal{F}_+^{\gamma_1}(x) = \mathcal{F}^{\gamma_2}(x)$ in $\mathcal{M}(x)$ implies $\mathcal{F}_{+,x}^{\gamma_1} = \mathcal{F}_x^{\gamma_2}$ by Nakayama's Lemma, therefore x belongs to U_1 . Repeating the argument, we find that indeed x belongs to U_r . \square

Remark 25. The whole proof becomes much simpler over a Noetherian base.

Lemma 26. *Let \mathcal{M}_α be a finite collection of locally free sheaves of finite rank on X and for each α , let $(\mathcal{F}_\alpha^\gamma)_{\gamma \in \Gamma}$ be a non-increasing collection of quasi-coherent subsheaves of \mathcal{M}_α . Set $\mathcal{M} = \bigoplus \mathcal{M}_\alpha$ and $\mathcal{F}^\gamma = \bigoplus \mathcal{F}_\alpha^\gamma$. Then $(\mathcal{M}, (\mathcal{F}^\gamma))$ satisfies the above equivalent conditions if and only if each $(\mathcal{M}_\alpha, (\mathcal{F}_\alpha^\gamma))$ does.*

Proof. For every $\gamma \in \Gamma$ and $x \in X$, $\text{Gr}_{\mathcal{F}}^\gamma(\mathcal{M}) = \bigoplus_\alpha \text{Gr}_{\mathcal{F}_\alpha}^\gamma(\mathcal{M}_\alpha)$ and

$$\mathcal{M}(x) = \bigoplus_\alpha \mathcal{M}_\alpha(x), \quad \text{Gr}_{\mathcal{F}(x)}^\gamma(\mathcal{M}(x)) = \bigoplus_\alpha \text{Gr}_{\mathcal{F}_\alpha(x)}^\gamma(\mathcal{M}_\alpha(x)).$$

Moreover for every α and $x \in X$,

$$\dim_{k(x)} \mathcal{M}_\alpha(x) \geq \sum_\gamma \dim_{k(x)} \text{Gr}_{\mathcal{F}_\alpha(x)}^\gamma(\mathcal{M}_\alpha(x)).$$

The lemma easily follows. \square

3.3.4. Let $\text{Rep}^\circ(G) \rightarrow \text{Sch}/S$ be the substack of $\text{Rep}(G) \rightarrow \text{Sch}/S$ whose fiber over $T \rightarrow S$ is the exact, rigid, full sub- \otimes -category $\text{Rep}^\circ(G)(T)$ of $\text{Rep}(G)(T)$ whose objects are the representations of G_T on finite locally free sheaves on T . We write

$$V^\circ : \text{Rep}^\circ(G) \rightarrow \text{LF}/S$$

for the forgetful functor. For an S -scheme $T \rightarrow S$, we denote by

$$V_T^\circ : \text{Rep}^\circ(G_T) \rightarrow \text{LF}/T \quad \text{and} \quad \omega_T^\circ : \text{Rep}^\circ(G)(S) \rightarrow \text{LF}(T)$$

the induced morphism of fpqc stack over Sch/T and fiber functor. Note that ω_T° is now an exact \otimes -functor, since all short exact sequences in $\text{Rep}^\circ(G)(S)$ are pure.

3.3.5. We obtain yet another equivariant diagram of fpqc sheaves on S , namely

$$\begin{array}{ccc} \text{Aut}^\otimes(V^\circ) & & \mathbb{G}^\Gamma(V^\circ) \xrightarrow{\text{Fil}} \mathbb{F}^\Gamma(V^\circ) \\ \downarrow \text{res} & \text{acting on} & \downarrow \text{res} \\ \text{Aut}^\otimes(\omega^\circ) & & \mathbb{G}^\Gamma(\omega^\circ) \xrightarrow{\text{Fil}} \mathbb{F}^\Gamma(\omega^\circ) \end{array}$$

where everything is defined as before, using V° and ω° instead of V and ω . The only differences worth mentioning are as follows: for any S -scheme T , the Γ -graduations

or Γ -filtrations on ω_T° are automatically compatible with inner Homs and duals, and there γ -components are exact functors. We also have equivariant diagrams

$$\begin{array}{ccc} \text{Aut}^\otimes(V) & & \mathbb{G}^\Gamma(V) \xrightarrow{\text{Fil}} \mathbb{F}^\Gamma(V) \\ \downarrow \text{res} & \text{acting on} & \downarrow \text{res} \\ \text{Aut}^\otimes(V^\circ) & & \mathbb{G}^\Gamma(V^\circ) \xrightarrow{\text{Fil}} \mathbb{F}^\Gamma(V^\circ) \end{array}$$

and similarly for ω and ω° , where all the vertical maps are induced by pre-composition with the full embedding $\text{Rep}^\circ(G) \hookrightarrow \text{Rep}(G)$.

3.3.6. Finally, the definitions of $\text{Aut}^\otimes(\mathcal{G})$, $\text{Aut}^\otimes(\mathcal{F})$, $\text{Aut}^{\otimes!}(\mathcal{F})$ and $\text{Aut}^\otimes(\text{Gr}_{\mathcal{F}}^\bullet)$ given in section 3.2.8 carry over to the situation considered here.

3.4. **Skalar extensions.** The whole diagram at the beginning of this section has now been defined. It is covariantly functorial in G but not entirely compatible with base change on S : if $\tilde{S} \rightarrow S$ is any morphism, $\tilde{G} = G \times_S \tilde{S}$ and $\tilde{V}, \tilde{\omega} \dots$ are the relevant functors for \tilde{G} , then $\mathbb{G}^\Gamma(\tilde{G}) = \mathbb{G}^\Gamma(G)|_{\tilde{S}}$, $\mathbb{F}^\Gamma(\tilde{G}) = \mathbb{F}^\Gamma(G)|_{\tilde{S}}$ and

$$\text{Aut}^\otimes(\tilde{X}) = \text{Aut}^\otimes(X)|_{\tilde{S}}, \quad \mathbb{G}^\Gamma(\tilde{X}) = \mathbb{G}^\Gamma(X)|_{\tilde{S}} \quad \text{and} \quad \mathbb{F}^\Gamma(\tilde{X}) = \mathbb{F}^\Gamma(X)|_{\tilde{S}}$$

for $X \in \{V, V^\circ\}$, but the natural morphisms of fpqc sheaves on \tilde{S} ,

$$\text{Aut}^\otimes(\tilde{Y}) \rightarrow \text{Aut}^\otimes(Y)|_{\tilde{S}}, \quad \mathbb{G}^\Gamma(\tilde{Y}) \rightarrow \mathbb{G}^\Gamma(Y)|_{\tilde{S}} \quad \text{and} \quad \mathbb{F}^\Gamma(\tilde{Y}) \rightarrow \mathbb{F}^\Gamma(Y)|_{\tilde{S}}$$

may not be isomorphisms for $Y \in \{\omega, \omega^\circ\}$. We investigate this issue.

3.4.1. When \mathcal{C} is a category and \mathcal{B} is a ring object in \mathcal{C} , we can form the category $\mathcal{C}(\mathcal{B})$ of (left) \mathcal{B} -modules in \mathcal{C} . Here \mathcal{C} will be an additive \otimes -category and the ring object will be given by its multiplication morphism $\mu : \mathcal{B} \otimes \mathcal{B} \rightarrow \mathcal{B}$ and unit $1 \rightarrow \mathcal{B}$, where 1 is the neutral object for the tensor product, the abelian group structure on \mathcal{B} being provided by the additive structure of \mathcal{C} . Then $\mathcal{C}(\mathcal{B})$ is the category of pairs (\mathcal{M}, ν) where \mathcal{M} is an object of \mathcal{C} and $\nu : \mathcal{B} \otimes \mathcal{M} \rightarrow \mathcal{M}$ is a morphism in \mathcal{C} subject to certain natural conditions. There is an adjunction

$$f^* : \mathcal{C} \leftrightarrow \mathcal{C}(\mathcal{B}) : f_* \quad \text{given by} \quad f_*(\mathcal{M}, \nu) = \mathcal{M} \quad \text{and} \quad f^*(\mathcal{N}) = (\mathcal{B} \otimes \mathcal{N}, \mu \otimes \text{Id}).$$

In many cases, it is also possible to equip $\mathcal{C}(\mathcal{B})$ with a \otimes -product inherited from the \otimes -product on \mathcal{C} , with (\mathcal{B}, μ) as neutral object. Instead of trying to develop this formal theory more rigorously, let us list some of the relevant examples:

$\mathcal{C} = \text{QCoh}(S)$ and $\mathcal{B} = f_* \mathcal{O}_T$ where $f : T \rightarrow S$ is an affine morphism. There is an equivalence of \otimes -categories $\mathcal{C}(\mathcal{B}) \simeq \text{QCoh}(T)$ which is compatible with the usual adjunctions $f^* : \text{QCoh}(S) \leftrightarrow \text{QCoh}(T) : f_*$, see [15, 1.4].

$\mathcal{C} = \text{Gr}^\Gamma \text{QCoh}(S)$ and \mathcal{B} as above with the trivial Γ -graduation. The first example induces an equivalence of \otimes -categories $\mathcal{C}(\mathcal{B}) \simeq \text{Gr}^\Gamma \text{QCoh}(T)$ which is again compatible with the natural adjunctions.

$\mathcal{C} = \text{Fil}^\Gamma \text{QCoh}(S)$ and \mathcal{B} as above with the trivial Γ -filtration. The first example now only induces a fully faithful exact \otimes -functor $\mathcal{C}(\mathcal{B}) \hookrightarrow \text{Fil}^\Gamma \text{QCoh}(T)$. The essential image is made of those Γ -filtered quasi-coherent sheaves $(\mathcal{M}, \mathcal{F})$ on T such that, locally on S (as opposed to T) for the fpqc topology, \mathcal{F} has a splitting.

$\mathbf{C} = \text{Rep}(G)(S)$ and \mathcal{B} as above with the trivial action of G . The first example again induces an equivalence of \otimes -categories $\mathbf{C}(\mathcal{B}) \simeq \text{Rep}(G)(T)$ which is compatible with the adjunctions given on the comodules by the following formulas

$$\begin{aligned} f^* \left(V(\rho) \xrightarrow{c_\rho} V(\rho) \otimes_{\mathcal{O}_S} \mathcal{A}(G) \right) &= \left(V(f^*\rho) \xrightarrow{c_{f^*\rho}} V(f^*\rho) \otimes_{\mathcal{O}_T} \mathcal{A}(G_T) \right), \\ f_* \left(V(\rho) \xrightarrow{c_\rho} V(\rho) \otimes_{\mathcal{O}_T} \mathcal{A}(G_T) \right) &= \left(V(f_*\rho) \xrightarrow{c_{f_*\rho}} V(f_*\rho) \otimes_{\mathcal{O}_S} \mathcal{A}(G) \right). \end{aligned}$$

$\mathbf{C} = \text{LF}(S)$ and $\mathcal{B} = f_*\mathcal{O}_T$ where $f : T \rightarrow S$ is a finite étale morphism. The first example induces an equivalence of \otimes -categories $\mathbf{C}(\mathcal{B}) \simeq \text{LF}(T)$. We have to show that for a quasi-coherent sheaf \mathcal{M} on T , \mathcal{M} is a finite locally free \mathcal{O}_T -module if and only if $f_*\mathcal{M}$ is a finite locally free \mathcal{O}_S -module (the direct implication is easy, and only requires f to be finite and locally free). By [17, 2.5.2], our claim is local in the fpqc topology on S . But, locally on S for the étale topology, our finite étale morphism f is simply a finite disjoint union of open and closed embeddings (this follows from [19, 17.9.3]), for which the claim is now obvious.

Combining this last example with the previous three, we obtain:

$\mathbf{C} = \text{Gr}^\Gamma \text{LF}(S)$ and \mathcal{B} as above with the trivial Γ -graduation. Then $\mathbf{C}(\mathcal{B}) \simeq \text{Gr}^\Gamma \text{LF}(T)$.

$\mathbf{C} = \text{Fil}^\Gamma \text{LF}(S)$ and \mathcal{B} as above with the trivial Γ -filtration. Then $\mathbf{C}(\mathcal{B}) \simeq \text{Fil}^\Gamma \text{LF}(T)$.

$\mathbf{C} = \text{Rep}^\circ(G)(S)$ and \mathcal{B} as above with the trivial action. Then $\mathbf{C}(\mathcal{B}) \simeq \text{Rep}^\circ(G)(T)$.

3.4.2. The point of this abstract nonsense is that, if $\alpha : \mathbf{C} \rightarrow \mathbf{D}$ is a \otimes -functor and \mathcal{B} is a ring object in \mathbf{C} , then $\alpha(\mathcal{B})$ is a ring object in \mathbf{D} and α extends to a \otimes -functor $\alpha(\mathcal{B}) : \mathbf{C}(\mathcal{B}) \rightarrow \mathbf{D}(\alpha(\mathcal{B}))$ which we call the skalar extension of α . Similarly, if η is a \otimes -isomorphism of α such that $\eta_{\mathcal{B}}$ is the identity of $\alpha(\mathcal{B})$, then η extends to a \otimes -isomorphism $\eta(\mathcal{B})$ of $\alpha(\mathcal{B})$ which we call the skalar extension of η .

Proposition 27. (1) Let $f : \tilde{S} \rightarrow S$ be a finite étale morphism and denote by $\tilde{\omega}$ the fiber functors for $\tilde{G} = G_{\tilde{S}}$. Then we have isomorphisms of fpqc sheaves on \tilde{S} :

$$\text{Aut}^\otimes(\omega^\circ)|_{\tilde{S}} = \text{Aut}^\otimes(\tilde{\omega}^\circ), \quad \mathbb{G}^\Gamma(\omega^\circ)|_{\tilde{S}} = \mathbb{G}^\Gamma(\tilde{\omega}^\circ) \quad \text{and} \quad \mathbb{F}^\Gamma(\omega^\circ)|_{\tilde{S}} = \mathbb{F}^\Gamma(\tilde{\omega}^\circ).$$

(2) If f is merely affine, then $\mathbb{F}^\Gamma(\omega)|_{\tilde{S}} = \mathbb{F}^\Gamma(\tilde{\omega})$.

Proof. (1) Let T be an \tilde{S} -scheme. We have to define mutually inverse maps

$$\alpha : \begin{array}{ccc} \text{Aut}^\otimes(\tilde{\omega}^\circ)(T) & \longleftrightarrow & \text{Aut}^\otimes(\omega^\circ)(T) \\ \mathbb{G}^\Gamma(\tilde{\omega}^\circ)(T) & \longleftrightarrow & \mathbb{G}^\Gamma(\omega^\circ)(T) \\ \mathbb{F}^\Gamma(\tilde{\omega}^\circ)(T) & \longleftrightarrow & \mathbb{F}^\Gamma(\omega^\circ)(T) \end{array} : \beta$$

functorial in T . The α maps are induced by precomposition with the base change map $\text{Rep}^\circ(G)(S) \rightarrow \text{Rep}^\circ(G)(\tilde{S})$. The β maps are defined by composing the skalar extension maps with the base change maps for the \tilde{S} -section $\iota : T \rightarrow \tilde{T}$ of the projection $f_T : \tilde{T} = T \times_S \tilde{S} \rightarrow T$ given by the structural morphism $T \rightarrow \tilde{S}$:

$$\beta : \begin{array}{ccccc} \text{Aut}^\otimes(\omega^\circ)(T) & \longrightarrow & \text{Aut}^\otimes(\tilde{\omega}^\circ)(\tilde{T}) & \longrightarrow & \text{Aut}^\otimes(\tilde{\omega}^\circ)(T) \\ \mathbb{G}^\Gamma(\omega^\circ)(T) & \longrightarrow & \mathbb{G}^\Gamma(\tilde{\omega}^\circ)(\tilde{T}) & \longrightarrow & \mathbb{G}^\Gamma(\tilde{\omega}^\circ)(T) \\ \mathbb{F}^\Gamma(\omega^\circ)(T) & \longrightarrow & \mathbb{F}^\Gamma(\tilde{\omega}^\circ)(\tilde{T}) & \longrightarrow & \mathbb{F}^\Gamma(\tilde{\omega}^\circ)(T) \end{array}$$

Explicitly, for η, \mathcal{G} and \mathcal{F} in the source sets and $\tilde{\rho} \in \text{Rep}^\circ(\tilde{G})(\tilde{S})$, we first view $f_*\tilde{\rho}$ as a \mathcal{B} -module in $\text{Rep}^\circ(G)(S)$ where $\mathcal{B} = f_*\mathcal{O}_{\tilde{S}}$ with trivial G -action. Then:

- $\eta_{f_*\tilde{\rho}}$ is a \mathcal{B}_T -linear isomorphism of $\omega_T^\circ(f_*\tilde{\rho}) = (f_T)_*\tilde{\omega}_T^\circ(\tilde{\rho})$. It thus corresponds to an isomorphism of $\tilde{\omega}_T^\circ(\tilde{\rho})$ whose pull-back to $\iota^*\tilde{\omega}_T^\circ(\tilde{\rho}) = \tilde{\omega}_T^\circ(\tilde{\rho})$ is an isomorphism $\beta(\eta)_{\tilde{\rho}}$. By construction, there is a commutative diagram

$$\begin{array}{ccc} \omega_T^\circ(f_*\tilde{\rho}) = \tilde{\omega}_T^\circ(f_*f_*\tilde{\rho}) & \longrightarrow & \tilde{\omega}_T^\circ(\tilde{\rho}) \\ \eta_{f_*\tilde{\rho}} \downarrow & & \downarrow \beta(\eta)_{\tilde{\rho}} \\ \omega_T^\circ(f_*\tilde{\rho}) = \tilde{\omega}_T^\circ(f_*f_*\tilde{\rho}) & \longrightarrow & \tilde{\omega}_T^\circ(\tilde{\rho}) \end{array}$$

where the horizontal map comes from the adjunction morphism $f^*f_*\tilde{\rho} \rightarrow \tilde{\rho}$.

- $\mathcal{G}(f_*\tilde{\rho})$ is a \mathcal{B}_T -stable Γ -graduation on $(f_T)_*\tilde{\omega}_T^\circ(\tilde{\rho})$, giving a Γ -graduation on $\tilde{\omega}_T^\circ(\tilde{\rho})$ whose pull-back is a Γ -graduation $\beta(\mathcal{G})(\tilde{\rho})$ on $\tilde{\omega}_T^\circ(\tilde{\rho})$. Thus $\beta(\mathcal{G})_\gamma(\tilde{\rho})$ is the image of $\mathcal{G}_\gamma(f_*\tilde{\rho})$ under the adjunction morphism $\omega_T^\circ(f_*\tilde{\rho}) \rightarrow \tilde{\omega}_T^\circ(\tilde{\rho})$.
- $\mathcal{F}(f_*\tilde{\rho})$ is a \mathcal{B}_T -stable Γ -filtration on $(f_T)_*\tilde{\omega}_T^\circ(\tilde{\rho})$, giving a Γ -filtration on $\tilde{\omega}_T^\circ(\tilde{\rho})$ whose pull-back is a Γ -filtration $\beta(\mathcal{F})(\tilde{\rho})$ on $\tilde{\omega}_T^\circ(\tilde{\rho})$. Thus $\mathcal{B}(\mathcal{F})^\gamma(\tilde{\rho})$ is the image of $\mathcal{F}^\gamma(f_*\tilde{\rho})$ under the adjunction morphism $\omega_T^\circ(f_*\tilde{\rho}) \rightarrow \tilde{\omega}_T^\circ(\tilde{\rho})$.

One checks easily that $\alpha \circ \beta = \text{Id}$ and $\beta \circ \alpha = \text{Id}$. The proof of (2) is similar. \square

Remark 28. We have not mentioned $\text{Aut}^\otimes(\omega)$ and $\mathbb{G}^\Gamma(\omega)$ in part (2) of the above Proposition, because we will establish a stronger result for them in the next section.

3.5. The regular representation. The single most important representation of G is the regular representation ρ_{reg} . We shall use it to establish the classical:

Theorem 29. *The above morphisms of fpqc sheaves induce isomorphisms*

$$G \simeq \text{Aut}^\otimes(V) \simeq \text{Aut}^\otimes(\omega) \quad \text{and} \quad \mathbb{G}^\Gamma(G) \simeq \mathbb{G}^\Gamma(V) \simeq \mathbb{G}^\Gamma(\omega).$$

3.5.1. The regular representation ρ_{reg} of G on $V(\rho_{\text{reg}}) = \mathcal{A}(G)$ is defined by

$$(g \cdot a)(h) = a(hg)$$

for $T \rightarrow S$, $a \in \Gamma(T, \mathcal{A}(G)_T) = \Gamma(G_T, \mathcal{O}_{G_T})$ and $g, h \in G(T)$. The corresponding $\mathcal{A}(G)$ -comodule structure morphism is the comultiplication map:

$$\left(V(\rho_{\text{reg}}) \xrightarrow{c_{\text{reg}}} V(\rho_{\text{reg}}) \otimes_{\mathcal{O}_S} \mathcal{A}(G) \right) = \left(\mathcal{A}(G) \xrightarrow{\mu^h} \mathcal{A}(G) \otimes_{\mathcal{O}_S} \mathcal{A}(G) \right)$$

The \mathcal{O}_S -algebra structure morphisms on $\mathcal{A}(G)$, namely the unit $\mathcal{O}_S \rightarrow \mathcal{A}(G)$ and the multiplication $\mathcal{A}(G) \otimes \mathcal{A}(G) \rightarrow \mathcal{A}(G)$ correspond to G -equivariant morphisms

$$1_S \rightarrow \rho_{\text{reg}} \quad \text{and} \quad \rho_{\text{reg}} \otimes \rho_{\text{reg}} \rightarrow \rho_{\text{reg}}.$$

For any $\rho \in \text{Rep}(G)(S)$, we denote by $\rho_0 \in \text{Rep}(G)(S)$ the trivial representation of G on $V(\rho_0) = V(\rho)$. We may then view the $\mathcal{A}(G)$ -comodule structure morphism $c_\rho : V(\rho) \rightarrow V(\rho) \otimes_{\mathcal{O}_S} \mathcal{A}(G)$ of ρ as a G -equivariant morphism in $\text{Rep}(G)(S)$

$$c_\rho : \rho \rightarrow \rho_0 \otimes \rho_{\text{reg}}$$

The underlying morphism of quasi-coherent sheaves on S is a split monomorphism since $(\text{Id} \otimes 1_G^h) \circ c_\rho = \text{Id}$ on $V(\rho)$ where $1_G^h : \mathcal{A}(G) \rightarrow \mathcal{O}_S$ is the counit of $\mathcal{A}(G)$.

3.5.2. It follows that any $\eta \in \text{Aut}^\otimes(\omega_T)$, $\mathcal{G} \in \mathbb{G}^\Gamma(\omega_T)$ or $\mathcal{F} \in \mathbb{F}^\Gamma(\omega_T)$ is uniquely determined by its value η_{reg} , \mathcal{G}_{reg} or \mathcal{F}_{reg} on ρ_{reg} . Indeed for any $\rho \in \text{Rep}(G)(S)$, η_ρ , $\mathcal{G}(\rho)$ and $\mathcal{F}(\rho)$ will then be the automorphism, Γ -graduation and Γ -filtration on

$$\omega_T(\rho) \xrightarrow{\omega_T(c_\rho)} \omega_T(\rho_0) \otimes \omega_T(\rho_{\text{reg}})$$

which are respectively induced by the corresponding objects for $\rho_0 \otimes \rho_{\text{reg}}$, namely

$$\begin{aligned} \eta_{\rho_0 \otimes \rho_{\text{reg}}} &= \text{Id} \otimes \eta_{\text{reg}}, \\ \mathcal{G}(\rho_0 \otimes \rho_{\text{reg}}) &= \omega_T(\rho_0) \otimes \mathcal{G}_{\text{reg}}, \\ \mathcal{F}(\rho_0 \otimes \rho_{\text{reg}}) &= \omega_T(\rho_0) \otimes \mathcal{F}_{\text{reg}}. \end{aligned}$$

We have here used the defining axioms (A1) and (A2) for η , (G1) and (G2) for \mathcal{G} and (F1) and (F2) for \mathcal{F} , as well as the fact that for every $\gamma \in \Gamma$, the functors \mathcal{G}_γ and $\mathcal{F}^\gamma : \text{Rep}(G)(S) \rightarrow \text{QCoh}(T)$ both preserve pure short exact sequences.

3.5.3. By the same token, we find that the morphisms of fpqc sheaves

$$\text{Aut}^\otimes(V) \rightarrow \text{Aut}^\otimes(\omega), \quad \mathbb{G}^\Gamma(V) \rightarrow \mathbb{G}^\Gamma(\omega) \quad \text{and} \quad \mathbb{F}^\Gamma(V) \rightarrow \mathbb{F}^\Gamma(\omega)$$

are monomorphisms. For instance if $\eta \in \text{Aut}^\otimes(V_T)$ induces the identity of ω_T , then for any $f : X \rightarrow T$ and $\rho \in \text{Rep}(G)(X)$, η_ρ is the identity of $V(\rho)$ because

$$\eta_{\rho_0 \otimes \rho_{\text{reg}, X}} = \eta_{\rho_0} \otimes \eta_{\rho_{\text{reg}, X}} = \text{Id}_{V(\rho_0)} \otimes f^*(\eta_{\rho_{\text{reg}, T}})$$

and $\eta_{\rho_{\text{reg}, T}}$ is the trivial automorphism of $V(\rho_{\text{reg}, T}) = \omega_T(\rho_{\text{reg}})$.

3.5.4. We now show that $G = \text{Aut}^\otimes(\omega)$. Let T be an S -scheme and $\eta \in \text{Aut}^\otimes(\omega_T)$. Recall that η_{reg} is the \mathcal{O}_T -linear automorphism of $\omega_T(\rho_{\text{reg}}) = \mathcal{A}(G_T)$ induced by η . Since $\eta_{1_S} = \text{Id}_{\mathcal{O}_T}$ on $\omega_T(1_S) = \mathcal{O}_T$ by (A2) and $\eta_{\rho_{\text{reg}} \otimes \rho_{\text{reg}}} = \eta_{\text{reg}} \otimes \eta_{\text{reg}}$ on

$$\omega_T(\rho_{\text{reg}} \otimes \rho_{\text{reg}}) = \mathcal{A}(G_T) \otimes \mathcal{A}(G_T)$$

by (A1), the functoriality of η applied to $1_S \rightarrow \rho_{\text{reg}}$ and $\rho_{\text{reg}} \otimes \rho_{\text{reg}} \rightarrow \rho_{\text{reg}}$ implies that η_{reg} is an automorphism of the quasi-coherent \mathcal{O}_T -algebra $\mathcal{A}(G_T)$. Similarly for any $\rho \in \text{Rep}(G)(S)$, the G -equivariant morphism $c_\rho : \rho \rightarrow \rho_0 \otimes \rho_{\text{reg}}$ induces a commutative diagram of quasi-coherent \mathcal{O}_T -modules

$$\begin{array}{ccc} \omega_T(\rho) & \xrightarrow{(c_\rho)_T} & \omega_T(\rho_0) \otimes_{\mathcal{O}_T} \mathcal{A}(G_T) \\ \eta_\rho \downarrow & & \downarrow \text{Id} \otimes \eta_{\text{reg}} \\ \omega_T(\rho) & \xrightarrow{(c_\rho)_T} & \omega_T(\rho_0) \otimes_{\mathcal{O}_T} \mathcal{A}(G_T) \end{array}$$

Composing η_{reg} with the counit $1_{G,T}^\natural : \mathcal{A}(G)_T \rightarrow \mathcal{O}_T$, we obtain a morphism of \mathcal{O}_T -algebras $s(\eta)^\natural : \mathcal{A}(G)_T \rightarrow \mathcal{O}_T$, i.e. a T -valued point $s(\eta) \in G(T)$. Now for any $g \in G(T)$ corresponding to $g^\natural : \mathcal{A}(G)_T \rightarrow \mathcal{O}_T$ and mapping to $\iota(g) \in \text{Aut}^\otimes(\omega_T)$, the automorphism $\iota(g)_\rho = \rho_T(g)$ of $\omega_T(\rho)$ is obtained by composing $(c_\rho)_T$ with

$$\text{Id} \otimes g^\natural : \omega_T(\rho) \otimes_{\mathcal{O}_T} \mathcal{A}(G)_T \rightarrow \omega_T(\rho).$$

We thus find that $s \circ \iota(g) = g$ since

$$s \circ \iota(g)^\natural = 1_{G,T}^\natural \circ \iota(g)_{\text{reg}} = 1_{G,T}^\natural \circ (\text{Id} \otimes g^\natural) \circ \mu_T^\natural = (1_{G,T}^\natural \otimes g^\natural) \circ \mu_T^\natural = (1_{G_T} \cdot g)^\natural = g^\natural.$$

On the other hand, $\iota \circ s(\eta) = \eta$ since for any $\rho \in \text{Rep}(G)(S)$,

$$\begin{aligned} (\iota \circ s)(\eta)_\rho &= \left(\text{Id} \otimes 1_{G,T}^\natural \right) \circ (\text{Id} \otimes \eta_{\text{reg}}) \circ (c_\rho)_T \\ &= \left(\text{Id} \otimes 1_{G,T}^\natural \right) \circ (c_\rho)_T \circ \eta_\rho \\ &= \rho(1_G)_T^\natural \circ \eta_\rho = \eta_\rho. \end{aligned}$$

Thus $G = \text{Aut}^\otimes(\omega)$ and by 3.5.3, also $G = \text{Aut}^\otimes(V)$.

3.5.5. We now show that $\mathbb{G}^\Gamma(G) = \mathbb{G}^\Gamma(\omega)$. Let T be an S -scheme, $\mathcal{G} \in \mathbb{G}^\Gamma(\omega_T)$. Then for any T -scheme X , the Γ -graduation \mathcal{G} on ω_T and the \otimes -equivalence

$$\text{Gr}^\Gamma \text{QCoh}(T) \simeq \text{Rep}(\mathbb{D}_T(\Gamma))(T)$$

together induce a factorization

$$\omega_X^1 : \text{Rep}(G)(S) \xrightarrow{\mathcal{G}'} \text{Rep}(\mathbb{D}_T(\Gamma))(T) \xrightarrow{\omega_X^2} \text{QCoh}(X)$$

of the fiber functor ω_X^1 for the group scheme G over S through the fiber functor ω_X^2 for the group scheme $\mathbb{D}_T(\Gamma)$ over T . Moreover \mathcal{G}' is a \otimes -isomorphism preserving trivial representations by (G1) and (G2). It thus induces a group homomorphism

$$\mathbb{D}_T(\Gamma)(X) \stackrel{3.5.4}{\simeq} \text{Aut}^\otimes(\omega_X^2) \rightarrow \text{Aut}^\otimes(\omega_X^1) \stackrel{3.5.4}{\simeq} G(X).$$

The latter being functorial in X gives a morphism $s(\mathcal{G}) : \mathbb{D}_T(\Gamma) \rightarrow G_T$ of group schemes over T , i.e. an element $s(\mathcal{G})$ of $\mathbb{G}^\Gamma(G)(T)$. Since $\mathcal{G} \mapsto s(\mathcal{G})$ is itself functorial in T , it gives a morphism of fpqc sheaves $s : \mathbb{G}^\Gamma(\omega) \rightarrow \mathbb{G}^\Gamma(G)$ which is the inverse of $\iota : \mathbb{G}^\Gamma(G) \rightarrow \mathbb{G}^\Gamma(\omega)$. Thus $\mathbb{G}^\Gamma(G) = \mathbb{G}^\Gamma(\omega)$ and by 3.5.3, also $\mathbb{G}^\Gamma(G) = \mathbb{G}^\Gamma(V)$.

3.6. Relating $\text{Rep}(G)(S)$ and $\text{Rep}^\circ(G)(S)$.

3.6.1. While $\text{Rep}(G)(S)$ already contains the interesting regular representation, it could be that $\text{Rep}^\circ(G)(S)$ contains no representations beyond the trivial ones, in which case $\text{Aut}^\otimes(\omega^\circ)$, $\mathbb{G}^\Gamma(\omega^\circ)$ and $\mathbb{F}^\Gamma(\omega^\circ)$ are the trivial sheaves represented by S .

For instance, let S be one of the two curves considered in [1, X 6.4], whose enlarged fundamental group equals \mathbb{Z} . Let $n \geq 2$ and $A \in GL_n(\mathbb{Z})$ be any matrix with no roots of unity as eigenvalue. Then by [1, X 7.1], this determines an n -dimensional torus G over S , and all representations $\rho \in \text{Rep}^\circ(G)(S)$ are trivial because \mathbb{Z}^n contains no finite A -orbit except $\{0\}$.

3.6.2. When S is quasi-compact, we also consider the full subcategory $\text{Rep}'(G)(S)$ of $\text{Rep}(G)(S)$ whose objects are the representations ρ for which $\rho = \varinjlim \tau$ where τ runs through the partially ordered set $X(\rho)$ of all subrepresentations of ρ which belong to $\text{Rep}^\circ(G)(S)$. For such ρ 's, $V(\rho) = \varinjlim V(\tau)$ is a flat \mathcal{O}_S -module and the quasi-compactness of S implies that $X(\rho)$ is a filtered set. This subcategory is stable under tensor product and the $\rho \mapsto \rho_0$ construction, it contains $\text{Rep}^\circ(G)(S)$ as a full subcategory, and for any $\rho_1, \rho_2 \in \text{Rep}'(G)(S)$,

$$(3.1) \quad \text{Hom}_{\text{Rep}(G)}(\rho_1, \rho_2) = \varprojlim_{\tau_1 \in X(\rho_1)} \varinjlim_{\tau_2 \in X(\rho_2)} \text{Hom}_{\text{Rep}^\circ(G)}(\tau_1, \tau_2).$$

We denote by $\omega'_T : \text{Rep}'(G)(S) \rightarrow \text{QCoh}(T)$ the restriction of ω_T to $\text{Rep}'(G)(S)$ and define the fpqc sheaf $\text{Aut}^\otimes(\omega') : (\text{Sch}/S)^\circ \rightarrow \text{Group}$ as before, with automorphisms of ω'_T satisfying the axioms (A1) and (A2), thus obtaining a factorization

$$\text{Aut}^\otimes(\omega) \rightarrow \text{Aut}^\otimes(\omega') \rightarrow \text{Aut}^\otimes(\omega^\circ).$$

On the other hand, it is obvious that $\mathrm{Aut}^{\otimes}(\omega') = \mathrm{Aut}^{\otimes}(\omega^{\circ})$.

3.6.3. The following assumption implies that $\mathrm{Rep}^{\circ}(G)(S)$ is sufficiently big:

HYP(ω°) There exists a covering $\{S_i \rightarrow S\}$ by finite étale morphisms such that for every i , G_{S_i}/S_i satisfies **HYP'**(ω°) where:

HYP'(ω°) S is quasi-compact and ρ_{reg} belongs to $\mathrm{Rep}'(G)(S)$.

Proposition 30. *If G/S satisfies **HYP**(ω°), then*

$$G = \mathrm{Aut}^{\otimes}(\omega^{\circ}), \quad \mathbb{G}^{\Gamma}(G) = \mathbb{G}^{\Gamma}(\omega^{\circ}) \quad \text{and} \quad \mathbb{F}^{\Gamma}(\omega) \subset \mathbb{F}^{\Gamma}(\omega^{\circ}).$$

Proof. These being fpqc sheaves on S , it is sufficient to establish the Proposition for their restriction to the S_i 's, which by Proposition 27 reduces us to the case where S is quasi-compact and ρ_{reg} belongs to $\mathrm{Rep}'(G)(S)$. The proof of Theorem 29 then shows that $G = \mathrm{Aut}^{\otimes}(\omega_T)$. Thus $G = \mathrm{Aut}^{\otimes}(\omega^{\circ})$. To prove that $\mathbb{G}^{\Gamma}(G) = \mathbb{G}^{\Gamma}(\omega^{\circ})$, we may test this on quasi-compact schemes, and then the proof of section 3.5.5 carries over to this case. Finally: a Γ -filtration \mathcal{F} on ω_T is uniquely determined by its value on ρ_{reg} by 3.5.3, thus $\mathbb{F}^{\Gamma}(\omega) \subset \mathbb{F}^{\Gamma}(\omega^{\circ})$ since $\rho_{\mathrm{reg}} \in \mathrm{Rep}'(G)(S)$. \square

3.6.4. For the V° variants of these, one needs a weaker assumption:

HYP(V°) Locally on S for the fpqc topology, ρ_{reg} belongs to $\mathrm{Rep}'(G)(S)$.

Proposition 31. *If G/S satisfies **HYP**(V°), then*

$$G = \mathrm{Aut}^{\otimes}(V^{\circ}), \quad \mathbb{G}^{\Gamma}(G) = \mathbb{G}^{\Gamma}(V^{\circ}) \quad \text{and} \quad \mathbb{F}^{\Gamma}(V) \subset \mathbb{F}^{\Gamma}(V^{\circ}).$$

Proof. This being local in the fpqc topology on S , we may assume that S is quasi-compact and ρ_{reg} is in $\mathrm{Rep}'(G)(S)$, then G_T/T satisfies **HYP'**(ω°) for every quasi-compact T over S and the proposition easily follows from the previous one. \square

3.6.5. It remains to give some cases where our assumptions are met.

Definition 32. A reductive group G over S is called isotrivial if and only if there exists a covering $\{S_i \rightarrow S\}$ by finite étale morphisms such that each G_{S_i} is splittable.

For tori, this definition is slightly more general than that given in [1, IX 1.1], which requires a single finite étale cover $S' \rightarrow S$. If S is quasi-compact, both notions coincide. For arbitrary reductive groups, [12, XXIV 4.1] only defines local and semi-local isotriviality. If S is local, these two notions coincide with ours.

Proposition 33. *If S is local and geometrically unibranch, then G is isotrivial.*

Proof. We may assume that G is a torus by [12, XXIV 4.1.5]. We then have to show that the connected components of $R = \underline{\mathrm{Hom}}_S(G, \mathbb{G}_{m,S})$ are open and finite over S by [13, X 5.11], and this follows from Proposition 3 and Lemma 4. \square

Proposition 34. (1) *If $S = \mathrm{Spec}(A)$ for a Prüfer domain A and $\rho \in \mathrm{Rep}(G)(S)$,*

$$\rho \in \mathrm{Rep}'(G)(S) \iff V(\rho) \text{ is a flat } \mathcal{O}_S\text{-module.}$$

- (2) *A split reductive group over a quasi-compact S satisfies **HYP'**(ω°)*
- (3) *An isotrivial reductive group over a quasi-compact S satisfies **HYP**(ω°)*
- (4) *A reductive group over any S satisfies **HYP**(V°)*

Proof. (1) is exactly [36, Corollary 5.10]. For (2), we may assume that G is of constant type [12, XXII 2.8], thus isomorphic [12, XXIII 5.2] to the base change of a reductive group G_0 over $\mathrm{Spec}(\mathbb{Z})$ [12, XXV 1.2] to which (1) now applies. Obviously (2) \Rightarrow (3), and (2) \Rightarrow (4) by [12, XXII 2.3]. \square

3.6.6. Together with Theorem 29, Proposition 30 and 31 give many cases where automorphisms or Γ -graduations automatically extend from ω° or V° to ω or V . Assuming that S is quasi-compact, we will now do something similar for Γ -filtrations.

3.6.7. Let \mathcal{F} be a Γ -filtration on ω_T° . For each $\gamma \in \Gamma$, we may extend

$$\mathcal{F}^\gamma : \text{Rep}^\circ(G)(S) \rightarrow \text{LF}(T) \quad \text{to} \quad \mathcal{F}^\gamma : \text{Rep}'(G)(S) \rightarrow \text{QCoh}(T)$$

by the formula $\mathcal{F}^\gamma(\rho) = \varinjlim \mathcal{F}^\gamma(\tau)$, where τ runs through $X(\rho)$. It defines a functor by (3.1), and gives back $\mathcal{F}^\gamma(\rho) = \mathcal{F}^\gamma(\tau)$ when $\rho = \tau$ belongs to $\text{Rep}^\circ(G)(S)$. In general, $\mathcal{F}^\gamma(\rho)$ is a pure quasi-coherent subsheaf of $V(\rho)_T = \varinjlim V(\tau)_T$ since filtered colimits are exact and commute with base change. While $\gamma \rightarrow \mathcal{F}^\gamma(\rho)$ is non-increasing, it may not be a Γ -filtration on $V(\rho)_T$ in our sense. However:

Lemma 35. *We have the following properties:*

(F1) *For every $\rho_1, \rho_2 \in \text{Rep}'(G)(S)$ and $\gamma \in \Gamma$,*

$$\mathcal{F}^\gamma(\rho_1 \otimes \rho_2) = \sum_{\gamma_1 + \gamma_2 = \gamma} \mathcal{F}^{\gamma_1}(\rho_1) \otimes \mathcal{F}^{\gamma_2}(\rho_2).$$

(F2) *For the trivial representation $\rho \in \text{Rep}'(G)(S)$ on $\mathcal{M} \in \text{QCoh}(S)$,*

$$\mathcal{F}^\gamma(\rho) = \mathcal{M} \text{ if } \gamma \leq 0 \quad \text{and} \quad \mathcal{F}^\gamma(\rho) = 0 \text{ if } \gamma > 0.$$

(F3r) *If $\rho \twoheadrightarrow \tau$ is an epimorphism with $\rho \in \text{Rep}'(G)(S)$ and $\tau \in \text{Rep}^\circ(G)(S)$, then $\mathcal{F}^\gamma(\rho) \twoheadrightarrow \mathcal{F}^\gamma(\tau)$ is an epimorphism in $\text{QCoh}(T)$ for every $\gamma \in \Gamma$.*

(F3l) *If ρ_{reg} belongs to $\text{Rep}'(G)(S)$ and $\rho_1 \hookrightarrow \rho_2$ is a pure monomorphism in $\text{Rep}'(G)(S)$, then $\mathcal{F}^\gamma(\rho_1) = \mathcal{F}^\gamma(\rho_2) \cap V_T(\rho_1)$ in $V_T(\rho_2)$ for every $\gamma \in \Gamma$.*

Proof. (F2) is obvious and (F1), (F3r) follow from the eponymous properties of \mathcal{F} on ω_T° because, since S is quasi-compact, $\{\tau_1 \otimes \tau_2 : (\tau_1, \tau_2) \in X(\rho_1) \times X(\rho_2)\}$ and $\{\tau' \in X(\rho) : \tau' \twoheadrightarrow \tau\}$ are respectively cofinal in $X(\rho_1 \otimes \rho_2)$ and $X(\rho)$. For (F3l), we first treat the special case of the pure monomorphism $c_\rho : \rho \hookrightarrow \rho_0 \otimes \rho_{\text{reg}}$ for an arbitrary $\rho \in \text{Rep}'(G)(S)$. Given (F1) and (F2), we have to show that

$$\mathcal{F}^\gamma(\rho) = \ker \left[\omega_T(\rho) \xrightarrow{\omega_T(c_\rho)} \omega_T(\rho_0) \otimes (\omega_T(\rho_{\text{reg}})/\mathcal{F}^\gamma(\rho_{\text{reg}})) \right].$$

Since both sides are filtered limits over $\tau \in X(\rho)$, we may assume that ρ belongs to $\text{Rep}^\circ(G)(S)$. The right hand side is then the filtered limit of

$$\ker \left[\omega_T(\rho) \xrightarrow{\omega_T(c_{\rho, \tau})} \omega_T(\rho_0) \otimes (\omega_T(\tau)/\mathcal{F}^\gamma(\tau)) \right] = \mathcal{F}^\gamma(\rho, \tau)$$

where τ ranges through the cofinal set X' of $X(\rho_{\text{reg}})$ defined by

$$X' = \left\{ \tau : c_\rho \text{ factors as } \rho \xrightarrow{c_{\rho, \tau}} \rho_0 \otimes \tau \hookrightarrow \rho_0 \otimes \rho_{\text{reg}} \right\}.$$

Note that $\rho_0 \otimes \tau \hookrightarrow \rho_0 \otimes \rho_{\text{reg}}$ since $V(\rho_0)$ is a flat \mathcal{O}_S -module. For each τ in X' , the cokernel $\sigma_{\rho, \tau}$ of $c_{\rho, \tau} : \rho \hookrightarrow \rho_0 \otimes \tau$ is an object of $\text{Rep}^\circ(G)(S)$: the counit $1_G^\natural : \mathcal{A}(G) \rightarrow \mathcal{O}_S$ gives a retraction of $V(c_{\rho, \tau})$, whose kernel is a direct factor of $V(\rho_0 \otimes \tau)$ isomorphic to $V(\sigma_{\rho, \tau})$. Since \mathcal{F}^γ is exact on $\text{Rep}^\circ(G)(S)$, it follows that

$$\mathcal{F}^\gamma(\rho) = \ker \left[\omega_T(\rho) \xrightarrow{\omega_T(c_{\rho, \tau})} \omega_T(\rho_0 \otimes \tau)/\mathcal{F}^\gamma(\rho_0 \otimes \tau) \right] = \mathcal{F}^\gamma(\rho, \tau)$$

for every $\tau \in X'$, which proves our claim. For any morphism $\rho_1 \rightarrow \rho_2$ in $\text{Rep}'(G)(S)$ and any $\gamma \in \Gamma$, we now have a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathcal{F}^\gamma(\rho_1) & \rightarrow & \omega_T(\rho_1) & \rightarrow & \omega_T(\rho_{1,0}) \otimes \omega_T(\rho_{\text{reg}}) / \mathcal{F}^\gamma(\rho_{\text{reg}}) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & \mathcal{F}^\gamma(\rho_2) & \rightarrow & \omega_T(\rho_2) & \rightarrow & \omega_T(\rho_{2,0}) \otimes \omega_T(\rho_{\text{reg}}) / \mathcal{F}^\gamma(\rho_{\text{reg}}) \end{array}$$

If $V(\rho_1) \rightarrow V(\rho_2)$ is a pure monomorphism, the vertical maps are monomorphisms, therefore $\mathcal{F}^\gamma(\rho_1) = \mathcal{F}^\gamma(\rho_2) \cap \omega_T(\rho_1)$ in $\omega_T(\rho_2)$: this proves (F3l). \square

3.6.8. As before, for every $\rho \in \text{Rep}'(G)(S)$ and $\gamma \in \Gamma$, we define

$$\mathcal{F}_+^\gamma(\rho) = \cup_{\eta > \gamma} \mathcal{F}^\eta(\rho) \quad \text{and} \quad \text{Gr}_{\mathcal{F}}^\gamma(\rho) = \mathcal{F}^\gamma(\rho) / \mathcal{F}_+^\gamma(\rho).$$

Since again filtered limits are exact, we find that

$$\mathcal{F}_+^\gamma(\rho) = \varinjlim \mathcal{F}_+^\gamma(\tau) \quad \text{and} \quad \text{Gr}_{\mathcal{F}}^\gamma(\rho) = \varinjlim \text{Gr}_{\mathcal{F}}^\gamma(\tau)$$

where τ ranges through $X(\rho)$. In particular, the formula

$$\text{Gr}_{\mathcal{F}}^\gamma(\rho_1 \otimes \rho_2) \simeq \oplus_{\gamma_1 + \gamma_2 = \gamma} \text{Gr}_{\mathcal{F}}^{\gamma_1}(\rho_1) \otimes \text{Gr}_{\mathcal{F}}^{\gamma_2}(\rho_2)$$

also holds for ρ_1 and ρ_2 in $\text{Rep}'(G)(S)$. All of the above constructions commute with arbitrary base change on T . Finally if the original Γ -filtration \mathcal{F} on ω_T° already was the restriction of some Γ -filtration \mathcal{F}' on ω_T , the restriction of the latter is equal to the extension of the former on ω'_T since \mathcal{F}'^γ commutes with arbitrary colimits.

3.6.9. We first use the above device to show that:

Proposition 36. *If G/S satisfies HYP(ω°), then $\mathbb{F}^\Gamma(V^\circ) \hookrightarrow \mathbb{F}^\Gamma(\omega^\circ)$.*

Proof. By Proposition 27, we may assume: S is quasi-compact, $\rho_{\text{reg}} \in \text{Rep}'(G)(S)$. We have to show that for an S -scheme T and $\mathcal{F} \in \mathbb{F}^\Gamma(V_T^\circ)$ with image $\tilde{\mathcal{F}} \in \mathbb{F}^\Gamma(\omega_T^\circ)$, for any $U \rightarrow T$, the Γ -filtration \mathcal{F}_U on $\text{Rep}^\circ(G_U)(U) \rightarrow \text{LF}(U)$ induced by \mathcal{F} is determined by $\tilde{\mathcal{F}}$. We may assume that T and U are quasi-compact. Then: \mathcal{F}_U is determined by its extension to $\text{Rep}'(G_U)(U) \rightarrow \text{QCoh}(U)$, which itself is determined by its value on $\rho_{\text{reg},U} \in \text{Rep}'(G_U)(U)$ thanks to (F1-2) and (F3l) applied to the pure monomorphisms $c_\rho : \rho \rightarrow \rho_0 \otimes \rho_{\text{reg},U}$ for all ρ 's in $\text{Rep}'(G_U)(U)$. Since U is quasi-compact, $X(\rho_{\text{reg}})_U = \{\tau_U : \tau \in X(\rho_{\text{reg}})\}$ is cofinal in $X(\rho_{\text{reg},U})$, thus $\mathcal{F}_U(\rho_{\text{reg},U})$ is determined by the restriction of \mathcal{F}_U to $X(\rho_{\text{reg}})_U$. By the axiom (F0) for \mathcal{F} , the latter is determined by the values of \mathcal{F}_T on $X(\rho_{\text{reg}})_T$, which are the values of $\tilde{\mathcal{F}}$ on $X(\rho_{\text{reg}})$. Thus $\tilde{\mathcal{F}}$ determines \mathcal{F}_U and \mathcal{F} uniquely. \square

3.6.10. Here is another useful assumption: we say that G/S is linear if there exists $\tau \in \text{Rep}^\circ(G)(S)$ inducing a closed immersion $\tau : G \hookrightarrow GL(V(\tau))$. Note that upon replacing τ with $\tau \oplus (\det \tau)^{-1}$, we may then also assume that $\det \tau = 1$.

Lemma 37. *The affine and flat group G over S is linear in the following cases:*

- (1) G is of finite type over a noetherian regular S with $\dim S \leq 2$.
- (2) HYP(ω°) holds and S is quasi-compact and quasi-separated.
- (3) G is an isotrivial reductive group over a quasi-compact S .
- (4) G is a reductive group of adjoint type over any S .

Proof. (1) is [13, VIB 13.2]. For (2), let $f : S' \rightarrow S$ be a finite étale cover such that $\text{HYP}'(\omega^\circ)$ holds for $G' = G_{S'}$. Then S' is also quasi-compact and quasi-separated, thus by [16, 1.7.9], the finitely generated quasi-coherent $\mathcal{O}_{S'}$ -algebra $\mathcal{A}(G')$ is generated by a finitely generated quasi-coherent $\mathcal{O}_{S'}$ -submodule \mathcal{E} . By assumption $\text{HYP}'(\omega^\circ)$ for G' , we may replace \mathcal{E} by a larger $V(\tau')$ for some τ' in $X(f^*\rho_{\text{reg}})$. The proof of [13, VIB 13.2] then shows that $\tau' : G' \rightarrow GL(V(\tau'))$ is a closed immersion. Put $\tau = f_*\tau'$, so that τ belongs to $\text{Rep}^\circ(G)(S)$. Then $\tau : G \rightarrow GL(V(\tau))$ is a closed immersion. Indeed, it is sufficient to show that $f^*\tau : G' \rightarrow GL(V(f^*\tau))$ is a closed immersion by [17, 2.7.1]. But $f^*\tau = \rho \otimes \tau'$ in $\text{Rep}^\circ(G')(S')$, where $\rho = f^*f_*1_{G'}$ is the trivial representation on $V(\rho) = f^*f_*\mathcal{O}_{S'}$, i.e. $f^*\tau$ is the composition

$$G' \xrightarrow{\rho'} GL(V(\tau')) \xrightarrow{\text{Id} \otimes -} GL(V(\rho) \otimes V(\tau'))$$

of two closed immersions, therefore itself a closed immersion. For (3): it is well-known that the Chevalley groups over $\text{Spec } \mathbb{Z}$ are linear (a complete overkill: see (1)), so are therefore also the split reductive groups over any base by [12, XXII 2.8, XXIII 5.2 and XXV 1.2], to which one reduces as in (2). For (4), simply take τ to be the adjoint representation ρ_{ad} of G on its Lie algebra $\text{Lie}(G) = \mathfrak{g} = V(\rho_{\text{ad}})$. \square

3.7. The stabilizer of a Γ -filtration, I.

3.7.1. Let now G be a reductive group over S and let $\rho_{\text{ad}} \in \text{Rep}^\circ(G)(S)$ be the adjoint representation of G on $V(\rho_{\text{ad}}) = \mathfrak{g} = \text{Lie}(G)$. Let T be an S -scheme.

Theorem 38. *Let \mathcal{F} be a Γ -filtration on V_T . Then $\text{Aut}^\otimes(\mathcal{F})$ is a parabolic subgroup $P_{\mathcal{F}}$ of G_T with unipotent radical $U_{\mathcal{F}} \subset \text{Aut}^{\otimes!}(\mathcal{F})$. Moreover,*

$$\text{Lie}(U_{\mathcal{F}}) = \mathcal{F}_+^0(\rho_{\text{ad}}) \quad \text{and} \quad \text{Lie}(P_{\mathcal{F}}) = \mathcal{F}^0(\rho_{\text{ad}}) \quad \text{in} \quad V_T(\rho_{\text{ad}}) = \mathfrak{g}_T.$$

Remark 39. Let $\chi : \mathbb{D}_T(\Gamma) \rightarrow G_T$ be a morphism, \mathcal{G} the corresponding Γ -graduation and \mathcal{F} the induced Γ -filtration. Let $P_\chi = U_\chi \rtimes L_\chi$ be the subgroups of G_T defined in Proposition 14. Since $\text{Aut}^\otimes(\mathcal{F}) = \text{Aut}^{\otimes!}(\mathcal{F}) \rtimes \text{Aut}^\otimes(\mathcal{G})$ with $\text{Aut}^\otimes(\mathcal{G})$ equal to L_χ and isomorphic to $\text{Aut}^\otimes(\text{Gr}_\bullet^{\mathcal{F}})$ (because $\mathcal{G} \simeq \text{Gr}_\bullet^{\mathcal{F}}$), the theorem implies that

$$P_\chi = \text{Aut}^\otimes(\mathcal{F}), \quad U_\chi = \text{Aut}^{\otimes!}(\mathcal{F}) \quad \text{and} \quad P_\chi/U_\chi \simeq \text{Aut}^\otimes(\text{Gr}_\bullet^{\mathcal{F}}).$$

Corollary 40. *The quotients $\text{Fil} : \mathbb{G}^\Gamma(G) \rightarrow \mathbb{F}^\Gamma(G)$ of $\mathbb{G}^\Gamma(G)$ defined in sections 2.2 and 3.2.9 are canonically isomorphic, and for any $\mathcal{F} \in \mathbb{F}^\Gamma(G)(T)$,*

$$P_{\mathcal{F}} = \text{Aut}^\otimes(\iota\mathcal{F}), \quad U_{\mathcal{F}} = \text{Aut}^{\otimes!}(\iota\mathcal{F}) \quad \text{and} \quad P_{\mathcal{F}}/U_{\mathcal{F}} \simeq \text{Aut}^\otimes(\text{Gr}_{\iota\mathcal{F}}^\bullet)$$

where $\iota\mathcal{F}$ is the image of \mathcal{F} in $\mathbb{F}^\Gamma(V_T)$.

Proof. For the first assertion, we only have to show that for $\chi_1, \chi_2 : \mathbb{D}_T(\Gamma) \rightarrow G_T$,

$$\chi_1 \sim_{\text{Par}} \chi_2 \iff \text{Fil} \circ \iota(\chi_1) = \text{Fil} \circ \iota(\chi_2) \quad \text{in} \quad \mathbb{F}^\Gamma(V_T).$$

Put $\mathcal{G}_i = \iota(\chi_i)$, $\mathcal{F}_i = \text{Fil}(\mathcal{G}_i)$ and $P_i = \text{Aut}^\otimes(\mathcal{F}_i) = P_{\chi_i}$. If $\chi_1 \sim_{\text{Par}} \chi_2$, then $\chi_2 = \text{Int}(p) \circ \chi_1$ for some $p \in P_1(T)$, thus $\mathcal{F}_2 = p\mathcal{F}_1 = \mathcal{F}_1$. If $\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}$, then $P_1 = P_2 = P$ and the canonical isomorphism $\mathcal{G}_1 \simeq \text{Gr}_\bullet^{\mathcal{F}} \simeq \mathcal{G}_2$ gives an element of $\text{Aut}^\otimes(V_T)$ preserving \mathcal{F} and mapping \mathcal{G}_1 to \mathcal{G}_2 , i.e. an element $p \in P(T)$ such that $\chi_2 = \text{Int}(p) \circ \chi_1$, thus $\chi_1 \sim_{\text{Par}} \chi_2$. The remaining assertions are local in the fpqc topology on T and thus follow from the above remark. \square

3.7.2. For Γ -filtrations on ω_T , we need a technical assumption on G/S :

TA There exists an fpqc cover $\{f_i : S_i \rightarrow S\}$ such that (a) each f_i is an affine morphism, and (b) each $G_i = G_{S_i}$ is linear (3.6.10).

This is true for *any* reductive group G over a *separated* S : starting from a Zariski covering of S by affine U_i 's, we pick fpqc covers $\{U_{i,j} \rightarrow U_i\}$ splitting G_{U_i} , and again cover the $U_{i,j}$'s by affine $U_{i,j,k}$'s. The resulting fpqc cover $\{U_{i,j,k} \rightarrow S\}$ satisfies our assumption: $U_{i,j,k} \rightarrow U_i$ is affine as a morphism between affine schemes, $U_i \hookrightarrow S$ is affine because S is separated, and $G_{U_{i,j,k}}$ is linear by Lemma 37 since it is split.

Theorem 41. *Under this assumption, let \mathcal{F} be a Γ -filtration on ω_T . Then $\text{Aut}^\otimes(\mathcal{F})$ is a parabolic subgroup $P_{\mathcal{F}}$ of G_T with unipotent radical $U_{\mathcal{F}} \subset \text{Aut}^{\otimes 1}(\mathcal{F})$. Moreover,*

$$\text{Lie}(U_{\mathcal{F}}) = \mathcal{F}_+^0(\rho_{\text{ad}}) \quad \text{and} \quad \text{Lie}(P_{\mathcal{F}}) = \mathcal{F}^0(\rho_{\text{ad}}) \quad \text{in} \quad V_T(\rho_{\text{ad}}) = \mathfrak{g}_T.$$

3.7.3. If \mathcal{F}' is a Γ -filtration on V_T and \mathcal{F} is the induced Γ -filtration on ω_T , then $\text{Aut}^\otimes(\mathcal{F}') = \text{Aut}^\otimes(\mathcal{F})$ as subsheaves of G_T by 3.5.3 and Theorem 29. Therefore: (a) Theorem 41 holds without the technical assumption for such filtrations on ω_T , and (b) Theorem 38, which is local on S , follows from Theorem 41 applied to any affine cover of S . We thus only have to consider the case of a Γ -filtration \mathcal{F} on ω_T . The technical assumption will be used only once below, in section 3.7.9.

3.7.4. The adjoint-regular representation ρ_{adj} of G on $V(\rho_{\text{adj}}) = \mathcal{A}(G)$ is given by

$$(g \cdot a)(h) = a(g^{-1}hg)$$

for $T \rightarrow S$, $a \in \Gamma(T, \mathcal{A}(G_T))$ and $g, h \in G(T)$. The unit, counit 1_G^\natural , multiplication, comultiplication μ^\natural and inversion inv^\natural of $\mathcal{A}(G)$ define morphisms in $\text{Rep}(G)(S)$:

$$1_S \rightarrow \rho_{\text{adj}}, \quad \rho_{\text{adj}} \rightarrow 1_S, \quad \rho_{\text{adj}} \otimes \rho_{\text{adj}} \rightarrow \rho_{\text{adj}}, \quad \rho_{\text{adj}} \rightarrow \rho_{\text{adj}} \otimes \rho_{\text{adj}}, \quad \rho_{\text{adj}} \rightarrow \rho_{\text{adj}}.$$

For any ρ in $\text{Rep}(G)(S)$, we may also view c_ρ as a split monomorphism

$$c_\rho : \rho \rightarrow \rho \otimes \rho_{\text{adj}} \quad \text{in} \quad \text{Rep}(G)(S).$$

If τ belongs to $\text{Rep}^\circ(G)(S)$, c_τ gives a morphism $\tau^\vee \otimes \tau \rightarrow \rho_{\text{adj}}$ which induces a G -equivariant morphism of quasi-coherent $G - \mathcal{O}_S$ -algebras

$$\text{Sym}^\bullet(\tau^\vee \otimes \tau) \rightarrow \rho_{\text{adj}}$$

whose underlying morphism of quasi-coherent \mathcal{O}_S -algebras is given by

$$\text{Sym}_{\mathcal{O}_S}^\bullet(V(\tau)^\vee \otimes V(\tau)) \hookrightarrow \text{Sym}_{\mathcal{O}_S}^\bullet(\underline{\text{End}}_{\mathcal{O}_S}(\tau)) \left[\frac{1}{\det} \right] = \mathcal{A}(GL(V(\tau))) \xrightarrow{\tau^\natural} \mathcal{A}(G)$$

where τ^\natural is the morphism attached to $\tau : G \rightarrow GL(V(\rho))$. In particular, if the latter is a closed embedding and $\det(\tau) = 1$, then $\text{Sym}^\bullet(\tau^\vee \otimes \tau) \rightarrow \rho_{\text{adj}}$ is an epimorphism.

3.7.5. Let ρ_{adj}° the kernel of $1_G^\natural : \rho_{\text{adj}} \rightarrow 1_S$. Thus $\rho_{\text{adj}} = \rho_{\text{adj}}^\circ \oplus 1_S$ and $V(\rho_{\text{adj}}^\circ)$ is the augmentation ideal $\mathcal{I}(G)$ of $\mathcal{A}(G)$. For any $n \geq 1$, the multiplication map $\mathcal{I}(G)^{\otimes n+1} \rightarrow \mathcal{I}(G)$ defines a morphism $(\rho_{\text{adj}}^\circ)^{\otimes n+1} \rightarrow \rho_{\text{adj}}^\circ$ in $\text{Rep}(G)(S)$. We denote by $\rho^n \in \text{Rep}^\circ(G)(S)$ its cokernel, a representation of G on $V(\rho^n) = \mathcal{I}(G)/\mathcal{I}(G)^{n+1}$, and by $\rho_n = (\rho^n)^\vee \in \text{Rep}^\circ(G)(S)$ the dual of ρ^n . Thus $\rho_1 = \rho_{\text{ad}}$, the adjoint representation of G on $V(\rho_{\text{ad}}) = \mathfrak{g}$.

3.7.6. Let now $\mathcal{I}(\mathcal{F})$ and $\mathcal{J}(\mathcal{F})$ be the quasi-coherent ideals of $\mathcal{A}(G_T)$ which are respectively generated by the quasi-coherent subsheaves $\mathcal{F}_+^0(\rho_{\text{adj}}^\circ)$ and $\mathcal{F}^0(\rho_{\text{adj}}^\circ)$ of the augmentation ideal $\mathcal{I}(G_T) = \omega_T(\rho_{\text{adj}}^\circ)$ of $\mathcal{A}(G_T)$. Then

$$U_{\mathcal{F}} \stackrel{\text{def}}{=} \text{Spec}(\mathcal{A}(G_T)/\mathcal{J}(\mathcal{F})) \hookrightarrow P_{\mathcal{F}} \stackrel{\text{def}}{=} \text{Spec}(\mathcal{A}(G_T)/\mathcal{I}(\mathcal{F}))$$

are closed subgroup schemes of G_T , because $\mathcal{J}(\mathcal{F})$ and $\mathcal{I}(\mathcal{F})$ are compatible with the comultiplication μ_T^{\natural} and inversion inv_T^{\natural} of $\mathcal{A}(G_T)$, since $\mu^{\natural} : \rho_{\text{adj}} \rightarrow \rho_{\text{adj}} \otimes \rho_{\text{adj}}$ and $\text{inv}^{\natural} : \rho_{\text{adj}} \rightarrow \rho_{\text{adj}}$ are morphisms in $\text{Rep}(G)(S)$. It follows from their definition that the formation of $U_{\mathcal{F}}$ and $P_{\mathcal{F}}$ commutes with arbitrary base change on T .

3.7.7. Let $N(U_{\mathcal{F}})$ and $N(P_{\mathcal{F}})$ be the normalizers of $U_{\mathcal{F}}$ and $P_{\mathcal{F}}$ in G_T . Then

$$P_{\mathcal{F}} \subset \text{Aut}^{\otimes}(\mathcal{F}) \subset N(U_{\mathcal{F}}), N(P_{\mathcal{F}}) \quad \text{and} \quad U_{\mathcal{F}} \subset \text{Aut}^{\otimes!}(\mathcal{F})$$

as fpqc subsheaves of G_T . We have to check this on sections over an arbitrary T -scheme X , but we may assume that $X = T$. Since $G = \text{Aut}^{\otimes}(\omega)$ by Theorem 29,

$$\text{Aut}^{\otimes}(\mathcal{F})(T) = \{g \in G(T) \mid \forall \rho, \gamma : \rho(g) (\mathcal{F}^{\gamma}(\rho)) = \mathcal{F}^{\gamma}(\rho)\}.$$

On the other hand, for any ρ in $\text{Rep}(G)(S)$, the morphism $c_{\rho} : \rho \rightarrow \rho \otimes \rho_{\text{adj}}$ gives a morphism $\omega_T(c_{\rho}) : \omega_T(\rho) \rightarrow \omega_T(\rho) \otimes \omega_T(\rho_{\text{adj}})$ in $\text{QCoh}(T)$ mapping $\mathcal{F}^{\gamma}(\rho)$ into

$$\mathcal{F}^{\gamma}(\rho \otimes \rho_{\text{adj}}) = \sum_{\alpha+\beta=\gamma} \mathcal{F}^{\alpha}(\rho) \otimes \mathcal{F}^{\beta}(\rho_{\text{adj}}).$$

(a) For g in $\text{Aut}^{\otimes}(\mathcal{F})(T)$, $\rho_{\text{adj}}^{\circ}(g)$ fixes $\mathcal{F}_+^0(\rho_{\text{adj}}^{\circ}) = \cup_{\gamma>0} \mathcal{F}^{\gamma}(\rho_{\text{adj}}^{\circ})$ as well as the $\mathcal{A}(G_T)$ -ideal $\mathcal{I}(\mathcal{F})$ which it spans. It follows that the inner automorphism of G_T defined by g fixes $P_{\mathcal{F}}$. Thus g belongs to $N(P_{\mathcal{F}})(T)$. Similarly, $g \in N(U_{\mathcal{F}})(T)$.

(b) For g in $P_{\mathcal{F}}(T)$, $g^{\natural} : \mathcal{A}(G_T) \rightarrow \mathcal{O}_T$ is trivial on $\mathcal{F}^{\beta}(\rho_{\text{adj}})$ for every $\beta > 0$ and thus $\rho(g) = (\text{Id} \otimes g^{\natural}) \circ \omega_T(c_{\rho})$ maps $\mathcal{F}^{\gamma}(\rho)$ into $\sum_{\alpha \geq \gamma} \mathcal{F}^{\alpha}(\rho) = \mathcal{F}^{\gamma}(\rho)$. Since g^{-1} also belongs to $P_{\mathcal{F}}(T)$, $\rho(g)$ fixes $\mathcal{F}^{\gamma}(\rho)$. Thus g belongs to $\text{Aut}^{\otimes}(\mathcal{F})(T)$.

(c) For g in $U_{\mathcal{F}}(T)$, $g^{\natural} - 1^{\natural} : \mathcal{A}(G_T) \rightarrow \mathcal{O}_T$ is trivial on $\mathcal{F}^0(\rho_{\text{adj}}) = \mathcal{O}_T \oplus \mathcal{F}^0(\rho_{\text{adj}}^{\circ})$, thus $\rho(g) - \rho(1) = (\text{Id} \otimes (g^{\natural} - 1^{\natural})) \circ \omega_T(c_{\rho})$ maps $\mathcal{F}^{\gamma}(\rho)$ into $\sum_{\alpha > \gamma} \mathcal{F}^{\alpha}(\rho) = \mathcal{F}_+^{\gamma}(\rho)$. Therefore g belongs to $\text{Aut}^{\otimes!}(\mathcal{F})(T)$.

3.7.8. We will establish below that the neutral components [13, VIB 3.1] $U_{\mathcal{F}}^{\circ}$ and $P_{\mathcal{F}}^{\circ}$ of $U_{\mathcal{F}}$ and $P_{\mathcal{F}}$ are smooth over S , using the following criterion:

Proposition 42. *Let G be affine and smooth over S , $\mathcal{A} = \mathcal{A}(G)$ and $\mathcal{I} = \mathcal{I}(G)$. Let $H \subset G$ be a closed subgroup defined by a quasi-coherent ideal $\mathcal{J} \subset \mathcal{I}$ of \mathcal{A} such that*

- (1) \mathcal{J} is finitely generated,
- (2) $\mathcal{J} \cap \mathcal{I}^2 = \mathcal{I} \cdot \mathcal{J}$ in \mathcal{A} , and
- (3) $\mathcal{I}/\mathcal{J} + \mathcal{I}^2$ is finite locally free on S .

Then H° is representable by a smooth open subgroup scheme of H .

Proof. By [13, VIB 3.10], we have to show that H is smooth at all points of its unit section. Let thus $x \in H$ be the image of $s \in S$ under $1_H : S \rightarrow H$. By [16, 1.4.3 and 1.4.5], we already know from (1) that H is locally of finite presentation over S . Thus by [19, 17.5.1] and the Jacobian criterion [16, 0_{IV} 22.6.4], we have to show that $\mathcal{J}_x/\mathcal{J}_x^2 \otimes_{\mathcal{O}_{H,x}} k \rightarrow \Omega_{\mathcal{O}_{G,x}/\mathcal{O}_{S,s}}^1 \otimes_{\mathcal{O}_{G,x}} k$ is injective, where k is the common residue field of s and x , and the morphism is induced by the universal derivation $d : \mathcal{O}_{G,x} \rightarrow \Omega_{\mathcal{O}_{G,x}/\mathcal{O}_{S,s}}^1$. This map factors through the corresponding

map for \mathcal{I}_x , namely $\mathcal{I}_x/\mathcal{I}_x^2 \otimes_{\mathcal{O}_{S,s}} k \rightarrow \Omega_{\mathcal{O}_{G,x}/\mathcal{O}_{S,s}}^1 \otimes_{\mathcal{O}_{G,x}} k$, which is injective (because $\mathcal{O}_{G,x}/\mathcal{I}_x = \mathcal{O}_{S,s}$ is formally smooth over itself!). We thus have to show that

$$\mathcal{J}_x/\mathcal{J}_x^2 \otimes_{\mathcal{O}_{H,x}} k = \mathcal{J}_x/m_x \mathcal{J}_x \rightarrow \mathcal{I}_x/m_x \mathcal{I}_x = \mathcal{I}_x/\mathcal{I}_x^2 \otimes_{\mathcal{O}_{S,s}} k$$

is injective, where m_x is the maximal ideal of $\mathcal{O}_{G,x}$. The latter map is base-changed from the morphism $\mathcal{J}_x/\mathcal{J}_x \mathcal{I}_x \rightarrow \mathcal{I}_x/\mathcal{I}_x^2$, which itself is the localization at x of the morphism $\mathcal{J}/\mathcal{J}\mathcal{I} \rightarrow \mathcal{I}/\mathcal{I}^2$, which is a pure monomorphism by assumption. \square

3.7.9. We now show that $\mathcal{I}(\mathcal{F})$ and $\mathcal{J}(\mathcal{F})$ are finitely generated, focusing on $\mathcal{I}(\mathcal{F})$ to simplify the exposition. Let $\{S_i \rightarrow S\}$ be an fpqc cover as in assumption (TA), $\{f_i : T_i \rightarrow T\}$ the corresponding fpqc cover of T , ω_i the fiber functor for $G_i = G_{S_i}$ and \mathcal{F}_i the extension of \mathcal{F}_{T_i} to a Γ -filtration on ω_{i,T_i} – which exists by Proposition 27 since f_i is affine. By [17, 2.5.2], it is sufficient to show that $f_i^* \mathcal{I}(\mathcal{F})$ is finitely generated. Since f_i is flat, $f_i^* \mathcal{I}(\mathcal{F}) = \mathcal{I}(\mathcal{F}_{T_i})$ and obviously $\mathcal{I}(\mathcal{F}_{T_i}) = \mathcal{I}(\mathcal{F}_i)$. We may thus assume that G is linear over S : there exists $\tau \in \text{Rep}^\circ(G)(S)$ inducing a closed embedding $\tau : G \hookrightarrow GL(V(\tau))$ with $\det \tau \equiv 1$, thus also an epimorphism $S^\bullet(\tau) = \text{Sym}^\bullet(\tau^\vee \otimes \tau) \twoheadrightarrow \rho_{\text{adj}}$ of quasi-coherent G - \mathcal{O}_S -algebras. By the axiom (F3) for \mathcal{F} , $\mathcal{I}(\mathcal{F})$ is the image of the ideal $\mathcal{I}(\tau)$ spanned by $\mathcal{F}_+^0(S^\bullet(\tau))$ in $V(S^\bullet(\tau))_T$. Using Proposition 24, we may assume that there is a splitting $V(\tau^\vee \otimes \tau)_T = \bigoplus_{\gamma} \mathcal{G}_\gamma$ of \mathcal{F} on $\tau^\vee \otimes \tau$. By the axioms (F1) and (F3), it induces a splitting of \mathcal{F} on $S^\bullet(\tau)$,

$$V(S^n(\tau))_T = \bigoplus_{\gamma} \bigoplus_{\gamma_1 + \dots + \gamma_n = \gamma} \mathcal{G}_{\gamma_1} \cdots \mathcal{G}_{\gamma_n}.$$

It follows easily that $\mathcal{I}(\tau)$ is spanned by the finite locally free subsheaf $\bigoplus_{\gamma > 0} \mathcal{G}_\gamma$ of $V(S^1(\tau))_T = V(\tau^\vee \otimes \tau)_T$, therefore $\mathcal{I}(\tau)$ and $\mathcal{I}(\mathcal{F})$ are indeed finitely generated.

3.7.10. We now show that $\mathcal{I}(\mathcal{F}) \cap \mathcal{I}(G_T)^2 = \mathcal{I}(\mathcal{F}) \cdot \mathcal{I}(G_T)$ – the proof for $\mathcal{J}(\mathcal{F})$ is similar. Plainly, $\mathcal{I}(\mathcal{F}) \cdot \mathcal{I}(G_T) \subset \mathcal{I}(\mathcal{F}) \cap \mathcal{I}(G_T)^2$. For the other inclusion, we may assume that T is affine and work with global sections. Let thus s be a (global) section of $\mathcal{I}(\mathcal{F})$, so that $s = a + b$ with a a section of $\mathcal{F}_+^0(\rho_{\text{adj}}^\circ)$ and b a section of

$$\mathcal{I}(G_T) \cdot \mathcal{F}_+^0(\rho_{\text{adj}}^\circ) \subset \mathcal{I}(G_T) \cdot \mathcal{I}(\mathcal{F}) \subset \mathcal{I}(G_T)^2.$$

Then s belongs to $\mathcal{I}(G_T)^2$ if and only a does, i.e. a is a section of $\mathcal{F}_+^0(\rho_{\text{adj}}^\circ) \cap \mathcal{I}(G_T)^2$. The pure short exact sequence and epimorphism of quasi-coherent sheaves on S

$$0 \rightarrow \mathcal{I}(G)^2 \rightarrow \mathcal{I}(G) \rightarrow \mathcal{I}(G)/\mathcal{I}(G)^2 \rightarrow 0 \quad \text{and} \quad \mathcal{I}(G)^{\otimes 2} \twoheadrightarrow \mathcal{I}(G)^2$$

correspond to a pure short exact sequence and epimorphism in $\text{Rep}(G)(S)$,

$$0 \rightarrow \rho_{\text{adj}}^{\circ(2)} \rightarrow \rho_{\text{adj}}^\circ \rightarrow \rho^1 \rightarrow 0 \quad \text{and} \quad (\rho_{\text{adj}}^\circ)^{\otimes 2} \twoheadrightarrow \rho_{\text{adj}}^{\circ(2)}$$

which together give, using the axioms (F1) and (F3) for \mathcal{F} ,

$$\mathcal{F}_+^0(\rho_{\text{adj}}^\circ) \cap \mathcal{I}(G_T)^2 = \mathcal{F}_+^0(\rho_{\text{adj}}^{\circ(2)}) = \sum_{\gamma_1 + \gamma_2 > 0} \mathcal{F}^{\gamma_1}(\rho_{\text{adj}}^\circ) \cdot \mathcal{F}^{\gamma_2}(\rho_{\text{adj}}^\circ)$$

which is contained in $\mathcal{I}(\mathcal{F}) \cdot \mathcal{I}(G_T)$, thus $\mathcal{I}(\mathcal{F}) \cap \mathcal{I}(G_T)^2 \subset \mathcal{I}(\mathcal{F}) \cdot \mathcal{I}(G_T)$.

3.7.11. We show that $\mathcal{I}(G_T)/\mathcal{I}(\mathcal{F}) + \mathcal{I}(G_T)^2$ is finite locally free – the proof for $\mathcal{J}(\mathcal{F})$ is similar. By the axiom (F3), $\mathcal{I}(\mathcal{F}) + \mathcal{I}(G_T)^2/\mathcal{I}(G_T)^2$ is the $\mathcal{A}(G_T)$ -submodule of $\mathcal{I}(G_T)/\mathcal{I}(G_T)^2 = \omega_T(\rho^1)$ generated by $\mathcal{F}_+^0(\rho^1)$, i.e. this \mathcal{O}_T -submodule itself since $\mathcal{A}(G_T)$ acts on $\mathcal{I}(G_T)/\mathcal{I}(G_T)^2$ through \mathcal{O}_T . We are thus claiming that $\omega_T(\rho^1)/\mathcal{F}_+^0(\rho^1)$ is finite locally free, which follows from Proposition 24.

3.7.12. We have just established that $U_{\mathcal{F}}^{\circ}$ and $P_{\mathcal{F}}^{\circ}$ are representable by smooth open subschemes of $U_{\mathcal{F}}$ and $P_{\mathcal{F}}$. They are also finitely presented over T : they are separated over T as compositions of affine morphisms and open immersions, and they are quasi-compact over T by [13, VIB 3.9], since $U_{\mathcal{F}}$ and $P_{\mathcal{F}}$ are finitely presented over S , being locally of finite presentation by 3.7.9 and [16, 1.4.5], and affine by definition. From 3.7.7, we obtain the following chain of inclusions

$$\begin{array}{ccccccc} U_{\mathcal{F}}^{\circ} & \subset & U_{\mathcal{F}} & \subset & \text{Aut}^{\otimes 1}(\mathcal{F}) & & \text{Aut}^{\otimes}(\mathcal{F}) \subset N(P_{\mathcal{F}}) \subset N(P_{\mathcal{F}}^{\circ}) \\ \cap & & \cap & & \cap & \text{and} & \parallel \\ P_{\mathcal{F}}^{\circ} & \subset & P_{\mathcal{F}} & \subset & \text{Aut}^{\otimes}(\mathcal{F}) & & \text{Aut}^{\otimes}(\mathcal{F}) \subset N(U_{\mathcal{F}}) \subset N(U_{\mathcal{F}}^{\circ}) \end{array}$$

The Lie algebras of $U_{\mathcal{F}}^{\circ} \subset U_{\mathcal{F}}$ and $P_{\mathcal{F}}^{\circ} \subset P_{\mathcal{F}}$ are respectively given by

$$\begin{aligned} \text{Lie}(U_{\mathcal{F}}^{\circ}) &= \text{Lie}(U_{\mathcal{F}}) = (\mathcal{I}(G_T)/\mathcal{J}(\mathcal{F}) + \mathcal{I}(G_T)^2)^{\vee} \\ \text{and } \text{Lie}(P_{\mathcal{F}}^{\circ}) &= \text{Lie}(P_{\mathcal{F}}) = (\mathcal{I}(G_T)/\mathcal{I}(\mathcal{F}) + \mathcal{I}(G_T)^2)^{\vee} \end{aligned}$$

As quasi-coherent \mathcal{O}_T -submodules of

$$\text{Lie}(G_T) = \mathfrak{g}_T = (\mathcal{I}(G_T)/\mathcal{I}(G_T)^2)^{\vee}$$

they correspond to the \mathcal{O}_T -linear forms on $\omega_T(\rho^1) = \mathcal{I}(G_T)/\mathcal{I}(G_T)^2$ vanishing on

$$\mathcal{F}^0(\rho_1) = \mathcal{J}(\mathcal{F}) + \mathcal{I}(G_T)^2/\mathcal{I}(G_T)^2 \quad \text{and} \quad \mathcal{F}_+^0(\rho_1) = \mathcal{I}(\mathcal{F}) + \mathcal{I}(G_T)^2/\mathcal{I}(G_T)^2$$

respectively. We thus find that, as \mathcal{O}_T -submodules of $\mathfrak{g}_T = \omega_T(\rho_{\text{ad}})$,

$$\text{Lie}(U_{\mathcal{F}}^{\circ}) = \text{Lie}(U_{\mathcal{F}}) = \mathcal{F}_+^0(\rho_{\text{ad}}) \quad \text{and} \quad \text{Lie}(P_{\mathcal{F}}^{\circ}) = \text{Lie}(P_{\mathcal{F}}) = \mathcal{F}^0(\rho_{\text{ad}}).$$

3.7.13. We show that $P_{\mathcal{F}}^{\circ}$ is a parabolic subgroup of G_T with unipotent radical $U_{\mathcal{F}}^{\circ}$. Since both groups are finitely presented and smooth over T with $U_{\mathcal{F}}^{\circ} \subset N(P_{\mathcal{F}}^{\circ})$, we may assume that $T = \text{Spec}(k)$ for some algebraically closed field k by [12, XXVI 1.1 and 1.6]. Since then $T \rightarrow S$ is affine, we may *also* assume that $S = \text{Spec}(k)$ by part (2) of Proposition 27, in which case G is linear by lemma 37. Using the criterion of [30, IV 2.4.3.1], we now have to verify that

$$(a) \dim U_{\mathcal{F}}^{\circ} = \dim G/P_{\mathcal{F}}^{\circ} \quad \text{and} \quad (b) U_{\mathcal{F}}^{\circ} \text{ is unipotent.}$$

The equality of dimensions follows from Proposition 43 below since

$$\dim U_{\mathcal{F}}^{\circ} = \dim_k \text{Lie}(U_{\mathcal{F}}^{\circ}) = \dim_k \mathcal{F}_+^0(\rho_{\text{ad}}) = \sum_{\gamma > 0} \dim_k \text{Gr}_k^{\gamma}(\rho_{\text{ad}})$$

and

$$\dim G/P_{\mathcal{F}}^{\circ} = \dim_k \mathfrak{g}/\mathcal{F}^0(\rho_{\text{ad}}) = \dim_k \mathcal{F}_+^0(\rho_{\text{ad}}^{\vee}) = \sum_{\gamma > 0} \dim_k \text{Gr}_k^{\gamma}(\rho_{\text{ad}}).$$

For (b), pick a finite dimensional faithful representation τ of G . Then

$$U_{\mathcal{F}}^{\circ} \subset U_{\mathcal{F}} \subset \text{Aut}^{\otimes 1}(\mathcal{F}) \subset U(\mathcal{F}(\tau))$$

where $U(\mathcal{F}(\tau))$ is the unipotent subgroup of $GL(V(\tau))$ defined by the Γ -filtration $\mathcal{F}(\tau)$ on $V(\tau)$. Therefore $U_{\mathcal{F}}^{\circ}$ is unipotent by [1, XVII 2.2.ii].

3.7.14. By [12, XXII 5.8.5], $P_{\mathcal{F}}^{\circ} = N(P_{\mathcal{F}}^{\circ})$, therefore also

$$P_{\mathcal{F}}^{\circ} = P_{\mathcal{F}} = \text{Aut}^{\otimes}(\mathcal{F}) = N(P_{\mathcal{F}}) = N(P_{\mathcal{F}}^{\circ}).$$

On the other hand, the above proof of (b) shows that $U_{\mathcal{F}}$ has unipotent – thus connected – geometric fibers, therefore also $U_{\mathcal{F}} = U_{\mathcal{F}}^{\circ}$, and this finishes the proof of our theorem. Note that we can not say much more about $\text{Aut}^{\otimes 1}(\mathcal{F})$ at this point – we do not even know that it is actually representable.

3.8. Grothendieck groups. Let again G be affine and flat over S . Let T be an S -scheme and let \mathcal{F} be a Γ -filtration on ω_T° . Since \mathcal{F} and Gr are exact \otimes -functors,

$$\text{Gr}_{\mathcal{F}}^\bullet : \text{Rep}^\circ(G)(S) \xrightarrow{\mathcal{F}} \text{Fil}^\Gamma \text{LF}(T) \xrightarrow{\text{Gr}} \text{Gr}^\Gamma \text{LF}(T)$$

is also an exact \otimes -functor. It thus induces a morphism between the Grothendieck ring $K_0(G)$ of $\text{Rep}^\circ(G)(S)$ and the Grothendieck ring of $\text{Gr}^\Gamma \text{LF}(T)$. The rank function on finite locally free sheaves over T defines a morphism from the latter ring to the ring $\mathcal{C}(T, \mathbb{Z}[\Gamma])$ of locally constant functions on T with values in the group ring $\mathbb{Z}[\Gamma]$ of Γ . The Γ -filtration \mathcal{F} on ω_T° thus defines a ring homomorphism

$$\kappa(\mathcal{F}) : K_0(G) \rightarrow \mathcal{C}(T, \mathbb{Z}[\Gamma])$$

which maps the class of $\rho \in \text{Rep}^\circ(G)(S)$ in $K_0(G)$ to the function

$$t \mapsto \sum_{\gamma \in \Gamma} \dim_{k(t)}(\text{Gr}_{\mathcal{F}}^\gamma(\rho) \otimes k(t)) \cdot e^\gamma$$

where e^γ is the basis element of $\mathbb{Z}[\Gamma]$ corresponding to γ . We have:

$$\forall z \in K_0(G) : \quad \kappa(\mathcal{F})(z^\vee) = \kappa(\mathcal{F})(z)^\vee$$

where the involutions $z \mapsto z^\vee$ are induced by the duality $\rho \mapsto \rho^\vee$ on $\text{Rep}^\circ(G)(S)$ and by $\sum x_\lambda e^\lambda \mapsto \sum x_\lambda e^{-\lambda}$ on $\mathbb{Z}[\Gamma]$. When G is smooth over S , we define

$$\begin{aligned} \kappa(G) &= [\rho_{\text{ad}}] - [\rho_{\text{ad}}^\vee] \in K_0(G) \\ \text{and } \kappa(G, \mathcal{F}) &= \text{image of } \kappa(G) \text{ in } \mathcal{C}(T, \mathbb{Z}[\Gamma]) \end{aligned}$$

The formation of $\kappa(G, \mathcal{F})$ is compatible with arbitrary base change on T .

Proposition 43. *If (1) G is an isotrivial reductive group over a quasi-compact S , or if (2) G is a reductive group over S and \mathcal{F} comes from a filtration on ω_T , then*

$$\kappa(G, \mathcal{F}) = 0 \quad \text{in } \mathcal{C}(T, \mathbb{Z}[\Gamma]).$$

Proof. (1) Let $\{S_i \rightarrow S\}$ be a covering of S by finite étale morphisms such that each $G_i = G_{S_i}$ splits. Let $\{T_i \rightarrow T\}$ be the corresponding covering of T . By part (1) of Proposition 27, \mathcal{F}_{T_i} extends to a Γ -filtration \mathcal{F}_i on $\omega_i^\circ : \text{Rep}^\circ(G_i)(S_i) \rightarrow \text{LF}(T_i)$, and obviously $\kappa(G_i, \mathcal{F}_i) = \kappa(G, \mathcal{F}) \circ (T_i \rightarrow T)$. We may thus assume that G splits over S , in which case the Proposition follows from Lemma 44 below. The proof of (2) is similar: let $\{t \rightarrow T\}$ be a covering of T by geometric points, thus G_t splits. By part (2) of Proposition 27, \mathcal{F}_t extends to a Γ -filtration on $\omega_t^\circ : \text{Rep}(G_t)(t) \rightarrow \text{LF}(t)$ which we also denote by \mathcal{F}_t , and obviously $\kappa(G_t, \mathcal{F}_t) = \kappa(G, \mathcal{F}) \circ (t \rightarrow T)$. \square

Lemma 44. *If G is a split reductive group over a quasi-compact S , then $\kappa(G) = 0$.*

Proof. By [12, XXII 2.8], there is a decomposition $S = \coprod_{i \in I} S_i$ into open and closed subschemes $S_i \neq \emptyset$ of S such that for each $i \in I$, G_{S_i} is of constant type, thus isomorphic [12, XXIII 5.2] to the base change of a split reductive group $G_{0,i}$ over $\text{Spec}(\mathbb{Z})$ [12, XXV 1.2]. Since S is quasi-compact, the indexing set I is finite and $K_0(G) \simeq \otimes_{i \in I} K_0(G_{S_i})$ with $\kappa(G) = \sum_{i \in I} \kappa(G)_i$ where $\kappa(G)_i$ is the image of $\kappa(G_{0,i})$ under $K_0(G_{0,i}) \rightarrow K_0(G_{S_i}) \rightarrow K_0(G)$. We may thus assume that $S = \text{Spec}(A)$ where A a principal ideal domain. By [33, Théorème 5], we may even assume that $A = K$ is a field. Let H be a split maximal torus in G , with character group M and Weyl group W . The restriction $\text{Rep}^\circ(G) \rightarrow \text{Rep}^\circ(H)$ induces a ring homomorphism $K_0(G) \rightarrow K_0(H) \simeq \mathbb{Z}[M]$ which yields an isomorphism from $K_0(G)$ to $\mathbb{Z}[M]^W$ by [33, Théorème 4]. Let $R \subset M$ be the set of roots of H in the Lie algebra $\mathfrak{g} = V(\rho_{\text{ad}})$. The weight decomposition of $\rho_{\text{ad}}|_H$ is then given by $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$

with $\dim_K \mathfrak{g}_\alpha = 1$ for every $\alpha \in R$ (and $\mathfrak{g}_0 = \mathfrak{h}$ is the Lie algebra of H). Since $R = -R$, we find that $\rho_{\text{ad}}|_H \simeq \rho_{\text{ad}}^\vee|_H$. Therefore indeed $\kappa(G) = 0$ in $K_0(G)$. \square

3.9. The stabilizer of a Γ -filtration, II. We have the following variant of Theorem 38 and 41. Let G be an isotrivial reductive group over a quasi-compact S .

Theorem 45. *For an S -scheme T and a Γ -filtration \mathcal{F} on V_T° or ω_T° , $\text{Aut}^\otimes(\mathcal{F})$ is a parabolic subgroup $P_{\mathcal{F}}$ of G_T with unipotent radical $U_{\mathcal{F}} \subset \text{Aut}^{\otimes!}(\mathcal{F})$. Moreover,*

$$\text{Lie}(U_{\mathcal{F}}) = \mathcal{F}_+^0(\rho_{\text{ad}}) \quad \text{and} \quad \text{Lie}(P_{\mathcal{F}}) = \mathcal{F}^0(\rho_{\text{ad}}) \quad \text{in} \quad V_T(\rho_{\text{ad}}) = \mathfrak{g}_T.$$

Corollary 46. *For any S -scheme T and $\mathcal{F} \in \mathbb{F}^\Gamma(G)(T)$,*

$$P_{\mathcal{F}} = \text{Aut}^\otimes(\iota\mathcal{F}), \quad U_{\mathcal{F}} = \text{Aut}^{\otimes!}(\iota\mathcal{F}) \quad \text{and} \quad P_{\mathcal{F}}/U_{\mathcal{F}} = \text{Aut}^\otimes(\text{Gr}_{\iota\mathcal{F}}^\bullet)$$

where $\iota\mathcal{F}$ stands for the image of \mathcal{F} in either $\mathbb{F}^\Gamma(V_T^\circ)$ or $\mathbb{F}^\Gamma(\omega_T^\circ)$.

The proof of the Corollary is identical to that of its earlier counterpart.

3.9.1. By Propositions 34, 30, 31 and 36, it is sufficient to establish the Theorem for a Γ -filtration \mathcal{F} on ω_T° . For any X -scheme T , we have

$$\begin{aligned} \text{Aut}^\otimes(\mathcal{F})(X) &= \{g \in G(X) \mid \forall \tau, \gamma \in \text{Rep}^\circ(G)(S) \times \Gamma : \rho_X(g)(\mathcal{F}^\gamma(\tau)_X) = \mathcal{F}^\gamma(\tau)_X\}, \\ &= \{g \in G(X) \mid \forall \rho, \gamma \in \text{Rep}'(G)(S) \times \Gamma : \rho_X(g)(\mathcal{F}^\gamma(\rho)_X) = \mathcal{F}^\gamma(\rho)_X\}. \end{aligned}$$

We have to show that the fpqc subsheaf $\text{Aut}^\otimes(\mathcal{F}) : (\text{Sch}/T)^0 \rightarrow \text{Set}$ of G_T is representable by a parabolic subgroup with the specified Lie algebra: this is a local question in the fpqc topology on T . Let $\{S_i \rightarrow S\}$ be a covering of S by finite étale morphisms such that $G_i = G_{S_i}$ is split, let $\{T_i \rightarrow T\}$ be the induced covering of T , let ω_i denote the fiber functors for G_i and let \mathcal{F}_i be the unique extension of \mathcal{F}_{T_i} to a Γ -graduation on ω_{i,T_i}° . Going back to its actual definition in the proof of Proposition 27, one checks easily that $\text{Aut}^\otimes(\mathcal{F})|_{T_i} = \text{Aut}^\otimes(\mathcal{F}_{T_i})$ is equal to $\text{Aut}^\otimes(\mathcal{F}_i)$ as a subsheaf of $G|_{T_i} = \text{Aut}^\otimes(\omega^\circ)|_{T_i} = \text{Aut}^\otimes(\omega_i^\circ)|_{T_i}$. We may (and do) therefore assume that G is a split reductive group over a quasi-compact S . By [12, XXII 2.8, XXIII 5.2 and XXV 1.2], we then have a finite partition of $S = \coprod S_i$ into open and closed subschemes such that each $G_i = G_{S_i}$ arises from a split group over $\text{Spec}(\mathbb{Z})$, and repeating the above argument with that covering, we may thus also assume that G is the base change of a split reductive group G_0 over $\text{Spec}(\mathbb{Z})$.

3.9.2. In particular, the proof of part (2) of Proposition 34 now shows that with ρ_{reg} , also ρ_{adj} and ρ_{adj}° belong to $\text{Rep}'(G)(S)$, to which we have extended \mathcal{F} in section 3.6.6. We may thus define subschemes $U_{\mathcal{F}}$ and $P_{\mathcal{F}}$ of G_T as in section 3.7.6, and try to follow from there on the subsequent steps of the proof of Theorem 38. Of course, we have to check that we are only using our filtration where it is defined, namely on $\text{Rep}'(G)(S)$, and that whenever the axiom (F3) was used, we could have replaced it with the weaker left and right axioms (F3l) or (F3r).

3.9.3. In 3.7.9 and 3.7.10, we used the right exactness of \mathcal{F} for (respectively)

$$A : S^\bullet(\tau) = \text{Sym}^\bullet(\tau^\vee \otimes \tau) \twoheadrightarrow \rho_{\text{adj}} \quad \text{and} \quad B : (\rho_{\text{adj}}^\circ)^{\otimes 2} \twoheadrightarrow \rho_{\text{adj}}^{\circ(2)}.$$

To deal with the first one, it would be sufficient to know that there is a cofinal set $\Sigma \in X(\rho_{\text{adj}})$ such that for all $\sigma \in \Sigma$, $A^{-1}(\sigma)$ is still in $\text{Rep}'(G)(S)$: then

$$\begin{aligned} \mathcal{F}^\gamma(\rho_{\text{adj}}) &= \varinjlim \mathcal{F}^\gamma(\sigma) \stackrel{\text{F3r}}{=} \varinjlim A(\mathcal{F}^\gamma(A^{-1}(\sigma))) \\ &= A(\mathcal{F}^\gamma(\varinjlim A^{-1}(\sigma))) = A(\mathcal{F}^\gamma(S^\bullet(\sigma))). \end{aligned}$$

Over a Dedekind domain, we have Wedhorn's criterion: a ρ is in $\text{Rep}'(G)(S)$ if and only if $V(\rho)$ is flat, i.e. torsion free: thus over such a domain, $A^{-1}(\sigma)$ still belongs to $\text{Rep}'(G)(S)$ for any $\sigma \in X(\rho_{\text{adj}})$. Applying this to G_0 and choosing τ in 3.7.9 also be defined over $\text{Spec}(\mathbb{Z})$ settles the case of A , and that of B is similar.

3.9.4. Everything then goes through up to 3.7.13: $U_{\mathcal{F}}^{\circ}$ and $P_{\mathcal{F}}^{\circ}$ are smooth subgroups of G_T with the good Lie algebras, etc. . . In 3.7.13, we may still reduce to the case where $T = \text{Spec}(k)$ for some algebraically closed field k and use the criterion of [30, IV 2.4.3.1], but we can not change S to $\text{Spec}(k)$. However, since we have already reduced to the split case, Proposition 43 (or Lemma 44) deals perfectly well with condition (a), and Lemma 37 with condition (b).

3.10. Splitting filtrations. We now come to the main statement of Theorem 21. Let thus G be a reductive isotrivial group over a quasi-compact S , let T be an S -scheme and let \mathcal{F} be a Γ -filtration on ω_T° . We will then show that: locally on T for the étale topology, \mathcal{F} has a splitting $\chi : \mathbb{D}_T(\Gamma) \rightarrow G_T$.

3.10.1. Let $f : \tilde{S} \rightarrow S$ be a finite étale cover splitting G and denote by $\tilde{\mathcal{F}}$ the unique extension of $\mathcal{F}_{\tilde{T}}$ to a Γ -filtration on $\tilde{\omega}_{\tilde{T}}^{\circ}$ (see Proposition 27), where $\tilde{T} = T_{\tilde{S}}$ and $\tilde{\omega}$ is the fiber functor for $\tilde{G} = G_{\tilde{S}}$. If $\chi : \mathbb{D}_{\tilde{T}}(\Gamma) \rightarrow G_{\tilde{T}}$ is a splitting of $\tilde{\mathcal{F}}$, it is *a fortiori* a splitting of $\mathcal{F}_{\tilde{T}}$: we may thus assume that G splits over S .

3.10.2. For a positive integer k , there is a cartesian diagram of fpqc sheaves on S ,

$$\begin{array}{ccccc} \mathbb{G}^{\Gamma}(G) & \xrightarrow{\text{Prop. 30}} & \mathbb{G}^{\Gamma}(\omega^{\circ}) & \xrightarrow{\text{Fil}} & \mathbb{F}^{\Gamma}(\omega^{\circ}) \\ k_1 \downarrow & & k_2 \downarrow & & k_3 \downarrow \\ \mathbb{G}^{\Gamma}(G) & \xrightarrow{\text{Prop. 30}} & \mathbb{G}^{\Gamma}(\omega^{\circ}) & \xrightarrow{\text{Fil}} & \mathbb{F}^{\Gamma}(\omega^{\circ}) \end{array}$$

where the k_i 's map χ , \mathcal{G} and \mathcal{F} to respectively $k_1(\chi) = \chi \circ \mathbb{D}_T(k) = \chi^k$,

$$k_2(\mathcal{G})_{\gamma}(\rho) = \begin{cases} 0 & \text{if } \gamma \notin k\Gamma, \\ \mathcal{G}_{\eta}(\rho) & \text{if } \gamma = k\eta, \end{cases} \quad \text{and} \quad k_3(\mathcal{F})^{\gamma}(\rho) = \cup_{k\eta \geq \gamma} \mathcal{F}^{\eta}(\rho).$$

They are all obviously well-defined monomorphisms, and the image of k_2 is the subsheaf of $\mathbb{G}^{\Gamma}(\omega^{\circ})$ made of those Γ -graduation \mathcal{G}' for which $\mathcal{G}'_{\gamma} \equiv 0$ for $\gamma \notin k\Gamma$. The diagram is cartesian because if \mathcal{G}' splits $k_3(\mathcal{F})$, then $\mathcal{G}'_{\gamma} \simeq \text{Gr}_{k_3(\mathcal{F})}^{\gamma} \equiv 0$ for $\gamma \notin k\Gamma$, thus $\mathcal{G}' = k_2(\mathcal{G})$ for a unique \mathcal{G} , which has to also split \mathcal{F} since

$$k_3(\mathcal{F}) = \text{Fil}(\mathcal{G}') = \text{Fil}(k_2(\mathcal{G})) = k_3(\text{Fil}(\mathcal{G})).$$

3.10.3. For a central isogeny $f : G \rightarrow G'$, there is a commutative diagram

$$\begin{array}{ccccc} \mathbb{G}^{\Gamma}(G) & \xrightarrow{\text{Prop. 30}} & \mathbb{G}^{\Gamma}(\omega^{\circ}) & \xrightarrow{\text{Fil}} & \mathbb{F}^{\Gamma}(\omega^{\circ}) \\ f_1 \downarrow & & f_2 \downarrow & & f_3 \downarrow \\ \mathbb{G}^{\Gamma}(G') & \xrightarrow{\text{Prop. 30}} & \mathbb{G}^{\Gamma}(\omega'^{\circ}) & \xrightarrow{\text{Fil}} & \mathbb{F}^{\Gamma}(\omega'^{\circ}) \end{array}$$

where $\omega' = \omega \circ f^*$ denotes the fiber functor for G' and the f_i 's map χ , \mathcal{G} and \mathcal{F} to respectively $f_1(\chi) = f \circ \chi$, $f_2(\mathcal{G}) = \mathcal{G} \circ f^*$ and $f_3(\mathcal{F}) = \mathcal{F} \circ f^*$, with

$$f^* : \text{Rep}(G')(S) \rightarrow \text{Rep}(G)(S) \quad f^*(\rho) = \rho \circ f.$$

We claim that (1) all f_i 's are monomorphisms, and (2) the diagram is cartesian. This is local in the finite étale topology on S by Proposition 27, and we may thus assume that the kernel C of f is isomorphic to $\mathbb{D}_S(X)$ for some finite commutative group X . We fix an S -scheme T and consider sections of the above sheaves over T . If $f \circ \chi_1 = f \circ \chi_2$, then $\chi_1^{-1}\chi_2$ is a morphism $\mathbb{D}_T(\Gamma) \rightarrow C_T$, which has to be trivial since X is finite and Γ torsion free: f_1 is injective. Any $\rho \in \text{Rep}^\circ(G)(S)$ has a finite sum decomposition $\rho = \bigoplus \rho(x)$ according to the characters $x \in X$ of C , and C acts trivially on $\rho(x)^{\otimes k(x)}$ where $k(x) \geq 1$ is the order of x in X . If two Γ -filtrations \mathcal{F}_1 and \mathcal{F}_2 on ω_T° induce the same Γ -filtration on ω_T° , then $\mathcal{F}_1(\rho) = \mathcal{F}_2(\rho)$ for every ρ on which C acts trivially, thus $\mathcal{F}_1(\rho(x)) = \mathcal{F}_2(\rho(x))$ for every ρ and x by Lemma 47 below, therefore $\mathcal{F}_1(\rho) = \mathcal{F}_2(\rho)$ since $\rho = \bigoplus \rho(x)$: f_3 is injective. Similarly: f_2 is injective. Finally, suppose that \mathcal{G}' splits $f_3(\mathcal{F})$. Let $\chi' : \mathbb{D}_T(\Gamma) \rightarrow G'_T$ be the corresponding morphism. Fix $k \geq 1$ such that $k_1(\chi')$ lifts to $\chi_k : \mathbb{D}_T(\Gamma) \rightarrow G_T$, giving a Γ -graduation \mathcal{G}_k and a Γ -filtration \mathcal{F}_k on ω_T° . They respectively map to

$$f_2(\mathcal{G}_k) = f_2 \circ \iota(\chi_k) = \iota \circ f_1(\chi_k) = \iota \circ k_1(\chi') = k_2 \circ \iota(\chi') = k_2(\mathcal{G}')$$

where ι is the isomorphism $\mathbb{G}^\Gamma(G) \simeq \mathbb{G}^\Gamma(\omega^\circ)$, and

$$f_3(\mathcal{F}_k) = f_3 \circ \text{Fil}(\mathcal{G}_k) = \text{Fil} \circ f_2(\mathcal{G}_k) = \text{Fil} \circ k_2(\mathcal{G}') = k_3 \circ \text{Fil}(\mathcal{G}') = k_3 \circ f_3(\mathcal{F}).$$

Thus $f_3(\mathcal{F}_k) = f_3 \circ k_3(\mathcal{F})$ and $\mathcal{F}_k = k_3(\mathcal{F})$ since f_3 is a monomorphism. Since \mathcal{G}_k splits $k_3(\mathcal{F})$, there is a unique \mathcal{G} such that $\mathcal{F} = \text{Fil}(\mathcal{G})$ and $k_2(\mathcal{G}) = \mathcal{G}_k$ by the cartesian diagram of the previous subsection. Moreover $f_2(\mathcal{G}) = \mathcal{G}'$ since

$$k_2 \circ f_2(\mathcal{G}) = f_2 \circ k_2(\mathcal{G}) = f_2(\mathcal{G}_k) = k_2(\mathcal{G}')$$

and k_2 is a monomorphism: our diagram is indeed cartesian.

Lemma 47. *Let \mathcal{M} be a finite locally free sheaf on a scheme S , $k \geq 1$.*

(1) *Let \mathcal{F}_1 and \mathcal{F}_2 be local direct factors of \mathcal{M} . Then:*

$$\mathcal{F}_1^{\otimes k} = \mathcal{F}_2^{\otimes k} \text{ in } \mathcal{M}^{\otimes k} \implies \mathcal{F}_1 = \mathcal{F}_2 \text{ in } \mathcal{M}.$$

(2) *Let \mathcal{F}_1 and \mathcal{F}_2 be Γ -filtrations on \mathcal{M} . Then:*

$$\mathcal{F}_1^{\otimes k} = \mathcal{F}_2^{\otimes k} \text{ on } \mathcal{M}^{\otimes k} \implies \mathcal{F}_1 = \mathcal{F}_2 \text{ on } \mathcal{M}.$$

Proof. (1) Fix $s \in S$ with residue field $k(s)$. We have to show that $\mathcal{F}_1 = \mathcal{F}_2$ in a neighborhood of s . Shrinking S if necessary, we may assume that \mathcal{F}_1 and \mathcal{F}_2 are free of constant rank n_1 and n_2 . By assumption, $n_1^k = n_2^k$, therefore $n_1 = n_2 = n$. If $n = 0$, $\mathcal{F}_1 = 0 = \mathcal{F}_2$ and we are done. Suppose $n > 0$, and choose a linear form $f : \mathcal{M}(s) \rightarrow k(s)$ which is non-zero on $\mathcal{F}_1(s)$ and $\mathcal{F}_2(s)$. Shrinking S further, we may lift f to an \mathcal{O}_S -linear map $f : \mathcal{M} \rightarrow \mathcal{O}_S$ such that $f(\mathcal{F}_1) = \mathcal{O}_S = f(\mathcal{F}_2)$. Then for the \mathcal{O}_S -linear map $F = \text{Id} \otimes f^{k-1} : \mathcal{M}^{\otimes k} \rightarrow \mathcal{M}$, we have

$$\mathcal{F}_1 = F(\mathcal{F}_1^{\otimes k}) = F(\mathcal{F}_2^{\otimes k}) = \mathcal{F}_2.$$

(2) The question is local for the Zariski topology on S . By Proposition 24, we may thus assume that both filtrations split, say

$$\mathcal{F}_1^\gamma = \bigoplus_{\eta \geq \gamma} \mathcal{G}_1^\eta \quad \text{and} \quad \mathcal{F}_2^\gamma = \bigoplus_{\eta \geq \gamma} \mathcal{G}_2^\eta$$

with \mathcal{G}_i^γ locally free of constant rank n_i^γ for every $i \in \{1, 2\}$ and $\gamma \in \Gamma$. We then argue by induction on the constant rank $n = \sum n_1^\gamma = \sum n_2^\gamma$ of \mathcal{M} . For $n = 0$, there is nothing to prove. Suppose $n > 0$. By assumption, for every $\gamma \in \Gamma$,

$$\sum_{a_1 + \dots + a_k = \gamma} \mathcal{F}_1^{a_1} \otimes \dots \otimes \mathcal{F}_1^{a_k} = \sum_{a_1 + \dots + a_k = \gamma} \mathcal{F}_2^{a_1} \otimes \dots \otimes \mathcal{F}_2^{a_k}$$

which means that

$$\bigoplus_{a_1+\dots+a_k \geq \gamma} \mathcal{G}_1^{a_1} \otimes \dots \otimes \mathcal{G}_1^{a_k} = \bigoplus_{a_1+\dots+a_k \geq \gamma} \mathcal{G}_2^{a_1} \otimes \dots \otimes \mathcal{G}_2^{a_k}$$

Let γ_i be the largest element of the (non-empty!) finite set $\{a : \mathcal{G}_i^a \neq 0\}$. Then

$$\bigoplus_{a_1+\dots+a_k \geq \gamma} \mathcal{G}_i^{a_1} \otimes \dots \otimes \mathcal{G}_i^{a_k} = \begin{cases} 0 & \text{if } \gamma > k\gamma_i, \\ \mathcal{G}_i^{\gamma_i} \otimes \dots \otimes \mathcal{G}_i^{\gamma_i} & \text{for } \gamma = k\gamma_i, \\ \neq 0 & \text{if } \gamma \leq k\gamma_i. \end{cases}$$

Thus $k\gamma_1 = k\gamma_2$, $\gamma_1 = \gamma_2 = \gamma_0$ and $\mathcal{G}_1^{\gamma_0} \otimes \dots \otimes \mathcal{G}_1^{\gamma_0} = \mathcal{G}_2^{\gamma_0} \otimes \dots \otimes \mathcal{G}_2^{\gamma_0}$ in $\mathcal{M}^{\otimes k}$, therefore $\mathcal{F}_1^{\gamma_0} = \mathcal{G}_1^{\gamma_0} = \mathcal{G}_2^{\gamma_0} = \mathcal{F}_2^{\gamma_0} = \mathcal{N}$ in \mathcal{M} by the previous lemma. We conclude by our induction hypothesis applied to the images of \mathcal{F}_1 and \mathcal{F}_2 in \mathcal{M}/\mathcal{N} . \square

3.10.4. Suppose that $G = G_1 \times_S G_2$. Let \mathcal{F} be a Γ -filtration on ω_T° . Then \mathcal{F} induces a Γ -filtration \mathcal{F}_i on the fiber functor $\omega_{i,T}^\circ$ for G_i by the formulas:

$$\mathcal{F}_1^\gamma(\rho_1) = \mathcal{F}^\gamma(\rho_1 \boxtimes 1_{G_2}) \quad \text{and} \quad \mathcal{F}_2^\gamma(\rho_2) = \mathcal{F}^\gamma(1_{G_1} \boxtimes \rho_2)$$

We claim that if χ_i splits \mathcal{F}_i , then $\chi = (\chi_1, \chi_2)$ splits \mathcal{F} . Indeed, we may as above assume that G_1 and G_2 are split, and we extend \mathcal{F} to $\text{Rep}'(G)(S)$. We then have to show that the Γ -filtration \mathcal{F}' associated to χ equals \mathcal{F} on ρ_{reg} . Since

$$\rho_{\text{reg}} = \rho_{1,\text{reg}} \boxtimes \rho_{2,\text{reg}} = \varinjlim \tau_1 \boxtimes \tau_2$$

where $\rho_{i,\text{reg}}$ is the regular representation of G_i and the colimit is over $\tau_i \in X(\rho_{i,\text{reg}})$, it is also sufficient to establish that \mathcal{F}' equals \mathcal{F} on $\rho = \tau_1 \boxtimes \tau_2$, $\tau_i \in \text{Rep}^\circ(G)(S)$. Note that $\rho = \rho_1 \otimes \rho_2$ where $\rho_1 = \tau_1 \boxtimes 1_{G_2}$ and $\rho_2 = 1_{G_1} \boxtimes \tau_2$. We thus find

$$\begin{aligned} \mathcal{F}^\gamma(\rho) &= \sum_{\gamma_1+\gamma_2=\gamma} \mathcal{F}^{\gamma_1}(\rho_1) \otimes \mathcal{F}^{\gamma_2}(\rho_2) \\ &= \sum_{\gamma_1+\gamma_2=\gamma} \mathcal{F}_1^{\gamma_1}(\tau_1) \otimes \mathcal{F}_2^{\gamma_2}(\tau_2) \\ &= \bigoplus_{\gamma_1+\gamma_2 \geq \gamma} \mathcal{G}_1^{\gamma_1}(\tau_1) \otimes \mathcal{G}_2^{\gamma_2}(\tau_2) \\ &= \bigoplus_{\eta \geq \gamma} \mathcal{G}^\eta(\tau_1 \boxtimes \tau_2) \\ &= \mathcal{F}'^\gamma(\rho) \end{aligned}$$

where \mathcal{G} and the \mathcal{G}_i 's are the Γ -graduations induced by χ and the χ_i 's.

3.10.5. Applying 3.10.1, 3.10.3 with the central isogeny from G to the product of its adjoint group and its coradical, and finally 3.10.4, we may assume that G is either a split torus or a split reductive group of adjoint type.

3.10.6. Let thus first $G = \mathbb{D}_S(M)$ for some $M \simeq \mathbb{Z}^d$ and let \mathcal{F} be a Γ -filtration on ω_T° for an S -scheme T , which we may assume to be (absolutely) affine. Let ρ_m be the representation of G on $V(\rho_m) = \mathcal{O}_S$ on which G acts by the character $m \in M$. By Proposition 24, there exists a Γ -graduation $\mathcal{O}_T = \bigoplus_\gamma \mathcal{I}_\gamma(m)$ such that

$$\forall \gamma \in \Gamma : \quad \mathcal{F}^\gamma(\rho_m) = \bigoplus_{\eta \geq \gamma} \mathcal{I}_\eta(m).$$

Let $T_\gamma(m)$ be the support of $\mathcal{I}_\gamma(m)$. Thus $T = \coprod_\gamma T_\gamma(m)$ and $T_\gamma(m)$ is open and closed in T . For $t \in T$ and $m \in M$, we denote by $f(t)(m)$ the unique element γ in Γ such that t belongs to $T_\gamma(m)$. Thus $\mathcal{F}_t^\gamma(\rho_m) = k(t)$ if $\gamma \leq f(t)(m)$ and 0 otherwise, where $k(t)$ is the residue field at t . Since $\rho_0 = 1_G$, $f(t)(0) = 0$ by the axiom (F2) for \mathcal{F} . Since $\rho_{m_1} \otimes \rho_{m_2} = \rho_{m_1+m_2}$, $f(t)(m_1+m_2) = f(t)(m_1) + f(t)(m_2)$ by the axiom (F1) for \mathcal{F} . Thus $f(t) : M \rightarrow \Gamma$ is a group homomorphism. Since M is finitely

generated, $f : T \rightarrow \text{Hom}_{\text{Group}}(M, \Gamma)$ is locally constant, and thus corresponds to a global section $\chi : \mathbb{D}_T(\Gamma) \rightarrow G_T$ of the locally constant sheaf (see [1, VIII 1.5])

$$\text{Hom}(M, \Gamma)_T = \underline{\text{Hom}}(M_T, \Gamma_T) = \underline{\text{Hom}}(\mathbb{D}_T(\Gamma), \mathbb{D}_T(M)) = \underline{\text{Hom}}(\mathbb{D}_T(\Gamma), G_T).$$

Let \mathcal{F}' be the corresponding Γ -filtration on ω_T . For any morphism $\phi : M \rightarrow \Gamma$, let $T(\phi)$ be the open and closed subset of T where $f \equiv \phi$. Thus $T = \coprod T(\phi)$ and

$$T(\phi) = \cap_{m \in M} T_{\phi(m)}(m) = \cap_{i=1}^r T_{\phi(m_i)}(m_i)$$

if $\{m_1, \dots, m_r\} \subset M$ spans M . On $T(\phi)$, we find that

$$\mathcal{F}'_{T(\phi)}{}^\gamma(\rho_m) = \begin{cases} \mathcal{O}_{T(\phi)} & \text{if } \gamma \leq \phi(m) \\ 0 & \text{if } \gamma > \phi(m) \end{cases} = \mathcal{F}_{T(\phi)}^\gamma(\rho_m).$$

Thus $\mathcal{F}'(\rho_m) = \mathcal{F}(\rho_m)$ for every m . Extending \mathcal{F} as in 3.6.6, also $\mathcal{F}'(\rho_{\text{reg}}) = \mathcal{F}(\rho_{\text{reg}})$ since $\rho_{\text{reg}} = \oplus_{m \in M} \rho_m$. Finally $\mathcal{F}'(\rho) = \mathcal{F}(\rho)$ for any ρ by (F3l) applied to c_ρ . Therefore χ is a splitting of \mathcal{F} – it is in fact the unique such splitting.

3.10.7. Suppose finally that G is a split reductive group of adjoint type over S , let T be an S -scheme, and let \mathcal{F} be a Γ -filtration on ω_T° . We have just recalled that \mathcal{F} is uniquely determined by the value of its extension to $\text{Rep}'(G)(S)$ on ρ_{reg} , but we now also have this: there is at most one Γ -filtration \mathcal{F}' on ω_T which equals \mathcal{F} on the adjoint representation ρ_{ad} of G on $V(\rho_{\text{ad}}) = \mathfrak{g} = \text{Lie}(G)$. In particular, any morphism $\chi : \mathbb{D}_T(\Gamma) \rightarrow G_T$ inducing \mathcal{F} on ρ_{ad} is a splitting of \mathcal{F} . To establish our claim, we consider the G -equivariant epimorphism of quasi-coherent G - \mathcal{O}_S -algebras

$$f : \text{Sym}_{\mathcal{O}_S}^\bullet(\rho_{\text{ad},0}^\vee \otimes \rho_{\text{ad}}) \rightarrow \rho_{\text{reg}}$$

which is defined as in section 3.7.4, starting from $c_{\text{ad}} : \rho_{\text{ad}} \rightarrow \rho_{\text{ad},0} \otimes \rho_{\text{reg}}$ for the closed embedding $\rho_{\text{ad}} : G \rightarrow GL(\mathfrak{g})$. If \mathcal{F}' equals \mathcal{F} on ρ_{ad} , they are also equal on $\text{Sym}^\bullet(\rho_{\text{ad},0}^\vee \otimes \rho_{\text{ad}})$ by the axioms (F1-3) for Γ -filtrations on ω_T° , thus also

$$\mathcal{F}'^\gamma(\rho_{\text{reg}}) \subset \mathcal{F}^\gamma(\rho_{\text{reg}})$$

for every $\gamma \in \Gamma$ by the axiom (F3) for the Γ -filtration \mathcal{F}' on ω_T — it is not yet known to be satisfied by the extension of \mathcal{F} to $\text{Rep}'(G)(S)$, unless we appeal to the arguments of section 3.9.3, which is not necessary: then $\mathcal{F}'^\gamma(\rho) \subset \mathcal{F}^\gamma(\rho)$ for every ρ in $\text{Rep}^\circ(G)(S)$ by (F3l) with c_ρ , therefore also $\mathcal{F}'_+(\rho) \subset \mathcal{F}_+(\rho)$; applying the latter inclusion to ρ^\vee and dualizing gives $\mathcal{F}^\gamma(\rho) \subset \mathcal{F}'^\gamma(\rho)$. Thus $\mathcal{F} = \mathcal{F}'$ on ω_T° .

3.10.8. By Theorem 45, $P_{\mathcal{F}} = \text{Aut}^\otimes(\mathcal{F})$ is a parabolic subgroup of G_T . Since our problem is local for the étale topology on T , we may assume that T is affine and the pair $(G_T, P_{\mathcal{F}})$ has an épingle $\mathcal{E} = (H, M, R, \dots)$ [12, XXVI 1.14]. Thus $H = \mathbb{D}_T(M)$ is a trivialized split maximal torus of G_T contained in $P_{\mathcal{F}}$, $R \subset M$ is the set of roots of H in \mathfrak{g}_T and if $\mathfrak{g}_T = \mathfrak{g}_0 \oplus \oplus_{\alpha \in R} \mathfrak{g}_\alpha$ is the corresponding weight decomposition (so that $\mathfrak{g}_0 = \text{Lie}(H)$), then $\text{Lie}(P_{\mathcal{F}}) = \mathfrak{g}_0 \oplus \oplus_{\alpha \in R'} \mathfrak{g}_\alpha$ for some subset R' of R as in [12, XXVI.1.4]. The maximal torus $H \subset P_{\mathcal{F}}$ gives rise to a Levi decomposition $P_{\mathcal{F}} = U_{\mathcal{F}} \rtimes L_{\mathcal{F}}$ with $H \subset L_{\mathcal{F}}$, $\text{Lie}(L_{\mathcal{F}}) = \mathfrak{g}_0 \oplus \oplus_{\alpha \in R'_1} \mathfrak{g}_\alpha$ and $\text{Lie}(U_{\mathcal{F}}) = \oplus_{\alpha \in R'_2} \mathfrak{g}_\alpha$ where $R'_1 = \{\alpha \in R' : -\alpha \in R'\}$ and $R'_2 = \{\alpha \in R' : -\alpha \notin R'\}$ [12, XXII.5.11.3]. We will then show that \mathcal{F} has a splitting $\chi : \mathbb{D}_T(\Gamma) \rightarrow G_T$.

3.10.9. Since $H \subset P_{\mathcal{F}} = \text{Aut}^{\otimes}(\mathcal{F})$, the Γ -filtration \mathcal{F} is stable under H and

$$\forall \gamma \in \Gamma, \rho \in \text{Rep}^{\circ}(G)(S) : \quad \mathcal{F}^{\gamma}(\rho) = \bigoplus_{m \in M} \mathcal{F}_m^{\gamma}(\rho)$$

where $\mathcal{F}_m^{\gamma}(\rho)$ is the m -th eigenspace of $\mathcal{F}^{\gamma}(\rho)$, viewed as a representation of H . Since $\text{Lie}(U_{\mathcal{F}}) = \mathcal{F}_+^0(\rho_{\text{ad}})$ and $\text{Lie}(P_{\mathcal{F}}) = \mathcal{F}^0(\rho_{\text{ad}})$ by Theorem 45, $\mathcal{F}_{\alpha}^{\gamma}(\rho_{\text{ad}}) = 0$ for ($\gamma > 0$ and $\alpha \notin R'_2$) or ($\gamma = 0$ and $\alpha \notin R' \cup \{0\}$) while $\mathcal{F}_{\alpha}^{\gamma}(\rho_{\text{ad}}) = \mathfrak{g}_{\alpha}$ when $\gamma \leq 0$ and $\alpha \in R' \cup \{0\}$. This determines $\mathcal{F}_{\alpha}^{\gamma}(\rho_{\text{ad}})$ for $\alpha \in R'_1 \cup \{0\}$:

$$\forall \alpha \in \pm R'_1 \cup \{0\} : \quad \mathcal{F}_{\alpha}^{\gamma}(\rho_{\text{ad}}) = \begin{cases} \mathfrak{g}_{\alpha} & \text{if } \gamma \leq 0, \\ 0 & \text{if } \gamma > 0. \end{cases}$$

For the remaining α 's (those in $\pm R'_2$), \mathfrak{g}_{α} is free of rank 1. Using lemma 26, we obtain a partition $T = \coprod T(f)$ into non-empty open and closed subschemes $T(f)$ of T indexed by certain functions $f : \pm R'_2 \rightarrow \Gamma$ such that, over $T(f)$,

$$\forall \alpha \in \pm R'_2 : \quad \mathcal{F}_{\alpha}^{\gamma}(\rho_{\text{ad}}) = \begin{cases} \mathfrak{g}_{\alpha} & \text{if } \gamma \leq f(\alpha), \\ 0 & \text{if } \gamma > f(\alpha). \end{cases}$$

We extend these functions to $R \cup \{0\}$ by setting $f(R'_1 \cup \{0\}) = 0$. Thus over $T(f)$,

$$\mathcal{F}^{\gamma}(\rho_{\text{ad}}) = \bigoplus_{\alpha \in R \cup \{0\} : f(\alpha) \geq \gamma} \mathfrak{g}_{\alpha}$$

Moreover $f(\alpha) > 0$ (resp. < 0) if and only if $\alpha \in R'_2$ (resp. $-R'_2$).

3.10.10. We will establish below that each of these f 's extends to a group homomorphism $f : M \rightarrow \Gamma$. The locally constant function $T \rightarrow \text{Hom}(M, \Gamma)$ mapping $t \in T(f)$ to f thus defines a morphism $\chi : \mathbb{D}_T(\Gamma) \rightarrow \mathbb{D}_T(M) = H \hookrightarrow G_T$. By construction, χ splits \mathcal{F} on ρ_{ad} , therefore χ splits \mathcal{F} everywhere by 3.10.7.

3.10.11. To show that f extends to a group homomorphism $f : M \rightarrow \Gamma$, we may assume that $T = T_f = \text{Spec}(k)$ where k is a field. By the definition of adjoint groups in [12, XXII.4.3.3] and using [12, XXI.3.5.5], we have to show that

- (1) $f(-\alpha) = -f(\alpha)$ for every $\alpha \in R$ and
- (2) $f(\alpha + \beta) = f(\alpha) + f(\beta)$ for every $\alpha, \beta \in R$ such that also $\alpha + \beta \in R$.

3.10.12. Since again $H \subset P_{\mathcal{F}} = \text{Aut}^{\otimes}(\mathcal{F})$ fixes \mathcal{F} , there is a factorization of $\text{Gr}_{\mathcal{F}}^{\bullet}$:

$$\text{Rep}^{\circ}(G)(S) \longrightarrow \text{Gr}^{\Gamma} \text{Rep}^{\circ}(H)(k) \longrightarrow \text{Gr}^{\Gamma} \text{LF}(k)$$

where $\text{Gr}^{\Gamma} \text{Rep}^{\circ}(H)(k)$ is the abelian \otimes -category of Γ -graded objects in $\text{Rep}^{\circ}(H)(k)$. Both functors are exact \otimes -functors, and we thus obtain a factorization of $\kappa(\mathcal{F})$:

$$K_0(G) \xrightarrow{\kappa} K_0(H)[\Gamma] = \mathbb{Z}[M][\Gamma] \longrightarrow \mathbb{Z}[\Gamma]$$

The morphism κ maps the class of $\rho \in \text{Rep}^{\circ}(G)(S)$ to

$$\kappa[\rho] = \sum_{m, \gamma} x_m^{\gamma}[\rho] \cdot e^m e^{\gamma}$$

where $e^m \in \mathbb{Z}[M]$ and $e^{\gamma} \in \mathbb{Z}[\Gamma]$ are the basis elements corresponding to $m \in M$ and $\gamma \in \Gamma$ and $x_m^{\gamma}[\rho]$ is the dimension of the m -th eigenspace of $\text{Gr}_{\mathcal{F}}^{\gamma}(\rho)$. Thus

$$\kappa[\rho_{\text{ad}}] = \left(\dim_k(\mathfrak{g}_0) \cdot e^0 + \sum_{\alpha \in R'_1} e^{\alpha} \right) \cdot e^0 + \sum_{\alpha \in \pm R'_2} e^{\alpha} e^{f(\alpha)}.$$

Since the above functors are compatible with dualities,

$$\begin{aligned} \kappa[\rho_{\text{ad}}^{\vee}] &= \left(\dim_k(\mathfrak{g}_0) \cdot \epsilon^0 + \sum_{\alpha \in R'_1} \epsilon^{-\alpha} \right) \cdot e^0 + \sum_{\alpha \in \pm R'_2} \epsilon^{-\alpha} e^{-f(\alpha)} \\ &= \left(\dim_k(\mathfrak{g}_0) \cdot \epsilon^0 + \sum_{\alpha \in R'_1} \epsilon^{\alpha} \right) \cdot e^0 + \sum_{\alpha \in \pm R'_2} \epsilon^{\alpha} e^{-f(-\alpha)}. \end{aligned}$$

Since $[\rho_{\text{ad}}] = [\rho_{\text{ad}}^{\vee}]$ in $K_0(G)$ by Lemma 44, $f(-\alpha) = -f(\alpha)$ for every $\alpha \in R$.

3.10.13. We have already defined the dual ρ_n of $\rho^n = \text{Coker}((\rho_{\text{adj}}^{\circ})^{\otimes n+1} \rightarrow \rho_{\text{adj}}^{\circ})$ in section 3.7.5. These representations act compatibly (as n varies), functorially (as ρ varies) and G -equivariantly on any representation $\rho \in \text{Rep}(G)(S)$ by

$$\rho_n \otimes \rho \xrightarrow{\text{Id} \otimes c_{\rho}} \rho_n \otimes \rho \otimes \rho_{\text{adj}} \xrightarrow{\text{Id} \otimes \text{proj}} \rho_n \otimes \rho \otimes \rho_{\text{adj}}^{\circ} \xrightarrow{\text{Id} \otimes \text{proj}} \rho_n \otimes \rho \otimes \rho^n \xrightarrow{\text{eval}_n} \rho$$

For $n = 1$, we retrieve the usual adjoint G -equivariant action

$$\text{ad}(\rho) : \rho_{\text{ad}} \otimes \rho \rightarrow \rho$$

of \mathfrak{g} on $V(\rho)$, which for $\rho = \rho_{\text{ad}}$ is nothing but the usual Lie bracket

$$[-, -] : \rho_{\text{ad}} \otimes \rho_{\text{ad}} \rightarrow \rho_{\text{ad}}.$$

We also denote by $[-, -] : \rho_n \otimes \rho_{\text{ad}} \rightarrow \rho_{\text{ad}}$ the above actions on ρ_{ad} . Thus

$$\forall \gamma \in \Gamma, \forall \alpha, \beta \in M : \quad [\mathcal{F}_{\alpha}^{\gamma}(\rho_n), \mathfrak{g}_{\beta}] \subset \mathcal{F}_{\alpha+\beta}^{\gamma+f(\beta)}(\rho_{\text{ad}}).$$

In particular, $[\mathcal{F}_{\alpha}^{\gamma}(\rho_n), \mathfrak{g}_{\beta}] \neq 0$ implies $\alpha + \beta, \beta \in R \cup \{0\}$ and

$$f(\alpha + \beta) \geq \gamma + f(\beta).$$

3.10.14. Suppose that α, β and $\alpha + \beta$ all belong to R , with $\ell(\alpha) \leq \ell(\beta)$ where ℓ is the length. Let q and p be the positive integers (with $2 \leq p + q \leq 4$) such that

$$\{\beta + n\alpha \in R : n \in \mathbb{Z}\} = \{\beta - (p-1)\alpha, \dots, \beta, \beta + \alpha, \dots, \beta + q\alpha\}$$

see [12, XXI 2.3.5 and 1]. By Chevalley's rule [12, XXIII 6.5],

$$[\mathfrak{g}_{\alpha}, \mathfrak{g}_{\beta}] = p\mathfrak{g}_{\alpha+\beta} \quad \text{and} \quad [\mathfrak{g}_{-\alpha}, \mathfrak{g}_{-\beta}] = p\mathfrak{g}_{-\alpha-\beta}.$$

Thus if $p \neq 0$ in k , $[\mathfrak{g}_{\alpha}, \mathfrak{g}_{\beta}] \neq 0$ and $[\mathfrak{g}_{-\alpha}, \mathfrak{g}_{-\beta}] \neq 0$, therefore

$$\left. \begin{array}{l} f(\alpha + \beta) \geq f(\alpha) + f(\beta) \\ f(-\alpha - \beta) \geq f(-\alpha) + f(-\beta) \end{array} \right\} \xrightarrow{(1)} f(\alpha + \beta) = f(\alpha) + f(\beta).$$

If $q = 1$, Chevalley's rule gives $[\mathfrak{g}_{\alpha}, \mathfrak{g}_{-\alpha-\beta}] \neq 0$ and $[\mathfrak{g}_{-\alpha}, \mathfrak{g}_{\alpha+\beta}] \neq 0$, thus again $f(\alpha + \beta) = f(\alpha) + f(\beta)$. This leaves a single case: $p = q = 2 = \text{char}(k)$, where the same method already gives $f(\beta) = f(\beta - \alpha) + f(\alpha)$. We will see below that also

$$[\mathcal{F}_{2\alpha}^{2f(\alpha)}(\rho_2), \mathfrak{g}_{\beta-\alpha}] = \mathfrak{g}_{\alpha+\beta} \quad \text{and} \quad [\mathcal{F}_{-2\alpha}^{-2f(\alpha)}(\rho_2), \mathfrak{g}_{\alpha+\beta}] = \mathfrak{g}_{\beta-\alpha}.$$

Therefore $f(\alpha + \beta) = 2f(\alpha) + f(\beta - \alpha)$, thus also $f(\alpha + \beta) = f(\alpha) + f(\beta)$.

3.10.15. The pure short exact sequences of finite locally free sheaves on S

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathrm{Sym}_{\mathcal{O}_S}^2 \left(\frac{\mathcal{I}(G)}{\mathcal{I}(G)^2} \right) & \rightarrow & \frac{\mathcal{I}(G)}{\mathcal{I}(G)^3} & \rightarrow & \frac{\mathcal{I}(G)}{\mathcal{I}(G)^2} \rightarrow 0 \\ 0 & \rightarrow & \ker & \rightarrow & \left(\frac{\mathcal{I}(G)}{\mathcal{I}(G)^2} \right)^{\otimes 2} & \rightarrow & \mathrm{Sym}_{\mathcal{O}_S}^2 \left(\frac{\mathcal{I}(G)}{\mathcal{I}(G)^2} \right) \rightarrow 0 \end{array}$$

give rise to pure short exact sequences in $\mathrm{Rep}^\circ(G)(S)$ which dualize to

$$\begin{array}{ccccccc} 0 & \rightarrow & \rho_{\mathrm{ad}} & \rightarrow & \rho_2 & \rightarrow & \Gamma^2(\rho_{\mathrm{ad}}) \rightarrow 0 \\ 0 & \rightarrow & \Gamma^2(\rho_{\mathrm{ad}}) & \rightarrow & \rho_{\mathrm{ad}}^{\otimes 2} & \rightarrow & \Lambda^2(\rho_{\mathrm{ad}}) \rightarrow 0 \end{array}$$

where $\Gamma^2(\rho) = \mathrm{Sym}^2(\rho^\vee)^\vee = \ker(\rho^{\otimes 2} \rightarrow \Lambda^2(\rho))$. Therefore

$$[\rho_2] = [\rho_{\mathrm{ad}}] + [\rho_{\mathrm{ad}}]^2 - [\Lambda^2(\rho_{\mathrm{ad}})] \quad \text{in } K_0(G).$$

Since $\mathfrak{g}_{2\alpha} = 0 = \Lambda^2(\mathfrak{g})_{2\alpha}$, the coefficients of $\epsilon^{2\alpha}$ in $\kappa[\rho_2]$ and $\kappa[\rho_{\mathrm{ad}}^{\otimes 2}] = \kappa[\rho_{\mathrm{ad}}]^2$ are both equal to $e^{2f(\alpha)}$. Thus if $\mathfrak{d} = \bigoplus \mathfrak{d}_m$ is the weight decomposition of $\mathfrak{d} = \omega_k^\circ(\rho_2)$, then $\mathfrak{d}_{2\alpha}$ is 1-dimensional and contained in $\mathcal{F}^\gamma(\rho_2)$ if and only if $\gamma \leq 2f(\alpha)$. In particular, $\mathcal{F}_{2\alpha}^{2f(\alpha)}(\rho_2) = \mathfrak{d}_{2\alpha}$, and similarly for $-\alpha$. We thus want:

$$[\mathfrak{d}_{2\alpha}, \mathfrak{g}_{\beta-\alpha}] = \mathfrak{g}_{\beta+\alpha} \quad \text{and} \quad [\mathfrak{d}_{-2\alpha}, \mathfrak{g}_{\beta+\alpha}] = \mathfrak{g}_{\beta-\alpha}.$$

3.10.16. This now only involves the split group G_k and its épinglage, all of which descends to $\mathrm{Spec}(\mathbb{Z})$ by [12, XXIII 5.1 and XXV 1.2]. We may thus assume that G and $\mathcal{E} = (H, M, R, \dots)$ are defined over $S = \mathrm{Spec}(\mathbb{Z})$. The épinglage comes along with simple roots $\Delta \subset R$ and, for each $\alpha \in R$, a basis X_α of \mathfrak{g}_α , which extends to a Chevalley system $\{X_\alpha : \alpha \in R\}$ by [12, XXIII 6.2], giving rise to isomorphisms $u_\alpha(t) = \exp(tX_\alpha)$ from \mathbb{G}_a to the subgroup U_α determined by $\alpha \in R$. As a linear form on $\mathcal{I}(G)/\mathcal{I}(G)^2$, X_α corresponds to the composition of $u_\alpha^1 : \mathcal{I}(G) \rightarrow \mathcal{I}(\mathbb{G}_a)$ with the linear form on $\mathcal{I}(\mathbb{G}_a) = t\mathbb{Z}[t]$ defined by the coefficient of t . If instead we take the coefficient of t^2 , we obtain a linear form on $\mathcal{I}(G)/\mathcal{I}(G)^3$ which is a basis $X_{2\alpha}$ of $\mathfrak{d}_{2\alpha}$. The action of X_α on the regular representation is given by

$$\mathcal{A}(G) \rightarrow \mathcal{A}(G \times \mathbb{G}_a) = \mathcal{A}(G)[t] \rightarrow \mathcal{A}(G)$$

where the first map takes f in $\mathcal{A}(G)$ to the function $(g, t) \mapsto f(u_\alpha(t)gu_\alpha^{-1}(t))$, and the second takes the coefficient of t (or evaluates $\frac{d}{dt}$ at $t = 0$). The action of $X_{2\alpha}$ is obtained by replacing the second map with the coefficient of t^2 , thus $2X_{2\alpha} = X_\alpha^2$ on ρ_{reg} , therefore $2X_{2\alpha} = X_\alpha^2$ on all ρ 's. Let us now return to our chain of roots

$$\{\beta - \alpha, \beta, \beta + \alpha, \beta + 2\alpha\} \subset R.$$

By Chevalley's rule [12, XXIII 6.5]

$$[X_\alpha, X_{\beta-\alpha}] = \pm X_\beta \quad \text{and} \quad [X_\alpha, X_\beta] = \pm 2X_{\beta+\alpha}.$$

Therefore $[X_{2\alpha}, X_{\beta-\alpha}] = \pm X_{\beta+\alpha}$ since (we are now over \mathbb{Z} !)

$$2[X_{2\alpha}, X_{\beta-\alpha}] = [2X_{2\alpha}, X_{\beta-\alpha}] = [X_\alpha, [X_\alpha, X_{\beta-\alpha}]] = \pm 2X_{\beta+\alpha}.$$

Similarly, $[X_{-2\alpha}, X_{\beta+\alpha}] = \pm X_{\beta-\alpha}$, and this completes our proof.

3.11. **Consequences.** Let G be a reductive group over S .

3.11.1. *Proof of Theorem 21.* The assertions concerning automorphisms and Γ -graduations follow from Theorem 29 and Propositions 30, 31 and 34. If G is an isotrivial reductive group over a quasi-compact S , we have monomorphisms

$$\begin{array}{ccccc}
 & & & \mathbb{F}^\Gamma(V^\circ) & \hookrightarrow \\
 & & \text{Prop. 31} & \nearrow & \text{Prop. 36} \\
 \mathbb{F}^\Gamma(G) & \hookrightarrow & \text{Cor. 40} & \mathbb{F}^\Gamma(V) & \hookrightarrow \\
 & & & \searrow & \mathbb{F}^\Gamma(\omega^\circ) \\
 & & & \mathbb{F}^\Gamma(\omega) & \hookrightarrow \\
 & & \text{3.5.3} & \nwarrow & \text{Prop. 30}
 \end{array}$$

and we have just seen that $\mathbb{G}^\Gamma(G) \rightarrow \mathbb{F}^\Gamma(G) \rightarrow \mathbb{F}^\Gamma(\omega^\circ)$ is an epimorphism, therefore

$$\mathbb{F}^\Gamma(G) = \mathbb{F}^\Gamma(V) = \mathbb{F}^\Gamma(V^\circ) = \mathbb{F}^\Gamma(\omega) = \mathbb{F}^\Gamma(\omega^\circ)$$

in this case, from which easily follows that also

$$\mathbb{F}^\Gamma(G) = \mathbb{F}^\Gamma(V) = \mathbb{F}^\Gamma(V^\circ)$$

for any reductive group over any S – and this is contained in $\mathbb{F}^\Gamma(\omega)$ by 3.5.3.

3.11.2. Since the S -scheme $\mathbb{G}^\Gamma(G)$ and $\mathbb{F}^\Gamma(G)$ of section 2 represent the functors indicated in theorem 21, there is a universal Γ -graduation $\mathcal{G}_{\text{univ}}$ on $V_{\mathbb{G}^\Gamma(G)}$ (inducing universal Γ -graduations on $V_{\mathbb{G}^\Gamma(G)}^\circ$, $\omega_{\mathbb{G}^\Gamma(G)}$ and $\omega_{\mathbb{G}^\Gamma(G)}^\circ$) and a universal Γ -filtration $\mathcal{F}_{\text{univ}}$ on $V_{\mathbb{F}^\Gamma(G)}$ (inducing universal Γ -filtrations on $V_{\mathbb{F}^\Gamma(G)}^\circ$, $\omega_{\mathbb{F}^\Gamma(G)}$ and $\omega_{\mathbb{F}^\Gamma(G)}^\circ$) from which all other Γ -graduations or Γ -filtrations over some base T can be retrieved by pull-back through unique morphisms $T \rightarrow \mathbb{G}^\Gamma(G)$ or $T \rightarrow \mathbb{F}^\Gamma(G)$ – for the ω° variants, we have to assume that G is isotrivial and S quasi-compact, or that the Γ -graduations or Γ -filtrations (over T) extend to ω or V° . The S -scheme $\mathbb{C}^\Gamma(G)$ is a coarse moduli scheme for either Γ -graduations or Γ -filtrations (on the various fiber functors): two such objects (over T) are fpqc locally (on T) isomorphic if and only if the induced morphisms $T \rightarrow \mathbb{C}^\Gamma(G)$ are equal.

3.11.3. From this perspective, we may either deduce non-trivial properties of the S -schemes constructed in section 2 from obvious properties of Γ -graduations and Γ -filtrations, or non-trivial properties of the latter from already established properties of the former. We propose the following samples:

Corollary 48. *The sequence $\mathbb{G}^\Gamma(G) \xrightarrow{\text{Fil}} \mathbb{F}^\Gamma(G) \xrightarrow{F} \mathbb{C}^\Gamma(G)$ is functorial on the fibered category of reductive groups over schemes.*

Proof. We have to show that for any morphism $\varphi : G_1 \rightarrow f^*G_2$ over $f : T_1 \rightarrow T_2$ in the latter category, there is a canonical commutative diagram of schemes

$$\begin{array}{ccccccc}
 \mathbb{G}^\Gamma(G_1) & \xrightarrow{\text{Fil}} & \mathbb{F}^\Gamma(G_1) & \xrightarrow{F} & \mathbb{C}^\Gamma(G_1) & \xrightarrow{\text{struct}} & T_1 \\
 \downarrow \varphi & & \downarrow \varphi & & \downarrow \varphi & & \downarrow f \\
 \mathbb{G}^\Gamma(G_2) & \xrightarrow{\text{Fil}} & \mathbb{F}^\Gamma(G_2) & \xrightarrow{F} & \mathbb{C}^\Gamma(G_2) & \xrightarrow{\text{struct}} & T_2
 \end{array}$$

In the Tannakian point of view, the first two vertical morphisms are induced by precomposition with the restriction functor $\text{Rep}(f^*G_2) \rightarrow \text{Rep}(G_1)$ which maps τ to $\tau \circ \varphi$. For the third one: if T is a T_1 -scheme and x is a T -valued point of $\mathbb{C}^\Gamma(G_1)$, it lifts to a Γ -filtration over an fpqc covering $\{T_i \rightarrow T\}$ of T , and two such lifts become isomorphic over a common refinement of the corresponding fpqc

coverings. The image of these lifts in $\mathbb{F}^\Gamma(G_2)$ thus yield a well-defined morphism $\varphi(x) : T \rightarrow \mathbb{C}^\Gamma(G_2)$, and this defines the morphism $\varphi : \mathbb{C}^\Gamma(G_1) \rightarrow \mathbb{C}^\Gamma(G_2)$. \square

Corollary 49. *Suppose that S is affine. Then every Γ -filtration \mathcal{F} on V_S, V_S° or ω_S splits over S , and so do the Γ -filtrations on ω_S° if G is isotrivial.*

Proof. Given Theorem 21, this follows from [12, XXVI 2.2] as in section 2.2.7. \square

3.12. Appendix: pure subsheaves.

Lemma 50. *For $\mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{C}$ in $\text{QCoh}(X)$, consider the following conditions:*

(1) *For every quasi-coherent sheaf \mathcal{F} on X ,*

$$0 \rightarrow \mathcal{A} \otimes \mathcal{F} \rightarrow \mathcal{B} \otimes \mathcal{F} \rightarrow \mathcal{C} \otimes \mathcal{F} \rightarrow 0 \quad \text{is exact in } \text{QCoh}(X).$$

(2) *For every morphism $f : Y \rightarrow X$,*

$$0 \rightarrow f^* \mathcal{A} \rightarrow f^* \mathcal{B} \rightarrow f^* \mathcal{C} \rightarrow 0 \quad \text{is exact in } \text{QCoh}(Y)$$

(3) *For every morphism $f : Y \rightarrow X$ and quasi-coherent sheaf \mathcal{F} on Y ,*

$$0 \rightarrow f^* \mathcal{A} \otimes \mathcal{F} \rightarrow f^* \mathcal{B} \otimes \mathcal{F} \rightarrow f^* \mathcal{C} \otimes \mathcal{F} \rightarrow 0 \quad \text{is exact in } \text{QCoh}(Y)$$

Then (1) \Leftrightarrow (2) \Leftrightarrow (3) and (1) \Leftrightarrow (2) \Leftrightarrow (3) if X is quasi-separated.

Proof. Obviously (3) \Rightarrow (1) and (2). Suppose (2) holds. Let $f : Y \rightarrow X$ be a morphism, \mathcal{F} a quasi-coherent sheaf on Y , $g : Z \rightarrow Y$ the structural morphism of $Z = \text{Spec}(\mathcal{O}_Y[\mathcal{F}])$ where $\mathcal{O}_Y[\mathcal{F}] = \mathcal{O}_Y \oplus \mathcal{F}$ is the quasi-coherent \mathcal{O}_Y -algebra defined by $\mathcal{F} \cdot \mathcal{F} = 0$. By assumption, $0 \rightarrow h^* \mathcal{A} \rightarrow h^* \mathcal{B} \rightarrow h^* \mathcal{C} \rightarrow 0$ is an exact sequence of quasi-coherent sheaves on Z , where $h = f \circ g$. Since g is affine,

$$0 \rightarrow g_* h^* \mathcal{A} \rightarrow g_* h^* \mathcal{B} \rightarrow g_* h^* \mathcal{C} \rightarrow 0$$

is an exact sequence of quasi-coherent sheaves on Y . But

$$g_* h^* \mathcal{X} = g_* g^* f^* \mathcal{X} = f^* \mathcal{X} \oplus f^* \mathcal{X} \otimes \mathcal{F}$$

for any \mathcal{X} in $\text{QCoh}(X)$, therefore

$$0 \rightarrow f^* \mathcal{A} \otimes \mathcal{F} \rightarrow f^* \mathcal{B} \otimes \mathcal{F} \rightarrow f^* \mathcal{C} \otimes \mathcal{F} \rightarrow 0$$

is exact and (2) \Rightarrow (3). Suppose now that X is quasi-separated and (1) holds. Let $f : Y \rightarrow X$ be any morphism. Let $\{X_i\}$ and $\{Y_{i,j}\}$ be open coverings of X and Y by affine schemes such that $f(Y_{i,j}) \subset X_i$ and let $f_{i,j} : Y_{i,j} \rightarrow X_i$ be the induced morphism. Since $(f^* \mathcal{X})|_{Y_{i,j}} = f_{i,j}^*(\mathcal{X}|_{X_i})$ for every $\mathcal{X} \in \text{QCoh}(X)$, we have to show that $0 \rightarrow f_{i,j}^*(\mathcal{A}_i) \rightarrow f_{i,j}^*(\mathcal{B}_i) \rightarrow f_{i,j}^*(\mathcal{C}_i) \rightarrow 0$ is exact on $Y_{i,j}$ for every i, j , with $\mathcal{X}_i = \mathcal{X}|_{X_i}$. Since $Y_{i,j}$ and X_i are affine, this amounts to showing that

$$0 \rightarrow \mathcal{A}_i \otimes \mathcal{O}_{i,j} \rightarrow \mathcal{B}_i \otimes \mathcal{O}_{i,j} \rightarrow \mathcal{C}_i \otimes \mathcal{O}_{i,j} \rightarrow 0$$

is exact on X_i for every i, j , for the quasi-coherent sheaf $\mathcal{O}_{i,j} = (f_{i,j})_* \mathcal{O}_{Y_{i,j}}$ on X_i . Since X is quasi-separated, the immersion $\iota_i : X_i \hookrightarrow X$ is quasi-compact and quasi-separated by [16, 1.2.2.i & 1.2.7.b], thus $\mathcal{F}_{i,j} = (\iota_i)_* \mathcal{O}_{i,j}$ is a quasi-coherent sheaf on X by [16, 1.7.4] and $0 \rightarrow \mathcal{A} \otimes \mathcal{F}_{i,j} \rightarrow \mathcal{B} \otimes \mathcal{F}_{i,j} \rightarrow \mathcal{C} \otimes \mathcal{F}_{i,j} \rightarrow 0$ is an exact sequence on X by assumption. Pulling back through the exact restriction functor $\iota_i^* : \text{QCoh}(X) \rightarrow \text{QCoh}(X_i)$ yields the desired result. \square

Definition 51. We say that the sequence $0 \rightarrow \mathcal{A} \xrightarrow{\iota} \mathcal{B} \rightarrow \mathcal{C} \rightarrow 0$ is pure exact, or that ι is a pure monomorphism, or that $\iota(\mathcal{A})$ is a pure (quasi-coherent) subsheaf of \mathcal{B} if the above condition (2) holds.

Lemma 52. *Let \mathcal{B} be a quasi-coherent sheaf on X . Then*

$$\mathcal{P} : (\text{Sch}/X)^\circ \rightarrow \text{Set} \quad T \mapsto \{\text{pure quasi-coherent subsheaves } \mathcal{A} \text{ of } \mathcal{B}_T\}$$

is an fpqc sheaf on Sch/X .

Proof. It's a functor: if $\mathcal{A} \in \mathcal{P}(T)$ and $\alpha : T' \rightarrow T$ is an X -morphism, the monomorphism $\alpha^*(\mathcal{A} \hookrightarrow \mathcal{B}_T)$ identifies $\alpha^*(\mathcal{A})$ with a quasi-coherent subsheaf of $\alpha^*(\mathcal{B}_T) = \mathcal{B}_{T'}$, which is pure since for any morphism $f' : Y \rightarrow T'$, if $f = \alpha \circ f'$, then $f'^* \circ \alpha^*(\mathcal{A} \hookrightarrow \mathcal{B}_T) = f'^*(\mathcal{A} \hookrightarrow \mathcal{B}_T)$ is a monomorphism of quasi-coherent sheaves on Y since \mathcal{A} is pure in \mathcal{B}_T . It's an fpqc sheaf: if $\{T_i \rightarrow T\}$ is an fpqc cover and $\mathcal{A}_i \in \mathcal{P}(T_i)$ have the same image $\mathcal{A}_{i,j} \in \mathcal{P}(T_i \times_T T_j)$, then the quasi-coherent subsheaves \mathcal{A}_i of \mathcal{B}_{T_i} glue to a quasi-coherent subsheaf \mathcal{A} of \mathcal{B}_T which is pure since for any $f : Y \rightarrow T$, $f^*(\mathcal{A} \hookrightarrow \mathcal{B}_T)$ is a monomorphism of quasi-coherent sheaves on Y as it becomes so in the fpqc cover $\{Y \times_T T_i \rightarrow Y\}$ of Y . \square

Lemma 53. *Let \mathcal{A} be a quasi-coherent subsheaf of \mathcal{B} .*

- (1) *Suppose that locally on X for the fpqc topology, \mathcal{A} is a direct factor of \mathcal{B} . Then \mathcal{A} is a pure subsheaf of \mathcal{B} .*
- (2) *Suppose that \mathcal{A} is a pure subsheaf of \mathcal{B} and $C = \mathcal{B}/\mathcal{A}$ is finitely presented. Then locally on X for the Zariski topology, \mathcal{A} is a direct factor of \mathcal{B} .*

Proof. (1) A direct factor being obviously pure, this follows from the previous lemma. As for (2): the assumptions are local in the Zariski topology by the previous lemma, we may thus assume that $X = \text{Spec}(R)$ for some ring R . Then $A = \Gamma(X, \mathcal{A})$ is a pure R -submodule of $B = \Gamma(X, \mathcal{B})$ in the sense of [23, Appendix to §7] by (2) \Rightarrow (1) of lemma 50, and $C = B/A$ is a finitely presented R -module. Therefore A is a direct factor of B by [23, Theorem 7.14], i.e. \mathcal{A} is a direct factor of \mathcal{B} . \square

4. THE BUILDINGS

4.1. The Tits vectorial building $\mathbf{F}^\Gamma(G)$.

4.1.1. We say that a morphism of posets $f : (\mathbf{X}, \leq) \rightarrow (\mathbf{Y}, \leq)$ is nice if

$$\forall x, y \in \mathbf{X} \times \mathbf{Y} \text{ with } f(x) \leq y, \text{ there is a unique } x' \in f^{-1}(y) \text{ with } x \leq x'.$$

We say that it is very nice if also

$$\forall x, y \in \mathbf{X} \times \mathbf{Y} \text{ with } f(x) \geq y, \text{ there is an } x' \in f^{-1}(y) \text{ with } x \geq x'.$$

4.1.2. Let \mathcal{O} be a local ring, G a reductive group over $\text{Spec}(\mathcal{O})$. We shall here define an $\text{Aut}(G)$ -equivariant sequence of nice surjective morphisms of posets

$$\mathbf{SBP}(G) \xrightarrow{a} \mathbf{SP}(G) \xrightarrow{b} \mathbf{OPP}(G) \xrightarrow{p_1} \mathbf{P}(G) \xrightarrow{t} \mathbf{O}(G)$$

The group $\mathbf{G} = G(\mathcal{O})$ acts on it through $\text{Int} : \mathbf{G} \rightarrow \text{Aut}(G)$, and we will see that

$$\mathbf{G} \backslash \mathbf{SBP}(G) = \mathbf{G} \backslash \mathbf{SP}(G) = \mathbf{G} \backslash \mathbf{OPP}(G) = \mathbf{G} \backslash \mathbf{P}(G) = \mathbf{O}(G).$$

4.1.3. We first define our posets. We will use the following notations:

$$\begin{aligned}
 \mathbf{S}(G) &= \{S : \text{maximal split torus of } G\} \\
 \mathbf{B}(G) &= \{B : \text{minimal parabolic subgroup of } G\} \\
 \mathbf{P}(G) &= \{P : \text{parabolic subgroup of } G\} \\
 \mathbf{SP}(G) &= \{(S, P) : Z_G(S) \subset P\} \\
 \mathbf{SBP}(G) &= \{(S, B, P) : Z_G(S) \subset B \subset P\} \\
 \mathbf{OPP}(G) &= \{(P, P') : \text{opposed parabolic subgroups of } G\}
 \end{aligned}$$

Thus $\mathbf{P}(G) = \mathbb{P}(G)(\mathcal{O})$ and $\mathbf{OPP}(G) = \mathbb{OPP}(G)(\mathcal{O})$. In addition, we set

$$\mathbf{O}(G) = \text{image of } t : \mathbb{P}(G)(\mathcal{O}) \rightarrow \mathbb{O}(G)(\mathcal{O}).$$

We endow $\mathbf{P}(G)$ and $\mathbf{O}(G)$ with their natural partial orders and the remaining three sets $\mathbf{SBP}(G)$, $\mathbf{SP}(G)$ and $\mathbf{OPP}(G)$ with the following ones:

$$\begin{aligned}
 (S_1, B_1, P_1) \leq (S_2, B_2, P_2) &\iff S_1 = S_2, B_1 = B_2 \text{ and } P_1 \subset P_2 \\
 (S_1, P_1) \leq (S_2, P_2) &\iff S_1 = S_2 \text{ and } P_1 \subset P_2 \\
 (P_1, P'_1) \leq (P_2, P'_2) &\iff P_1 \subset P_2 \text{ and } P'_1 \subset P'_2
 \end{aligned}$$

4.1.4. The morphism $t : \mathbf{P}(G) \rightarrow \mathbf{O}(G)$ maps P to its type $t(P)$. It is plainly a morphism of posets. It is surjective by definition of $\mathbf{O}(G)$, nice by [12, XXVI 3.8] and even very nice by [12, XXVI 5.5]. The group G acts trivially on $\mathbf{O}(G)$, and $G \cdot P = t^{-1}t(P)$ by [12, XXVI 5.2], thus $G \backslash \mathbf{P}(G) = \mathbf{O}(G)$.

4.1.5. The morphism $p_1 : \mathbf{OPP}(G) \rightarrow \mathbf{P}(G)$ maps (P, P') to P . It is plainly a morphism of posets, and surjective by [12, XXVI 2.3 & 4.3.2]. Let $(P, P') \in \mathbf{OPP}(G)$, $Q \in \mathbf{P}(G)$ and suppose first that $P \subset Q$. Since t is nice, there is a unique $Q' \in \mathbf{P}(G)$ with $P' \subset Q'$ and $t(Q') = \iota t(Q)$, where ι is the opposition involution of $\mathbf{O}(G)$. We have $(Q, Q') \in \mathbf{OPP}(G)$ by [12, XXVI 4.3.2 & 4.2.1], thus p_1 is nice. If $Q \subset P$, then $Q_L = Q \cap L$ is a parabolic subgroup of $L = P \cap P'$ and its Levi subgroups are the Levi subgroups of Q contained in L by [12, XXVI 1.20]. Since $p_1 : \mathbf{OPP}(L) \rightarrow \mathbf{P}(L)$ is surjective, there is a parabolic subgroup Q'_L of L opposed to Q_L . Then $Q'_L = Q' \cap L$ for a unique parabolic subgroup Q' of G contained in P' , and $(Q, Q') \in \mathbf{OPP}(G)$ since $Q \cap Q' = Q_L \cap Q'_L$ is a Levi subgroup of Q_L and Q'_L , thus also of Q and Q' . Therefore p_1 is very nice. Finally, the stabilizer of P in G is $P = P(\mathcal{O})$ by [12, XXVI 1.2], and $P \cdot (P, P') = p_1^{-1}(P)$ by [12, XXVI 1.8 & 4.3.2], thus $G \backslash \mathbf{OPP}(G) = G \backslash \mathbf{P}(G)$.

4.1.6. The morphism $b : \mathbf{SP}(G) \rightarrow \mathbf{OPP}(G)$ maps (S, P) to $(P, \iota_S P)$, where $\iota_S P$ is defined in the following lemma, which also says that b is a morphism of posets.

Lemma 54. *For $S \in \mathbf{S}(G)$ and $P \in \mathbf{P}(G)$ with $Z_G(S) \subset P$, there exists a unique Levi subgroup L of P and a unique parabolic subgroup $\iota_S P$ of G opposed to P with $Z_G(S) \subset L, \iota_S P$. Moreover $L = P \cap \iota_S P$ and $P \mapsto \iota_S P$ preserves inclusions.*

Proof. By [1, XIV 3.20], there is a maximal torus T in $Z_G(S)$. It is also maximal in G and P . By [12, XXVI 1.6], there is a unique Levi subgroup L of P with $T \subset L$. We have to show that $Z_G(S) \subset L$. By [12, XXVI 6.11], this is equivalent to $R_{sp}(L) \subset S$, where $R_{sp}(L)$ is the split radical of L , i.e. the maximal split subtorus $R(L)_{sp}$ of the radical $R(L)$ of L . Since T is a maximal torus in L , $R(L)$ is contained in T , thus $R_{sp}(L)$ is contained in the maximal split subtorus T_{sp} of T , which obviously

contains S and in fact equals S by maximality of S . This proves the existence and uniqueness of L . That of $\iota_S P$ follows from [12, XXVI 4.3.2] which also shows that $L = P \cap \iota_S P$. If $P \subset Q$, there is a unique $(Q, Q') \in \mathbf{OPP}(G)$ with $\iota_S P \subset Q'$ because p_1 is nice, and obviously $\iota_S Q = Q'$, thus $\iota_S P \subset \iota_S Q$. \square

Starting with $(P, P') \in \mathbf{OPP}(G)$ put $L = P \cap P'$ and let S be a maximal split torus in G containing the split radical $R_{sp}(L)$ of L . Then $Z_G(S) \subset Z_G(R_{sp}(L))$ which equals L by [12, XXVI 6.11], thus $(S, P) \in \mathbf{SP}(G)$ and $b(S, P) = (P, P')$, i.e. b is surjective. It is obviously nice, although not *very* nice. The stabilizer of $b(S, P)$ in \mathbf{G} is $\mathbf{L} = L(\mathcal{O})$ where $L = P \cap \iota_S P$, and $\mathbf{L} \cdot (S, P) = b^{-1}b(S, P)$ by [12, XXVI 6.16], thus $\mathbf{G} \backslash \mathbf{SP}(G) = \mathbf{G} \backslash \mathbf{OPP}(G)$.

4.1.7. The morphism $a : \mathbf{SBP}(G) \rightarrow \mathbf{SP}(G)$ maps (S, B, P) to (S, P) . It is plainly a nice morphism of poset, although not *very* nice. Starting with $(S, P) \in \mathbf{SP}(G)$, let $L = P \cap \iota_S P$. Then [12, XXVI 1.20] sets up a bijection between: the set of minimal parabolic subgroup B of G with $Z_G(S) \subset B \subset P$ (the fiber $a^{-1}(S, P)$) and the set of minimal parabolic subgroups $B_L = B \cap L$ of L with $Z_G(S) \subset B_L$. The latter set is not empty by [12, XXVI 6.16], thus a is surjective. The stabilizer of (S, P) in \mathbf{G} equals $\mathbf{N}_L(S) = N_L(S)(\mathcal{O})$ and $\mathbf{N}_L(S) \cdot (S, B, P) = a^{-1}(S, P)$ by [12, XXVI 7.2] applied to $Z_G(S) \subset L$, thus $\mathbf{G} \backslash \mathbf{SBP}(G) = \mathbf{G} \backslash \mathbf{SP}(G)$. The stabilizer of (S, B, P) in \mathbf{G} is the stabilizer of (S, B) , namely $Z_G(S) = Z_G(S)(\mathcal{O})$ since $Z_G(S) = B \cap \iota_S B$.

4.1.8. By [12, XXVI 5.7], there is a smallest element \circ in $\mathbf{O}(G)$. For

$$\mathbf{X} \text{ in } \{\mathbf{SBP}(G), \mathbf{SP}(G), \mathbf{OPP}(G), \mathbf{P}(G)\},$$

the morphism $f : \mathbf{X} \rightarrow \mathbf{O}(G)$ is very nice. We've proved it already in the last two cases. Since f is nice, our assertion is equivalent to: $\mathbf{X}_{\min} = f^{-1}(\circ)$ where \mathbf{X}_{\min} is the set of minimal elements in \mathbf{X} . This is obvious for $\mathbf{SBP}(G)$, and also for $\mathbf{SP}(G)$ since a is surjective. For any $x \in \mathbf{X}_{\min} = f^{-1}(\circ)$, there is then a unique section

$$(\mathbf{X}, \leq) \xrightleftharpoons[f]{s_x} (\mathbf{O}(G), \leq)$$

with $s_x(\circ) = x$, and these sections cover \mathbf{X} .

4.1.9. Let now $\Gamma = (\Gamma, +, \leq)$ be a non-trivial totally ordered commutative group and form the $\text{Aut}(G)$ -equivariant cartesian diagram of sets:

$$\begin{array}{ccccccccc} \mathbf{ACF}^\Gamma(G) & \xrightarrow{a} & \mathbf{AF}^\Gamma(G) & \xrightarrow{b} & \mathbf{G}^\Gamma(G) & \xrightarrow{\text{Fil}} & \mathbf{F}^\Gamma(G) & \xrightarrow{t} & \mathbf{C}^\Gamma(G) \\ \downarrow F & & \downarrow F & & \downarrow F & & \downarrow F & & \downarrow F \\ \mathbf{SBP}(G) & \xrightarrow{a} & \mathbf{SP}(G) & \xrightarrow{b} & \mathbf{OPP}(G) & \xrightarrow{p_1} & \mathbf{P}(G) & \xrightarrow{t} & \mathbf{O}(G) \end{array}$$

where $\mathbf{C}^\Gamma(G)$ is the inverse image of $\mathbf{O}(G)$ under $F : \mathbb{C}^\Gamma(G)(\mathcal{O}) \rightarrow \mathbb{O}(G)(\mathcal{O})$. Of course we may and do identify $\mathbf{F}^\Gamma(G)$ with $\mathbb{F}^\Gamma(G)(\mathcal{O})$ and $\mathbf{G}^\Gamma(G)$ with $\mathbb{G}^\Gamma(G)(\mathcal{O})$,

see section 2.2.6. With these identifications, we find:

$$\begin{aligned}
 \mathbf{AF}^\Gamma(G) &= \{(S, \mathcal{F}) \in \mathbf{S}(G) \times \mathbf{F}^\Gamma(G) \text{ with } Z_G(S) \subset P_{\mathcal{F}}\} \\
 &= \{(S, \mathcal{G}) \in \mathbf{S}(G) \times \mathbf{G}^\Gamma(G) \text{ with } Z_G(S) \subset L_{\mathcal{G}}\} \\
 &= \{(S, \mathcal{G}) : S \in \mathbf{S}(G), \mathcal{G} \in \mathbf{G}^\Gamma(S)\} \\
 \mathbf{ACF}^\Gamma(G) &= \{(S, B, \mathcal{F}) : \mathbf{S}(G) \times \mathbf{B}(G) \times \mathbf{F}^\Gamma(G) \text{ with } Z_G(S) \subset B \subset P_{\mathcal{F}}\} \\
 &\quad \{(S, B, \mathcal{G}) : S \in \mathbf{S}(G), \mathcal{G} \in \mathbf{G}^\Gamma(S) \text{ with } Z_G(S) \subset B \subset P_{\mathcal{G}}\}.
 \end{aligned}$$

4.1.10. Fix $S \in \mathbf{S}(G)$. Let M be its group of characters, $R \subset M$ the roots of S in $\mathfrak{g} = \text{Lie}(G)$ and $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$ the corresponding decomposition of \mathfrak{g} . Put

$$W = (N_G(S)/Z_G(S))(\mathcal{O}) = N_G(S)(\mathcal{O})/Z_G(S)(\mathcal{O}).$$

By [12, XXVI 7.4], there exists a unique root datum $\mathcal{R} = (M, R, M^*, R^*)$ with Weyl group W and a W -equivariant bijection $B \leftrightarrow R_+$ between the set of all $B \in \mathbf{B}(G)$ with $Z_G(S) \subset B$ and the set of all systems of positive roots $R_+ \subset R$, given by

$$\text{Lie}(B) = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in R_+} \mathfrak{g}_\alpha.$$

Fix one such B and let $\Delta \subset R_+$ be the corresponding set of simple roots. By [12, XXVI 7.7], there is an inclusion preserving bijection $P \leftrightarrow A$ between the set of all $P \in \mathbf{P}(G)$ with $B \subset P$ and the set of all subsets A of Δ , given by

$$\text{Lie}(P) = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in R_A} \mathfrak{g}_\alpha$$

where $R_A = R_+ \amalg (\mathbb{Z}A \cap R_-)$ is the set of roots in $R = R_+ \amalg R_-$ which are either positive or in the group spanned by A . We write P_A for the parabolic associated to A . Since $f : \mathbf{SBP}(G) \rightarrow \mathbf{O}(G)$ is (very) nice, we obtain a poset bijection

$$f_{S,B} : (\{A \subset \Delta\}, \subset) \rightarrow (\mathbf{O}(G), \leq), \quad A \mapsto t(P_A).$$

Fix one such $P = P_A$. Then the fiber of $F : \mathbf{ACF}^\Gamma(G) \rightarrow \mathbf{SBP}(G)$ above (S, B, P) is the set of all (S, B, \mathcal{G}) with $\mathcal{G} \in \mathbf{G}^\Gamma(S) = \text{Hom}(M, \Gamma)$ such that

$$\forall \alpha \in \Delta : \begin{cases} \mathcal{G}(\alpha) = 0 & \text{if } \alpha \in A, \\ \mathcal{G}(\alpha) > 0 & \text{if } \alpha \notin A. \end{cases}$$

Since the elements of Δ are linearly independent and Γ is non-trivial, this fiber is not empty and $F : \mathbf{ACF}^\Gamma(G) \rightarrow \mathbf{SBP}(G)$ is surjective.

4.1.11. It follows that the five F 's in our diagram are surjective. Their fibers are called facets, the type of a facet is its image in $\mathbf{O}(G)$, and all facets of the same type are canonically isomorphic. The facets of type \circ are called chambers. For any $f' : \mathbf{X}' \rightarrow \mathbf{C}^\Gamma(G)$ over $f : \mathbf{X} \rightarrow \mathbf{O}(G)$ in our diagram, the closed facet of $x \in \mathbf{X}$ is $F^{-1}(\bar{x}) \subset \mathbf{X}'$ where $\bar{x} = \{y \geq x\}$. It is a disjoint union of finitely many facets. Since $x = \min FF^{-1}(x)$, closed facets have a well-defined type and those of the same type are canonically isomorphic. We equip the set of closed facets with the partial order given by inclusion, which is opposite to the partial order on \mathbf{X} . A closed chamber is a maximal closed facet, and the set of all closed chambers equals

$\mathbf{X}_{\min} = f^{-1}(\circ)$. Since f is nice, every $x \in \mathbf{X}_{\min}$ defines compatible sections

$$\begin{array}{ccc} \mathbf{X}' & \begin{array}{c} \xleftarrow{s_x} \\ \xrightarrow{f'} \end{array} & \mathbf{C}^\Gamma(G) \\ F \downarrow & & \downarrow F \\ \mathbf{X} & \begin{array}{c} \xleftarrow{s_x} \\ \xrightarrow{f} \end{array} & \mathbf{O}(G) \end{array}$$

and the closed chamber $F^{-1}(\bar{x})$ is the image of $s_x : \mathbf{C}^\Gamma(G) \rightarrow \mathbf{X}'$. Since f is very nice, any $x' \in \mathbf{X}'$ belongs to some closed chamber. Since $\mathbf{G} \backslash \mathbf{X}' = \mathbf{C}^\Gamma(G)$, any closed chamber is a fundamental domain for the action of \mathbf{G} on \mathbf{X}' .

4.1.12. The facets which are minimal among the set of non-minimal facets are called panels. A panel $F^{-1}(x)$ bounds a chamber $F^{-1}(y)$ if $F^{-1}(x) \subset F^{-1}(\bar{y})$, i.e. $y \leq x$. Any panel bounds at least 3 chambers. Indeed, this means that a non-minimal parabolic subgroup P of G contains at least 3 minimal parabolic subgroups. To establish this, fix a Levi subgroup L of P – which exists by [12, XXVI 2.3] or the surjectivity of p_1 . Then $Q \mapsto L \cap Q$ yields a bijection between the parabolic subgroups Q of G contained in P and the parabolic subgroups of L , by [12, XXVI 1.20]. Since P is non-minimal, L is not a minimal parabolic subgroup of itself. By [12, XXVI 5.11], it contains at least 3 such subgroups, and so does P .

4.1.13. The apartment attached to $S \in \mathbf{S}(G)$ is the subset $\mathbf{F}^\Gamma(S)$ of all \mathcal{F} 's in $\mathbf{F}^\Gamma(G)$ such that $Z_G(S) \subset P_{\mathcal{F}}$. It is canonically isomorphic to $\mathbf{G}^\Gamma(S)$ by the map which sends $\mathcal{G} : \mathbb{D}_{\mathcal{O}}(\Gamma) \rightarrow S$ to $\text{Fil}(\mathcal{G})$. Our notations are thus consistent since

$$\mathbf{F}^\Gamma(S) = \mathbf{G}^\Gamma(S) = \mathbb{G}^\Gamma(S)(\mathcal{O}) = \mathbb{F}^\Gamma(S)(\mathcal{O}).$$

Since $F : \mathbf{A}\mathbf{F}^\Gamma(G) \rightarrow \mathbf{S}\mathbf{P}$ is surjective, $\mathbf{F}^\Gamma(S)$ is the disjoint union of the facets $F^{-1}(P)$ with $Z_G(S) \subset P$. Since $Z_G(S) = B \cap B'$ for some pair of opposed minimal parabolic subgroups of G , $\mathbf{F}^\Gamma(S)$ determines $Z_G(S) = \bigcap_{F^{-1}(P) \subset \mathbf{F}^\Gamma(S)} P$ and its split radical S . Thus $S \mapsto \mathbf{F}^\Gamma(S)$ is an $\text{Aut}(G)$ -equivariant bijection from $\mathbf{S}(G)$ onto the set $\mathbf{A}(G)$ of apartments in $\mathbf{F}^\Gamma(G)$. In particular,

$$\begin{aligned} \mathbf{A}\mathbf{F}^\Gamma(G) &= \{(A, \mathcal{F}) : A \in \mathbf{A}(G), \mathcal{F} \in A\} \\ \mathbf{A}\mathbf{C}\mathbf{F}^\Gamma(G) &= \{(A, C, \mathcal{F}) : \mathcal{F} \in C = \text{closed chamber of } A \in \mathbf{A}(G)\}. \end{aligned}$$

Since $\mathbf{A}\mathbf{F}^\Gamma(G) \rightarrow \mathbf{F}^\Gamma(G)$ is surjective, every $\mathcal{F} \in \mathbf{F}^\Gamma(G)$ belongs to some $A \in \mathbf{A}(G)$. The stabilizer of $\mathbf{F}^\Gamma(S)$ in $\mathbf{G} = G(\mathcal{O})$ equals $N_G(S) = N_G(S)(\mathcal{O})$ and its pointwise stabilizer equals $Z_G(S) = Z_G(S)(\mathcal{O})$. Thus $W_G(S) = N_G(S)/Z_G(S)$ acts on $\mathbf{F}^\Gamma(S)$, and this gives the usual action of $W_G(S)$ on $\mathbf{F}^\Gamma(S) = \mathbf{G}^\Gamma(S) = \text{Hom}(\mathbb{D}_{\mathcal{O}}(\Gamma), S)$.

4.1.14. A panel bounds exactly two chambers in any apartment which contains it. Indeed, let $F^{-1}(Q)$ be a panel in $\mathbf{F}^\Gamma(S)$. Given [12, XXVI 1.20], we have to show that there are exactly two minimal parabolic subgroups of $L = Q \cap \iota_S Q$ containing $Z_G(S)$. By assumption, $\mathbf{O}(L) = \{\circ, t(L)\}$. Our claim then follows from 4.1.10.

4.1.15. Suppose now that \mathcal{O} is a Henselian local ring with residue field k .

Proposition 55. *The specialization from \mathcal{O} to k induces a map from the diagram of section 4.1.9 for G to the similar diagram for G_k . In the resulting commutative diagram, all the specialization maps $\mathbf{X}(G) \rightarrow \mathbf{X}(G_k)$ are surjective, all the squares involving two F 's are cartesian, and $\mathbf{O}(G) \simeq \mathbf{O}(G_k)$, $\mathbf{C}^\Gamma(G) \simeq \mathbf{C}^\Gamma(G_k)$.*

Proof. Since \mathbb{G}^Γ , \mathbb{F}^Γ , \mathbb{C}^Γ , \mathbb{OPP} , \mathbb{P} and \mathbb{O} are smooth over $\text{Spec}(\mathcal{O})$, the specialization from \mathcal{O} to k induces a map from the last two squares of our diagram for G to the last two squares of the analogous diagram for G_k , in which all specialization maps $\mathbf{X}(G) \rightarrow \mathbf{X}(G_k)$ are surjective by [19, 18.5.17]. Since \mathbb{O} is finite étale over $\text{Spec}(\mathcal{O})$, $\mathbf{O}(G) \rightarrow \mathbf{O}(G_k)$ is also injective by [19, 18.5.4-5], i.e. $\mathbf{O}(G) = \mathbf{O}(G_k)$. It follows that $\mathbf{P}(G) \rightarrow \mathbf{P}(G_k)$ induces $\mathbf{B}(G) \rightarrow \mathbf{B}(G_k)$. If S is a maximal split torus in G , then $Z_G(S)$ is a Levi subgroup of a minimal parabolic subgroup B of G , S is the maximal split subtorus of the radical R of $Z_G(S)$, thus R/S is an anisotropic torus, i.e. $\text{Hom}(\mathbb{G}_{m,\mathcal{O}}, R/S) = 0$. Then by Proposition 3, Lemma 4 and [19, 18.5.4-5], also $\text{Hom}(\mathbb{G}_{m,k}, R_k/S_k) = 0$, thus S_k is the maximal split subtorus of the radical R_k of the Levi subgroup $Z_G(S)_k = Z_{G_k}(S_k)$ of the minimal parabolic subgroup B_k of G_k , in particular S_k is a maximal split subtorus of G_k and the specialization map $\mathbf{S}(G) \rightarrow \mathbf{S}(G_k)$ is well-defined. It is surjective: starting with \bar{S} in $\mathbf{S}(G_k)$, choose $\bar{B} \in \mathbf{B}(G_k)$ containing $Z_{G_k}(\bar{S})$, lift \bar{B} to some $B \in \mathbf{B}(G)$, choose $S' \in \mathbf{S}(G)$ with $Z_G(S') \subset B$, write $\bar{S} = \text{Int}(\bar{b})(S'_k)$ for some $b \in B(k)$, lift \bar{b} to some $b \in B(\mathcal{O})$ using [19, 18.5.17] and set $S = \text{Int}(b)(S')$. Then $S \in \mathbf{S}(G)$ and $S_k = \bar{S}$. The same argument shows that $\mathbf{SBP}(G) \rightarrow \mathbf{SBP}(G_k)$ and $\mathbf{SP}(G) \rightarrow \mathbf{SP}(G_k)$ are well-defined and surjective, from which follows that also $\mathbf{ACF}^\Gamma(G) \rightarrow \mathbf{ACF}^\Gamma(G_k)$ and $\mathbf{AF}^\Gamma(G) \rightarrow \mathbf{AF}^\Gamma(G_k)$ are well-defined. To establish all of the remaining claims, it is sufficient to show that $\mathbf{C}^\Gamma(G) \rightarrow \mathbf{C}^\Gamma(G_k)$ is also injective. Fix (S, B) as above and let $s : \mathbf{C}^\Gamma(G) \hookrightarrow \mathbf{F}^\Gamma(G)$ and $s_k : \mathbf{C}^\Gamma(G_k) \hookrightarrow \mathbf{F}^\Gamma(G_k)$ be the corresponding sections. They are compatible with the specialization maps and their images are respectively contained in the apartments $\mathbf{F}^\Gamma(S)$ of $\mathbf{F}^\Gamma(G)$ and $\mathbf{F}^\Gamma(S_k)$ of $\mathbf{F}^\Gamma(G_k)$. Since $\mathbf{G}^\Gamma(S) \simeq \mathbf{G}^\Gamma(S_k)$, the specialization map $\mathbf{F}^\Gamma(G) \rightarrow \mathbf{F}^\Gamma(G_k)$ restricts to a bijection $\mathbf{F}^\Gamma(S) \simeq \mathbf{F}^\Gamma(S_k)$, therefore $\mathbf{C}^\Gamma(G) \rightarrow \mathbf{C}^\Gamma(G_k)$ is also injective. \square

4.1.16. Suppose now that \mathcal{O} is a valuation ring with fraction field K .

Proposition 56. *The generization from \mathcal{O} to K induces a map from the diagram of section 4.1.9 for G to the similar diagram for G_K . In the resulting commutative diagram, all the generization maps $\mathbf{X}(G) \rightarrow \mathbf{X}(G_K)$ are injective, they are bijective for $\mathbf{X} \in \{\mathbf{F}^\Gamma, \mathbf{C}^\Gamma, \mathbf{P}, \mathbf{O}\}$ and all the squares involving two F 's are cartesian.*

Proof. Since \mathbb{G}^Γ , \mathbb{F}^Γ , \mathbb{C}^Γ , \mathbb{OPP} , \mathbb{P} and \mathbb{O} are separated over $\text{Spec}(\mathcal{O})$, the generization from \mathcal{O} to K induces a map from the last two squares of our diagram for G to the last two squares of the analogous diagram for G_K , in which all generization maps $\mathbf{X}(G) \rightarrow \mathbf{X}(G_K)$ are injective. Since \mathbb{O} and \mathbb{P} are proper over $\text{Spec}(\mathcal{O})$, the maps $\mathbf{P}(G) \rightarrow \mathbf{P}(G_K)$ and $\mathbf{O}(G) \rightarrow \mathbf{O}(G_K)$ are in fact bijective. It follows that $\mathbf{P}(G) \simeq \mathbf{P}(G_K)$ induces $\mathbf{B}(G) \simeq \mathbf{B}(G_K)$. If S is a maximal split torus in G , then $Z_G(S)$ is a Levi subgroup of a minimal parabolic subgroup B of G , S is the maximal split subtorus of the radical R of $Z_G(S)$, thus R/S is an anisotropic torus, i.e. $\text{Hom}(\mathbb{G}_{m,\mathcal{O}}, R/S) = 0$. Then by Proposition 3 and Lemma 4, also $\text{Hom}(\mathbb{G}_{m,K}, R_K/S_K) = 0$, thus S_K is the maximal split subtorus of the radical R_K of the Levi subgroup $Z_G(S)_K = Z_{G_K}(S_K)$ of the minimal parabolic subgroup B_K of G_K , in particular S_K is a maximal split subtorus of G_K and the specialization map $\mathbf{S}(G) \rightarrow \mathbf{S}(G_K)$ is well-defined. It is injective by [12, XXII 5.8.3], so are $\mathbf{SBP}(G) \rightarrow \mathbf{SBP}(G_K)$ and $\mathbf{SP}(G) \rightarrow \mathbf{SP}(G_K)$, while $\mathbf{ACF}^\Gamma(G) \rightarrow \mathbf{ACF}^\Gamma(G_K)$ and $\mathbf{AF}^\Gamma(G) \rightarrow \mathbf{AF}^\Gamma(G_K)$ are well-defined. To establish the remaining claims, it is sufficient to show that $\mathbf{C}^\Gamma(G) \hookrightarrow \mathbf{C}^\Gamma(G_K)$ is also surjective. Fix (S, B) as above and let $s : \mathbf{C}^\Gamma(G) \hookrightarrow \mathbf{F}^\Gamma(G)$ and $s_K : \mathbf{C}^\Gamma(G_K) \hookrightarrow \mathbf{F}^\Gamma(G_K)$ be the corresponding

sections. They are compatible with the generalization maps and there images are respectively contained in the apartments $\mathbf{F}^\Gamma(S)$ of $\mathbf{F}^\Gamma(G)$ and $\mathbf{F}^\Gamma(S_K)$ of $\mathbf{F}^\Gamma(G_K)$. Since $\mathbf{G}^\Gamma(S) \simeq \mathbf{G}^\Gamma(S_K)$, the generalization map $\mathbf{F}^\Gamma(G) \rightarrow \mathbf{F}^\Gamma(G_K)$ restricts to a bijection $\mathbf{F}^\Gamma(S) \simeq \mathbf{F}^\Gamma(S_K)$, therefore $\mathbf{C}^\Gamma(G) \hookrightarrow \mathbf{C}^\Gamma(G_K)$ is also surjective. \square

Remark 57. Under the identifications of Theorem 21 (note that G is isotrivial over the valuation ring \mathcal{O} by Proposition 33), the inverse of the map $\mathbf{F}(G) \rightarrow \mathbf{F}(G_K)$ maps a Γ -filtration \mathcal{F}_K on ω_K° to the Γ -filtration \mathcal{F} on ω° defined by

$$\forall \tau \in \text{Rep}^\circ(G)(\mathcal{O}), \gamma \in \Gamma : \quad \mathcal{F}^\gamma(\tau) = \mathcal{F}_K^\gamma(\tau) \cap V(\tau) \text{ in } V_K(\tau).$$

It is not at all obvious that this formula indeed defines a right exact functor!

4.2. Distances and angles. Suppose for this section that Γ is a subring of \mathbb{R} with the induced total order on the underlying commutative group.

4.2.1. For any $\mathcal{F}_1, \mathcal{F}_2 \in \mathbf{F}^\Gamma(G)$, there is an apartment $A \in \mathbf{A}(G)$ containing \mathcal{F}_1 and \mathcal{F}_2 if and only if $P_{\mathcal{F}_1}$ and $P_{\mathcal{F}_2}$ are in standard position [12, XXVI 4.5]. Indeed if $\mathcal{F}_1, \mathcal{F}_2 \in \mathbf{F}^\Gamma(S)$ for some $S \in \mathbf{A}(G)$, then $Z_G(S)$ contains a maximal torus T by [1, XIV 3.20], thus $T \subset Z_G(S) \subset P_{\mathcal{F}_1} \cap P_{\mathcal{F}_2}$. If conversely $T \subset P_{\mathcal{F}_1} \cap P_{\mathcal{F}_2}$ for some maximal torus T of G , then $\mathcal{F}_1, \mathcal{F}_2 \in \mathbf{F}^\Gamma(S)$ for any $S \in \mathbf{S}(G)$ containing the maximal split torus T_{sp} of T : if R_i is the split radical of the unique Levi subgroup L_i of $P_{\mathcal{F}_i}$ containing T [12, XXVI 1.6], then $R_i \subset T_{sp} \subset S$, therefore $Z_G(S) \subset Z_G(R_i) = L_i \subset P_{\mathcal{F}_i}$ by [12, XXVI 6.11]. We denote by $\mathbf{Std}(G)$ and $\mathbf{Std}^\Gamma(G) = F^{-1}(\mathbf{Std}(G))$ the corresponding subsets of $\mathbf{P}(G)^2$ and $\mathbf{F}^\Gamma(G)^2$.

4.2.2. Recall from Theorem 21 that for $\tau \in \text{Rep}^\circ(G)(\mathcal{O})$, any $\mathcal{F} \in \mathbf{F}^\Gamma(G)$ defines a Γ -filtration $\mathcal{F}(\tau)$ on the (free) \mathcal{O} -module $V(\tau)$. For any $(\mathcal{F}_1, \mathcal{F}_2) \in \mathbf{Std}^\Gamma(G)$ and $\gamma_1, \gamma_2 \in \Gamma$, the \mathcal{O} -module

$$\text{Gr}_{\mathcal{F}_1, \mathcal{F}_2}^{\gamma_1, \gamma_2}(\tau) = \frac{\mathcal{F}_1(\tau)^{\gamma_1} \cap \mathcal{F}_2(\tau)^{\gamma_2}}{\mathcal{F}_1(\tau)_+^{\gamma_1} \cap \mathcal{F}_2(\tau)^{\gamma_2} + \mathcal{F}_1(\tau)^{\gamma_1} \cap \mathcal{F}_2(\tau)_+^{\gamma_2}}$$

is free of finite rank: if $\mathcal{F}_i = \text{Fil}(\mathcal{G}_i)$ with $\mathcal{G}_i \in \mathbf{G}^\Gamma(S) = \text{Hom}(M, \Gamma)$ for some S in $\mathbf{S}(G)$ with $M = \text{Hom}(S, \mathbb{G}_{m, \mathcal{O}})$, then $\mathcal{F}_i(\tau)^\gamma = \oplus_{\mathcal{G}_i(m) \geq \gamma} V(\tau)_m$ for any $i \in \{1, 2\}$ and $\gamma \in \Gamma$ where $V(\tau) = \oplus_{m \in M} V(\tau)_m$ is the eigenspace decomposition of $\tau|_S$, thus

$$\text{Gr}_{\mathcal{F}_1, \mathcal{F}_2}^{\gamma_1, \gamma_2}(\tau) = \oplus_{m: \mathcal{G}_i(m) = \gamma_i} V(\tau)_m.$$

4.2.3. Since Γ is a subring of \mathbb{R} :

- Any apartment is endowed with a canonical structure of free Γ -module, and these structures are preserved by the action of $\text{Aut}(G)$ on $\mathbf{F}^\Gamma(G)$. Indeed,

$$\mathbf{F}^\Gamma(S) = \mathbf{G}^\Gamma(S) = \text{Hom}(M(S), \Gamma) \quad \text{with} \quad M(S) = \text{Hom}(S, \mathbb{G}_{m, \mathcal{O}}).$$

- Any $\tau \in \text{Rep}^\circ(G)(\mathcal{O})$ defines a \mathbf{G} -invariant function

$$\langle -, - \rangle_\tau : \mathbf{Std}^\Gamma(G) \rightarrow \Gamma \quad \langle \mathcal{F}_1, \mathcal{F}_2 \rangle_\tau = \sum_{\gamma_1, \gamma_2} \text{rank}_{\mathcal{O}}(\text{Gr}_{\mathcal{F}_1, \mathcal{F}_2}^{\gamma_1, \gamma_2}(\tau)) \cdot \gamma_1 \gamma_2$$

whose restriction to $\mathbf{F}^\Gamma(S)$ is bilinear, symmetric and non-negative, given by

$$\langle \mathcal{F}_1, \mathcal{F}_2 \rangle_\tau = \sum_{m \in M(S)} \text{rank}_{\mathcal{O}}(V(\tau)_m) \cdot \mathcal{G}_1(m) \mathcal{G}_2(m)$$

if $\mathcal{F}_i \in \mathbf{F}^\Gamma(S)$ corresponds to $\mathcal{G}_i \in \text{Hom}(M(S), \Gamma)$. Its kernel equals $\mathbf{G}^\Gamma(\ker(\tau|_S))$, thus $\langle -, - \rangle_\tau$ is positive definite when τ is a faithful representation of G .

• Write $\|\mathcal{F}\|_\tau = \langle \mathcal{F}, \mathcal{F} \rangle_\tau^{1/2}$. Thus $\|-\|_\tau : \mathbf{F}^\Gamma(G) \rightarrow \Gamma^{1/2}$ is a \mathbf{G} -invariant function and it descends to a \mathbf{G} -invariant function $\ell_\tau : \mathbf{C}^\Gamma(G) \rightarrow \Gamma^{1/2}$ with $\|\mathcal{F}\|_\tau = \ell_\tau(t(\mathcal{F}))$. We have the Cauchy-Schwartz inequality

$$\forall (\mathcal{F}_1, \mathcal{F}_2) \in \mathbf{Std}^\Gamma(G) : \quad |\langle \mathcal{F}_1, \mathcal{F}_2 \rangle_\tau| \leq \|\mathcal{F}_1\|_\tau \|\mathcal{F}_2\|_\tau.$$

• We may thus also define a \mathbf{G} -invariant angle

$$\angle_\tau(-, -) : \mathbf{Std}^\Gamma(G) \rightarrow [0, \pi] \quad \langle \mathcal{F}_1, \mathcal{F}_2 \rangle_\tau = \cos(\angle_\tau(\mathcal{F}_1, \mathcal{F}_2)) \cdot \|\mathcal{F}_1\|_\tau \|\mathcal{F}_2\|_\tau$$

and a \mathbf{G} -invariant function

$$d_\tau(-, -) : \mathbf{Std}^\Gamma(G) \rightarrow \mathbb{R}_+ \quad d_\tau(\mathcal{F}_1, \mathcal{F}_2) = \sqrt{\|\mathcal{F}_1\|_\tau^2 + \|\mathcal{F}_2\|_\tau^2 - 2\langle \mathcal{F}_1, \mathcal{F}_2 \rangle_\tau}$$

inducing the distance $d_\tau(\mathcal{F}_1, \mathcal{F}_2) = \|\mathcal{F}_2 - \mathcal{F}_1\|_\tau$ on any apartment.

• If τ is faithful, then $d_\tau(\mathcal{F}_1, \mathcal{F}_2) = 0$ if and only if $\mathcal{F}_1 = \mathcal{F}_2$ and the map $\mathcal{G} \mapsto (\text{Fil}(\mathcal{G}), \text{Fil}(\mathcal{U}\mathcal{G}))$ induces a \mathbf{G} -equivariant bijection

$$\mathbf{G}^\Gamma(G) \simeq \left\{ (\mathcal{F}_1, \mathcal{F}_2) \in \mathbf{Std}^\Gamma(G) : \|\mathcal{F}_1\| = \|\mathcal{F}_2\| \text{ and } \angle_\tau(\mathcal{F}_1, \mathcal{F}_2) = \pi \right\}.$$

From now on, we fix one such faithful τ .

4.2.4. For $x, y \in \mathbf{O}(G)$, there is a single \mathbf{G} -orbit of (P, Q) 's in $t^{-1}(x) \times t^{-1}(y)$ with the property that $P \cap Q$ is a parabolic subgroup of G [12, XXVI 5.4-5], and this orbit is contained in $\mathbf{Std}(G)$. Thus for any $x, y \in \mathbf{C}^\Gamma(G)$, there is a single \mathbf{G} -orbit of $(\mathcal{F}_1, \mathcal{F}_2)$'s in $t^{-1}(x) \times t^{-1}(y)$ with the property that $P_{\mathcal{F}_1} \cap P_{\mathcal{F}_2}$ is a parabolic subgroup of G , and it is contained in $\mathbf{Std}^\Gamma(G)$. We set $\angle_\tau(x, y) = \angle_\tau(\mathcal{F}_1, \mathcal{F}_2)$ and $\langle x, y \rangle_\tau = \langle \mathcal{F}_1, \mathcal{F}_2 \rangle_\tau$, thus obtaining another pair of symmetric functions

$$\angle_\tau(-, -) : \mathbf{C}^\Gamma(G) \times \mathbf{C}^\Gamma(G) \rightarrow [0, \pi] \quad \text{and} \quad \langle -, - \rangle_\tau : \mathbf{C}^\Gamma(G) \times \mathbf{C}^\Gamma(G) \rightarrow \mathbb{R}.$$

Fix $(S, B) \in \mathbf{S}(G) \times \mathbf{B}(G)$ with $Z_G(S) \subset B$ and let $s : \mathbf{C}^\Gamma(G) \hookrightarrow \mathbf{ACF}^\Gamma(G)$ be the corresponding section of $\mathbf{ACF}^\Gamma(G) \twoheadrightarrow \mathbf{C}^\Gamma(G)$. Since $P_{s(x)} \cap P_{s(y)}$ contains B , it is a parabolic subgroup of G . Thus for every $x, y \in \mathbf{C}^\Gamma(G)$,

$$\angle_\tau(x, y) = \angle_\tau(s(x), s(y)) \quad \text{and} \quad \langle x, y \rangle_\tau = \langle s(x), s(y) \rangle_\tau.$$

In particular, the ‘‘scalar product’’ is compatible with the monoid structure:

$$\langle x_1 + x_2, y \rangle_\tau = \langle x_1, y \rangle_\tau + \langle x_2, y \rangle_\tau \quad \text{and} \quad \langle x, y_1 + y_2 \rangle_\tau = \langle x, y_1 \rangle_\tau + \langle x, y_2 \rangle_\tau.$$

The following lemma is related to the angle rigidity axiom of [20, 4.1.2].

Lemma 58. *For any $x, y \in \mathbf{C}^\Gamma(G)$, the set*

$$D_\tau(x, y) = \left\{ \angle_\tau(\mathcal{F}_1, \mathcal{F}_2) : (\mathcal{F}_1, \mathcal{F}_2) \in \mathbf{Std}^\Gamma(G) \cap t^{-1}(x) \times t^{-1}(y) \right\}$$

is finite and $\angle_\tau(x, y) = \min D_\tau(x, y)$.

Proof. Fix (S, B) and $s : \mathbf{C}^\Gamma(G) \hookrightarrow \mathbf{ACF}^\Gamma(G)$ as above. Then any pair

$$(\mathcal{F}_1, \mathcal{F}_2) \in \mathbf{Std}^\Gamma(G) \cap t^{-1}(x) \times t^{-1}(y)$$

is \mathbf{G} -conjugated to some pair in $W_G(S) \cdot s(x) \times W_G(S) \cdot s(y) \subset \mathbf{F}^\Gamma(S)^2$, thus

$$\begin{aligned} D_\tau(x, y) &= \left\{ \angle_\tau(w_1 \cdot s(x), w_2 \cdot s(y)) : (w_1, w_2) \in W_G(S)^2 \right\} \\ &= \left\{ \angle_\tau(s(x), w \cdot s(y)) : w \in W_G(S) \right\} \end{aligned}$$

is finite. To establish our final claim, we have to show that

$$\langle s(x), s(y) \rangle_\tau \geq \langle s(x), w \cdot s(y) \rangle_\tau$$

for every $w \in W_G(S)$, which follows from [6, Proposition 18]. \square

4.2.5. Let us use the above notions to show that

Proposition 59. *For a facet F , a chamber C and apartments A_1, A_2 in $\mathbf{F}^\Gamma(G)$ with $F \cup C \subset A_1 \cap A_2$, there exists $g \in G$ with $gA_1 = A_2$ and $g \equiv 1$ on $\overline{F} \cup \overline{C}$.*

Proof. In group theoretical terms, this means that for $P \in \mathbf{P}(G)$, $B \in \mathbf{B}(G)$ and $S_1, S_2 \in \mathbf{S}(G)$ with $Z_G(S_i) \subset B \cap P$, there is a $g \in G$ such that $\text{Int}(g)(S_1) = S_2$ and $g \in B \cap P$ with $\mathbf{B} = B(\mathcal{O})$, $\mathbf{P} = P(\mathcal{O})$. This does not depend upon Γ , and we may thus assume that $\Gamma = \mathbb{R}$. Since (S_1, B) and $(S_2, B) \in \mathbf{SP}(G)$ have the same image in $\mathbf{O}(G)$, there exists an element $g \in G$ with $g(S_1, B) = (S_2, B)$, i.e. $\text{Int}(g)(S_1) = S_2$ and $g \in B$. We will show that also $g \in P$, i.e. $g\mathcal{F} = \mathcal{F}$ for any $\mathcal{F} \in F^{-1}(P) \subset \mathbf{F}^{\mathbb{R}}(G)$. Note that \mathcal{F} , $g\mathcal{F}$ and the chamber $C = F^{-1}(B)$ are all contained in the apartment $\mathbf{F}^{\mathbb{R}}(S_2)$. Fix a faithful $\tau \in \text{Rep}^\circ(G)(\mathcal{O})$. Then

$$\langle \mathcal{F}, \mathcal{F}' \rangle_\tau = \langle g\mathcal{F}, g\mathcal{F}' \rangle_\tau = \langle g\mathcal{F}, \mathcal{F}' \rangle_\tau$$

for all $\mathcal{F}' \in F^{-1}(B)$, thus $\mathcal{F} = g\mathcal{F}$ because $F^{-1}(B)$ is a non-empty open subset of the Euclidean space $(\mathbf{F}^{\mathbb{R}}(S_2), \langle -, - \rangle_\tau)$. \square

4.2.6. Suppose for this and the next section that our local ring $\mathcal{O} = k$ is a field.

Theorem 60. [12, XXVI 4.1.1] $\text{Std}(G) = \mathbf{P}(G)^2$ and thus also $\text{Std}^\Gamma(G) = \mathbf{F}^\Gamma(G)^2$.

Corollary 61. *For any apartments A_1, A_2 in $\mathbf{F}^\Gamma(G)$ and facets F_1, F_2 in $A_1 \cap A_2$, there exists $g \in G$ mapping A_1 to A_2 with $g \equiv 1$ on $\overline{F}_1 \cup \overline{F}_2$.*

Proof. Fix closed chambers $F_1 \subset \overline{C}_1 \subset A_1$ and $F_2 \subset \overline{C}_2 \subset A_2$ and choose an apartment A_3 containing C_1 and C_2 . The previous proposition shows that there exists elements $g_1, g_2 \in G$ such that $g_1A_1 = A_3 = g_2A_2$, $g_1 \equiv 1$ on $\overline{C}_1 \cup \overline{F}_2$ and $g_2 \equiv 1$ on $\overline{C}_2 \cup \overline{F}_1$. Then $g = g_2^{-1}g_1$ maps A_1 to A_2 and $g \equiv 1$ on $\overline{F}_1 \cup \overline{F}_2$. \square

Corollary 62. *For a monomorphism $f : G_1 \rightarrow G_2$ of reductive group over k , the induced map $f : \mathbf{F}^\Gamma(G_1) \rightarrow \mathbf{F}^\Gamma(G_2)$ is injective.*

Proof. Fix a faithful $\tau \in \text{Rep}^\circ(G_2)(k)$. Then $\tau \circ f \in \text{Rep}^\circ(G_1)(k)$ if faithful and

$$\forall \mathcal{F}, \mathcal{F}' \in \mathbf{F}^\Gamma(G_1) : \quad \langle f(\mathcal{F}), f(\mathcal{F}') \rangle_\tau = \langle \mathcal{F}, \mathcal{F}' \rangle_{f(\tau)}.$$

Thus also $d_\tau(f(\mathcal{F}), f(\mathcal{F}')) = d_{f(\tau)}(\mathcal{F}, \mathcal{F}')$ and $f(\mathcal{F}) = f(\mathcal{F}')$ implies $\mathcal{F} = \mathcal{F}'$. \square

Corollary 63. *Let P be a parabolic subgroup of G with unipotent radical U and Levi subgroup L . Then $\mathbf{F}^\Gamma(L)$ is a fundamental domain for the action of $U(k)$ on $\mathbf{F}^\Gamma(G)$. Let $r = r_{P,L} : \mathbf{F}^\Gamma(G) \rightarrow \mathbf{F}^\Gamma(L)$ be the corresponding retraction. Then*

$$\forall x, y \in \mathbf{F}^\Gamma(G) : \quad d_\tau(rx, ry) \leq d_\tau(x, y).$$

Corollary 64. *The function $d_\tau : \mathbf{F}^\Gamma(G) \times \mathbf{F}^\Gamma(G) \rightarrow \mathbb{R}_+$ is a distance:*

$$\forall x, y, z \in \mathbf{F}^\Gamma(G) : \quad d_\tau(x, y) \leq d_\tau(x, z) + d_\tau(z, y).$$

Proof. Fix $S_0 \in \mathbf{S}(L)$. The $\mathbf{P} = P(k)$ and $\mathbf{L} = L(k)$ orbits of S_0 in $\mathbf{S}(G)$ are respectively equal to $\mathbf{S}(G, P) = \{S \in \mathbf{S}(G) : Z_G(S) \subset P\}$ and $\mathbf{S}(L)$. Since any $\mathcal{F} \in \mathbf{F}^\Gamma(G)$ belongs to $\mathbf{F}^\Gamma(S)$ for some $S \in \mathbf{S}(G, P)$, we find that with $\mathbf{U} = U(k)$,

$$\mathbf{F}^\Gamma(G) = \cup_{S \in \mathbf{S}(G, P)} \mathbf{F}^\Gamma(S) = \mathbf{P} \cdot \mathbf{F}^\Gamma(S_0) = \mathbf{U} \cdot \cup_{S \in \mathbf{S}(L)} \mathbf{F}^\Gamma(S) = \mathbf{U} \cdot \mathbf{F}^\Gamma(L).$$

Suppose that $\mathcal{F}, u\mathcal{F} \in \mathbf{F}^\Gamma(L)$ for some $u \in \mathbf{U}$, and choose an $S \in \mathbf{S}(L)$ with $\mathcal{F}, u\mathcal{F} \in \mathbf{F}^\Gamma(S)$. Since $Z_G(S) \subset L \subset P$, there is a $B \in \mathbf{B}(G)$ with $Z_G(S) \subset B \subset P$. Let $C = F^{-1}(B)$ be the corresponding (G -)chamber in $A = \mathbf{F}^\Gamma(S)$. Since $U \subset B$, $uC = C$ and $\mathcal{F}, C \in A \cap u^{-1}A$. Choose $g \in \mathbf{G}$ with $gu^{-1}A = A$, $g\mathcal{F} = \mathcal{F}$ and $gC = C$. Then g belongs to $\mathbf{B} = B(k)$, thus gu^{-1} belongs to $\mathbf{B} \cap \mathbf{N}_G(S) = Z_G(S)$ which acts trivially on A . Therefore $u\mathcal{F} = gu^{-1}u\mathcal{F} = g\mathcal{F} = \mathcal{F}$ and $\mathbf{F}^\Gamma(L)$ is a fundamental domain for the action of \mathbf{U} on $\mathbf{F}^\Gamma(G)$.

For $A \in \mathbf{A}(G)$ containing $F^{-1}(P)$, there is a unique $A_L \in \mathbf{A}(L) \cap \mathbf{U} \cdot \{A\}$ such that $r(x) = ux$ for any $x \in A$ and $u \in \mathbf{U}$ such that $uA = A_L$. Indeed, there is a $p = lu$ in $\mathbf{P} = \mathbf{L}\mathbf{U}$ such that pA is an apartment of $\mathbf{F}^\Gamma(L)$, then $uA = l^{-1}pA \subset \mathbf{F}^\Gamma(L)$ and $r(x) = ux$ for every $x \in A$. Thus for $x, y \in A$, $d_\tau(rx, ry) = d_\tau(x, y)$.

For the remaining claims, we may assume that $\Gamma = \mathbb{R}$ and use induction on the semi-simple rank s of G . If $s = 0$ everything is obvious. If $s > 0$ but $G = L$, then r is the identity thus $d_\tau(rx, ry) = d_\tau(x, y)$ for every $x, y \in \mathbf{F}^\mathbb{R}(G)$. If $G \neq L$, choose an apartment A in $\mathbf{F}^\mathbb{R}(G)$ containing x and y , let $[x, y]$ be the corresponding segment of A , and write $[x, y] = \cup_{i=0}^{n-1} [x_i, x_{i+1}]$ for consecutive points $x_i \in [x, y]$ with $x_0 = x$, $x_n = y$ and $]x_i, x_{i+1}[$ contained in a facet $F_i \subset A$. Then there is an apartment containing $F^{-1}(P)$ and $\{x_i, x_{i+1}\} \subset \overline{F}_i$, thus $d_\tau(rx_i, rx_{i+1}) = d_\tau(x_i, x_{i+1})$ for every $i \in \{0, \dots, n-1\}$. Since d_τ is a distance on $\mathbf{F}^\mathbb{R}(L)$ by our induction hypothesis,

$$d_\tau(rx, ry) \leq \sum_{i=0}^{n-1} d_\tau(rx_i, rx_{i+1}) = \sum_{i=0}^{n-1} d_\tau(x_i, x_{i+1}) = d_\tau(x, y).$$

Finally for $x, y, z \in \mathbf{F}^\mathbb{R}(G)$, choose an apartment $\mathbf{F}^\mathbb{R}(S)$ containing x, y and a chamber $F^{-1}(B)$, let $r = r_{B, Z_G(S)}$ be the corresponding retraction. Then

$$d_\tau(x, y) = d_\tau(rx, ry) \leq d_\tau(rx, rz) + d_\tau(rz, ry) \leq d_\tau(x, z) + d_\tau(z, y).$$

This finishes the proof of corollaries 63 and 64. \square

Remark 65. We could pursue here with many further corollaries, but our knowledgeable readers will recognize that already with corollary 61, we have established that $\mathbf{F}^\mathbb{R}(G)$, together with its collections of apartments and facets (and the function d_τ for some choice of a faithful τ), is a (discrete) Euclidean building in the sense of [29, 6.1]. It is the vectorial (Tits) building defined in [29, 10.6]. But the construction given there singles out a pair $Z_G(S) \subset B$ and uses more of the finest results from [5]: $\mathbf{F}^\mathbb{R}(G)$ is the building associated to the saturated Tits system $(\mathbf{G}, \mathbf{B}, \mathbf{N}) = (G, B, N_G(S))(k)$. By contrast, we may retrieve some of the results of [5] using the strongly transitive and strongly type-preserving action of \mathbf{G} on our globally constructed building $\mathbf{F}^\mathbb{R}(G)$, for instance the fact that $(\mathbf{G}, \mathbf{B}, \mathbf{N})$ is indeed a saturated Tits system [29, 8.6]. Among the many nice properties of buildings, let us mention that the distance d_τ turns $\mathbf{F}^\mathbb{R}(G)$ into a CAT(0)-space, there is a well-defined notion of convexity, bounded subsets have a unique circumcenter etc. . .

4.2.7. If τ' is another faithful representation of G , the distances $d_{\tau'}$ and d_τ are equivalent. One checks it first on a fixed apartment A , thus obtaining constants $c, C > 0$ such that $cd_\tau(x, y) \leq d_{\tau'}(x, y) \leq Cd_\tau(x, y)$ for $x, y \in A$. Then this holds true for every $x, y \in \mathbf{F}^\Gamma(G)$, since any such pair is \mathbf{G} -conjugated to one in A . We thus obtain a canonical metrizable \mathbf{G} -invariant topology on $\mathbf{F}^\Gamma(G)$.

The \mathbf{G} -invariant functions of section 4.2.3 are continuous with respect to the canonical topology. The apartments and the ‘‘closed facets’’ of section 4.1.11 are topologically closed, being complete for the induced topology.

The canonical topology on $\mathbf{C}^\Gamma(G)$ is the quotient topology of the canonical topology on $\mathbf{G}^\Gamma(G)$. Since $\mathbf{G} \setminus \mathbf{G}^\Gamma(G) = \mathbf{C}^\Gamma(G)$, the sections defined by the ‘‘closed chambers’’ are homeomorphisms. In particular, the canonical topology on $\mathbf{C}^\Gamma(G)$ is also metrizable, it is compatible with the monoid structure of $\mathbf{C}^\Gamma(G)$ and the functions defined in section 4.2.4 are continuous.

4.2.8. Suppose now that our local ring \mathcal{O} is an integral domain with fraction field K and residue field k , giving rise to morphisms of cartesian squares

$$\begin{array}{ccccc}
 \mathbf{F}^\Gamma(G_K) & \longleftarrow & \mathbf{F}^\Gamma(G) & \longrightarrow & \mathbf{F}^\Gamma(G_k) \\
 \downarrow F & \searrow t & \downarrow F & \searrow t & \downarrow F \\
 \mathbf{C}^\Gamma(G_K) & \longleftarrow & \mathbf{C}^\Gamma(G) & \longrightarrow & \mathbf{C}^\Gamma(G_k) \\
 \downarrow F & \searrow t & \downarrow F & \searrow t & \downarrow F \\
 \mathbf{P}(G_K) & \longleftarrow & \mathbf{P}(G) & \longrightarrow & \mathbf{P}(G_k) \\
 \downarrow F & \searrow t & \downarrow F & \searrow t & \downarrow F \\
 \mathbf{O}(G_K) & \longleftarrow & \mathbf{O}(G) & \longrightarrow & \mathbf{O}(G_k)
 \end{array}$$

Proposition 66. *For any faithful $\tau \in \text{Rep}^\circ(G)(\mathcal{O})$ and $x, y \in \mathbf{F}^\Gamma(G)$,*

$$\begin{aligned}
 \langle x_k, y_k \rangle_{\tau_k} &\geq \langle x_K, y_K \rangle_{\tau_K} \\
 \angle_{\tau_k}(x_k, y_k) &\leq \angle_{\tau_K}(x_K, y_K) \\
 d_{\tau_k}(x_k, y_k) &\leq d_{\tau_K}(x_K, y_K) \\
 \|x_k\|_{\tau_k} &= \|x_K\|_{\tau_K}
 \end{aligned}$$

Proof. We may assume that $\Gamma = \mathbb{R}$. For $(x, y) \in \mathbf{Std}^{\mathbb{R}}(G)$, one checks easily that all of the above inequalities are in fact equalities – in particular $\|x_k\|_{\tau_k} = \|x_K\|_{\tau_K}$ for all $x \in \mathbf{F}^\Gamma(G)$. For an arbitrary pair (x, y) in $\mathbf{F}^{\mathbb{R}}(G)$, the facet decomposition of $\mathbf{F}^{\mathbb{R}}(G)$ induces a decomposition of the segment $[x, y] = \cup_{i=0}^{n-1} [x_i, x_{i+1}]$ as in the proof of corollary 63, with $(x_i, x_{i+1}) \in \mathbf{Std}^{\mathbb{R}}(G)$ for every i . Thus

$$d_{\tau_K}(x_K, y_K) = \sum_{i=0}^{n-1} d_{\tau_K}(x_{i,K}, x_{i+1,K}) = \sum_{i=0}^{n-1} d_{\tau_k}(x_{i,k}, x_{i+1,k}) \geq d_{\tau_k}(x_k, y_k)$$

and the other two inequalities easily follow. \square

4.3. Affine $\mathbf{F}(G)$ -buildings. Let G be a reductive group over a field K . From now on, we take $\Gamma = \mathbb{R}$ and drop it from our notations. We also fix a faithful finite dimensional representation τ of G and drop it from the notations of section 4.2.3. We denote by \mathbf{X} the K -valued points of a K -scheme X . For $S \in \mathbf{S}(G)$, we denote by $\Phi(S, G)$ the set of roots of S in $\mathfrak{g} = \text{Lie}(G)$. For $a \in \Phi(S, G)$, we denote by $U_a \subset G$ the corresponding root subgroup, which is the unipotent radical of the parabolic subgroup of $Z_G(S_a)$ containing $Z_G(S)$ with Lie algebra $\mathfrak{g}_0 \oplus \bigoplus_{b \in \mathbb{N}a \cap \Phi(S, G)} \mathfrak{g}_b$, where $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{b \in \Phi(S, G)} \mathfrak{g}_b$ is the weight decomposition of \mathfrak{g} and S_a is the neutral component of $\ker(a : S \rightarrow \mathbb{G}_{m, K})$. Thus if $2a \in \Phi(S, G)$, then $U_{2a} \subset U_a$.

4.3.1. We define a partial order on $\mathbf{C}(G)$ (the dominance order) as follows:

$$x \leq y \iff \forall z \in \mathbf{C}(G) : \langle x, z \rangle \leq \langle y, z \rangle.$$

If we choose a minimal pair $(S, B) \in \mathbf{SP}(G)$ and let $s : \mathbf{C}(G) \hookrightarrow \mathbf{AF}(G)$ be the corresponding section, then $x \leq y$ if and only if $s(y) - s(x)$ belongs to the dual cone C^* of $C = s(\mathbf{C}(G))$, defined by $C^* = \{t \in \mathbf{G}(S) : \forall c \in \mathbf{C}(G), \langle t, c \rangle \geq 0\}$. We also have the following characterization [4, 12.14]: $x \leq y$ if and only if $s(x)$

belongs to the convex hull of the Weyl orbit $W_G(S) \cdot s(y)$ in the real vector space $\mathbf{F}(S) = \mathbf{G}(S)$. In particular, this partial order does not depend upon the chosen scalar product (= chosen τ). It is compatible with the monoid structure on $\mathbf{C}(G)$ and it is related to the decomposition $\mathbf{C}(G) = \mathbf{C}^r(G) \times \mathbf{G}(Z)$ of section 2.2.9 (where $\mathbf{C}^r(G) = \mathbb{C}(G)^r(K)$ and $\mathbf{G}(Z) = \mathbb{C}(G)^c(K)$) as follows: for $x = (x^r, x^c)$ and $y = (y^r, y^c)$ in $\mathbf{C}^r(G) \times \mathbf{G}(Z)$, $x \leq y$ if and only if $x^r \leq y^r$ and $x^c = y^c$. The poset $(\mathbf{C}(G), \leq)$ is a lattice and $\mathbf{G}(Z) \subset \mathbf{C}(G)$ is its subset of minimal elements.

4.3.2. An affine $\mathbf{F}(G)$ -space is a triple $\mathbf{X}(G) = (\mathbf{X}(G), \mathbf{X}(-), +)$ where $\mathbf{X}(G)$ is a set with an action of \mathbf{G} while $\mathbf{X}(-)$ and $+$ are \mathbf{G} -equivariant maps

$$\mathbf{X}(-) : \mathbf{S}(G) \rightarrow \mathcal{P}(\mathbf{X}(G)) \quad \text{and} \quad + : \mathbf{X}(G) \times \mathbf{F}(G) \rightarrow \mathbf{X}(G)$$

such that for every $S \in \mathbf{S}(G)$, the $+$ -map turns $\mathbf{X}(S)$ into an affine $\mathbf{G}(S)$ -space and $\mathbf{X}(G) = \cup_S \mathbf{X}(S)$. We refer to $\mathbf{X}(S)$ as the apartment of S in $\mathbf{X}(G)$.

4.3.3. The group $\mathbf{N}_G(S)$ thus acts on $\mathbf{X}(S)$ by affine morphisms, the vectorial part of this action equals $\nu_S^v : \mathbf{N}_G(S) \rightarrow W_G(S) \subset \text{Aut}(\mathbf{G}(S))$ and the kernel $Z_G(S)$ of ν_S^v acts on $\mathbf{X}(S)$ by translations, through a $W_G(S)$ -equivariant morphism $\nu_{\mathbf{X},S} : Z_G(S) \rightarrow \mathbf{G}(S)$. For any other $S' \in \mathbf{S}(G)$, there is commutative diagram

$$\begin{array}{ccc} Z_G(S) & \xrightarrow{\nu_{\mathbf{X},S}} & \mathbf{G}(S) \\ \simeq \downarrow & & \downarrow \simeq \\ Z_G(S') & \xrightarrow{\nu_{\mathbf{X},S'}} & \mathbf{G}(S') \end{array}$$

where the vertical maps are induced by $\text{Int}(g)$ for any $g \in \mathbf{G}$ with $\text{Int}(g)(S) = S'$. The type of $\mathbf{X}(G)$ is the $W_G = \varprojlim W_G(S)$ -equivariant morphism

$$\nu_{\mathbf{X}} = \varprojlim \nu_{\mathbf{X},S} : \varprojlim Z_G(S) \rightarrow \varprojlim \mathbf{G}(S)$$

which is obtained from these diagrams by taking the limits over all $S \in \mathbf{S}(G)$.

4.3.4. An affine $\mathbf{F}(G)$ -building is an affine $\mathbf{F}(G)$ -space $\mathbf{X}(G)$ such that:

- A1** For $x, y \in \mathbf{X}(G)$, there is an $S \in \mathbf{S}(G)$ with $x, y \in \mathbf{X}(S)$.
- A2** For $(x, \mathcal{F}) \in \mathbf{X}(G) \times \mathbf{F}(G)$, there is an $S \in \mathbf{S}(G)$ with $(x, \mathcal{F}) \in \mathbf{X}(S) \times \mathbf{F}(S)$.
- A3** For $S, S' \in \mathbf{S}(G)$ and $x, y \in \mathbf{X}(S) \cap \mathbf{X}(S')$, there is a $g \in \mathbf{G}$ with

$$\text{Int}(g)(S') = S, \quad gx = x \quad \text{and} \quad gy = y.$$

These axioms already imply that there is a unique \mathbf{G} -equivariant map

$$\mathbf{d} : \mathbf{X}(G) \times \mathbf{X}(G) \rightarrow \mathbf{C}(G)$$

such that $\mathbf{d}(x, s + \mathcal{F}) = t(\mathcal{F})$ for every $x \in \mathbf{X}(G)$ and $\mathcal{F} \in \mathbf{F}(G)$. We also require:

- A4** The \mathbf{d} -map is continuous: for sequences (x_n) , (y_n) and points x, y in $\mathbf{X}(G)$,

$$\lim_{n \rightarrow \infty} \mathbf{d}(x_n, x) = 0 = \lim_{n \rightarrow \infty} \mathbf{d}(y_n, y) \implies \lim_{n \rightarrow \infty} \mathbf{d}(x_n, y_n) = \mathbf{d}(x, y).$$

- A5** The addition map is non-expanding: for $x, y \in \mathbf{X}(G)$ and $\mathcal{F} \in \mathbf{F}(G)$,

$$\mathbf{d}(x + \mathcal{F}, y + \mathcal{F}) \leq \mathbf{d}(x, y) \quad \text{in} \quad \mathbf{C}(G).$$

- A6** For $x \in \mathbf{X}(G)$, $\mathcal{F} \in \mathbf{F}(G)$ and $u \in \cup_{\mathcal{F}} \mathbf{X}(\mathcal{F})$,

$$\lim_{t \rightarrow \infty} \mathbf{d}(x + t\mathcal{F}, ux + t\mathcal{F}) = 0 \quad \text{in} \quad \mathbf{C}(G).$$

Proposition 67. *Let $\mathbf{X}(G)$ be an affine $\mathbf{F}(G)$ -building. Then:*

(1) The function \mathbf{d} satisfies the triangle inequality: for every $x, y, z \in \mathbf{X}(G)$,

$$\mathbf{d}(x, z) \leq \mathbf{d}(x, y) + \mathbf{d}(y, z) \quad \text{in } \mathbf{C}(G).$$

(2) For $P \in \mathbf{P}(G)$ with Levi decomposition $P = U \rtimes L$, $\mathbf{X}(L) = \cup_{S \in \mathbf{S}(L)} \mathbf{X}(S)$ is a fundamental domain for the action of $\mathbf{U} = U(K)$ on $\mathbf{X}(G)$ and the induced retraction $r_{P,L} : \mathbf{X}(G) \rightarrow \mathbf{X}(L)$ is non-expanding: for every $x, y \in \mathbf{X}(G)$,

$$\mathbf{d}(r_{P,L}(x), r_{P,L}(y)) \leq \mathbf{d}(x, y) \quad \text{in } \mathbf{C}(G).$$

Proof. One checks easily that the triangle inequality holds whenever x, y, z belong to $\mathbf{X}(S)$ for some $S \in \mathbf{S}(G)$, using [6, VI §1 Proposition 18]. For a general triple $x, y, z \in \mathbf{X}(G)$, choose $S \in \mathbf{S}(G)$ with $x, z \in \mathbf{X}(S)$ (using **A1**) and pick a minimal parabolic subgroup B with Levi subgroup $L = Z_G(S)$. Then, assuming (2) for the pair (B, L) , we obtain the triangle inequality of (1) as follows with $r = r_{B,L}$:

$$\mathbf{d}(x, z) = \mathbf{d}(r(x), r(z)) \leq \mathbf{d}(r(x), r(y)) + \mathbf{d}(r(y), r(z)) \leq \mathbf{d}(x, y) + \mathbf{d}(y, z).$$

For the proof of (2), first note that $\mathbf{X}(G) = \mathbf{U} \cdot \mathbf{X}(L)$ by **A2**. Fix $\mathcal{G} \in \mathbf{G}(G)$ with $(P_{\mathcal{G}}, L_{\mathcal{G}}) = (P, L)$, set $\mathcal{F} = \text{Fil}(\mathcal{G})$, $\mathcal{F}' = \text{Fil}(\iota\mathcal{G})$ and $r_t(x) = (x + t\mathcal{F}) + t\mathcal{F}'$ for $t \geq 0$ and $x \in \mathbf{X}(G)$. Thus r_t is non-expanding by **A5**, $r_t(x) = x$ for all $x \in \mathbf{X}(L)$, and $\lim_{t \rightarrow \infty} \mathbf{d}(r_t(x), r_t(ux)) = 0$ for all $x \in \mathbf{X}(G)$, $u \in \mathbf{U}$ by **A5** and **A6**. It follows that $\mathbf{X}(L)$ is indeed a fundamental domain for the action of \mathbf{U} on $\mathbf{X}(G)$. Let $r = r_{P,L}$ be the corresponding retraction. Then $\lim_{t \rightarrow \infty} \mathbf{d}(r_t(x), r(x)) = 0$ for all $x \in \mathbf{X}(G)$, since $r(x) = ux = r_t(ux)$ for some $u \in \mathbf{U}$ with $ux \in \mathbf{X}(L)$. Thus $\lim_{t \rightarrow \infty} \mathbf{d}(r_t(x), r_t(y)) = \mathbf{d}(r(x), r(y))$ by **A4** and $\mathbf{d}(r(x), r(y)) \leq \mathbf{d}(x, y)$ for all $x, y \in \mathbf{X}(G)$, which finishes the proof of (2). \square

4.3.5. For an affine $\mathbf{F}(G)$ -building $\mathbf{X}(G)$, we denote by

$$\mathbf{d}^r : \mathbf{X}(G) \times \mathbf{X}(G) \rightarrow \mathbf{C}^r(G) \quad \text{and} \quad \mathbf{d}^c : \mathbf{X}(G) \times \mathbf{X}(G) \rightarrow \mathbf{G}(Z)$$

the components of \mathbf{d} . These are \mathbf{G} -invariant functions. For $x, y, z \in \mathbf{X}(G)$,

$$\mathbf{d}^r(x, z) \leq \mathbf{d}^r(x, y) + \mathbf{d}^r(y, z) \quad \text{and} \quad \mathbf{d}^c(x, z) = \mathbf{d}^c(x, y) + \mathbf{d}^c(y, z).$$

The function $g \mapsto \mathbf{d}^c(x, gx)$ thus does not depend upon x and defines a morphism

$$\nu_{\mathbf{X}}^c : \mathbf{G} \rightarrow \mathbf{G}(Z).$$

4.3.6. Composing \mathbf{d} with the length function $\ell : \mathbf{C}(G) \rightarrow \mathbb{R}_+$ attached to our chosen τ , we obtain a \mathbf{G} -invariant distance $d : \mathbf{X}(G) \times \mathbf{X}(G) \rightarrow \mathbb{R}_+$, and the maps $x \mapsto x + \mathcal{F}$ are non-expanding by **A5**. By 4.2.7, another choice of τ yields an equivalent distance, and the resulting topology on $\mathbf{X}(G)$ does not depend upon any choice. We call it the canonical topology of $\mathbf{X}(G)$. The $+$ -map and the function \mathbf{d} are continuous with respect to the canonical topologies. Being complete for the induced metric, the apartments of $\mathbf{X}(G)$ are closed subset of $\mathbf{X}(G)$.

4.3.7. A morphism of affine $\mathbf{F}(G)$ -spaces $f : \mathbf{X}(G) \rightarrow \mathbf{Y}(G)$ is a \mathbf{G} -equivariant map between the underlying sets such that $f(\mathbf{X}(S)) \subset \mathbf{Y}(S)$ for every $S \in \mathbf{S}(G)$ and $f(x + \mathcal{F}) = f(x) + \mathcal{F}$ for every $x \in \mathbf{X}(G)$ and $\mathcal{F} \in \mathbf{F}(G)$.

4.3.8. Any morphism of affine $\mathbf{F}(G)$ -buildings is an isomorphism: it is bijective on any apartment, thus globally bijective by **A1**. An automorphism θ of an affine $\mathbf{F}(G)$ -building $\mathbf{X}(G)$ acts on the apartment $\mathbf{X}(S)$ by an $\mathbf{N}_G(S)$ -equivariant translation, thus given by a vector θ_S in $\mathbf{G}(Z) = \mathbf{G}(S)^{\mathbf{W}_G(S)}$. The \mathbf{G} -equivariance of θ implies that $S \mapsto \theta_S$ is also \mathbf{G} -equivariant, therefore constant. It follows that

$$\mathrm{Aut}(\mathbf{X}(G)) = \mathbf{G}(Z).$$

4.3.9. For an affine $\mathbf{F}(G)$ -building $\mathbf{X}(G)$, we define

$$\mathbf{X}^r(G) = \mathbf{X}(G)/\mathbf{G}(Z) \quad \text{and} \quad \mathbf{X}^e(G) = \mathbf{X}^r(G) \times \mathbf{G}(Z).$$

The group \mathbf{G} acts: on the quotient $\mathbf{X}^r(G)$ of $\mathbf{X}(G)$, on $\mathbf{G}(Z)$ by translations through the morphism $\nu_{\mathbf{X}}^c : \mathbf{G} \rightarrow \mathbf{G}(Z)$, and on $\mathbf{X}^e(G)$ diagonally. Then, the formulas

$$\mathbf{X}^r(S) = \mathbf{X}(S)/\mathbf{G}(Z) \quad \text{and} \quad \mathbf{X}^e(S) = \mathbf{X}^r(S) \times \mathbf{G}(Z)$$

yield \mathbf{G} -equivariant maps $\mathbf{X}^r : \mathbf{S}(G) \rightarrow \mathcal{P}(\mathbf{X}^r(G))$ and $\mathbf{X}^e : \mathbf{S}(G) \rightarrow \mathcal{P}(\mathbf{X}^e(G))$, the $+$ -map on $\mathbf{X}(G)$ descends to a \mathbf{G} -equivariant map $+$: $\mathbf{X}^r(S) \times \mathbf{F}^r(S) \rightarrow \mathbf{X}^r(S)$, which together with the addition map on $\mathbf{G}(Z)$ yields a \mathbf{G} -equivariant map

$$+ : \mathbf{X}^e(G) \times \mathbf{F}(G) \rightarrow \mathbf{X}^e(G) \quad ([x], \theta) + \mathcal{F} = ([x] + \mathcal{F}^r, \theta + \mathcal{F}^e).$$

The resulting triple $\mathbf{X}^e(G)$ is yet another affine $\mathbf{F}(G)$ -building, with $\nu_{\mathbf{X}^e} = \nu_{\mathbf{X}}$. In fact, any point $x_0 \in \mathbf{X}(G)$ defines an isomorphism of affine $\mathbf{F}(G)$ -buildings

$$\mathbf{X}(G) \simeq \mathbf{X}^e(G) \quad x \mapsto ([x], \mathbf{d}^c(x_0, x)).$$

Thus $\mathbf{X}^e(G)$ appears as a rigidified version of $\mathbf{X}(G)$: there are no non-trivial automorphisms of $\mathbf{X}^e(G)$ preserving the subspace $\mathbf{X}^r(G) \simeq \mathbf{X}^r(G) \times \{0\}$ of $\mathbf{X}^e(G)$.

4.3.10. We shall also consider the following strengthenings of **A2**, **A3** and **A6**:

A2! For $x \in \mathbf{X}(G)$ and $\mathcal{F}, \mathcal{G} \in \mathbf{F}(G)$, there is an $S \in \mathbf{S}(G)$ and some $\epsilon > 0$ with

$$\mathcal{F} \in \mathbf{F}(S) \quad \text{and} \quad x + \eta\mathcal{G} \in \mathbf{X}(S) \quad \text{for all } \eta \in [0, \epsilon].$$

A3! For $S, S' \in \mathbf{X}(G)$, there is a $g \in \mathbf{G}$ with

$$\mathrm{Int}(g)(S') = S \quad \text{and} \quad g \equiv \mathrm{Id} \quad \text{on } \mathbf{X}(S) \cap \mathbf{X}(S').$$

A6! For $x \in \mathbf{X}(G)$, $\mathcal{F} \in \mathbf{F}(G)$ and $u \in \mathbf{U}_{\mathcal{F}}$,

$$u(x + t\mathcal{F}) = x + t\mathcal{F} \quad \text{for all } t \gg 0.$$

Lemma 68. *The axioms **A1**, **A2!**, **A3** and **A6** together imply **A4** and **A5**.*

Proof. Since $\mathbf{X}(G)$ satisfies **A1** – **3**, the map \mathbf{d} is well-defined. We will use several times the following consequence of our assumptions:

For $x, y \in \mathbf{X}(G)$ and $\mathcal{F} \in \mathbf{F}(G)$, choose $S \in \mathbf{S}(G)$ with $x, y \in \mathbf{X}(S)$ using **A1**, write $y = x + \mathcal{G}$ with $\mathcal{G} \in \mathbf{G}(S)$ and set $x_t = x + t\mathcal{G}$ for $t \in \mathbb{R}$. By **A2!**, there exist $\epsilon_t > 0$ and $S_t^\pm \in \mathbf{S}(G)$ such that for all $\pm\eta \in [0, \epsilon_t]$, $\mathcal{F} \in \mathbf{F}(S_t^\pm)$ and $x_{t \pm \eta} \in \mathbf{X}(S_t^\pm)$. It follows that there is a sequence $0 = t_0 < t_1 < \dots < t_{n+1} = 1$ and tori $S_i \in \mathbf{S}(G)$ such that for all $t_i \leq t \leq t_{i+1}$, $x_t \in \mathbf{X}(S_i)$ and $\mathcal{F} \in \mathbf{F}(S_i)$.

As a first application, we obtain our axiom **A5** as follows, with $x_i = x_{t_i}$:

$$\mathbf{d}(x, y) = \sum_{i=0}^n \mathbf{d}(x_i, x_{i+1}) = \sum_{i=0}^n \mathbf{d}(x_i + \mathcal{F}, x_{i+1} + \mathcal{F}) \geq \mathbf{d}(x + \mathcal{F}, y + \mathcal{F})$$

provided we know that the triangle inequality holds for \mathbf{d} . To establish the latter, it is again sufficient to prove that the retraction $r_{B,L} : \mathbf{X}(G) \rightarrow \mathbf{X}(S)$ corresponding to a minimal parabolic subgroup $B = U \rtimes L$ with Levi $L = Z_G(S)$ (exists and) is non-expanding. For its existence, note first that $\mathbf{X}(G) = \mathbf{U} \cdot \mathbf{X}(S)$ by **A2**. If x and ux belong to $\mathbf{X}(S)$ for some $u \in \mathbf{U}$, then $\mathbf{d}(x, ux) = \mathbf{d}(x + \mathcal{G}, ux + \mathcal{G})$ for all $\mathcal{G} \in \mathbf{F}(S)$, thus taking $\mathcal{G} = t\mathcal{F}$ with $\mathcal{F} \in F^{-1}(B)$ and $t \gg 0$, we find that $\mathbf{d}(x, ux) = 0$ by **A6**, i.e. $x = ux$ and our retraction $r : \mathbf{X}(G) \rightarrow \mathbf{X}(S)$ is well-defined. For $x, y \in \mathbf{X}(G)$, let us apply the above observation to some $\mathcal{F} \in F^{-1}(B)$, thus obtaining S_i 's with $F^{-1}(B) \subset \mathbf{F}(S_i)$. Then, there is a $u_i \in \mathbf{U}$ with $\text{Int}(u_i)(S_i) = S$, in which case $r(z) = u_i z$ for all $z \in \mathbf{X}(S_i)$. We obtain the desired inequality as follows:

$$\mathbf{d}(x, y) = \sum_{i=0}^n \mathbf{d}(x_i, x_{i+1}) = \sum_{i=0}^n \mathbf{d}(r(x_i), r(x_{i+1})) \geq \mathbf{d}(r(x), r(y))$$

because the triangle inequality holds on $\mathbf{X}(S)$. In particular, we now know that \mathbf{d} is a distance, from which **A4** immediately follows. \square

4.3.11. Let $\mathbf{X}(G)$ be an affine $\mathbf{F}(G)$ -space.

For $S \in \mathbf{S}(G)$, $a \in \Phi(S, G)$ and $u \in \mathbf{U}_a \setminus \{1\}$, there exists a unique triple $(u_1, u_2, m(u))$ with $u_1 u u_2 = m(u)$, $u_1, u_2 \in \mathbf{U}_{-a}$ and $m(u) \in \mathbf{N}_G(S)$; moreover $\nu_S^v(m(u))$ is the symmetry $s_a \in \mathbf{W}_G(S)$ attached to a . This follows from [5, §5] by [7, 6.1.2.2 & 6.1.3.c]. In particular, there is a unique affine hyperplane $\mathbf{X}(S, u)$ in $\mathbf{X}(S)$ which is preserved by $m(u)$, the underlying vector space is the fixed point set $\mathbf{G}(\ker(a))$ of s_a in $\mathbf{G}(S)$, and $m(u)$ acts on $\mathbf{X}(S, u)$ as $x \mapsto x + \nu_{\mathbf{X}}(S, u)$ for some $\nu_{\mathbf{X}}(S, u) \in \mathbf{G}(\ker(a))$. Of course $m(u)$ fixes $\mathbf{X}(S, u)$ if and only if $\nu_{\mathbf{X}}(S, u) = 0$, and this happens whenever $m(u)$ already has finite order in $\mathbf{N}_G(S)$, which holds true for any $u \in \mathbf{U}_a \setminus \{1\}$ if $2a \notin \Phi(S, G)$. Indeed, set $\Phi' = \{b \in \Phi(S, G) : 2b \notin \Phi(S, G)\}$. This is again a root system and $U_b \simeq \mathbb{G}_{a,K}^{n(b)}$ for some $n(b) \geq 1$ for all $b \in \Phi'(S, G)$. Choose a set of simple roots Δ' of Φ' containing a and choose for each $b \in \Delta'$ a 1-dimensional K -subspace \mathbf{U}'_b in $U_b \simeq K^{n(b)}$, with $u \in \mathbf{U}'_a$. Then by [5, 7.2], there is a unique split reductive subgroup G' of G containing S with $\Phi(S, G') = \Phi'(S, G)$ such that the root subgroup U'_b of $b \in \Delta'$ in G' is the subgroup of U_b determined by \mathbf{U}'_b , i.e. $U'_b = U'_b(K)$. Let now S_a be the neutral component of $\ker(a)$. Then $(Z_{G'}(S_a), S, a)$ is an elementary system in the sense of [12, XX 1.3] by [12, XIX 3.9]. Let $f : S_{\mathcal{L}} \rightarrow Z_{G'}(S_a)$ be the corresponding morphism constructed in [12, XX 5.8] and let $X \neq 0$ be the unique element of $\mathcal{L} = \text{Lie}(U'_a)$ with $\exp(X) = u$. Since

$$\begin{pmatrix} 1 & 0 \\ -X^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -X^{-1} & 1 \end{pmatrix} = \begin{pmatrix} 0 & X \\ -X^{-1} & 0 \end{pmatrix}$$

in $S_{\mathcal{L}}(K)$, we find that

$$m(u) = f \begin{pmatrix} 0 & X \\ -X^{-1} & 0 \end{pmatrix}, \quad m(u)^2 = f \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad m(u)^4 = 1.$$

On the other hand if $2a \in \Phi(G, S)$, then [34, 1.15] provides examples where $m(u)$ has infinite order. Note also that $m(u)$ fixes $H(S, u)$ when there is a $z \in Z_G(S)$ such that $zuz^{-1} = u^{-1}$: since $m(u^{-1}) = m(u)^{-1}$ and $m(zuz^{-1}) = zm(u)z^{-1}$,

$$\begin{aligned} \mathbf{X}(S, u^{-1}) &= \mathbf{X}(S, u) & \text{and} & & \mathbf{X}(S, zuz^{-1}) &= \mathbf{X}(S, u) + \nu_{\mathbf{X}, S}(z) \\ \nu_{\mathbf{X}}(S, u^{-1}) &= -\nu_{\mathbf{X}}(S, u) & & & \nu_{\mathbf{X}}(S, zuz^{-1}) &= \nu_{\mathbf{X}}(S, u) \end{aligned}$$

therefore $zuz^{-1} = u^{-1}$ implies $\nu_{\mathbf{X}}(S, u) = 0$. Note also that since

$$uu_2m(u)^{-1}u_1m(u) = m(u) \quad \text{and} \quad m(u)u_2m(u)^{-1}u_1u = m(u)$$

we find that $m(u_1) = m(u_2) = m(u)$, therefore

$$\mathbf{X}(S, u_1) = \mathbf{X}(S, u_2) = \mathbf{X}(S, u) \quad \text{and} \quad \nu_{\mathbf{X}}(S, u_1) = \nu_{\mathbf{X}}(S, u_2) = \nu_{\mathbf{X}}(S, u).$$

We define

$$\mathbf{X}^+(S, u) = \mathbf{X}(S, u) + \{\mathcal{F} \in \mathbf{F}(S) : a(\mathcal{F}) \geq 0\}.$$

4.3.12. The following axiom is related to alcove-based retractions, see [25, 1.4].

A7 For $S_1, S_2, S_3 \in \mathbf{S}(G)$, if $\mathbf{X}(S_i) \cap \mathbf{X}(S_j)$ contains an half-subspace of $\mathbf{X}(S_i)$ for every $(i, j) \in \{1, 2, 3\}^2$, then $\mathbf{X}(S_1) \cap \mathbf{X}(S_2) \cap \mathbf{X}(S_3) \neq \emptyset$.

Lemma 69. *Suppose that the affine $\mathbf{F}(G)$ -space $\mathbf{X}(G)$ satisfies **A6!** and **A7**. Then for every $S \in \mathbf{S}(G)$, $a \in \Phi(S, G)$ and $u \in U_a \setminus \{1\}$,*

$$\{x \in \mathbf{X}(S) : ux \in \mathbf{X}(S)\} = \{x \in \mathbf{X}(S) : ux = x\} = \mathbf{X}^+(S, u).$$

Moreover $\mathbf{X}(S, u) = \{x \in \mathbf{X}(S) : m(u)(x) = x\}$, i.e. $\nu_{\mathbf{X}}(S, u) = 0$.

Proof. The first equality already follows from **A6!**: if x and ux both belong to $\mathbf{X}(S)$, pick $\mathcal{F} \in \mathbf{F}(S)$ with $u \in U_{\mathcal{F}}$, then $x + t\mathcal{F} = u(x + t\mathcal{F}) = ux + t\mathcal{F}$ for $t \gg 0$, thus $x = ux$ since $\mathbf{X}(S)$ is an affine $\mathbf{F}(S)$ -space. For $t \in \mathbb{R}$, put

$$\begin{aligned} \mathbf{X}(S, u, t) &= \mathbf{X}(S, u) + \{\mathcal{F} \in \mathbf{F}(S) : a(\mathcal{F}) = t\} \\ \mathbf{X}^+(S, u, t) &= \mathbf{X}(S, u, t) + \{\mathcal{F} \in \mathbf{F}(S) : a(\mathcal{F}) \geq 0\} \\ \mathbf{X}^-(S, u, t) &= \mathbf{X}(S, u, t) + \{\mathcal{F} \in \mathbf{F}(S) : a(\mathcal{F}) \leq 0\} \end{aligned}$$

If u fixes some $x \in \mathbf{X}(S, u, t)$, then also $u \equiv \text{Id}$ on $\mathbf{X}^+(S, u, t)$ since

$$\forall \mathcal{F} \in \mathbf{F}(S) : \quad a(\mathcal{F}) \geq 0 \iff U_a \subset P_{\mathcal{F}}.$$

By **A6!**, u fixes some point in $\mathbf{X}(S)$, therefore $u \equiv \text{Id}$ on $\mathbf{X}^+(S, u, t)$ for $t \gg 0$. Similarly, u_1 and u_2 fix $\mathbf{X}^-(S, u, t)$ for $t \ll 0$. Choose $T > 0$ such that $u \equiv \text{Id}$ on $\mathbf{X}^+(S, u, T)$ and $u_1 \equiv u_2 \equiv \text{Id}$ on $\mathbf{X}^-(S, u, -T)$. Then: $\mathbf{X}(S)$ and $u\mathbf{X}(S)$ share the half-subspace $\mathbf{X}^+(S, u, T)$, $\mathbf{X}(S)$ and $u_1^{-1}\mathbf{X}(S)$ share the half-subspace $\mathbf{X}^-(S, u, -T)$, while $u\mathbf{X}(S)$ and $u_1^{-1}\mathbf{X}(S)$ share the half-subspace

$$u\mathbf{X}^-(S, u, -T) = uu_2\mathbf{X}^-(S, u, -T) = u_1^{-1}m(u)\mathbf{X}^-(S, u, -T) = u_1^{-1}\mathbf{X}^+(S, u, T).$$

Thus by **A7**, there is a point $x \in \mathbf{X}(S) \cap u\mathbf{X}(S) \cap u_1^{-1}\mathbf{X}(S)$. Any such point is fixed by u^{-1} and u_1 , thus also by $m(u)u_2^{-1}$. In particular $u_2^{-1}(x) = m(u)^{-1}(x)$ also belongs to $\mathbf{X}(S)$, so that again x is fixed by u_2 , as well as $m(u)$. It follows that u, u_1, u_2 and $m(u)$ act trivially on $\mathbf{X}(S, u)$. If u fixes $y \in \mathbf{X}(S, u, t)$ for some $t < 0$, then y would also belong to $\mathbf{X}(S) \cap u\mathbf{X}(S) \cap u_1^{-1}\mathbf{X}(S)$, which equals $\mathbf{X}(S, u)$, a contradiction. Thus $\mathbf{X}^+(S, u) = \{x \in \mathbf{X}(S) : ux = x\}$, which proves our claims. \square

4.3.13. For a subset $\Omega \neq \emptyset$ of $\mathbf{X}(S)$, we denote by \mathbf{G}_{Ω} the pointwise stabilizer of Ω in \mathbf{G} and by $\mathbf{G}_{S, \Omega}$ the subgroup of \mathbf{G} spanned by $\mathbf{N}_G(S)_{\Omega} = \mathbf{G}_{\Omega} \cap \mathbf{N}_G(S)$ and

$$\{u \in U_a \setminus \{1\} : a \in \Phi(G, S), \Omega \subset \mathbf{X}^+(S, u)\}.$$

When $\Omega = \{x\}$, we simply write $\mathbf{G}_x = \mathbf{G}_{\{x\}}$ and $\mathbf{G}_{S, x} = \mathbf{G}_{S, \{x\}}$. The previous lemma implies that under **A6!** and **A7**, for every $S \in \mathbf{X}(G)$ and $\Omega \subset \mathbf{X}(S)$, we have $\mathbf{G}_{S, \Omega} \subset \mathbf{G}_{\Omega}$. This justifies the following sharp strengthening of **A6!**:

A8 For some (or every) $S \in \mathbf{S}(G)$, $\mathbf{G}_{S, x} = \mathbf{G}_x$ for all $x \in \mathbf{X}(S)$.

A tight $\mathbf{F}(G)$ -building is an affine $\mathbf{F}(G)$ -building $\mathbf{X}(G)$ which satisfies **A8**. It then also satisfies **A6!** and the conclusion of lemma 69, and it is determined by its type:

Lemma 70. *Suppose that $\mathbf{X}(G)$ is a tight $\mathbf{F}(G)$ -building and $\mathbf{Y}(G)$ is an affine $\mathbf{F}(G)$ -building which satisfies the conclusion of lemma 69. Then*

$$\nu_{\mathbf{X}} = \nu_{\mathbf{Y}} \iff \mathbf{X}(G) \simeq \mathbf{Y}(G).$$

Proof. We have to show that $\nu_{\mathbf{X}} = \nu_{\mathbf{Y}}$ implies $\mathbf{X}^e(G) \simeq \mathbf{Y}^e(G)$. For $S \in \mathbf{S}(G)$, let $\theta_S : \mathbf{X}^e(S) \rightarrow \mathbf{Y}^e(S)$ be the unique $\mathbf{N}_G(S)$ -equivariant isomorphism of affine $\mathbf{F}(S)$ -spaces mapping $\mathbf{X}^r(S)$ to $\mathbf{Y}^r(S)$ – its existence follows from the proof of [28, 2.1.9], showing that $\mathbf{N}_G(S)$ is an inessential extension of $W_G(S)$. If $\text{Int}(g)(S) = S'$, then $g \circ \theta_S = \theta_{S'} \circ g$. For $x \in \mathbf{X}^e(S) \cap \mathbf{X}^e(S')$, there is such a g in \mathbf{G}_x by **A3** for $\mathbf{X}(G)$. Thus g belongs to $\mathbf{G}_{S,x}$ by **A8**, which equals $\mathbf{G}_{S,\theta_S(x)}$ by definition. Then $g \in \mathbf{G}_{\theta_S(x)}$ by assumption on $\mathbf{Y}(G)$, thus $\theta_{S'}(x) = \theta_{S'}(gx) = g\theta_S(x) = \theta_S(x)$. Our isomorphisms θ_S thus glue to $\theta : \mathbf{X}^e(G) \rightarrow \mathbf{Y}^e(G)$, which is the desired isomorphism. \square

Remark 71. A tight $\mathbf{F}(G)$ -building $\mathbf{X}(G)$ (or more generally an affine $\mathbf{F}(G)$ -space which satisfies **A3** and **A8**) can be retrieved from any apartment $\mathbf{X}(S)$ with its $\mathbf{N}_G(S)$ -action: it is the quotient of $\mathbf{G} \times \mathbf{X}(S)$ for the equivalence relation induced by $(g, x) \mapsto gx$, which indeed only depends upon the apartment: $(g, x) \sim (g', x')$ if and only if $g' = gkn$ and $x' = n^{-1}x$ for some $k \in \mathbf{G}_{S,x}$ and $n \in \mathbf{N}_G(S)$.

4.3.14. We denote by $\mathbf{RX}(G)$ the set of all functions $f : \mathbb{R}^+ \rightarrow \mathbf{X}(G)$ of the form $f(t) = x + t\mathcal{F}$ for some $(x, \mathcal{F}) \in \mathbf{X}(G) \times \mathbf{F}(G)$. The tangent space $\mathbf{TX}(G)$ of $\mathbf{X}(G)$ is the quotient of $\mathbf{RX}(G)$ by the equivalence relation defined by $f_1 \sim f_2$ if and only if there is an $\epsilon > 0$ such that $f_1(t) = f_2(t)$ for all $t \in [0, \epsilon]$. The group \mathbf{G} acts on $\mathbf{RX}(G)$ and $\mathbf{TX}(G)$ and the obvious maps $\mathbf{RX}(G) \rightarrow \mathbf{TX}(G) \rightarrow \mathbf{X}(G)$ are \mathbf{G} -equivariant. We denote by $\mathbf{T}_x\mathbf{X}(G)$ the fiber of $\mathbf{TX}(G) \rightarrow \mathbf{X}(G)$ over $x \in \mathbf{X}(G)$, and by $\text{loc}_x : \mathbf{F}(G) \rightarrow \mathbf{T}_x\mathbf{X}(G)$ the \mathbf{G}_x -equivariant map which sends \mathcal{F} to the germ of $x + t\mathcal{F}$. If x belongs to $\mathbf{X}(S)$, then the restriction of loc_x to $\mathbf{F}(S)$ is injective. We denote by $\mathbf{T}_x\mathbf{X}(S)$ the corresponding apartment in $\mathbf{T}_x\mathbf{X}(G)$. If the axiom **A2!** holds for $\mathbf{X}(G)$, then so does the axiom **A1** for $\mathbf{T}_x\mathbf{X}(G)$: any two elements of $\mathbf{T}_x\mathbf{X}(G)$ belong to $\mathbf{T}_x\mathbf{X}(S)$ for some $S \in \mathbf{S}(G)$ with $x \in \mathbf{X}(S)$.

4.3.15. We have $\mathbf{RX}(G) = \mathbf{X}(G) \times \mathbf{F}(G)$ for any affine $\mathbf{F}(G)$ -space $\mathbf{X}(G)$ which satisfies **A2**, **A3!** and the following weakening of **A8**:

A8^b If g fixes $x + t\mathcal{F}$ for all $t \gg 0$, then g fixes \mathcal{F} .

Indeed, if $f(t) = x_1 + t\mathcal{F}_1 = x_2 + t\mathcal{F}_2$ for all t , then $x_1 = x_2 = x = f(0)$. Now choose $S_i \in \mathbf{S}(G)$ with $x \in \mathbf{X}(S_i)$ and $\mathcal{F}_i \in \mathbf{F}(S_i)$, using **A2**. Then by **A3!**, there is a $g \in \mathbf{G}$ fixing f with $\text{Int}(g)(S_2) = S_1$. Since g fixes $x + t\mathcal{F}_2$ for all t , it fixes x and \mathcal{F}_2 by **A8^b**. Thus $g(x + t\mathcal{F}_2) = gx + tg\mathcal{F}_2 = x + t\mathcal{F}_2 = x + t\mathcal{F}_1$ with $x \in \mathbf{X}(S)$, \mathcal{F}_1 and \mathcal{F}_2 in $\mathbf{F}(S)$, therefore $\mathcal{F}_1 = \mathcal{F}_2$ since $\mathbf{X}(S)$ is an affine $\mathbf{F}(S)$ -space.

Under the above identification, the $+$ map just becomes $f \mapsto f(1)$.

4.4. Example: $\mathbf{F}(G)$ is a tight affine $\mathbf{F}(G)$ -building. The apartment map is given by $S \mapsto \mathbf{F}(S)$. The $+$ -map is most easily defined using the identification $\mathbf{F}(G) = \mathbf{F}(\omega_K^\circ)$ of theorem 21: for $\mathcal{F}_1, \mathcal{F}_2 \in \mathbf{F}(G)$, $\tau \in \text{Rep}^\circ(G)(K)$ and $\gamma \in \mathbb{R}$,

$$(\mathcal{F}_1 + \mathcal{F}_2)^\gamma(\tau) = \sum_{\gamma_1 + \gamma_2 = \gamma} \mathcal{F}_1^{\gamma_1}(\tau) \cap \mathcal{F}_2^{\gamma_2}(\tau).$$

The formula indeed defines an \mathbb{R} -filtration on ω_K° : choose $\mathbf{F}(S)$ containing \mathcal{F}_1 and \mathcal{F}_2 , let \mathcal{G}_1 and \mathcal{G}_2 be the corresponding splittings in $\mathbf{G}(S)$, put $M = \text{Hom}(S, \mathbb{G}_{m,K})$ and let $\mathcal{G}_i^* : M \rightarrow \mathbb{R}$ be the group homomorphism corresponding to \mathcal{G}_i , so that

$$\mathcal{F}_i^\gamma(\tau) = \bigoplus_{\gamma' \geq \gamma} \mathcal{G}_{i,\gamma'}(\tau) = \bigoplus_{m: \mathcal{G}_i^*(m) \geq \gamma} V(\tau)_m$$

where $V(\tau)_m$ is the m -th eigenspace of $\tau|_S$. Then

$$\begin{aligned} \mathcal{F}_1^{\gamma_1}(\tau) \cap \mathcal{F}_2^{\gamma_2}(\tau) &= \bigoplus_{m: \mathcal{G}_1^*(m) \geq \gamma_1 \text{ and } \mathcal{G}_2^*(m) \geq \gamma_2} V(\tau)_m \\ \text{and } (\mathcal{F}_1 + \mathcal{F}_2)^\gamma(\tau) &= \bigoplus_{m: (\mathcal{G}_1^* + \mathcal{G}_2^*)(m) \geq \gamma} V(\tau)_m \end{aligned}$$

thus $\mathcal{F}_1 + \mathcal{F}_2$ is split by $\mathcal{G}_1 + \mathcal{G}_2$. It follows that our addition is indeed well-defined. It is plainly \mathbf{G} -equivariant and commutative (but not associative).

The above computation already shows that $\mathbf{F}(G) = (\mathbf{F}(G), \mathbf{F}(-), +)$ is an affine $\mathbf{F}(G)$ -space. It satisfies the axioms **A1** = **A2** by Theorem 60 and **A3** by Corollary 61. Actually for $\mathcal{F}, \mathcal{G} \in \mathbf{F}(G)$, choosing $S \in \mathbf{S}(G)$ with $\mathcal{F}, \mathcal{G} \in \mathbf{F}(S)$, we find that $\mathcal{F} + \eta\mathcal{G}$ belongs to a fixed closed alcove of $\mathbf{F}(S)$ for all sufficiently small $\eta \geq 0$, from which the stronger axiom **A2!** easily follows. For $S \in \mathbf{S}(G)$ and $\mathcal{F} \in \mathbf{F}(S)$, $\mathbb{G}_{S,\mathcal{F}}$ is the group spanned by $\mathbf{N}_G(S) \cap P_{\mathcal{F}}$ and the U_a 's for $a \in \Phi(G, S)$, $a(\mathcal{F}) \geq 0$; the group $\mathbf{N}_G(S) \cap P_{\mathcal{F}} = \mathbf{N}_L(S)$ and the U_a 's with $a(\mathcal{F}) = 0$ together span $\mathbf{L} = L(K)$ where L is the Levi subgroup of $P_{\mathcal{F}}$ which contains $Z_G(S)$ – this is the Bruhat decomposition of \mathbf{L} , see [5, 5.15]; the remaining U_a 's span the unipotent radical $\mathbf{U}_{\mathcal{F}}$ of $P_{\mathcal{F}}$, therefore $\mathbb{G}_{S,\mathcal{F}} = P_{\mathcal{F}}$ and $\mathbf{F}(G)$ satisfies **A8**, thus also **A6!**, as well as **A4** and **A5** by lemma 68. It satisfies **A7** by Corollary 63 (or proposition 67) and [25, 1.4].

The trivial point $0 \in \mathbf{F}(G)$ is fixed by \mathbf{G} , and it follows from lemma 70 that $\mathbf{F}(G)$ is the unique affine $\mathbf{F}(G)$ -building with a fixed point: any such building trivially satisfies the conclusion of lemma 69 and has trivial type.

4.5. Example: a symmetric space. Let $K = \mathbb{R}$ and $G = GL(V)$, where V is an \mathbb{R} -vector space of dimension $n \in \mathbb{N}$. The action of G on the set $\mathbb{P}^1(V)$ of \mathbb{R} -lines in V identifies $\mathbf{S}(G)$ with $\mathbf{S}(V) = \{S \subset \mathbb{P}^1(V) : V = \bigoplus_{L \in S} L\}$. The action of G on V identifies $\mathbf{F}(G)$ with the set $\mathbf{F}(V)$ of all \mathbb{R} -filtrations on V . We denote by $\mathbf{F}(S)$ the apartment of $\mathbf{F}(V)$ corresponding to $S \in \mathbf{S}(V)$, thus $\mathcal{F} \in \mathbf{F}(S)$ if and only if

$$\forall \gamma \in \mathbb{R} : \quad \mathcal{F}^\gamma = \bigoplus_{L \in S, \gamma(L) \geq \gamma} L \quad \text{where} \quad \gamma(L) = \sup\{\lambda : L \subset \mathcal{F}^\lambda\}.$$

We also identify $\mathbf{C}(G)$ with $\mathbb{R}_{\geq}^n = \{\gamma_1 \leq \dots \leq \gamma_n : \gamma_i \in \mathbb{R}\}$ by the map which sends $t(\mathcal{F})$ to $\underline{t}(\mathcal{F}) = (t_i(\mathcal{F}))_{i=1}^n$, with $\#\{i : t_i(\mathcal{F}) = \gamma\} = \dim_{\mathbb{R}} \text{Gr}_{\mathcal{F}}^\gamma(V)$ for $\gamma \in \mathbb{R}$. The dominance order on $\mathbf{C}(G)$ defined in section 4.3.1 corresponds to

$$(\gamma_i)_{i=1}^n \leq (\gamma'_i)_{i=1}^n \iff \begin{cases} \sum_{j=1}^n \gamma_j = \sum_{j=1}^n \gamma'_j & \text{and} \\ \sum_{j=i}^n \gamma_j \leq \sum_{j=i}^n \gamma'_j & \text{for } 2 \leq i \leq n. \end{cases}$$

The exponential map $\exp : \mathbb{R} \rightarrow \mathbb{R}^\times$ defines an \mathbb{R} -valued section \exp of $\mathbb{D}(\mathbb{R})$, whose evaluation at the character $\gamma \in \mathbb{R}$ of $\mathbb{D}(\mathbb{R})$ is given by $\gamma(\exp) = \exp(\gamma) \in \mathbb{R}^\times$.

4.5.1. Let $\mathbf{B}(V)$ be the space of Euclidean norms $\alpha : V \rightarrow \mathbb{R}_+$, i.e. α^2 is a positive definite quadratic form on V . The group \mathbf{G} acts on $\mathbf{B}(V)$ by $(g \cdot \alpha)(v) = \alpha(g^{-1}v)$. For $S \in \mathbf{S}(V)$, we denote by $\mathbf{B}(S)$ the set of all α 's in $\mathbf{B}(V)$ for which $V = \bigoplus_{L \in S} L$ is an orthogonal decomposition. For $\alpha \in \mathbf{B}(V)$ and $\mathcal{F} \in \mathbf{F}(V)$, we denote by $\mathcal{G}_\alpha(\mathcal{F})_\gamma$ the α -orthogonal complement of \mathcal{F}_+^γ in \mathcal{F}^γ . Thus $\mathcal{G}_\alpha(\mathcal{F})$ is a splitting of \mathcal{F} which is orthogonal for α . We define $\alpha + \mathcal{F} \in \mathbf{B}(V)$ by the following formula:

$$(\alpha + \mathcal{F})(v) = \alpha \left(\sum_{\gamma} e^{-\gamma} v_{\gamma} \right) : \quad v = \sum_{\gamma} v_{\gamma}, \quad v_{\gamma} \in \mathcal{G}_\alpha(\mathcal{F})_{\gamma}.$$

Thus $\alpha + \mathcal{F} = g_\alpha(\mathcal{F}) \cdot \alpha$ with $g_\alpha(\mathcal{F}) = \mathcal{G}_\alpha(\mathcal{F})(\exp)$ in $\mathbf{G} = G(\mathbb{R})$. For $\mathcal{S} \in \mathbf{S}(V)$, $\alpha \in \mathbf{B}(\mathcal{S})$ and $\mathcal{F} \in \mathbf{F}(\mathcal{S})$, we find that

$$(\alpha + \mathcal{F})^2(v) = \sum_{L \in \mathcal{S}} (\alpha + \mathcal{F})^2(v_L) = \sum_{L \in \mathcal{S}} (e^{-\gamma(L)} \alpha)^2(v_L)$$

where $v = \sum_{L \in \mathcal{S}} v_L$ with $v_L \in L$, thus also $\alpha + \mathcal{F} \in \mathbf{B}(\mathcal{S})$.

4.5.2. The above formulas already show that $\mathbf{B}(V) = (\mathbf{B}(V), \mathbf{B}(-), +)$ is an affine $\mathbf{F}(V)$ -space. It is well-known that it satisfies **A1**, and **A2** follows from the existence of the α -orthogonal splittings. For **A3**, recall that the Fischer-Courant theory tells us that the \mathbf{G} -orbits in $\mathbf{B}(V) \times \mathbf{B}(V)$ are classified by a \mathbf{G} -equivariant map

$$\underline{\mathbf{d}} : \mathbf{B}(V) \times \mathbf{B}(V) \rightarrow \mathbb{R}_{\leq}^n$$

whose i -th component $\mathbf{d}_i : \mathbf{B}(V) \times \mathbf{B}(V) \rightarrow \mathbb{R}$ is given by

$$\mathbf{d}_i(\alpha, \beta) = -\log \left(\sup \left\{ \inf \left\{ \frac{\beta(x)}{\alpha(x)} : x \in W \setminus \{0\} \right\} : W \subset V, \dim_{\mathbb{R}} W = i \right\} \right).$$

Suppose that $\alpha, \beta \in \mathbf{B}(\mathcal{S}) \cap \mathbf{B}(\mathcal{S}')$ and choose \mathbb{R} -basis $e = (e_i)_{i=1}^n$ and $e' = (e'_i)_{i=1}^n$ of V such that $\mathcal{S} = \{\mathbb{R}e_i : i = 1, \dots, n\}$, $\mathcal{S}' = \{\mathbb{R}e'_i : i = 1, \dots, n\}$, e and e' are orthonormal for α , and $\beta(e_1) \geq \dots \geq \beta(e_n)$, $\beta(e'_1) \geq \dots \geq \beta(e'_n)$. Then necessarily

$$\forall i \in \{1, \dots, n\} : \quad \beta(e_i) = \exp(-\mathbf{d}_i(\alpha, \beta)) = \beta(e'_i)$$

The element $g \in \mathbf{G}$ mapping e to e' satisfies $g\mathcal{S} = \mathcal{S}'$, $g\alpha = \alpha$ and $g\beta = \beta$, which proves **A3**. The resulting map \mathbf{d} equals $\underline{\mathbf{d}}$ under the identification $\mathbf{C}(G) \simeq \mathbb{R}_{\leq}^n$:

$$\underline{\mathbf{d}}(\alpha, \alpha + \mathcal{F}) = \underline{\mathbf{t}}(\mathcal{F}).$$

4.5.3. Define $\mathbf{d}^i(\alpha, \beta) = \sum_{j=0}^{i-1} \mathbf{d}_{n-j}(\alpha, \beta)$, thus

$$\begin{aligned} \mathbf{d}^i(\alpha, \beta) &= \sup \{ \mathbf{d}^i(\alpha|W, \beta|W) : W \subset V, \dim_{\mathbb{R}} W = i \} \\ &= \log \sup \left\{ \frac{\Lambda^i(\alpha)(v)}{\Lambda^i(\beta)(v)} : v \in \Lambda^i(V) \setminus \{0\} \right\} \end{aligned}$$

where $\Lambda^i(\alpha)$ is the Euclidean norm on $\Lambda^i(V)$ induced by α . We have

$$\mathbf{d}^n(\alpha, \beta) = \log \left(\frac{\int_{\beta(v) \leq 1} dv}{\int_{\alpha(v) \leq 1} dv} \right)$$

for any Borel measure dv on V , thus

$$\begin{aligned} \mathbf{d}^n(\alpha, \gamma) &= \mathbf{d}^n(\alpha, \beta) + \mathbf{d}^n(\beta, \gamma), \\ \mathbf{d}^n(\alpha, g\alpha) &= \log |\det(g)|, \\ \mathbf{d}^n(\alpha, \alpha + \mathcal{F}) &= \sum_{\gamma} \gamma \dim_{\mathbb{R}} \text{Gr}_{\mathcal{F}}^{\gamma}. \end{aligned}$$

In particular, if $\mathbf{d}^i(\alpha, \gamma) = \mathbf{d}^i(\alpha|W, \gamma|W)$ for some $W \subset V$, $\dim_{\mathbb{R}} W = i$, then

$$\mathbf{d}^i(\alpha, \gamma) = \mathbf{d}^i(\alpha|W, \beta|W) + \mathbf{d}^i(\beta|W, \gamma|W) \leq \mathbf{d}^i(\alpha, \beta) + \mathbf{d}^i(\beta, \gamma)$$

i.e. $\underline{\mathbf{d}}$ satisfies the triangle inequality, from which **A4** easily follows.

4.5.4. We next show that for any $\alpha, \beta \in \mathbf{B}(V)$ and $\mathcal{F}, \mathcal{G} \in \mathbf{F}(V)$, the function

$$\mathbb{R}_+ \ni t \mapsto \underline{\mathbf{d}}(\alpha + t\mathcal{F}, \beta + t\mathcal{G}) \in \mathbb{R}_{\leq}^n$$

is convex, i.e. if $x(t) = \alpha + t\mathcal{F}$ and $y(t) = \beta + t\mathcal{G}$, then for $0 \leq t_0 \leq t \leq t_1$,

$$\underline{\mathbf{d}}(x(t), y(t)) \leq \frac{t-t_0}{t_1-t_0} \cdot \underline{\mathbf{d}}(x(t_1), y(t_1)) + \frac{t_1-t}{t_1-t_0} \cdot \underline{\mathbf{d}}(x(t_0), y(t_0)) \quad \text{in } \mathbb{R}_{\leq}^n.$$

By a standard procedure, we may assume that $(t_0, t, t_1) = (0, 1, 2)$. We want:

$$\underline{\mathbf{d}}(x(1), y(1)) \leq \frac{1}{2} (\underline{\mathbf{d}}(x(0), y(0)) + \underline{\mathbf{d}}(x(2), y(2))).$$

Choose $\mathcal{S}, \mathcal{S}' \in \mathbf{S}(V)$ with $\beta \in \mathbf{B}(\mathcal{S}), \mathcal{G} \in \mathbf{F}(\mathcal{S}), \alpha \in \mathbf{B}(\mathcal{S}')$ and $\gamma = \beta + 2\mathcal{G} \in \mathbf{B}(\mathcal{S}')$. Write $\gamma = \alpha + 2\mathcal{H}$ with $\mathcal{H} \in \mathbf{F}(\mathcal{S}')$ and set $z(t) = \alpha + t\mathcal{H}, z'(t) = \gamma + t\mathcal{S}'\mathcal{H}, y'(t) = \gamma + t\mathcal{S}\mathcal{G}$, so that $z(t) = z'(2-t)$ and $y(t) = y'(2-t)$ for $t \in [0, 2]$. Suppose that we have established our claim for (x, z) and (z', y') . Then

$$\begin{aligned} \underline{\mathbf{d}}(x(1), y(1)) &\leq \underline{\mathbf{d}}(x(1), z(1)) + \underline{\mathbf{d}}(z'(1), y'(1)) \\ &\leq \frac{1}{2} (\underline{\mathbf{d}}(x(2), z(2)) + \underline{\mathbf{d}}(z'(2), y'(2))) \\ &= \frac{1}{2} (\underline{\mathbf{d}}(x(2), y(2)) + \underline{\mathbf{d}}(x(0), y(0))) \end{aligned}$$

which reduces us further to the case where $\alpha = x(0) = y(0) = \beta$. We now want:

$$2 \cdot \underline{\mathbf{d}}(\alpha + \mathcal{F}, \alpha + \mathcal{G}) \leq \underline{\mathbf{d}}(\alpha + 2\mathcal{F}, \alpha + 2\mathcal{G}).$$

Put $f = g_\alpha(\mathcal{F}), g = g_\alpha(\mathcal{G})$. Then $f^2 = g_\alpha(2\mathcal{F}), g^2 = g_\alpha(2\mathcal{G})$ and we have to show

$$2 \cdot \underline{\mathbf{d}}(f\alpha, g\alpha) \leq \underline{\mathbf{d}}(f^2\alpha, g^2\alpha) \quad \text{i.e.} \quad 2 \cdot \underline{\mathbf{d}}(g^{-1}f\alpha, \alpha) \leq \underline{\mathbf{d}}(g^{-2}f^2\alpha, \alpha).$$

For $1 \leq i \leq n$, we have

$$\begin{aligned} \exp(2\mathbf{d}^i(g^{-1}f\alpha, \alpha)) &= \sup \left\{ \frac{\langle f^{-1}gx, f^{-1}gx \rangle_{\alpha, i}}{\langle x, x \rangle_{\alpha, i}} : x \in \Lambda^i(V) \right\}, \\ &= \sup \left\{ \frac{\langle gf^{-2}gx, x \rangle_{\alpha, i}}{\langle x, x \rangle_{\alpha, i}} : x \in \Lambda^i(V) \right\}. \end{aligned}$$

where $\langle -, - \rangle_{\alpha, i}$ is the symmetric bilinear form on $\Lambda^i(V)$ attached to the Euclidean norm $\Lambda^i(\alpha)$, for which plainly f and g are indeed self-adjoint. Therefore

$$\exp(2\mathbf{d}^i(g^{-1}f\alpha, \alpha)) = r^i(gf^{-2}g) = r^i(f^{-2}g^2)$$

where $r^i(h)$ is the spectral radius of $h \in \mathbf{G}$ acting on $\Lambda^i(V)$, i.e. the largest absolute value of the (real) eigenvalues of h . But then obviously

$$r^i(f^{-2}g^2) \leq \exp(\mathbf{d}^i(g^{-2}f^2\alpha, \alpha))$$

with equality for $i = n$, which finally proves our claim.

4.5.5. For $\alpha \in \mathbf{B}(V)$ and $\mathcal{F} \in \mathbf{F}(V)$, we denote by $\text{Gr}_{\mathcal{F}}(\alpha)$ the Euclidean norm on $\text{Gr}_{\mathcal{F}}(V)$ induced by α through the isomorphism $V \simeq \text{Gr}_{\mathcal{F}}(V)$ provided by the α -orthogonal splitting $\mathcal{G}_\alpha(\mathcal{F})$ of \mathcal{F} . We claim that for every $\alpha, \beta \in \mathbf{B}(V)$,

$$\lim_{t \rightarrow \infty} \underline{\mathbf{d}}(\alpha + t\mathcal{F}, \beta + t\mathcal{F}) = \underline{\mathbf{d}}(\text{Gr}_{\mathcal{F}}(\alpha), \text{Gr}_{\mathcal{F}}(\beta)) \quad \text{in } \mathbb{R}_{\leq}^n.$$

Indeed, choosing an isomorphism $(\mathcal{G}_\alpha(\mathcal{F})_\gamma, \alpha|\mathcal{G}_\alpha(\mathcal{F})_\gamma) \simeq (\mathcal{G}_\beta(\mathcal{F})_\gamma, \beta|\mathcal{G}_\beta(\mathcal{F})_\gamma)$ for every $\gamma \in \mathbb{R}$, we obtain an element $g \in \mathbf{G}$ which fixes \mathcal{F} and maps α to β . It then also maps $\alpha + t\mathcal{F} = g_\alpha(t\mathcal{F}) \cdot \alpha$ to $\beta + t\mathcal{F} = gg_\alpha(t\mathcal{F}) \cdot \alpha$, so that

$$\underline{\mathbf{d}}(\alpha + t\mathcal{F}, \beta + t\mathcal{F}) = \underline{\mathbf{d}}(\alpha, g_\alpha^{-1}(t\mathcal{F})gg_\alpha(t\mathcal{F}) \cdot \alpha).$$

Let $L_\alpha(\mathcal{F})$ be the centralizer of $\mathcal{G}_\alpha(\mathcal{F})$, so that $P_{\mathcal{F}} = U_{\mathcal{F}} \rtimes L_\alpha(\mathcal{F})$. Write $g = u \cdot \ell$ with $u \in U_{\mathcal{F}}$ and $\ell \in L_\alpha(\mathcal{F})$, so that $g_\alpha^{-1}(t\mathcal{F})gg_\alpha(t\mathcal{F}) = g_\alpha^{-1}(t\mathcal{F})ug_\alpha(t\mathcal{F}) \cdot \ell$. Let then $\mathfrak{u}_{\mathcal{F}} = \bigoplus_{\gamma > 0} \mathfrak{u}_\gamma$ be the weight decomposition of $\mathfrak{u}_{\mathcal{F}} = \text{Lie}(U_{\mathcal{F}})(\mathbb{R})$ induced by

$$\text{ad} \circ \mathcal{G}_\alpha(\mathcal{F}) : \mathbb{D}(\mathbb{R}) \rightarrow G \rightarrow GL(\mathfrak{g})$$

where $\mathfrak{g} = \text{Lie}(G)(\mathbb{R})$. Then $g_\alpha(t\mathcal{F})$ acts on \mathfrak{u}_γ by $\exp(t\gamma)$, from which easily follows that $g_\alpha^{-1}(t\mathcal{F})ug_\alpha(t\mathcal{F})$ converges to 1 in $U_{\mathcal{F}}$ (for the real topology). It follows that

$$\lim_{t \rightarrow \infty} \mathbf{d}(\alpha + t\mathcal{F}, \beta + t\mathcal{F}) = \mathbf{d}(\alpha, \ell\alpha) = \mathbf{d}(\text{Gr}_{\mathcal{F}}(\alpha), \text{Gr}_{\mathcal{F}}(\beta)).$$

Taking $\beta = u\alpha$ with $u \in U_{\mathcal{F}}$, we obtain **A6**. On the other hand for any β , since

$$\mathbb{R}_+ \ni t \mapsto \mathbf{d}(\alpha + t\mathcal{F}, \beta + t\mathcal{F}) \in \mathbb{R}_+^+$$

is convex and bounded, it is non-increasing, which proves **A5**.

4.5.6. We have thus established that $(\mathbf{B}(V), \mathbf{B}(-), +)$ is an affine $\mathbf{F}(V)$ -building. If $S \in \mathbf{S}(G)$ corresponds to $\mathcal{S} \in \mathbf{S}(V)$, the type map $\nu_{\mathbf{B}, \mathcal{S}} : \mathbf{S} \rightarrow \mathbf{G}(\mathcal{S})$ maps $s \in \mathbf{S}$ to the unique morphism $\mathbb{D}_{\mathbb{R}}(\mathbb{R}) \rightarrow S$ whose composite with the character χ_L through which S acts on $L \in \mathcal{S}$ is the character $\log |\chi_L(s)| \in \mathbb{R}$ of $\mathbb{D}(\mathbb{R})$.

4.6. **Example: Bruhat-Tits buildings.** Let K be a field with a non-trivial, non-archimedean absolute value $|\cdot| : K \rightarrow \mathbb{R}^+$.

4.6.1. *The Bruhat-Tits building of $GL(V)$.* Let $G = GL(V)$, where V is a K -vector space of dimension $n \in \mathbb{N}$. There are again \mathbf{G} -equivariant bijections

$$\begin{aligned} \mathbf{S}(G) &\simeq \mathbf{S}(V) = \{\mathcal{S} \subset \mathbb{P}^1(V)(K) : V = \bigoplus_{L \in \mathcal{S}} L\} \\ \mathbf{F}(G) &\simeq \mathbf{F}(V) = \{\mathbb{R}\text{-filtrations on } V\} \end{aligned}$$

A K -norm on V is a function $\alpha : V \rightarrow \mathbb{R}_+$ such that (1) $\alpha(v) = 0 \iff v = 0$, (2) $\alpha(\lambda v) = |\lambda| \alpha(v)$ for every $\lambda \in K$ and $v \in V$, and (3) $\alpha(u + v) \leq \max\{\alpha(u), \alpha(v)\}$ for every $u, v \in V$. The K -norm α is split by $\mathcal{S} \in \mathbf{S}(V)$ if and only if

$$\forall v \in V : \alpha(v) = \max\{\alpha(v_L) : L \in \mathcal{S}\} \quad \text{where } v = \sum_{L \in \mathcal{S}} v_L, v_L \in L.$$

It is splittable if it is split by \mathcal{S} for some $\mathcal{S} \in \mathbf{S}(V)$. If K is locally compact, every K -norm on V is splittable [14, Proposition 1.1]. We denote by $\mathbf{B}(V)$ the set of all splittable norms, by $\mathbf{B}(\mathcal{S})$ the subset of all norms split by \mathcal{S} . We let \mathbf{G} act on $\mathbf{B}(V)$ by $(g \cdot \alpha)(v) = \alpha(g^{-1}v)$, and define $+$: $\mathbf{B}(V) \times \mathbf{F}(V) \rightarrow \mathbf{B}(V)$ by

$$(\alpha + \mathcal{F})(v) = \inf \left\{ \max \{e^{-\gamma} \alpha(v_\gamma) : \gamma \in \mathbb{R}\} : v = \sum_{\gamma \in \mathbb{R}} v_\gamma, v_\gamma \in \mathcal{F}^\gamma \right\}$$

where the sums $\sum_{\gamma \in \mathbb{R}} v_\gamma$ have finite support. We have to verify that this operation is well-defined. Note first that **A2** follows from the second proof of [9, 1.5.ii]: for $\alpha \in \mathbf{B}(V)$ and $\mathcal{F} \in \mathbf{F}(V)$, there is an $\mathcal{S} \in \mathbf{S}(V)$ with $\alpha \in \mathbf{B}(\mathcal{S})$ and $\mathcal{F} \in \mathbf{F}(\mathcal{S})$. Let us identify $\mathbf{F}(\mathcal{S})$ with $\mathbb{R}^{\mathcal{S}}$ by $\mathcal{F} \mapsto (\gamma_L(\mathcal{F}))_{L \in \mathcal{S}}$ where $\mathcal{F}^\gamma = \bigoplus_{\gamma_L(\mathcal{F}) \geq \gamma} L$ for all $\gamma \in \mathbb{R}$. Then for $v = \sum_{L \in \mathcal{S}} v_L$ in $V = \bigoplus_{L \in \mathcal{S}} L$, we find that

$$\inf \left\{ \max \{e^{-\gamma} \alpha(v_\gamma) : \gamma \in \mathbb{R}\} \mid \begin{matrix} v = \sum_{\gamma \in \mathbb{R}} v_\gamma \\ v_\gamma \in \mathcal{F}^\gamma \end{matrix} \right\} = \max \left\{ e^{-\gamma_L(\mathcal{F})} \alpha(v_L) : L \in \mathcal{S} \right\}.$$

Indeed for $v = \sum_{\gamma} v_{\gamma}$ with $v_{\gamma} = \sum_L v_{\gamma,L}$, $v_{\gamma,L} \in L$ and $v_{\gamma,L} = 0$ if $\gamma > \gamma_L(\mathcal{F})$,

$$\begin{aligned} \max \{e^{-\gamma} \alpha(v_{\gamma}) : \gamma \in \mathbb{R}\} &= \max \{e^{-\gamma} \alpha(v_{\gamma,L}) : \gamma \in \mathbb{R}, L \in \mathcal{S}\} \\ &\geq \max \{e^{-\gamma_L(\mathcal{F})} \alpha(v_{\gamma,L}) : \gamma \in \mathbb{R}, L \in \mathcal{S}\} \\ &\geq \max \{e^{-\gamma_L(\mathcal{F})} \alpha(v_L) : L \in \mathcal{S}\} \end{aligned}$$

since $\alpha \in \mathbf{B}(\mathcal{S})$ (for the first equality) and $v_L = \sum_{\gamma} v_{\gamma,L}$ (for the last inequality), which provides the non-trivial required inequality in the displayed formula. Thus

$$(4.1) \quad (\alpha + \mathcal{F})(v) = \max \{e^{-\gamma_L(\mathcal{F})} \alpha(v_L) : L \in \mathcal{S}\}$$

from which follows that $\alpha + \mathcal{F}$ is well-defined and again belongs to $\mathbf{B}(\mathcal{S})$. The apartment and $+$ -maps are plainly G -equivariant, and the above formula shows that the latter turns $\mathbf{B}(\mathcal{S})$ into an affine $\mathbf{F}(\mathcal{S})$ -space. If $S \leftrightarrow \mathcal{S}$, the type map

$$\nu_{\mathbf{B},S} : S \rightarrow \mathbf{G}(S)$$

maps s to the unique $\mathcal{F} \in \mathbf{F}(S)$ with $\gamma_L(\mathcal{F}) = \log |\chi_L(s)|$ for all $L \in \mathcal{S}$, where $\chi_L : S \rightarrow \mathbb{G}_{m,k}$ is the character through which S acts on L .

In [25, §3], Parreau shows that a closely related set Δ is an affine building in the sense of [25, 1.1] (see also [9, 14]). The axioms **A1**, **A3!** and **A7** for $\mathbf{B}(V)$ respectively follow from the axioms A3, A2 and A5 for Δ in [25]. The axiom **A2!** is a consequence of Proposition 1.8 of [25], and then **A4** and **A5** follow from lemma 68 together with **A6!**: for $\alpha \in \mathbf{B}(V)$, $\mathcal{F} \in \mathbf{F}(V)$ and $u \in \mathbf{U}_{\mathcal{F}}$, $u(x + t\mathcal{F}) = x + t\mathcal{F}$ for $t \gg 0$, which is proved as follows. We may assume that $\alpha \in \mathbf{B}(\mathcal{S})$, $\mathcal{F} \in \mathbf{F}(\mathcal{S})$. Write $\mathcal{S} = \{Kv_1, \dots, Kv_n\}$ with $i \mapsto \gamma_i = \gamma_{Kv_i}(\mathcal{F})$ non-increasing, identify $\mathbf{B}(\mathcal{S})$ with \mathbb{R}^n via $\alpha \mapsto (\alpha_1, \dots, \alpha_n)$ where $\alpha_i = -\log(\alpha(v_i))$. Then $t\mathcal{F}$ acts on it by

$$(\alpha_1, \dots, \alpha_n) \mapsto (\alpha_1 + t\gamma_1, \dots, \alpha_n + t\gamma_n)$$

and the matrix $(u_{i,j})$ of $u \in \mathbf{U}_{\mathcal{F}}$ in the basis (v_1, \dots, v_n) of V satisfies $u_{i,i} = 1$ and $u_{i,j} \neq 0$ if and only if $\gamma_i > \gamma_j$ for $i \neq j$. It fixes α if and only if $\alpha_j - \alpha_i \leq -\log |u_{i,j}|$ for all $1 \leq i, j \leq n$ by [25, 3.5]. It thus fixes $\alpha + t\mathcal{F}$ for all $t \gg 0$.

Let $\mathbf{G}(Z) \simeq \mathbb{R}$ be the isomorphism which maps \mathcal{G} to the unique weight $\gamma(\mathcal{G})$ of the corresponding representation of $\mathbb{D}_K(\mathbb{R})$ on V . The projection $\mathbf{G}(S) \twoheadrightarrow \mathbf{G}(Z)$ maps \mathcal{F} to the unique \mathcal{G} with $\gamma(\mathcal{G}) = \frac{1}{n} \sum_{L \in \mathcal{S}} \gamma_L(\mathcal{F})$. It follows that the projection

$$\mathbf{d}^c : \mathbf{B}(V) \times \mathbf{B}(V) \rightarrow \mathbf{G}(Z)$$

of the distance $\mathbf{d} : \mathbf{B}(V) \times \mathbf{B}(V) \rightarrow \mathbf{C}(G)$ maps (α, β) to the unique \mathcal{G} with

$$\gamma(\mathcal{G}) = \frac{1}{n} \sum_{i=1}^n \log \alpha(v_i) - \log \beta(v_i)$$

for any K -basis (v_1, \dots, v_n) of V such that $\alpha, \beta \in \mathbf{B}(\mathcal{S})$ with $\mathcal{S} = \{Kv_1, \dots, Kv_n\}$. From [25, 3.2], we deduce that the morphism

$$\nu_{\mathbf{B}}^c : \mathbf{G} \rightarrow \mathbf{G}(Z_G)$$

maps g to the unique \mathcal{G} with $\gamma(\mathcal{G}) = \frac{1}{n} \log |\det g|$. In particular, $|\det \mathbf{G}_{\alpha}| = 1$ for every $\alpha \in \mathbf{B}(V)$, and then [25, 3.5] implies **A8**: $\mathbf{G}_{\alpha} = \mathbf{G}_{S,\alpha}$ for all $\alpha \in \mathbf{B}(S)$.

Therefore $\mathbf{B}(V)$ is a tight $\mathbf{F}(V)$ -building. For a more general case, see [9].

4.6.2. *The Bruhat-Tits building of G .* For a general reductive group G over K , we have to make some assumption on the triple $(G, K, |\cdot|)$: the existence of a *valuation* on the root datum $(Z_G(S), (U_a)_{a \in \Phi(G,S)})$ of $\mathbf{G} = G(K)$, in the sense of [7, 6.2.1]. Let then $\mathbf{B}^r(G)$ and $\mathbf{B}^e(G) = \mathbf{B}^r(G) \times \mathbf{G}(Z)$ be respectively the reduced and extended Bruhat-Tits buildings of G , as defined in [7, §7] and [8, 4.2.16 & 5.1.29].

These two sets have compatible actions of \mathbf{G} , they are covered by apartments $\mathbf{B}^r(S)$ and $\mathbf{B}^e(S) = \mathbf{B}^r(S) \times \mathbf{G}(Z)$ which are \mathbf{G} -equivariantly parametrized by $\mathbf{S}(G)$, $\mathbf{B}^e(S)$ is an affine $\mathbf{F}(S)$ -space on which $\mathbf{N}_G(S)$ acts by affine transformations with linear part $\nu_S^v : \mathbf{N}_G(S) \rightarrow \mathbf{W}_G(S)$ and the resulting action of $Z_G(S)$ is given by a morphism $\nu_{\mathbf{B},S} : Z_G(S) \rightarrow \mathbf{G}(S)$ which is uniquely characterized by the following property: for every morphism $\chi : Z_G(S) \rightarrow \mathbb{G}_{m,K}$, the induced morphism

$$\mathbf{G}(\chi|_S) \circ \nu_{\mathbf{B},S} : Z_G(S) \rightarrow \mathbf{G}(\mathbb{G}_{m,K})$$

maps z in $Z_G(S)$ to $\log |\chi(z)|$ in $\mathbb{R} = \mathbf{G}(\mathbb{G}_{m,K})$. Similarly, the action of \mathbf{G} on $\mathbf{G}(Z)$ is given by a morphism $\nu_{\mathbf{B}}^c : \mathbf{G} \rightarrow \mathbf{G}(Z)$ which is uniquely characterized by the following property: for every morphism $\chi : G \rightarrow \mathbb{G}_{m,K}$, the induced morphism

$$\mathbf{G}(\chi|Z) \circ \nu_{\mathbf{B}}^c : \mathbf{G} \rightarrow \mathbf{G}(\mathbb{G}_{m,K})$$

maps g in \mathbf{G} to $\log |\chi(g)|$ in $\mathbb{R} = \mathbf{G}(\mathbb{G}_{m,K})$. There is a \mathbf{G} -equivariant distance

$$d : \mathbf{B}^e(G) \times \mathbf{B}^e(G) \rightarrow \mathbb{R}^+$$

inducing an Euclidean distance on each apartment, which turns $\mathbf{B}^e(G)$ into a CAT(0)-space. Finally, $\mathbf{B}^e(G)$ already satisfies our axiom **A1** by [7, 7.4.18.i] as well as the following strong form of **A8** and **A3!**:

For every subset $\Omega \neq \emptyset$ of $\mathbf{B}^e(S)$, the pointwise stabilizer $\mathbf{G}_\Omega \subset \mathbf{G}$ of Ω equals $\mathbf{G}_{S,\Omega}$ by [7, 7.4.4], and it acts transitively on the set of apartments containing Ω by [7, 7.4.9].

We denote by $+_S : \mathbf{B}^e(S) \times \mathbf{F}(S) \rightarrow \mathbf{B}^e(S)$ the given structure of affine $\mathbf{F}(S)$ -space on $\mathbf{B}^e(S)$. These maps are compatible in the following sense:

$$g \cdot (x +_S \mathcal{F}) = (g \cdot x +_{g \cdot S} g \cdot \mathcal{F}).$$

Let us first prove **A2**: starting with $x \in \mathbf{B}^e(G)$ and $\mathcal{F} \in \mathbf{F}(G)$, choose a minimal parabolic subgroup $B' \subset P_{\mathcal{F}}$ with Levi $Z_G(S')$, pick $c \in \mathbf{B}^e(S')$ and form the sector $C' = c +_{S'} F^{-1}(B')$ in $\mathbf{B}^e(S')$. By [7, 7.4.18.ii], there is another apartment $\mathbf{B}^e(S)$ containing x and a subsector C of C' , which *a priori* is of the form $C = c +_S F^{-1}(B)$ for some minimal parabolic B with Levi $Z_G(S)$. Since $C \subset \mathbf{B}^e(S) \cap \mathbf{B}^e(S')$, there is a $g \in \mathbf{G}$ fixing C (thus also \overline{C}) and mapping S to S' . Since $\mathbf{G}_C = \mathbf{G}_{S,C} \subset \mathbf{B} \cap \mathbf{G}_c$, $\text{Int}(g)(B) = B$ and $Z_G(S') \subset B$. Moreover $C = gC = c +_{S'} F^{-1}(B)$, thus actually $B = B'$ because $c +_{S'} F^{-1}(B) \subset c' +_{S'} F^{-1}(B')$ in the affine $\mathbf{F}(S)$ -space $\mathbf{B}^e(S')$. Now $x \in \mathbf{B}^e(S)$ and $\mathcal{F} \in \mathbf{F}(S)$ since $Z_G(S) \subset B \subset P_{\mathcal{F}}$, which proves **A2**. Note also that for any \mathcal{G} in the closure of $F^{-1}(B)$, $c +_S \mathcal{G} = g(c +_S \mathcal{G}) = c +_{S'} \mathcal{G}$. In particular, $c +_S t\mathcal{F} = c +_{S'} t\mathcal{F}'$ for every $t \in \mathbb{R}_+$.

Suppose now $x \in \mathbf{B}^e(S_1) \cap \mathbf{B}^e(S_2)$ and $\mathcal{F} \in \mathbf{F}(S_1) \cap \mathbf{F}(S_2)$ with $S_1, S_2 \in \mathbf{S}(G)$. We would like to show that $x +_{S_1} \mathcal{F} = x +_{S_2} \mathcal{F}$. Put $x_i(t) = x +_{S_i} t\mathcal{F}$ for $t \geq 0$. The CAT(0)-property of d implies that $t \mapsto d(x_1(t), x_2(t))$ is a (continuous) convex function, and it is therefore sufficient to show that it is also bounded. Let us choose minimal parabolic subgroups $Z_G(S_i) \subset B_i \subset P_{\mathcal{F}}$, and form the corresponding sectors $C_i = x +_{S_i} F^{-1}(B_i)$ in $\mathbf{B}^e(S_i)$. By [7, 7.4.18.iii], there is another appartement $\mathbf{B}^e(S)$ which contains subsectors $C'_1 \subset C_1$ and $C'_2 \subset C_2$. We have just

seen that then $Z_G(S) \subset B_i$ (thus $\mathcal{F} \in \mathbf{F}(S)$) and $C'_i = y_i +_S F^{-1}(B_i)$ in $\mathbf{B}^e(S)$ for some y_i 's in $\mathbf{B}^e(S)$, with moreover $y_i(t) = y_i +_S t\mathcal{F} = y_i +_{S_i} t\mathcal{F}$ for $t \geq 0$. Then $t \mapsto d(y_1(t), y_2(t))$ and $t \mapsto d(x_i(t), y_i(t))$ are constant, thus $t \mapsto d(x_1(t), x_2(t))$ is indeed bounded.

We may at last define our $+$ map: for $x \in \mathbf{B}^e(G)$ and $\mathcal{F} \in \mathbf{F}(G)$, choose $S \in \mathbf{S}(G)$ with $x \in \mathbf{B}^e(S)$ and $\mathcal{F} \in \mathbf{F}(S)$ and set $x + \mathcal{F} = x +_S \mathcal{F}$. It is plainly G -equivariant, and induces the given structure of affine $\mathbf{F}(S)$ -space on $\mathbf{B}^e(S)$.

Now for $x \in \mathbf{B}^e(G)$ and $\mathcal{F}, \mathcal{G} \in \mathbf{F}(G)$, choose $S \in \mathbf{S}(G)$ with $x \in \mathbf{B}^e(S)$, $\mathcal{F} \in \mathbf{F}(S)$, let F be the ‘‘facet’’ in $\mathbf{B}^e(S)$ denoted by $\gamma(x, E)$ in [5, 7.2.4] with $E = \{t\mathcal{F} : t > 0\}$, let C be a ‘‘chamber’’ of $\mathbf{B}^e(S)$ containing F in its closure. Using [7, 7.4.18.ii] as above, we find that there is an apartment $\mathbf{B}^e(S')$ containing C with $\mathcal{G} \in \mathbf{F}(S')$. It then also contains F by [7, 7.4.8], which means that for some $\epsilon > 0$, it contains $x + \eta\mathcal{G}$ for every $\eta \in [0, \epsilon]$: this proves **A2!**

We already have **A3!**, **A8** (thus **A6!**). Then **A4** and **A5** come along by lemma 68: $\mathbf{B}^e(G)$ is a tight affine $\mathbf{F}(G)$ -building. Finally **A7** also holds, by [25, 1.4] and [7, 7.4.19]. For $G = GL(V)$, $\mathbf{B}^e(G) \simeq \mathbf{B}(V)$ by lemma 70 (see also [14, 9, 25]).

4.6.3. Suppose that the valuation ring $\mathcal{O} = \{x \in K : |x| \leq 1\}$ is Henselian. Then for every algebraic extension L of K , there is a unique extension of $|\cdot| : K \rightarrow \mathbb{R}_+$ to an absolute value $|\cdot| : L \rightarrow \mathbb{R}_+$ on L , and the corresponding valuation ring $\mathcal{O}_L = \{x \in L : |x| \leq 1\}$ is the integral closure of \mathcal{O} in L . We say that L/K has a property \mathcal{P} over \mathcal{O} if the corresponding morphism $\text{Spec}(\mathcal{O}_L) \rightarrow \text{Spec}(\mathcal{O})$ does.

Proposition 72. *Let G be a reductive group over \mathcal{O} .*

- (1) *There is an extension L/K , finite étale and Galois over \mathcal{O} , splitting G .*
- (2) *The Bruhat-Tits building $\mathbf{B}^e(G_K)$ exists and contains a canonical point*

$$\circ_{G,K}^e = \circ_G^e = (\circ_G^r, 0) \in \mathbf{B}^e(G_K) = \mathbf{B}^r(G_K) \times \mathbf{G}(Z(G_K))$$

with stabilizer $G(\mathcal{O})$ in $G(K)$. The projection \circ_G^r of \circ_G^e is the unique fixed point of $G(\mathcal{O})$ in $\mathbf{B}^r(G_K)$ if the residue field of \mathcal{O} is neither \mathbb{F}_2 nor \mathbb{F}_3 .

- (3) *The apartments of $\mathbf{B}^e(G_K)$ containing \circ_G^e are the $\mathbf{B}^e(S_K)$'s for $S \in \mathbf{S}(G)$.*

Proof. Let S be a maximal split torus of G and let T be a maximal torus of $Z_G(S)$ [1, XIV 3.20]. Then G and T are isotrivial by Proposition 33, thus split by a finite étale cover of $\text{Spec}(\mathcal{O})$ which we may assume to be connected and Galois, i.e. of the form $\text{Spec}(\mathcal{O}_L) \rightarrow \text{Spec}(\mathcal{O})$ where \mathcal{O}_L is the normalization of \mathcal{O} in a finite étale Galois extension L/K over \mathcal{O} by [19, 18.10.12]. Since \mathcal{O} is Henselian, \mathcal{O}_L is also the valuation ring of $(L, |\cdot|)$. Let (x_α) be a Chevalley system for $(G_{\mathcal{O}_L}, T_{\mathcal{O}_L})$, as defined in [12, XXIII 6.2], giving rise to a Chevalley valuation φ_L for G_L , as explained in [7, 6.2.3.b] and [8, 4.2.1], thus also to the reduced Bruhat-Tits building $\mathbf{B}^r(G_L)$ with its distinguished apartment $\mathbf{B}^r(T_L)$ and the distinguished point $\circ_G^r \equiv \varphi_L$ in $\mathbf{B}^r(T_L)$, as defined in [7, §7]. For $f = 0$, the group schemes $\mathfrak{G}_f^0 \subset \mathfrak{G}_f \subset \hat{\mathfrak{G}}_f \subset \mathfrak{G}_f^\dagger$ constructed in [8, 4.3-6] are all equal to $G_{\mathcal{O}_L}$ [8, 4.6.22]. Thus by [8, 4.6.28], $G(\mathcal{O}_L)$ is the stabilizer of the distinguished point $\circ_G^e = (\circ_G^r, 0)$ of $\mathbf{B}^e(T_L) \subset \mathbf{B}^e(G_L)$ in $G(L)$, and \circ_G^r is the unique fixed point of $G(\mathcal{O}_L)$ in $\mathbf{B}^r(G_L)$ by [8, 5.1.39] if the residue field of \mathcal{O}_L is not equal to \mathbb{F}_2 or \mathbb{F}_3 , which we can always assume.

The pair (G_K, K) satisfies the conditions of the pair denoted by (H, K^\natural) in [8, 5.1.1]. The Galois group $\Sigma = \text{Gal}(L/K)$ acts compatibly on $G(L)$ and $\mathbf{B}^e(G_L)$. It therefore fixes \circ_G^e , which thus belongs to $\mathbf{B}^e(T_L)^\Sigma = \mathbf{B}^e(S_K)$. Applying this to $Z_G(S)$ instead of G , we see that (G_K, K) also satisfies the assumption (DE) of [8,

5.1.5]. Then by [8, 5.1.20], the valuation φ_L descends to a valuation φ for G_K . The corresponding building $\mathbf{B}^e(G_K)$ is the fixed point set of Σ in $\mathbf{B}^e(G_L)$ by [8, 5.1.25]. The stabilizer of $\circ_G^e \in \mathbf{B}^e(G_K)$ in $G(K)$ equals $G(\mathcal{O}) = G(K) \cap G(\mathcal{O}_L)$ and again by [8, 5.1.39], \circ_G^e is the unique fixed point of $G(\mathcal{O})$ in $\mathbf{B}^r(G_K)$ if the residue field of \mathcal{O} is not equal to \mathbb{F}_2 or \mathbb{F}_3 . By construction, \circ_G^e belongs to $\mathbf{B}^e(S_K)$. Therefore [7, 7.4.9] proves our last claim, since $G(\mathcal{O})$ also acts transitively on $\mathbf{S}(G)$. \square

4.6.4. We denote by $\mathbf{B}^e(G, K, |-\!|)$ the pointed affine $\mathbf{F}(G_K)$ -building

$$\mathbf{B}^e(G, K, |-\!|) = (\mathbf{B}^e(G_K), \circ_G^e)$$

attached to a reductive group G over \mathcal{O} . It easily follows from [8, 5.1.41] that this construction is functorial in the Henselian pair $(K, |-\!|)$. More precisely, let \mathbf{HV} be the category whose objects are pairs $(K, |-\!|)$ where K is a field and $|-\!| : K \rightarrow \mathbb{R}_+$ is a non-trivial, non-archimedean absolute value whose valuation ring \mathcal{O}_K is Henselian. Then for every morphism $f : (K, |-\!|) \rightarrow (L, |-\!|)$ in \mathbf{HV} and every reductive group G over \mathcal{O}_K , there is a canonical morphism $f : \mathbf{B}^e(G_K) \rightarrow \mathbf{B}^e(G_L)$ such that

$$f(\circ_G^e) = \circ_G^e, \quad f(gx) = f(g)f(x) \quad \text{and} \quad f(x + \mathcal{F}) = f(x) + f(\mathcal{F})$$

for every $x \in \mathbf{B}^e(G_K)$, $g \in G(K)$ and $\mathcal{F} \in \mathbf{F}(G_K)$. The first and last property already determine f uniquely: by the axiom **A1** for $\mathbf{B}^e(G_K)$, any element x of $\mathbf{B}^e(G_K)$ equals $\circ_G^e + \mathcal{F}$ for some $\mathcal{F} \in \mathbf{F}(G_K)$.

Remark 73. The above functoriality amounts to saying that the mapping

$$\mathbf{B}^e(G_K) \ni \circ_G^e + \mathcal{F} \mapsto \circ_G^e + f(\mathcal{F}) \in \mathbf{B}^e(G_L)$$

is well-defined and equivariant with respect to $G(K) \rightarrow G(L)$. This indeed implies the equivariance with respect to $f : \mathbf{F}(G_K) \rightarrow \mathbf{F}(G_L)$ as follows. For $S \in \mathbf{S}(G)$ mapping into $S' \in \mathbf{S}(G_{\mathcal{O}_L})$, the above mapping restricts to a well-defined map $\mathbf{B}^e(S_K) \rightarrow \mathbf{B}^e(S'_L)$ which is equivariant with respect to $f : \mathbf{F}(S_K) \rightarrow \mathbf{F}(S'_K)$; by the axiom **A2** for $\mathbf{B}^e(G_K)$ and Proposition 56, any pair (x, \mathcal{F}) in $\mathbf{B}^e(G_K) \times \mathbf{F}(G_K)$ is conjugated by some $g \in G(K)$ to one in $\mathbf{B}^e(S_K) \times \mathbf{F}(S_K)$, thus

$$f(x + \mathcal{F}) = f(g^{-1})f(gx + g\mathcal{F}) = f(g^{-1})(f(gx) + f(g\mathcal{F})) = f(x) + f(\mathcal{F}).$$

Theorem 74. *The pointed affine $\mathbf{F}(G)$ -building $\mathbf{B}^e(G, K, |-\!|)$ is also functorial in the reductive group G over \mathcal{O}_K : for every morphism $f : G \rightarrow H$ of reductive groups over \mathcal{O}_K , there is a unique morphism $f : \mathbf{B}^e(G_K) \rightarrow \mathbf{B}^e(H_K)$ such that*

$$f(\circ_G^e) = \circ_H^e, \quad f(gx) = f(g)f(x) \quad \text{and} \quad f(x + \mathcal{F}) = f(x) + f(\mathcal{F})$$

for every $x \in \mathbf{B}^e(G_K)$, $g \in G(K)$ and $\mathcal{F} \in \mathbf{F}(G_K)$.

This essentially follows from Landvogt's work in [21], which has no assumptions on the reductive groups over K but requires $(K, |-\!|)$ to be quasi-local, in particular discrete. The main difficulty there is the construction of base points with good properties, which is here trivialized by the given points \circ_G^e and \circ_H^e . Note that again, the uniqueness of $f : \mathbf{B}^e(G_K) \rightarrow \mathbf{B}^e(H_K)$ follows from the first and last displayed requirements, and its existence amounts to showing that the mapping

$$\mathbf{B}^e(G_K) \ni \circ_G^e + \mathcal{F} \mapsto \circ_H^e + f(\mathcal{F}) \in \mathbf{B}^e(H_K)$$

is well-defined and equivariant with respect to $f : G(K) \rightarrow H(K)$. Given the identification $\mathbf{B}^e(GL(V)) \simeq \mathbf{B}(V)$, the above theorem is closely related to the Tannakian theorem 77 below. We will prove the former as a corollary of the latter.

4.6.5. For every $\nu > 0$, there is a $G(K)$ -equivariant commutative diagram

$$\begin{array}{ccccc} \mathbf{B}^e(G, K, |-|) & \times & \mathbf{F}(G_K) & \xrightarrow{+} & \mathbf{B}^e(G, K, |-|) \\ a \downarrow & & b \downarrow & & a \downarrow \\ \mathbf{B}^e(G, K, |-|^\nu) & \times & \mathbf{F}(G_K) & \xrightarrow{+} & \mathbf{B}^e(G, K, |-|^\nu) \end{array}$$

where a is a canonical $G(K)$ -equivariant map and $b(\mathcal{F}) = \nu\mathcal{F}$.

4.7. A Tannakian formalism for Bruhat-Tits buildings.

4.7.1. Let again $(K, |-|)$ be a field with a non-trivial, non-archimedean absolute value $|-| : K \rightarrow \mathbb{R}_+$, with valuation ring $\mathcal{O} = \mathcal{O}_K$ and residue field k . We denote by $\text{Norm}^\circ(K, |-|)$ the category whose objects are pairs (V, α) where V is a finite dimensional K -vector space and $\alpha : V \rightarrow \mathbb{R}^+$ is a splittable K -norm on V . A morphism $f : (V, \alpha) \rightarrow (V', \alpha')$ is a K -linear morphism $f : V \rightarrow V'$ such that $\alpha'(f(x)) \leq \alpha(x)$ for every $x \in V$. This is an \mathcal{O} -linear rigid \otimes -category with neutral object $1_K = (K, |-|)$. The \otimes -products, inner homs and duals

$$\begin{aligned} (V_1, \alpha_1) \otimes (V_2, \alpha_2) &= (V_1 \otimes V_2, \alpha_1 \otimes \alpha_2) \\ \text{Hom}((V_1, \alpha_1), (V_2, \alpha_2)) &= (\text{Hom}(V_1, V_2), \text{Hom}(\alpha_1, \alpha_2)) \\ (V, \alpha)^* &= (V^*, \alpha^*) \end{aligned}$$

are respectively defined by : $\alpha_1 \otimes \alpha_2 = \text{Hom}(\alpha_1^*, \alpha_2)$ under $V_1 \otimes V_2 = \text{Hom}(V_1^*, V_2)$,

$$\begin{aligned} \text{Hom}(\alpha_1, \alpha_2)(f) &= \sup \left\{ \frac{\alpha_2(f(x))}{\alpha_1(x)} : x \in V_1 \setminus \{0\} \right\}, \\ \alpha^*(f) &= \sup \left\{ \frac{|f(x)|}{\alpha(x)} : x \in V \setminus \{0\} \right\}. \end{aligned}$$

In addition, $\text{Norm}^\circ(K, |-|)$ is an exact category in Quillen's sense: a short sequence

$$(V_1, \alpha_1) \xrightarrow{f_1} (V_2, \alpha_2) \xrightarrow{f_2} (V_3, \alpha_3)$$

is exact precisely when the underlying sequence of K -vector spaces is exact and

$$\alpha_1(x) = \alpha_2(f_1(x)), \quad \alpha_3(z) = \inf \{ \alpha_2(y) : y \in f_2^{-1}(z) \}$$

for every $x \in V_1$ and $z \in V_3$. For $\gamma \in \mathbb{R}$ and $(V, \alpha) \in \text{Norm}^\circ(K, |-|)$, we set

$$\begin{aligned} B(\alpha, \gamma) &= \{x \in V : \alpha(x) < \exp(-\gamma)\} \\ \overline{B}(\alpha, \gamma) &= \{x \in V : \alpha(x) \leq \exp(-\gamma)\} \end{aligned}$$

These are \mathcal{O} -submodules of V and the functors $(V, \alpha) \mapsto B(\alpha, \gamma)$ are easily seen to be exact. However, $(V, \alpha) \mapsto \overline{B}(\alpha, \gamma)$ is *also* exact, because in fact every exact sequence in $\text{Norm}^\circ(K)$ is split by [9, 1.5.ii + Appendix]! If M is an \mathcal{O} -lattice in V (by which we mean a finitely generated, thus free, \mathcal{O} -submodule spanning V), we denote by α_M the splittable K -norm on V with $\overline{B}(\alpha_M, 0) = M$ defined by

$$\alpha_M(x) = \inf \{ |\lambda| : \lambda \in K, x \in \lambda M \}.$$

4.7.2. If $(K, |-|) \rightarrow (L, |-|)$ is any morphism, there is an exact \mathcal{O} -linear \otimes -functor

$$- \otimes L : \text{Norm}^\circ(K, |-|) \rightarrow \text{Norm}^\circ(L, |-|)$$

defined by $(V, \alpha) \otimes L = (V_L, \alpha_L)$ where $V_L = V \otimes L$ and

$$\alpha_L(v) = \inf \{ \max\{|x_k| \alpha(v_k)\} : v = \sum v_k \otimes x_k, v_k \in V, x_k \in L \}.$$

For $(V, \alpha) \in \text{Norm}^\circ(K, |-|)$, $\gamma \in \mathbb{R}$ and $x \in V$,

$$B(\alpha_L, \gamma) = B(\alpha, \gamma) \otimes \mathcal{O}_L, \quad \overline{B}(\alpha_L, \gamma) = \overline{B}(\alpha, \gamma) \otimes \mathcal{O}_L \quad \text{and} \quad \alpha = \alpha_L|_V.$$

If M is an \mathcal{O} -lattice in V , then $\alpha_{M,L} = \alpha_{M \otimes \mathcal{O}_L}$.

4.7.3. We shall also consider the category $\text{Norm}'(K)$ whose objects are triples (V, α, M) where (V, α) is an object of $\text{Norm}^\circ(K)$ and M is an \mathcal{O} -lattice in V , with the obvious morphisms. It is again an \mathcal{O} -linear \otimes -category. The formula

$$\text{loc}^\gamma(V, \alpha, M) = \text{image of } \overline{B}(\alpha, \gamma) \cap M \text{ in } M_k = M \otimes_{\mathcal{O}} k$$

defines an \mathcal{O} -linear \otimes -functor with values in $\text{Fil}(k) = \text{Fil}^{\mathbb{R}} \text{LF}(k)$,

$$\text{loc} : \text{Norm}'(K) \rightarrow \text{Fil}(k).$$

Indeed by the axiom **A1** for $\mathbf{B}(V)$, we may find an \mathcal{O} -basis (e_1, \dots, e_n) of M which is adapted to α , thus $\alpha(\sum x_i e_i) = \max\{|x_i| e^{-\gamma_i}\}$ where $\gamma_i = -\log \alpha(e_i)$ and

$$\text{loc}^\gamma(V, \alpha, M) = \oplus_{\gamma_i \geq \gamma} k e_i$$

from which easily follows that loc is well-defined and compatible with \otimes -products.

4.7.4. For an extension $(K, |-|) \rightarrow (L, |-|)$ and a reductive group G over \mathcal{O}_K , we denote by $\mathbf{B}'(\omega_G^\circ, L, |-|)$ or simply $\mathbf{B}'(\omega_G^\circ, L)$ the set of all factorizations

$$\text{Rep}^\circ(G)(\mathcal{O}_K) \xrightarrow{\alpha} \text{Norm}^\circ(L, |-|) \xrightarrow{\text{forg}} \text{Vect}(L)$$

of the fiber functor $\omega_{G,L}^\circ$ through an \mathcal{O}_K -linear \otimes -functor α . For $\tau \in \text{Rep}^\circ(G)(\mathcal{O}_K)$ and $\alpha \in \mathbf{B}'(\omega_G^\circ, L)$, we denote by $\alpha(\tau)$ the corresponding L -norm on $V_L(\tau)$.

4.7.5. For $g \in G(L)$ and $\mathcal{F} \in \mathbf{F}(G_L)$, the following formulas

$$(g \cdot \alpha)(\tau) = \tau_L(g) \cdot \alpha(\tau) \quad \text{and} \quad (\alpha + \mathcal{F})(\tau) = \alpha(\tau) + \mathcal{F}(\tau)$$

respectively define an action of $G(L)$ on $\mathbf{B}'(\omega_G^\circ, L)$ and a $G(L)$ -equivariant map

$$+ : \mathbf{B}'(\omega_G^\circ, L) \times \mathbf{F}(G_L) \rightarrow \mathbf{B}'(\omega_G^\circ, L).$$

4.7.6. We define the canonical L -norm $\alpha_{G,L}$ on $\omega_{G,L}^\circ$ by the formula

$$\alpha_{G,L}(\tau) = \alpha_{V_{\mathcal{O}_L}(\tau)} = \alpha_{V(\tau),L}.$$

By Propositions 30 and 33, $G(\mathcal{O}_L)$ is the stabilizer of $\alpha_{G,L}$ in $G(L)$. We set

$$\mathbf{B}(\omega_G^\circ, L) \stackrel{\text{def}}{=} \alpha_{G,L} + \mathbf{F}(G_L).$$

This is a $G(\mathcal{O}_L)$ -stable subset of $\mathbf{B}'(\omega_G^\circ, L)$ equipped with a $G(\mathcal{O}_L)$ -equivariant map

$$\text{can} : \mathbf{F}(G_L) \rightarrow \mathbf{B}(\omega_G^\circ, L).$$

4.7.7. Any L -norm α on $\omega_{G,L}^\circ$ induces an \mathcal{O} -linear \otimes -functor

$$\alpha' : \text{Rep}^\circ(G)(\mathcal{O}) \rightarrow \text{Norm}'(L)$$

by the formula $\alpha'(\tau) = (V_L(\tau), \alpha(\tau), V_{\mathcal{O}_L}(\tau))$, thus also an \mathcal{O} -linear \otimes -functor

$$\text{loc}(\alpha) : \text{Rep}^\circ(G)(\mathcal{O}) \rightarrow \text{Fil}(k_L), \quad \text{loc}(\alpha) = \text{loc} \circ \alpha'$$

where k_L is the residue field of \mathcal{O}_L . We may thus define

$$\mathbf{B}^2(\omega_G^\circ, L) = \{\alpha \in \mathbf{B}'(\omega_G^\circ, L) : \text{loc}(\alpha) \text{ is exact}\}.$$

This is a $G(\mathcal{O}_L)$ -stable subset of $\mathbf{B}'(\omega_G^\circ, L)$ equipped with a $G(\mathcal{O}_L)$ -equivariant map

$$\text{loc} : \mathbf{B}^2(\omega_G^\circ, L) \rightarrow \mathbf{F}(G_{k_L}).$$

4.7.8. All of the above constructions are functorial in G , $(K, |-\cdot|)$ and $(L, |-\cdot|)$, using pre- or post-composition with the obvious exact \otimes -functors

$$\begin{array}{ccc} \text{Rep}^\circ(G_2)(\mathcal{O}_K) & \longrightarrow & \text{Rep}^\circ(G_1)(\mathcal{O}_K) & & G_1 & \longrightarrow & G_2 \\ \text{Rep}^\circ(G)(\mathcal{O}_{K_1}) & \longrightarrow & \text{Rep}^\circ(G)(\mathcal{O}_{K_2}) & \text{ for } & (K_1, |-\cdot|_1) & \longrightarrow & (K_2, |-\cdot|_2) \\ \text{Norm}^\circ(L_1, |-\cdot|_1) & \longrightarrow & \text{Norm}^\circ(L_2, |-\cdot|_2) & & (L_1, |-\cdot|_1) & \longrightarrow & (L_2, |-\cdot|_2) \end{array}$$

Lemma 75. *For any reductive group G over \mathcal{O}_K , we have*

$$\mathbf{B}(\omega_G^\circ, L) \subset \mathbf{B}^2(\omega_G^\circ, L) \subset \mathbf{B}'(\omega_G^\circ, L)$$

and the composition $\text{loc} \circ \text{can} : \mathbf{F}(G_L) \rightarrow \mathbf{F}(G_{k_L})$ is the reduction map

$$\mathbf{F}(G_L) \xleftarrow{\simeq} \mathbf{F}(G_{\mathcal{O}_L}) \xrightarrow{\text{red}} \mathbf{F}(G_{k_L}).$$

For any $S \in \mathbf{S}(G_{\mathcal{O}_L})$, the functorial map $\mathbf{B}'(\omega_S^\circ, L) \rightarrow \mathbf{B}'(\omega_G^\circ, L)$ is injective.

Proof. By proposition 56, any $\mathcal{F} \in \mathbf{F}(G_L)$ belongs to $\mathbf{F}(S_L)$ for some $S \in \mathbf{S}(G_{\mathcal{O}_L})$. Pre-composing with $\text{Rep}^\circ(G)(\mathcal{O}_K) \rightarrow \text{Rep}^\circ(S)(\mathcal{O}_L)$ yields a commutative diagram

$$\begin{array}{ccccccc} \mathbf{F}(S_L) & \xrightarrow{\text{can}} & \mathbf{B}(\omega_S^\circ, L) & \hookrightarrow & \mathbf{B}'(\omega_S^\circ, L) & \longleftarrow & \mathbf{B}^2(\omega_S^\circ, L) \xrightarrow{\text{loc}} \mathbf{F}(S_{k_L}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathbf{F}(G_L) & \xrightarrow{\text{can}} & \mathbf{B}(\omega_G^\circ, L) & \hookrightarrow & \mathbf{B}'(\omega_G^\circ, L) & \longleftarrow & \mathbf{B}^2(\omega_G^\circ, L) \xrightarrow{\text{loc}} \mathbf{F}(G_{k_L}) \end{array}$$

which reduces us to the case $K = L$, $G = S$ treated below. \square

Lemma 76. *Suppose that $G = S$ is a split torus. Then all maps in*

$$\mathbf{F}(S_L) \xrightarrow{\text{can}} \mathbf{B}(\omega_S^\circ, L) \hookrightarrow \mathbf{B}'(\omega_S^\circ, L) \longleftarrow \mathbf{B}^2(\omega_S^\circ, L) \xrightarrow{\text{loc}} \mathbf{F}(S_{k_L})$$

are isomorphisms of pointed affine $\mathbf{G}(S)$ -spaces. Moreover, $S(L)$ acts on

$$\mathbf{B}(\omega_S^\circ, L) = \mathbf{B}^2(\omega_S^\circ, L) = \mathbf{B}'(\omega_S^\circ, L)$$

by translations through the morphism

$$\nu_{\mathbf{B}, S} : S(L) \rightarrow \mathbf{G}(S)$$

which maps $s \in S(L)$ to the unique morphism $\nu_{\mathbf{B}, S}(s) : \mathbb{D}_{\mathcal{O}}(\mathbb{R}) \rightarrow S$ whose composite with any character χ of S is the character $\log |\chi(z)| \in \mathbb{R}$ of $\mathbb{D}_{\mathcal{O}}(\mathbb{R})$.

Proof. Put $M = \text{Hom}(S, \mathbb{G}_{m, \mathcal{O}})$ and let ρ_m be the representation of S on $\mathcal{O} = \mathcal{O}_K$ given by the character $m \in M$. For $\tau \in \text{Rep}^\circ(S)(\mathcal{O})$, let $\tau = \oplus \tau_m$ be the weight decompositions of τ . Recall from section 3.10.6 that the formulas

$$\mathcal{F}^\gamma(\tau) = \oplus_{\mathcal{F}^\#(m) \geq \gamma} V(\tau_m), \quad \mathcal{F}^\#(m) = \sup\{\gamma : \mathcal{F}^\gamma(\rho_m) \neq 0\}$$

yield isomorphisms between $\mathbf{F}(S) = \mathbf{G}(S)$ and $\text{Hom}(M, \mathbb{R})$. Similarly, the formulas

$$\alpha(\tau)(x) = \max \left\{ e^{-\alpha^\#(m)} \alpha_{V_{\mathcal{O}_L}(\tau_m)}(x_m) : m \in M \right\}, \quad \alpha^\#(m) = -\log \alpha(\rho_m)(1_{\mathcal{O}})$$

where $x = \sum x_m$ is the decomposition of x in $V_L(\tau) = \oplus V_L(\tau_m)$ yield isomorphisms between $\mathbf{B}'(\omega_S^\circ, L)$ and $\text{Hom}(M, \mathbb{R})$. One then checks easily that

$$\alpha_{S, L}^\# = 0, \quad (\alpha + \mathcal{F})^\# = \alpha^\# + \mathcal{F}^\# \quad \text{and} \quad s \cdot \alpha = \alpha + \nu_{\mathbf{B}, S}(s)$$

as well as $\text{loc}^\gamma(\alpha)(\tau) = \oplus_{\alpha^\#(m) \geq \gamma} V_{k_L}(\tau_m)$, from which the lemma follows. \square

4.7.9. For $S \in \mathbf{S}(G_{\mathcal{O}_L})$, we identify $\mathbf{B}(\omega_S^\circ, L)$ with its image in $\mathbf{B}(\omega_G^\circ, L)$ and call it the apartment attached to S . The $+$ -map on $\mathbf{B}'(\omega_G^\circ, L)$ thus induces a structure of affine $\mathbf{F}(S_L)$ -space on $\mathbf{B}(\omega_S^\circ, L)$, and the action of $G(L)$ on $\mathbf{B}'(\omega_G^\circ, L)$ restricts to an action of $S(L)$ on $\mathbf{B}(\omega_S^\circ, L)$, by translations through the above morphism $\nu_{\mathbf{B}, S}$.

4.7.10. We now restrict our attention to Henselian fields, so that $\mathbf{B}^e(G, L, |-|)$ is also well-defined, functorial in $(L, |-|)$, and equal to $\circ_G^e + \mathbf{F}(G_L)$ by **A1**. Given the functorial properties of $\mathbf{B}(\omega_G^\circ, L)$, theorem 74 immediately follows from:

Theorem 77. *The formula $\circ_G^e + \mathcal{F} \mapsto \alpha_{G, L} + \mathcal{F}$ defines a functorial bijection*

$$\alpha : \mathbf{B}^e(G, L, |-|) \rightarrow \mathbf{B}(\omega_G^\circ, L, |-|)$$

such that for every $x \in \mathbf{B}^e(G_L)$, $g \in G(L)$ and $\mathcal{F} \in \mathbf{F}(G_L)$,

$$\alpha(\circ_G^e) = \alpha_G, \quad \alpha(g \cdot x) = g \cdot \alpha(x) \quad \text{and} \quad \alpha(x + \mathcal{F}) = \alpha(x) + \mathcal{F}.$$

Proof. Fix $(L, |-|) \rightarrow (L', |-|)$ with valuation ring \mathcal{O}' such that $G' = G_{\mathcal{O}'}$ splits and consider the following diagram, where $\mathcal{F} \in \mathbf{F}(G_L)$ and $\mathcal{F}' \in \mathbf{F}(G_{L'}) = \mathbf{F}(G_{L'})$:

$$\begin{array}{ccccc} \circ_G^e + \mathcal{F} & \mathbf{B}^e(G, L) & \xrightarrow{\quad} & \mathbf{B}^e(G', L') & \circ_{G'}^e + \mathcal{F}' \\ \downarrow ? & \downarrow \alpha & \searrow \beta & \swarrow \beta' & \downarrow ? \\ \alpha_{G, L} + \mathcal{F} & \mathbf{B}'(\omega_G^\circ, L) & \xrightarrow{-\otimes L'} & \mathbf{B}'(\omega_G^\circ, L') & \leftarrow^{\text{Res}} & \mathbf{B}'(\omega_{G'}^\circ, L') & \downarrow \alpha' & \alpha_{G', L'} + \mathcal{F}' \end{array}$$

The bottom maps are respectively induced by post and pre-composition with

$$-\otimes L' : \text{Norm}^\circ(L) \rightarrow \text{Norm}^\circ(L') \quad \text{and} \quad -\otimes \mathcal{O}' : \text{Rep}^\circ(G)(\mathcal{O}) \rightarrow \text{Rep}^\circ(G')(\mathcal{O}').$$

If α' is well-defined and equivariant with respect to the operations of $G(L')$ and $\mathbf{F}(G_{L'})$, so is β' . Then β is well-defined and equivariant with respect to the operations of $G(L)$ and $\mathbf{F}(G_L)$. But $\mathbf{B}'(\omega_G^\circ, L) \rightarrow \mathbf{B}'(\omega_G^\circ, L')$ is injective, thus α is also well-defined and equivariant with respect to the operations of $G(L)$ and $\mathbf{F}(G_L)$. Its image equals $\mathbf{B}(\omega_G^\circ, L)$ by definition, which thus is stable under the operations of $G(L)$ and $\mathbf{F}(G_L)$ on $\mathbf{B}'(\omega_G^\circ, L)$. Since $\text{loc}(\alpha_{G, L} + \mathcal{F}) = \mathcal{F}_{k_L}$ for every $\mathcal{F} \in \mathbf{F}(G_L)$, the restriction of α to any apartment $\mathbf{B}^e(S_L) = \circ_G^e + \mathbf{F}(S_L)$ for $S \in \mathbf{S}(G_{\mathcal{O}_L})$ is injective. Since any pair of points in $\mathbf{B}^e(G, L)$ is $G(L)$ -conjugated to one in such an apartment by the axiom **A1** for $\mathbf{B}^e(G, L)$, $\alpha : \mathbf{B}^e(G, L) \rightarrow \mathbf{B}(\omega_G^\circ, L)$ is a bijection. This reduces us to the case where G is split over $K = L$.

Suppose that $\circ_G^e + \mathcal{F}_1 = \circ_G^e + \mathcal{F}_2 = x$ in $\mathbf{B}^e(G_K)$ for some $\mathcal{F}_1, \mathcal{F}_2 \in \mathbf{F}(G_K)$, choose $S_i \in \mathbf{S}(G_K)$ such that $\mathcal{F}_i \in \mathbf{F}(S_i)$ and $\circ_G^e \in \mathbf{B}^e(S_i)$ using **A2** for $\mathbf{B}^e(G_K)$, and then choose $g \in G(K)$ fixing \circ_G^e and x such that $\text{Int}(g)(S_1) = S_2$ using **A3** for $\mathbf{B}^e(G_K)$. Then $S_i \in \mathbf{S}(G)$ and $g \in G(\mathcal{O})$ by Proposition 72, moreover $g\mathcal{F}_1 = \mathcal{F}_2$ since $\mathbf{B}^e(S_2)$ is an affine $\mathbf{F}(S_2)$ -space. Thus $g(\alpha_G + \mathcal{F}_1) = \alpha_G + \mathcal{F}_2$ in $\mathbf{B}'(\omega_G^\circ, K)$, since $G(\mathcal{O})$ fixes α_G . But g fixes the point $x = \circ_G^e + \mathcal{F}_1$ of $\mathbf{B}^e(S_1)$, thus g fixes $\alpha_G + \mathcal{F}_1$ in $\mathbf{B}'(\omega_G^\circ, K)$ by Proposition 78 below, therefore $\alpha_G + \mathcal{F}_1 = \alpha_G + \mathcal{F}_2$ and our map $\alpha : \mathbf{B}^e(G, K) \rightarrow \mathbf{B}'(\omega_G^\circ, K)$ is indeed well-defined.

It is plainly $G(\mathcal{O})$ -equivariant. For any $S \in \mathbf{S}(G)$, the $G(K)$ -equivariant map α_S of Proposition 78 below coincides with α on $\mathbf{B}^e(S_K)$, thus α equals α_S everywhere since every point of $\mathbf{B}^e(G_K)$ is conjugated to one in $\mathbf{B}^e(S_K)$ by some element in $G(\mathcal{O})$. Therefore α is $G(K)$ -equivariant. Since every pair in $\mathbf{B}^e(G_K) \times \mathbf{F}(G_K)$ is conjugated to one in $\mathbf{B}^e(S_K) \times \mathbf{F}(S_K)$ by some element in $G(K)$, α is also compatible with the operations of $\mathbf{F}(G_K)$. \square

Proposition 78. *Suppose that G is split. Then for every $S \in \mathbf{S}(G)$, there exists a unique map $\alpha_S : \mathbf{B}^e(S_L) \rightarrow \mathbf{B}'(\omega_G^\circ, L)$ such that for all $x \in \mathbf{B}^e(S_L)$ and $\mathcal{F} \in \mathbf{F}(S_L)$,*

$$\alpha_S(\circ_G^e) = \alpha_{G,L} \quad \text{and} \quad \alpha_S(x + \mathcal{F}) = \alpha_S(x) + \mathcal{F}$$

It is $N_G(S)(L)$ -equivariant and extends uniquely to a $G(L)$ -equivariant map

$$\alpha_S : \mathbf{B}^e(G_L) \rightarrow \mathbf{B}'(\omega_G^\circ, L).$$

Proof. The uniqueness of both maps is obvious, the existence and equivariance under $S(L)$ of the first one follows from lemma 76. Since $G(\mathcal{O}_L)$ fixes \circ_G^e and $\alpha_{G,L}$, $\alpha_S : \mathbf{B}^e(S_L) \rightarrow \mathbf{B}'(\omega_G^\circ, L)$ is also equivariant for $N_G(S)(L) = N_G(S)(\mathcal{O}_L) \cdot S(L)$. In view of Remark 71, it remains to establish the following claim:

For any $\mathcal{F} \in \mathbf{F}(S_L)$, the stabilizer of $x = \circ_G^e + \mathcal{F} \in \mathbf{B}^e(G_L)$ in $G(L)$ is contained in the stabilizer of $\alpha = \alpha_{G,L} + \mathcal{F} \in \mathbf{B}'(\omega_G^\circ, L)$ in $G(L)$.

This is true for $\mathcal{F} = 0$, where both stabilizers equal $G(\mathcal{O}_L)$. This is therefore also true when $\mathcal{F} = \nu_{\mathbf{B},S}(s)$ for some $s \in S(L)$, since then $x = s \cdot \circ_G^e$ and $\alpha = s \cdot \alpha_{G,L}$. To clarify the proof, note that the base change maps from K to L identify

$$\begin{aligned} F &= \mathbf{F}(S_K) & \text{with} & & \mathbf{F}(S_L) &\subset & \mathbf{F}(G_L) \\ A &= \mathbf{B}^e(S_K) & \text{with} & & \mathbf{B}^e(S_L) &\subset & \mathbf{B}^e(G_L) \\ B &= \mathbf{B}(\omega_S^\circ, K) & \text{with} & & \mathbf{B}(\omega_S^\circ, L) &\subset & \mathbf{B}'(\omega_G^\circ, L) \end{aligned}$$

and the isomorphism of affine F -space $\alpha_S : A \rightarrow B$ also does not depend upon L . What does depend upon L is the subset $\Lambda(L) = \circ_G^e + \nu_{\mathbf{B},S}(\mathbf{S}(L))$ of A on which we already know the validity of our claim. So let us fix x and α as above, as well as some $g \in G(L)$ such that $gx = x$ and choose an extension $(L, |-|) \rightarrow (L', |-|)$ in HV such that $|L'^\times| = \mathbb{R}!$ Then $\Lambda(L') = A$, thus $g\alpha = \alpha$ in $\mathbf{B}'(\omega_G^\circ, L')$ since $gx = x$ in $\mathbf{B}^e(G_{L'})$. Since $\mathbf{B}'(\omega_G^\circ, L) \rightarrow \mathbf{B}'(\omega_G^\circ, L')$ is injective, also $g\alpha = \alpha$ in $\mathbf{B}'(\omega_G^\circ, L)$. \square

4.7.11. The above theorem implies various properties of $\mathbf{B}(\omega_G^\circ, K)$, for instance: $\mathbf{B}(\omega_G^\circ, K)$ is a tight affine $\mathbf{F}(G_K)$ -building. For an extension $(K, |-|) \rightarrow (L, |-|)$, the map $\mathbf{B}(\omega_{G_{\mathcal{O}_L}}^\circ, L) \rightarrow \mathbf{B}(\omega_G^\circ, L)$ is an isomorphism of affine $\mathbf{F}(G_L)$ -buildings. For a closed immersion $G_1 \hookrightarrow G_2$, the map $\mathbf{B}(\omega_{G_1}^\circ, K) \rightarrow \mathbf{B}(\omega_{G_2}^\circ, K)$ is injective (this follows from the axiom **A1** on the source, and the obvious injectivity on apartments). For a central isogeny $G_1 \twoheadrightarrow G_2$, the map $\mathbf{B}(\omega_{G_1}^\circ, K) \rightarrow \mathbf{B}(\omega_{G_2}^\circ, K)$ is an isomorphism. Thus $\mathbf{B}(\omega_G^\circ, K)$ has decompositions analogous to those of section 2.2.9.

4.7.12. Fix $\mathcal{F}_1, \mathcal{F}_2 \in \mathbf{F}(G_K)$. Suppose that for some $\epsilon > 0$, $\alpha_G + t\mathcal{F}_1 = \alpha_G + t\mathcal{F}_2$ for all $t \in [0, \epsilon]$. Then the reductions $\mathcal{F}_{1,k}$ and $\mathcal{F}_{2,k}$ are equal in $\mathbf{F}(G_k)$ by Lemma 75. Suppose conversely that $\mathcal{F}_{1,k} = \mathcal{F}_{2,k}$, and choose an apartment $\mathbf{B}^e(S)$ in $\mathbf{B}^e(G_K)$ containing the germs of $t \mapsto \circ_G^e + t\mathcal{F}_i$ for $i \in \{1, 2\}$ – in particular, S belongs to $\mathbf{S}(G)$ since \circ_G^e belongs to $\mathbf{B}^e(S)$. Then there are unique \mathcal{F}'_i in $\mathbf{F}(S)$ such that, for some $\epsilon > 0$, $\circ_G^e + t\mathcal{F}_i = \circ_G^e + t\mathcal{F}'_i$ in $\mathbf{B}^e(G_K)$ for all $t \in [0, \epsilon]$. But then also $\alpha_G + t\mathcal{F}_i = \alpha_G + t\mathcal{F}'_i$ in $\mathbf{B}(\omega_G^\circ, K)$, thus $\mathcal{F}_{i,k} = \mathcal{F}'_{i,k}$ in $\mathbf{F}(G_k)$, therefore $\mathcal{F}'_{1,k} = \mathcal{F}'_{2,k}$ and $\mathcal{F}'_1 = \mathcal{F}'_2$ since the reduction map is injective on $\mathbf{F}(S)$, thus again $\alpha_G + t\mathcal{F}_1 = \alpha_G + t\mathcal{F}_2$ for all $t \in [0, \epsilon]$. This yields canonical identifications

$$\begin{array}{ccc} & \mathbf{F}(G_K) & \\ \text{loc}_{\circ_G^e} \swarrow & \downarrow \text{loc}_{\alpha_G} & \searrow \text{red} \\ \mathbf{T}_{\circ_G^e} \mathbf{B}^e(G_K) & \xlongequal{\quad} \mathbf{T}_{\alpha_G} \mathbf{B}(\omega_G^\circ, K) & \xlongequal{\quad} \mathbf{F}(G_k) \end{array}$$

between the localization maps of 4.3.14 and the reduction map on $\mathbf{F}(G_K)$.

4.7.13. For every $\nu > 0$, there is a $G(K)$ -equivariant commutative diagram

$$\begin{array}{ccccc} \mathbf{B}(\omega_G^\circ, K, |-|) & \times & \mathbf{F}(G_K) & \xrightarrow{+} & \mathbf{B}(\omega_G^\circ, K, |-|) \\ a \downarrow & & b \downarrow & & a \downarrow \\ \mathbf{B}(\omega_G^\circ, K, |-|^\nu) & \times & \mathbf{F}(G_K) & \xrightarrow{+} & \mathbf{B}(\omega_G^\circ, K, |-|^\nu) \end{array}$$

where $a(\alpha) = \alpha^\nu$ and $b(\mathcal{F}) = \nu\mathcal{F}$. It is compatible with the analogous diagram of section 4.6.5 via the relevant α -maps.

4.7.14. For $x \in \mathbf{B}^e(G_K)$, the K -norm $\alpha(x) \in \mathbf{B}(\omega_G^\circ, K)$ is exact and extends to a K -norm on $\mathbf{B}(\omega_G^\circ, K)$ as in 3.6.6. Thus by Proposition 34, it yields a K -norm $\alpha(x)(\rho)$ on $V_K(\rho)$ for every representation ρ of G on a flat \mathcal{O} -module $V(\rho)$. We set

$$\alpha_{\text{ad}}(x) = \alpha(x)(\rho_{\text{ad}}), \quad \alpha_{\text{reg}}(x) = \alpha(x)(\rho_{\text{reg}}) \quad \text{and} \quad \alpha_{\text{adj}}(x) = \alpha(x)(\rho_{\text{adj}}).$$

Proposition 79. *Suppose that $(K, |-|)$ is discrete and write $|K^\times| = q^{\mathbb{Z}}$ with $q > 1$. Let $(\mathfrak{g}_{x,r})_{r \in \mathbb{R}}$ be the Moy-Prasad filtration attached to x on $\mathfrak{g}_K = \text{Lie}(G_K)$. Then*

$$\forall x \in \mathbb{R} : \quad \mathfrak{g}_{x,r} = \{v \in \mathfrak{g}_K : \alpha_{\text{ad}}(x)(v) \leq q^{-r}\}.$$

Proof. Given the definition of $\mathfrak{g}_{x,r}$ (by étale descent from the quasi-split case) and Proposition 72, we may assume that G splits over \mathcal{O} . Changing $|-|$ to $|-|^\nu$ with $\nu = \frac{1}{\log q}$, we may also assume that $q = e$. Fix $S \in \mathbf{S}(G)$ with x in $\mathbf{B}^e(S_K)$ and write $x = \circ_G^e + \mathcal{F}$ for some $\mathcal{F} \in \mathbf{F}(S_K)$, so that also $\alpha(x) = \alpha_G + \mathcal{F}$. Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\beta \in \Phi(G,S)} \mathfrak{g}_\beta$ be the weight decomposition of \mathfrak{g} and $\mathcal{F}^\sharp : M \rightarrow \mathbb{R}$ the morphism corresponding to \mathcal{F} , where $M = \text{Hom}(S, \mathbb{G}_{m,\mathcal{O}})$. Then for every $r \in \mathbb{R}$,

$$\overline{\mathbf{B}}(\alpha_{\text{ad}}(x), r) = \mathfrak{g}_{0,r} \oplus \bigoplus_{\beta \in \Phi(G,S)} \mathfrak{g}_{\beta,r}$$

where $\mathfrak{g}_{\beta,r} = \overline{\mathbf{B}}(\alpha_{\mathfrak{g}_\beta}, r - \mathcal{F}^\sharp(\beta))$ for $\beta \in \Phi(G,S) \cup \{0\}$. For $r = 0$, this is the Lie algebra \mathfrak{g}_x of the group scheme \mathfrak{G}_x over \mathcal{O} attached to x in [8]. Comparing now this formula with the definition of $\mathfrak{g}_{x,r}$ in [3, 2.1.3] proves our claim. \square

Let G_K^{an} be the analytic Berkovich space attached to G_K . In [27, 2.2], the authors construct a canonical map $\vartheta : \mathbf{B}^e(G_K) \rightarrow G_K^{\text{an}}$, thus attaching to every $x \in \mathbf{B}^e(G_K)$ a multiplicative K -semi-norm $\vartheta(x)$ on $\mathcal{A}(G_K)$.

Proposition 80. *For every $x \in \mathbf{B}^e(G_K)$, $\alpha_{\text{adj}}(x) = \vartheta(x)$. In particular, the K -norm $\alpha_{\text{adj}}(x)$ on $\mathcal{A}(G_K)$ is multiplicative (and $\vartheta(x)$ is a norm).*

Proof. If we equip G_K^{an} with the action of $G(K)$ induced by ρ_{adj} , then $x \mapsto \vartheta(x)$ is $G(K)$ -equivariant and compatible with extensions $(K, |-|) \rightarrow (L, |-|)$ in the sense that $\vartheta(x) = \vartheta(x_L)|_{\mathcal{A}(G_K)}$ for every $x \in \mathbf{B}^e(G_K)$ [27, Proposition 2.8]. The map $x \mapsto \alpha_{\text{adj}}(x)$ has the same properties. We may thus assume that G splits over \mathcal{O} , and again choosing L with $\log |L^\times| = \mathbb{R}$, we merely have to show that $\vartheta(\circ_G^e) = \alpha_{\text{adj}}(\circ_G^e) = \alpha_{\mathcal{A}(G)}$. By definition: $\{\vartheta(x)\}$ is the Shilov boundary of a K -affinoid subgroup G_x of G_K^{an} . For $x = \circ_G^e$, G_x is the affinoid group G^{an} attached to G , and its Shilov boundary is the gauge norm attached to $\mathcal{A}(G)$, i.e. $\alpha_{\mathcal{A}(G)}$. \square

Since the multiplication on $\mathcal{A}(G)$ is a morphism $\rho_{\text{reg}} \otimes \rho_{\text{reg}} \rightarrow \rho_{\text{reg}}$ in $\text{Rep}'(G)(\mathcal{O})$, the K -norm $\alpha_{\text{reg}}(x)$ on $\mathcal{A}(G_K)$ is sub-multiplicative. Since for every $\tau \in \text{Rep}^\circ(G)(\mathcal{O})$, the co-module map $V(\tau) \rightarrow V(\tau) \otimes \mathcal{A}(G)$ is a pure monomorphism $\tau \hookrightarrow \tau_0 \otimes \rho_{\text{reg}}$ in $\text{Rep}'(G)(\mathcal{O})$, $\alpha(x)(\tau)$ is the restriction of $\alpha_{V(\tau_0)} \otimes \alpha_{\text{reg}}(x)$ to $V_K(\tau)$, thus $\alpha_{\text{reg}}(x)$ determines $\alpha(x)$ and α_{reg} is a $G(K)$ -equivariant embedding of $\mathbf{B}^e(G_K)$ into the space of sub-multiplicative K -norms on $\mathcal{A}(G)$ (equipped with the regular action).

4.7.15. We expect that $\mathbf{B}(\omega_G^\circ, K) = \mathbf{B}^?(\omega_G^\circ, K)$, or perhaps even that every norm in $\mathbf{B}'(\omega_G^\circ, K)$ which is exact (they obviously form a $G(K)$ -stable subset) is already in $\mathbf{B}(\omega_G^\circ, K)$. Also, for a parabolic subgroup P of G with Levi L , the retraction $r_{P,L}$ of Proposition 67 should map a norm α on ω_G° to the norm on ω_L° which equips $V_K(\tau)$ with the norm induced by $\alpha(\text{Ind}_P^G(\tau))$ via the adjunction morphism.

4.7.16. Suppose that \mathcal{O} is a valuation ring of height > 1 , with fraction field K . Then $\Gamma = K^\times / \mathcal{O}^\times$ is a totally ordered commutative group which can not be embedded into \mathbb{R} . Let G be a reductive group over \mathcal{O} . Replacing \mathbb{R} with Γ in the above constructions, it should be possible to define a ‘‘Bruhat-Tits’’ building $\mathbf{B}(\omega_G^\circ, K)$ with compatible actions of $G(K)$ and $\mathbf{F}^\Gamma(G_K)$, made of factorizations of the fiber functor $\omega_{G,K}^\circ : \text{Rep}^\circ(G)(\mathcal{O}) \rightarrow \text{Vect}(K)$ through a suitable category of norms. The type maps should be the tautological morphisms $\nu : S(K) \rightarrow \mathbf{G}^\Gamma(S)$ mapping $s \in S(K)$ to the unique morphism $\nu(s) : \mathbb{D}_K(\Gamma) \rightarrow S$ whose composite with a character χ of S is the image of $\chi(s)$ in $\Gamma = K^\times / \mathcal{O}^\times$.

4.7.17. There should be a similar Tannakian formalism for the symmetric spaces of reductive groups over \mathbb{R} , with factorizations of fiber functors through the category of Euclidean spaces, using compact forms of the adjoint groups as base point.

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