Filtrations and Buildings

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En hommage à Alexander Grothendieck

2000 *Mathematics Subject Classification.* 14L15, 18D10, 20E42, 20G05, 51E24.

Key words and phrases. Filtrations, Buildings and the Tannakian formalism.

Abstract. We construct and study a scheme theoretical version of the Tits vectorial building, relate it to filtrations on fiber functors, and use them to clarify various constructions pertaining to affine Bruhat-Tits buildings, for which we also provide a Tannakian description.

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CHAPTER 1

Introduction

Spherical, affine and vectorial buildings are covered by apartments which are respectively spheres, affine and vector spaces. They interact with each other as spheres, affine and vector spaces do.

The *combinatorial* Tits building of a reductive group G over a field K reflects the incidence relations between the parabolic subgroups of G. The *spherical* Tits building is the geometric realization of the combinatorial one, obtained by gluing spheres along common spherical sectors. Both buildings were defined by Tits in [**46**], they only depend upon the adjoint group of G, and they are not functorial in G. The *vectorial* Tits building was defined by Rousseau in [**40**], and it does depend functorially upon G. For a semi-simple group G, it may by defined as the cone on the spherical building of G, obtained by gluing vector spaces along common sectors. For a torus G, it is the group of K-rational cocharacters of G, tensored with $\mathbb{R}$. This vectorial Tits building is the unifying theme of our somewhat eclectic paper.

When K is a non-archimedean local field, there is also an *affine* building attached to G: the (extended) Bruhat-Tits building of G, as defined in [**9, 10**]. It is obtained by gluing affine spaces along common alcoves and it reflects the incidence relations between bounded open subgroups of G. The combinatorial Tits building of G encodes the geometry of this affine Bruhat-Tits building at infinity, and the combinatorial Tits buildings of the maximal reductive quotients of the special fibers of various (non-necessarily reductive) integral models of G similarly encode the local geometry of the affine Bruhat-Tits building of G.

For classical groups, which come equipped with a standard faithful representation, a global construction of the Bruhat-Tits building is given in [**11, 12**], generalizing the pioneering work of Goldman and Iwahori in [**22**], which dealt with the case of a general linear group and served as a model for the development of the whole Bruhat-Tits theory. For these classical groups, the Bruhat-Tits building of G is cut out from the space of all non-archimedean K-norms on the standard representation. My initial intention was to expand such a global construction to arbitrary reductive groups, and to clarify and canonify the aforementioned relations between the Bruhat-Tits building of G and the Tits buildings of various related groups.

Let me try to explain how I came to be interested in these questions. Grassmannians, flag manifolds and their affine counterparts are essentially orbits of G acting on its related buildings, and they show up in many branches of mathematics. In particular, integral p-adic Hodge theory makes extensive use of filtrations and lattices, which are respectively parametrized by flag manifolds and affine Grassmannians, or by the corresponding larger ambient spaces: the vectorial Tits building and the affine Bruhat-Tits building. Working on classical results in p-adic Hodge theory [**16, 15, 14**], I eventually realized that a natural transitive action of the former on the latter explained many features of the interplay between filtrations and lattices in this area of mathematics. For $G = GL_n$, this circle of ideas was already more or less implicit in various works, for instance in Laffaille [**31**] or Fontaine and Rapoport [**19**]. But for more general groups, sound foundations seemed to be lacking or scattered in the existing literature: I needed a flexible and extensive dictionary connecting building-theoretical, geometric and metric notions and tools to properties of the relevant objects in linear algebra, filtrations and lattices.

For instance the aforementioned action itself was most definitely well-known to authors working in metric geometry. Here, the cone $\mathcal{C}(\partial X)$ on the visual boundary ∂X of a $CAT(0)$-metric space (X, d) (see [**8**] for these notions) acts on X by non-expanding maps as follows: an element of the cone is a pair (ξ, ℓ) where $\xi \in \partial X$ is an asymptotic class of unit speed geodesic rays in X and $\ell \geq 0$ is some length (or speed); it acts on a point x of X by moving it at distance ℓ along the unique geodesic ray in ξ emanating from x. Taking X to be the Bruhat-Tits building of G over K, it was also known since Bruhat and Tits that X can be equipped with a non-canonical metric d which turns it into a $CAT(0)$-space, and for which the visual boundary ∂X is a realization of the spherical building of G over K – this geometric statement was initially encapsulated in the group-theoretical notion of double Tits systems, see [**9**, 5.1.33] and [**40**, 11.10]. But there were no clear-cut identifications between filtrations for G and the cone $\mathcal{C}(\partial X)$, or between lattices (and norms) for G and its Bruhat-Tits building X; moreover, having to rely on the artificial choice of a metric to define the action furthermore blurred its naturality. This paper provides proper definitions of these notions, all the required canonical identifications, an explicit formula for the action of filtrations (which are elements of the vectorial Tits building) on norms (which are elements of the Bruhat-Tits building), along with many properties of these objects and constructions.

There is a general Grothendieckian recipe to pass from $G = GL_n$ to more general algebraic groups: replace the natural standard representation by the entire category of all algebraic representations. This has become a very common method in p-adic Hodge theory [**30, 17**], and as far as filtrations are concerned, it was already implemented for reductive groups over arbitrary fields in the foundational work on Tannakian categories, Saavedra Rivano's thesis [**41**]. For norms and lattices, it was completed more recently by Haines's student Wilson [**50**] for split reductive groups over complete discrete valuation fields. This Tannakian formalism also has many advantages: it has build-in functorialities, it works for arbitrary affine groups over arbitrary base schemes, it provides a conceptual framework for many algebraic constructions, and it gives rise to various interesting representable sheaves.

In chapter 2, we thus actually start with a reductive group G over an arbitrary base scheme S. For a totally ordered commutative group $\Gamma = (\Gamma, +, \leq)$, we introduce there our fundamental G-equivariant cartesian diagram of S-schemes

$$\begin{tikzcd}
\mathbb{G}^\Gamma(G) \arrow[r, "\mathrm{Fil}"] \arrow[d, "F"'] & \mathbb{F}^\Gamma(G) \arrow[r, "t"] \arrow[d, "F"'] & \mathbb{C}^\Gamma(G) \arrow[d, "F"'] \\
\mathbb{OPP}(G) \arrow[r, "p_1"] & \mathbb{P}(G) \arrow[r, "t"] & \mathbb{O}(G)
\end{tikzcd}$$

where $\mathbb{P}(G)$ and $\mathbb{OPP}(G)$ are respectively the S-schemes of parabolic subgroups P of G and pairs of opposed parabolic subgroups (P, P') of G, $\mathbb{O}(G)$ is the S-scheme of G-orbits in $\mathbb{P}(G)$ or $\mathbb{OPP}(G)$, $\mathbb{G}^\Gamma(G) = \underline{\mathrm{Hom}}(\mathbb{D}_S(\Gamma), G)$ where $\mathbb{D}_S(\Gamma)$ is the diagonalized multiplicative group over S with character group Γ, while $\mathbb{F}^\Gamma(G)$ and

$\mathbb{C}^\Gamma(G)$ are suitable quotients of $\mathbb{G}^\Gamma(G)$. The *facet* morphisms F are surjective and locally constant in the étale topology on their base, the p_1 and Fil morphisms are affine smooth surjective with geometrically connected fibers and the *type* morphisms t are projective smooth surjective with geometrically connected fibers. Since $\mathbb{O}(G)$ is finite étale over S, all of the above schemes are smooth, separated and surjective over S. We also equip $\mathbb{C}^\Gamma(G)$ and $\mathbb{O}(G)$ with *S-monoid structures*, and the facet map $F : \mathbb{C}^\Gamma(G) \to \mathbb{O}(G)$ is compatible with them. We finally define two related partial orders on the S-monoid $\mathbb{C}^\Gamma(G)$, the *weak and strong dominance orders*.

For $\Gamma = \mathbb{Z}$, $\mathbb{D}_S(\Gamma) = \mathbb{G}_{m,S}$ and $\mathbb{G}^\Gamma(G)$ is the S-scheme of cocharacters of G, whose conjugacy classes are classified by $\mathbb{C}^\Gamma(G)$. For $\Gamma = \mathbb{R}$, $\mathbb{F}^\Gamma(G)$ is a scheme theoretical version of the Tits vectorial building defined by Rousseau in [**40**] and $\mathbb{C}^\Gamma(G)$ is a scheme theoretical version of a closed Weyl chamber. More general Γ's, for instance the valuation groups of valuation fields of height > 1 may also be useful in connection with recent developements in p-adic geometry.

In chapter 3, we show that $\mathbb{G}^\Gamma(G)$ and $\mathbb{F}^\Gamma(G)$ represent functors respectively related to Γ-graduations and Γ-filtrations on a variety of fiber functors. The main difficulty here is to show that the Γ-filtrations split fpqc-locally on the base scheme. For $\Gamma = \mathbb{Z}$, this was essentially established in the thesis of Saavedra Rivano [**41**], at least when S is the spectrum of a field. We strictly follow Saavedra's proof (which he attributes to Deligne), adding a considerable amount of details and some patch when needed. We advise our reader to read both texts side by side, only switching to ours when he feels uncomfortable with the necessary generalizations of Saavedra's arguments[1]. Various constructions of chapter 2 have counterparts in this Tannakian framework, which are reviewed in section 3.11. In particular, we show that the first line of our fundamental diagram is functorial in the reductive group G over S. The weak dominance order on $\mathbb{C}^\Gamma(G)$ is compatible with this functoriality, but we would like to already emphasize here that the monoid structure is not.

In chapter 4, we study the sections of our schemes over a local ring $\mathcal{O}$. We first equip $\mathbf{F}^\Gamma(G) = \mathbb{F}^\Gamma(G)(\mathcal{O})$ with a collection of *apartments* $\mathbf{F}^\Gamma(S)$ indexed by the maximal split subtori S of G, and with the collection of *facets* $F^{-1}(P)$ indexed by the parabolic subgroups P of G. The key properties of the resulting combinatorial structure are well-known when $\mathcal{O}$ is a field and $\Gamma = \mathbb{R}$, in which case $\mathbf{F}^\Gamma(G)$ is the Tits vectorial building, but most of them carry over to this more general situation, thanks to the wonderful last chapter of SGA3 [**21**]. We describe the behavior of these auxiliary structures under specialization (when $\mathcal{O}$ is Henselian) or generization

[1]For $\Gamma = \mathbb{Z}$, Ziegler recently established the fpqc-splitting of $\mathbb{Z}$-filtrations on fiber functors on arbitrary Tannakian categories [**51**], thereby proving a conjecture which was left open after Saavedra's thesis. In particular, the $\mathbb{Z}$-filtrations we consider have fpqc-splittings even when G is not reductive, but defined over a field. In the reductive case, the final arguments in Ziegler's proof simplify those of Saavedra's, but rely more on the Saavedra-Deligne theorem that all fiber functors on Tannakian categories are fpqc-locally isomorphic [**18**]. According to D. Schäppi, it follows from his own work [**42, 43**] and Lurie's note on Tannaka duality that the same result holds for any $\otimes$-functor $\mathsf{Rep}^{fp}(G)(S) \to \mathsf{QCoh}(T)$ where: S is affine, T is an S-scheme, G is affine flat over S, $\mathsf{Rep}^{fp}(G)(S)$ is the $\otimes$-category of algebraic representations of G on finitely presented $\mathcal{O}_S$-modules, and G has the resolution property: any finitely presented algebraic representation of G is covered by another one which is locally free. It then seems likely that Ziegler's proof could yield a common generalization of his result ($\Gamma = \mathbb{Z}$, G affine over a field) and ours (Γ and S arbitrary, but G reductive) on the existence of fpqc-splittings of Γ-filtrations, using a hefty dose of the stack formalism. We have chosen to stick to the constructive, down-to-earth original proof of Saavedra/Deligne – and to reductive groups as well.

(when $\mathcal{O}$ is a valuation ring). When Γ is a subring of $\mathbb{R}$, we also attach to every finite free faithful representation τ of G a partially defined *scalar product* on $\mathbf{F}^{\Gamma}(G)$ and the corresponding *distance* and *angle* functions, and we study their basic properties. When $\mathcal{O}$ is a field, a theorem of Borel and Tits [**9**] implies that these functions are defined everywhere, and one thus retrieves the aforementioned non-canonical distances on the vectorial Tits building $\mathbf{F}(G) = \mathbf{F}^{\mathbb{R}}(G)$.

Over a field K and with $\Gamma = \mathbb{R}$, we next define a notion of *affine* $\mathbf{F}(G)$*-spaces*, which interact with the vectorial Tits building $\mathbf{F}(G)$ as affine spaces do with their underlying vector space. Strongly influenced by the formalism set up by Rousseau in [**40**] and Parreau in [**36**], we introduce various axioms that these spaces may satisfy, leading to the more restricted class of *affine* $\mathbf{F}(G)$*-buildings*. Most of the abstract definitions of affine buildings that have already been proposed [**29, 37, 40**] also add a euclidean metric into the picture, and involve a covering atlas of charts, which are isometries from a given fixed euclidean affine space onto subsets of the building (its apartments) subject to various conditions. Our definition also involves a covering by apartments, but their affine structure is inherited from a globally defined $G(K)$-equivariant transitive operation $(x, \mathcal{F}) \mapsto x + \mathcal{F}$ of the vectorial building $\mathbf{F}(G)$ on the given affine $\mathbf{F}(G)$-space. It is therefore essentially a boundary-based formalism for affine buildings, as opposed to the more usual apartment-based formalism.

Eventhough our affine $\mathbf{F}(G)$-buildings have no fixed metric, they are equipped with a canonical metrizable topology and a canonical vector valued convex distance $\mathbf{d}$, taking values in $\mathbf{C}(G) = t(\mathbf{F}(G))$. The choice of a faithful representation τ of G eventually equips them with a convex distance $d_\tau = \|\mathbf{d}\|_\tau$ in the usual sense, for which they often become $CAT(0)$-metric spaces as defined in [**8**]. But the finer and canonical vectorial distance $\mathbf{d}$ really is a key feature of our buildings: it retrieves and generalizes many classical invariants in various set-up (such as types of filtrations and relative positions of lattices or quadrics), and its formal properties imply most, if not all, of the known inequalities among these invariants.

Of course $\mathbf{F}(G)$ is itself an affine $\mathbf{F}(G)$-building, with a distinguished point. When K is equipped with a non-trivial, non-archimedean absolute value, we show in chapter 6 that the (extended) affine building $\mathbf{B}^e(G)$ constructed by Bruhat and Tits [**9, 10**] is canonically equipped with a structure of affine $\mathbf{F}(G)$-building in our sense. This is our precise formalization of the "combinatorial" assertion that the visual boundary of the Bruhat-Tits building is a geometric realization of the combinatorial Tits building. This being done, we may fix a base point $\circ_G$ in $\mathbf{B}^e(G)$ and try to describe the whole building as a quotient of $\mathbf{F}(G)$ using the surjective map $\mathbf{F}(G) \ni \mathcal{F} \mapsto \circ_G + \mathcal{F} \in \mathbf{B}^e(G)$. We do this in the last section, assuming that our base point $\circ_G$ is hyperspecial, i.e. corresponds to a reductive group G over the valuation ring $\mathcal{O}$ of K, which we also assume to be Henselian. Note that the existence of an hyperspecial point amounts to an assumption on G_K [**47**, 2.4].

More precisely, we first define a space of K-norms on the fiber functor

$$\omega_G^\circ : \mathsf{Rep}^\circ(G)(\mathcal{O}) \to \mathsf{Vect}(K)$$

where $\mathsf{Rep}^\circ(G)(\mathcal{O})$ is the category of algebraic representations of G on finite free $\mathcal{O}$-modules. This space is equipped with a $G(K)$-action, an explicit $G(K)$-equivariant operation of $\mathbf{F}(G_K)$ and a base point α_G fixed by $G(\mathcal{O})$. We show that the map $\circ_G + \mathcal{F} \mapsto \alpha_G + \mathcal{F}$ is well-defined, injective, $G(K)$-equivariant and compatible with the operations of $\mathbf{F}(G_K)$. It thus defines an isomorphism $\boldsymbol{\alpha}$ of affine $\mathbf{F}(G_K)$-buildings from $\mathbf{B}^e(G_K)$ to a set $\mathbf{B}(\omega_G^\circ, K) = \alpha_G + \mathbf{F}(G_K)$ of K-norms on ω_G°.

This Tannakian description of the extended Bruhat-Tits building immediately implies that the assignment $G \mapsto \mathbf{B}^e(G_K)$ is functorial in the reductive group G over $\mathcal{O}$. Such a functoriality was already established by Landvogt [**32**], with fewer assumptions on G_K but more assumptions on K. It also suggests a possible definition of Bruhat-Tits buildings for reductive groups over valuation rings of height greater than 1, as well as a similar Tannakian description of symmetric spaces (in the archimedean case, see 5.8). It is related to previous constructions as follows.

Our canonical isomorphism $\boldsymbol{\alpha} : \mathbf{B}^e(G_K) \to \mathbf{B}(\omega_G^\circ, K)$ assigns to a point x in $\mathbf{B}^e(G_K)$ and to any algebraic representation τ of G on a flat $\mathcal{O}$-module $V(\tau)$ a K-norm $\boldsymbol{\alpha}(x)(\tau)$ on $V_K(\tau) = V(\tau) \otimes K$. For the adjoint representation τ_{ad} of G on $\mathfrak{g} = \mathrm{Lie}(G)$, the adjoint-regular and regular representations ρ_{adj} and ρ_{reg} of G on $\mathcal{A}(G) = \Gamma(G, \mathcal{O}_G)$, we obtain respectively: a K-norm $\alpha_{\mathrm{ad}}(x)$ on $\mathfrak{g}_K = \mathrm{Lie}(G_K)$ whose closed balls give the Moy-Prasad filtration of x on $\mathfrak{g}_K$ [**35**], the K-norm $\alpha_{\mathrm{adj}}(x)$ in G_K^{an} constructed in [**38**], and an embedding $x \mapsto \alpha_{\mathrm{reg}}(x)$ of the extended Bruhat-Tits building in the analytic Berkovich space G_K^{an} attached to G_K. Our isomorphism $\boldsymbol{\alpha}$ also induces an explicit $G(\mathcal{O})$-equivariant identification between the "tangent space" of $\mathbf{B}^e(G_K)$ at $\circ_G$ and the vectorial Tits building $\mathbf{F}(G_k)$ of the special fiber G_k of G over the residue field k of $\mathcal{O}$, as expected from [**10**, 4.6.35-45].

In Wilson's Tannakian formalism for Bruhat-Tits buildings [**50**], alcoves and their parahorics played the leading role. His Moy-Prasad filtrations are the lattice chains of closed balls of our norms. We owe to his work the essential shape of our formalism, if not the very idea that such a formalism was indeed possible: we were first naively looking for a base-point free description of the Bruhat-Tits buildings. His point of view is more adapted to the study of the simplicial structure of these buildings, but only covers split groups over discrete valuation rings. Our approach covers unramified groups over fields equipped with a Henselian absolute value. It lacks an intrinsic description of (1) the equivalence relation on $\mathbf{F}(G_K)$ defined by $\mathcal{F} \sim \mathcal{F}' \iff \circ_G + \mathcal{F} = \circ_G + \mathcal{F}'$, and of (2) the image $\mathbf{B}(\omega_G^\circ, K) = \alpha_G + \mathbf{F}(G_K)$ of $\boldsymbol{\alpha}$ in the larger space of all K-norms on the fiber functor ω_G°.

Finally, we would like to mention that some of our results should extend to more general fiber functors, using the Schäppi/Lurie generalization of Deligne's theorem as mentioned in the previous footnote.

This work grew out of a question by J-F. Dat and many discussions with D. Mauger on buildings and cocharacters. I am very grateful to G. Rousseau and A. Parreau, who always had answers to my numerous questions. Apart from the emphasis on the boundary, most of the definitions and results of chapter 5 are either taken from his survey [**40**] or from her preprint [**36**]. P. Deligne kindly provided the patch at the very end of the proof of the splitting theorem, dealing with groups of type $G2$ in characteristic 2, and M. Hils the proof of lemma 134.

CHAPTER 2

The group theoretical formalism

For a reductive group scheme G over an arbitrary base scheme S, we will define and study a cartesian diagram of smooth and separated schemes over S,

$$\begin{tikzcd} \mathbb{G}^{\Gamma}(G) \arrow[r, two heads, "\mathrm{Fil}"] \arrow[d, "F"'] & \mathbb{F}^{\Gamma}(G) \arrow[r, two heads, "t"] \arrow[d, "F"'] & \mathbb{C}^{\Gamma}(G) \arrow[d, "F"'] \\ \mathbb{OPP}(G) \arrow[r, two heads, "p_1"] & \mathbb{P}(G) \arrow[r, two heads, "t"] & \mathbb{O}(G) \end{tikzcd}$$

Our main background reference for this chapter is SGA3 [**20, 1, 21**].

2.1. Γ-graduations on smooth affine groups

THEOREM 1. *Let H and G be group schemes over a base scheme S, with H of multiplicative type and quasi-isotrivial, G smooth and affine. Then the functor*

$$\underline{\mathrm{Hom}}_{S-\mathsf{Group}}(H,G) : (\mathsf{Sch}/S)^{\circ} \to \mathsf{Set}, \qquad T \mapsto \mathrm{Hom}_{T-\mathsf{Group}}(H_T, G_T)$$

is representable by a smooth and separated scheme over S.

REMARK 2. When H is of finite type, it is quasi-isotrivial by [**1**, X 4.5]. The theorem is then due to Grothendieck, see [**1**, XI 4.2]. The proof given there relies on the density theorem of [**1**, IX 4.7], definitely a special feature of finite type multiplicative groups. When H is trivial, we may still reduce the proof of the above theorem to the finite type case, as explained in remark 12 below. For the general case, we have to find another road through SGA3, passing through [**1**, X 5.6] which has no finite type assumption on H but requires H and G to be of multiplicative type and quasi-isotrivial:

PROPOSITION 3. *Let H and G be group schemes of multiplicative type over S, with H quasi-isotrivial and G of finite type. Then $\underline{\mathrm{Hom}}_{S-\mathsf{Group}}(H,G)$ is representable by a quasi-isotrivial twisted constant group scheme X over S.*

PROOF. This is [**1**, X 5.6], since G is also quasi-isotrivial by [**1**, X 4.5]. □

LEMMA 4. *Let X be a quasi-isotrivial twisted constant scheme over S. Then X is separated and étale over S, satisfies the valuative criterion of properness, and:*

(1) *If S is irreducible and geometrically unibranch with generic point η, then*

$$X = \textstyle\coprod_{\lambda \in X_\eta} X(\lambda) \quad \textit{with} \quad X(\lambda) = \overline{\{\lambda\}} \textit{ open and closed in } X,$$

each $X(\lambda)$ is a connected finite étale cover of S and $\Gamma(X/S) = \Gamma(X_\eta/\eta)$.

(2) *If S is local henselian with closed point s, then*

$$X = \textstyle\coprod_{x \in X_s} X(x) \quad \textit{with} \quad X(x) = \operatorname{Spec} \mathcal{O}_{X,x} \textit{ open and closed in } X,$$

each $X(x)$ is a connected finite étale cover of S, and $\Gamma(X/S) = \Gamma(X_s/s)$.

PROOF. The morphism $X \to S$ is separated by [**26**, 2.7.1] and étale by [**28**, 17.7.3]. Since valuation rings are normal integral domains, thus irreducible and geometrically unibranch, it remains to establish (1) and (2).

Suppose first that S is irreducible and geometrically unibranch with generic point η. Then by [**28**, 18.10.7] applied to $X \to S$,

$$X = \coprod_{\lambda \in X_\eta} X(\lambda) \quad \text{with} \quad X(\lambda) = \overline{\{\lambda\}} \text{ open and closed in } X,$$

thus $X(\lambda)$ is already étale over S. Fix an étale covering $\{S_i \to S\}$ trivializing X, so that $X \times_S S_i = Q_{i,S_i}$ for some set Q_i. Using [**28**, 18.10.7] again, we may assume that each S_i is connected, in which case we obtain decompositions

$$Q_i = \coprod_{\lambda \in X_\eta} Q_i(\lambda) \quad \text{with} \quad X(\lambda) \times_S S_i = Q_i(\lambda)_{S_i}.$$

Since the generic fiber $\lambda \to \eta$ of $X(\lambda) \to S$ is finite of degree $n(\lambda) = [k(\lambda) : k(\eta)]$, each $Q_i(\lambda)$ is a finite subset of Q_i of order $n(\lambda)$, therefore $X(\lambda) \times_S S_i$ is finite over S_i and $X(\lambda)$ is finite over S by [**26**, 2.7.1]. Being finite and étale over the connected S, $X(\lambda)$ is a finite étale cover of S. Being irreducible, it is also connected. By [**28**, 17.4.9], the map which sends a section g of $X \to S$ to its image $g(S)$ identifies $\Gamma(X/S)$ with the set of connected components $X(\lambda)$ of X for which $X(\lambda) \to S$ is an isomorphism, i.e. such that $n(\lambda) = 1$. Therefore $\Gamma(X/S) = \Gamma(X_\eta/\eta)$.

Suppose next that S is local henselian with closed point s. Since $X \to S$ is quasi-finite at every $x \in X_s$ by [**28**, 17.6.1], it follows from [**28**, 18.5.11.c] that

$$X \supset X' = \coprod_{x \in X_s} X(x) \quad \text{with} \quad X(x) = \operatorname{Spec} \mathcal{O}_{X,x} \text{ open and closed in } X,$$

and $X(x)$ is finite and étale over S. By assumption, there is a surjective étale morphism $S_0 \to S$ trivializing X, so that $X \times_S S_0 = Q_{S_0}$ for some set Q. Using [**28**, 18.5.11.c] again, we may assume that S_0 is a local scheme, finite and étale over S, say with closed point s_0 lying above s. Since $X' \times_S S_0$ is open in $X \times_S S_0$ and contains its special fiber X_{s_0}, we have $X' \times_S S_0 = X \times_S S_0$, thus actually $X' = X$ by [**26**, 2.7.1]. Finally $\Gamma(X/S) = \Gamma(X_s/s)$ by [**28**, 18.5.12]. □

LEMMA 5. *Let $f : H \to G$ be a morphism of group schemes over S, with H of multiplicative type and G separated of finite presentation. Then there is a unique closed multiplicative subgroup Q of G such that f factors through a faithfully flat morphism $f' : H \to Q$. Moreover f' is also uniquely determined by f.*

PROOF. Everything being local for the fpqc topology, we may assume that S is affine and $H = \mathbb{D}_S(M)$ for some abstract commutative group M. Then $M = \varinjlim M'$ where M' runs through the filtered set $\mathcal{F}(M)$ of finitely generated subgroups of M, thus also $\mathbb{D}_S(M) = \varprojlim \mathbb{D}_S(M')$. Since $\mathbb{D}_S(M')$ is affine for all M' and $G \to S$ is locally of finite presentation, it follows from [**27**, 8.13.1] that f factors through $f_1 : \mathbb{D}_S(M') \to G$ for some $M' \in \mathcal{F}(M)$. Applying [**1**, IX 6.8] to f_1 yields a closed multiplicative subgroup Q of G such that f_1 factors through a faithfully flat (and affine) morphism $f_1' : \mathbb{D}_S(M') \to Q$, whose composite with the faithfully flat (and affine) morphism $\mathbb{D}_S(M) \to \mathbb{D}_S(M')$ is the desired factorization. Since Q is then also the image of f in the category of fpqc sheaves on Sch/S, it is already unique as a subsheaf of G. Since $Q \to G$ is a monomorphism, also f' is unique. □

DEFINITION 6. We call Q the image of f and denote it by $Q = \operatorname{im}(f)$.

LEMMA 7. *Let $f : H \to G$ be a morphism of group schemes over S, with H of multiplicative type and G smooth and affine. Then the centralizer of f is equal to the centralizer of its image, and is representable by a closed smooth subgroup of G.*

PROOF. Let $f = \iota \circ f'$ be the factorization of the previous lemma. Since f' is faithfully flat (and quasi-compact, being a morphism between affine S-schemes, therefore even affine), it is an epimorphism in the category of schemes. It then follows from the definitions in [**20**, VIB §6] that the centralizers of f, ι and $\mathrm{im}(f)$ are equal. By [**1**, XI 5.3], the centralizer of ι is a closed smooth subgroup of G. □

LEMMA 8. *Let $f : H \to Q$ be a morphism of group schemes of multiplicative type over S, with Q of finite type. Define $U = \{s \in S : f_s$ is faithfully flat$\}$. Then U is open and closed in S and $f_U : H_U \to Q_U$ is faithfully flat.*

PROOF. Let I be the image of f. Then U is the set of points $s \in S$ where $I_s = Q_s$. Now apply [**1**, IX 2.9] to $I \hookrightarrow Q$. □

We may now prove theorem 1. Define presheaves A, B, C on Sch/S by

$$\begin{aligned} C(S') &= \{\text{multiplicative subgroups } Q \text{ of } G_{S'}\}, \\ B(S') &= \{(Q, f') : Q \in C(S') \text{ and } f : H_{S'} \to Q \text{ is a morphism}\}, \\ A(S') &= \{(Q, f') \in B(S') \text{ with } f' \text{ faithfully flat}\}. \end{aligned}$$

Then C is representable, smooth and separated by [**1**, XI 4.1], $B \to C$ is relatively representable by étale and separated morphisms by proposition 3 and lemma 4, $A \to B$ is relatively representable by open and closed immersions by lemma 8 and finally A is isomorphic to $\underline{\mathrm{Hom}}_{S-\mathsf{Group}}(H, G)$ by lemma 5, which is therefore indeed representable by a smooth and separated scheme over S.

DEFINITION 9. For an abstract commutative group $\Gamma = (\Gamma, +)$ and a smooth and affine group scheme G over S, we set $\mathbb{G}^{\Gamma}(G) = \underline{\mathrm{Hom}}_{S-\mathsf{Group}}(\mathbb{D}_S(\Gamma), G)$. Thus

$$\mathbb{G}^{\Gamma}(G) : (\mathsf{Sch}/S)^{\circ} \to \mathsf{Set}$$

is representable by a smooth and separated scheme over S.

PROPOSITION 10. *Let $f : \mathbb{D}_S(\Gamma) \to G$ be a morphism of group schemes over S, with G separated and of finite presentation. Then for each s in S,*

$$\Gamma(s) = \{\gamma \in \Gamma : \gamma \text{ is trivial on } \ker(f_s)\}$$

belongs to the set $\mathcal{F}(\Gamma)$ of finitely generated subgroups of Γ. For each $\Lambda \in \mathcal{F}(\Gamma)$,

$$S(\Lambda) = \{s \in S : \Gamma(s) = \Lambda\}$$

is open and closed in S, and finally

$$\ker(f)_{S(\Lambda)} = \mathbb{D}_{S(\Lambda)}(\Gamma/\Lambda) \quad \text{and} \quad \mathrm{im}(f)_{S(\Lambda)} = \mathbb{D}_{S(\Lambda)}(\Lambda).$$

PROOF. We may assume that S is affine and G is of multiplicative type (using lemma 5 for the latter). Since $\mathbb{D}_S(\Gamma) = \varprojlim \mathbb{D}_S(\Lambda)$, it follows again from [**27**, 8.13.1] that there is some Λ in $\mathcal{F}(\Gamma)$ such that f factors through $g : \mathbb{D}_S(\Lambda) \to G$, i.e. $\mathbb{D}_S(\Gamma/\Lambda) \subset \ker(f)$. But then $\Gamma(s) \subset \Lambda$ for every $s \in S$, which proves the first claim. Applying now [**1**, IX 2.11 (i)] to g gives a finite partition of S into open and closed subsets S_i, together with a collection of distinct subgroups Λ_i of Λ such that $\ker(g)_{S_i} = \mathbb{D}_{S_i}(\Lambda/\Lambda_i)$ and $\mathrm{im}(g)_{S_i} \simeq \mathbb{D}_{S_i}(\Lambda_i)$. But then $\ker(f)_{S_i} = \mathbb{D}_{S_i}(\Gamma/\Lambda_i)$, $\mathrm{im}(f)_{S_i} \simeq \mathbb{D}_{S_i}(\Lambda_i)$ and $S_i = S(\Lambda_i)$, which proves the remaining claims. □

COROLLARY 11. *If* Γ *is torsion free,* $\mathrm{im}(f)$ *is a locally trivial subtorus of* G.

REMARK 12. The above proposition suggests another proof of theorem 1 when $H = \mathbb{D}_S(\Gamma)$. It shows indeed that the Zariski sheaf $\mathbb{G}^\Gamma(G)$ is the disjoint union of relatively open and closed subsheaves $\mathbb{G}^\Gamma(G)(\Lambda)$, indexed by $\Lambda \in \mathcal{F}(\Gamma)$. Moreover, $\mathbb{G}^\Gamma(G)(\Lambda)$ is isomorphic to the subsheaf $\mathbb{G}^\Lambda(G)(\Lambda)$ of $\mathbb{G}^\Lambda(G)$, which is representable by a smooth and separated scheme over S by [**1**, XI 4.2].

2.2. Γ-filtrations on reductive groups

Let S be a scheme, G a reductive group over S, $\mathfrak{g} = \mathrm{Lie}(G)$ its Lie algebra. Let $\Gamma = (\Gamma, +, \leq)$ be a non-trivial totally ordered commutative group.

2.2.1. Recall from [**21**, XXVI 3.5] that the sheaf

$$\mathbb{P}(G) : (\mathsf{Sch}/S)^\circ \to \mathsf{Set}$$

whose sections over an S-scheme T are given by

$$\mathbb{P}(G)(T) = \{\text{parabolic subgroups } P \text{ of } G_T\}$$

is representable, smooth and projective over S, with Stein factorization

$$\mathbb{P}(G) \xrightarrow{t} \mathbb{O}(G) \to S$$

where $\mathbb{O}(G)$ is the S-scheme of open and closed subschemes of the Dynkin S-scheme $\mathbb{DYN}(G)$ of the reductive group G/S, see [**21**, XXIV 3.3]. Both $\mathbb{DYN}(G)$ and $\mathbb{O}(G)$ are twisted constant finite schemes over S, thus finite étale over S by [**26**, 2.7.1.xv] and [**28**, 17.7.3], and $\mathbb{O}(G)$ is actually a finite étale cover of S. The morphism t is smooth, projective, with non-empty geometrically connected fibers; it classifies the parabolic subgroups of G in the following sense: two parabolic subgroups P_1 and P_2 of G are conjugated locally in the fpqc topology on S if and only if $t(P_1) = t(P_2)$.

2.2.2. For a parabolic subgroup P of G with unipotent radical U, we denote by $\overline{R}(P)$ the radical of P/U [**21**, XXII 4.3.6]. For the universal parabolic subgroup P_u of $G_{\mathbb{P}(G)}$, we obtain a $\mathbb{P}(G)$-torus $R_{\mathbb{P}(G)} = \overline{R}(P_u)$. We claim that it descends canonically to an $\mathbb{O}(G)$-torus $R_{\mathbb{O}(G)}$ over $\mathbb{O}(G)$. Since t is faithfully flat and quasi-compact, it is a morphism of effective descent for affine group schemes by [**2**, VIII 2.1], thus also for tori by definition [**1**, IX 1.3]. Our claim now follows from:

LEMMA 13. *There exists a canonical descent datum on* $R_{\mathbb{P}(G)}$ *with respect to* t.

PROOF. We have to show that for any $T \to S$ and any pair of parabolic subgroups P_1 and P_2 of G_T such that $t(P_1) = t(P_2)$, there exists a canonical isomorphism $\overline{R}(P_1) \simeq \overline{R}(P_2)$. Let $M_i = P_i/U_i$ be the maximal reductive quotient of P_i, so that $R_i = \overline{R}(P_i)$ is the radical of M_i. We may assume that $T = S$ and, by a descent argument, that $P_2 = \mathrm{Int}(g)(P_1)$ for some $g \in G(S)$. Then $\mathrm{Int}(g)$ induces isomorphisms $P_1 \to P_2$, $M_1 \to M_2$ and $R_1 \to R_2$. Since g is well-defined up to right multiplication by an element of $P_1(S)$ thanks to [**21**, XXVI 1.2], $M_1 \to M_2$ is well-defined up to an inner automorphism of M_1 and $R_1 \to R_2$ does not depend upon any choice: this is our canonical isomorphism. □

2.2.3. By [**21**, XXVI 4.3.4 and 4.3.5], the formula

$$\mathbb{OPP}(G)(T) = \{(P_1, P_2) \text{ pair of opposed parabolic subgroups of } G_T\}$$

defines an open subscheme $\mathbb{OPP}(G)$ of $\mathbb{P}(G)^2$ and the two projections

$$p_1, p_2 : \mathbb{OPP}(G) \to \mathbb{P}(G)$$

are isomorphic U_u-torsors, thus affine smooth surjective morphisms with geometrically connected fibers. Here U_u is the unipotent radical of the universal parabolic subgroup P_u of $G_{\mathbb{P}(G)}$, it acts by conjugation on the fibers, and the isomorphism is the involution $\iota(P_1, P_2) = (P_2, P_1)$ of the S-scheme $\mathbb{OPP}(G)$. We denote by $(P_u^1, P_u^2) = (p_1^* P_u, p_2^* P_u)$ the universal pair of opposed parabolic subgroups of $G_{\mathbb{OPP}(G)}$, by $U_u^i = p_i^* U_u$ the unipotent radical of P_u^i, and by $R_{\mathbb{OPP}(G)}$ the radical of the corresponding universal Levi subgroup $L_u = P_u^1 \cap P_u^2$ of $G_{\mathbb{OPP}(G)}$. Thus

$$L_u \simeq P_u^i / U_u^i \simeq p_i^*(P_u/U_u) \quad \text{and} \quad R_{\mathbb{OPP}(G)} \simeq p_i^* R_{\mathbb{P}(G)}.$$

We also denote by ι the opposition involution on $\mathbb{O}(G)$, see [**21**, XXVI 4.3.1]. Thus

$$t \circ p_2 = t \circ p_1 \circ \iota = \iota \circ t \circ p_1.$$

2.2.4. The S-scheme $\mathbb{G}^\Gamma(R_{\mathbb{OPP}(G)})$ represents the functor mapping $T \to S$ to the set of triples (P_1, P_2, f) where (P_1, P_2) is a pair of opposed parabolic subgroups of G_T with Levi subgroup $L = P_1 \cap P_2$, and $f : \mathbb{D}_T(\Gamma) \to L$ is a central morphism. The next proposition uses the total ordering on $\Gamma = (\Gamma, +, \leq)$ to define a section

$$\mathbb{G}^\Gamma(G) \hookrightarrow \mathbb{G}^\Gamma(R_{\mathbb{OPP}(G)}), \qquad f \mapsto (P_f, P_{\iota f}, f)$$

of the obvious forgetful morphism of S-schemes

$$\mathbb{G}^\Gamma(R_{\mathbb{OPP}(G)}) \to \mathbb{G}^\Gamma(G), \qquad (P_1, P_2, f) \mapsto f.$$

PROPOSITION 14. *Let $f : \mathbb{D}_S(\Gamma) \to G$ be a morphism and write $\mathfrak{g} = \oplus_\gamma \mathfrak{g}_\gamma$ for the corresponding weight decomposition of $\mathrm{ad} \circ f : \mathbb{D}_S(\Gamma) \to GL_S(\mathfrak{g})$. There exists a unique parabolic subgroup P_f of G containing the centralizer L_f of f such that*

$$\mathrm{Lie}(P_f) = \oplus_{\gamma \geq 0} \mathfrak{g}_\gamma.$$

Moreover L_f is a Levi subgroup of P_f, thus $P_f = U_f \rtimes L_f$ where U_f is the unipotent radical of P_f. For $\iota f = f^{-1}$, $P_{\iota f}$ is opposed to P_f, $L_f = P_f \cap P_{\iota f}$ and

$$\begin{array}{ll} \mathrm{Lie}(P_f) = \oplus_{\gamma \geq 0} \mathfrak{g}_\gamma & \mathrm{Lie}(P_{\iota f}) = \oplus_{\gamma \leq 0} \mathfrak{g}_\gamma \\ \mathrm{Lie}(U_f) = \oplus_{\gamma > 0} \mathfrak{g}_\gamma & \mathrm{Lie}(U_{\iota f}) = \oplus_{\gamma < 0} \mathfrak{g}_\gamma \end{array} \quad \textit{and} \quad \mathrm{Lie}(L_f) = \mathfrak{g}_0.$$

PROOF. Let Q be the image of f. Then L_f is the centralizer of Q by lemma 7 and Q is a locally trivial subtorus of G by proposition 10 (since Γ is torsion free). We may assume that Q is trivial, i.e. $Q \simeq \mathbb{D}_S(\Lambda)$ for some finitely generated subgroup Λ of Γ. The proposition then follows from [**21**, XXVI 6.1]. □

PROPOSITION 15. *The morphism $\mathbb{G}^\Gamma(G) \to \mathbb{G}^\Gamma(R_{\mathbb{OPP}(G)})$ is an open and closed immersion, and $\mathbb{G}^\Gamma(G) \to \mathbb{OPP}(G)$ is a quasi-isotrivial twisted constant morphism.*

PROOF. The second assertion follows from the first one by Grothendieck's proposition 3. Given a section (P_1, P_2, f) of $\mathbb{G}^\Gamma(R_{\mathbb{OPP}(G)})$ over some S-scheme T, we have to show that the condition $(P_1, P_2) = (P_f, P_{\iota f})$ is representable by an open and closed subscheme of T. It is plainly representable by the inverse image of the diagonal of $\mathbb{OPP}(G)$ under the S-morphism $T \to \mathbb{OPP}(G)^2$ defined by our two pairs (P_1, P_2) and $(P_f, P_{\iota f})$, which is a closed subscheme of T since $\mathbb{OPP}(G)$ is separated over S. On the other hand, since the Levi subgroup $L = P_1 \cap P_2$ of G is contained in $L_f = P_f \cap P_{\iota f}$, our condition $(P_1, P_2) = (P_f, P_{\iota f})$ is equivalent to $(\mathfrak{p}_1, \mathfrak{p}_2) = (\oplus_{\gamma \geq 0} \mathfrak{g}_\gamma, \oplus_{\gamma \leq 0} \mathfrak{g}_\gamma)$ where $\mathfrak{p}_i = \mathrm{Lie}(P_i)$: this last claim is local in the fpqc topology on T, we may thus assume that L contains a maximal torus of G and then apply [**21**, XXII 5.3.5]. Now write $\mathfrak{u}_i = \oplus \mathfrak{u}_{i,\gamma}$ for the weight decomposition of the Lie algebra of the unipotent radical of P_i, and set $\mathfrak{u}_i^{\pm} = \oplus_{\pm\gamma \geq 0} \mathfrak{u}_{i,\gamma}$. Then our Lie algebra condition is equivalent to the vanishing of the locally free sheaf $\mathfrak{u}_1^- \oplus \mathfrak{u}_2^+$, and it is therefore representable by the open complement of its support. □

Remark 16. This gives yet another proof of theorem 1 (using Grothendieck's proposition 3), when G is reductive and Γ torsion free (using [**33**] to construct a total order $\leq$ on Γ).

2.2.5. The cartesian diagram (in the fibered category of tori over schemes):

$$\begin{tikzcd} R_{\mathbb{OPP}(G)} \arrow[r, two heads, "p_1"] \arrow[d] & R_{\mathbb{P}(G)} \arrow[r, two heads, "t"] \arrow[d] & R_{\mathbb{O}(G)} \arrow[d] \\ \mathbb{OPP}(G) \arrow[r, two heads, "p_1"] & \mathbb{P}(G) \arrow[r, two heads, "t"] & \mathbb{O}(G) \end{tikzcd}$$

induces an analogous cartesian diagram (in the fibered category of quasi-isotrivial twisted constant group schemes over schemes):

$$\begin{tikzcd} \mathbb{G}^\Gamma(R_{\mathbb{OPP}(G)}) \arrow[r, two heads, "\mathrm{Fil}"] \arrow[d, "F"'] & \mathbb{G}^\Gamma(R_{\mathbb{P}(G)}) \arrow[r, two heads, "t"] \arrow[d, "F"'] & \mathbb{G}^\Gamma(R_{\mathbb{O}(G)}) \arrow[d, "F"'] \\ \mathbb{OPP}(G) \arrow[r, two heads, "p_1"] & \mathbb{P}(G) \arrow[r, two heads, "t"] & \mathbb{O}(G) \end{tikzcd}$$

which is given on T-valued points by the following formulas:

$$\begin{tikzcd} (P_1, P_2, f) \arrow[r, mapsto, "\mathrm{Fil}"] \arrow[d, mapsto, "F"'] & (P_1, \overline{f}) \arrow[r, mapsto, "t"] \arrow[d, mapsto, "F"'] & (t(P_1), \overline{f}) \arrow[d, mapsto, "F"'] \\ (P_1, P_2) \arrow[r, mapsto, "p_1"] & P_1 \arrow[r, mapsto, "t"] & t(P_1) \end{tikzcd}$$

Here $\overline{f} : \mathbb{D}_T(\Gamma) \to \overline{R}(P_1)$ is defined by the diagram

$$\begin{tikzcd} \mathbb{D}_T(\Gamma) \arrow[r, "f"] \arrow[dr, "\overline{f}"'] & R(L) \arrow[r, hook] \arrow[d, "\simeq"] & L \arrow[d, "\simeq"] \\ & \overline{R}(P_1) \arrow[r, hook] & P_1/U_1 \end{tikzcd}$$

where $L = P_1 \cap P_2$ and U_1 is the unipotent radical of P_1.

Lemma 17. *The open and closed subscheme $\mathbb{G}^\Gamma(G)$ of $\mathbb{G}^\Gamma(R_{\mathbb{OPP}(G)})$ is saturated with respect to $\mathbb{G}^\Gamma(R_{\mathbb{OPP}(G)}) \to \mathbb{G}^\Gamma(R_{\mathbb{P}(G)})$ and $\mathbb{G}^\Gamma(R_{\mathbb{OPP}(G)}) \to \mathbb{G}^\Gamma(R_{\mathbb{O}(G)})$.*

Proof. It is sufficient to establish that it is saturated with respect to the second map. We have to show: for an S-scheme T, a morphism $f : \mathbb{D}_T(\Gamma) \to G_T$, a pair of opposed parabolic subgroups (P_1, P_2) of G_T with Levi $L = P_1 \cap P_2$, and a central morphism $h : \mathbb{D}_T(\Gamma) \to L$, if $(P_f, P_{\iota f}, f)$ and (P_1, P_2, h) have the same image in $\mathbb{G}^\Gamma(R_{\mathbb{O}(G)})(T)$, then $(P_1, P_2) = (P_h, P_{\iota h})$. This is local in the fpqc topology on

T. Since $t(P_f) = t(P_1)$ by assumption, we may assume that there is a $g \in G(T)$ such that $\mathrm{Int}(g)(P_f, P_{\iota f}) = (P_1, P_2)$ by [**21**, 4.3.4 iii]. But then also $\mathrm{Int}(g) \circ f = h$ (by assumption), thus $(P_1, P_2) = \mathrm{Int}(g)(P_f, P_{\iota f}) = (P_h, P_{\iota h})$. □

2.2.6. By an elementary case of fpqc descent (along p_1 and t), we thus obtain a cartesian diagram of open and closed embeddings of smooth S-schemes,

$$\begin{tikzcd} \mathbb{G}^\Gamma(G) \arrow[r, two heads, "\mathrm{Fil}"] \arrow[d, hook] & \mathbb{F}^\Gamma(G) \arrow[r, two heads, "t"] \arrow[d, hook] & \mathbb{C}^\Gamma(G) \arrow[d, hook] \\ \mathbb{G}^\Gamma(R_{\mathbb{OPP}(G)}) \arrow[r, two heads, "\mathrm{Fil}"] & \mathbb{G}^\Gamma(R_{\mathbb{P}(G)}) \arrow[r, two heads, "t"] & \mathbb{G}^\Gamma(R_{\mathbb{O}(G)}) \end{tikzcd}$$

which in turns gives our fundamental cartesian diagram of smooth S-schemes

$$\begin{tikzcd} \mathbb{G}^\Gamma(G) \arrow[r, two heads, "\mathrm{Fil}"] \arrow[d, "F"'] & \mathbb{F}^\Gamma(G) \arrow[r, two heads, "t"] \arrow[d, "F"'] & \mathbb{C}^\Gamma(G) \arrow[d, "F"'] \\ \mathbb{OPP}(G) \arrow[r, two heads, "p_1"] & \mathbb{P}(G) \arrow[r, two heads, "t"] & \mathbb{O}(G) \end{tikzcd}$$

The S-group scheme G acts on both diagrams by conjugation and their last column are the quotients of the other two columns by the action of G in the category of fpqc sheaves on S. The morphism $\mathrm{Fil} : \mathbb{G}^\Gamma(G) \to \mathbb{F}^\Gamma(G)$ is a $U_{\mathbb{F}^\Gamma(G)}$-torsor, where $U_{\mathbb{F}^\Gamma(G)}$ is the unipotent radical of the pull-back $P_{\mathbb{F}^\Gamma(G)}$ of the universal parabolic subgroup P_u of $G_{\mathbb{P}(G)}$. In particular, it is affine smooth surjective with geometrically connected fibers. The morphism $t : \mathbb{F}^\Gamma(G) \to \mathbb{C}^\Gamma(G)$ is projective smooth surjective with geometrically connected fibers. The three *facet* morphisms F are quasi-isotrivial twisted constant (i.e. locally constant in the étale topology on their base), in particular they are separated and étale by lemma 4. We will see in due time that they are also surjective (4.1.11). Since $\mathbb{O}(G)$ is a finite étale cover of S, everyone is smooth, surjective and separated over S. We denote by

$$0 : S \to \mathbb{G}^\Gamma(G), \quad 0 : S \to \mathbb{F}^\Gamma(G) \quad \text{and} \quad 0 : S \to \mathbb{C}^\Gamma(G)$$

the element of $\mathbb{G}^\Gamma(G)(S)$ corresponding to the trivial morphism $\mathbb{D}_S(\Gamma) \to G$ or its images in $\mathbb{F}^\Gamma(G)(S)$ or $\mathbb{C}^\Gamma(G)(S)$. They respectively map to $(G, G) \in \mathbb{OPP}(G)(S)$, $G \in \mathbb{P}(G)$ and $\mathbb{DYN}(G) \in \mathbb{O}(G)$. Being sections of separated S-schemes, these 0-sections are closed immersions and the last one is also open.

If S is irreducible and geometrically unibranch or local henselian, then so are the connected components of $\mathbb{O}(G)$ by [**28**, 18.10.1 and 18.5.10]. Over each of them, the facet map $F : \mathbb{C}^\Gamma(G) \to \mathbb{O}(G)$ is then merely an infinite disjoint union of connected finite étale covers, see lemma 4. The same decomposition then also holds for its pull-backs over $\mathbb{P}(G)$ or $\mathbb{OPP}(G)$.

2.2.7. For an S-scheme T and morphisms $x, y : \mathbb{D}_T(\Gamma) \to G_T$, we have

$$\begin{aligned} \mathrm{Fil}(x) = \mathrm{Fil}(y) \quad \text{in} \quad \mathbb{F}^\Gamma(G)(T) \quad &\iff \quad \exists p \in P_x(T) : \ \mathrm{Int}(p)(x) = y \\ &\iff \quad \exists u \in U_x(T) : \ \mathrm{Int}(u)(x) = y \end{aligned}$$

and then such a u is unique. This equivalence relation is known as the Par-equivalence and denoted by $x \sim_{\mathrm{Par}} y$. If T is an (absolutely) affine scheme, then

$$\mathbb{F}^\Gamma(G)(T) = \mathbb{G}^\Gamma(G)(T)/\sim_{\mathrm{Par}}$$

by [**21**, XXVI 2.2]. On the other hand,

$$\begin{array}{rl} & t\circ \mathrm{Fil}(x) = t\circ \mathrm{Fil}(y) \quad \text{in} \quad \mathbb{C}^{\Gamma}(G)(T) \\ \iff & \text{fpqc locally on } T,\ \exists g\in G(T):\ \mathrm{Int}(g)(x) = y. \end{array}$$

If T is semi-local, then by [**21**, XXVI 5.2],

$$\begin{array}{rl} & t\circ \mathrm{Fil}(x) = t\circ \mathrm{Fil}(y) \quad \text{in} \quad \mathbb{C}^{\Gamma}(G)(T) \\ \iff & \exists g\in G(T):\ \mathrm{Int}(g)(x) = y. \end{array}$$

2.2.8. For an S-scheme T and $\mathcal{F}$ in $\mathbb{F}^{\Gamma}(G)(T)$, we denote by $(P_{\mathcal{F}}, \overline{\mathcal{F}})$ the image of $\mathcal{F}$ in $\mathbb{G}^{\Gamma}(R_{\mathbb{P}(G)})(T)$. Thus $P_{\mathcal{F}} = F(\mathcal{F})$ is a parabolic subgroup of G_T, equal to the stabilizer of $\mathcal{F}$ in G_T by [**21**, XXVI 1.2] and $\overline{\mathcal{F}}:\mathbb{D}_T(\Gamma)\to \overline{R}(P_{\mathcal{F}})$ is a morphism of tori over T. We write $U_{\mathcal{F}} = R^u P_{\mathcal{F}}$ for the unipotent radical of $P_{\mathcal{F}}$, so that $\overline{R}(P_{\mathcal{F}})$ is the radical of $P_{\mathcal{F}}/U_{\mathcal{F}}$. If L is a Levi subgroup of $P_{\mathcal{F}} = U_{\mathcal{F}}\rtimes L$ and $f:\mathbb{D}_T(\Gamma)\to L$ is the corresponding central morphism lifting $\overline{\mathcal{F}}$, then

$$\mathcal{F} = \mathrm{Fil}(f) \quad \text{and} \quad L_f = L.$$

The inversion $f\mapsto f^{-1}$ yields compatible involutions on $\mathbb{G}^{\Gamma}(G)$ and $\mathbb{C}^{\Gamma}(G)$, which we shall both denote by ι. By proposition 14, they are also compatible with the eponymous involutions on $\mathbb{OPP}(G)$ and $\mathbb{O}(G)$:

$$F\circ\iota = \iota\circ F \quad \text{on} \quad \mathbb{G}^{\Gamma}(G) \text{ or } \mathbb{C}^{\Gamma}(G).$$

2.2.9. Functoriality. The formation of our fundamental diagram

$$\begin{tikzcd} \mathbb{G}^{\Gamma}(G) \arrow[r, two heads, "\mathrm{Fil}"] \arrow[d, "F"'] & \mathbb{F}^{\Gamma}(G) \arrow[r, two heads, "t"] \arrow[d, "F"'] & \mathbb{C}^{\Gamma}(G) \arrow[d, "F"'] \\ \mathbb{OPP}(G) \arrow[r, two heads, "p_1"] & \mathbb{P}(G) \arrow[r, two heads, "t"] & \mathbb{O}(G) \end{tikzcd}$$

is plainly compatible with base change on S. We will see later on (corollary 64) that the first line is covariantly functorial in Γ and G. This is obvious for $\mathbb{G}^{\Gamma}(G)$ and easy for $\mathbb{C}^{\Gamma}(G) = G\backslash\mathbb{G}^{\Gamma}(G)$, but not so for $\mathbb{F}^{\Gamma}(G)$: the Γ-functoriality of $\mathbb{G}^{\Gamma}(G)$ is not compatible with the facet maps, and the second line of our diagram is simply not functorial in G. To showcase the first (bad) behavior, note that we will eventually have an action of the set $\mathrm{End}(\Gamma,+,\leq)$ of non-decreasing homomorphisms of Γ by morphisms of S-schemes on the first line, simply denoted by $(\lambda, x)\mapsto \lambda\cdot x$. Then $x\mapsto 0\cdot x$ is nothing but the structural morphism of the S-scheme $\mathbb{G}^{\Gamma}(G)$, $\mathbb{F}^{\Gamma}(G)$ or $\mathbb{C}^{\Gamma}(G)$, followed by the corresponding 0-section. Thus $F(0\cdot x) = \mathbb{DYN}(G)$ in $\mathbb{O}(G)$ for any $x\in\mathbb{C}^{\Gamma}(G)$. However, for a monomorphism $\gamma:(\Gamma_1,+,\leq)\hookrightarrow(\Gamma_2,+,\leq)$, the induced morphisms γ in the commutative diagram

$$\begin{tikzcd} \mathbb{G}^{\Gamma_1}(G) \arrow[r, two heads, "\mathrm{Fil}"] \arrow[d, hook, "\gamma"'] & \mathbb{F}^{\Gamma_1}(G) \arrow[r, two heads, "t"] \arrow[d, hook, "\gamma"'] & \mathbb{C}^{\Gamma_1}(G) \arrow[d, hook, "\gamma"'] \\ \mathbb{G}^{\Gamma_2}(G) \arrow[r, two heads, "\mathrm{Fil}"] & \mathbb{F}^{\Gamma_2}(G) \arrow[r, two heads, "t"] & \mathbb{C}^{\Gamma_2}(G) \end{tikzcd}$$

are open and closed immersions which commute with the facet maps: this follows from proposition 10 and 14, given the construction of our fundamental diagram.

2.2.10. $\mathbb{C}^\Gamma(G)$ is a commutative monoid. There is natural structure of commutative monoid on the S-scheme $\mathbb{O}(G)$, given by the intersection morphism

$$\cap : \mathbb{O}(G) \times_S \mathbb{O}(G) \to \mathbb{O}(G) \qquad (a,b) \mapsto a \cap b$$

Let $\mathbb{O}'(G)$ be the open and closed subscheme of $\mathbb{O}(G) \times_S \mathbb{O}(G)$ on which $a \cap b = a$, i.e. $a \subset b$. Let p_1 and $p_2 : \mathbb{O}'(G) \to \mathbb{O}(G)$ be the two projections. We claim:

LEMMA 18. *There exists a canonical morphism $p_2^* R_{\mathbb{O}(G)} \to p_1^* R_{\mathbb{O}(G)}$.*

PROOF. Let $\mathbb{P}'(G)$ be the inverse image of $\mathbb{O}'(G)$ in $\mathbb{P}(G) \times_S \mathbb{P}(G)$, and denote by q_1 and $q_2 : \mathbb{P}'(G) \to \mathbb{P}(G)$ the two projections. Then $q_i^*(R_{\mathbb{P}(G)}) = (p_i^* R_{\mathbb{O}(G)})_{\mathbb{P}'(G)}$ for $i \in \{1,2\}$. We have to show that there is a canonical morphism

$$q_2^* R_{\mathbb{P}(G)} \to q_1^* R_{\mathbb{P}(G)}$$

compatible with the descent data on both sides. This boils down to: for any $S' \to S$ and $(P_1, P_2) \in \mathbb{P}'(G)(S')$, there exists a canonical morphism $\overline{R}(P_2) \to \overline{R}(P_1)$. We may assume that $S' = S$. Since $t(P_1) \subset t(P_2)$, there exists by [**21**, XXVI 3.8] a unique parabolic subgroup P_2' of G, containing P_1, such that $t(P_2) = t(P_2')$. Using the canonical isomorphism $\overline{R}(P_2') \simeq \overline{R}(P_2)$, we may thus assume that $P_2' = P_2$, i.e. $P_1 \subset P_2$. Let U_i be the unipotent radical of P_i, so that $U_2 \subset U_1$ is a normal subgroup of P_1. Then P_1/U_2 is a parabolic subgroup of P_2/U_2 with maximal reductive quotient P_1/U_1, which reduces us further to the case where $G = P_2$. Then P_1 contains the radical $\overline{R}(G) = R(G)$ of G, and $P_1 \to P_1/U_1$ maps $R(G)$ to the radical $\overline{R}(P_1)$ of P_1/U_1. This yields our canonical morphism $\overline{R}(P_2) \to \overline{R}(P_1)$. $\square$

Pulling back the above morphism through

$$\begin{array}{rcl} \mathbb{O}(G) \times_S \mathbb{O}(G) & \to & \mathbb{O}'(G) \\ (a,b) & \mapsto & (a \cap b, b) \end{array}$$

we obtain a morphism $p_2^* R_{\mathbb{O}(G)} \to (\cap)^* R_{\mathbb{O}(G)}$ of tori over $\mathbb{O}(G) \times_S \mathbb{O}(G)$. By symmetry, there is also a morphism $p_1^* R_{\mathbb{O}(G)} \to (\cap)^* R_{\mathbb{O}(G)}$. The product of these two yields a morphism in the fibered category of tori over Sch/S,

$$\begin{tikzcd} R_{\mathbb{O}(G)} \times_S R_{\mathbb{O}(G)} \arrow[r] \arrow[d] & R_{\mathbb{O}(G)} \times_{\mathbb{O}(G)} R_{\mathbb{O}(G)} \arrow[d] \\ \mathbb{O}(G) \times_S \mathbb{O}(G) \arrow[r, "\cap"] & \mathbb{O}(G) \end{tikzcd}$$

Composing it with the multiplication map on the $\mathbb{O}(G)$-torus $R_{\mathbb{O}(G)}$, we obtain yet another such morphism, namely

$$\begin{tikzcd} R_{\mathbb{O}(G)} \times_S R_{\mathbb{O}(G)} \arrow[r] \arrow[d] & R_{\mathbb{O}(G)} \arrow[d] \\ \mathbb{O}(G) \times_S \mathbb{O}(G) \arrow[r, "\cap"] & \mathbb{O}(G) \end{tikzcd}$$

Applying now the $\mathbb{G}^\Gamma(-)$ construction to the latter diagram yields a morphism

$$\begin{tikzcd} \mathbb{G}^\Gamma(R_{\mathbb{O}(G)}) \times_S \mathbb{G}^\Gamma(R_{\mathbb{O}(G)}) \arrow[r] \arrow[d] & \mathbb{G}^\Gamma(R_{\mathbb{O}(G)}) \arrow[d] \\ \mathbb{O}(G) \times_S \mathbb{O}(G) \arrow[r, "\cap"] & \mathbb{O}(G) \end{tikzcd}$$

in the fibered category of commutative group schemes over Sch/S. The top map of this diagram defines a commutative monoid structure on the S-scheme $\mathbb{G}^\Gamma(R_{\mathbb{O}(G)})$. By construction, the structural morphism $\mathbb{G}^\Gamma(R_{\mathbb{O}(G)}) \to \mathbb{O}(G)$ is compatible with the monoid structures on both sides.

LEMMA 19. *The S-scheme $\mathbb{C}^\Gamma(G)$ is a submonoid of $\mathbb{G}^\Gamma(R_{\mathbb{O}(G)})$.*

PROOF. Using additive notations, we have to show that for $S' \to S$ and c_1, c_2 in $\mathbb{C}^\Gamma(G)(S')$, there exists an fpqc cover $S'' \to S'$ and an element $f \in \mathbb{G}^\Gamma(G)(S'')$ such that $c_1 + c_2 = t \circ \mathrm{Fil}(f)$ in $\mathbb{G}^\Gamma(R_{\mathbb{O}(G)})$. We may assume that $S' = S$ and $c_i = t \circ \mathrm{Fil}(f_i)$ for some $f_i : \mathbb{D}_S(\Gamma) \to G$. Using [**21**, XXVI 1.14 and XXIV 1.5], we may also assume that there is an épinglage $(G, T, \Delta, \cdots)$ which is adapted to $P_1 = P_{f_1}$ and $P_2 = P_{f_2}$. Then by [**21**, XXVI 1.6 and 1.8], we may assume that $L_1 = L_{f_1}$ and $L_2 = L_{f_2}$ both contain the maximal torus T of G, so that both f_1 and f_2 factor through T. Let $f = f_1 + f_2 : \mathbb{D}_S(\Gamma) \to T \hookrightarrow G$ and $P = P_f$. We claim that $c_1 + c_2 = t \circ \mathrm{Fil}(f)$. By [**21**, XXVI 3.2], $t(P_i) = \Delta(P_i)_S$ where $\Delta(P_i) \subset \Delta \subset \mathrm{Hom}(T, \mathbb{G}_{m,S})$ is the set of simple roots occurring in $\mathrm{Lie}(L_i)$, i.e. $\Delta(P_i) = \{\alpha \in \Delta : \alpha \circ f_i = 0 \in \Gamma\}$. By construction, $\alpha \circ f_i \geq 0$ in Γ for every $\alpha \in \Delta$, thus also $\alpha \circ f = \alpha \circ f_1 + \alpha \circ f_2 \geq 0$ in Γ for every $\alpha \in \Delta$, with $\alpha \circ f = 0$ if and only if $\alpha \circ f_1 = 0 = \alpha \circ f_2$. It follows that our épinglage is also adapted to P, with $\Delta(P) = \Delta(P_1) \cap \Delta(P_2)$, i.e. $t(P) = t(P_1) \cap t(P_2)$ in $\mathbb{O}(G)(S)$. The inclusion $P \subset P_i$ induces the canonical morphism $\mathrm{can}_i : \overline{R}(P_i) \to \overline{R}(P)$ and one checks easily that $\overline{f} = \mathrm{can}_1 \circ \overline{f}_1 + \mathrm{can}_2 \circ \overline{f}_2$. Thus by definition,

$$c_1 + c_2 = t(P, \mathrm{can}_1 \circ \overline{f}_1 + \mathrm{can}_2 \circ \overline{f}_2) = t(P, \overline{f}) = t \circ \mathrm{Fil}(f)$$

as was to be shown. □

REMARK 20. The 0-section of $\mathbb{C}^\Gamma(G)$ is the identity element of its monoid structure. The latter is compatible with functoriality in Γ, but not with functoriality in G: if H is a subtorus of G and f is a section of $\mathbb{G}^\Gamma(H) = \mathbb{F}^\Gamma(H) = \mathbb{C}^\Gamma(H)$, then $f + \iota f$ is trivial in $\mathbb{C}^\Gamma(H)$, but not necessarily in $\mathbb{C}^\Gamma(G)$.

2.2.11. The split case. Suppose that (G, T, M, R) is a split reductive group over S [**21**, XXII 1.13]: G is a reductive group over S, M is a finite free $\mathbb{Z}$-module, $T \subset G$ is a maximal subtorus of G equipped with an isomorphism $T \simeq \mathbb{D}_S(M)$, $R \subset M$ is a set of roots of T in G and for each $\alpha \in R$, the corresponding quasi-coherent sub-sheaf $\mathfrak{g}_\alpha$ of $\mathfrak{g} = \mathrm{Lie}(G)$ is a free $\mathcal{O}_S$-module (of rank 1). Let

$$\mathscr{R} = (M, R, M^*, R^*)$$

be the induced (reduced) root system [**21**, XXII 1.14] with Weyl group $W = W(\mathscr{R})$ in $\mathrm{Aut}(M)$. Let $W_G(T) = N_G(T)/Z_G(T)$ be the Weyl group of T in G, a constant group scheme over S identified with W_S through its action on T [**21**, XXII 3.4]. The composition of the isomorphism of group schemes over S

$$(\mathrm{Hom}\,(M, \Gamma))_S \simeq \mathrm{Hom}_{S-\mathsf{Group}}\,(\mathbb{D}_S(\Gamma), \mathbb{D}_S(M)) \simeq \mathbb{G}^\Gamma(T)$$

from [**20**, VIII 1.5] with the morphism of S-schemes

$$\mathbb{G}^\Gamma(T) \hookrightarrow \mathbb{G}^\Gamma(G) \xrightarrow{\mathrm{Fil}} \mathbb{F}^\Gamma(G) \xrightarrow{t} \mathbb{C}^\Gamma(G)$$

thus factors through a morphism of étale S-schemes,

$$(W \backslash \mathrm{Hom}\,(M, \Gamma))_S \to \mathbb{C}^\Gamma(G).$$

We claim that the latter is an isomorphism. Since both sides are étale over S, it is sufficient to establish that for any geometric point $\mathrm{Spec}(k) \to S$, the induced map

$$W\backslash\mathrm{Hom}\,(M,\Gamma) \to \mathbb{C}^{\Gamma}(G)(k) = G(k)\backslash\mathrm{Hom}_{k-\mathsf{Group}}\,(\mathbb{D}_k(\Gamma), G_k)$$

is a bijection. Any $f : \mathbb{D}_k(\Gamma) \to G_k$ factors through a maximal torus T' of G_k by corollary 11, and $T' = \mathrm{Int}(g)(T_k)$ for some $g \in G(k)$ by [**1**, XII 6.6.a]: our map is surjective. For $\varphi, \varphi' : M \to \Gamma$ giving $f, f' : \mathbb{D}_k(\Gamma) \to T_k$ and $g \in G(k)$ such that $\mathrm{Int}(g) \circ f = f'$, $\mathrm{Int}(g)(T_k)$ and T_k are maximal tori of $L_{f'}$, thus $\mathrm{Int}(hg)(T_k) = T_k$ for some $h \in L_{f'}(k)$; but then $n = hg \in N_G(T)(k)$ and $\mathrm{Int}(n) \circ f = \mathrm{Int}(h) \circ f' = f'$, thus $\varphi' = w\varphi$ where w is the image of n in $W = W_G(T)(k)$: our map is injective.

Fix a system of positive roots $R_+ \subset R$ [**21**, XXI 3.2.1], which corresponds to a Borel subgroup B of G by [**21**, XXII 5.5.1]. By lemma 28 below, the submonoid

$$\mathrm{Hom}^+(M,\Gamma) = \{f \in \mathrm{Hom}(M,\Gamma) : \forall \alpha \in R_+,\ f(\alpha) \geq 0\}$$

is a fundamental domain for the action of W on $\mathrm{Hom}(M,\Gamma)$. The isomorphism

$$\left(\mathrm{Hom}^+\,(M,\Gamma)\right)_S \simeq \left(W\backslash\mathrm{Hom}\,(M,\Gamma)\right)_S \simeq \mathbb{C}^{\Gamma}(G)$$

is then easily seen to be compatible with the S-monoid structures.

2.2.12. $\mathbb{C}^{\Gamma}(G)$ is a partially ordered commutative monoid. A partial order $\leq$ on an S-scheme X is a subscheme $\mathcal{R} = \mathcal{R}(\leq)$ of $X \times_S X$ such that for every S-scheme Y, the subset $\mathcal{R}(Y)$ of $X(Y) \times X(Y)$ defines a partial order (also denoted by $\leq$) on $X(Y)$. We say that the partial order is open (resp. closed) if $\mathcal{R} \hookrightarrow X \times_S X$ is an open (resp. closed) immersion. A partial order on an S-monoid $(X, \cdot)$ is a partial order on the underlying S-scheme such that for any S-scheme Y and f_1, f_2, g in $X(Y)$, $f_1 \leq f_2$ implies $f_1 \cdot g \leq f_2 \cdot g$ and $g \cdot f_1 \leq g \cdot f_2$.

If $\mathscr{R} = (M, R, M^*, R^*)$ is a (not necessarily reduced) root system and $R_+ \subset R$ is a system of positive roots, the weak ($\leq$) and strong ($\preceq$) partial orders on the abstract monoid $\mathrm{Hom}^+(M,\Gamma)$ defined in section 2.4 below induce open and closed partial orders on the constant S-monoid $\left(\mathrm{Hom}^+\,(M,\Gamma)\right)_S$. If $\mathscr{R} = \mathscr{R}(G,T,M,R)$ is the root system of a split reductive group (G,T,M,R), we thus obtain open and closed partial orders on the S-monoid $\mathbb{C}^{\Gamma}(G)$. These partial orders then do not depend upon the chosen auxiliary data $(T, M, R; R^+)$: this may be checked on geometric points, where all such data are indeed conjugated. Since every reductive group G over S is locally splittable in the étale topology on S [**21**, XXII 2.3], we finally obtain by étale descent: the S-monoid $\mathbb{C}^{\Gamma}(G)$ is canonically equipped with weak ($\leq$) and strong ($\preceq$) partial orders, both open and closed.

The weak and strong partial orders are functorial in Γ, and coincide if Γ is divisible. We will see later on that the weak partial order is also functorial in G.

2.2.13. Behavior under isogenies. Suppose that the (torsion free) commutative group Γ is (uniquely) divisible, i.e. that it is a $\mathbb{Q}$-vector space.

PROPOSITION 21. *The fundamental cartesian diagram*

$$\begin{tikzcd}
\mathbb{G}^{\Gamma}(G) \arrow[r, "\mathrm{Fil}"] \arrow[d, "F"] & \mathbb{F}^{\Gamma}(G) \arrow[r, "t"] \arrow[d, "F"] & \mathbb{C}^{\Gamma}(G) \arrow[d, "F"] \\
\mathbb{OPP}(G) \arrow[r, "p_1"] & \mathbb{P}(G) \arrow[r, "t"] & \mathbb{O}(G)
\end{tikzcd}$$

is invariant under central isogenies.

PROOF. The bottom line only depends upon the adjoint group $G^{\mathrm{ad}} = G/Z(G)$: this is true for $\mathbb{O}(G)$ because $\mathbb{DYN}(G) = \mathbb{DYN}(G^{\mathrm{ad}})$ by definition of the Dynkin S-scheme [**21**, XXIV 3.3] in view of [**21**, XXII 4.3.7], and the maps $P \mapsto P/Z(G)$ and $P^{\mathrm{ad}} \mapsto \mathrm{ad}^{-1}(P^{\mathrm{ad}})$ (where $\mathrm{ad} : G \to G^{\mathrm{ad}}$ is the quotient map) induce mutually inverse bijections between parabolic subgroups of G and parabolic subgroups of G^{ad}, which are compatible with the type maps and with opposition. For the top line, let $f : G_1 \to G_2$ be a central isogeny [**21**, XXII 4.2.9]. We first claim that composition with f yields an isomorphism $\mathbb{G}^{\Gamma}(G_1) \to \mathbb{G}^{\Gamma}(G_2)$: for split tori, this immediately follows from [**20**, VIII 1.5] and our assumption on Γ; for tori, our claim is local in the fpqc topology on S by [**26**, 2.7.1], which reduces us to the previous case; for arbitrary reductive groups, use lemma 5 and [**1**, XVII 7.1.1]. If now P_1 is a parabolic subgroup of G_1 with image P_2 in G_2, then f induces an isogeny $\overline{R}(P_1) \to \overline{R}(P_2)$. Thus f yields an isogeny $R_f : R_{\mathbb{P}(G_1)} \to R_{\mathbb{P}(G_2)}$ of tori over $\mathbb{P}(G_1) \simeq \mathbb{P}(G_2)$. The induced isomorphism $\mathbb{G}^{\Gamma}(R_{\mathbb{P}(G_1)}) \simeq \mathbb{G}^{\Gamma}(R_{\mathbb{P}(G_2)})$ is compatible with the morphisms $\mathbb{G}^{\Gamma}(G_i) \to \mathbb{G}^{\Gamma}(R_{\mathbb{P}(G_i)})$, therefore also $\mathbb{F}^{\Gamma}(G_1) \simeq \mathbb{F}^{\Gamma}(G_2)$. Since R_f is also compatible with the canonical descent data of lemma 13, it descends to an isogeny $R_f : R_{\mathbb{O}(G_1)} \to R_{\mathbb{O}(G_2)}$ of tori over $\mathbb{O}(G_1) \simeq \mathbb{O}(G_2)$. The induced isomorphism $\mathbb{G}^{\Gamma}(R_{\mathbb{O}(G_1)}) \simeq \mathbb{G}^{\Gamma}(R_{\mathbb{O}(G_2)})$ is again compatible with the morphisms $\mathbb{F}^{\Gamma}(G_i) \to \mathbb{G}^{\Gamma}(R_{\mathbb{O}(G_i)})$, therefore also $\mathbb{C}^{\Gamma}(G_1) \simeq \mathbb{C}^{\Gamma}(G_2)$. □

Plainly, the above diagrams are also compatible with products. Considering the canonical diagram of central isogenies [**21**, XXII 4.3 & 6.2]

$$R(G) \times G^{\mathrm{der}} \to G \to G^{\mathrm{ab}} \times G^{\mathrm{ss}} \to G^{\mathrm{ab}} \times G^{\mathrm{ad}}$$

where $R(G)$ is the radical of G, G^{der} its derived group, $G^{\mathrm{ab}} = G/G^{\mathrm{der}}$ its coradical, $G^{\mathrm{ss}} = G/R(G)$ its semi-simplification and $G^{\mathrm{ad}} = G/Z(G)$ its adjoint group, we obtain compatible canonical decompositions

$$\begin{array}{ccccc} \mathbb{G}^{\Gamma}(G) & = & \mathbb{G}^{\Gamma}(G)^r & \times & \mathbb{G}^{\Gamma}(G)^c \\ \mathbb{F}^{\Gamma}(G) & = & \mathbb{F}^{\Gamma}(G)^r & \times & \mathbb{F}^{\Gamma}(G)^c \\ \mathbb{C}^{\Gamma}(G) & = & \mathbb{C}^{\Gamma}(G)^r & \times & \mathbb{C}^{\Gamma}(G)^c \end{array}$$

with $\mathbb{G}^{\Gamma}(G)^c = \mathbb{F}^{\Gamma}(G)^c = \mathbb{C}^{\Gamma}(G)^c = \mathbb{G}^{\Gamma}(R(G)) = \mathbb{G}^{\Gamma}(G^{\mathrm{ab}}) = \mathbb{G}^{\Gamma}(Z(G))$ and

$$\begin{array}{ccccccc} \mathbb{G}^{\Gamma}(G)^r & = & \mathbb{G}^{\Gamma}(G^{\mathrm{der}}) & = & \mathbb{G}^{\Gamma}(G^{\mathrm{ss}}) & = & \mathbb{G}^{\Gamma}(G^{\mathrm{ad}}), \\ \mathbb{F}^{\Gamma}(G)^r & = & \mathbb{F}^{\Gamma}(G^{\mathrm{der}}) & = & \mathbb{F}^{\Gamma}(G^{\mathrm{ss}}) & = & \mathbb{F}^{\Gamma}(G^{\mathrm{ad}}), \\ \mathbb{C}^{\Gamma}(G)^r & = & \mathbb{C}^{\Gamma}(G^{\mathrm{der}}) & = & \mathbb{C}^{\Gamma}(G^{\mathrm{ss}}) & = & \mathbb{C}^{\Gamma}(G^{\mathrm{ad}}). \end{array}$$

The decomposition of $\mathbb{C}^{\Gamma}(G)$ is compatible with the partially ordered (weak=strong) monoid structures: for $x = (x^r, x^c)$ and $y = (y^r, y^c)$ in $\mathbb{C}^{\Gamma}(G) = \mathbb{C}^{\Gamma}(G)^r \times \mathbb{C}^{\Gamma}(G)$,

$$x + y = (x^r + y^r, x^c + y^c) \quad \text{and} \quad x \leq y \iff (x^r \leq y^r \text{ and } x^c = y^c)\,.$$

This is easily checked by reduction to the split case, cf. section 2.4.10 below.

2.3. Relative positions of Γ-filtrations

Let G be a reductive group over S.

2.3.1. Standard positions. Recall that two parabolic subgroups P_1 and P_2 of G are said to be in standard (relative) position if and only if they satisfy the equivalent conditions of [**21**, XXVI 4.5.1], in particular: (i) $P_1 \cap P_2$ is smooth over S, or (ii) $P_1 \cap P_2$ is a subgroup of type (R) of G, or (iv) $P_1 \cap P_2$ contains, locally on S for the Zariski topology, a maximal torus of G. Then all such maximal tori are, locally on S for the étale topology, conjugated in $P_1 \cap P_2$ [**1**, XII 7.1]. In any case, $P_1 \cap P_2$ has geometrically connected fibers [**5**, 4.5]. For an S-scheme T, we set

$$\mathbb{STD}(G)(T) = \left\{(P_1, P_2) \in \mathbb{P}(G)^2(T) : \ P_1 \text{ and } P_2 \text{ are in standard position}\right\}.$$

By [**21**, XXVI 4.5.3], this defines a representable subsheaf of $\mathbb{P}(G)^2$ with Stein factorization

$$\mathbb{STD}(G) \xrightarrow{t_2} \mathbb{TSTD}(G) \longrightarrow S$$

fitting in a commutative (but not cartesian) diagram

$$\begin{tikzcd} \mathbb{STD}(G) \arrow[r, "t_2"] \arrow[d, hook] & \mathbb{TSTD}(G) \arrow[d, "q"'] \\ \mathbb{P}(G)^2 \arrow[r, "t^2"] & \mathbb{O}(G)^2 \end{tikzcd}$$

where q is a finite étale surjective morphism while t_2 is a smooth, surjective, finitely presented morphism with geometrically connected fibers which is a quotient of $\mathbb{STD}(G)$ by the diagonal action of G in the category of fpqc sheaves on S. By [**21**, XXVI 4.2.5 & 4.4.3], the morphism q has two canonical sections

$$tr, os : \mathbb{O}(G)^2 \to \mathbb{TSDT}(G)$$

corresponding respectively to the transverse and osculatory (standard) positions. By [**21**, XXVI 4.2.4], $t_2^{-1}(\mathrm{im}(tr))$ is a relatively dense open S-subscheme $\mathbb{GEN}(G)$ of $\mathbb{P}(G)^2$. Pulling back everything through the surjective étale facet morphism $F^2 : \mathbb{C}^\Gamma(G)^2 \to \mathbb{O}(G)^2$, we thus obtain a commutative diagram

$$\begin{tikzcd} \mathbb{STD}^\Gamma(G) \arrow[r, "t_2"] \arrow[d, hook] & \mathbb{TSTD}^\Gamma(G) \arrow[d, shift right, "q"'] \\ \mathbb{F}^\Gamma(G)^2 \arrow[r, "t^2"] & \mathbb{C}^\Gamma(G)^2 \arrow[u, shift right, "tr,os"'] \end{tikzcd}$$

where t_2 and q still have the properties listed above, together with a relatively dense open S-subscheme $\mathbb{GEN}^\Gamma(G)$ of $\mathbb{F}^\Gamma(G)^2$. For a scheme Z over $\mathbb{P}(G)^2$, we set

$$\mathbb{STD}(Z) = Z \times_{\mathbb{P}(G)^2} \mathbb{STD}(G).$$

For instance, $\mathbb{STD}^\Gamma(G) = \mathbb{STD}\left(\mathbb{F}^\Gamma(G) \times_S \mathbb{F}^\Gamma(G)\right)$.

REMARK 22. The monomorphisms $\mathbb{STD}(G) \hookrightarrow \mathbb{P}(G)^2$ and $\mathbb{STD}^\Gamma(G) \hookrightarrow \mathbb{F}^\Gamma(G)^2$ are surjective. More precisely, for any S-scheme $T = \mathrm{Spec}(k)$ with k a field,

$$\mathbb{STD}(G)(k) = \mathbb{P}(G)^2(k) \quad \text{and} \quad \mathbb{STD}^\Gamma(G)(k) = \mathbb{F}^\Gamma(G)^2(k)$$

by Bruhat's theorem [**21**, XXVI 4.1.1].

2.3.2. The addition map on Γ-filtrations.

PROPOSITION 23. *There is an S-morphism*

$$+ : \mathbb{STD}^{\Gamma}(G) \to \mathbb{F}^{\Gamma}(G), \qquad (\mathcal{F}, \mathcal{G}) \mapsto \mathcal{F} + \mathcal{G}$$

such that for every S-scheme T, $(\mathcal{F}, \mathcal{G}) \in \mathbb{STD}^{\Gamma}(G)(T)$ and $\mathcal{H} \in \mathbb{F}^{\Gamma}(G)(T)$,

$$\mathcal{F} + \mathcal{G} = \mathcal{G} + \mathcal{F} \quad \text{and} \quad \mathcal{H} + 0 = 0 + \mathcal{H} = \mathcal{H} \quad \text{in} \quad \mathbb{F}^{\Gamma}(G)(T).$$

PROOF. Since $(P_{\mathcal{F}}, P_{\mathcal{G}}) \in \mathbb{STD}(G)(T)$, there is, locally on T for the Zariski topology, a maximal torus H of G_T inside $P_{\mathcal{F}} \cap P_{\mathcal{G}}$ [**21**, XXVI 4.5.1]. Let $L_{\mathcal{F}}$ and $L_{\mathcal{G}}$ be the Levi subgroups of $P_{\mathcal{F}}$ and $P_{\mathcal{G}}$ containing H [**21**, XXVI 1.6], let $f : \mathbb{D}_T(\Gamma) \to L_{\mathcal{F}}$ and $g : \mathbb{D}_T(\Gamma) \to L_{\mathcal{G}}$ be the corresponding central morphisms lifting $\overline{\mathcal{F}}$ and $\overline{\mathcal{G}}$. Then f and g both factor through H, and their product $f + g$ in the commutative group H is a group homomorphism $\mathbb{D}_T(\Gamma) \to G_T$. We claim that $\mathcal{F} + \mathcal{G} = \mathrm{Fil}(f + g)$ does not depend upon the choice of the maximal torus H – the whole construction is then indeed local in the Zariski topology on T as well as functorial in the S-scheme T, and the resulting S-morphism $+ : \mathbb{STD}^{\Gamma}(G) \to \mathbb{F}^{\Gamma}(G)$ obviously has the required properties. Let thus H' be another maximal torus of G_T inside $K = P_{\mathcal{F}} \cap P_{\mathcal{G}}$, giving rise to f', g' and $f' + g' : \mathbb{D}_T(\Gamma) \to H' \subset G$. Then, locally on T for the étale topology, there is a $k \in K(T)$ such that $H' = \mathrm{Int}(k)(H)$ by [**1**, XII 7.1], in which case also $f' = \mathrm{Int}(k) \circ f$, $g' = \mathrm{Int}(k) \circ g$ and $f' + g' = \mathrm{Int}(k) \circ (f + g)$. It is thus sufficient to establish that $K \subset P_{f+g}$. This second claim is again local in the étale topology on T, which reduces us further to the following case: (G, H, M, R) is a split group over $S = T$ (i.e. $H \simeq \mathbb{D}_S(M)$ and $R \subset M$ is the set of roots of H in $\mathrm{Lie}(G)$) with f and g respectively induced by morphisms $f^{\sharp}$ and $g^{\sharp} : M \to \Gamma$, so that $f + g$ is induced by $(f + g)^{\sharp} = f^{\sharp} + g^{\sharp}$. For a closed subset R' of R, we denote by $H_{R'} \supset H$ the corresponding subgroup of G of type (R), as in [**21**, XXII 5.4] (thus $H = H_{\emptyset}$ and $G = H_R$). Then $P_{\mathcal{F}} = H_{R(f)}$, $P_{\mathcal{G}} = H_{R(g)}$ and $P_{f+g} = H_{R(f+g)}$ where $R(h) = \{\alpha \in R : h^{\sharp}(\alpha) \geq 0\}$ by definition of these parabolic subgroups of G. Thus $K = H_{R(f) \cap R(g)}$ is contained in $H_{R(f+g)} = P_{f+g}$ by [**21**, XXII 5.4.5]. □

PROPOSITION 24. *For any S-scheme T and $(\mathcal{F}, \mathcal{G}) \in \mathbb{STD}^{\Gamma}(G)(T)$,*

$$t(\mathcal{F} + \mathcal{G}) \preceq t(\mathcal{F}) + t(\mathcal{G}) \quad \text{in} \quad \mathbb{C}^{\Gamma}(G)(T)$$

with equality if $\mathcal{F}$ and $\mathcal{G}$ are in osculatory position.

PROOF. We may assume that $T = s$ is a geometric point, with $\mathcal{F}$ and $\mathcal{G}$ lifting to morphisms $f, g : \mathbb{D}_s(\Gamma) \to H$ for some maximal (split) subtorus $H \simeq \mathbb{D}_s(M)$ of G_s, corresponding to morphisms $f^{\sharp}, g^{\sharp} : M \to \Gamma$ as above. Let $R \subset M$ be the roots of H in $\mathrm{Lie}(G_s)$. By [**21**, XXI 3.3.6], there is a system of positive roots $R_+ \subset R$ such that $(f^{\sharp} + g^{\sharp})(R_+) \subset \Gamma_+$, i.e. $f^{\sharp} + g^{\sharp} \in \mathrm{Hom}^+(M, \Gamma)$ in the notations of section 2.4. Thus if $\vartheta : \mathrm{Hom}(M, \Gamma) \twoheadrightarrow \mathrm{Hom}^+(M, \Gamma)$ is the retraction from lemma 28, then $\vartheta(f^{\sharp} + g^{\sharp}) = f^{\sharp} + g^{\sharp}$ and $f^{\sharp} \preceq \vartheta(f^{\sharp})$, $g^{\sharp} \preceq \vartheta(g^{\sharp})$ in $\mathrm{Hom}(M, \Gamma)$ by lemma 29, therefore $\vartheta(f^{\sharp} + g^{\sharp}) \preceq \vartheta(f^{\sharp}) + \vartheta(g^{\sharp})$ in $\mathrm{Hom}^+(M, \Gamma)$, i.e. $t(\mathcal{F} + \mathcal{G}) \preceq t(\mathcal{F}) + t(\mathcal{G})$ in $\mathbb{C}^{\Gamma}(G)(s)$ by definition. If $\mathcal{F}$ and $\mathcal{G}$ are in osculatory position, there is a Borel subgroup B of G_s inside $P_{\mathcal{F}} \cap P_{\mathcal{G}}$ [**21**, XXVI 4.4.1]. We may then take H inside B and for R_+, the roots of H in $\mathrm{Lie}(B)$, so that $f^{\sharp}$ and $g^{\sharp}$ already belong to $\mathrm{Hom}^+(M, \Gamma)$, $\vartheta(f^{\sharp} + g^{\sharp}) = \vartheta(f^{\sharp}) + \vartheta(g^{\sharp})$ and indeed $t(\mathcal{F} + \mathcal{G}) = t(\mathcal{F}) + t(\mathcal{G})$. □

2.3.3. We record here a special case of the functoriality of $\mathbb{F}^\Gamma(-)$.

PROPOSITION 25. *Let L be a Levi subgroup of a parabolic subgroup P of G with unipotent radical U. Then:* (1) *for $f : \mathbb{D}_S(\Gamma) \to L$ inducing $g : \mathbb{D}_S(\Gamma) \to G$ and $h : \mathbb{D}_S(\Gamma) \to P/U$, the parabolic subgroups P and P_g of G are in standard relative position, $K = P \cap P_g$ is a smooth subgroup scheme of G, $K \cdot U$ is a parabolic subgroup of G with Levi L_f, $K \cap L = (K \cdot U) \cap L = P_f$ and $K \cdot U/U = P_h$ in P/U.* (2) *There is a unique morphism $\iota : \mathbb{F}^\Gamma(L) \to \mathbb{F}^\Gamma(G)$ such that the diagram*

$$\begin{tikzcd} \mathbb{G}^\Gamma(L) \arrow[r, hook] \arrow[d, "\mathrm{Fil}"'] & \mathbb{G}^\Gamma(G) \arrow[d, "\mathrm{Fil}"] \\ \mathbb{F}^\Gamma(L) \arrow[r, "\iota"] & \mathbb{F}^\Gamma(G) \end{tikzcd}$$

is commutative.

PROOF. Everything in (1) is local for the fpqc topology on S. We may thus assume that $L_f = Z_L(f)$ and G are split with respect to a maximal torus H of G contained in L_f with H trivial, i.e. $H = \mathbb{D}_S(M)$ for some finitely generated abelian group M [**21**, XXII 2.3]. Then $H \subset L_f = L \cap L_g \subset P \cap P_g$, thus P and P_g are in standard relative position, $K = P \cap P_g$ is a smooth subgroup of G and $K \cdot U$ is a parabolic subgroup of G by [**21**, XXVI 4.5.1] and its proof. More precisely, let $R \subset M$ be the roots of H in $\mathrm{Lie}(G)$, so that $R = R_L \coprod R_U \coprod -R_U$ where R_L and R_U are respectively the roots of H in $\mathrm{Lie}(L)$ and $\mathrm{Lie}(U)$. For X in $\{\emptyset, L, U\}$ let $R_X = R_X^0 \coprod R_X^+ \coprod R_X^-$ be the decomposition of R_X induced by g, i.e.

$$R_X^\pm = \{\alpha \in R_X : \pm\alpha \circ g > 0 \text{ in } \Gamma\} \quad \text{and} \quad R_X^0 = \{\alpha \in R_X : \alpha \circ g = 0 \text{ in } \Gamma\}.$$

This yields a decomposition of R in nine pieces, as shown in the following table:

	L_g	U_g	$U_{\iota g}$
L	R_L^0	R_L^+	$R_L^- = -R_L^+$
U	R_U^0	R_U^+	R_U^-
	$-R_U^0$	$-R_U^-$	$-R_U^+$

For a closed subset R' of R, let $H(R')$ be the subgroup scheme of G of type (R) which is determined by R', see [**21**, XXII 5.4.2-7]. Thus $L = H(R_L)$, $P = H(R_L \cup R_U)$, $L_g = H(R^0)$, $P_g = H(R^0 \cup R^+)$, $L_f = H(R_L^0)$ and $P_f = H(R_L^0 \cup R_L^+)$ while

$$\begin{aligned} K &= H(R_L^0 \cup R_L^+ \cup R_U^0 \cup R_U^+) \\ \text{and} \quad K \cdot U &= H(R_L^0 \cup R_L^+ \cup R_U^0 \cup R_U^+ \cup R_U^-). \end{aligned}$$

By [**21**, XXVI 6.1], $L_f = H(R_L^0)$ is a Levi subgroup of $K \cdot U$. By [**21**, XXII 5.4.5], $P_f \subset K$, thus $P_f \subset K \cap L \subset (K \cdot U) \cap L$. But $(K \cdot U) \cap L$ is a parabolic subgroup of L with Levi L_f, thus $P_f = K \cap L = (K \cdot U) \cap L$ and $P_f \cdot U = K \cdot U$ by repeated applications of [**21**, XXVI 1.20]. Finally, P_f maps to $P_h = P_f \cdot U/U = K \cdot U/U$ under the isomorphism $L \simeq P/U$, which finishes the proof of (1). Then (2) easily follows: if (f, f') induce (g, g') and $\mathrm{Fil}(f) = \mathrm{Fil}(f')$, then $f' = \mathrm{Int}(p) \circ f$ for some $p \in P_f(S)$, thus also $g' = \mathrm{Int}(p) \circ g$ and $\mathrm{Fil}(g') = \mathrm{Fil}(g)$ since $P_f = L \cap P_g \subset P_g$. $\square$

2.3.4. Let $G' = P_u/U_u$ where $P_u \subset G_{\mathbb{P}(G)}$ is the universal parabolic subgroup with unipotent radical $U_u = R^u(P_u)$. Thus G' is a reductive group over $\mathbb{P}(G)$ and

$$\mathbb{F}^\Gamma(G')(T) = \left\{(P,\mathcal{F}) : P \in \mathbb{P}(G)(T),\ \mathcal{F} \in \mathbb{F}^\Gamma(P/U)(T),\ U = R^u(P)\right\}$$

for any S-scheme T.

PROPOSITION 26. *There is a canonical morphism of schemes over* $\mathbb{P}(G)$*,*

$$\mathbb{STD}\left(\mathbb{P}(G) \times_S \mathbb{F}^\Gamma(G)\right) \to \mathbb{F}^\Gamma(G') \qquad (P,\mathcal{F}) \mapsto (P, \mathrm{Gr}_P(\mathcal{F})).$$

PROOF. Start with $(P,\mathcal{F}) \in \mathbb{STD}\left(\mathbb{P}(G) \times_S \mathbb{F}^\Gamma(G)\right)(T)$ and put $K = P \cap P_{\mathcal{F}}$. Then K is a smooth subgroup scheme of G_T which contains, locally on T for the Zariski topology, a maximal subtorus H of G_T [**21**, XXVI 4.5.1]. Let L and $L_{\mathcal{F}}$ be the Levi subgroups of P and $P_{\mathcal{F}}$ containing H [**21**, XXVI 1.6]. Let $f : \mathbb{D}_T(\Gamma) \to L_{\mathcal{F}}$ be the central morphism lifting $\overline{\mathcal{F}} : \mathbb{D}_T(\Gamma) \to \overline{R}(P_{\mathcal{F}})$, so that $\mathcal{F} = \mathrm{Fil}(f)$ and $L_f = L_{\mathcal{F}}$. Then f factors through the maximal subtorus H of $L_{\mathcal{F}}$, which is also a maximal subtorus of L. Let $h : \mathbb{D}_T(\Gamma) \to P/U$ be the induced morphism. By the previous proposition, $P_h = K \cdot U/U$, thus K fixes $\mathrm{Fil}(h) \in \mathbb{F}^\Gamma(P/U)(T)$. If H' is another maximal subtorus of G contained in K, then, locally on T for the étale topology, $H' = \mathrm{Int}(k)(H)$ for some $k \in K(T)$ by [**1**, XII 7.1]. But then $L' = \mathrm{Int}(k)(L)$, $L'_{\mathcal{F}} = \mathrm{Int}(k)(L_{\mathcal{F}})$, $f' = \mathrm{Int}(k) \circ f$ and $h' = \mathrm{Int}(k) \circ h$ are the objects associated to H' as above, thus $\mathrm{Fil}(h') = k \cdot \mathrm{Fil}(h) = \mathrm{Fil}(h)$ since K fixes $\mathrm{Fil}(h)$. It follows that $\mathrm{Gr}_P(\mathcal{F}) = \mathrm{Fil}(h)$ does not depend upon the choice of H, and also that the whole construction is indeed local in the Zariski topology on T. □

REMARK 27. The pull-back of this morphism through $p_1 : \mathbb{OPP}(G) \to \mathbb{P}(G)$ has a canonical section: for an S-scheme T, the latter is given by the formula

$$(P_1, P_2, \mathcal{F}) \mapsto (P_1, P_2, \iota(\mathcal{F}_L))$$

where $(P_1, P_2) = (U_1 \rtimes L, U_2 \rtimes L)$ is a pair of opposed parabolic subgroups of G_T with common Levi subgroup $L = P_1 \cap P_2$, $\mathcal{F}$ is an element of $\mathbb{F}^\Gamma(P_1/U_1)(T)$, $\mathcal{F}_L$ is its unique lift in $\mathbb{F}^\Gamma(L)(T)$, and $\iota : \mathbb{F}^\Gamma(L) \to \mathbb{F}^\Gamma(G_T)$ is the morphism of proposition 25 (thus indeed P_1 and $P_{\iota(\mathcal{F}_L)}$ are in standard relative position).

2.4. Interlude on the dominance partial orders

Let $\Gamma = (\Gamma, +, \leq)$ be a non-trivial totally ordered commutative group. We set

$$\Gamma_+ = \{\gamma \in \Gamma : \gamma \geq 0\}.$$

2.4.1. Let $\mathscr{R} = (M, R, M^*, R^*)$ be a root system [**21**, XXI 1.1.1] with Weyl group $W = W(\mathscr{R})$ [**21**, XXI 1.1.8]. Fix a system of positive roots $R_+ \subset R$ [**21**, XXI 3.2.1] and let $\Delta \subset R_+$ be the corresponding simple roots [**21**, XXI 3.2.8]. Then

LEMMA 28. *The submonoid of dominant morphisms in* $\mathrm{Hom}(M,\Gamma)$*,*

$$\begin{aligned}\mathrm{Hom}^+(M,\Gamma) &= \{f \in \mathrm{Hom}(M,\Gamma) : \forall \alpha \in R_+,\ f(\alpha) \geq 0\} \\ &= \{f \in \mathrm{Hom}(M,\Gamma) : \forall \alpha \in \Delta,\ f(\alpha) \geq 0\}\end{aligned}$$

is a fundamental domain for the action of W *on* $\mathrm{Hom}(M,\Gamma)$*.*

PROOF. For any morphism $f : M \to \Gamma$, define

$$\begin{aligned} R_{f\geq 0} &= \{\alpha \in R : f(\alpha) \geq 0\}, \\ R_{f>0} &= \{\alpha \in R : f(\alpha) > 0\}, \\ R_{f=0} &= \{\alpha \in R : f(\alpha) = 0\}. \end{aligned}$$

Thus $R_{f=0}$ is closed and symmetric, $R = R_{f>0} \coprod R_{f=0} \coprod -R_{f>0}$ and

$$f \text{ is dominant } \iff R_+ \subset R_{f\geq 0} \iff R_{f>0} \subset R_+.$$

By [**21**, XXI 3.3.6], there exists $w \in W$ such that

$$R_+ \subset wR_{f\geq 0} = R_{wf\geq 0},$$

therefore wf is dominant. If f and wf are dominant, then

$$R_+ = R_{f>0} \coprod R_+^1 \quad \text{and} \quad w^{-1}R_+ = R_{f>0} \coprod R_+^2$$

where $R_+^1 = R_+ \cap R_{f=0}$ and $R_+^2 = w^{-1}R_+ \cap R_{f=0}$ are systems of positive roots in the closed symmetric subset $R_{f=0}$ of R. Thus by [**21**, XXI 3.4.1 and 3.3.7], there is an w_0 in the Weyl group $W_f \subset W$ of $R_{f=0}$ such that $w_0 R_+^2 = R_+^1$. Now W_f is spanned by the reflections $\{s_\alpha : \alpha \in R_{f=0}\}$ and for any $m \in M$ and $\alpha \in R_{f=0}$,

$$(s_\alpha f)(m) = f(s_\alpha m) = f(m - \langle m, \alpha^* \rangle \alpha) = f(m) - \langle m, \alpha^* \rangle f(\alpha) = f(m),$$

thus w_0 fixes f, stabilizes $R_{f>0}$ and maps $w^{-1}R_+$ to R_+. But then $w = w_0$ by [**21**, XXI 5.4] hence $wf = w_0 f = f$, which proves the lemma. □

2.4.2. Applying the lemma to the dual root system $\mathscr{R}^* = (M^*, R^*, M, R)$ with $\Gamma = \mathbb{Z}$, we obtain the well known fact that the cone of dominant weights

$$\begin{aligned} M_d &= \{m \in M : \forall \alpha \in R_+, \langle m, \alpha^* \rangle \geq 0\} \\ &= \{m \in M : \forall \delta \in \Delta, \langle m, \delta^* \rangle \geq 0\} \end{aligned}$$

is a fundamental domain for the action of W on M.

2.4.3. The coroot cone and coroot lattice defined by

$$\begin{aligned} \Gamma_+ R_+^* &= \left\{ m \mapsto \textstyle\sum_{\alpha \in R_+} \langle m, \alpha^* \rangle \gamma_\alpha : \forall \alpha \in R_+, \gamma_\alpha \in \Gamma_+ \right\} \\ \Gamma R^* &= \left\{ m \mapsto \textstyle\sum_{\alpha \in R} \langle m, \alpha^* \rangle \gamma_\alpha : \forall \alpha \in R, \gamma_\alpha \in \Gamma \right\} \end{aligned}$$

are a submonoid and a subgroup of $\mathrm{Hom}(M, \Gamma)$, and so are their saturations

$$\begin{aligned} \left(\Gamma_+ R_+^*\right)_{sat} &= \left\{ f \in \mathrm{Hom}(M, \Gamma) : \exists n \in \mathbb{N}^\times \text{ such that } nf \in \Gamma_+ R_+^* \right\} \\ \left(\Gamma R^*\right)_{sat} &= \left\{ f \in \mathrm{Hom}(M, \Gamma) : \exists n \in \mathbb{N}^\times \text{ such that } nf \in \Gamma R^* \right\} \end{aligned}$$

in $\mathrm{Hom}(M, \Gamma)$. Inside $\mathrm{Hom}(M, \Gamma \otimes \mathbb{Q})$, any $f \in (\Gamma R^*)_{sat}$ can be written as

$$f(-) = \textstyle\sum_{\delta \in \Delta} \langle -, \mathrm{ind}(\delta^*) \rangle \gamma_\delta$$

for a unique $(\gamma_\delta) \in (\Gamma \otimes \mathbb{Q})^\Delta$, where $\mathrm{ind}(\delta^*)$ is the simple coroot corresponding to $\delta \in \Delta$, namely $\mathrm{ind}(\delta^*) = \delta^*$ if $2\delta \notin R$ and $\mathrm{ind}(\delta^*) = \frac{1}{2}\delta^* = (2\delta)^*$ otherwise. Then

$$\begin{aligned} f \in (\Gamma_+ R_+^*)_{sat} &\iff \exists n \in \mathbb{N}^\times \text{ such that } n(\gamma_\delta) \in \Gamma_+^\Delta, \\ f \in \Gamma R^* &\iff (\gamma_\delta) \in \Gamma^\Delta, \\ f \in \Gamma_+ R_+^* &\iff (\gamma_\delta) \in \Gamma_+^\Delta. \end{aligned}$$

In particular, $(\Gamma_+ R_+^*)_{sat} \cap -(\Gamma_+ R_+^*)_{sat} = \{0\}$. Moreover by duality,

$$\left(\Gamma_+ R_+^*\right)_{sat} = \{ f \in \mathrm{Hom}(M, \Gamma) : \forall m \in M_d, f(m) \geq 0 \}.$$

2.4.4. The weak dominance partial order $\leq$ on $\mathrm{Hom}(M,\Gamma)$ is defined by

$$\begin{aligned} f_1 \leq f_2 \quad &\iff \quad \forall m \in M_d : \quad f_1(m) \leq f_2(m), \\ &\iff \quad f_2 - f_1 \in \left(\Gamma_+ R_+^*\right)_{sat}. \end{aligned}$$

The strong dominance partial order $\preceq$ on $\mathrm{Hom}(M,\Gamma)$ is defined by

$$f_1 \preceq f_2 \iff f_2 - f_1 \in \Gamma_+ R_+^*.$$

They are both compatible with the addition map: for $f_1, f_2, g_1, g_2 \in \mathrm{Hom}(M,\Gamma)$,

$$\begin{aligned} (f_1 \leq g_1 \text{ and } f_2 \leq g_2) \quad &\Longrightarrow \quad f_1 + f_2 \leq g_1 + g_2, \\ (f_1 \preceq g_1 \text{ and } f_2 \preceq g_2) \quad &\Longrightarrow \quad f_1 + f_2 \preceq g_1 + g_2. \end{aligned}$$

They are related as follows: for any $f_1, f_2 \in \mathrm{Hom}(M,\Gamma)$, we have

$$f_1 \preceq f_2 \quad \iff \quad f_1 \leq f_2 \quad \text{and} \quad \pi(f_1) = \pi(f_2)$$

where $\pi : \mathrm{Hom}(M,\Gamma) \twoheadrightarrow \mathrm{Hom}(M,\Gamma)/\Gamma R^*$ is the projection. Note that

$$f_1 \leq f_2 \Longrightarrow \pi(f_2 - f_1) \in (\Gamma R^*)_{sat}/\Gamma R^*.$$

In particular since $(\Gamma R^*)_{sat}/\Gamma R^*$ is torsion,

$$f_1 \leq f_2 \iff \exists n \in \mathbb{N}^\times : \quad n f_1 \preceq n f_2.$$

2.4.5. Since $W M_d = M$, both partial orders restrict to the identity on the fixed point set of W in $\mathrm{Hom}(M,\Gamma)$: for any W-invariant $f_1, f_2 \in \mathrm{Hom}(M,\Gamma)$,

$$\begin{tikzcd} f_1 \preceq f_2 \arrow[r, Rightarrow] & f_1 \leq f_2 \arrow[r, Rightarrow] & \forall m \in M_d : \ f_1(m) \leq f_2(m) \arrow[d, Rightarrow] \\ f_1 = f_2 \arrow[u, Rightarrow] & & \forall m \in M : \ f_1(m) \leq f_2(m) \arrow[ll, Rightarrow] \end{tikzcd}$$

2.4.6. These partial orders yield the following characterization of $\mathrm{Hom}^+(M,\Gamma)$:

LEMMA 29. *The projection π is W-invariant and for every $f \in \mathrm{Hom}(M,\Gamma)$,*

$$\begin{aligned} f \in \mathrm{Hom}^+(M,\Gamma) \quad &\iff \quad \forall w \in W : \quad wf \leq f, \\ &\iff \quad \forall w \in W : \quad wf \preceq f. \end{aligned}$$

In particular, $\mathrm{Hom}(M,\Gamma)^W \subset \mathrm{Hom}^+(M,\Gamma)$.

PROOF. For any $f \in \mathrm{Hom}(M,\Gamma)$ and $\alpha \in R$,

$$f - s_\alpha f = \langle -, \alpha^* \rangle \, f(\alpha) \quad \text{in} \quad \mathrm{Hom}(M,\Gamma).$$

Thus π is W-invariant, $wf \leq f \iff wf \preceq f$ for $w \in W$ and

$$f \in \mathrm{Hom}^+(M,\Gamma) \quad \iff \quad \forall \alpha \in R_+ : \quad s_\alpha f \preceq f.$$

It remains to establish that

$$\forall \alpha \in R_+ : \quad s_\alpha f \preceq f \quad \Longrightarrow \quad \forall w \in W : \quad wf \preceq f$$

and we argue by induction on the length $\ell(w)$ of w in the coxeter group $(W, (s_\alpha)_{\alpha \in \Delta})$. If $\ell(w) > 1$, then $w = w' s_\alpha$ for some $\alpha \in \Delta$, $w' \in W$ with $\ell(w') < \ell(w)$. Thus

$$f - wf = (f - w'f) + w' (f - s_\alpha f) = (f - w'f) + \langle -, w'\alpha^* \rangle \, f(\alpha).$$

Now $f - w'f \in \Gamma_+ R_+^*$ by induction, $f(\alpha) \geq 0$ by assumption and $w'\alpha = -w\alpha \in R^+$ by [**7**, VI, §1, n°1.6, Corollaire 2], therefore $f - wf \in \Gamma_+ R_+^*$, i.e. $wf \preceq f$. □

2.4.7. If Γ is (uniquely) divisible, the weak and strong dominance order coincide. Moreover, for any $f \in \mathrm{Hom}^+(M,\Gamma)$, lemma 29 implies that $f^\flat \leq f$ where

$$f^\flat = \tfrac{1}{\sharp Wf} \textstyle\sum_{f' \in Wf} f' \in \mathrm{Hom}(M,\Gamma)^W \subset \mathrm{Hom}^+(M,\Gamma).$$

Thus $\mathrm{Hom}(M,\Gamma)^W$ is then precisely the set of minimal elements in $\mathrm{Hom}^+(M,\Gamma)$. For $\Gamma = \mathbb{Z}$ and $\mathscr{R}$ reduced, semi-simple and adjoint (i.e. $\mathbb{Z}R = M$), the strong dominance order on $\mathrm{Hom}^+(M,\mathbb{Z})$ is studied in [**45**]. Its minimal elements are the linear forms $f : M \to \mathbb{Z}$ such that $f(R_+) \in \{0,1\}$.

2.4.8. Applying lemma 29 to the dual root system $\mathscr{R}^*$ with $\Gamma = \mathbb{Z}$, we obtain: (1) Let M' be the kernel of the coinvariant map $M \twoheadrightarrow M_W$. Then $M' \subset \mathbb{Z}R$. Since also $\alpha \equiv -\alpha$ in M_W for every $\alpha \in R$, actually $2\mathbb{Z}R \subset M' \subset \mathbb{Z}R$, therefore M' and $\mathbb{Z}R$ have the same saturation in M. And: (2) For every $m \in M$,

$$m \in M_d \iff \forall w \in W : \quad m - wm \in \mathbb{N}R_+ \quad \text{(or: } (\mathbb{N}R_+)_{sat}).$$

Returning to the original root system $\mathscr{R}$ and the general Γ, we thus find:

$$\mathrm{Hom}(M,\Gamma)^W = \mathrm{Hom}(M/M',\Gamma) = \mathrm{Hom}(M/\mathbb{Z}R,\Gamma) = \mathrm{Hom}(M/(\mathbb{Z}R)_{sat},\Gamma)$$

and for every $f \in \mathrm{Hom}^+(M,\Gamma)$ and $m \in M_d$,

$$f(m) = \max f(Wm) \quad \text{in} \quad \Gamma.$$

2.4.9. This last property yields the following characterisation of the restriction of the weak order to the cone $\mathrm{Hom}^+(M,\Gamma)$. Any morphism $f : M \to \Gamma$ induces a ring homomorphism $f : \mathbb{Z}[M] \to \mathbb{Z}[\Gamma]$. For $x \in \mathbb{Z}[\Gamma]$, we denote by $\max(x) \in \Gamma$ the largest element in the finite support of x if $x \neq 0$, and set $\max(0) = 0$. Then:

LEMMA 30. *For* $f_1, f_2 \in \mathrm{Hom}^+(M,\Gamma)$,

$$f_1 \leq f_2 \quad \iff \quad \forall x \in \mathbb{N}[M]^W, : \quad \max(f_1(x)) \leq \max(f_2(x)).$$

PROOF. Since M_d is a fundamental domain for the action of W on M,

$$\mathbb{N}[M]^W = \left\{x = \textstyle\sum_{m \in M_d} x_m e_m : x_m \in \mathbb{N},\ \{x_m \neq 0\} \text{ finite}\right\}$$

where $e_m = \sum_{m' \in Wm} m'$. For any $f : M \to \Gamma$ and $x \in \mathbb{N}[M]^W$ with $x \neq 0$, $f(x)$ is also nonzero with support $\cup_{m \in M_d, x_m \neq 0} f(Wm)$. Thus if f is moreover dominant,

$$\max(f(x)) = \max\{f(m) : m \in M_d,\ x_m \neq 0\}.$$

The lemma easily follows. □

2.4.10. Let $\mathscr{R}_{ss} = (M_{ss}, R_{ss}, M^*_{ss}, R^*_{ss})$ be the semi-simplification of $\mathscr{R}$, as defined in [**21**, XXI 6.5]. Thus $M_{ss} = \mathbb{Z}R_{sat}$, $R_{ss} = R$, M^*_{ss} is the dual of M_{ss} and R^*_{ss} is the image of R^* under the transpose map $M^* \twoheadrightarrow M^*_{ss}$. The restriction map $f \mapsto f_{ss} = f|M_{ss}$ yields an epimorphism $\mathrm{Hom}(M,\Gamma) \twoheadrightarrow \mathrm{Hom}(M_{ss},\Gamma)$ with kernel $\mathrm{Hom}(M,\Gamma)^W$, inducing epimorphism of monoids $\mathrm{Hom}^+(M,\Gamma) \twoheadrightarrow \mathrm{Hom}^+(M_{ss},\Gamma)$ and $\Gamma_+ R^*_+ \twoheadrightarrow \Gamma_+ R^*_{ss,+}$ – the former is therefore also compatible with the weak and strong partial orders. If Γ is divisible, the average map $f \mapsto f^\flat$ of section 2.4.7 gives a retraction of $\mathrm{Hom}(M,\Gamma)^W \hookrightarrow \mathrm{Hom}(M,\Gamma)$, and it follows that $f \mapsto (f_{ss}, f^\flat)$ yields an isomorphism of partially ordered monoids

$$\mathrm{Hom}^+(M,\Gamma) \simeq \mathrm{Hom}^+(M_{ss},\Gamma) \times \mathrm{Hom}(M,\Gamma)^W.$$

The (weak=strong) partial order on the product is then given by

$$(f_{ss}, f^\flat) \leq (g_{ss}, g^\flat) \iff f_{ss} \leq g_{ss} \quad \text{and} \quad f^\flat = g^\flat.$$

2.4.11. If $\Gamma = \mathbb{R}$, then $\mathrm{Hom}^+(M, \mathbb{R})$ is a closed cone in the finite dimensional $\mathbb{R}$-vector space $\mathrm{Hom}(M, \mathbb{R})$. The (weak=strong) dominance partial order then has the following intrinsic characterisation: for every $f_1, f_2 \in \mathrm{Hom}^+(M, \mathbb{R})$,

$$f_1 \leq f_2 \iff f_1 \text{ lies in the convex hull of } W \cdot f_2.$$

Indeed, suppose first that $f_1 \leq f_2$. If f_1 does not belong to the convex hull of $W \cdot f_2$, there is a linear form F on $\mathrm{Hom}(M, \mathbb{R})$ such that $F(f_1) > F(wf_2)$ for every $w \in W$, which means that there is an x in $M \otimes \mathbb{R}$ such that $f_1(x) > f_2(wx)$ for every $w \in W$. Since $M \otimes \mathbb{Q}$ is dense in $\mathbb{R}$, we may assume that $x \in M \otimes \mathbb{Q}$, and then rescaling that actually $x \in M$. Let $y = wx$ be the unique element in $Wx \cap M_d$ (2.4.2). Then $f_1(y) \geq f_1(x)$ since $f_1 \in \mathrm{Hom}^+(M, \mathbb{R})$ (2.4.8) and $f_1(x) > f_2(y)$ by construction, thus $f_1(y) > f_2(y)$ with $y \in M_d$, a contradiction. Suppose conversely that

$$f_1 = \textstyle\sum_{w \in W} \lambda_w w f_2 \text{ in } \mathrm{Hom}(M, \mathbb{R}) \text{ with } \lambda_w \in [0,1], \textstyle\sum_{w \in W} \lambda_w = 1.$$

Since $f_2 \in \mathrm{Hom}^+(M, \mathbb{R})$, $wf_2 \leq f_2$ for every $w \in W$ by lemma 29, thus

$$f_1 = \textstyle\sum_{w \in W} \lambda_w w f_2 \leq \textstyle\sum_{w \in W} \lambda_w f_2 = f_2 \text{ in } \mathrm{Hom}(M, \mathbb{R}).$$

2.4.12. The partial orders on $W \backslash \mathrm{Hom}(M, \Gamma)$ which are induced by the restriction of $\preceq$ and $\leq$ to the fundamental domain $\mathrm{Hom}^+(M, \Gamma) \simeq W \backslash \mathrm{Hom}(M, \Gamma)$ of lemma 28 do not depend upon the chosen system of positive roots R^+ – indeed, all such systems are conjugated under W. The weak order even does not depend upon the root system giving rise to W: any orbit $[f] \in W \backslash \mathrm{Hom}(M, \Gamma)$ yields a well-defined function $[f] : \mathbb{Z}[M]^W \to \mathbb{Z}[\Gamma]$, and for every $[f_1], [f_2] \in W \backslash \mathrm{Hom}(M, \Gamma)$,

$$[f_1] \leq [f_2] \iff \forall x \in \mathbb{N}[M]^W : \quad \max[f_1](x) \leq \max[f_2](x)$$

by lemma 30. The strong order moreover depends upon ΓR^*:

$$[f_1] \preceq [f_2] \iff [f_1] \leq [f_2] \quad \text{and} \quad \pi[f_1] = \pi[f_2]$$

where $\pi : W \backslash \mathrm{Hom}(M, \Gamma) \twoheadrightarrow \mathrm{Hom}(M, \Gamma)/\Gamma R^*$ is the projection from lemma 29.

2.4.13. We record here a technical result comparing the partial orders attached to, respectively, the relative and absolute root systems of a reductive group G over a field k. Let S a maximal split torus in G, T a maximal torus in the centralizer $Z_G(S)$ of S, k^s a separable closure of k, $\mathrm{Gal}_k = \mathrm{Gal}(k^s/k)$. Denote by

$$\overline{\mathscr{R}} = \mathscr{R}(G_{k^s}, T_{k^s}) = \left(\overline{M}, \overline{R}, \overline{M}^*, \overline{R}^*\right) \quad \text{and} \quad \mathscr{R} = \mathscr{R}(G, S) = (M, R, M^*, R^*)$$

the absolute and relative root systems [**21**, XXVI 7.12], with Weyl groups

$$\overline{W} = W(\overline{\mathscr{R}}) = W(G_{k^s}, T_{k^s}) \quad \text{and} \quad W = W(\mathscr{R}) = W(G, S).$$

We will also consider the subgroups $\overline{W}_S^0 \subset \overline{W}_S \subset \overline{W}$ defined by

$$\mathcal{N}_G(T) \cap \mathcal{Z}_G(S)/\mathcal{Z}_G(T) \subset \mathcal{N}_G(T) \cap \mathcal{N}_G(S)/\mathcal{Z}_G(T) \subset \mathcal{N}_G(T)/\mathcal{Z}_G(T).$$

The embedding $S \hookrightarrow T$ induces a pair of dual morphisms

$$\begin{tikzcd} \overline{M} \times \overline{M}^* \arrow[r, "{\langle -,- \rangle}"] \arrow[d, shift right=6, "\mathrm{res}"'] & \mathbb{Z} \arrow[d, equal] \\ M \times M^* \arrow[r, "{\langle -,- \rangle}"] \arrow[u, hook, shift left=6, "\mathrm{res}^*"'] & \mathbb{Z} \end{tikzcd}$$

with $R \subset \mathrm{res}(\overline{R}) \subset R \cup \{0\}$. Set $\overline{R}(0) = \left\{\overline{\alpha} \in \overline{R} : \mathrm{res}(\overline{\alpha}) = 0\right\}$, so that

$$\overline{\mathscr{R}}(0) = \mathscr{R}\left(Z_G(S)_{k^s}, T_{k^s}\right) = \left(\overline{M}, \overline{R}(0), \overline{M}^*, \overline{R}(0)^*\right)$$

is the absolute root system of $Z_G(S)$, with Weyl group

$$\overline{W}_S^0 = W\left(\overline{\mathscr{R}}(0)\right) = W\left(Z_G(S)_{k^s}, T_{k^s}\right).$$

By [**5**, 5.5], the natural map $\overline{W}_S \to W$ identifies W with $\overline{W}_S/\overline{W}_S^0$. Since the restriction $\mathrm{res} : \overline{M} \twoheadrightarrow M$ is equivariant with respect to $\overline{W}_S \twoheadrightarrow W$, it induces a map

$$W\backslash\mathrm{Hom}\left(M, \Gamma\right) \hookrightarrow \overline{W}_S\backslash\mathrm{Hom}\left(\overline{M}, \Gamma\right) \twoheadrightarrow \overline{W}\backslash\mathrm{Hom}\left(\overline{M}, \Gamma\right).$$

Proposition 31. *The map* $W\backslash\mathrm{Hom}\left(M, \Gamma\right) \to \overline{W}\backslash\mathrm{Hom}\left(\overline{M}, \Gamma\right)$ *is injective. Moreover for any* $f, g \in W\backslash\mathrm{Hom}\left(M, \Gamma\right)$ *with image* $\overline{f}, \overline{g} \in \overline{W}\backslash\mathrm{Hom}\left(\overline{M}, \Gamma\right)$,

$$f \preceq g \Longrightarrow \overline{f} \preceq \overline{g} \Longrightarrow \overline{f} \leq \overline{g} \iff f \leq g.$$

Remark 32. We do not know if $\overline{f} \preceq \overline{g}$ implies $f \preceq g$. The proof given below relates this to the following question. Recall that G is simply connected if and only if $\mathbb{Z}\overline{R}^* = \overline{M}^*$. Is it true that then also $\mathbb{Z}R^* = M^*$? The dual question has a positive answer: G is adjoint if and only if $\mathbb{Z}\overline{R} = \overline{M}$, in which case also $\mathbb{Z}R = M$.

2.4.14. Let $G_{der} \subset G$ be the derived group of G [**21**, XXII 6.2], $\pi : G_{sc} \twoheadrightarrow G_{der}$ the simply connected cover of G_{der} [**13**, A.4.11]. Then $T_{der} = T \cap G_{der}$ is a maximal torus in G_{der} [**21**, XXII 6.2.8] and $T_{sc} = \pi^{-1}(T_{der})$ is a maximal torus in G_{sc} [**1**, XVII 7.1.1]. Let $S_{der} \subset T_{der}$ and $S_{sc} \subset T_{sc}$ be their maximal split subtori and denote by $\mathscr{R}_{der}$, $\mathscr{R}_{sc}$, $\overline{\mathscr{R}}_{der}$ and $\overline{\mathscr{R}}_{sc}$ the corresponding relative and absolute root systems. By [**21**, XXII 6.2.7] and the definition of G_{sc}, the morphisms $T_{sc} \twoheadrightarrow T_{der} \hookrightarrow T$ induce compatible bijections $\overline{R} \simeq \overline{R}_{der} \simeq \overline{R}_{sc}$ and $\overline{R}_{sc}^* \simeq \overline{R}_{der}^* \simeq \overline{R}^*$. They also induce morphisms $S_{sc} \twoheadrightarrow S_{der} \hookrightarrow S$, and $S = S_{der} \cdot R(G)_{sp}$ since $T = T_{der} \cdot R(G)$ [**21**, XXII 6.2.8], where $R(G)_{sp}$ is the maximal split subtorus of the radical $R(G)$ of G. Since R, R_{der} and R_{sc} are the nonzero restrictions of the elements of $\overline{R}$, $\overline{R}_{der}$ and $\overline{R}_{sc}$ to respectively S, S_{der} and S_{sc}, it follows that our morphisms also induce bijections $R \simeq R_{der} \simeq R_{sc}$. Finally, the morphisms $G_{sc} \twoheadrightarrow G_{der} \hookrightarrow G$ induce embeddings $W_{sc} \hookrightarrow W_{der} \hookrightarrow W$ between the Weyl groups of the maximal tori $S_{sc} \subset G_{sc}$, $S_{der} \subset G_{der}$ and $S \subset G$. It then follows from the unicity of the relative coroots, or from their actual construction in [**21**, XXVI 7.4], that $S_{sc} \twoheadrightarrow S_{der} \hookrightarrow S$ also induces compatible bijections $R_{sc}^* \simeq R_{der}^* \simeq R^*$ (and the Weyl groups maps are bijective). Since composition with $S_{sc} \hookrightarrow T_{sc}$ maps $\mathbb{Z}R_{sc}^*$ into $X_*(T_{sc}) = \mathbb{Z}\overline{R}_{sc}^*$, we obtain

$$\mathrm{res}^*(\mathbb{Z}R^*) \subset \mathbb{Z}\overline{R}^*.$$

2.4.15. Fix a minimal parabolic subgroup $Z_G(S) \subset P \subset G$, a Borel subgroup $T_{k^s} \subset B \subset P_{k^s} \subset G_{k^s}$, let $\Delta \subset R_+ \subset R$ and $\overline{\Delta} \subset \overline{R}_+ \subset \overline{R}$ be the corresponding simple and positive roots, $\overline{\Delta}(0) \subset \overline{R}(0)_+ \subset \overline{R}(0)$ the simple and positive roots attached to $Z_G(S)_{k^s} \cap B$, so that $\overline{\Delta}(0) = \overline{\Delta} \cap \overline{R}(0)$, $\overline{R}(0)_+ = \overline{R}_+ \cap \overline{R}(0)$,

$$R_+ \subset \mathrm{res}\left(\overline{R}_+\right) \subset R_+ \cup \{0\} \quad \text{and} \quad \Delta \subset \mathrm{res}\left(\overline{\Delta}\right) \subset \Delta \cup \{0\}.$$

In particular, the morphism $\mathrm{res}_\Gamma^* : \mathrm{Hom}\left(M, \Gamma\right) \hookrightarrow \mathrm{Hom}\left(\overline{M}, \Gamma\right)$ maps $\mathrm{Hom}^+\left(M, \Gamma\right)$ to $\mathrm{Hom}^+\left(\overline{M}, \Gamma\right)$. The first assertion of Proposition 31 thus follows from Lemma 28.

For the remaining claims, we have to establish the following inclusions:

$$\Gamma_+ R_+^* \subset (\mathrm{res}_\Gamma^*)^{-1}\left(\Gamma_+ \overline{R}_+^*\right) \subset (\mathrm{res}_\Gamma^*)^{-1}\left(\left(\Gamma_+ \overline{R}_+^*\right)_{sat}\right) = \left(\Gamma_+ R_+^*\right)_{sat}.$$

We may assume that $\Gamma = \mathbb{Z}$, in which case $\mathrm{res}_\Gamma^* = \mathrm{res}^* : M^* \hookrightarrow \overline{M}^*$ and we want:

$$\mathbb{N}R_+^* \subset (\mathrm{res}^*)^{-1}\left(\mathbb{N}\overline{R}_+^*\right) \subset (\mathrm{res}^*)^{-1}\left(\left(\mathbb{N}\overline{R}_+^*\right)_{sat}\right) = \left(\mathbb{N}R_+^*\right)_{sat}.$$

The central inclusion is obvious. Since we already know that $\mathrm{res}^*(\mathbb{Z}R^*) \subset \mathbb{Z}\overline{R}^*$ and

$$\mathbb{N}R_+^* = \mathbb{Z}R^* \cap \left(\mathbb{N}R_+^*\right)_{sat}, \qquad \mathbb{N}\overline{R}_+^* = \mathbb{Z}\overline{R}^* \cap \left(\mathbb{N}\overline{R}_+^*\right)_{sat}$$

it only remains to establish the following lemma.

LEMMA 33. *With notations as above,*

$$(\mathrm{res}^*)^{-1}\left((\mathbb{N}\overline{R}_+^*)_{sat}\right) = (\mathbb{N}R_+^*)_{sat} \quad \textit{and} \quad \left(\mathrm{res}\left(\overline{M}_d\right)\right)_{sat} = M_d.$$

Note that the second formula follows from the first one by duality.

2.4.16. The dual and bidual cones of the coroot cones $\mathbb{N}R_+^*$ and $\mathbb{N}\overline{R}_+^*$ are respectively equal to the cones of dominant weights M_d and $\overline{M}_d$, and to their own saturations $(\mathbb{N}R_+^*)_{sat}$ and $(\mathbb{N}\overline{R}_+^*)_{sat}$. By 2.4.8, the restriction map $\mathrm{res} : \overline{M} \twoheadrightarrow M$ sends $\overline{M}_d$ into M_d. Indeed for $m = \mathrm{res}(\overline{m})$ with $\overline{m} \in \overline{M}_d$ and any $w \in W$ lifting to $\overline{w} \in \overline{W}_S$, $m - wm = \mathrm{res}(\overline{m} - \overline{w}\overline{m})$ belongs to $\mathbb{N}R_+$ since $\overline{m} - \overline{w}\overline{m}$ belongs to $\mathbb{N}\overline{R}_+$. Passing to the bidual cones, we thus obtain the easiest inclusion:

$$\mathrm{res}^*\left(\mathbb{N}R_+^*\right)_{sat} \subset \left(\mathbb{N}\overline{R}_+^*\right)_{sat}.$$

2.4.17. For the opposite inclusion, we will need a few more notations:

(1) Recall from [**21**, XXI 1.2.1] that the formulas

$$p(x) = \textstyle\sum_{\overline{\alpha} \in \overline{R}} \langle x, \overline{\alpha}^* \rangle\, \overline{\alpha}^* \quad \text{and} \quad \ell(x) = \langle x, p(x) \rangle \quad (x \in \overline{M})$$

define a morphism $p : \overline{M} \to \overline{M}^*$ and a map $\ell : \overline{M} \to \mathbb{N}$ such that

$$\forall \overline{\alpha} \in \overline{R} : \qquad \ell(\overline{\alpha}) > 0 \quad \text{and} \quad 2p(\overline{\alpha}) = \ell(\overline{\alpha})\overline{\alpha}^*,$$

(2) The Galois group Gal_k acts on $\overline{M}$, $\overline{M}^*$, $\overline{W}$, $\overline{W}_S$ and $\overline{W}_S^0$, the morphisms res and res* are $\overline{W}_S \rtimes \mathrm{Gal}_k$-equivariant, the latter identifies M^* with $(\overline{M}^*)^{\mathrm{Gal}_k}$, the subset $\overline{R} \subset \overline{M}$ and $\overline{R}^* \subset \overline{M}^*$ are $\overline{W} \rtimes \mathrm{Gal}_k$-stable, and

$$* : \overline{R} \to \overline{R}^*, \quad p : \overline{M} \to \overline{M}^*, \quad \ell : \overline{M} \to \mathbb{N}$$

are also $\overline{W} \rtimes \mathrm{Gal}_k$-equivariant (with the trivial action on $\mathbb{N}$).

(3) For every $\gamma \in \mathrm{Gal}_k$, there is a unique $w_\gamma \in \overline{W}_S^0$ such that

$$w_\gamma \gamma \overline{R}(0)_+ = \overline{R}(0)_+$$

by [**21**, XXI 3.3.7], in which case also $w_\gamma \gamma \overline{R}_+ = \overline{R}_+$ since

$$\overline{R}_+ \setminus \overline{R}(0)_+ = \overline{R} \cap \mathrm{res}^{-1}(R_+)$$

is already stable under $\overline{W}_S^0 \rtimes \mathrm{Gal}_k$. The twisted action of Gal_k on $\overline{\mathscr{R}}$ [**5**, 6.2] is given by $\gamma \cdot = w_\gamma \gamma$. The above maps are equivariant for the twisted action, which moreover preserves $\overline{\Delta}$ and $\overline{\Delta}(0)$. For every $\alpha \in \Delta$, the twisted action is transitive on $\overline{\Delta}(\alpha) = \mathrm{res}^{-1}(\alpha) \cap \overline{\Delta}$ by [**5**, 6.4.2 & 6.8].

(4) A root $\overline{\alpha} \in \overline{R}$ maps to $\alpha \in \Delta$ if and only if it is the sum of a (unique) simple root $\overline{\delta} \in \overline{\Delta}(\alpha)$ and some element of $\mathbb{N}\overline{R}(0)_+$. This yields a partition of $\overline{R}(\alpha) = \overline{R} \cap \mathrm{res}^{-1}(\alpha)$ indexed by $\overline{\Delta}(\alpha)$, whose parts are permuted transitively by the twisted action of Gal_k. It follows that

$$\textstyle\sum_{\overline{\alpha}\in\overline{R}(\alpha)}\overline{\alpha} = n_\alpha \cdot \sum_{\overline{\delta}\in\overline{\Delta}(\alpha)}\overline{\delta} + \widetilde{\alpha}_0 \quad \text{in} \quad \overline{M}$$

with $n_\alpha \in \mathbb{N}^\times$ and $\widetilde{\alpha}_0 \in \mathbb{N}\overline{R}(0)_+$. Applying the morphism $2p$, we obtain

$$\textstyle\sum_{\overline{\alpha}\in\overline{R}(\alpha)}\ell(\overline{\alpha})\overline{\alpha}^* = n_\alpha\ell_\alpha \cdot \sum_{\overline{\delta}\in\overline{\Delta}(\alpha)}\overline{\delta}^* + 2p(\widetilde{\alpha}_0) \quad \text{in} \quad \overline{M}^*$$

with $p(\widetilde{\alpha}_0) \in \mathbb{N}\overline{R}(0)^*_+$ and $\ell_\alpha = \ell(\overline{\delta}) \in \mathbb{N}^\times$ for any $\overline{\delta} \in \overline{\Delta}(\alpha)$.

2.4.18. Fix $\alpha \in \Delta$ and change (G, S, T, P, B) to $(H, S, T, P \cap H, B \cap H_{k^s})$ where H is the unique reductive subgroup of G containing $Z_G(S)$ with

$$\mathrm{Lie}(H) = \mathrm{Lie}(G)_0 \oplus \oplus_{\beta\in\mathbb{Z}\alpha\cap R}\mathrm{Lie}(G)_\beta$$

This changes our absolute and relative based root data to respectively

$$\left(\overline{M}, \overline{R}_\alpha, \overline{M}^*, \overline{R}^*_\alpha; \overline{\Delta}(\alpha) \cup \overline{\Delta}(0)\right) \quad \text{and} \quad (M, \mathbb{Z}\alpha \cap R, M^*, (\mathbb{Z}\alpha \cap R)^*; \{\alpha\})$$

where $\overline{R}_\alpha = \left\{\overline{\beta} \in \overline{R} : \mathrm{res}(\overline{\beta}) \in \mathbb{Z}\alpha\right\}$. We thus already know that

$$\mathrm{res}^*\left(\mathrm{ind}(\alpha^*)\right) = \textstyle\sum_{\overline{\delta}\in\overline{\Delta}(\alpha)}\lambda_{\overline{\delta}}\overline{\delta}^* + \widetilde{\alpha}^*_0 \quad \text{in} \quad \mathbb{N}\overline{R}_+ \subset \overline{M}^*$$

with $\lambda_{\overline{\delta}} \in \mathbb{N}$ and $\widetilde{\alpha}^*_0 \in \mathbb{N}\overline{R}(0)^*_+$. Since $\mathrm{res}^*(\mathrm{ind}(\alpha^*))$ is fixed by the twisted action, the coefficient map $\overline{\delta} \mapsto \lambda_{\overline{\delta}}$ is constant on the (twisted) Gal_k-orbit $\Delta(\alpha)$, thus

$$\mathrm{res}^*\left(\mathrm{ind}(\alpha^*)\right) = \lambda_\alpha \cdot \textstyle\sum_{\overline{\delta}\in\overline{\Delta}(\alpha)}\overline{\delta}^* + \widetilde{\alpha}^*_0 \quad \text{in} \quad \overline{M}^* \tag{2.4.1}$$

with $\lambda_\alpha \in \mathbb{N}$, therefore also

$$n_\alpha\ell_\alpha \cdot \mathrm{res}^*\left(\mathrm{ind}(\alpha^*)\right) = \lambda_\alpha \cdot \textstyle\sum_{\overline{\alpha}\in\overline{R}(\alpha)}\ell(\overline{\alpha})\overline{\alpha}^* + \left(n_\alpha\ell_\alpha \cdot \widetilde{\alpha}^*_0 - 2\lambda_\alpha \cdot p(\widetilde{\alpha}_0)\right).$$

Since $\mathrm{res}^*\left(\mathrm{ind}(\alpha^*)\right)$ and $\sum_{\overline{\alpha}\in\overline{R}(\alpha)}\ell(\overline{\alpha})\overline{\alpha}^*$ are fixed by the usual (untwisted) action of Gal_k on $\overline{M}^*$, so is the remaining term, which thus belongs to $\mathrm{res}^*(M^*)$. But

$$\mathrm{res}^*(M^*) \cap \mathbb{Z}\overline{R}(0)^* = 0$$

since any element of $\mathrm{res}^*(M^*)$ pairs trivially with all of $\overline{R}(0)$ while the restriction of the pairing $\overline{M} \times \overline{M}^* \to \mathbb{Z}$ to $\mathbb{Z}\overline{R}(0) \times \mathbb{Z}\overline{R}(0)^*$ is non-degenerate by [**21**, XXI 1.2.5]. We thus obtain the following equalities in $\overline{M}^*$: $n_\alpha\ell_\alpha \cdot \widetilde{\alpha}^*_0 = 2\lambda_\alpha \cdot p(\widetilde{\alpha}_0)$ and

$$n_\alpha\ell_\alpha \cdot \mathrm{res}^*\left(\mathrm{ind}(\alpha^*)\right) = \lambda_\alpha \cdot \textstyle\sum_{\overline{\alpha}\in\overline{R}(\alpha)}\ell(\overline{\alpha})\overline{\alpha}^*.$$

In particular, $\lambda_\alpha \in \mathbb{N}^\times$ since $\mathrm{res}^*\left(\mathrm{ind}(\alpha^*)\right) \neq 0$.

2.4.19. Suppose now that $x \in M^* \otimes \mathbb{Q}$ is such that

$$\mathrm{res}^*(x) = \textstyle\sum_{\overline{\delta}\in\overline{\Delta}}y_{\overline{\delta}} \cdot \overline{\delta}^* \quad \text{with} \quad y_{\overline{\delta}} \in \mathbb{Q}_+.$$

Since the left hand side is invariant under the twisted action of Gal_k,

$$\mathrm{res}^*(x) = \textstyle\sum_{\alpha\in\Delta}y_\alpha \cdot \sum_{\overline{\delta}\in\overline{\Delta}(\alpha)}\overline{\delta}^* + \widetilde{x}^*_0 \quad \text{with} \quad y_\alpha \in \mathbb{N}, \quad \widetilde{x}^*_0 \in \mathbb{Q}_+\overline{R}(0)^*_+.$$

Using (2.4.1) and $\mathbb{Q}\overline{R}(0)^* \cap \mathrm{res}^*(M^* \otimes \mathbb{Q}) = 0$, we obtain

$$\left(\widetilde{x}^*_0 - \textstyle\sum_{\alpha\in\Delta}y_\alpha\lambda_\alpha^{-1} \cdot \widetilde{\alpha}^*_0\right) = \mathrm{res}^*\left(x - \sum_{\alpha\in\Delta}y_\alpha\lambda_\alpha^{-1} \cdot \mathrm{ind}(\alpha^*)\right) = 0$$

thus $x = \sum_{\alpha \in \Delta} y_\alpha \lambda_\alpha^{-1} \cdot \mathrm{ind}(\alpha^*)$ belongs to $\mathbb{Q}_+ R_+$. It follows that

$$(\mathrm{res}^*)^{-1}\left(\left(\mathbb{N}\overline{R}_+^*\right)_{sat}\right) \subset \left(\mathbb{N}R_+^*\right)_{sat} \quad \text{in} \quad M^*,$$

which completes the proof of lemma 33 and proposition 31.

CHAPTER 3

The Tannakian formalism

Let G be an affine and flat group scheme over S and let $\Gamma = (\Gamma, +, \leq)$ be a non-trivial, totally ordered commutative group. We will define below an equivariant diagram of fpqc sheaves $(\mathsf{Sch}/S)^\circ \to \mathsf{Group}$ or $(\mathsf{Sch}/S)^\circ \to \mathsf{Set}$:

$$\begin{tikzcd}
G \arrow[d, "\iota"] & \text{acting on} & \mathbb{G}^\Gamma(G) \arrow[r, "\mathrm{Fil}"] \arrow[d, "\iota"] & \mathbb{F}^\Gamma(G) \arrow[d, "\iota"] \\
\mathrm{Aut}^\otimes(V) \arrow[d] & \cdots & \mathbb{G}^\Gamma(V) \arrow[r, "\mathrm{Fil}"] \arrow[d] & \mathbb{F}^\Gamma(V) \arrow[d] \\
\mathrm{Aut}^\otimes(V^\circ) \text{ or } \mathrm{Aut}^\otimes(\omega) \arrow[d] & \cdots & \mathbb{G}^\Gamma(V^\circ) \text{ or } \mathbb{G}^\Gamma(\omega) \arrow[r, "\mathrm{Fil}"] \arrow[d] & \mathbb{F}^\Gamma(V^\circ) \text{ or } \mathbb{F}^\Gamma(\omega) \arrow[d] \\
\mathrm{Aut}^\otimes(\omega^\circ) & \cdots & \mathbb{G}^\Gamma(\omega^\circ) \arrow[r, "\mathrm{Fil}"] & \mathbb{F}^\Gamma(\omega^\circ)
\end{tikzcd}$$

The main result of this chapter will then be the following theorem:

Theorem 34. *If G is a reductive group over S, then*

$$\begin{array}{ccccccc}
G & = & \mathrm{Aut}^\otimes(V) & = & \mathrm{Aut}^\otimes(V^\circ) & = & \mathrm{Aut}^\otimes(\omega) \\
\mathbb{G}^\Gamma(G) & = & \mathbb{G}^\Gamma(V) & = & \mathbb{G}^\Gamma(V^\circ) & = & \mathbb{G}^\Gamma(\omega) \\
\mathbb{F}^\Gamma(G) & = & \mathbb{F}^\Gamma(V) & = & \mathbb{F}^\Gamma(V^\circ) & \subset & \mathbb{F}^\Gamma(\omega)
\end{array}$$

If moreover G is isotrivial and S quasi-compact, then also

$$G = \mathrm{Aut}^\otimes(\omega^\circ), \quad \mathbb{G}^\Gamma(G) = \mathbb{G}^\Gamma(\omega^\circ) \quad \textit{and} \quad \mathbb{F}^\Gamma(G) = \mathbb{F}^\Gamma(\omega) = \mathbb{F}^\Gamma(\omega^\circ).$$

More precisely, we will first show that for any affine flat group scheme G over S,

$$\begin{array}{ccccc}
G & = & \mathrm{Aut}^\otimes(V) & = & \mathrm{Aut}^\otimes(\omega) \\
\mathbb{G}^\Gamma(G) & = & \mathbb{G}^\Gamma(V) & = & \mathbb{G}^\Gamma(\omega) \\
\mathbb{F}^\Gamma(G) & \subset & \mathbb{F}^\Gamma(V) & \subset & \mathbb{F}^\Gamma(\omega)
\end{array}$$

Then, under technical assumptions which are satisfied by all reductive groups (resp. all isotrivial reductive groups over quasi-compact bases), we will also establish that

$$\begin{array}{ccc}
\mathrm{Aut}^\otimes(V) & = & \mathrm{Aut}^\otimes(V^\circ) \\
\mathbb{G}^\Gamma(V) & = & \mathbb{G}^\Gamma(V^\circ) \\
\mathbb{F}^\Gamma(V) & \subset & \mathbb{F}^\Gamma(V^\circ)
\end{array}
\quad
\left(\text{resp.} \quad \begin{array}{ccc}
\mathrm{Aut}^\otimes(\omega) & = & \mathrm{Aut}^\otimes(\omega^\circ) \\
\mathbb{G}^\Gamma(\omega) & = & \mathbb{G}^\Gamma(\omega^\circ) \\
\mathbb{F}^\Gamma(\omega), \mathbb{F}^\Gamma(V^\circ) & \subset & \mathbb{F}^\Gamma(\omega^\circ)
\end{array}\right).$$

We will finally show that for G reductive and isotrivial over a quasi-compact S, the morphism $\mathbb{G}^\Gamma(G) \to \mathbb{F}^\Gamma(\omega^\circ)$ is an epimorphism of fpqc sheaves on S. Thus

$$\mathbb{F}^\Gamma(G) = \mathbb{F}^\Gamma(V) = \mathbb{F}^\Gamma(V^\circ) = \mathbb{F}^\Gamma(\omega) = \mathbb{F}^\Gamma(\omega^\circ)$$

in this case, and the remaining statement, namely

$$\mathbb{F}^\Gamma(G) = \mathbb{F}^\Gamma(V) = \mathbb{F}^\Gamma(V^\circ)$$

for a reductive group G over an arbitrary S easily follows.

REMARK 35. As will be clear from the definitions below, the assertions about ω° and V correspond to the two extreme cases of a variety of possible statements about filtrations on fiber functors. These cases were not clearly distinguished in [**41**, Chapitre IV], which lead us to revisit its proofs. Our definition of isotriviality for reductive groups in section 3.6.3 is tailor-made to fit the ω°-case: it is not even local for the Zariski topology on the base. The corresponding Zariski-local notion was defined in [**21**, XXIV 4.1.2]. For a locally isotrivial reductive group, the above theorem works with a suitably (Zariski) localized version of the fiber functor ω°.

3.1. Γ-graduations and Γ-filtrations on quasi-coherent sheaves

3.1.1. Let $\mathcal{M}$ be a quasi-coherent sheaf on a scheme X.

DEFINITION 36. A Γ-graduation on $\mathcal{M}$ is a collection $\mathcal{G} = (\mathcal{G}_\gamma)_{\gamma\in\Gamma}$ of quasi-coherent subsheaves of $\mathcal{M}$ such that $\mathcal{M} = \oplus_{\gamma\in\Gamma}\mathcal{G}_\gamma$. A Γ-filtration on $\mathcal{M}$ is a collection $\mathcal{F} = (\mathcal{F}^\gamma)_{\gamma\in\Gamma}$ of quasi-coherent subsheaves of $\mathcal{M}$ such that, locally on X for the fpqc topology, there exists a Γ-graduation $\mathcal{G} = (\mathcal{G}_\gamma)_{\gamma\in\Gamma}$ on $\mathcal{M}$ for which $\mathcal{F}^\gamma = \oplus_{\eta\geq\gamma}\mathcal{G}_\eta$. We call any such $\mathcal{G}$ a splitting of $\mathcal{F}$ and write $\mathcal{F} = \mathrm{Fil}(\mathcal{G})$. We set

$$\mathcal{F}_+^\gamma = \cup_{\eta>\gamma}\mathcal{F}^\eta \quad \text{and} \quad \mathrm{Gr}_{\mathcal{F}}^\gamma\mathcal{M} = \mathcal{F}^\gamma/\mathcal{F}_+^\gamma.$$

LEMMA 37. *Let $\mathcal{F}$ be a Γ-filtration on $\mathcal{M}$. Then $\gamma\mapsto\mathcal{F}^\gamma$ is non-increasing, exhaustive ($\cup\mathcal{F}^\gamma = \mathcal{M}$), separated ($\cap\mathcal{F}^\gamma = 0$), and for every $\gamma\in\Gamma$,*

$$0\to\mathcal{F}^\gamma\to\mathcal{M}\to\mathcal{M}/\mathcal{F}^\gamma\to 0 \quad \textit{and} \quad 0\to\mathcal{F}_+^\gamma\to\mathcal{F}^\gamma\to\mathrm{Gr}_{\mathcal{F}}^\gamma(\mathcal{M})\to 0$$

are pure exact sequences of quasi-coherent sheaves (see 3.13).

PROOF. This is local in the fpqc topology on X, trivial if $\mathcal{F}$ has a splitting. □

3.1.2. These definitions give rise to a diagram of fpqc stacks over Sch

$$\mathsf{Gr}^\Gamma\mathsf{QCoh} \underset{\mathrm{Gr}}{\overset{\mathrm{Fil}}{\rightleftarrows}} \mathsf{Fil}^\Gamma\mathsf{QCoh} \xrightarrow{\mathrm{forg}} \mathsf{QCoh}$$

whose fiber over a scheme X is the diagram of exact $\otimes$-functors

$$\mathsf{Gr}^\Gamma\mathsf{QCoh}(X) \underset{\mathrm{Gr}}{\overset{\mathrm{Fil}}{\rightleftarrows}} \mathsf{Fil}^\Gamma\mathsf{QCoh}(X) \xrightarrow{\mathrm{forg}} \mathsf{QCoh}(X)$$

where $\mathsf{QCoh}(X)$ is the abelian $\otimes$-category of quasi-coherent sheaves $\mathcal{M}$ on X, $\mathsf{Gr}^\Gamma\mathsf{QCoh}(X)$ is the abelian $\otimes$-category of Γ-graded quasi-coherent sheaves $(\mathcal{M},\mathcal{G})$ on X, and $\mathsf{Fil}^\Gamma\mathsf{QCoh}(X)$ is the exact (in Quillen's sense) $\otimes$-category of Γ-filtered quasi-coherent sheaves $(\mathcal{M},\mathcal{F})$ on X. The morphisms in these last two categories are the morphisms of the underlying quasi-coherent sheaves which preserve the given collections of subsheaves, and the $\otimes$-products are given by the usual formulas

$$\begin{aligned}(\mathcal{M}_1,\mathcal{G}_1)\otimes(\mathcal{M}_2,\mathcal{G}_2) = (\mathcal{M}_1\otimes\mathcal{M}_2,\mathcal{G}) \quad &\text{with} \quad \mathcal{G}_\gamma = \oplus_{\gamma_1+\gamma_2=\gamma}\mathcal{G}_{1,\gamma_1}\otimes\mathcal{G}_{2,\gamma_2},\\ (\mathcal{M}_1,\mathcal{F}_1)\otimes(\mathcal{M}_2,\mathcal{F}_2) = (\mathcal{M}_1\otimes\mathcal{M}_2,\mathcal{F}) \quad &\text{with} \quad \mathcal{F}^\gamma = \textstyle\sum_{\gamma_1+\gamma_2=\gamma}\mathcal{F}_1^{\gamma_1}\otimes\mathcal{F}_2^{\gamma_2}.\end{aligned}$$

The second formula makes sense by the purity mentioned above, and indeed defines a Γ-filtration on $\mathcal{M}_1\otimes\mathcal{M}_2$: if $\mathcal{G}_i$ splits $\mathcal{F}_i$ for $i\in\{1,2\}$, then $\mathcal{G}$ splits $\mathcal{F}$. We have

$$\begin{aligned}\mathcal{F}_+^\gamma \quad &= \quad \textstyle\sum_{\gamma_1+\gamma_2>\gamma}\mathcal{F}_1^{\gamma_1}\otimes\mathcal{F}_2^{\gamma_2}\\ \text{and} \quad \mathrm{Gr}_{\mathcal{F}}^\gamma(\mathcal{M}_1\otimes\mathcal{M}_2) \quad &\simeq \quad \oplus_{\gamma_1+\gamma_2=\gamma}\mathrm{Gr}_{\mathcal{F}_1}^{\gamma_1}(\mathcal{M}_1)\otimes\mathrm{Gr}_{\mathcal{F}_2}^{\gamma_2}(\mathcal{M}_2).\end{aligned}$$

The first formula is trivial and gives the morphism (from right to left) in the second formula, which is easily seen to be an isomorphism by localization to an fpqc cover of X over which $\mathcal{F}_1$ and $\mathcal{F}_2$ both acquire a splitting. The neutral objects for $\otimes$ are

$$1_X = (\mathcal{O}_X, \mathcal{G} \text{ of } \mathcal{F}) \text{ with } \mathcal{G}_\gamma = \begin{cases} \mathcal{O}_X & \text{for } \gamma = 0, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \mathcal{F}^\gamma = \begin{cases} \mathcal{O}_X & \text{for } \gamma \leq 0, \\ 0 & \text{otherwise.} \end{cases}$$

A morphism $(\mathcal{M}_1, \mathcal{F}_1) \to (\mathcal{M}_2, \mathcal{F}_2)$ is strict if $\mathrm{Im}(\mathcal{F}_1^\gamma) = \mathcal{F}_2^\gamma \cap \mathrm{Im}(\mathcal{M}_1)$ in $\mathcal{M}_2$ for every $\gamma \in \Gamma$. The short exact sequences of $\mathsf{Fil}^\Gamma\mathsf{QCoh}(X)$ are those made of strict arrows whose underlying sequence of sheaves is short exact. The formulas

$$\mathrm{Fil}(\mathcal{M}, \mathcal{G}) = (\mathcal{M}, \mathrm{Fil}(\mathcal{G})), \quad \mathrm{Gr}(\mathcal{M}, \mathcal{F}) = \oplus_\gamma \mathrm{Gr}^\gamma_{\mathcal{F}} \mathcal{M} \quad \text{and} \quad \mathrm{forg}(\mathcal{M}, -) = \mathcal{M}$$

define the exact $\otimes$-functors between our three categories. Finally the "base change functors" defining the fibered category structures on $\mathsf{Gr}^\Gamma\mathsf{QCoh}$ and $\mathsf{Fil}^\Gamma\mathsf{QCoh}$ are induced by the base change functors on QCoh (thanks to the purity of the subsheaves). It is well-known that QCoh is an fpqc stack over Sch (see for instance [**48**, Theorem 4.23]) and it follows rather formally from their definitions that the other two fibered categories are also fpqc stacks over Sch. We denote by

$$\mathsf{Gr}^\Gamma\mathsf{QCoh}/S \underset{\mathrm{Gr}}{\overset{\mathrm{Fil}}{\rightleftarrows}} \mathsf{Fil}^\Gamma\mathsf{QCoh}/S \xrightarrow{\mathrm{forg}} \mathsf{QCoh}\ /S$$

the corresponding stacks over Sch/S where S is any base scheme.

3.2. Γ-graduations and Γ-filtrations on fiber functors

3.2.1. Let $s : G \to S$ be an affine and flat group scheme. We denote by $\mathsf{Rep}(G)$ the fpqc stack over Sch/S whose fiber over $T \to S$ is the abelian $\otimes$-category $\mathsf{Rep}(G)(T)$ of quasi-coherent G_T-$\mathcal{O}_T$-modules as defined in [**20**, I 4.7.1]. Then

$$\mathcal{A}(G) = s_*\mathcal{O}_G$$

is a quasi-coherent Hopf algebra over S and $\mathsf{Rep}(G)(T)$ is $\otimes$-equivalent to the category of quasi-coherent $\mathcal{A}(G_T)$-comodules where $\mathcal{A}(G_T) = \mathcal{A}(G)_T$. Let

$$V : \mathsf{Rep}(G) \to \mathsf{QCoh}/S$$

be the forgetful functor. For any S-scheme $q : T \to S$, we denote by

$$V_T : \mathsf{Rep}(G_T) \to \mathsf{QCoh}/T \quad \text{and} \quad \omega_T : \mathsf{Rep}(G)(S) \to \mathsf{QCoh}(T)$$

the induced morphism of fpqc stack over Sch/T and fiber functor. Note that ω_T is a right exact $\otimes$-functor. It also commutes with arbitrary colimits and preserves pure monomorphisms and pure short exact sequences, where purity in $\mathsf{Rep}(G)(S)$ refers to purity of the underlying objects in $\mathsf{QCoh}(S)$.

3.2.2. A Γ-graduation $\mathcal{G}$ on $V_T : \mathsf{Rep}(G_T) \to \mathsf{QCoh}/T$ is a factorization

$$\mathsf{Rep}(G_T) \xrightarrow{\mathcal{G}} \mathsf{Gr}^\Gamma\mathsf{QCoh}/T \xrightarrow{\mathrm{forg}} \mathsf{QCoh}/T$$

of V_T such that if $\mathcal{G}_\gamma : \mathsf{Rep}(G_T) \to \mathsf{QCoh}/T$ is the γ-component of $\mathcal{G}$,

(G0) For every T-morphism $f : X \to Y$, $\rho \in \mathsf{Rep}(G)(Y)$ and $\gamma \in \Gamma$,

$$f^*(\mathcal{G}_\gamma(\rho)) = \mathcal{G}_\gamma(f^*\rho).$$

(G1) For every T-scheme $X \to T$, $\rho_1, \rho_2 \in \mathsf{Rep}(G)(X)$ and $\gamma \in \Gamma$,

$$\mathcal{G}_\gamma(\rho_1 \otimes \rho_2) = \oplus_{\gamma_1 + \gamma_2 = \gamma} \mathcal{G}_{\gamma_1}(\rho_1) \otimes \mathcal{G}_{\gamma_2}(\rho_2).$$

Thus (G0) says that each $\mathcal{G}_\gamma$ is a morphism of fibered categories over Sch/T. Then (G1) implies that $\mathcal{G}_0(\rho) = \mathcal{M}$ and $\mathcal{G}_\gamma(\rho) = 0$ for $\gamma \neq 0$ when ρ is the trivial representation of G_X on $\mathcal{M} \in \mathsf{QCoh}(X)$ (one proves it first for $\mathcal{M} = \mathcal{O}_X$).

3.2.3. A Γ-graduation $\mathcal{G}$ on $\omega_T : \mathsf{Rep}(G)(S) \to \mathsf{QCoh}(T)$ is a factorization

$$\mathsf{Rep}(G)(S) \xrightarrow{\mathcal{G}} \mathsf{Gr}^\Gamma\mathsf{QCoh}(T) \xrightarrow{\text{forg}} \mathsf{QCoh}(T)$$

of ω_T such that if $\mathcal{G}_\gamma : \mathsf{Rep}(G)(S) \to \mathsf{QCoh}(T)$ is the γ-component of $\mathcal{G}$,

(G1) For every $\rho_1, \rho_2 \in \mathsf{Rep}(G)(S)$ and $\gamma \in \Gamma$,

$$\mathcal{G}_\gamma(\rho_1 \otimes \rho_2) = \oplus_{\gamma_1+\gamma_2=\gamma}\mathcal{G}_{\gamma_1}(\rho_1) \otimes \mathcal{G}_{\gamma_2}(\rho_2).$$

(G2) For the trivial representation ρ of G on $\mathcal{M} \in \mathsf{QCoh}(S)$,

$$\mathcal{G}_0(\rho) = \mathcal{M} \quad \text{and} \quad \mathcal{G}_\gamma(\rho) = 0 \text{ if } \gamma \neq 0.$$

Note that each $\mathcal{G}_\gamma$ is right exact, commutes with arbitrary colimits and preserves pure monomorphisms and pure short exact sequences.

3.2.4. A Γ-filtration $\mathcal{F}$ on $V_T : \mathsf{Rep}(G_T) \to \mathsf{QCoh}/T$ is a factorization

$$\mathsf{Rep}(G_T) \xrightarrow{\mathcal{F}} \mathsf{Fil}^\Gamma\mathsf{QCoh}/T \xrightarrow{\text{forg}} \mathsf{QCoh}/T$$

of V_T such that if $\mathcal{F}^\gamma : \mathsf{Rep}(G_T) \to \mathsf{QCoh}/T$ is the γ-component of $\mathcal{F}$,

(F0) For every T-morphism $f : X \to Y$, $\rho \in \mathsf{Rep}(G)(Y)$ and $\gamma \in \Gamma$,

$$f^*(\mathcal{F}^\gamma(\rho)) = \mathcal{F}^\gamma(f^*\rho).$$

(F1) For every $X \to T$, $\rho_1, \rho_2 \in \mathsf{Rep}(G)(X)$ and $\gamma \in \Gamma$,

$$\mathcal{F}^\gamma(\rho_1 \otimes \rho_2) = \textstyle\sum_{\gamma_1+\gamma_2=\gamma}\mathcal{F}^{\gamma_1}(\rho_1) \otimes \mathcal{F}^{\gamma_2}(\rho_2).$$

(F3) For every $X \to T$ and $\gamma \in \Gamma$, $\mathcal{F}^\gamma : \mathsf{Rep}(G)(X) \to \mathsf{QCoh}(X)$ is exact.

Thus (F0) says that each $\mathcal{F}^\gamma$ is a morphism of fibered categories over Sch/T. Then again (F1) and (F3) imply that $\mathcal{F}^\gamma(\rho) = \mathcal{M}$ for $\gamma \leq 0$ and $\mathcal{F}^\gamma(\rho) = 0$ for $\gamma > 0$ when ρ is the trivial representation of G on $\mathcal{M} \in \mathsf{QCoh}(X)$.

3.2.5. A Γ-filtration $\mathcal{F}$ on $\omega_T : \mathsf{Rep}(G)(S) \to \mathsf{QCoh}(T)$ is a factorization

$$\mathsf{Rep}(G)(S) \xrightarrow{\mathcal{F}} \mathsf{Fil}^\Gamma\mathsf{QCoh}(T) \xrightarrow{\text{forg}} \mathsf{QCoh}(T)$$

of ω_T such that if $\mathcal{F}^\gamma : \mathsf{Rep}(G)(S) \to \mathsf{QCoh}(T)$ is the γ-component of $\mathcal{F}$,

(F1) For every $\rho_1, \rho_2 \in \mathsf{Rep}(G)(S)$ and $\gamma \in \Gamma$,

$$\mathcal{F}^\gamma(\rho_1 \otimes \rho_2) = \textstyle\sum_{\gamma_1+\gamma_2=\gamma}\mathcal{F}^{\gamma_1}(\rho_1) \otimes \mathcal{F}^{\gamma_2}(\rho_2).$$

(F2) For the trivial representation ρ of G on $\mathcal{M} \in \mathsf{QCoh}(S)$,

$$\mathcal{F}^\gamma(\rho) = \mathcal{M} \text{ if } \gamma \leq 0 \quad \text{and} \quad \mathcal{F}^\gamma(\rho) = 0 \text{ if } \gamma > 0.$$

(F3) For every $\gamma \in \Gamma$, $\mathcal{F}^\gamma : \mathsf{Rep}(G)(S) \to \mathsf{QCoh}(T)$ is right exact.

Since $\mathcal{F}^\gamma$ preserves arbitrary direct sums (as a subfunctor of ω_T which does), this last axiom implies that $\mathcal{F}^\gamma$ commutes with arbitrary colimits. It also preserves pure monomorphisms and pure short exact sequences.

3.2.6. We may now introduce a diagram of fpqc sheaves $(\mathsf{Sch}/S)^\circ \to \mathsf{Set}$,

$$\begin{tikzcd} \mathbb{G}^\Gamma(V) \arrow[r, "\mathrm{res}"] \arrow[d, "\mathrm{Fil}"'] & \mathbb{G}^\Gamma(\omega) \arrow[d, "\mathrm{Fil}"'] \\ \mathbb{F}^\Gamma(V) \arrow[r, "\mathrm{res}"] & \mathbb{F}^\Gamma(\omega) \end{tikzcd}$$

The four presheaves map an S-scheme T to the corresponding set of Γ-graduations or Γ-filtrations on V_T or ω_T, the Fil-morphisms are given by post-composition with the eponymous functors, and the res morphisms map $\mathcal{G}$ or $\mathcal{F}$ on V_T to

$$\mathsf{Rep}(G)(S) \to \mathsf{Rep}(G)(T) \xrightarrow{\mathcal{G}_T} \mathsf{Gr}^\Gamma\mathsf{QCoh}(T) \quad \text{or} \quad \cdots \xrightarrow{\mathcal{F}_T} \mathsf{Fil}^\Gamma\mathsf{QCoh}(T).$$

The fact that all four presheaves are actually fpqc sheaves on S is essentially a formal consequence of the fact that the corresponding fibered categories of Γ-graded and Γ-filtered quasi-coherent sheaves are fpqc stacks over Sch/S.

3.2.7. The above diagram is equivariant with respect to a morphism

$$\mathrm{Aut}^\otimes(V) \xrightarrow{\mathrm{res}} \mathrm{Aut}^\otimes(\omega)$$

of fpqc sheaves of groups on Sch/S, with $\mathrm{Aut}^\otimes(\star)$ acting on $\mathbb{G}^\Gamma(\star)$ and $\mathbb{F}^\Gamma(\star)$ and mapping an S-scheme T to a group $\mathrm{Aut}^\otimes(\star_T)$ defined as follows: $\mathrm{Aut}^\otimes(V_T)$ is the group of all automorphisms $\eta : V_T \to V_T$ such that:

(A0) For every T-morphism $f : X \to Y$ and $\rho \in \mathsf{Rep}(G)(Y)$,

$$\eta_{f^*(\rho)} = f^*(\eta_\rho).$$

(A1) For every T-scheme $X \to T$ and $\rho_1, \rho_2 \in \mathsf{Rep}(G)(X)$,

$$\eta_{\rho_1 \otimes \rho_2} = \eta_{\rho_1} \otimes \eta_{\rho_2}.$$

These conditions imply as above that $\eta_\rho = \mathrm{Id}_\mathcal{M}$ when ρ is the trivial representation of G_X on a quasi-coherent $\mathcal{O}_X$-module $\mathcal{M}$. Similarly, $\mathrm{Aut}^\otimes(\omega_T)$ is the group of all automorphisms $\eta : \omega_T \to \omega_T$ such that:

(A1) For every $\rho_1, \rho_2 \in \mathsf{Rep}(G)(S)$,

$$\eta_{\rho_1 \otimes \rho_2} = \eta_{\rho_1} \otimes \eta_{\rho_2}.$$

(A2) For the trivial representation ρ of G on $\mathcal{M} \in \mathsf{QCoh}(S)$,

$$\eta_\rho = \mathrm{Id}_\mathcal{M}.$$

The fact that these two presheaves are actually fpqc sheaves on S is essentially a formal consequence of the fact that QCoh/S is a stack over Sch/S. The morphism between them sends $\eta \in \mathrm{Aut}^\otimes(V_T)$ to the automorphism of ω_T which maps ρ in $\mathsf{Rep}(G)(S)$ to the automorphism η_{ρ_T} of $V(\rho_T) = \omega_T(\rho)$, the actions mentioned above are the obvious ones, and the claimed equivariance is equally straightforward.

3.2.8. For $\star \in \{V, \omega\}$ and $\mathcal{X} \in \mathbb{G}^\Gamma(\star)(T)$ or $\mathbb{F}^\Gamma(\star)(T)$, we denote by

$$\mathrm{Aut}^\otimes(\mathcal{X}) : (\mathsf{Sch}/T)^\circ \to \mathsf{Group}$$

the stabilizer of $\mathcal{X}$ in the restriction $\mathrm{Aut}^\otimes(\star)|_T$ of $\mathrm{Aut}^\otimes(\star)$ to Sch/T. It is an fpqc subsheaf of $\mathrm{Aut}^\otimes(\star)|_T$. For $\mathcal{X} = \mathcal{F}$ in $\mathbb{F}^\Gamma(\star)(T)$, there is also a morphism

$$\mathrm{Gr}^\bullet : \mathrm{Aut}^\otimes(\mathcal{F}) \to \mathrm{Aut}^\otimes(\mathrm{Gr}^\bullet_\mathcal{F}).$$

Here $\mathrm{Aut}^{\otimes}(\mathrm{Gr}^{\bullet}_{\mathcal{F}})$ is an fpqc sheaf of groups on Sch/T which maps $X \to T$ to a group of automorphisms of $\mathrm{Gr}^{\bullet}_{\mathcal{F}_X} = \mathrm{Gr}^{\bullet} \circ \mathcal{F}_X$ subject to conditions whose precise formulation will be left to the reader. The kernel of this morphism is an fpqc sheaf

$$\mathrm{Aut}^{\otimes !}(\mathcal{F}) : (\mathsf{Sch}/T)^{\circ} \to \mathsf{Group}.$$

If $\mathcal{G}$ is a splitting of $\mathcal{F}$, then $\mathrm{Gr}^{\bullet}_{\mathcal{F}} \simeq \mathcal{G}$, thus $\mathrm{Aut}^{\otimes}(\mathrm{Gr}^{\bullet}_{\mathcal{F}}) \simeq \mathrm{Aut}^{\otimes}(\mathcal{G})$ and

$$\mathrm{Aut}^{\otimes}(\mathcal{F}) \simeq \mathrm{Aut}^{\otimes !}(\mathcal{F}) \rtimes \mathrm{Aut}^{\otimes}(\mathcal{G}).$$

3.2.9. There is finally another equivariant diagram of fpqc sheaves on S,

$$\begin{tikzcd} G \arrow[d, "\iota"] \\ \mathrm{Aut}^{\otimes}(V) \end{tikzcd} \quad \text{acting on} \quad \begin{tikzcd} \mathbb{G}^{\Gamma}(G) \arrow[r, "\mathrm{Fil}"] \arrow[d, "\iota"] & \mathbb{F}^{\Gamma}(G) \arrow[d, "\iota"] \\ \mathbb{G}^{\Gamma}(V) \arrow[r, "\mathrm{Fil}"] & \mathbb{F}^{\Gamma}(V) \end{tikzcd}$$

The morphism $\iota : G \to \mathrm{Aut}^{\otimes}(V)$ sends $g \in G(T)$ to the automorphism $\iota(g)$ of V_T which maps $\rho \in \mathsf{Rep}(G)(X)$ to the automorphism $\rho(g_X)$ of $V(\rho)$ – for an S-scheme T and a T-scheme X. The morphism $\iota : \mathbb{F}^{\Gamma}(G) \hookrightarrow \mathbb{F}^{\Gamma}(V)$ is the image of

$$\mathbb{G}^{\Gamma}(G) \xrightarrow{\iota} \mathbb{G}^{\Gamma}(V) \xrightarrow{\mathrm{Fil}} \mathbb{F}^{\Gamma}(V)$$

where $\iota : \mathbb{G}^{\Gamma}(G) \to \mathbb{G}^{\Gamma}(V)$ is defined as follows. Recall from [**20**, I 4.7.3] that the fpqc stacks $\mathsf{Gr}^{\Gamma}\mathsf{QCoh}$ and $\mathsf{Rep}\,\mathbb{D}(\Gamma)$ over Sch are $\otimes$-equivalent: A Γ-graded quasi-coherent sheaf $\mathcal{M} = \oplus_{\gamma\in\Gamma}\mathcal{G}_{\gamma}$ on a scheme X is mapped to the unique representation ρ of $\mathbb{D}_X(\Gamma)$ on $\mathcal{M}$ such that for every $f : Y \to X$ and $\alpha : \Gamma \to \Gamma(Y, \mathcal{O}_Y^*)$ in $\mathbb{D}_X(\Gamma)(Y)$, $\rho(\alpha)(x)$ equals $\alpha(\gamma)\cdot x$ for every $\gamma \in \Gamma$ and $x \in \Gamma(Y, f^*\mathcal{G}_{\gamma})$. Conversely, a representation ρ of $\mathbb{D}_X(\Gamma)$ on a quasi-coherent $\mathcal{O}_X$-module $\mathcal{M}$ is sent to the Γ-grading on $\mathcal{M}$ defined by the eigenspace decomposition of ρ. Then ι maps a morphism $\chi : \mathbb{D}_T(\Gamma) \to G_T$ in $\mathbb{G}^{\Gamma}(G)(T)$ to the Γ-graduation on V_T defined by

$$\mathsf{Rep}(G_T) \xrightarrow{-\circ\chi} \mathsf{Rep}(\mathbb{D}_T(\Gamma)) \simeq \mathsf{Gr}^{\Gamma}\mathsf{QCoh}/T \xrightarrow{\mathrm{forg}} \mathsf{QCoh}/T.$$

REMARK 38. We will show in corollary 55 that for a *reductive* group G, the definition of the fpqc sheaf $\mathbb{F}^{\Gamma}(G)$ on Sch/S given here (image of $\mathbb{G}^{\Gamma}(G) \to \mathbb{F}^{\Gamma}(V)$) coincides with the definition of section 2.2 (image of $\mathbb{G}^{\Gamma}(G) \to \mathbb{G}^{\Gamma}(R_{\mathbb{P}(G)})$).

3.3. The subcategories of rigid objects

We briefly discuss the $-^{\circ}$ variants of the above definitions, mostly mentioning the new features.

3.3.1. Finite locally free sheaves. Let $\mathsf{LF} \to \mathsf{Sch}$ be the fibered category whose fiber over X is the full subcategory $\mathsf{LF}(X)$ of $\mathsf{QCoh}(X)$ whose objects are the finite locally free sheaves on X. Then LF is a substack of QCoh by [**26**, 2.5.2]. Pulling back through $\mathsf{LF} \hookrightarrow \mathsf{QCoh}$, we obtain a diagram of fpqc stacks over Sch,

$$\mathsf{Gr}^{\Gamma}\mathsf{LF} \underset{\mathrm{Gr}}{\overset{\mathrm{Fil}}{\rightleftarrows}} \mathsf{Fil}^{\Gamma}\mathsf{LF} \xrightarrow{\mathrm{forg}} \mathsf{LF}$$

whose fiber over a scheme X is a diagram of exact (in Quillen's sense) $\otimes$-functors

$$\mathsf{Gr}^{\Gamma}\mathsf{LF}(X) \underset{\mathrm{Gr}}{\overset{\mathrm{Fil}}{\rightleftarrows}} \mathsf{Fil}^{\Gamma}\mathsf{LF}(X) \xrightarrow{\mathrm{forg}} \mathsf{LF}(X).$$

An alternative and useful description of the objects of $\mathsf{Fil}^{\Gamma}\mathsf{LF}(X)$ is provided by proposition 39 below, which also implies that the Gr functor is indeed well-defined. Over a base scheme S, there is the corresponding diagram of fpqc stacks:

$$\mathsf{Gr}^{\Gamma}\mathsf{LF}/S \underset{\mathrm{Gr}}{\overset{\mathrm{Fil}}{\rightleftarrows}} \mathsf{Fil}^{\Gamma}\mathsf{LF}/S \xrightarrow{\mathrm{forg}} \mathsf{LF}/S$$

3.3.2. These categories have compatible inner Hom's and duals given by

$$\underline{\mathrm{Hom}}(x,y) = x^{\vee}\otimes y \quad \text{with} \quad (\mathcal{M},\mathcal{G})^{\vee} = (\mathcal{M}^{\vee},\mathcal{G}^{\vee}) \quad \text{and} \quad (\mathcal{M},\mathcal{F})^{\vee} = (\mathcal{M}^{\vee},\mathcal{F}^{\vee})$$

where $\mathcal{M}^{\vee}$ is the dual of $\mathcal{M}$, $(\mathcal{G}^{\vee})_{\gamma} = (\mathcal{G}_{-\gamma})^{\vee}$ and $(\mathcal{F}^{\vee})^{\gamma} = (\mathcal{F}_{+}^{-\gamma})^{\perp} = (\mathcal{M}/\mathcal{F}_{+}^{-\gamma})^{\vee}$. Thus if $\mathcal{G}$ is a splitting of $\mathcal{F}$, then $\mathcal{G}^{\vee}$ is a splitting of $\mathcal{F}^{\vee}$. Moreover, we have

$$(\mathcal{F}^{\vee})_{+}^{\gamma} = (\mathcal{F}^{-\gamma})^{\perp} \simeq (\mathcal{M}/\mathcal{F}^{-\gamma})^{\vee} \quad \text{and} \quad \mathrm{Gr}_{\mathcal{F}^{\vee}}^{\gamma}(\mathcal{M}^{\vee}) \simeq \mathrm{Gr}_{\mathcal{F}}^{-\gamma}(\mathcal{M})^{\vee}.$$

For the inner Homs, we obtain the following formula:

$$\mathrm{Gr}_{\mathcal{F}}^{\gamma}\left(\underline{\mathrm{Hom}}(\mathcal{M}_1,\mathcal{M}_2)\right) \simeq \oplus_{\gamma_2-\gamma_1=\gamma}\underline{\mathrm{Hom}}\left(\mathrm{Gr}_{\mathcal{F}_1}^{\gamma_1}(\mathcal{M}_1),\mathrm{Gr}_{\mathcal{F}_2}^{\gamma_2}(\mathcal{M}_2)\right).$$

3.3.3. Γ-filtrations on finite locally free sheaves.

PROPOSITION 39. *Let $\mathcal{M}$ be a finite locally free sheaf on X. Let $(\mathcal{F}^{\gamma})_{\gamma\in\Gamma}$ be a non-increasing collection of quasi-coherent subsheaves of $\mathcal{M}$. Then the following conditions are equivalent:*

(1) *For every affine open subset U of X, there is a Γ-graduation*

$$\mathcal{M}_U = \oplus_{\gamma\in\Gamma}\mathcal{G}_{\gamma}$$

such that $\mathcal{F}_U^{\gamma} = \oplus_{\eta\geq\gamma}\mathcal{G}_{\eta}$ for every $\gamma\in\Gamma$.

(2) *Locally on X for the Zariski topology, there is a Γ-graduation*

$$\mathcal{M} = \oplus_{\gamma\in\Gamma}\mathcal{G}_{\gamma}$$

such that $\mathcal{F}^{\gamma} = \oplus_{\eta\geq\gamma}\mathcal{G}_{\eta}$ for every $\gamma\in\Gamma$.

(3) *Locally on X for the fpqc topology, there exists a Γ-graduation*

$$\mathcal{M} = \oplus_{\gamma\in\Gamma}\mathcal{G}_{\gamma}$$

such that $\mathcal{F}^{\gamma} = \oplus_{\eta\geq\gamma}\mathcal{G}_{\eta}$ for every $\gamma\in\Gamma$, i.e. $\mathcal{F}$ is a Γ-filtration on $\mathcal{M}$.

(4) *For every $\gamma\in\Gamma$, $\mathrm{Gr}_{\mathcal{F}}^{\gamma}(\mathcal{M})$ is finite locally free and for every $x\in X$,*

$$\dim_{k(x)}\mathcal{M}(x) = \textstyle\sum_{\gamma}\dim_{k(x)}\mathrm{Gr}_{\mathcal{F}(x)}^{\gamma}(\mathcal{M}(x)).$$

In (4), $\mathcal{F}(x)$ is the image of $\mathcal{F}$ in $\mathcal{M}(x) = \mathcal{M}\otimes k(x)$ and $\mathrm{Gr}_{\mathcal{F}}^{\gamma}(\mathcal{M})$, $\mathrm{Gr}_{\mathcal{F}(x)}^{\gamma}(\mathcal{M}(x))$ are defined as usual. Under the above equivalent conditions, for all $\gamma\in\Gamma$: $\mathcal{F}^{\gamma}$, $\mathcal{F}_{+}^{\gamma}$ and $\mathrm{Gr}_{\mathcal{F}}^{\gamma}(\mathcal{M})$ are finite locally free sheaves on X and for every $x\in X$,

$$\mathcal{F}^{\gamma}(x) \simeq \mathcal{F}^{\gamma}\otimes k(x), \quad \mathcal{F}_{+}^{\gamma}(x) \simeq \mathcal{F}_{+}^{\gamma}\otimes k(x), \quad \mathrm{Gr}_{\mathcal{F}(x)}^{\gamma}(\mathcal{M}(x)) \simeq \mathrm{Gr}_{\mathcal{F}}^{\gamma}(\mathcal{M})\otimes k(x).$$

PROOF. Plainly (1) $\Rightarrow$ (2) $\Rightarrow$ (3). Moreover (3) $\Rightarrow$ (4) is easy (using [**26**, 2.5.2.iii]) and the last assertions follow from (1). To prove that (4) $\Rightarrow$ (1), we may assume that $X = U$ is affine. Since $\mathrm{Gr}_{\mathcal{F}}^{\gamma}(\mathcal{M})$ is finite locally free by assumption, it is then projective in $\mathsf{QCoh}(X)$ by [**34**, Corollary of 7.12]. Therefore, there exists a quasi-coherent subsheaf $\mathcal{G}_{\gamma}$ of $\mathcal{F}_{\gamma}$ such that $\mathcal{F}^{\gamma} = \mathcal{G}_{\gamma}\oplus\mathcal{F}_{+}^{\gamma}$. We will show that

$$\mathcal{M} = \oplus_{\gamma\in\Gamma}\mathcal{G}_{\gamma} \quad \text{and} \quad \forall\gamma : \mathcal{F}^{\gamma} = \oplus_{\eta\geq\gamma}\mathcal{G}_{\eta}.$$

This being now a local question in the Zariski topology of X, we may assume that the rank of $\mathcal{M}$ is constant on X, and also nonzero. Fix $x \in X$ and define

$$\Gamma(x) = \{\gamma : \mathrm{Gr}^{\gamma}_{\mathcal{F}(x)}(\mathcal{M}(x)) \neq 0\} = \{\gamma_1 < \cdots < \gamma_r\}.$$

Define $U_0 = \mathrm{Supp}(\mathcal{M}/\mathcal{F}^{\gamma_1})^c$, $U_i = \mathrm{Supp}(\mathcal{F}^{\gamma_i}_+/\mathcal{F}^{\gamma_{i+1}})^c \cap U_{i-1}$ for $0 < i < r$ and $U_r = \mathrm{Supp}(\mathcal{F}^{\gamma_r}_+)^c \cap U_{r-1}$. Since $\mathcal{M}$ is finite locally free, $\mathcal{M}/\mathcal{F}^{\gamma_1}$ is finitely generated and U_0 is open in X. Since $\mathcal{M} = \mathcal{F}^{\gamma_1}$ over U_0 and $\mathcal{F}^{\gamma_1} = \mathcal{F}^{\gamma_1}_+ \oplus \mathcal{G}_{\gamma_1}$ over X, $\mathcal{M} = \mathcal{F}^{\gamma_1}_+ \oplus \mathcal{G}_{\gamma_1}$ over U_0. Therefore $\mathcal{F}^{\gamma_1}_+$ is finite locally free over U_0. Repeating this argument successively with $(\mathcal{M}, X)$ replaced by $(\mathcal{F}^{\gamma_1}_+, U_0)$, $(\mathcal{F}^{\gamma_2}_+, U_1)$ etc... we obtain: U_r is open in X, $\mathcal{M} = \oplus_i \mathcal{G}_{\gamma_i}$ and $\mathcal{F}^{\gamma} = \oplus_{i:\gamma_i \geq \gamma} \mathcal{G}_{\gamma_i}$ over U_r for every $\gamma \in \Gamma$, with everyone finite locally free over U_r. All we have to do now is to show that the formula of (4) implies that x belongs to U_r. The formula is equivalent to:

$$\mathcal{F}^{\gamma}(x) = \begin{cases} \mathcal{M}(x) & \text{if } \gamma \leq \gamma_1, \\ \mathcal{F}^{\gamma_{i+1}}(x) & \text{if } \gamma \in]\gamma_i, \gamma_{i+1}], \\ 0 & \text{if } \gamma > \gamma_r. \end{cases}$$

Since $\mathcal{M}$ is finitely generated over X, $\mathcal{F}^{\gamma_1}(x) = \mathcal{M}(x)$ implies $\mathcal{F}^{\gamma_1}_x = \mathcal{M}_x$ by Nakayama's lemma, thus x belongs to U_0. Since $\mathcal{M} = \mathcal{F}^{\gamma_1} = \mathcal{F}^{\gamma_1}_+ \oplus \mathcal{G}_{\gamma_1}$ over U_0, $\mathcal{F}^{\gamma_1}_+(x) = \mathcal{F}^{\gamma_2}(x)$ in $\mathcal{M}(x)$ implies $\mathcal{F}^{\gamma_1}_{+,x} = \mathcal{F}^{\gamma_2}_x$ by Nakayama's lemma, therefore x belongs to U_1. Repeating the argument, we find that indeed x belongs to U_r. □

REMARK 40. The whole proof becomes much simpler over a Noetherian base.

LEMMA 41. *Let $\mathcal{M}_\alpha$ be a finite collection of locally free sheaves of finite rank on X and for each α, let $(\mathcal{F}^{\gamma}_\alpha)_{\gamma \in \Gamma}$ be a non-increasing collection of quasi-coherent subsheaves of $\mathcal{M}_\alpha$. Set $\mathcal{M} = \oplus \mathcal{M}_\alpha$ and $\mathcal{F}^{\gamma} = \oplus \mathcal{F}^{\gamma}_\alpha$. Then $(\mathcal{M}, (\mathcal{F}^{\gamma}))$ satisfies the above equivalent conditions if and only if each $(\mathcal{M}_\alpha, (\mathcal{F}^{\gamma}_\alpha))$ does.*

PROOF. For every $\gamma \in \Gamma$ and $x \in X$, $\mathrm{Gr}^{\gamma}_{\mathcal{F}}(\mathcal{M}) = \oplus_\alpha \mathrm{Gr}^{\gamma}_{\mathcal{F}^\alpha}(\mathcal{M}_\alpha)$ and

$$\mathcal{M}(x) = \oplus_\alpha \mathcal{M}_\alpha(x), \quad \mathrm{Gr}^{\gamma}_{\mathcal{F}(x)}(\mathcal{M}(x)) = \oplus_\alpha \mathrm{Gr}^{\gamma}_{\mathcal{F}_\alpha(x)}(\mathcal{M}_\alpha(x)).$$

Moreover for every α and $x \in X$,

$$\dim_{k(x)} \mathcal{M}_\alpha(x) \geq \textstyle\sum_\gamma \dim_{k(x)} \mathrm{Gr}^{\gamma}_{\mathcal{F}_\alpha(x)}(\mathcal{M}_\alpha(x)).$$

The lemma easily follows. □

3.3.4. Let $\mathsf{Rep}^\circ(G) \to \mathsf{Sch}/S$ be the substack of $\mathsf{Rep}(G) \to \mathsf{Sch}/S$ whose fiber over $T \to S$ is the exact, rigid, full sub-$\otimes$-category $\mathsf{Rep}^\circ(G)(T)$ of $\mathsf{Rep}(G)(T)$ whose objects are the representations of G_T on finite locally free sheaves on T. We write

$$V^\circ : \mathsf{Rep}^\circ(G) \to \mathsf{LF}/S$$

for the forgetful functor. For an S-scheme $T \to S$, we denote by

$$V^\circ_T : \mathsf{Rep}^\circ(G_T) \to \mathsf{LF}/T \quad \text{and} \quad \omega^\circ_T : \mathsf{Rep}^\circ(G)(S) \to \mathsf{LF}(T)$$

the induced morphism of fpqc stack over Sch/T and fiber functor. Note that ω°_T is now an exact $\otimes$-functor, since all short exact sequences in $\mathsf{Rep}^\circ(G)(S)$ are pure.

3.3.5. We obtain yet another equivariant diagram of fpqc sheaves on S,

$$\begin{array}{ccc} \mathrm{Aut}^{\otimes}(V^\circ) & & \mathbb{G}^\Gamma(V^\circ) \xrightarrow{\mathrm{Fil}} \mathbb{F}^\Gamma(V^\circ) \\ \downarrow{\scriptstyle \mathrm{res}} & \text{acting on} & \downarrow{\scriptstyle \mathrm{res}} \qquad\quad \downarrow{\scriptstyle \mathrm{res}} \\ \mathrm{Aut}^{\otimes}(\omega^\circ) & & \mathbb{G}^\Gamma(\omega^\circ) \xrightarrow{\mathrm{Fil}} \mathbb{F}^\Gamma(\omega^\circ) \end{array}$$

where everything is defined as before, using V° and ω° instead of V and ω. The only differences worth mentioning are as follows: for any S-scheme T, the Γ-graduations or Γ-filtrations on ω_T° are automatically compatible with inner Homs and duals, and there γ-components are exact functors. We also have equivariant diagrams

$$\begin{array}{ccc} \mathrm{Aut}^{\otimes}(V) & & \mathbb{G}^\Gamma(V) \xrightarrow{\mathrm{Fil}} \mathbb{F}^\Gamma(V) \\ \downarrow{\scriptstyle \mathrm{res}} & \text{acting on} & \downarrow{\scriptstyle \mathrm{res}} \qquad\quad \downarrow{\scriptstyle \mathrm{res}} \\ \mathrm{Aut}^{\otimes}(V^\circ) & & \mathbb{G}^\Gamma(V^\circ) \xrightarrow{\mathrm{Fil}} \mathbb{F}^\Gamma(V^\circ) \end{array}$$

and similarly for ω and ω°, where all the vertical maps are induced by pre-composition with the full embedding $\mathsf{Rep}^\circ(G) \hookrightarrow \mathsf{Rep}(G)$.

3.3.6. Finally, the definitions of $\mathrm{Aut}^{\otimes}(\mathcal{G})$, $\mathrm{Aut}^{\otimes}(\mathcal{F})$, $\mathrm{Aut}^{\otimes !}(\mathcal{F})$ and $\mathrm{Aut}^{\otimes}(\mathrm{Gr}^\bullet_{\mathcal{F}})$ given in section 3.2.8 carry over to the situation considered here.

3.4. Skalar extensions

The whole diagram at the beginning of this section has now been defined. It is covariantly functorial in G but not entirely compatible with base change on S: if $\tilde{S} \to S$ is any morphism, $\tilde{G} = G \times_S \tilde{S}$ and $\tilde{V}$, $\tilde{\omega}$ ... are the relevant functors for $\tilde{G}$, then $\mathbb{G}^\Gamma(\tilde{G}) = \mathbb{G}^\Gamma(G)|_{\tilde{S}}$, $\mathbb{F}^\Gamma(\tilde{G}) = \mathbb{F}^\Gamma(G)|_{\tilde{S}}$ and

$$\mathrm{Aut}^{\otimes}(\tilde{X}) = \mathrm{Aut}^{\otimes}(X)|_{\tilde{S}}, \quad \mathbb{G}^\Gamma(\tilde{X}) = \mathbb{G}^\Gamma(X)|_{\tilde{S}} \quad \text{and} \quad \mathbb{F}^\Gamma(\tilde{X}) = \mathbb{F}^\Gamma(X)|_{\tilde{S}}$$

for $X \in \{V, V^\circ\}$, but the natural morphisms of fpqc sheaves on $\tilde{S}$,

$$\mathrm{Aut}^{\otimes}(\tilde{Y}) \to \mathrm{Aut}^{\otimes}(Y)|_{\tilde{S}}, \quad \mathbb{G}^\Gamma(\tilde{Y}) \to \mathbb{G}^\Gamma(Y)|_{\tilde{S}} \quad \text{and} \quad \mathbb{F}^\Gamma(\tilde{Y}) \to \mathbb{F}^\Gamma(Y)|_{\tilde{S}}$$

may not be isomorphisms for $Y \in \{\omega, \omega^\circ\}$. We investigate this issue.

3.4.1. When C is a category and $\mathcal{B}$ is a ring object in C, we can form the category $\mathsf{C}(\mathcal{B})$ of (left) $\mathcal{B}$-modules in C. Here C will be an additive $\otimes$-category and the ring object will be given by its multiplication morphism $\mu : \mathcal{B} \otimes \mathcal{B} \to \mathcal{B}$ and unit $1 \to \mathcal{B}$, where 1 is the neutral object for the tensor product, the abelian group structure on $\mathcal{B}$ being provided by the additive structure of C. Then $\mathsf{C}(\mathcal{B})$ is the category of pairs $(\mathcal{M}, \nu)$ where $\mathcal{M}$ is an object of C and $\nu : \mathcal{B} \otimes \mathcal{M} \to \mathcal{M}$ is a morphism in C subject to certain natural conditions. There is an adjunction

$$f^* : \mathsf{C} \leftrightarrow \mathsf{C}(\mathcal{B}) : f_* \quad \text{given by} \quad f_*(\mathcal{M}, \nu) = \mathcal{M} \text{ and } f^*(\mathcal{N}) = (\mathcal{B} \otimes \mathcal{N}, \mu \otimes \mathrm{Id}).$$

In many cases, it is also possible to equip $\mathsf{C}(\mathcal{B})$ with a $\otimes$-product inherited from the $\otimes$-product on C, with $(\mathcal{B}, \mu)$ as neutral object. Instead of trying to develop this formal theory more rigorously, let us list some of the relevant examples:

$\mathsf{C} = \mathsf{QCoh}(S)$ **and** $\mathcal{B} = f_*\mathcal{O}_T$ **where** $f : T \to S$ **is an affine morphism.** There is an equivalence of $\otimes$-categories $\mathsf{C}(\mathcal{B}) \simeq \mathsf{QCoh}(T)$ which is compatible with the usual adjunctions $f^* : \mathsf{QCoh}(S) \leftrightarrow \mathsf{QCoh}(T) : f_*$, see [**24**, 1.4].

$\mathsf{C} = \mathsf{Gr}^\Gamma\mathsf{QCoh}(S)$ and $\mathcal{B}$ as above with the trivial Γ-graduation. The first example induces an equivalence of $\otimes$-categories $\mathsf{C}(\mathcal{B}) \simeq \mathsf{Gr}^\Gamma\mathsf{QCoh}(T)$ which is again compatible with the natural adjunctions.

$\mathsf{C} = \mathsf{Fil}^\Gamma\mathsf{QCoh}(S)$ and $\mathcal{B}$ as above with the trivial Γ-filtration. The first example now only induces a fully faithful exact $\otimes$-functor $\mathsf{C}(\mathcal{B}) \hookrightarrow \mathsf{Fil}^\Gamma\mathsf{QCoh}(T)$. The essential image is made of those Γ-filtered quasi-coherent sheaves $(\mathcal{M}, \mathcal{F})$ on T such that, locally on S (as opposed to T) for the fpqc topology, $\mathcal{F}$ has a splitting.

$\mathsf{C} = \mathsf{Rep}(G)(S)$ and $\mathcal{B}$ as above with the trivial action of G. The first example again induces an equivalence of $\otimes$-categories $\mathsf{C}(\mathcal{B}) \simeq \mathsf{Rep}(G)(T)$ which is compatible with the adjunctions given on the comodules by the following formulas:

$$\begin{aligned} f^*\left(V(\rho) \xrightarrow{c_\rho} V(\rho) \otimes_{\mathcal{O}_S} \mathcal{A}(G)\right) &= \left(V(f^*\rho) \xrightarrow{c_{f^*\rho}} V(f^*\rho) \otimes_{\mathcal{O}_T} \mathcal{A}(G_T)\right), \\ f_*\left(V(\rho) \xrightarrow{c_\rho} V(\rho) \otimes_{\mathcal{O}_T} \mathcal{A}(G_T)\right) &= \left(V(f_*\rho) \xrightarrow{c_{f_*\rho}} V(f_*\rho) \otimes_{\mathcal{O}_S} \mathcal{A}(G)\right). \end{aligned}$$

$\mathsf{C} = \mathsf{LF}(S)$ and $\mathcal{B} = f_*\mathcal{O}_T$ where $f : T \to S$ is a finite étale morphism. The first example induces an equivalence of $\otimes$-categories $\mathsf{C}(\mathcal{B}) \simeq \mathsf{LF}(T)$. We have to show that for a quasi-coherent sheaf $\mathcal{M}$ on T, $\mathcal{M}$ is a finite locally free $\mathcal{O}_T$-module if and only if $f_*\mathcal{M}$ is a finite locally free $\mathcal{O}_S$-module (the direct implication is easy, and only requires f to be finite and locally free). By [**26**, 2.5.2], our claim is local in the fpqc topology on S. But, locally on S for the étale topology, our finite étale morphism f is simply a finite disjoint union of open and closed embeddings (this follows from [**28**, 17.9.3]), for which the claim is now obvious.

Combining this last example with the previous three, we obtain:

$\mathsf{C} = \mathsf{Gr}^\Gamma\mathsf{LF}(S)$ and $\mathcal{B}$ as above with the trivial Γ-graduation. Then

$$\mathsf{C}(\mathcal{B}) \simeq \mathsf{Gr}^\Gamma\mathsf{LF}(T).$$

$\mathsf{C} = \mathsf{Fil}^\Gamma\mathsf{LF}(S)$ and $\mathcal{B}$ as above with the trivial Γ-filtration. Then

$$\mathsf{C}(\mathcal{B}) \simeq \mathsf{Fil}^\Gamma\mathsf{LF}(T).$$

$\mathsf{C} = \mathsf{Rep}^\circ(G)(S)$ and $\mathcal{B}$ as above with the trivial action. Then

$$\mathsf{C}(\mathcal{B}) \simeq \mathsf{Rep}^\circ(G)(T).$$

3.4.2. The point of this abstract nonsense is that, if $\alpha : \mathsf{C} \to \mathsf{D}$ is a $\otimes$-functor and $\mathcal{B}$ is a ring object in C, then $\alpha(\mathcal{B})$ is a ring object in D and α extends to a $\otimes$-functor $\alpha(\mathcal{B}) : \mathsf{C}(\mathcal{B}) \to \mathsf{D}(\alpha(\mathcal{B}))$ which we call the skalar extension of α. Similarly, if η is a $\otimes$-automorphism of α such that $\eta_{\mathcal{B}}$ is the identity of $\alpha(\mathcal{B})$, then η extends to a $\otimes$-automorphism $\eta(\mathcal{B})$ of $\alpha(\mathcal{B})$ which we call the skalar extension of η.

PROPOSITION 42. (1) *Let $f : \widetilde{S} \to S$ be a finite étale morphism and denote by $\tilde{\omega}$ the fiber functors for $\tilde{G} = G_{\tilde{S}}$. Then we have isomorphisms of fpqc sheaves on $\tilde{S}$:*

$$\mathrm{Aut}^\otimes(\omega^\circ)|_{\widetilde{S}} = \mathrm{Aut}^\otimes(\tilde{\omega}^\circ), \quad \mathbb{G}^\Gamma(\omega^\circ)|_{\widetilde{S}} = \mathbb{G}^\Gamma(\tilde{\omega}^\circ) \quad \textit{and} \quad \mathbb{F}^\Gamma(\omega^\circ)|_{\widetilde{S}} = \mathbb{F}^\Gamma(\tilde{\omega}^\circ).$$

(2) *If f is merely affine, then $\mathbb{F}^\Gamma(\omega)|_{\tilde{S}} = \mathbb{F}^\Gamma(\tilde{\omega})$.*

PROOF. (1) Let T be an $\tilde{S}$-scheme. We have to define mutually inverse maps

$$\alpha : \begin{array}{ccc} \mathrm{Aut}^{\otimes}(\tilde{\omega}^{\circ})(T) & \longleftrightarrow & \mathrm{Aut}^{\otimes}(\omega^{\circ})(T) \\ \mathbb{G}^{\Gamma}(\tilde{\omega}^{\circ})(T) & \longleftrightarrow & \mathbb{G}^{\Gamma}(\omega^{\circ})(T) \\ \mathbb{F}^{\Gamma}(\tilde{\omega}^{\circ})(T) & \longleftrightarrow & \mathbb{F}^{\Gamma}(\omega^{\circ})(T) \end{array} : \beta$$

functorial in T. The α maps are induced by precomposition with the base change map $\mathsf{Rep}^{\circ}(G)(S) \to \mathsf{Rep}^{\circ}(G)(\widetilde{S})$. The β maps are defined by composing the skalar extension maps with the base change maps for the $\tilde{S}$-section $\iota : T \to \tilde{T}$ of the projection $f_T : \tilde{T} = T \times_S \tilde{S} \to T$ given by the structural morphism $T \to \tilde{S}$:

$$\beta : \begin{array}{ccccc} \mathrm{Aut}^{\otimes}(\omega^{\circ})(T) & \longrightarrow & \mathrm{Aut}^{\otimes}(\tilde{\omega}^{\circ})(\tilde{T}) & \longrightarrow & \mathrm{Aut}^{\otimes}(\tilde{\omega}^{\circ})(T) \\ \mathbb{G}^{\Gamma}(\omega^{\circ})(T) & \longrightarrow & \mathbb{G}^{\Gamma}(\tilde{\omega}^{\circ})(\tilde{T}) & \longrightarrow & \mathbb{G}^{\Gamma}(\tilde{\omega}^{\circ})(T) \\ \mathbb{F}^{\Gamma}(\omega^{\circ})(T) & \longrightarrow & \mathbb{F}^{\Gamma}(\tilde{\omega}^{\circ})(\tilde{T}) & \longrightarrow & \mathbb{F}^{\Gamma}(\tilde{\omega}^{\circ})(T) \end{array}$$

Explicitly, for η, $\mathcal{G}$ and $\mathcal{F}$ in the source sets and $\tilde{\rho} \in \mathsf{Rep}^{\circ}(\tilde{G})(\tilde{S})$, we first view $f_*\tilde{\rho}$ as a $\mathcal{B}$-module in $\mathsf{Rep}^{\circ}(G)(S)$ where $\mathcal{B} = f_*\mathcal{O}_{\tilde{S}}$ with trivial G-action. Then:

- $\eta_{f_*\tilde{\rho}}$ is a $\mathcal{B}_T$-linear isomorphism of $\omega_T^{\circ}(f_*\tilde{\rho}) = (f_T)_*\tilde{\omega}_{\tilde{T}}^{\circ}(\tilde{\rho})$. It thus corresponds to an isomorphism of $\tilde{\omega}_{\tilde{T}}^{\circ}(\tilde{\rho})$ whose pull-back to $\iota^*\tilde{\omega}_{\tilde{T}}^{\circ}(\tilde{\rho}) = \tilde{\omega}_T^{\circ}(\tilde{\rho})$ is an isomorphism $\beta(\eta)_{\tilde{\rho}}$. By construction, there is a commutative diagram

$$\begin{tikzcd} \omega_T^{\circ}(f_*\tilde{\rho}) = \tilde{\omega}_T^{\circ}(f^*f_*\tilde{\rho}) \arrow[r, two heads] \arrow[d, "\eta_{f_*\tilde{\rho}}"'] & \tilde{\omega}_T^{\circ}(\tilde{\rho}) \arrow[d, "\beta(\eta)_{\tilde{\rho}}"] \\ \omega_T^{\circ}(f_*\tilde{\rho}) = \tilde{\omega}_T^{\circ}(f^*f_*\tilde{\rho}) \arrow[r, two heads] & \tilde{\omega}_T^{\circ}(\tilde{\rho}) \end{tikzcd}$$

  where the horizontal map comes from the adjunction morphism

$$f^*f_*\tilde{\rho} \to \tilde{\rho}.$$

- $\mathcal{G}(f_*\tilde{\rho})$ is a $\mathcal{B}_T$-stable Γ-graduation on $(f_T)_*\tilde{\omega}_{\tilde{T}}^{\circ}(\tilde{\rho})$, giving a Γ-graduation on $\tilde{\omega}_{\tilde{T}}^{\circ}(\tilde{\rho})$ whose pull-back is a Γ-graduation $\beta(\mathcal{G})(\tilde{\rho})$ on $\tilde{\omega}_T^{\circ}(\tilde{\rho})$. Thus $\beta(\mathcal{G})_{\gamma}(\tilde{\rho})$ is the image of $\mathcal{G}_{\gamma}(f_*\tilde{\rho})$ under the adjunction $\omega_T^{\circ}(f_*\tilde{\rho}) \twoheadrightarrow \tilde{\omega}_T^{\circ}(\tilde{\rho})$.
- $\mathcal{F}(f_*\tilde{\rho})$ is a $\mathcal{B}_T$-stable Γ-filtration on $(f_T)_*\tilde{\omega}_{\tilde{T}}^{\circ}(\tilde{\rho})$, giving a Γ-filtration on $\tilde{\omega}_{\tilde{T}}^{\circ}(\tilde{\rho})$ whose pull-back is a Γ-filtration $\beta(\mathcal{F})(\tilde{\rho})$ on $\tilde{\omega}_T^{\circ}(\tilde{\rho})$. Thus $\mathcal{B}(\mathcal{F})^{\gamma}(\tilde{\rho})$ is the image of $\mathcal{F}^{\gamma}(f_*(\tilde{\rho}))$ under the adjunction $\omega_T^{\circ}(f_*\tilde{\rho}) \twoheadrightarrow \tilde{\omega}_T^{\circ}(\tilde{\rho})$.

One checks easily that $\alpha \circ \beta = \mathrm{Id}$ and $\beta \circ \alpha = \mathrm{Id}$. The proof of (2) is similar. □

REMARK 43. We have not mentioned $\mathrm{Aut}^{\otimes}(\omega)$ and $\mathbb{G}^{\Gamma}(\omega)$ in part (2) of the proposition, because we will establish a stronger result for them in the next section.

3.5. The regular representation

The single most important representation of G is the regular representation ρ_{reg}. We shall use it to establish the classical:

THEOREM 44. *The above morphisms of fpqc sheaves induce isomorphisms*

$$G \simeq \mathrm{Aut}^{\otimes}(V) \simeq \mathrm{Aut}^{\otimes}(\omega) \quad \textit{and} \quad \mathbb{G}^{\Gamma}(G) \simeq \mathbb{G}^{\Gamma}(V) \simeq \mathbb{G}^{\Gamma}(\omega).$$

3.5.1. The regular representation ρ_{reg} of G on $V(\rho_{\mathrm{reg}}) = \mathcal{A}(G)$ is defined by

$$(g \cdot a)(h) = a(hg)$$

for $T \to S$, $a \in \Gamma(T, \mathcal{A}(G)_T) = \Gamma(G_T, \mathcal{O}_{G_T})$ and $g, h \in G(T)$. The corresponding $\mathcal{A}(G)$-comodule structure morphism is the comultiplication map:

$$\left(V(\rho_{\mathrm{reg}}) \xrightarrow{c_{\mathrm{reg}}} V(\rho_{\mathrm{reg}}) \otimes_{\mathcal{O}_S} \mathcal{A}(G)\right) = \left(\mathcal{A}(G) \xrightarrow{\mu^\natural} \mathcal{A}(G) \otimes_{\mathcal{O}_S} \mathcal{A}(G)\right)$$

The $\mathcal{O}_S$-algebra structure morphisms on $\mathcal{A}(G)$, namely the unit $\mathcal{O}_S \to \mathcal{A}(G)$ and the multiplication $\mathcal{A}(G) \otimes \mathcal{A}(G) \to \mathcal{A}(G)$ correspond to G-equivariant morphisms

$$1_S \to \rho_{\mathrm{reg}} \quad \text{and} \quad \rho_{\mathrm{reg}} \otimes \rho_{\mathrm{reg}} \to \rho_{\mathrm{reg}}.$$

For any $\rho \in \mathsf{Rep}(G)(S)$, we denote by $\rho_0 \in \mathsf{Rep}(G)(S)$ the trivial representation of G on $V(\rho_0) = V(\rho)$. We may then view the $\mathcal{A}(G)$-comodule structure morphism $c_\rho : V(\rho) \to V(\rho) \otimes_{\mathcal{O}_S} \mathcal{A}(G)$ of ρ as a G-equivariant morphism in $\mathsf{Rep}(G)(S)$

$$c_\rho : \rho \to \rho_0 \otimes \rho_{\mathrm{reg}}$$

The underlying morphism of quasi-coherent sheaves on S is a split monomorphism since $(\mathrm{Id} \otimes 1_G^\natural) \circ c_\rho = \mathrm{Id}$ on $V(\rho)$ where $1_G^\natural : \mathcal{A}(G) \to \mathcal{O}_S$ is the counit of $\mathcal{A}(G)$.

3.5.2. It follows that any $\eta \in \mathrm{Aut}^\otimes(\omega_T)$, $\mathcal{G} \in \mathbb{G}^\Gamma(\omega_T)$ or $\mathcal{F} \in \mathbb{F}^\Gamma(\omega_T)$ is uniquely determined by its value η_{reg}, $\mathcal{G}_{\mathrm{reg}}$ or $\mathcal{F}_{\mathrm{reg}}$ on ρ_{reg}. Indeed for any $\rho \in \mathsf{Rep}(G)(S)$, η_ρ, $\mathcal{G}(\rho)$ and $\mathcal{F}(\rho)$ will then be the automorphism, Γ-graduation and Γ-filtration on

$$\omega_T(\rho) \xhookrightarrow{\omega_T(c_\rho)} \omega_T(\rho_0) \otimes \omega_T(\rho_{\mathrm{reg}})$$

which are respectively induced by the corresponding objects for $\rho_0 \otimes \rho_{\mathrm{reg}}$, namely

$$\begin{aligned} \eta_{\rho_0 \otimes \rho_{\mathrm{reg}}} &= \mathrm{Id} \otimes \eta_{\mathrm{reg}}, \\ \mathcal{G}(\rho_0 \otimes \rho_{\mathrm{reg}}) &= \omega_T(\rho_0) \otimes \mathcal{G}_{\mathrm{reg}}, \\ \mathcal{F}(\rho_0 \otimes \rho_{\mathrm{reg}}) &= \omega_T(\rho_0) \otimes \mathcal{F}_{\mathrm{reg}}. \end{aligned}$$

We have here used the defining axioms (A1) and (A2) for η, (G1) and (G2) for $\mathcal{G}$ and (F1) and (F2) for $\mathcal{F}$, as well as the fact that for every $\gamma \in \Gamma$, the functors $\mathcal{G}_\gamma$ and $\mathcal{F}^\gamma : \mathsf{Rep}(G)(S) \to \mathsf{QCoh}(T)$ both preserve pure short exact sequences.

3.5.3. By the same token, we find that the morphisms of fpqc sheaves

$$\mathrm{Aut}^\otimes(V) \to \mathrm{Aut}^\otimes(\omega), \quad \mathbb{G}^\Gamma(V) \to \mathbb{G}^\Gamma(\omega) \quad \text{and} \quad \mathbb{F}^\Gamma(V) \to \mathbb{F}^\Gamma(\omega)$$

are monomorphisms. For instance if $\eta \in \mathrm{Aut}^\otimes(V_T)$ induces the identity of ω_T, then for any $f : X \to T$ and $\rho \in \mathsf{Rep}(G)(X)$, η_ρ is the identity of $V(\rho)$ because

$$\eta_{\rho_0 \otimes \rho_{\mathrm{reg},X}} = \eta_{\rho_0} \otimes \eta_{\rho_{\mathrm{reg},X}} = \mathrm{Id}_{V(\rho_0)} \otimes f^*(\eta_{\rho_{\mathrm{reg},T}})$$

and $\eta_{\rho_{\mathrm{reg},T}}$ is the trivial automorphism of $V(\rho_{\mathrm{reg},T}) = \omega_T(\rho_{\mathrm{reg}})$.

3.5.4. We show that $G = \mathrm{Aut}^{\otimes}(\omega)$. Fix an S-scheme T and $\eta \in \mathrm{Aut}^{\otimes}(\omega_T)$. Recall that η_{reg} is the $\mathcal{O}_T$-linear automorphism of $\omega_T(\rho_{\mathrm{reg}}) = \mathcal{A}(G_T)$ induced by η. Since $\eta_{1_S} = \mathrm{Id}_{\mathcal{O}_T}$ on $\omega_T(1_S) = \mathcal{O}_T$ by (A2) and $\eta_{\rho_{\mathrm{reg}}\otimes\rho_{\mathrm{reg}}} = \eta_{\mathrm{reg}} \otimes \eta_{\mathrm{reg}}$ on

$$\omega_T(\rho_{\mathrm{reg}} \otimes \rho_{\mathrm{reg}}) = \mathcal{A}(G_T) \otimes \mathcal{A}(G_T)$$

by (A1), the functoriality of η applied to $1_S \to \rho_{\mathrm{reg}}$ and $\rho_{\mathrm{reg}} \otimes \rho_{\mathrm{reg}} \to \rho_{\mathrm{reg}}$ implies that η_{reg} is an automorphism of the quasi-coherent $\mathcal{O}_T$-algebra $\mathcal{A}(G_T)$. Similarly for any $\rho \in \mathsf{Rep}(G)(S)$, the G-equivariant morphism $c_\rho : \rho \to \rho_0 \otimes \rho_{\mathrm{reg}}$ induces a commutative diagram of quasi-coherent $\mathcal{O}_T$-modules

$$\begin{tikzcd}
\omega_T(\rho) \arrow[r, "(c_\rho)_T"] \arrow[d, "\eta_\rho"'] & \omega_T(\rho_0) \otimes_{\mathcal{O}_T} \mathcal{A}(G_T) \arrow[d, "\mathrm{Id}\otimes\eta_{\mathrm{reg}}"] \\
\omega_T(\rho) \arrow[r, "(c_\rho)_T"] & \omega_T(\rho_0) \otimes_{\mathcal{O}_T} \mathcal{A}(G_T)
\end{tikzcd}$$

Composing η_{reg} with the counit $1^{\natural}_{G,T} : \mathcal{A}(G)_T \to \mathcal{O}_T$, we obtain a morphism of $\mathcal{O}_T$-algebras $s(\eta)^{\natural} : \mathcal{A}(G)_T \to \mathcal{O}_T$, i.e. a T-valued point $s(\eta) \in G(T)$. Now for any $g \in G(T)$ corresponding to $g^{\natural} : \mathcal{A}(G)_T \to \mathcal{O}_T$ and mapping to $\iota(g) \in \mathrm{Aut}^{\otimes}(\omega_T)$, the automorphism $\iota(g)_\rho = \rho_T(g)$ of $\omega_T(\rho)$ is obtained by composing $(c_\rho)_T$ with

$$\mathrm{Id} \otimes g^{\natural} : \omega_T(\rho) \otimes_{\mathcal{O}_T} \mathcal{A}(G)_T \to \omega_T(\rho).$$

We thus find that $s \circ \iota(g) = g$ since

$$s \circ \iota(g)^{\natural} = 1^{\natural}_{G,T} \circ \iota(g)_{\mathrm{reg}} = 1^{\natural}_{G_T} \circ (\mathrm{Id} \otimes g^{\natural}) \circ \mu^{\natural}_T = (1^{\natural}_{G_T} \otimes g^{\natural}) \circ \mu^{\natural}_T = (1_{G_T} \cdot g)^{\natural} = g^{\natural}.$$

On the other hand, $\iota \circ s(\eta) = \eta$ since for any $\rho \in \mathsf{Rep}(G)(S)$,

$$\begin{aligned}
(\iota \circ s)(\eta)_\rho &= \left(\mathrm{Id} \otimes 1^{\natural}_{G,T}\right) \circ (\mathrm{Id} \otimes \eta_{\mathrm{reg}}) \circ (c_\rho)_T \\
&= \left(\mathrm{Id} \otimes 1^{\natural}_{G,T}\right) \circ (c_\rho)_T \circ \eta_\rho \\
&= \rho(1_G)^{\natural}_T \circ \eta_\rho = \eta_\rho.
\end{aligned}$$

Thus $G = \mathrm{Aut}^{\otimes}(\omega)$ and by 3.5.3, also $G = \mathrm{Aut}^{\otimes}(V)$.

3.5.5. We show that $\mathbb{G}^{\Gamma}(G) = \mathbb{G}^{\Gamma}(\omega)$. Let T be an S-scheme, $\mathcal{G} \in \mathbb{G}^{\Gamma}(\omega_T)$. Then for any T-scheme X, the Γ-graduation $\mathcal{G}$ on ω_T and the $\otimes$-equivalence

$$\mathsf{Gr}^{\Gamma}\mathsf{QCoh}(T) \simeq \mathsf{Rep}(\mathbb{D}_T(\Gamma))(T)$$

together induce a factorization

$$\omega^1_X : \mathsf{Rep}(G)(S) \xrightarrow{\mathcal{G}'} \mathsf{Rep}(\mathbb{D}_T(\Gamma))(T) \xrightarrow{\omega^2_X} \mathsf{QCoh}(X)$$

of the fiber functor ω^1_X for the group scheme G over S through the fiber functor ω^2_X for the group scheme $\mathbb{D}_T(\Gamma)$ over T. Moreover $\mathcal{G}'$ is a $\otimes$-functor preserving trivial representations by (G1) and (G2). It thus induces a group homomorphism

$$\mathbb{D}_T(\Gamma)(X) \overset{??}{\simeq} \mathrm{Aut}^{\otimes}(\omega^2_X) \to \mathrm{Aut}^{\otimes}(\omega^1_X) \overset{??}{\simeq} G(X).$$

The latter being functorial in X gives a morphism $s(\mathcal{G}) : \mathbb{D}_T(\Gamma) \to G_T$ of group schemes over T, i.e. an element $s(\mathcal{G})$ of $\mathbb{G}^{\Gamma}(G)(T)$. Since $\mathcal{G} \mapsto s(\mathcal{G})$ is itself functorial in T, it gives a morphism of fpqc sheaves $s : \mathbb{G}^{\Gamma}(\omega) \to \mathbb{G}^{\Gamma}(G)$ which is the inverse of $\iota : \mathbb{G}^{\Gamma}(G) \to \mathbb{G}^{\Gamma}(\omega)$. Thus $\mathbb{G}^{\Gamma}(G) = \mathbb{G}^{\Gamma}(\omega)$ and by 3.5.3, also $\mathbb{G}^{\Gamma}(G) = \mathbb{G}^{\Gamma}(V)$.

3.6. Relating $\mathsf{Rep}(G)(S)$ and $\mathsf{Rep}^\circ(G)(S)$

While $\mathsf{Rep}(G)(S)$ already contains the interesting regular representation, it could be that $\mathsf{Rep}^\circ(G)(S)$ contains no representations beyond the trivial ones, in which case $\mathrm{Aut}^\otimes(\omega^\circ)$, $\mathbb{G}^\Gamma(\omega^\circ)$ and $\mathbb{F}^\Gamma(\omega^\circ)$ are the trivial sheaves represented by S. For instance, let S be one of the two curves considered in [**1**, X 6.4], whose enlarged fundamental group equals $\mathbb{Z}$. Let $n \geq 2$ and $A \in GL_n(\mathbb{Z})$ be any matrix with no roots of unity as eigenvalue. Then by [**1**, X 7.1], this determines an n-dimensional torus G over S, and all representations $\rho \in \mathsf{Rep}^\circ(G)(S)$ are trivial because $\mathbb{Z}^n$ contains no finite A-orbit except $\{0\}$.

When S is quasi-compact, we also consider the intermediate full subcategory

$$\mathsf{Rep}^\circ(G)(S) \subset \mathsf{Rep}'(G)(S) \subset \mathsf{Rep}(G)(S)$$

whose objects are the representations ρ for which $\rho = \varinjlim \tau$ where τ runs through the partially ordered set $X(\rho)$ of all subrepresentations of ρ which belong to $\mathsf{Rep}^\circ(G)(S)$. For such ρ's, $V(\rho) = \varinjlim V(\tau)$ is a flat $\mathcal{O}_S$-module and the quasi-compactness of S implies that $X(\rho)$ is a filtered set. This subcategory is stable under tensor product and the $\rho \mapsto \rho_0$ construction, it contains $\mathsf{Rep}^\circ(G)(S)$ as a full subcategory, and for any $\rho_1, \rho_2 \in \mathsf{Rep}'(G)(S)$,

$$\mathrm{Hom}_{\mathsf{Rep}(G)}(\rho_1, \rho_2) = \varprojlim_{\tau_1 \in X(\rho_1)} \varinjlim_{\tau_2 \in X(\rho_2)} \mathrm{Hom}_{\mathsf{Rep}(G)}(\tau_1, \tau_2). \tag{3.6.1}$$

We denote by $\omega'_T : \mathsf{Rep}'(G)(S) \to \mathsf{QCoh}(T)$ the restriction of ω_T to $\mathsf{Rep}'(G)(S)$ and define the fpqc sheaf $\mathrm{Aut}^\otimes(\omega') : (\mathsf{Sch}/S)^\circ \to \mathsf{Group}$ as before, with automorphisms of ω'_T satisfying the axioms (A1) and (A2), thus obtaining a factorization

$$\mathrm{Aut}^\otimes(\omega) \to \mathrm{Aut}^\otimes(\omega') \to \mathrm{Aut}^\otimes(\omega^\circ).$$

On the other hand, it is obvious that $\mathrm{Aut}^\otimes(\omega') = \mathrm{Aut}^\otimes(\omega^\circ)$.

3.6.1. The following assumption implies that $\mathsf{Rep}^\circ(G)(S)$ is sufficiently big:

HYP(ω°) There exists a covering $\{S_i \to S\}$ by finite étale morphisms such that for every i, G_{S_i}/S_i satisfies HYP$'(\omega^\circ)$ where:

HYP$'(\omega^\circ)$ S is quasi-compact and ρ_{reg} belongs to $\mathsf{Rep}'(G)(S)$.

PROPOSITION 45. *If G/S satisfies* HYP(ω°)*, then*

$$G = \mathrm{Aut}^\otimes(\omega^\circ), \quad \mathbb{G}^\Gamma(G) = \mathbb{G}^\Gamma(\omega^\circ) \quad \textit{and} \quad \mathbb{F}^\Gamma(\omega) \subset \mathbb{F}^\Gamma(\omega^\circ).$$

PROOF. These being fpqc sheaves on S, it is sufficient to establish the proposition for their restriction to the S_i's, which by proposition 42 reduces us to the case where S is quasi-compact and ρ_{reg} belongs to $\mathsf{Rep}'(G)(S)$. The proof of theorem 44 then shows that $G = \mathrm{Aut}^\otimes(\omega'_T)$. Thus $G = \mathrm{Aut}^\otimes(\omega^\circ)$. To prove that $\mathbb{G}^\Gamma(G) = \mathbb{G}^\Gamma(\omega^\circ)$, we may test this on quasi-compact schemes, and then the proof of section 3.5.5 carries over to this case. Finally: a Γ-filtration $\mathcal{F}$ on ω_T is uniquely determined by its value on ρ_{reg} by 3.5.3, thus $\mathbb{F}^\Gamma(\omega) \subset \mathbb{F}^\Gamma(\omega^\circ)$ since $\rho_{\mathrm{reg}} \in \mathsf{Rep}'(G)(S)$. □

3.6.2. For the V° variants of these, one needs a weaker assumption:

HYP(V°) Locally on S for the fpqc topology, ρ_{reg} belongs to $\mathsf{Rep}'(G)(S)$.

PROPOSITION 46. *If G/S satisfies* HYP(V°)*, then*

$$G = \mathrm{Aut}^\otimes(V^\circ), \quad \mathbb{G}^\Gamma(G) = \mathbb{G}^\Gamma(V^\circ) \quad \textit{and} \quad \mathbb{F}^\Gamma(V) \subset \mathbb{F}^\Gamma(V^\circ).$$

PROOF. This being local in the fpqc topology on S, we may assume that S is quasi-compact and ρ_{reg} is in $\mathsf{Rep}'(G)(S)$, then G_T/T satisfies $\mathrm{HYP}'(\omega^\circ)$ for every quasi-compact T over S and the proposition easily follows from the previous one. $\square$

3.6.3. It remains to give some cases where our assumptions are met.

DEFINITION 47. A reductive group G over S is called isotrivial if and only if there exists a covering $\{S_i \to S\}$ by finite étale morphisms such that each G_{S_i} is splittable.

For tori, this definition is slightly more general than that given in [**1**, IX 1.1], which requires a single finite étale cover $S' \to S$. If S is quasi-compact or connected, both notions coincide. For arbitrary reductive groups, [**21**, XXIV 4.1] only defines local and semi-local isotriviality. If S is local, these two notions coincide with ours.

PROPOSITION 48. *If S is local and either geometrically unibranch or henselian, then every reductive group G over S is isotrivial.*

PROOF. We may assume that G is a torus by [**21**, XXIV 4.1.5]. We then have to show that the connected components of $R = \underline{\mathrm{Hom}}_S(G, \mathbb{G}_{m,S})$ are open and finite over S by [**20**, X 5.11], and this follows from proposition 3 and lemma 4. The henselian case also follows directly from [**1**, X 4.6] or [**21**, XXIV 1.21]. $\square$

PROPOSITION 49. (1) *If $S = \mathrm{Spec}(A)$ for a Prüfer domain A and $\rho \in \mathsf{Rep}(G)(S)$,*

$$\rho \in \mathsf{Rep}'(G)(S) \iff V(\rho) \text{ is a flat } \mathcal{O}_S\text{-module.}$$

(2) *A split reductive group over a quasi-compact S satisfies* $\mathrm{HYP}'(\omega^\circ)$.
(3) *An isotrivial reductive group over a quasi-compact S satisfies* $\mathrm{HYP}(\omega^\circ)$.
(4) *A reductive group over any S satisfies* $\mathrm{HYP}(V^\circ)$.

PROOF. (1) is exactly [**49**, Corollary 5.10]. For (2), we may assume that G is of constant type [**21**, XXII 2.8], thus isomorphic [**21**, XXIII 5.2] to the base change of a reductive group G_0 over $\mathrm{Spec}(\mathbb{Z})$ [**21**, XXV 1.2] to which (1) now applies. Obviously (2) $\Rightarrow$ (3), and (2) $\Rightarrow$ (4) by [**21**, XXII 2.3]. $\square$

3.6.4. Together with theorem 44, proposition 45 and 46 give many cases where automorphisms or Γ-graduations automatically extend from ω° or V° to ω or V. Assuming that S is quasi-compact, we will now do something similar for Γ-filtrations.

3.6.5. Let $\mathcal{F}$ be a Γ-filtration on ω_T°. For each $\gamma \in \Gamma$, we may extend

$$\mathcal{F}^\gamma : \mathsf{Rep}^\circ(G)(S) \to \mathsf{LF}(T) \quad \text{to} \quad \mathcal{F}^\gamma : \mathsf{Rep}'(G)(S) \to \mathsf{QCoh}(T)$$

by the formula $\mathcal{F}^\gamma(\rho) = \varinjlim \mathcal{F}^\gamma(\tau)$, where τ runs through $X(\rho)$. It defines a functor by (3.6.1), and gives back $\mathcal{F}^\gamma(\rho) = \mathcal{F}^\gamma(\tau)$ when $\rho = \tau$ belongs to $\mathsf{Rep}^\circ(G)(S)$. In general, $\mathcal{F}^\gamma(\rho)$ is a pure quasi-coherent subsheaf of $V(\rho)_T = \varinjlim V(\tau)_T$ since filtered colimits are exact and commute with base change. While $\gamma \to \mathcal{F}^\gamma(\rho)$ is non-increasing, it may not be a Γ-filtration on $V(\rho)_T$ in our sense. However:

LEMMA 50. *We have the following properties:*

(F1) *For every $\rho_1, \rho_2 \in \mathsf{Rep}'(G)(S)$ and $\gamma \in \Gamma$,*

$$\mathcal{F}^\gamma(\rho_1 \otimes \rho_2) = \textstyle\sum_{\gamma_1+\gamma_2=\gamma} \mathcal{F}^{\gamma_1}(\rho_1) \otimes \mathcal{F}^{\gamma_2}(\rho_2).$$

(F2) *For a trivial representation $\rho \in \mathsf{Rep}'(G)(S)$ on $\mathcal{M} \in \mathsf{QCoh}(S)$,*

$$\mathcal{F}^\gamma(\rho) = \mathcal{M} \text{ if } \gamma \leq 0 \quad \text{and} \quad \mathcal{F}^\gamma(\rho) = 0 \text{ if } \gamma > 0.$$

(F3r) *If $\rho \twoheadrightarrow \tau$ is an epimorphism with $\rho \in \mathsf{Rep}'(G)(S)$ and $\tau \in \mathsf{Rep}^\circ(G)(S)$, then $\mathcal{F}^\gamma(\rho) \twoheadrightarrow \mathcal{F}^\gamma(\tau)$ is an epimorphism in $\mathsf{QCoh}(T)$ for every $\gamma \in \Gamma$.*

(F3l) *If ρ_{reg} belongs to $\mathsf{Rep}'(G)(S)$ and $\rho_1 \hookrightarrow \rho_2$ is a pure monomorphism in $\mathsf{Rep}'(G)(S)$, then $\mathcal{F}^\gamma(\rho_1) = \mathcal{F}^\gamma(\rho_2) \cap V_T(\rho_1)$ in $V_T(\rho_2)$ for every $\gamma \in \Gamma$.*

Proof. (F2) is obvious and (F1), (F3r) follow from the eponymous properties of $\mathcal{F}$ on ω_T° because, since S is quasi-compact, $\{\tau_1 \otimes \tau_2 : (\tau_1, \tau_2) \in X(\rho_1) \times X(\rho_2)\}$ and $\{\tau' \in X(\rho) : \tau' \twoheadrightarrow \tau\}$ are respectively cofinal in $X(\rho_1 \otimes \rho_2)$ and $X(\rho)$. For (F3l), we first treat the special case of the pure monomorphism $c_\rho : \rho \hookrightarrow \rho_0 \otimes \rho_{\mathrm{reg}}$ for an arbitrary $\rho \in \mathsf{Rep}'(G)(S)$. Given (F1) and (F2), we have to show that

$$\mathcal{F}^\gamma(\rho) = \ker\left[\omega_T(\rho) \stackrel{\omega_T(c_\rho)}{\longrightarrow} \omega_T(\rho_0) \otimes (\omega_T(\rho_{\mathrm{reg}})/\mathcal{F}^\gamma(\rho_{\mathrm{reg}}))\right].$$

Since both sides are filtered limits over $\tau \in X(\rho)$, we may assume that ρ belongs to $\mathsf{Rep}^\circ(G)(S)$. The right hand side is then the filtered limit of

$$\ker\left[\omega_T(\rho) \stackrel{\omega_T(c_{\rho,\tau})}{\longrightarrow} \omega_T(\rho_0) \otimes (\omega_T(\tau)/\mathcal{F}^\gamma(\tau))\right] = \mathcal{F}^\gamma(\rho, \tau)$$

where τ ranges through the cofinal set X' of $X(\rho_{\mathrm{reg}})$ defined by

$$X' = \left\{\tau : c_\rho \text{ factors as } \rho \stackrel{c_{\rho,\tau}}{\longrightarrow} \rho_0 \otimes \tau \hookrightarrow \rho_0 \otimes \rho_{\mathrm{reg}}\right\}.$$

Note that $\rho_0 \otimes \tau \hookrightarrow \rho_0 \otimes \rho_{\mathrm{reg}}$ since $V(\rho_0)$ is a flat $\mathcal{O}_S$-module. For each τ in X', the cokernel $\sigma_{\rho,\tau}$ of $c_{\rho,\tau} : \rho \hookrightarrow \rho_0 \otimes \tau$ is an object of $\mathsf{Rep}^\circ(G)(S)$: the counit $1_G^\natural : \mathcal{A}(G) \to \mathcal{O}_S$ gives a retraction of $V(c_{\rho,\tau})$, whose kernel is a direct factor of $V(\rho_0 \otimes \tau)$ isomorphic to $V(\sigma_{\rho,\tau})$. Since $\mathcal{F}^\gamma$ is exact on $\mathsf{Rep}^\circ(G)(S)$, it follows that

$$\mathcal{F}^\gamma(\rho) = \ker\left[\omega_T(\rho) \stackrel{\omega_T(c_{\rho,\tau})}{\longrightarrow} \omega_T(\rho_0 \otimes \tau)/\mathcal{F}^\gamma(\rho_0 \otimes \tau)\right] = \mathcal{F}^\gamma(\rho, \tau)$$

for every $\tau \in X'$, which proves our claim. For any morphism $\rho_1 \to \rho_2$ in $\mathsf{Rep}'(G)(S)$ and any $\gamma \in \Gamma$, we now have a commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 & \to & \mathcal{F}^\gamma(\rho_1) & \to & \omega_T(\rho_1) & \to & \omega_T(\rho_{1,0}) \otimes \omega_T(\rho_{\mathrm{reg}})/\mathcal{F}^\gamma(\rho_{\mathrm{reg}}) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \mathcal{F}^\gamma(\rho_2) & \to & \omega_T(\rho_2) & \to & \omega_T(\rho_{2,0}) \otimes \omega_T(\rho_{\mathrm{reg}})/\mathcal{F}^\gamma(\rho_{\mathrm{reg}})
\end{array}$$

If $V(\rho_1) \to V(\rho_2)$ is a pure monomorphism, the vertical maps are monomorphisms, therefore $\mathcal{F}^\gamma(\rho_1) = \mathcal{F}^\gamma(\rho_2) \cap \omega_T(\rho_1)$ in $\omega_T(\rho_2)$: this proves (F3l). □

3.6.6. As before, for every $\rho \in \mathsf{Rep}'(G)(S)$ and $\gamma \in \Gamma$, we define

$$\mathcal{F}_+^\gamma(\rho) = \cup_{\eta > \gamma} \mathcal{F}^\eta(\rho) \quad \text{and} \quad \mathrm{Gr}_{\mathcal{F}}^\gamma(\rho) = \mathcal{F}^\gamma(\rho)/\mathcal{F}_+^\gamma(\rho).$$

Since again filtered limits are exact, we find that

$$\mathcal{F}_+^\gamma(\rho) = \varinjlim \mathcal{F}_+^\gamma(\tau) \quad \text{and} \quad \mathrm{Gr}_{\mathcal{F}}^\gamma(\rho) = \varinjlim \mathrm{Gr}_{\mathcal{F}}^\gamma(\tau)$$

where τ ranges through $X(\rho)$. In particular, the formula

$$\mathrm{Gr}_{\mathcal{F}}^\gamma(\rho_1 \otimes \rho_2) \simeq \oplus_{\gamma_1 + \gamma_2 = \gamma} \mathrm{Gr}_{\mathcal{F}}^{\gamma_1}(\rho_1) \otimes \mathrm{Gr}_{\mathcal{F}}^{\gamma_2}(\rho_2)$$

also holds for ρ_1 and ρ_2 in $\mathsf{Rep}'(G)(S)$. All of the above constructions commute with arbitrary base change on T. Finally if the original Γ-filtration $\mathcal{F}$ on ω_T° already was the restriction of some Γ-filtration $\mathcal{F}'$ on ω_T, the restriction of the latter is equal to the extension of the former on ω'_T since $\mathcal{F}'^{\gamma}$ commutes with arbitrary colimits.

3.6.7. We first use the above device to show that:

PROPOSITION 51. *If G/S satisfies* $\mathrm{HYP}(\omega^\circ)$*, then* $\mathbb{F}^\Gamma(V^\circ) \hookrightarrow \mathbb{F}^\Gamma(\omega^\circ)$.

PROOF. By proposition 42, we may assume: S is quasi-compact and ρ_{reg} is in $\mathsf{Rep}'(G)(S)$. We have to show that for an S-scheme T and $\mathcal{F} \in \mathbb{F}^\Gamma(V_T^\circ)$ with image $\tilde{\mathcal{F}} \in \mathbb{F}^\Gamma(\omega_T^\circ)$, for any $U \to T$, the Γ-filtration $\mathcal{F}_U$ on $\mathsf{Rep}^\circ(G_U)(U) \to \mathsf{LF}(U)$ induced by $\mathcal{F}$ is determined by $\tilde{\mathcal{F}}$. We may assume that T and U are quasi-compact. Then: $\mathcal{F}_U$ is determined by its extension to $\mathsf{Rep}'(G_U)(U) \to \mathsf{QCoh}(U)$, which itself is determined by its value on $\rho_{\mathrm{reg},U} \in \mathsf{Rep}'(G_U)(U)$ thanks to (F1-2) and (F3l) applied to the pure monomorphisms $c_\rho : \rho \to \rho_0 \otimes \rho_{\mathrm{reg},U}$ for all ρ's in $\mathsf{Rep}'(G_U)(U)$. Since U is quasi-compact, $X(\rho_{\mathrm{reg}})_U = \{\tau_U : \tau \in X(\rho_{\mathrm{reg}})\}$ is cofinal in $X(\rho_{\mathrm{reg},U})$, thus $\mathcal{F}_U(\rho_{\mathrm{reg,U}})$ is determined by the restriction of $\mathcal{F}_U$ to $X(\rho_{\mathrm{reg}})_U$. By the axiom (F0) for $\mathcal{F}$, the latter is determined by the values of $\mathcal{F}_T$ on $X(\rho_{\mathrm{reg}})_T$, which are the values of $\tilde{\mathcal{F}}$ on $X(\rho_{\mathrm{reg}})$. Thus $\tilde{\mathcal{F}}$ determines $\mathcal{F}_U$ and $\mathcal{F}$ uniquely. □

3.6.8. Here is another useful assumption: we say that G/S is linear if there exists $\tau \in \mathsf{Rep}^\circ(G)(S)$ inducing a closed immersion $\tau : G \hookrightarrow GL(V(\tau))$. Note that upon replacing τ with $\tau \oplus (\det \tau)^{-1}$, we may then also assume that $\det \tau = 1$.

LEMMA 52. *The affine and flat group G over S is linear in the following cases:*

1. *G is of finite type over a noetherian regular S with* $\dim S \leq 2$.
2. $\mathrm{HYP}(\omega^\circ)$ *holds and S is quasi-compact and quasi-separated.*
3. *G is an isotrivial reductive group over a quasi-compact S.*
4. *G is a reductive group of adjoint type over any S.*

PROOF. (1) is [**20**, VIB 13.2]. For (2), let $f : S' \to S$ be a finite étale cover such that $\mathrm{HYP}'(\omega^\circ)$ holds for $G' = G_{S'}$. Then S' is also quasi-compact and quasi-separated, thus by [**25**, 1.7.9], the finitely generated quasi-coherent $\mathcal{O}_{S'}$-algebra $\mathcal{A}(G')$ is generated by a finitely generated quasi-coherent $\mathcal{O}_{S'}$-submodule $\mathcal{E}$. By assumption $\mathrm{HYP}'(\omega^\circ)$ for G', we may replace $\mathcal{E}$ by a larger $V(\tau')$ for some τ' in $X(f^*\rho_{\mathrm{reg}})$. The proof of [**20**, VIB 13.2] then shows that $\tau' : G' \to GL(V(\tau'))$ is a closed immersion. Put $\tau = f_*\tau'$, so that τ belongs to $\mathsf{Rep}^\circ(G)(S)$. Then $\tau : G \to GL(V(\tau))$ is a closed immersion. Indeed, it is sufficient to show that $f^*\tau : G' \to GL(V(f^*\tau))$ is a closed immersion by [**26**, 2.7.1]. But $f^*\tau = \rho \otimes \tau'$ in $\mathsf{Rep}^\circ(G')(S')$, where $\rho = f^*f_*1_{G'}$ is the trivial representation on $V(\rho) = f^*f_*\mathcal{O}_{S'}$, i.e. $f^*\tau$ is the composition

$$G' \xrightarrow{\rho'} GL(V(\tau')) \xrightarrow{\mathrm{Id}\otimes -} GL(V(\rho) \otimes V(\tau'))$$

of two closed immersions, therefore itself a closed immersion. For (3): it is well-known that the Chevalley groups over $\operatorname{Spec}\mathbb{Z}$ are linear (a complete overkill: use (1)), so are therefore also the split reductive groups over any base by [**21**, XXII 2.8, XXIII 5.2 and XXV 1.2], to which one reduces as in (2). For (4), simply take τ to be the adjoint representation ρ_{ad} of G on its Lie algebra $\mathrm{Lie}(G) = \mathfrak{g} = V(\rho_{\mathrm{ad}})$. □

3.7. The stabilizer of a Γ-filtration, I

3.7.1. Let now G be a reductive group over S and let $\rho_{\mathrm{ad}} \in \mathsf{Rep}^\circ(G)(S)$ be the adjoint representation of G on $V(\rho_{\mathrm{ad}}) = \mathfrak{g} = \mathrm{Lie}(G)$. Let T be an S-scheme.

THEOREM 53. *Let $\mathcal{F}$ be a Γ-filtration on V_T. Then* $\mathrm{Aut}^{\otimes}(\mathcal{F})$ *is a parabolic subgroup $P_{\mathcal{F}}$ of G_T with unipotent radical $U_{\mathcal{F}} \subset \mathrm{Aut}^{\otimes !}(\mathcal{F})$. Moreover,*

$$\mathrm{Lie}(U_{\mathcal{F}}) = \mathcal{F}^0_+(\rho_{\mathrm{ad}}) \quad \textit{and} \quad \mathrm{Lie}(P_{\mathcal{F}}) = \mathcal{F}^0(\rho_{\mathrm{ad}}) \quad \textit{in} \quad V_T(\rho_{\mathrm{ad}}) = \mathfrak{g}_T.$$

REMARK 54. Let $\chi : \mathbb{D}_T(\Gamma) \to G_T$ be a morphism, $\mathcal{G}$ the corresponding Γ-graduation and $\mathcal{F}$ the induced Γ-filtration. Let $P_\chi = U_\chi \rtimes L_\chi$ be the subgroups of G_T defined in proposition 14. Since $\mathrm{Aut}^{\otimes}(\mathcal{F}) = \mathrm{Aut}^{\otimes !}(\mathcal{F}) \rtimes \mathrm{Aut}^{\otimes}(\mathcal{G})$ with $\mathrm{Aut}^{\otimes}(\mathcal{G})$ equal to L_χ and isomorphic to $\mathrm{Aut}^{\otimes}(\mathrm{Gr}^\bullet_{\mathcal{F}})$ (because $\mathcal{G} \simeq \mathrm{Gr}^\bullet_{\mathcal{F}}$), the theorem implies

$$P_\chi = \mathrm{Aut}^{\otimes}(\mathcal{F}), \quad U_\chi = \mathrm{Aut}^{\otimes !}(\mathcal{F}) \quad \text{and} \quad P_\chi/U_\chi \simeq \mathrm{Aut}^{\otimes}(\mathrm{Gr}^\bullet_{\mathcal{F}}).$$

COROLLARY 55. *The quotients* $\mathrm{Fil} : \mathbb{G}^\Gamma(G) \twoheadrightarrow \mathbb{F}^\Gamma(G)$ *of* $\mathbb{G}^\Gamma(G)$ *defined in sections 2.2 and 3.2.9 are canonically isomorphic, and for any $\mathcal{F} \in \mathbb{F}^\Gamma(G)(T)$,*

$$P_{\mathcal{F}} = \mathrm{Aut}^{\otimes}(\iota\mathcal{F}), \quad U_{\mathcal{F}} = \mathrm{Aut}^{\otimes !}(\iota\mathcal{F}) \quad \textit{and} \quad P_{\mathcal{F}}/U_{\mathcal{F}} \simeq \mathrm{Aut}^{\otimes}(\mathrm{Gr}^\bullet_{\iota\mathcal{F}})$$

where $\iota\mathcal{F}$ is the image of $\mathcal{F}$ in $\mathbb{F}^\Gamma(V_T)$.

PROOF. For the first assertion, we have to show that for $\chi_1, \chi_2 : \mathbb{D}_T(\Gamma) \to G_T$,

$$\chi_1 \sim_{\mathrm{Par}} \chi_2 \iff \mathrm{Fil} \circ \iota(\chi_1) = \mathrm{Fil} \circ \iota(\chi_2) \text{ in } \mathbb{F}^\Gamma(V_T).$$

Put $\mathcal{G}_i = \iota(\chi_i)$, $\mathcal{F}_i = \mathrm{Fil}(\mathcal{G}_i)$ and $P_i = \mathrm{Aut}^{\otimes}(\mathcal{F}_i) = P_{\chi_i}$. If $\chi_1 \sim_{\mathrm{Par}} \chi_2$, then $\chi_2 = \mathrm{Int}(p) \circ \chi_1$ for some $p \in P_1(T)$, thus $\mathcal{F}_2 = p\mathcal{F}_1 = \mathcal{F}_1$. If $\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}$, then $P_1 = P_2 = P$ and the canonical isomorphism $\mathcal{G}_1 \simeq \mathrm{Gr}^\bullet_{\mathcal{F}} \simeq \mathcal{G}_2$ gives an element of $\mathrm{Aut}^{\otimes}(V_T)$ preserving $\mathcal{F}$ and mapping $\mathcal{G}_1$ to $\mathcal{G}_2$, i.e. an element $p \in P(T)$ such that $\chi_2 = \mathrm{Int}(p) \circ \chi_1$, thus $\chi_1 \sim_{\mathrm{Par}} \chi_2$. The remaining assertions are local in the fpqc topology on T and thus follow from the above remark. □

3.7.2. For Γ-filtrations on ω_T, we need a technical assumption on G/S:

TA There exists an fpqc cover $\{f_i : S_i \to S\}$ such that (a) each f_i is an affine morphism, and (b) each $G_i = G_{S_i}$ is linear (3.6.8).

This is true for *any* reductive group G over a *separated* S: starting from a Zariski covering of S by affine U_i's, we pick fpqc covers $\{U_{i,j} \to U_i\}$ splitting G_{U_i}, and again cover the $U_{i,j}$'s by affine $U_{i,j,k}$'s. The resulting fpqc cover $\{U_{i,j,k} \to S\}$ satisfies our assumption: $U_{i,j,k} \to U_i$ is affine as a morphism between affine schemes, $U_i \hookrightarrow S$ is affine because S is separated, and $G_{U_{i,j,k}}$ is linear by lemma 52 since it is split.

THEOREM 56. *Assume TA. Let $\mathcal{F}$ be a Γ-filtration on ω_T. Then* $\mathrm{Aut}^{\otimes}(\mathcal{F})$ *is a parabolic subgroup $P_{\mathcal{F}}$ of G_T with unipotent radical $U_{\mathcal{F}} \subset \mathrm{Aut}^{\otimes !}(\mathcal{F})$. Moreover,*

$$\mathrm{Lie}(U_{\mathcal{F}}) = \mathcal{F}^0_+(\rho_{\mathrm{ad}}) \quad \textit{and} \quad \mathrm{Lie}(P_{\mathcal{F}}) = \mathcal{F}^0(\rho_{\mathrm{ad}}) \quad \textit{in} \quad V_T(\rho_{\mathrm{ad}}) = \mathfrak{g}_T.$$

3.7.3. If $\mathcal{F}'$ is a Γ-filtration on V_T and $\mathcal{F}$ is the induced Γ-filtration on ω_T, then $\mathrm{Aut}^{\otimes}(\mathcal{F}') = \mathrm{Aut}^{\otimes}(\mathcal{F})$ as subsheaves of G_T by 3.5.3 and theorem 44. Therefore: (a) theorem 56 holds without the technical assumption for such filtrations on ω_T, and (b) theorem 53, which is local on S, follows from theorem 56 applied to any affine cover of S. We thus only have to consider the case of a Γ-filtration $\mathcal{F}$ on ω_T. The technical assumption will be used only once below, in section 3.7.9.

3.7.4. The adjoint-regular representation ρ_{adj} of G on $V(\rho_{\mathrm{adj}}) = \mathcal{A}(G)$ is given by

$$(g \cdot a)(h) = a(g^{-1}hg)$$

for $T \to S$, $a \in \Gamma(T, \mathcal{A}(G_T))$ and $g, h \in G(T)$. The unit, counit $1_G^\natural$, multiplication, comultiplication $\mu^\natural$ and inversion $\mathrm{inv}^\natural$ of $\mathcal{A}(G)$ define morphisms in $\mathsf{Rep}(G)(S)$:

$$1_S \to \rho_{\mathrm{adj}}, \quad \rho_{\mathrm{adj}} \to 1_S, \quad \rho_{\mathrm{adj}} \otimes \rho_{\mathrm{adj}} \to \rho_{\mathrm{adj}}, \quad \rho_{\mathrm{adj}} \to \rho_{\mathrm{adj}} \otimes \rho_{\mathrm{adj}}, \quad \rho_{\mathrm{adj}} \to \rho_{\mathrm{adj}}.$$

For any ρ in $\mathsf{Rep}(G)(S)$, we may also view c_ρ as a split monomorphism

$$c_\rho : \rho \to \rho \otimes \rho_{\mathrm{adj}} \quad \text{in} \quad \mathsf{Rep}(G)(S).$$

If τ belongs to $\mathsf{Rep}^\circ(G)(S)$, c_τ gives a morphism $\tau^\vee \otimes \tau \to \rho_{\mathrm{adj}}$ which induces a G-equivariant morphism of quasi-coherent $G - \mathcal{O}_S$-algebras

$$\mathrm{Sym}^\bullet(\tau^\vee \otimes \tau) \to \rho_{\mathrm{adj}}$$

whose underlying morphism of quasi-coherent $\mathcal{O}_S$-algebras is given by

$$\mathrm{Sym}^\bullet_{\mathcal{O}_S}(V(\tau)^\vee \otimes V(\tau)) \hookrightarrow \mathrm{Sym}^\bullet_{\mathcal{O}_S}\left(\underline{\mathrm{End}}_{\mathcal{O}_S}(\tau)\right)\left[\tfrac{1}{\det}\right] = \mathcal{A}\left(GL(V(\tau))\right) \xrightarrow{\tau^\natural} \mathcal{A}(G)$$

where $\tau^\natural$ is the morphism attached to $\tau : G \to GL(V(\rho))$. In particular, if the latter is a closed embedding and $\det(\tau) = 1$, then $\mathrm{Sym}^\bullet(\tau^\vee \otimes \tau) \twoheadrightarrow \rho_{\mathrm{adj}}$ is an epimorphism.

3.7.5. Let $\rho^\circ_{\mathrm{adj}}$ be the kernel of $1_G^\natural : \rho_{\mathrm{adj}} \to 1_S$. Thus $\rho_{\mathrm{adj}} = \rho^\circ_{\mathrm{adj}} \oplus 1_S$ and $V(\rho^\circ_{\mathrm{adj}})$ is the augmentation ideal $\mathcal{I}(G)$ of $\mathcal{A}(G)$. For any $n \geq 1$, the multiplication map $\mathcal{I}(G)^{\otimes n+1} \to \mathcal{I}(G)$ defines a morphism $(\rho^\circ_{\mathrm{adj}})^{\otimes n+1} \to \rho^\circ_{\mathrm{adj}}$ in $\mathsf{Rep}(G)(S)$. We denote by $\rho^n \in \mathsf{Rep}^\circ(G)(S)$ its cokernel, a representation of G on $V(\rho^n) = \mathcal{I}(G)/\mathcal{I}(G)^{n+1}$, and by $\rho_n = (\rho^n)^\vee \in \mathsf{Rep}^\circ(G)(S)$ the dual of ρ^n. Thus $\rho_1 = \rho_{\mathrm{ad}}$, the adjoint representation of G on $V(\rho_{\mathrm{ad}}) = \mathfrak{g}$.

3.7.6. Let now $\mathcal{I}(\mathcal{F})$ and $\mathcal{J}(\mathcal{F})$ be the quasi-coherent ideals of $\mathcal{A}(G_T)$ which are respectively generated by the quasi-coherent subsheaves $\mathcal{F}^0_+(\rho^\circ_{\mathrm{adj}})$ and $\mathcal{F}^0(\rho^\circ_{\mathrm{adj}})$ of the augmentation ideal $\mathcal{I}(G_T) = \omega_T(\rho^\circ_{\mathrm{adj}})$ of $\mathcal{A}(G_T)$. Then

$$U_\mathcal{F} \stackrel{\mathrm{def}}{=} \mathrm{Spec}\left(\mathcal{A}(G_T)/\mathcal{J}(\mathcal{F})\right) \hookrightarrow P_\mathcal{F} \stackrel{\mathrm{def}}{=} \mathrm{Spec}\left(\mathcal{A}(G_T)/\mathcal{I}(\mathcal{F})\right)$$

are closed subgroup schemes of G_T, because $\mathcal{J}(\mathcal{F})$ and $\mathcal{I}(\mathcal{F})$ are compatible with the comultiplication $\mu_T^\natural$ and inversion $\mathrm{inv}_T^\natural$ of $\mathcal{A}(G_T)$, since $\mu^\natural : \rho_{\mathrm{adj}} \to \rho_{\mathrm{adj}} \otimes \rho_{\mathrm{adj}}$ and $\mathrm{inv}^\natural : \rho_{\mathrm{adj}} \to \rho_{\mathrm{adj}}$ are morphisms in $\mathsf{Rep}(G)(S)$. It follows from their definition that the formation of $U_\mathcal{F}$ and $P_\mathcal{F}$ commutes with arbitrary base change on T.

3.7.7. Let $N(U_\mathcal{F})$ and $N(P_\mathcal{F})$ be the normalizers of $U_\mathcal{F}$ and $P_\mathcal{F}$ in G_T. Then

$$P_\mathcal{F} \subset \mathrm{Aut}^\otimes(\mathcal{F}) \subset N(U_\mathcal{F}), N(P_\mathcal{F}) \quad \text{and} \quad U_\mathcal{F} \subset \mathrm{Aut}^{\otimes !}(\mathcal{F})$$

as fpqc subsheaves of G_T. We have to check this on sections over an arbitrary T-scheme X, but we may assume that $X = T$. Since $G = \mathrm{Aut}^\otimes(\omega)$ by theorem 44,

$$\mathrm{Aut}^\otimes(\mathcal{F})(T) \quad = \quad \left\{g \in G(T) \,|\, \forall \rho, \gamma : \rho(g)\left(\mathcal{F}^\gamma(\rho)\right) = \mathcal{F}^\gamma(\rho)\right\}.$$

On the other hand, for any ρ in $\mathsf{Rep}(G)(S)$, the morphism $c_\rho : \rho \to \rho \otimes \rho_{\mathrm{adj}}$ gives a morphism $\omega_T(c_\rho) : \omega_T(\rho) \to \omega_T(\rho) \otimes \omega_T(\rho_{\mathrm{adj}})$ in $\mathsf{QCoh}(T)$ mapping $\mathcal{F}^\gamma(\rho)$ into

$$\mathcal{F}^\gamma(\rho \otimes \rho_{\mathrm{adj}}) = \textstyle\sum_{\alpha+\beta=\gamma} \mathcal{F}^\alpha(\rho) \otimes \mathcal{F}^\beta(\rho_{\mathrm{adj}}).$$

(a) For g in $\mathrm{Aut}^{\otimes}(\mathcal{F})(T)$, $\rho^{\circ}_{\mathrm{adj}}(g)$ fixes $\mathcal{F}^0_+(\rho^{\circ}_{\mathrm{adj}}) = \cup_{\gamma>0}\mathcal{F}^{\gamma}(\rho^{\circ}_{\mathrm{adj}})$ as well as the $\mathcal{A}(G_T)$-ideal $\mathcal{I}(\mathcal{F})$ which it spans. It follows that the inner automorphism of G_T defined by g fixes $P_{\mathcal{F}}$. Thus g belongs to $N(P_{\mathcal{F}})(T)$. Similarly, $g \in N(U_{\mathcal{F}})(T)$.

(b) For g in $P_{\mathcal{F}}(T)$, $g^{\natural} : \mathcal{A}(G_T) \to \mathcal{O}_T$ is trivial on $\mathcal{F}^{\beta}(\rho_{\mathrm{adj}})$ for every $\beta > 0$ and thus $\rho(g) = (\mathrm{Id} \otimes g^{\natural}) \circ \omega_T(c_\rho)$ maps $\mathcal{F}^{\gamma}(\rho)$ into $\sum_{\alpha\geq\gamma}\mathcal{F}^{\alpha}(\rho) = \mathcal{F}^{\gamma}(\rho)$. Since g^{-1} also belongs to $P_{\mathcal{F}}(T)$, $\rho(g)$ fixes $\mathcal{F}^{\gamma}(\rho)$. Thus g belongs to $\mathrm{Aut}^{\otimes}(\mathcal{F})(T)$.

(c) For g in $U_{\mathcal{F}}(T)$, $g^{\natural} - 1^{\natural} : \mathcal{A}(G_T) \to \mathcal{O}_T$ is trivial on $\mathcal{F}^0(\rho_{\mathrm{adj}}) = \mathcal{O}_T \oplus \mathcal{F}^0(\rho^{\circ}_{\mathrm{adj}})$, thus $\rho(g) - \rho(1) = \left(\mathrm{Id} \otimes \left(g^{\natural} - 1^{\natural}\right)\right) \circ \omega_T(c_\rho)$ maps $\mathcal{F}^{\gamma}(\rho)$ into $\sum_{\alpha>\gamma}\mathcal{F}^{\alpha}(\rho) = \mathcal{F}^{\gamma}_+(\rho)$. Therefore g belongs to $\mathrm{Aut}^{\otimes !}(\mathcal{F})(T)$.

3.7.8. We will establish below that the neutral components [**20**, VIB 3.1] $U^{\circ}_{\mathcal{F}}$ and $P^{\circ}_{\mathcal{F}}$ of $U_{\mathcal{F}}$ and $P_{\mathcal{F}}$ are smooth over S, using the following criterion:

PROPOSITION 57. *Let G be affine smooth over S, $\mathcal{A} = \mathcal{A}(G)$ and $\mathcal{I} = \mathcal{I}(G)$. Let $H \subset G$ be a closed subgroup defined by a quasi-coherent ideal $\mathcal{J}$ of $\mathcal{A}$ such that*

(1) *$\mathcal{J}$ is finitely generated,*
(2) *$\mathcal{J} \cap \mathcal{I}^2 = \mathcal{I} \cdot \mathcal{J}$ in $\mathcal{A}$, and*
(3) *$\mathcal{I}/\mathcal{J} + \mathcal{I}^2$ is finite locally free on S.*

Then H° is representable by a smooth open subgroup scheme of H.

PROOF. By [**20**, VIB 3.10], we have to show that H is smooth at all points of its unit section. Let thus $x \in H$ be the image of $s \in S$ under $1_H : S \to H$. By [**25**, 1.4.3 and 1.4.5], we already know from (1) that H is locally of finite presentation over S. Thus by [**28**, 17.5.1] and the Jacobian criterion [**25**, 0_{IV} 22.6.4], we have to show that $\mathcal{J}_x/\mathcal{J}_x^2 \otimes_{\mathcal{O}_{H,x}} k \to \Omega^1_{\mathcal{O}_{G,x}/\mathcal{O}_{S,s}} \otimes_{\mathcal{O}_{G,x}} k$ is injective, where k is the common residue field of s and x, and the morphism is induced by the universal derivation $d : \mathcal{O}_{G,x} \to \Omega^1_{\mathcal{O}_{G,x}/\mathcal{O}_{S,s}}$. This map factors through the corresponding map for $\mathcal{I}_x$, namely $\mathcal{I}_x/\mathcal{I}_x^2 \otimes_{\mathcal{O}_{S,s}} k \to \Omega^1_{\mathcal{O}_{G,x}/\mathcal{O}_{S,s}} \otimes_{\mathcal{O}_{G,x}} k$, which is injective (because $\mathcal{O}_{G,x}/\mathcal{I}_x = \mathcal{O}_{S,s}$ is formally smooth over itself!). We thus have to show that

$$\mathcal{J}_x/\mathcal{J}_x^2 \otimes_{\mathcal{O}_{H,x}} k = \mathcal{J}_x/m_x\mathcal{J}_x \to \mathcal{I}_x/m_x\mathcal{I}_x = \mathcal{I}_x/\mathcal{I}_x^2 \otimes_{\mathcal{O}_{S,s}} k$$

is injective, where m_x is the maximal ideal of $\mathcal{O}_{G,x}$. The latter map is base-changed from the morphism $\mathcal{J}_x/\mathcal{J}_x\mathcal{I}_x \to \mathcal{I}_x/\mathcal{I}_x^2$, which itself is the localization at x of the morphism $\mathcal{J}/\mathcal{J}\mathcal{I} \to \mathcal{I}/\mathcal{I}^2$, which is a pure monomorphism by assumption. □

3.7.9. We show that $\mathcal{I}(\mathcal{F})$ and $\mathcal{J}(\mathcal{F})$ are finitely generated, focusing on $\mathcal{I}(\mathcal{F})$ to simplify the exposition. Let $\{S_i \to S\}$ be an fpqc cover as in assumption (TA), $\{f_i : T_i \to T\}$ the corresponding fpqc cover of T, ω_i the fiber functor for $G_i = G_{S_i}$ and $\mathcal{F}_i$ the extension of $\mathcal{F}_{T_i}$ to a Γ-filtration on ω_{i,T_i} – which exists by proposition 42 since f_i is affine. By [**26**, 2.5.2], it is sufficient to show that $f_i^*\mathcal{I}(\mathcal{F})$ is finitely generated. Since f_i is flat, $f_i^*\mathcal{I}(\mathcal{F}) = \mathcal{I}(\mathcal{F}_{T_i})$ and obviously $\mathcal{I}(\mathcal{F}_{T_i}) = \mathcal{I}(\mathcal{F}_i)$. We may thus assume that G is linear over S: there exists $\tau \in \mathsf{Rep}^{\circ}(G)(S)$ inducing a closed embedding $\tau : G \hookrightarrow GL(V(\tau))$ with $\det\tau \equiv 1$, thus also an epimorphism $S^{\bullet}(\tau) = \mathrm{Sym}^{\bullet}(\tau^{\vee} \otimes \tau) \twoheadrightarrow \rho_{\mathrm{adj}}$ of quasi-coherent G-$\mathcal{O}_S$-algebras. By the axiom (F3) for $\mathcal{F}$, $\mathcal{I}(\mathcal{F})$ is the image of the ideal $\mathcal{I}(\tau)$ spanned by $\mathcal{F}^0_+(S^{\bullet}(\tau))$ in $V(S^{\bullet}(\tau))_T$. Using proposition 39, we may assume that there is a splitting $V(\tau^{\vee} \otimes \tau)_T = \oplus_\gamma \mathcal{G}_\gamma$ of $\mathcal{F}$ on $\tau^{\vee} \otimes \tau$. By the axioms (F1) and (F3), it induces a splitting of $\mathcal{F}$ on $S^{\bullet}(\tau)$,

$$V(S^n(\tau))_T = \oplus_\gamma \oplus_{\gamma_1+\cdots+\gamma_n=\gamma} \mathcal{G}_{\gamma_1} \cdots \mathcal{G}_{\gamma_n}.$$

It follows easily that $\mathcal{I}(\tau)$ is spanned by the finite locally free subsheaf $\oplus_{\gamma>0}\mathcal{G}_\gamma$ of $V(S^1(\tau))_T = V(\tau^\vee \otimes \tau)_T$, therefore $\mathcal{I}(\tau)$ and $\mathcal{I}(\mathcal{F})$ are indeed finitely generated.

3.7.10. We show that $\mathcal{I}(\mathcal{F}) \cap \mathcal{I}(G_T)^2 = \mathcal{I}(\mathcal{F}) \cdot \mathcal{I}(G_T)$ – the proof for $\mathcal{J}(\mathcal{F})$ is similar. Plainly, $\mathcal{I}(\mathcal{F}) \cdot \mathcal{I}(G_T) \subset \mathcal{I}(\mathcal{F}) \cap \mathcal{I}(G_T)^2$. For the other inclusion, we may assume that T is affine and work with global sections. Let thus s be a (global) section of $\mathcal{I}(\mathcal{F})$, so that $s = a + b$ with a a section of $\mathcal{F}^0_+(\rho^\circ_{\mathrm{adj}})$ and b a section of

$$\mathcal{I}(G_T) \cdot \mathcal{F}^0_+(\rho^\circ_{\mathrm{adj}}) \subset \mathcal{I}(G_T) \cdot \mathcal{I}(\mathcal{F}) \subset \mathcal{I}(G_T)^2.$$

Then s belongs to $\mathcal{I}(G_T)^2$ if and only a does, i.e. a is a section of $\mathcal{F}^0_+(\rho^\circ_{\mathrm{adj}}) \cap \mathcal{I}(G_T)^2$. The pure short exact sequence and epimorphism of quasi-coherent sheaves on S

$$0 \to \mathcal{I}(G)^2 \to \mathcal{I}(G) \to \mathcal{I}(G)/\mathcal{I}(G)^2 \to 0 \quad \text{and} \quad \mathcal{I}(G)^{\otimes 2} \twoheadrightarrow \mathcal{I}(G)^2$$

correspond to a pure short exact sequence and epimorphism in $\mathsf{Rep}(G)(S)$,

$$0 \to \rho^{\circ(2)}_{\mathrm{adj}} \to \rho^\circ_{\mathrm{adj}} \to \rho^1 \to 0 \quad \text{and} \quad (\rho^\circ_{\mathrm{adj}})^{\otimes 2} \twoheadrightarrow \rho^{\circ(2)}_{\mathrm{adj}}$$

which together give, using the axioms (F1) and (F3) for $\mathcal{F}$,

$$\mathcal{F}^0_+(\rho^\circ_{\mathrm{adj}}) \cap \mathcal{I}(G_T)^2 = \mathcal{F}^0_+(\rho^{\circ(2)}_{\mathrm{adj}}) = \textstyle\sum_{\gamma_1+\gamma_2>0} \mathcal{F}^{\gamma_1}(\rho^\circ_{\mathrm{adj}}) \cdot \mathcal{F}^{\gamma_2}(\rho^\circ_{\mathrm{adj}})$$

which is contained in $\mathcal{I}(\mathcal{F}) \cdot \mathcal{I}(G_T)$, thus $\mathcal{I}(\mathcal{F}) \cap \mathcal{I}(G_T)^2 \subset \mathcal{I}(\mathcal{F}) \cdot \mathcal{I}(G_T)$.

3.7.11. We show that $\mathcal{I}(G_T)/\mathcal{I}(\mathcal{F}) + \mathcal{I}(G_T)^2$ is finite locally free – the proof for $\mathcal{J}(\mathcal{F})$ is similar. By the axiom (F3), $\mathcal{I}(\mathcal{F}) + \mathcal{I}(G_T)^2/\mathcal{I}(G_T)^2$ is the $\mathcal{A}(G_T)$-submodule of $\mathcal{I}(G_T)/\mathcal{I}(G_T)^2 = \omega_T(\rho^1)$ generated by $\mathcal{F}^0_+(\rho^1)$, i.e. this $\mathcal{O}_T$-submodule itself since $\mathcal{A}(G_T)$ acts on $\mathcal{I}(G_T)/\mathcal{I}(G_T)^2$ through $\mathcal{O}_T$. We are thus claiming that $\omega_T(\rho^1)/\mathcal{F}^0_+(\rho^1)$ is finite locally free, which follows from proposition 39.

3.7.12. We have just established that $U^\circ_{\mathcal{F}}$ and $P^\circ_{\mathcal{F}}$ are representable by smooth open subschemes of $U_{\mathcal{F}}$ and $P_{\mathcal{F}}$. They are also finitely presented over T: they are separated over T as compositions of affine morphisms and open immersions, and they are quasi-compact over T by [**20**, VIB 3.9], since $U_{\mathcal{F}}$ and $P_{\mathcal{F}}$ are finitely presented over S, being locally of finite presentation by 3.7.9 and [**25**, 1.4.5], and affine by definition. From 3.7.7, we obtain the following chain of inclusions

$$\begin{array}{ccccc} U^\circ_{\mathcal{F}} & \subset & U_{\mathcal{F}} & \subset & \mathrm{Aut}^{\otimes !}(\mathcal{F}) \\ \cap & & \cap & & \cap \\ P^\circ_{\mathcal{F}} & \subset & P_{\mathcal{F}} & \subset & \mathrm{Aut}^{\otimes}(\mathcal{F}) \end{array} \quad \text{and} \quad \begin{array}{ccccc} \mathrm{Aut}^{\otimes}(\mathcal{F}) & \subset & N(P_{\mathcal{F}}) & \subset & N(P^\circ_{\mathcal{F}}) \\ \| & & & & \\ \mathrm{Aut}^{\otimes}(\mathcal{F}) & \subset & N(U_{\mathcal{F}}) & \subset & N(U^\circ_{\mathcal{F}}) \end{array}$$

The Lie algebras of $U^\circ_{\mathcal{F}} \subset U_{\mathcal{F}}$ and $P^\circ_{\mathcal{F}} \subset P_{\mathcal{F}}$ are respectively given by

$$\begin{array}{rcccl} \mathrm{Lie}(U^\circ_{\mathcal{F}}) & = & \mathrm{Lie}(U_{\mathcal{F}}) & = & \left(\mathcal{I}(G_T)/\mathcal{J}(\mathcal{F}) + \mathcal{I}(G_T)^2\right)^\vee \\ \text{and} \quad \mathrm{Lie}(P^\circ_{\mathcal{F}}) & = & \mathrm{Lie}(P_{\mathcal{F}}) & = & \left(\mathcal{I}(G_T)/\mathcal{I}(\mathcal{F}) + \mathcal{I}(G_T)^2\right)^\vee \end{array}$$

As quasi-coherent $\mathcal{O}_T$-submodules of

$$\mathrm{Lie}(G_T) = \mathfrak{g}_T = \left(\mathcal{I}(G_T)/\mathcal{I}(G_T)^2\right)^\vee$$

they correspond to the $\mathcal{O}_T$-linear forms on $\omega_T(\rho^1) = \mathcal{I}(G_T)/\mathcal{I}(G_T)^2$ vanishing on

$$\mathcal{F}^0(\rho^1) = \mathcal{J}(\mathcal{F}) + \mathcal{I}(G_T)^2/\mathcal{I}(G_T)^2 \quad \text{and} \quad \mathcal{F}^0_+(\rho^1) = \mathcal{I}(\mathcal{F}) + \mathcal{I}(G_T)^2/\mathcal{I}(G_T)^2$$

respectively. We thus find that, as $\mathcal{O}_T$-submodules of $\mathfrak{g}_T = \omega_T(\rho_{\mathrm{ad}}) = \omega_T(\rho_1)$,

$$\mathrm{Lie}(U^\circ_{\mathcal{F}}) = \mathrm{Lie}(U_{\mathcal{F}}) = \mathcal{F}^0_+(\rho_{\mathrm{ad}}) \quad \text{and} \quad \mathrm{Lie}(P^\circ_{\mathcal{F}}) = \mathrm{Lie}(P_{\mathcal{F}}) = \mathcal{F}^0(\rho_{\mathrm{ad}}).$$

3.7.13. We show that $P^\circ_{\mathcal{F}}$ is a parabolic subgroup of G_T with unipotent radical $U^\circ_{\mathcal{F}}$. Since both groups are finitely presented and smooth over T with $P^\circ_{\mathcal{F}} \subset N(U^\circ_{\mathcal{F}})$, we may assume that $T = \mathrm{Spec}(k)$ for some algebraically closed field k by [**21**, XXVI 1.1 and 1.6]. Since then $T \to S$ is affine, we may *also* assume that $S = \mathrm{Spec}(k)$ by part (2) of proposition 42, in which case G is linear by lemma 52. Using the criterion of [**41**, IV 2.4.3.1], we now have to verify that

$$(a)\ \dim U^\circ_{\mathcal{F}} = \dim G/P^\circ_{\mathcal{F}} \quad \text{and} \quad (b)\ U^\circ_{\mathcal{F}} \text{ is unipotent.}$$

The equality of dimensions follows from proposition 58 below since

$$\dim U^\circ_{\mathcal{F}} = \dim_k \mathrm{Lie}(U^\circ_{\mathcal{F}}) = \dim_k \mathcal{F}^0_+(\rho_{\mathrm{ad}}) = \textstyle\sum_{\gamma>0} \dim_k \mathrm{Gr}^\gamma_k(\rho_{\mathrm{ad}})$$

and

$$\dim G/P^\circ_{\mathcal{F}} = \dim_k \mathfrak{g}/\mathcal{F}^0(\rho_{\mathrm{ad}}) = \dim_k \mathcal{F}^0_+(\rho^\vee_{\mathrm{ad}}) = \textstyle\sum_{\gamma>0} \dim_k \mathrm{Gr}^\gamma_k(\rho_{\mathrm{ad}}).$$

For (b), pick a finite dimensional faithful representation τ of G. Then

$$U^\circ_{\mathcal{F}} \subset U_{\mathcal{F}} \subset \mathrm{Aut}^{\otimes !}(\mathcal{F}) \subset U(\mathcal{F}(\tau))$$

where $U(\mathcal{F}(\tau))$ is the unipotent subgroup of $GL(V(\tau))$ defined by the Γ-filtration $\mathcal{F}(\tau)$ on $V(\tau)$. Therefore $U^\circ_{\mathcal{F}}$ is unipotent by [**1**, XVII 2.2.ii].

3.7.14. By [**21**, XXII 5.8.5], $P^\circ_{\mathcal{F}} = N(P^\circ_{\mathcal{F}})$, therefore also

$$P^\circ_{\mathcal{F}} = P_{\mathcal{F}} = \mathrm{Aut}^{\otimes}(\mathcal{F}) = N(P_{\mathcal{F}}) = N(P^\circ_{\mathcal{F}}).$$

On the other hand, the above proof of (b) shows that $U_{\mathcal{F}}$ has unipotent geometric fibers, and then so does its quotient $U_{\mathcal{F}}/U^\circ_{\mathcal{F}}$ by [**1**, XVII 2.2.iii]. Since $U_{\mathcal{F}}/U^\circ_{\mathcal{F}}$ is also normal in the reductive group $P^\circ_{\mathcal{F}}/U^\circ_{\mathcal{F}}$, it must be trivial, thus $U_{\mathcal{F}} = U^\circ_{\mathcal{F}}$ and this finishes the proof of our theorem. Note that we can not say much more about $\mathrm{Aut}^{\otimes !}(\mathcal{F})$ at this point – we do not even know that it is actually representable.

3.8. Grothendieck groups

Let again G be affine and flat over S. Let T be an S-scheme and let $\mathcal{F}$ be a Γ-filtration on ω°_T. Since $\mathcal{F}$ and Gr are exact $\otimes$-functors,

$$\mathrm{Gr}^\bullet_{\mathcal{F}} : \mathsf{Rep}^\circ(G)(S) \xrightarrow{\mathcal{F}} \mathsf{Fil}^\Gamma\mathsf{LF}(T) \xrightarrow{\mathrm{Gr}} \mathsf{Gr}^\Gamma\mathsf{LF}(T)$$

is also an exact $\otimes$-functor. It thus induces a morphism between the Grothendieck ring $K_0(G)$ of $\mathsf{Rep}^\circ(G)(S)$ and the Grothendieck ring of $\mathsf{Gr}^\Gamma\mathsf{LF}(T)$. The rank function on finite locally free sheaves over T defines a morphism from the latter ring to the ring $\mathcal{C}(T, \mathbb{Z}[\Gamma])$ of locally constant functions on T with values in the group ring $\mathbb{Z}[\Gamma]$ of Γ. The Γ-filtration $\mathcal{F}$ on ω°_T thus defines a ring homomorphism

$$\kappa(\mathcal{F}) : K_0(G) \to \mathcal{C}(T, \mathbb{Z}[\Gamma])$$

which maps the class of $\rho \in \mathsf{Rep}^\circ(G)(S)$ in $K_0(G)$ to the function

$$t \mapsto \textstyle\sum_{\gamma\in\Gamma} \dim_{k(t)} \left(\mathrm{Gr}^\gamma_{\mathcal{F}}(\rho) \otimes k(t)\right) \cdot e^\gamma$$

where e^γ is the basis element of $\mathbb{Z}[\Gamma]$ corresponding to γ. We have:

$$\forall z \in K_0(G) : \qquad \kappa(\mathcal{F})(z^\vee) = \kappa(\mathcal{F})(z)^\vee$$

where the involutions $z \mapsto z^\vee$ are induced by the duality $\rho \mapsto \rho^\vee$ on $\mathsf{Rep}^\circ(G)(S)$ and by $\sum x_\lambda e^\lambda \mapsto \sum x_\lambda e^{-\lambda}$ on $\mathbb{Z}[\Gamma]$. When G is smooth over S, we define

$$\begin{array}{rcl} \kappa(G) & = & [\rho_{\mathrm{ad}}] - [\rho_{\mathrm{ad}}^\vee] \in K_0(G) \\ \text{and} \quad \kappa(G, \mathcal{F}) & = & \text{image of } \kappa(G) \text{ in } \mathcal{C}(T, \mathbb{Z}[\Gamma]) \end{array}$$

The formation of $\kappa(G, \mathcal{F})$ is compatible with arbitrary base change on T.

Proposition 58. *If* (1) *G is an isotrivial reductive group over a quasi-compact S, or* (2) *G is a reductive group over S and $\mathcal{F}$ comes from a filtration on ω_T, then*

$$\kappa(G, \mathcal{F}) = 0 \quad \textit{in} \quad \mathcal{C}(T, \mathbb{Z}[\Gamma]).$$

Proof. (1) Let $\{S_i \to S\}$ be a covering of S by finite étale morphisms such that each $G_i = G_{S_i}$ splits. Let $\{T_i \to T\}$ be the corresponding covering of T. By part (1) of proposition 42, $\mathcal{F}_{T_i}$ extends to a Γ-filtration $\mathcal{F}_i$ on $\omega_i^\circ : \mathsf{Rep}^\circ(G_i)(S_i) \to \mathsf{LF}(T_i)$, and obviously $\kappa(G_i, \mathcal{F}_i) = \kappa(G, \mathcal{F}) \circ (T_i \to T)$. We may thus assume that G splits over S, in which case the proposition follows from lemma 59 below. The proof of (2) is similar: let $\{t \to T\}$ be a covering of T by geometric points, thus G_t splits. By part (2) of proposition 42, $\mathcal{F}_t$ extends to a Γ-filtration on $\omega_t^\circ : \mathsf{Rep}(G_t)(t) \to \mathsf{LF}(t)$ which we also denote by $\mathcal{F}_t$, and obviously $\kappa(G_t, \mathcal{F}_t) = \kappa(G, \mathcal{F}) \circ (t \to T)$. □

Lemma 59. *If G is a split reductive group over a quasi-compact S, then*

$$\kappa(G) = 0 \quad \textit{in} \quad K_0(G).$$

Proof. By [**21**, XXII 2.8], there is a decomposition $S = \coprod_{i \in I} S_i$ into open and closed subschemes $S_i \neq \emptyset$ of S such that for each $i \in I$, G_{S_i} is of constant type, thus isomorphic [**21**, XXIII 5.2] to the base change of a split reductive group $G_{0,i}$ over $\mathrm{Spec}(\mathbb{Z})$ [**21**, XXV 1.2]. Since S is quasi-compact, the indexing set I is finite and $K_0(G) \simeq \otimes_{i \in I} K_0(G_{S_i})$ with $\kappa(G) = \sum_{i \in I} \kappa(G)_i$ where $\kappa(G)_i$ is the image of $\kappa(G_{0,i})$ under $K_0(G_{0,i}) \to K_0(G_{S_i}) \to K_0(G)$. We may thus assume that $S = \mathrm{Spec}(A)$ where A a principal ideal domain. By [**44**, Théorème 5], we may even assume that $A = K$ is a field. Let H be a split maximal torus in G, with character group M and Weyl group W. The restriction $\mathsf{Rep}^\circ(G) \to \mathsf{Rep}^\circ(H)$ induces a ring homomorphism $K_0(G) \to K_0(H) \simeq \mathbb{Z}[M]$ which yields an isomorphism from $K_0(G)$ to $\mathbb{Z}[M]^W$ by [**44**, Théorème 4]. Let $R \subset M$ be the set of roots of H in the Lie algebra $\mathfrak{g} = V(\rho_{\mathrm{ad}})$. The weight decomposition of $\rho_{\mathrm{ad}}|H$ is then given by $\mathfrak{g} = \mathfrak{g}_0 \oplus \oplus_{\alpha \in R} \mathfrak{g}_\alpha$ with $\dim_K \mathfrak{g}_\alpha = 1$ for $\alpha \in R$ and $\mathfrak{g}_0 = \mathfrak{h}$ is the Lie algebra of H. Since $R = -R$, we find that $\rho_{\mathrm{ad}}|H \simeq \rho_{\mathrm{ad}}^\vee|H$. Thus indeed $\kappa(G) = 0$ in $K_0(G)$. □

3.9. The stabilizer of a Γ-filtration, II

We have the following variant of theorem 53 and 56. Let G be an isotrivial reductive group over a quasi-compact S.

Theorem 60. *For an S-scheme T and a Γ-filtration $\mathcal{F}$ on V_T° or ω_T°, $\mathrm{Aut}^\otimes(\mathcal{F})$ is a parabolic subgroup $P_\mathcal{F}$ of G_T with unipotent radical $U_\mathcal{F} \subset \mathrm{Aut}^{\otimes !}(\mathcal{F})$. Moreover,*

$$\mathrm{Lie}(U_\mathcal{F}) = \mathcal{F}_+^0(\rho_{\mathrm{ad}}) \quad \textit{and} \quad \mathrm{Lie}(P_\mathcal{F}) = \mathcal{F}^0(\rho_{\mathrm{ad}}) \quad \textit{in} \quad V_T(\rho_{\mathrm{ad}}) = \mathfrak{g}_T.$$

Corollary 61. *For any S-scheme T and $\mathcal{F} \in \mathbb{F}^\Gamma(G)(T)$,*

$$P_\mathcal{F} = \mathrm{Aut}^\otimes(\iota\mathcal{F}), \quad U_\mathcal{F} = \mathrm{Aut}^{\otimes !}(\iota\mathcal{F}) \quad \textit{and} \quad P_\mathcal{F}/U_\mathcal{F} = \mathrm{Aut}^\otimes(\mathrm{Gr}^\bullet_{\iota\mathcal{F}})$$

where $\iota\mathcal{F}$ stands for the image of $\mathcal{F}$ in either $\mathbb{F}^\Gamma(V_T^\circ)$ or $\mathbb{F}^\Gamma(\omega_T^\circ)$.

The proof of the corollary is identical to that of its earlier counterpart.

3.9.1. By propositions 49, 45, 46 and 51, it is sufficient to establish the theorem for a Γ-filtration $\mathcal{F}$ on ω_T°. For any T-scheme X, we have

$$\begin{aligned}\mathrm{Aut}^{\otimes}(\mathcal{F})(X) &= \{g \in G(X) | \forall \tau, \gamma \in \mathsf{Rep}^\circ(G)(S) \times \Gamma : \rho_X(g)\left(\mathcal{F}^\gamma(\tau)_X\right) = \mathcal{F}^\gamma(\tau)_X\}, \\ &= \left\{g \in G(X) | \forall \rho, \gamma \in \mathsf{Rep}'(G)(S) \times \Gamma : \rho_X(g)\left(\mathcal{F}^\gamma(\rho)_X\right) = \mathcal{F}^\gamma(\rho)_X\right\}.\end{aligned}$$

We have to show that the fpqc subsheaf $\mathrm{Aut}^{\otimes}(\mathcal{F}) : (\mathsf{Sch}/T)^0 \to \mathsf{Set}$ of G_T is representable by a parabolic subgroup with the specified Lie algebra: this is a local question in the fpqc topology on T. Let $\{S_i \to S\}$ be a covering of S by finite étale morphisms such that $G_i = G_{S_i}$ is split, let $\{T_i \to T\}$ be the induced covering of T, let ω_i denote the fiber functors for G_i and let $\mathcal{F}_i$ be the unique extension of $\mathcal{F}_{T_i}$ to a Γ-graduation on ω_{i,T_i}°. Going back to its actual definition in the proof of proposition 42, one checks easily that $\mathrm{Aut}^{\otimes}(\mathcal{F})|_{T_i} = \mathrm{Aut}^{\otimes}(\mathcal{F}_{T_i})$ is equal to $\mathrm{Aut}^{\otimes}(\mathcal{F}_i)$ as a subsheaf of $G|_{T_i} = \mathrm{Aut}^{\otimes}(\omega^\circ)|_{T_i} = \mathrm{Aut}^{\otimes}(\omega_i^\circ)|_{T_i}$. We may (and do) therefore assume that G is a split reductive group over a quasi-compact S. By [**21**, XXII 2.8, XXIII 5.2 and XXV 1.2], we then have a finite partition of $S = \coprod S_i$ into open and closed subschemes such that each $G_i = G_{S_i}$ arises from a split group over $\mathrm{Spec}(\mathbb{Z})$, and repeating the above argument with that covering, we may thus also assume that G is the base change of a split reductive group G_0 over $\mathrm{Spec}(\mathbb{Z})$.

3.9.2. In particular, the proof of part (2) of proposition 49 now shows that with ρ_{reg}, also ρ_{adj} and $\rho_{\mathrm{adj}}^\circ$ belong to $\mathsf{Rep}'(G)(S)$, to which we have extended $\mathcal{F}$ in section 3.6.4. We may thus define subschemes $U_{\mathcal{F}}$ and $P_{\mathcal{F}}$ of G_T as in section 3.7.6, and try to follow from there on the subsequent steps of the proof of theorem 53. Of course, we have to check that we are only using our filtration where it is defined, namely on $\mathsf{Rep}'(G)(S)$, and that whenever the axiom (F3) was used, we could have replaced it with the weaker left and right properties (F3l) or (F3r).

3.9.3. In 3.7.9 and 3.7.10, we used the right exactness of $\mathcal{F}$ for (respectively)

$$A : S^\bullet(\tau) = \mathrm{Sym}^\bullet(\tau^\vee \otimes \tau) \twoheadrightarrow \rho_{\mathrm{adj}} \quad \text{and} \quad B : (\rho_{\mathrm{adj}}^\circ)^{\otimes 2} \twoheadrightarrow \rho_{\mathrm{adj}}^{\circ(2)}.$$

To deal with the first one, it would be sufficient to know that there is a cofinal set $\Sigma \in X(\rho_{\mathrm{adj}})$ such that for all $\sigma \in \Sigma$, $A^{-1}(\sigma)$ is still in $\mathsf{Rep}'(G)(S)$: then

$$\begin{aligned}\mathcal{F}^\gamma(\rho_{\mathrm{adj}}) &= \varinjlim \mathcal{F}^\gamma(\sigma) \overset{\mathrm{F3r}}{=} \varinjlim A(\mathcal{F}^\gamma(A^{-1}(\sigma))) \\ &= A(\mathcal{F}^\gamma(\varinjlim A^{-1}(\sigma))) = A(\mathcal{F}^\gamma(S^\bullet(\tau))).\end{aligned}$$

Over a Dedekind domain, we have Wedhorn's criterion: a ρ is in $\mathsf{Rep}'(G)(S)$ if and only $V(\rho)$ is flat, i.e. torsion free: thus over such a domain, $A^{-1}(\sigma)$ still belongs to $\mathsf{Rep}'(G)(S)$ for any $\sigma \in X(\rho_{\mathrm{adj}})$. Applying this to G_0 and choosing τ in 3.7.9 to also be defined over $\mathrm{Spec}(\mathbb{Z})$ settles the case of A, and that of B is similar.

3.9.4. Everything then goes through up to 3.7.13: $U_{\mathcal{F}}^\circ$ and $P_{\mathcal{F}}^\circ$ are smooth subgroups of G_T with the good Lie algebras, etc… In 3.7.13, we may still reduce to the case where $T = \mathrm{Spec}(k)$ for some algebraically closed field k and use the criterion of [**41**, IV 2.4.3.1], but we can not change S to $\mathrm{Spec}(k)$. However, since we have already reduced to the split case, proposition 58 (or lemma 59) deals perfectly well with condition (a), and lemma 52 with condition (b).

3.10. Splitting filtrations

We now come to the main statement of theorem 34. Let thus G be a reductive isotrivial group over a quasi-compact S, let T be an S-scheme and let $\mathcal{F}$ be a Γ-filtration on ω_T°. We will then show that: locally on T for the étale topology, $\mathcal{F}$ has a splitting $\chi : \mathbb{D}_T(\Gamma) \to G_T$.

3.10.1. Let $f : \tilde{S} \to S$ be a finite étale cover splitting G and denote by $\tilde{\mathcal{F}}$ the unique extension of $\mathcal{F}_{\tilde{T}}$ to a Γ-filtration on $\tilde{\omega}_{\tilde{T}}^\circ$ (see proposition 42), where $\tilde{T} = T_{\tilde{S}}$ and $\tilde{\omega}$ is the fiber functor for $\tilde{G} = G_{\tilde{S}}$. If $\chi : \mathbb{D}_{\tilde{T}}(\Gamma) \to G_{\tilde{T}}$ is a splitting of $\tilde{\mathcal{F}}$, it is *a fortiori* a splitting of $\mathcal{F}_{\tilde{T}}$: we may thus assume that G splits over S.

3.10.2. For a positive integer k, there is a cartesian diagram of fpqc sheaves on S,

$$\begin{tikzcd} \mathbb{G}^\Gamma(G) \arrow[r, equal, "\text{Prop. 45}"] \arrow[d, "k_1"'] & \mathbb{G}^\Gamma(\omega^\circ) \arrow[r, "\mathrm{Fil}"] \arrow[d, "k_2"'] & \mathbb{F}^\Gamma(\omega^\circ) \arrow[d, "k_3"'] \\ \mathbb{G}^\Gamma(G) \arrow[r, equal, "\text{Prop. 45}"] & \mathbb{G}^\Gamma(\omega^\circ) \arrow[r, "\mathrm{Fil}"] & \mathbb{F}^\Gamma(\omega^\circ) \end{tikzcd}$$

where the k_i's map χ, $\mathcal{G}$ and $\mathcal{F}$ to respectively $k_1(\chi) = \chi \circ \mathbb{D}_T(k) = \chi^k$,

$$k_2(\mathcal{G})_\gamma(\rho) = \begin{cases} 0 & \text{if } \gamma \notin k\Gamma, \\ \mathcal{G}_\eta(\rho) & \text{if } \gamma = k\eta, \end{cases} \quad \text{and} \quad k_3(\mathcal{F})^\gamma(\rho) = \cup_{k\eta \geq \gamma} \mathcal{F}^\eta(\rho).$$

They are all obviously well-defined monomorphisms, and the image of k_2 is the subsheaf of $\mathbb{G}^\Gamma(\omega^\circ)$ made of those Γ-graduation $\mathcal{G}'$ for which $\mathcal{G}'_\gamma \equiv 0$ for $\gamma \notin k\Gamma$. The diagram is cartesian because if $\mathcal{G}'$ splits $k_3(\mathcal{F})$, then $\mathcal{G}'_\gamma \simeq \mathrm{Gr}^\gamma_{k_3(\mathcal{F})} \equiv 0$ for $\gamma \notin k\Gamma$, thus $\mathcal{G}' = k_2(\mathcal{G})$ for a unique $\mathcal{G}$, which has to also split $\mathcal{F}$ since

$$k_3(\mathcal{F}) = \mathrm{Fil}(\mathcal{G}') = \mathrm{Fil}\,(k_2(\mathcal{G})) = k_3(\mathrm{Fil}(\mathcal{G})).$$

3.10.3. For a central isogeny $f : G \to G'$, there is a commutative diagram

$$\begin{tikzcd} \mathbb{G}^\Gamma(G) \arrow[r, equal, "\text{Prop. 45}"] \arrow[d, "f_1"'] & \mathbb{G}^\Gamma(\omega^\circ) \arrow[r, "\mathrm{Fil}"] \arrow[d, "f_2"'] & \mathbb{F}^\Gamma(\omega^\circ) \arrow[d, "f_3"'] \\ \mathbb{G}^\Gamma(G') \arrow[r, equal, "\text{Prop. 45}"] & \mathbb{G}^\Gamma(\omega'^\circ) \arrow[r, "\mathrm{Fil}"] & \mathbb{F}^\Gamma(\omega'^\circ) \end{tikzcd}$$

where $\omega' = \omega \circ f^*$ denotes the fiber functor for G' and the f_i's map χ, $\mathcal{G}$ and $\mathcal{F}$ to respectively $f_1(\chi) = f \circ \chi$, $f_2(\mathcal{G}) = \mathcal{G} \circ f^*$ and $f_3(\mathcal{F}) = \mathcal{F} \circ f^*$, with

$$f^* : \mathsf{Rep}(G')(S) \to \mathsf{Rep}(G)(S) \qquad f^*(\rho) = \rho \circ f.$$

We claim that (1) all f_i's are monomorphisms, and (2) the diagram is cartesian. This is local in the finite étale topology on S by proposition 42, and we may thus assume that the kernel C of f is isomorphic to $\mathbb{D}_S(X)$ for some finite commutative group X. We fix an S-scheme T and consider sections of the above sheaves over T. If $f \circ \chi_1 = f \circ \chi_2$, then $\chi_1^{-1}\chi_2$ is a morphism $\mathbb{D}_T(\Gamma) \to C_T$, which has to be trivial since X is finite and Γ torsion free: f_1 is injective. Any $\rho \in \mathsf{Rep}^\circ(G)(S)$ has a finite sum decomposition $\rho = \oplus \rho(x)$ according to the characters $x \in X$ of C, and C acts trivially on $\rho(x)^{\otimes k(x)}$ where $k(x) \geq 1$ is the order of x in X. If two Γ-filtrations $\mathcal{F}_1$ and $\mathcal{F}_2$ on ω_T° induce the same Γ-filtration on ω'°_T, then $\mathcal{F}_1(\rho) = \mathcal{F}_2(\rho)$ for every ρ on which C acts trivially, thus $\mathcal{F}_1(\rho(x)) = \mathcal{F}_2(\rho(x))$ for every ρ and x by lemma 62 below, therefore $\mathcal{F}_1(\rho) = \mathcal{F}_2(\rho)$ since $\rho = \oplus\rho(x)$: f_3 is injective. Similarly: f_2 is injective. Finally, suppose that $\mathcal{G}'$ splits $f_3(\mathcal{F})$. Let $\chi' : \mathbb{D}_T(\Gamma) \to G'_T$ be the corresponding morphism. Fix $k \geq 1$ such that $k_1(\chi')$ lifts to $\chi_k : \mathbb{D}_T(\Gamma) \to G_T$, giving a Γ-graduation $\mathcal{G}_k$ and a Γ-filtration $\mathcal{F}_k$ on ω_T°. They respectively map to

$$f_2(\mathcal{G}_k) = f_2 \circ \iota(\chi_k) = \iota \circ f_1(\chi_k) = \iota \circ k_1(\chi') = k_2 \circ \iota(\chi') = k_2(\mathcal{G}')$$

where ι is the isomorphism $\mathbb{G}^\Gamma(G) \simeq \mathbb{G}^\Gamma(\omega^\circ)$, and

$$f_3(\mathcal{F}_k) = f_3 \circ \mathrm{Fil}(\mathcal{G}_k) = \mathrm{Fil} \circ f_2(\mathcal{G}_k) = \mathrm{Fil} \circ k_2(\mathcal{G}') = k_3 \circ \mathrm{Fil}(\mathcal{G}') = k_3 \circ f_3(\mathcal{F}).$$

Thus $f_3(\mathcal{F}_k) = f_3 \circ k_3(\mathcal{F})$ and $\mathcal{F}_k = k_3(\mathcal{F})$ since f_3 is a monomorphism. Since $\mathcal{G}_k$ splits $k_3(\mathcal{F})$, there is a unique $\mathcal{G}$ such that $\mathcal{F} = \mathrm{Fil}(\mathcal{G})$ and $k_2(\mathcal{G}) = \mathcal{G}_k$ by the cartesian diagram of the previous subsection. Moreover $f_2(\mathcal{G}) = \mathcal{G}'$ since

$$k_2 \circ f_2(\mathcal{G}) = f_2 \circ k_2(\mathcal{G}) = f_2(\mathcal{G}_k) = k_2(\mathcal{G}')$$

and k_2 is a monomorphism: our diagram is indeed cartesian.

LEMMA 62. *Let $\mathcal{M}$ be a finite locally free sheaf on a scheme S, $k \geq 1$.*

(1) *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be local direct factors of $\mathcal{M}$. Then:*

$$\mathcal{F}_1^{\otimes k} = \mathcal{F}_2^{\otimes k} \text{ in } \mathcal{M}^{\otimes k} \implies \mathcal{F}_1 = \mathcal{F}_2 \text{ in } \mathcal{M}.$$

(2) *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be Γ-filtrations on $\mathcal{M}$. Then:*

$$\mathcal{F}_1^{\otimes k} = \mathcal{F}_2^{\otimes k} \text{ on } \mathcal{M}^{\otimes k} \implies \mathcal{F}_1 = \mathcal{F}_2 \text{ on } \mathcal{M}.$$

PROOF. (1) Fix $s \in S$ with residue field $k(s)$. We have to show that $\mathcal{F}_1 = \mathcal{F}_2$ in a neighborhood of s. Shrinking S if necessary, we may assume that $\mathcal{F}_1$ and $\mathcal{F}_2$ are free of constant rank n_1 and n_2. By assumption, $n_1^k = n_2^k$, therefore $n_1 = n_2 = n$. If $n = 0$, $\mathcal{F}_1 = 0 = \mathcal{F}_2$ and we are done. Suppose $n > 0$, and choose a linear form $f : \mathcal{M}(s) \to k(s)$ which is non-zero on $\mathcal{F}_1(s)$ and $\mathcal{F}_2(s)$. Shrinking S further, we may lift f to an $\mathcal{O}_S$-linear map $f : \mathcal{M} \to \mathcal{O}_S$ such that $f(\mathcal{F}_1) = \mathcal{O}_S = f(\mathcal{F}_2)$. Then for the $\mathcal{O}_S$-linear map $F = \mathrm{Id} \otimes f^{k-1} : \mathcal{M}^{\otimes k} \to \mathcal{M}$, we have

$$\mathcal{F}_1 = F(\mathcal{F}_1^{\otimes k}) = F(\mathcal{F}_2^{\otimes k}) = \mathcal{F}_2.$$

(2) The question is local for the Zariski topology on S. By proposition 39, we may thus assume that both filtrations split, say

$$\mathcal{F}_1^\gamma = \oplus_{\eta \geq \gamma} \mathcal{G}_1^\eta \quad \text{and} \quad \mathcal{F}_2^\gamma = \oplus_{\eta \geq \gamma} \mathcal{G}_2^\eta$$

with $\mathcal{G}_i^\gamma$ locally free of constant rank n_i^γ for every $i \in \{1,2\}$ and $\gamma \in \Gamma$. We then argue by induction on the constant rank $n = \sum n_1^\gamma = \sum n_2^\gamma$ of $\mathcal{M}$. For $n = 0$, there is nothing to prove. Suppose $n > 0$. By assumption, for every $\gamma \in \Gamma$,

$$\sum_{a_1 + \cdots + a_k = \gamma} \mathcal{F}_1^{a_1} \otimes \cdots \otimes \mathcal{F}_1^{a_k} = \sum_{a_1 + \cdots + a_k = \gamma} \mathcal{F}_2^{a_1} \otimes \cdots \otimes \mathcal{F}_2^{a_k}$$

which means that

$$\bigoplus_{a_1 + \cdots + a_k \geq \gamma} \mathcal{G}_1^{a_1} \otimes \cdots \otimes \mathcal{G}_1^{a_k} = \bigoplus_{a_1 + \cdots + a_k \geq \gamma} \mathcal{G}_2^{a_1} \otimes \cdots \otimes \mathcal{G}_2^{a_k}$$

Let γ_i be the largest element of the (non-empty!) finite set $\{a : \mathcal{G}_i^a \neq 0\}$. Then

$$\oplus_{a_1 + \cdots + a_k \geq \gamma} \mathcal{G}_i^{a_1} \otimes \cdots \otimes \mathcal{G}_i^{a_k} = \begin{cases} 0 & \text{if } \gamma > k\gamma_i, \\ \mathcal{G}_i^{\gamma_i} \otimes \cdots \otimes \mathcal{G}_i^{\gamma_i} & \text{for } \gamma = k\gamma_i, \\ \neq 0 & \text{if } \gamma \leq k\gamma_i. \end{cases}$$

Thus $k\gamma_1 = k\gamma_2$, $\gamma_1 = \gamma_2 = \gamma_0$ and $\mathcal{G}_1^{\gamma_0} \otimes \cdots \otimes \mathcal{G}_1^{\gamma_0} = \mathcal{G}_2^{\gamma_0} \otimes \cdots \otimes \mathcal{G}_2^{\gamma_0}$ in $\mathcal{M}^{\otimes k}$, therefore $\mathcal{F}_1^{\gamma_0} = \mathcal{G}_1^{\gamma_0} = \mathcal{G}_2^{\gamma_0} = \mathcal{F}_2^{\gamma_0} = \mathcal{N}$ in $\mathcal{M}$ by the previous lemma. We conclude by our induction hypothesis applied to the images of $\mathcal{F}_1$ and $\mathcal{F}_2$ in $\mathcal{M}/\mathcal{N}$. □

3.10.4. Suppose that $G = G_1 \times_S G_2$. Let $\mathcal{F}$ be a Γ-filtration on ω_T°. Then $\mathcal{F}$ induces a Γ-filtration $\mathcal{F}_i$ on the fiber functor $\omega_{i,T}^\circ$ for G_i by the formulas:

$$\mathcal{F}_1^\gamma(\rho_1) = \mathcal{F}^\gamma(\rho_1 \boxtimes 1_{G_2}) \quad \text{and} \quad \mathcal{F}_2^\gamma(\rho_2) = \mathcal{F}^\gamma(1_{G_1} \boxtimes \rho_2)$$

We claim that if χ_i splits $\mathcal{F}_i$, then $\chi = (\chi_1, \chi_2)$ splits $\mathcal{F}$. Indeed, we may as above assume that G_1 and G_2 are split, and we extend $\mathcal{F}$ to $\mathsf{Rep}'(G)(S)$. We then have to show that the Γ-filtration $\mathcal{F}'$ associated to χ equals $\mathcal{F}$ on ρ_{reg}. Since

$$\rho_{\mathrm{reg}} = \rho_{1,\mathrm{reg}} \boxtimes \rho_{2,\mathrm{reg}} = \varinjlim \tau_1 \boxtimes \tau_2$$

where $\rho_{i,\mathrm{reg}}$ is the regular representation of G_i and the colimit is over $\tau_i \in X(\rho_{i,\mathrm{reg}})$, it is also sufficient to establish that $\mathcal{F}'$ equals $\mathcal{F}$ on $\rho = \tau_1 \boxtimes \tau_2$, $\tau_i \in \mathsf{Rep}^\circ(G)(S)$. Note that $\rho = \rho_1 \otimes \rho_2$ where $\rho_1 = \tau_1 \boxtimes 1_{G_2}$ and $\rho_2 = 1_{G_1} \boxtimes \tau_2$. We thus find

$$\begin{aligned}
\mathcal{F}^\gamma(\rho) &= \textstyle\sum_{\gamma_1+\gamma_2=\gamma} \mathcal{F}^{\gamma_1}(\rho_1) \otimes \mathcal{F}^{\gamma_2}(\rho_2) \\
&= \textstyle\sum_{\gamma_1+\gamma_2=\gamma} \mathcal{F}_1^{\gamma_1}(\tau_1) \otimes \mathcal{F}_2^{\gamma_2}(\tau_2) \\
&= \oplus_{\gamma_1+\gamma_2 \geq \gamma} \mathcal{G}_1^{\gamma_1}(\tau_1) \otimes \mathcal{G}_2^{\gamma_2}(\tau_2) \\
&= \oplus_{\eta \geq \gamma} \mathcal{G}^\eta(\tau_1 \boxtimes \tau_2) \\
&= \mathcal{F}'^\gamma(\rho)
\end{aligned}$$

where $\mathcal{G}$ and the $\mathcal{G}_i$'s are the Γ-graduations induced by χ and the χ_i's.

3.10.5. Applying 3.10.1, 3.10.3 with the central isogeny from G to the product of its adjoint group and its coradical, and finally 3.10.4, we may assume that G is either a split torus or a split reductive group of adjoint type.

3.10.6. Let thus first $G = \mathbb{D}_S(M)$ for some $M \simeq \mathbb{Z}^d$ and let $\mathcal{F}$ be a Γ-filtration on ω_T° for an S-scheme T, which we may assume to be (absolutely) affine. Let ρ_m be the representation of G on $V(\rho_m) = \mathcal{O}_S$ on which G acts by the character $m \in M$. By proposition 39, there exists a Γ-graduation $\mathcal{O}_T = \oplus_\gamma \mathcal{I}_\gamma(m)$ such that

$$\forall \gamma \in \Gamma : \qquad \mathcal{F}^\gamma(\rho_m) = \oplus_{\eta \geq \gamma} \mathcal{I}_\eta(m).$$

Let $T_\gamma(m)$ be the support of $\mathcal{I}_\gamma(m)$, so that $T = \coprod_\gamma T_\gamma(m)$ and $T_\gamma(m)$ is open and closed in T. For $t \in T$ and $m \in M$, we denote by $f(t)(m)$ the unique element γ in Γ such that t belongs to $T_\gamma(m)$. Thus $\mathcal{F}_t^\gamma(\rho_m) = k(t)$ if $\gamma \leq f(t)(m)$ and 0 otherwise, where $k(t)$ is the residue field at t. Since $\rho_0 = 1_G$, $f(t)(0) = 0$ by the axiom (F2) for $\mathcal{F}$. Since $\rho_{m_1} \otimes \rho_{m_2} = \rho_{m_1+m_2}$, $f(t)(m_1+m_2) = f(t)(m_1) + f(t)(m_2)$ by the axiom (F1) for $\mathcal{F}$. Therefore $f(t) : M \to \Gamma$ is a group homomorphism. Since M is finitely generated, $f : T \to \mathrm{Hom}_{\mathsf{Group}}(M, \Gamma)$ is locally constant, and thus corresponds to a global section $\chi : \mathbb{D}_T(\Gamma) \to G_T$ of the locally constant sheaf (see [**1**, VIII 1.5])

$$\mathrm{Hom}(M,\Gamma)_T = \underline{\mathrm{Hom}}(M_T, \Gamma_T) = \underline{\mathrm{Hom}}(\mathbb{D}_T(\Gamma), \mathbb{D}_T(M)) = \underline{\mathrm{Hom}}(\mathbb{D}_T(\Gamma), G_T).$$

Let $\mathcal{F}'$ be the corresponding Γ-filtration on ω_T. For any morphism $\phi : M \to \Gamma$, let $T(\phi)$ be the open and closed subset of T where $f \equiv \phi$, so that $T = \coprod T(\phi)$ and

$$T(\phi) = \cap_{m \in M} T_{\phi(m)}(m) = \cap_{i=1}^r T_{\phi(m_i)}(m_i)$$

if $\{m_1, \dots, m_r\} \subset M$ spans M. On $T(\phi)$, we find that

$$\mathcal{F}'^{\gamma}_{T(\phi)}(\rho_m) = \left\{ \begin{array}{ll} \mathcal{O}_{T(\phi)} & \text{if } \gamma \leq \phi(m) \\ 0 & \text{if } \gamma > \phi(m) \end{array} \right\} = \mathcal{F}^{\gamma}_{T(\phi)}(\rho_m).$$

Thus $\mathcal{F}'(\rho_m) = \mathcal{F}(\rho_m)$ for every m. Extending $\mathcal{F}$ as in 3.6.4, also $\mathcal{F}'(\rho_{\mathrm{reg}}) = \mathcal{F}(\rho_{\mathrm{reg}})$ since $\rho_{\mathrm{reg}} = \oplus_{m \in M} \rho_m$. Finally $\mathcal{F}'(\rho) = \mathcal{F}(\rho)$ for any ρ by (F3l) applied to c_ρ. Therefore χ is a splitting of $\mathcal{F}$ – it is in fact the unique such splitting.

3.10.7. Suppose finally that G is a split reductive group of adjoint type over S, let T be an S-scheme, and let $\mathcal{F}$ be a Γ-filtration on ω_T°. We have just recalled that $\mathcal{F}$ is uniquely determined by the value of its extension to $\mathsf{Rep}'(G)(S)$ on ρ_{reg}, but we now also have this: there is at most one Γ-filtration $\mathcal{F}'$ on ω_T which equals $\mathcal{F}$ on the adjoint representation ρ_{ad} of G on $V(\rho_{\mathrm{ad}}) = \mathfrak{g} = \mathrm{Lie}(G)$. In particular, any morphism $\chi : \mathbb{D}_T(\Gamma) \to G_T$ inducing $\mathcal{F}$ on ρ_{ad} is a splitting of $\mathcal{F}$. To establish our claim, we consider the G-equivariant epimorphism of quasi-coherent G-$\mathcal{O}_S$-algebras

$$f : \mathrm{Sym}^{\bullet}_{\mathcal{O}_S}(\rho_{\mathrm{ad},0}^{\vee} \otimes \rho_{\mathrm{ad}}) \twoheadrightarrow \rho_{\mathrm{reg}}$$

which is defined as in section 3.7.4, starting from $c_{\mathrm{ad}} : \rho_{\mathrm{ad}} \to \rho_{\mathrm{ad},0} \otimes \rho_{\mathrm{reg}}$ for the closed embedding $\rho_{\mathrm{ad}} : G \to GL(\mathfrak{g})$. If $\mathcal{F}'$ equals $\mathcal{F}$ on ρ_{ad}, they are also equal on $\mathrm{Sym}^{\bullet}\left(\rho_{\mathrm{ad},0}^{\vee} \otimes \rho_{\mathrm{ad}}\right)$ by the axioms (F1-3) for Γ-filtrations on ω_T°, thus also

$$\mathcal{F}'^{\gamma}(\rho_{\mathrm{reg}}) \subset \mathcal{F}^{\gamma}(\rho_{\mathrm{reg}})$$

for every $\gamma \in \Gamma$ by the axiom (F3) for the Γ-filtration $\mathcal{F}'$ on ω_T — it is not yet known to be satisfied by the extension of $\mathcal{F}$ to $\mathsf{Rep}'(G)(S)$, unless we appeal to the arguments of section 3.9.3, which is not necessary: then $\mathcal{F}'^{\gamma}(\rho) \subset \mathcal{F}^{\gamma}(\rho)$ for every ρ in $\mathsf{Rep}^{\circ}(G)(S)$ by (F3l) with c_ρ, therefore also $\mathcal{F}'^{\gamma}_{+}(\rho) \subset \mathcal{F}^{\gamma}_{+}(\rho)$; applying the latter inclusion to $\rho^{\vee}$ and dualizing gives $\mathcal{F}^{\gamma}(\rho) \subset \mathcal{F}'^{\gamma}(\rho)$. Thus $\mathcal{F} = \mathcal{F}'$ on ω_T°.

3.10.8. By theorem 60, $P_{\mathcal{F}} = \mathrm{Aut}^{\otimes}(\mathcal{F})$ is a parabolic subgroup of G_T. Since our problem is local for the étale topology on T, we may assume that T is affine and the pair $(G_T, P_{\mathcal{F}})$ has an épinglage $\mathcal{E} = (H, M, R, \cdots)$ [**21**, XXVI 1.14]. Thus $H = \mathbb{D}_T(M)$ is a trivialized split maximal torus of G_T contained in $P_{\mathcal{F}}$, $R \subset M$ is the set of roots of H in $\mathfrak{g}_T$ and if $\mathfrak{g}_T = \mathfrak{g}_0 \oplus \oplus_{\alpha \in R} \mathfrak{g}_\alpha$ is the corresponding weight decomposition (so that $\mathfrak{g}_0 = \mathrm{Lie}(H)$), then $\mathrm{Lie}(P_{\mathcal{F}}) = \mathfrak{g}_0 \oplus \oplus_{\alpha \in R'} \mathfrak{g}_\alpha$ for some subset R' of R as in [**21**, XXVI 1.4]. The maximal torus $H \subset P_{\mathcal{F}}$ gives rise to a Levi decomposition $P_{\mathcal{F}} = U_{\mathcal{F}} \rtimes L_{\mathcal{F}}$ with $H \subset L_{\mathcal{F}}$, $\mathrm{Lie}(L_{\mathcal{F}}) = \mathfrak{g}_0 \oplus \oplus_{\alpha \in R'_1} \mathfrak{g}_\alpha$ and $\mathrm{Lie}(U_{\mathcal{F}}) = \oplus_{\alpha \in R'_2} \mathfrak{g}_\alpha$ where $R'_1 = \{\alpha \in R' : -\alpha \in R'\}$ and $R'_2 = \{\alpha \in R' : -\alpha \notin R'\}$ [**21**, XXII 5.11.3]. We will then show that $\mathcal{F}$ has a splitting $\chi : \mathbb{D}_T(\Gamma) \to G_T$.

3.10.9. Since $H \subset P_{\mathcal{F}} = \mathrm{Aut}^{\otimes}(\mathcal{F})$, the Γ-filtration $\mathcal{F}$ is stable under H and

$$\forall \gamma \in \Gamma, \rho \in \mathsf{Rep}^{\circ}(G)(S) : \qquad \mathcal{F}^{\gamma}(\rho) = \oplus_{m \in M} \mathcal{F}^{\gamma}_m(\rho)$$

where $\mathcal{F}^{\gamma}_m(\rho)$ is the m-th eigenspace of $\mathcal{F}^{\gamma}(\rho)$, viewed as a representation of H. Since $\mathrm{Lie}(U_{\mathcal{F}}) = \mathcal{F}^{0}_{+}(\rho_{\mathrm{ad}})$ and $\mathrm{Lie}(P_{\mathcal{F}}) = \mathcal{F}^{0}(\rho_{\mathrm{ad}})$ by theorem 60, $\mathcal{F}^{\gamma}_{\alpha}(\rho_{\mathrm{ad}}) = 0$ for ($\gamma > 0$ and $\alpha \notin R'_2$) or ($\gamma = 0$ and $\alpha \notin R' \cup \{0\}$) while $\mathcal{F}^{\gamma}_{\alpha}(\rho_{\mathrm{ad}}) = \mathfrak{g}_\alpha$ when $\gamma \leq 0$ and $\alpha \in R' \cup \{0\}$. This determines $\mathcal{F}^{\gamma}_{\alpha}(\rho_{\mathrm{ad}})$ for $\alpha \in R'_1 \cup \{0\}$:

$$\forall \alpha \in R'_1 \cup \{0\} : \quad \mathcal{F}^{\gamma}_{\alpha}(\rho_{\mathrm{ad}}) = \begin{cases} \mathfrak{g}_\alpha & \text{if } \gamma \leq 0, \\ 0 & \text{if } \gamma > 0. \end{cases}$$

For the remaining α's (those in $\pm R'_2$), $\mathfrak{g}_\alpha$ is free of rank 1. Using lemma 41, we obtain a partition $T = \coprod T(f)$ into non-empty open and closed subschemes $T(f)$ of T indexed by certain functions $f : \pm R'_2 \to \Gamma$ such that, over $T(f)$,

$$\forall \alpha \in \pm R'_2 : \quad \mathcal{F}^\gamma_\alpha(\rho_{\mathrm{ad}}) = \begin{cases} \mathfrak{g}_\alpha & \text{if } \gamma \leq f(\alpha), \\ 0 & \text{if } \gamma > f(\alpha). \end{cases}$$

We extend these functions to $R \cup \{0\}$ by setting $f(R'_1 \cup \{0\}) = 0$. Thus over $T(f)$,

$$\mathcal{F}^\gamma(\rho_{\mathrm{ad}}) = \oplus_{\alpha \in R \cup \{0\} : f(\alpha) \geq \gamma} \mathfrak{g}_\alpha$$

Moreover $f(\alpha) > 0$ (resp. < 0) if and only if $\alpha \in R'_2$ (resp. $-R'_2$).

3.10.10. We will establish below that each of these f's extends to a group homomorphism $f : M \to \Gamma$. The locally constant function $T \to \mathrm{Hom}(M, \Gamma)$ mapping $t \in T(f)$ to f thus defines a morphism $\chi : \mathbb{D}_T(\Gamma) \to \mathbb{D}_T(M) = H \hookrightarrow G_T$. By construction, χ splits $\mathcal{F}$ on ρ_{ad}, therefore χ splits $\mathcal{F}$ everywhere by 3.10.7.

3.10.11. To show that f extends to a group homomorphism $f : M \to \Gamma$, we may assume that $T = T(f) = \mathrm{Spec}(k)$ where k is a field. By the definition of adjoint groups in [**21**, XXII 4.3.3] and using [**21**, XXI 3.5.5], we have to show that

(1) $f(-\alpha) = -f(\alpha)$ for every $\alpha \in R$ and
(2) $f(\alpha + \beta) = f(\alpha) + f(\beta)$ for every $\alpha, \beta \in R$ such that also $\alpha + \beta \in R$.

3.10.12. Since $H \subset P_{\mathcal{F}} = \mathrm{Aut}^{\otimes}(\mathcal{F})$ fixes $\mathcal{F}$, there is a factorization of $\mathrm{Gr}^\bullet_{\mathcal{F}}$:

$$\mathsf{Rep}^\circ(G)(S) \longrightarrow \mathsf{Gr}^\Gamma \mathsf{Rep}^\circ(H)(k) \longrightarrow \mathsf{Gr}^\Gamma \mathsf{LF}(k)$$

where $\mathsf{Gr}^\Gamma \mathsf{Rep}^\circ(H)(k)$ is the abelian $\otimes$-category of Γ-graded objects in $\mathsf{Rep}^\circ(H)(k)$. Both functors are exact $\otimes$-functors, and we thus obtain a factorization of $\kappa(\mathcal{F})$:

$$K_0(G) \xrightarrow{\kappa} K_0(H)[\Gamma] = \mathbb{Z}[M][\Gamma] \longrightarrow \mathbb{Z}[\Gamma]$$

The morphism κ maps the class of $\rho \in \mathsf{Rep}^\circ(G)(S)$ to

$$\kappa[\rho] = \textstyle\sum_{m,\gamma} x^\gamma_m[\rho] \cdot \epsilon^m e^\gamma$$

where $\epsilon^m \in \mathbb{Z}[M]$ and $e^\gamma \in \mathbb{Z}[\Gamma]$ are the basis elements corresponding to $m \in M$ and $\gamma \in \Gamma$ and $x^\gamma_m[\rho]$ is the dimension of the m-th eigenspace of $\mathrm{Gr}^\gamma_{\mathcal{F}}(\rho)$. Thus

$$\kappa[\rho_{\mathrm{ad}}] = \left(\dim_k(\mathfrak{g}_0) \cdot \epsilon^0 + \sum_{\alpha \in R'_1} \epsilon^\alpha \right) \cdot e^0 + \sum_{\alpha \in \pm R'_2} \epsilon^\alpha e^{f(\alpha)}.$$

Since the above functors are compatible with dualities,

$$\begin{aligned} \kappa[\rho^\vee_{\mathrm{ad}}] &= \left(\dim_k(\mathfrak{g}_0) \cdot \epsilon^0 + \sum_{\alpha \in R'_1} \epsilon^{-\alpha} \right) \cdot e^0 + \sum_{\alpha \in \pm R'_2} \epsilon^{-\alpha} e^{-f(\alpha)} \\ &= \left(\dim_k(\mathfrak{g}_0) \cdot \epsilon^0 + \sum_{\alpha \in R'_1} \epsilon^{\alpha} \right) \cdot e^0 + \sum_{\alpha \in \pm R'_2} \epsilon^{\alpha} e^{-f(-\alpha)}. \end{aligned}$$

Since $[\rho_{\mathrm{ad}}] = [\rho^\vee_{\mathrm{ad}}]$ in $K_0(G)$ by lemma 59, $f(-\alpha) = -f(\alpha)$ for every $\alpha \in R$.

3.10.13. We have already defined the dual ρ_n of

$$\rho^n = \mathrm{Coker}((\rho^\circ_{\mathrm{adj}})^{\otimes n+1} \to \rho^\circ_{\mathrm{adj}})$$

in section 3.7.5. These representations act compatibly (as n varies), functorialy (as ρ varies) and G-equivariantly on any representation $\rho \in \mathsf{Rep}(G)(S)$ by

$$\rho_n \otimes \rho \xrightarrow{\mathrm{Id}\otimes c_\rho} \rho_n \otimes \rho \otimes \rho_{\mathrm{adj}} \xrightarrow{\mathrm{Id}\otimes\mathrm{proj}} \rho_n \otimes \rho \otimes \rho^\circ_{\mathrm{adj}} \xrightarrow{\mathrm{Id}\otimes\mathrm{proj}} \rho_n \otimes \rho \otimes \rho^n \xrightarrow{\mathrm{eval}_n} \rho$$

For $n = 1$, we retrieve the usual adjoint G-equivariant action

$$\mathrm{ad}(\rho) : \rho_{\mathrm{ad}} \otimes \rho \to \rho$$

of $\mathfrak{g}$ on $V(\rho)$, which for $\rho = \rho_{\mathrm{ad}}$ is nothing but the usual Lie bracket

$$[-,-] : \rho_{\mathrm{ad}} \otimes \rho_{\mathrm{ad}} \to \rho_{\mathrm{ad}}.$$

We also denote by $[-,-] : \rho_n \otimes \rho_{\mathrm{ad}} \to \rho_{\mathrm{ad}}$ the above actions on ρ_{ad}. Thus

$$\forall \gamma \in \Gamma, \forall \alpha, \beta \in M : \qquad [\mathcal{F}^\gamma_\alpha(\rho_n), \mathfrak{g}_\beta] \subset \mathcal{F}^{\gamma+f(\beta)}_{\alpha+\beta}(\rho_{\mathrm{ad}}).$$

In particular, $[\mathcal{F}^\gamma_\alpha(\rho_n), \mathfrak{g}_\beta] \neq 0$ implies $\alpha+\beta, \beta \in R \cup \{0\}$ and

$$f(\alpha+\beta) \geq \gamma + f(\beta).$$

3.10.14. Suppose that α, β and $\alpha+\beta$ all belong to R, with $\ell(\alpha) \leq \ell(\beta)$ where ℓ is the length. Let q and p be the positive integers (with $2 \leq p+q \leq 4$) such that

$$\{\beta + n\alpha \in R : n \in \mathbb{Z}\} = \{\beta - (p-1)\alpha, \cdots, \beta, \beta+\alpha, \cdots, \beta + q\alpha\}$$

see [**21**, XXI 2.3.5 and 1]. By Chevalley's rule [**21**, XXIII 6.5],

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = p\mathfrak{g}_{\alpha+\beta} \quad \text{and} \quad [\mathfrak{g}_{-\alpha}, \mathfrak{g}_{-\beta}] = p\mathfrak{g}_{-\alpha-\beta}.$$

Thus if $p \neq 0$ in k, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \neq 0$ and $[\mathfrak{g}_{-\alpha}, \mathfrak{g}_{-\beta}] \neq 0$, therefore

$$f(\alpha+\beta) \geq f(\alpha) + f(\beta) \quad \text{and} \quad f(-\alpha-\beta) \geq f(-\alpha) + f(-\beta)$$

which implies (2) by (1), i.e.

$$f(\alpha+\beta) = f(\alpha) + f(\beta).$$

If $q = 1$, Chevalley's rule gives $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha-\beta}] \neq 0$ and $[\mathfrak{g}_{-\alpha}, \mathfrak{g}_{\alpha+\beta}] \neq 0$, thus again $f(\alpha+\beta) = f(\alpha) + f(\beta)$. This leaves a single case: $p = q = 2 = \mathrm{char}(k)$, where the same method already gives $f(\beta) = f(\beta-\alpha) + f(\alpha)$. We will see below that also

$$[\mathcal{F}^{2f(\alpha)}_{2\alpha}(\rho_2), \mathfrak{g}_{\beta-\alpha}] = \mathfrak{g}_{\alpha+\beta} \quad \text{and} \quad [\mathcal{F}^{-2f(\alpha)}_{-2\alpha}(\rho_2), \mathfrak{g}_{\alpha+\beta}] = \mathfrak{g}_{\beta-\alpha}.$$

Therefore $f(\alpha+\beta) = 2f(\alpha) + f(\beta-\alpha)$, thus again $f(\alpha+\beta) = f(\alpha) + f(\beta)$.

3.10.15. The pure short exact sequences of finite locally free sheaves on S

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{Sym}^2_{\mathcal{O}_S}\left(\frac{\mathcal{I}(G)}{\mathcal{I}(G)^2}\right) & \to & \frac{\mathcal{I}(G)}{\mathcal{I}(G)^3} & \to & \frac{\mathcal{I}(G)}{\mathcal{I}(G)^2} & \to & 0 \\ 0 & \to & \ker & \to & \left(\frac{\mathcal{I}(G)}{\mathcal{I}(G)^2}\right)^{\otimes 2} & \to & \mathrm{Sym}^2_{\mathcal{O}_S}\left(\frac{\mathcal{I}(G)}{\mathcal{I}(G)^2}\right) & \to & 0 \end{array}$$

give rise to pure short exact sequences in $\mathsf{Rep}^\circ(G)(S)$ which dualize to

$$\begin{array}{ccccccccc} 0 & \to & \rho_{\mathrm{ad}} & \to & \rho_2 & \to & \Gamma^2(\rho_{\mathrm{ad}}) & \to & 0 \\ 0 & \to & \Gamma^2(\rho_{\mathrm{ad}}) & \to & \rho_{\mathrm{ad}}^{\otimes 2} & \to & \Lambda^2(\rho_{\mathrm{ad}}) & \to & 0 \end{array}$$

where $\Gamma^2(\rho) = \mathrm{Sym}^2(\rho^\vee)^\vee = \ker\left(\rho^{\otimes 2} \to \Lambda^2(\rho)\right)$. Therefore

$$[\rho_2] = [\rho_{\mathrm{ad}}] + [\rho_{\mathrm{ad}}]^2 - [\Lambda^2(\rho_{\mathrm{ad}})] \quad \text{in} \quad K_0(G).$$

Since $\mathfrak{g}_{2\alpha} = 0 = \Lambda^2(\mathfrak{g})_{2\alpha}$, the coefficients of $\epsilon^{2\alpha}$ in $\kappa[\rho_2]$ and $\kappa[\rho_{\mathrm{ad}}^{\otimes 2}] = \kappa[\rho_{\mathrm{ad}}]^2$ are both equal to $e^{2f(\alpha)}$. Thus if $\mathfrak{d} = \oplus \mathfrak{d}_m$ is the weight decomposition of $\mathfrak{d} = \omega_k^\circ(\rho_2)$, then $\mathfrak{d}_{2\alpha}$ is 1-dimensional and contained in $\mathcal{F}^\gamma(\rho_2)$ if and only if $\gamma \leq 2f(\alpha)$. In particular, $\mathcal{F}_{2\alpha}^{2f(\alpha)}(\rho_2) = \mathfrak{d}_{2\alpha}$, and similarly for $-\alpha$. We thus want:

$$[\mathfrak{d}_{2\alpha}, \mathfrak{g}_{\beta-\alpha}] = \mathfrak{g}_{\beta+\alpha} \quad \text{and} \quad [\mathfrak{d}_{-2\alpha}, \mathfrak{g}_{\beta+\alpha}] = \mathfrak{g}_{\beta-\alpha}.$$

3.10.16. This now only involves the split group G_k and its épinglage, all of which descends to $\mathrm{Spec}(\mathbb{Z})$ by [**21**, XXIII 5.1 and XXV 1.2]. We may thus assume that G and $\mathcal{E} = (H, M, R, \cdots)$ are defined over $S = \mathrm{Spec}(\mathbb{Z})$. The épinglage comes along with simple roots $\Delta \subset R$ and, for each $\alpha \in R$, a basis X_α of $\mathfrak{g}_\alpha$, which extends to a Chevalley system $\{X_\alpha : \alpha \in R\}$ by [**21**, XXIII 6.2], giving rise to isomorphisms $u_\alpha(t) = \exp(tX_\alpha)$ from $\mathbb{G}_a = \mathrm{Spec}(\mathbb{Z}[t])$ to the root subgroup U_α of $\alpha \in R$. As a linear form on $\mathcal{I}(G)/I(G)^2$, X_α is the composition of $u_\alpha^\natural : \mathcal{I}(G) \to \mathcal{I}(\mathbb{G}_a)$ with the linear form on $\mathcal{I}(\mathbb{G}_a) = t\mathbb{Z}[t]$ given by the coefficient of t. If instead we take the coefficient of t^2, we obtain a linear form on $\mathcal{I}(G)/\mathcal{I}(G)^3$ which is a basis $X_{2\alpha}$ of $\mathfrak{d}_{2\alpha}$. The action of X_α on the regular representation is given by

$$\mathcal{A}(G) \to \mathcal{A}(G \times \mathbb{G}_a) = \mathcal{A}(G)[t] \to \mathcal{A}(G)$$

where the first map takes f in $\mathcal{A}(G)$ to the function $(g, t) \mapsto f\left(u_\alpha(t) g u_\alpha^{-1}(t)\right)$, and the second takes the coefficient of t (or evaluates $\frac{d}{dt}$ at $t = 0$). The action of $X_{2\alpha}$ is obtained by replacing the second map with the coefficient of t^2, thus $2X_{2\alpha} = X_\alpha^2$ on ρ_{reg}, therefore $2X_{2\alpha} = X_\alpha^2$ on all ρ's. Let us now return to our chain of roots

$$\{\beta - \alpha, \beta, \beta + \alpha, \beta + 2\alpha\} \subset R.$$

By Chevalley's rule [**21**, XXIII 6.5]

$$[X_\alpha, X_{\beta-\alpha}] = \pm X_\beta \quad \text{and} \quad [X_\alpha, X_\beta] = \pm 2X_{\beta+\alpha}.$$

Therefore $[X_{2\alpha}, X_{\beta-\alpha}] = \pm X_{\beta+\alpha}$ since (we are now over $\mathbb{Z}$!)

$$2[X_{2\alpha}, X_{\beta-\alpha}] = [2X_{2\alpha}, X_{\beta-\alpha}] = [X_\alpha, [X_\alpha, X_{\beta-\alpha}]] = \pm 2X_{\beta+\alpha}.$$

Similarly, $[X_{-2\alpha}, X_{\beta+\alpha}] = \pm X_{\beta-\alpha}$, and this completes our proof.

3.11. Consequences

Let G be a reductive group over S.

3.11.1. Proof of theorem 34. The assertions concerning automorphisms and Γ-graduations follow from theorem 44 and propositions 45, 46 and 49. If G is an isotrivial reductive group over a quasi-compact S, we have monomorphisms

$$\begin{tikzcd}
 & & \mathbb{F}^\Gamma(V^\circ) \arrow[dr, hook, "\text{Prop. 51}"] & \\
\mathbb{F}^\Gamma(G) \arrow[r, hook, "\text{Cor. 55}"] & \mathbb{F}^\Gamma(V) \arrow[ur, hook, "\text{Prop. 46}"] \arrow[dr, hook, "\text{??}"'] & & \mathbb{F}^\Gamma(\omega^\circ) \\
 & & \mathbb{F}^\Gamma(\omega) \arrow[ur, hook, "\text{Prop. 45}"'] &
\end{tikzcd}$$

and we have just seen that $\mathbb{G}^\Gamma(G) \to \mathbb{F}^\Gamma(G) \to \mathbb{F}^\Gamma(\omega^\circ)$ is an epimorphism, therefore

$$\mathbb{F}^\Gamma(G) = \mathbb{F}^\Gamma(V) = \mathbb{F}^\Gamma(V^\circ) = \mathbb{F}^\Gamma(\omega) = \mathbb{F}^\Gamma(\omega^\circ)$$

in this case, from which easily follows that also

$$\mathbb{F}^\Gamma(G) = \mathbb{F}^\Gamma(V) = \mathbb{F}^\Gamma(V^\circ)$$

for any reductive group over any S – and this is contained in $\mathbb{F}^\Gamma(\omega)$ by 3.5.3.

3.11.2. Since the S-scheme $\mathbb{G}^\Gamma(G)$ and $\mathbb{F}^\Gamma(G)$ of chapter 2 represent the functors indicated in theorem 34, there is a universal Γ-graduation $\mathcal{G}_{\mathrm{univ}}$ on $V_{\mathbb{G}^\Gamma(G)}$ (inducing universal Γ-graduations on $V^\circ_{\mathbb{G}^\Gamma(G)}$, $\omega_{\mathbb{G}^\Gamma(G)}$ and $\omega^\circ_{\mathbb{G}^\Gamma(G)}$) and a universal Γ-filtration $\mathcal{F}_{\mathrm{univ}}$ on $V_{\mathbb{F}^\Gamma(G)}$ (inducing universal Γ-filtrations on $V^\circ_{\mathbb{F}^\Gamma(G)}$, $\omega_{\mathbb{F}^\Gamma(G)}$ and $\omega^\circ_{\mathbb{F}^\Gamma(G)}$) from which all other Γ-graduations or Γ-filtrations over some base T can be retrieved by pull-back through unique morphisms $T \to \mathbb{G}^\Gamma(G)$ or $T \to \mathbb{F}^\Gamma(G)$ – for the ω or ω° variants, we have to assume that G is isotrivial and S quasi-compact, or that the Γ-graduations or Γ-filtrations (over T) extend to V or V°. The S-scheme $\mathbb{C}^\Gamma(G)$ is a coarse moduli scheme for either Γ-graduations or Γ-filtrations (on the various fiber functors): two such objects (over T) are fpqc locally (on T) isomorphic if and only if the induced morphisms $T \to \mathbb{C}^\Gamma(G)$ are equal.

3.11.3. From this perspective, we may either deduce non-trivial properties of the S-schemes constructed in chapter 2 from easier properties of Γ-graduations and Γ-filtrations, or non-trivial properties of the latter from already established properties of the former. For instance, theorem 34 implies that Γ-filtrations split over affine bases, a strengthening of the splitting results that we have established:

COROLLARY 63. *Suppose that S is affine. Then every Γ-filtration $\mathcal{F}$ on V_S or V_S° splits over S, and so do the Γ-filtrations on ω_S or ω_S° if G is isotrivial.*

PROOF. This follows from [**21**, XXVI 2.2] as in section 2.2.7. □

3.11.4. In the other direction, we obtain the expected functoriality.

COROLLARY 64. *The fundamental sequence of section 2.2.6*

$$\mathbb{G}^\Gamma(G) \xrightarrow{\mathrm{Fil}} \mathbb{F}^\Gamma(G) \xrightarrow{t} \mathbb{C}^\Gamma(G)$$

is covariantly functorial on the fibered category of reductive groups over schemes and covariantly functorial in the totally ordered commutative group Γ.

PROOF. We have to show that for a morphism $\varphi : G_1 \to f^*G_2$ over $f : T_1 \to T_2$ in the former category, there is a canonical commutative diagram of schemes

$$\begin{tikzcd}
\mathbb{G}^\Gamma(G_1) \arrow[r, "\mathrm{Fil}"] \arrow[d, "\varphi"] & \mathbb{F}^\Gamma(G_1) \arrow[r, "t"] \arrow[d, "\varphi"] & \mathbb{C}^\Gamma(G_1) \arrow[r, "\mathrm{struct}"] \arrow[d, "\varphi"] & T_1 \arrow[d, "f"] \\
\mathbb{G}^\Gamma(G_2) \arrow[r, "\mathrm{Fil}"] & \mathbb{F}^\Gamma(G_2) \arrow[r, "t"] & \mathbb{C}^\Gamma(G_2) \arrow[r, "\mathrm{struct}"] & T_2
\end{tikzcd}$$

In the Tannakian point of view, the first two vertical morphisms are induced by pre-composition with the restriction functor $\mathsf{Rep}(f^*G_2) \to \mathsf{Rep}(G_1)$ which maps τ to $\tau \circ \varphi$. For the third one: if T is a T_1-scheme and x is a T-valued point of $\mathbb{C}^\Gamma(G_1)$, it lifts to a Γ-filtration over an fpqc covering $\{T_i \to T\}$ of T, and two such lifts become isomorphic over a common refinement of the corresponding fpqc coverings. The image of these lifts in $\mathbb{F}^\Gamma(G_2)$ thus yield a well-defined morphism $\varphi(x) : T \to \mathbb{C}^\Gamma(G_2)$, and this defines the morphism $\varphi : \mathbb{C}^\Gamma(G_1) \to \mathbb{C}^\Gamma(G_2)$. The proof of the covariance in Γ is similar, using post-composition with the morphisms

$$\begin{tikzcd} \mathsf{Gr}^{\Gamma_1}\mathsf{QCoh} \arrow[r, "\mathrm{Fil}"] \arrow[d, "f"'] & \mathsf{Fil}^{\Gamma_1}\mathsf{QCoh} \arrow[d, "f"'] \\ \mathsf{Gr}^{\Gamma_2}\mathsf{QCoh} \arrow[r, "\mathrm{Fil}"] & \mathsf{Fil}^{\Gamma_2}\mathsf{QCoh} \end{tikzcd}$$

of fpqc stacks induced by $f : (\Gamma_1, +, \leq) \to (\Gamma_2, +, \leq)$, which are given by

$$f(\mathcal{G})_{\gamma_2} = \oplus_{f(\gamma_1)=\gamma_2} \mathcal{G}_{\gamma_1} \quad \text{and} \quad f(\mathcal{F})^{\gamma_2} = \textstyle\sum_{f(\gamma_1)\geq\gamma_2} \mathcal{F}^{\gamma_1}$$

for T over S, $\mathcal{G} \in \mathsf{Gr}^{\Gamma_1}\mathsf{QCoh}(T)$, $\mathcal{F} \in \mathsf{Fil}^{\Gamma_1}\mathsf{QCoh}(T)$ and $\gamma_2 \in \Gamma_2$. □

COROLLARY 65. *If $G = GL(\mathcal{V})$ for some $\mathcal{V} \in \mathsf{LF}(S)$ of rank $r \in \mathbb{N}^\times$, evaluation at the tautological representation τ of G on $\mathcal{V}$ identifies*

$$\mathbb{G}^\Gamma(G) \xrightarrow{\mathrm{Fil}} \mathbb{F}^\Gamma(G) \xrightarrow{t} \mathbb{C}^\Gamma(G)$$

with

$$\mathbb{G}^\Gamma(\mathcal{V}) \xrightarrow{\mathrm{Fil}} \mathbb{F}^\Gamma(\mathcal{V}) \xrightarrow{t} \mathbb{C}^\Gamma(\mathcal{V})$$

where for any S-scheme T,

$$\begin{array}{rcl} \mathbb{G}^\Gamma(\mathcal{V})(T) &=& \{\Gamma - \textit{graduations on } \mathcal{V}_T\} \\ \mathbb{F}^\Gamma(\mathcal{V})(T) &=& \{\Gamma - \textit{filtrations on } \mathcal{V}_T\} \\ \mathbb{C}^\Gamma(\mathcal{V})(T) &=& \left\{\textit{locally constant functions } f : T \to \Gamma^r_{\geq}\right\} \end{array}$$

where $\Gamma^r_{\geq} = \{(\gamma_1 \geq \cdots \geq \gamma_r) \in \Gamma^r\}$ and t sends a Γ-filtration $\mathcal{F}$ on $\mathcal{V}_T$ to the function which maps $x \in T$ to the r-tuple with $\dim_{k(x)} \mathrm{Gr}^\gamma_{\mathcal{F}}(x)$ copies of $\gamma \in \Gamma$.

PROOF. Evaluation at τ gives the morphisms τ_g, τ_f of the diagram

$$\begin{tikzcd} \mathbb{G}^\Gamma(G) \arrow[r, "\mathrm{Fil}"] \arrow[d, "\tau_g"'] & \mathbb{F}^\Gamma(G) \arrow[r, "t"] \arrow[d, "\tau_f"'] & \mathbb{C}^\Gamma(G) \arrow[d, "\tau_c"'] \\ \mathbb{G}^\Gamma(\mathcal{V}) \arrow[r, "\mathrm{Fil}"] & \mathbb{F}^\Gamma(\mathcal{V}) \arrow[r, "t"] & \mathbb{C}^\Gamma(\mathcal{V}) \end{tikzcd}$$

and the remaining morphism τ_c comes along by noting that $t \circ \tau_f$ is G-invariant. Plainly, τ_g is an isomorphism: a morphism $\mathbb{D}_T(\Gamma) \to G_T$ is nothing but a representation of $\mathbb{D}_T(\Gamma)$ on $\mathcal{V}_T$, i.e. a Γ-graduation on $\mathcal{V}_T$. Since every Γ-filtration on $\mathcal{V}_T$ splits locally for the fpqc topology on T by definition (and locally for the Zariski topology by proposition 39), $\mathrm{Fil} : \mathbb{G}^\Gamma(\mathcal{V}) \to \mathbb{F}^\Gamma(\mathcal{V})$ is an epimorphism of fpqc sheaves on Sch/S, and so is therefore also τ_f. If $\mathcal{F}_1, \mathcal{F}_2 \in \mathbb{F}^\Gamma(G)(T)$ induce the same filtration $\mathcal{F}_1(\tau) = \mathcal{F}_2(\tau)$ on $\mathcal{V}_T = V_T(\tau)$, they also have the same image at $\det\tau$ (a quotient of $\tau^{\otimes r}$) and $\tau' = \tau \oplus \det(\tau)^{-1}$. Arguing as in section 3.10.7, we obtain that both filtrations agree on ρ_{reg}, thus actually $\mathcal{F}_1 = \mathcal{F}_2$. It follows that τ_f is also a monomorphism, i.e. it is an isomorphism. One checks easily that $t : \mathbb{F}^\Gamma(\mathcal{V}) \to \mathbb{C}^\Gamma(\mathcal{V})$ is an epimorphism of fpqc sheaves on Sch/S, and so is therefore also $\mathbb{C}^\Gamma(G) \to \mathbb{C}^\Gamma(\mathcal{V})$. If $x, y \in \mathbb{C}^\Gamma(G)(T)$ have the same image in $\mathbb{C}^\Gamma(\mathcal{V})(T)$ and $\mathcal{X}, \mathcal{Y}$ are chosen lifts of x and y to $\mathbb{G}^\Gamma(G)(T')$ for some fpqc cover $T' \to T$, then, locally on T', $\mathcal{X}_\gamma(\tau)$ and $\mathcal{Y}_\gamma(\tau)$ are free with the same rank, thus isomorphic. Gluing these isomorphisms, we obtain a $g \in G(T')$ which maps $\mathcal{X}$ to $\mathcal{Y}$, thus

$x_{T'} = t \circ \mathrm{Fil}(\mathcal{X}) = t \circ \mathrm{Fil}(\mathcal{Y}) = y_{T'}$ in $\mathbb{C}^\Gamma(G)(T')$ by section 2.2.7. But then $x = y$ in $\mathbb{C}^\Gamma(G)(T)$, therefore τ_c is also a monomorphism, i.e. it is an isomorphism. $\square$

REMARK 66. For $G = GL(\mathcal{V})$ as above, the weak and strong dominance orders on $\mathbb{C}^\Gamma(G)$ are equal. They correspond to the following order on $\mathbb{C}^\Gamma(\mathcal{V})$: for an S-scheme T and locally constant functions $f_1, f_2 : T \to \Gamma^r_\geq$, we have

$$f_1 \leq f_2 \quad \text{in} \quad \mathbb{C}^\Gamma(\mathcal{V})(T) \quad \iff \quad \forall t \in T : \quad f_1(t) \leq f_2(t) \quad \text{in} \quad \Gamma^r_\geq$$

for the usual partial dominance order on $\Gamma^r_\geq$, given by

$$\begin{aligned}&(\gamma_1 \geq \cdots \geq \gamma_r) \leq (\gamma'_1 \geq \cdots \geq \gamma'_r)\\ \iff \quad &\begin{cases}\forall 1 \leq i \leq r-1: & \gamma_1 + \cdots + \gamma_i \leq \gamma'_1 + \cdots + \gamma'_i,\\ \text{and} & \gamma_1 + \cdots + \gamma_r = \gamma'_1 + \cdots + \gamma'_r.\end{cases}\end{aligned}$$

For a connected T, we will usually identify $\mathbb{C}^\Gamma(\mathcal{V})(T)$ and $\Gamma^r_\geq$.

3.11.5. We have already mentioned that the monoid structure on $\mathbb{C}^\Gamma(G)$ is not functorial in G. On the other hand, the weak dominance partial order $\leq$ on $\mathbb{C}^\Gamma(G)$ defined in section 2.2.12 is functorial in G.

PROPOSITION 67. *Let $\varphi : G \to H$ be a morphism of reductive group over S, let T be an S-scheme. Then for any $t_1, t_2 \in \mathbb{C}^\Gamma(G)(T)$,*

$$t_1 \leq t_2 \text{ in } \mathbb{C}^\Gamma(G)(T) \quad \Longrightarrow \quad \varphi(t_1) \leq \varphi(t_2) \text{ in } \mathbb{C}^\Gamma(H)(T).$$

In particular for any $\tau : G \to GL(\mathcal{V})$ in $\mathsf{Rep}^\circ(G)(S)$, with $\mathcal{V} = V(\tau) \in \mathsf{LF}(S)$,

$$t_1 \leq t_2 \text{ in } \mathbb{C}^\Gamma(G)(T) \quad \Longrightarrow \quad t_1(\tau) \leq t_2(\tau) \text{ in } \mathbb{C}^\Gamma(\mathcal{V})(T).$$

PROOF. Since $\leq$ is open in $\mathbb{C}^\Gamma(H)$, we may assume that T is a geometric point. The proposition then follows from the stronger proposition 68 below. $\square$

3.11.6. Suppose that G is isotrivial over a connected S. Then G is split by a single finite étale cover $\pi : S' \to S$, and we may assume that S' is connected and Galois over S with Galois group $\Theta = \mathrm{Aut}(S'/S)$. Let $\mathscr{R} = \mathscr{R}(G) = (M, R, M^*, R^*)$ be the constant type of G [**21**, XXII 6.8] with Weyl group $W = W(\mathscr{R})$ and fix a system of positive roots $R_+ \subset R$, giving rise to a based root data $\mathscr{R}_+ = (M, R, M^*, R^*; \Delta)$. Let $G_0 = G^{Ep}_{\mathrm{Spec}(\mathbb{Z})}(\mathscr{R}_+)$ be the corresponding pinned Chevalley group over $\mathrm{Spec}(\mathbb{Z})$ [**21**, XXV 1.2] and pick an isomorphism $\gamma : G_{0,S'} \simeq G_{S'}$. It exists by [**21**, XXIII 1.1] and corresponds to a pinning $\mathscr{E} = (T, \iota : \mathbb{D}_{S'}(M) \xrightarrow{\simeq} T, (X_\alpha)_{\alpha\in\Delta})$ of type $\mathscr{R}_+$ of $G_{S'}$ [**21**, XXIV 1.0] by [**21**, XXIV 1.20]. For any θ in Θ, the pull-back $\theta^*\gamma$ is another isomorphism $G_{0,S'} \simeq G_{S'}$, corresponding to the pinning $\theta^*\mathscr{E} = (\theta^*T, \theta^*\iota, (\theta^*X_\alpha)_{\alpha\in\Delta})$ of type $\mathscr{R}_+$ of $G_{S'}$. By [**21**, XXIV 1.5], there is a unique inner automorphism u_θ of $G_{S'}$ mapping $\theta^*\mathscr{E}$ back to $\mathscr{E}$, inducing an automorphism v_θ of $\mathscr{R}$. This defines an action of Θ on $\mathscr{R}_+$, analogous to the twisted action considered in section 2.4.17, which itself induces compatible actions of Θ on M_d, $\mathrm{Hom}^+(M, \Gamma)$, $\mathbb{N}[M]^W \ldots$

For $\tau \in \mathsf{Rep}^\circ(G)(S')$, we may use our fixed pinning $\mathscr{E}$ to view the restriction of τ to the maximal torus T of $G_{S'}$ as a representation of $\mathbb{D}_{S'}(M)$. We denote by $ch_{\mathscr{E}}(\tau)$ the corresponding element of $\mathbb{N}[M]^W$, i.e. $ch_{\mathscr{E}}(\tau) = \sum \mathrm{rank} V(\tau)_m \cdot e^m$ where e^m is the basis element of $\mathbb{Z}[M]$ corresponding to $m \in M$ and $V(\tau)_m$ is the m-th eigenspace of $\tau|T\circ\iota$. For any θ in Θ, the pull-back $\theta^*\tau$ also belongs to $\mathsf{Rep}^\circ(G)(S')$ and plainly $ch_{\theta^*\mathscr{E}}(\theta^*\tau)=ch_{\mathscr{E}}(\tau)$. On the other hand

$$ch_{\mathscr{E}}(\tau)=ch_{\mathscr{E}}(\tau\circ u_\theta)=v_\theta\left(ch_{\sigma^*\mathscr{E}}(\tau)\right)\quad\text{in}\quad\mathbb{N}[M]^W,$$

for every τ, thus $ch_{\mathscr{E}}(\theta^*\tau)=\theta\cdot ch_{\mathscr{E}}(\tau)$ in $\mathbb{N}[M]^W$. In particular $ch_{\mathscr{E}}(\tau)$ is fixed by Θ if $\theta^*\tau|T\simeq\tau|T$, for instance if τ comes from a representation in $\mathsf{Rep}^\circ(G)(S)$.

Let $\tau_{0,\lambda,\mathbb{Q}}\in\mathsf{Rep}^\circ(G_0)(\mathbb{Q})$ be the irreducible representation of $G_{0,\mathbb{Q}}$ with highest weight $\lambda\in M_d$ [**44**, Lemme 5], let $\tau_{0,\lambda}\in\mathsf{Rep}^\circ(G_0)(\mathbb{Z})$ be any extension of $\tau_{0,\lambda,\mathbb{Q}}$ to a representation of G_0 [**44**, Lemme 2], let $\tau'_\lambda\in\mathsf{Rep}^\circ(G)(S')$ be the corresponding representation of $G_{S'}$ and set $\tau_\lambda=\pi_*\tau'_\lambda\in\mathsf{Rep}^\circ(G)(S)$. Then

$$\tau_{\lambda,S'}=\pi^*\tau_\lambda\simeq\oplus_{\theta\in\Theta}\theta^*\tau'_\lambda\quad\text{in}\quad\mathsf{Rep}^\circ(G)(S').$$

Thus $ch_{\mathscr{E}}(\tau_{\lambda,S'})=\sum_{\theta\in\Theta}\theta\cdot ch_{\mathscr{E}}(\tau'_\lambda)$ with $ch_{\mathscr{E}}(\tau'_\lambda)=ch_{\mathscr{E}_0}(\tau_{0,\lambda})=ch_{\mathscr{E}_{0,\mathbb{Q}}}(\tau_{0,\lambda,\mathbb{Q}})$ in $\mathbb{N}[M]^W$, where $\mathscr{E}_0=(T_0,\iota_0,\cdots)$ is the pinning of G_0. Since the other weights of $\tau_{0,\lambda,\mathbb{Q}}$ are contained in $\lambda-\mathbb{N}\cdot R_+$, it follows that for any $f\in\mathrm{Hom}^+(M,\Gamma)$,

$$\max f\left(ch_{\mathscr{E}}(\tau_{\lambda,S'})\right)=\max\left\{f(\theta\cdot\lambda):\theta\in\Theta\right\}=\max\left\{(\theta\cdot f)(\lambda):\theta\in\Theta\right\}.$$

Our fixed pinning $\mathscr{E}$ also induces an isomorphism of partially ordered commutative S'-monoid between $\mathbb{C}^\Gamma(G_{S'})$ and $\mathrm{Hom}^+(M,\Gamma)_{S'}$, and the resulting isomorphism $\mathbb{C}^\Gamma(G)(S')\simeq\mathrm{Hom}^+(M,\Gamma)$ is Θ-equivariant, cf. section 2.2.11. If $t\in\mathbb{C}^\Gamma(G)(S')$ maps to $t_{\mathscr{E}}:M\to\Gamma$, then for every $\tau\in\mathsf{Rep}^\circ(G)(S')$, we have

$$t(\tau)=t_{\mathscr{E}}(ch_{\mathscr{E}}(\tau))\quad\text{in}\quad\Gamma_{\geq}^{r(\tau)}\subset\mathbb{N}[\Gamma]$$

under the natural identification of $\Gamma_{\geq}^{r(\tau)}$ with the subset of $\mathbb{N}[\Gamma]$ made of those elements which have degree $r(\tau)=\mathrm{rank}V(\tau)$ (if $\tau=0$, we set $\Gamma_{\geq}^{r(\tau)}=0$), thus also

$$\max t(\tau)=\max t_{\mathscr{E}}(ch_{\mathscr{E}}(\tau))\quad\text{in}\quad\Gamma.$$

For $\tau=\tau_{\lambda,S'}$ as above we therefore obtain

$$\max t(\tau_\lambda)=\max t(\tau_{\lambda,S'})=\max\left\{t_{\mathscr{E}}(\theta\cdot\lambda):\theta\in\Theta\right\}=\max\left\{(\theta\cdot t_{\mathscr{E}})(\lambda):\theta\in\Theta\right\}.$$

If moreover t belongs to $\mathbb{C}^\Gamma(G)(S)$, $\theta\cdot t_{\mathscr{E}}=t_{\mathscr{E}}$ for all $\theta\in\Theta$, thus

$$\max t(\tau_\lambda)=t_{\mathscr{E}}(\lambda)\quad\text{in}\quad\Gamma.$$

3.11.7. We may now prove the following strenghtening of Proposition 67.

PROPOSITION 68. *Suppose that G is isotrivial over a connected base scheme S. Then for every $t_1,t_2\in\mathbb{C}^\Gamma(G)(S)$, the following conditions are equivalent:*

(1) $t_1\leq t_2$ *in* $\mathbb{C}^\Gamma(G)(S)$.
(2) *For every* $\tau\in\mathsf{Rep}^\circ(G)(S)$, $t_1(\tau)\leq t_2(\tau)$ *in* $\Gamma_{\geq}^{r(\tau)}$.
(3) *For every* $\tau\in\mathsf{Rep}^\circ(G)(S)$, $\max t_1(\tau)\leq\max t_2(\tau)$ *in* Γ.

In (2), $r(\tau)$ *is the constant rank of* $V(\tau)$. *In* (3), $\max t(\tau)=0$ *if* $\tau=0$.

PROOF. Let $t_{\mathscr{E},i}$ be the Θ-invariant morphism in $\mathrm{Hom}^+(M,\Gamma)$ corresponding to the base change $t_{i,S'}\in\mathbb{C}^\Gamma(G)(S')$ of $t_i\in\mathbb{C}^\Gamma(G)(S)$. Then

$$\begin{aligned}
t_1\leq t_2\text{ in }\mathbb{C}^\Gamma(G)(S)\quad&\iff\quad t_{1,S'}\leq t_{2,S'}\text{ in }\mathbb{C}^\Gamma(G)(S')\\
&\iff\quad t_{\mathscr{E},1}\leq t_{\mathscr{E},2}\text{ in }\mathrm{Hom}^+(M,\Gamma)\\
&\iff\quad\forall\lambda\in M_d:\quad t_{\mathscr{E},1}(\lambda)\leq t_{\mathscr{E},2}(\lambda)\text{ in }\Gamma\\
&\iff\quad\forall x\in\mathbb{N}[M]^W:\quad\max t_{\mathscr{E},1}(x)\leq\max t_{\mathscr{E},2}(x)\text{ in }\Gamma
\end{aligned}$$

using lemma 30 for the last equivalence. Thus (1) $\Rightarrow$ (3) with $x = ch_{\mathscr{E}}(\tau_{S'})$ and (3) $\Rightarrow$ (1) with $\tau = \tau_\lambda$. Plainly (2) $\Rightarrow$ (3). Moreover, the equivalence (1) $\Leftrightarrow$ (3) already implies Proposition 67, from which (1) $\Rightarrow$ (2) immediately follows. $\square$

REMARK 69. For Γ-filtrations $\mathcal{F}_1, \mathcal{F}_2 \in \mathbb{F}^\Gamma(G)(S)$, the proposition implies:

$$\begin{aligned} t(\mathcal{F}_1) \leq t(\mathcal{F}_2) \text{ in } \mathbb{C}^\Gamma(G)(S) \quad &\iff \\ \forall \tau \in \mathsf{Rep}^\circ(G)(S), \forall \gamma \in \Gamma : \quad &\mathcal{F}_2^\gamma(\tau) = 0 \Rightarrow \mathcal{F}_1^\gamma(\tau) = 0 \end{aligned}$$

Indeed $\max t(\mathcal{F}_i)(\tau) = \max t(\mathcal{F}_i(\tau)) = \max\{\gamma : \mathcal{F}_i^\gamma(\tau) \neq 0\}$ if $\tau \neq 0$.

3.11.8. Still assuming that G is isotrivial over a connected base scheme S, suppose moreover that Γ is divisible. Let T be any connected S-scheme. We claim that the monomorphism $\mathbb{C}^\Gamma(G)(S) \hookrightarrow \mathbb{C}^\Gamma(G)(T)$ then has a canonical retraction

$$\sharp : \mathbb{C}^\Gamma(G)(T) \twoheadrightarrow \mathbb{C}^\Gamma(G)(S)$$

in the category of partially ordered commutative monoids, which is also functorial in T. To see this, we first fix a geometric point $s \to T$, giving rise to a morphism

$$\pi_1(T, s) \to \pi_1(S, s)$$

between the profinite étale fundamental group which classify the finite étale covers of T and S. Since $\mathbb{C}^\Gamma(G)$ becomes constant over the Galois cover S'/S, it is itself a disjoint union of finite étale covers of S (indexed by the orbits of Θ in $\mathrm{Hom}^+(M, \Gamma)$). Thus $\pi_1(S, s)$ (resp. $\pi_1(T, s)$) acts on $\mathbb{C}^\Gamma(G)(s)$ with finite orbits and fixed point set $\mathbb{C}^\Gamma(G)(S)$ (resp. $\mathbb{C}^\Gamma(G)(T)$). These actions respect the auxilliary structures, and averaging over the $\pi_1(S, s)$-orbits thus yields the desired retraction. If $s' \to T$ is another geometric point, there is a non-canonical equivariant diagram whose vertical maps are isomorphisms [**2**, V 7]

$$\begin{array}{ccccc} \pi_1(T, s') & \to & \pi_1(S, s') & & \mathbb{C}^\Gamma(G, s') \\ \simeq\downarrow & & \downarrow\simeq & \text{acting on} & \simeq\downarrow \\ \pi_1(T, s) & \to & \pi_1(S, s) & & \mathbb{C}^\Gamma(G, s) \end{array}$$

Our retraction is therefore independent of s, and thus also functorial in T.

PROPOSITION 70. *Suppose that Γ is divisible and G is isotrivial over a connected base scheme S. For every connected S-scheme T and $t_1, t_2 \in \mathbb{C}^\Gamma(G)(T)$, consider the following conditions:*

(1) $t_1^\sharp \leq t_2^\sharp$ *in* $\mathbb{C}^\Gamma(G)(S)$.
(2) *For every* $\tau \in \mathsf{Rep}^\circ(G)(S)$, $t_1(\tau_T) \leq t_2(\tau_T)$ *in* $\Gamma_{\geq}^{r(\tau)}$.
(3) *For every* $\tau \in \mathsf{Rep}^\circ(G)(S)$, $\max t_1(\tau_T) \leq \max t_2(\tau_T)$ *in* Γ.

Then (2) $\iff$ (3) $\Longrightarrow$ (1) *and* (1) $\iff$ (2) $\iff$ (3) *if* $t_1^\sharp = t_1$.

PROOF. We may assume that $T = s$ is a geometric point of the connected finite étale Galois cover S' of S splitting G, realizing $\Theta = \mathrm{Aut}(S'/S)$ as a quotient of $\pi_1(S, s)$ through which all of the above actions factor. Let $t_{\mathscr{E},i}$ be the image of t_i under $\mathbb{C}^\Gamma(G)(s) \simeq \mathbb{C}^\Gamma(G, S') \simeq \mathrm{Hom}^+(M, \Gamma)$. Then $t_i^\sharp$ maps to the average of the Θ-orbit of $t_{\mathscr{E},i}$. Plainly (2) $\Rightarrow$ (3) and conversely (3) $\Rightarrow$ (2) since

$$t_1(\tau_s) \leq t_2(\tau_s) \iff \begin{cases} \forall 1 \leq i \leq r(\tau) & \max t_1(\Lambda^i \tau_s) \leq \max t_2(\Lambda^i \tau_s), \\ \text{and} & \max t_1(\Lambda^{r(\tau)} \tau_s^\vee) \leq \max t_2(\Lambda^{r(\tau)} \tau_s^\vee). \end{cases}$$

For the remaining implications, note that

$$\begin{aligned} t_1^\sharp \leq t_2^\sharp \text{ in } \mathbb{C}^\Gamma(G)(S) &\iff t_{1,S'}^\sharp \leq t_{2,S'}^\sharp \text{ in } \mathbb{C}^\Gamma(G)(S') \\ &\iff t_{\mathscr{E},1}^\sharp \leq t_{\mathscr{E},2}^\sharp \text{ in } \mathrm{Hom}^+(M,\Gamma) \\ &\iff \forall\lambda\in M_d : t_{\mathscr{E},1}^\sharp(\lambda) \leq t_{\mathscr{E},2}^\sharp(\lambda) \text{ in } \Gamma \\ &\iff \forall\lambda\in M_d : t_{\mathscr{E},1}(\lambda^\sharp) \leq t_{\mathscr{E},2}(\lambda^\sharp) \text{ in } \Gamma \end{aligned}$$

where $\lambda^\sharp \in M\otimes\mathbb{Q}$ is the average of the Θ-orbit of λ, thus also

$$t_1^\sharp \leq t_2^\sharp \text{ in } \mathbb{C}^\Gamma(G)(S) \iff \forall\lambda\in M_d^\Theta : \quad t_{\mathscr{E},1}(\lambda) \leq t_{\mathscr{E},2}(\lambda) \text{ in } \Gamma$$

since $M_d^\Theta \subset M_d^\sharp \subset \mathbb{Q}_{\geq} M_d^\Theta$. Thus (3) $\Rightarrow$ (1) with $\tau = \tau_\lambda$ for $\lambda \in M_d^\Theta$, since

$$\max t_i(\tau_\lambda) = \max\{t_{\mathscr{E},i}(\theta\cdot\lambda) : \theta\in\Theta\} = t_{\mathscr{E},i}(\lambda) \text{ in } \Gamma.$$

Suppose finally that $t_1^\sharp = t_1$. Then using lemma 30, we have

$$\begin{aligned} t_1 \leq t_2^\sharp \text{ in } \mathbb{C}^\Gamma(G)(S) &\iff \forall x\in\mathbb{N}[M]^W : \max t_{\mathscr{E},1}(x) \leq \max t_{\mathscr{E},2}^\sharp(x) \text{ in } \Gamma \\ &\implies \forall x\in\mathbb{N}[M]^{W,\Theta} : \max t_{\mathscr{E},1}(x) \leq \max t_{\mathscr{E},2}(x) \text{ in } \Gamma \end{aligned}$$

since indeed for any $x \in \mathbb{N}[M]^W$ we have

$$\max t_{\mathscr{E},2}^\sharp(x) \leq \tfrac{1}{\sharp\Theta}\textstyle\sum_{\theta\in\Theta}\max(\theta\cdot t_{\mathscr{E},2})(x) = \tfrac{1}{\sharp\Theta}\textstyle\sum_{\theta\in\Theta}\max t_{\mathscr{E},2}(\theta\cdot x).$$

Thus (1) $\Rightarrow$ (3) if $t_1^\sharp = t_1$, with $x = ch_{\mathscr{E}}(\tau_s) \in \mathbb{N}[M]^{W,\Theta}$ for $\tau \in \mathsf{Rep}^\circ(G)(S)$. $\square$

3.11.9. The results of sections 3.11.5-3.11.8 were inspired by propositions 6.3.9 and 9.4.2 of [**17**]. However, the latter is contradicted by the following example, which shows that usually (1) does not imply (2) in proposition 70. Take

$$\Gamma = \mathbb{Q}, \quad S = \mathrm{Spec}K, \quad T = \mathrm{Spec}L \quad \text{and} \quad G = \mathrm{Res}_{L/K}\mathbb{G}_{m,L}$$

where L is a quadratic extension of a field K. Then $\mathbb{C}^\Gamma(G)(L) = \mathbb{Q}^2$ with the trivial partial order. The non-trivial element ι of $\mathrm{Gal}(L/K)$ acts by $(x,y)\mapsto(y,x)$, thus $(x,y)^\sharp \leq (x',y')^\sharp$ if and only if $x+y = x'+y'$. For $n,m\in\mathbb{Z}$, the formula $z\mapsto z^n(\iota z)^m$ defines a 2-dimensional representation $\tau_{n,m} : G \to GL_K(L)$ which is irreducible if $m \neq n$. It maps $(x,y)\in\mathbb{Q}^2$ to

$$(x,y)(\tau_{n,m}) = (\max,\min)\{xn+ym, yn+xm\} \in \mathbb{Q}^2_{\geq}$$

Thus for $t_1 = (1,-1)$ and $t_2 = (0,0)$, $t_1^\sharp = t_2^\sharp = 0$ in $\mathbb{C}^\Gamma(G)(K) = \mathbb{Q}$ but for every $n,m\in\mathbb{Z}$ with $n\neq m$, $t_1(\tau_{n,m}) = (|n-m|, -|n-m|) > (0,0) = t_2(\tau_{n,m})$ in $\mathbb{Q}^2_{\geq}$.

3.11.10. The addition map of section 2.3.2 has the following Tannakian description. For an S-scheme T, $(\mathcal{F},\mathcal{G}) \in \mathbb{STD}^\Gamma(G)(T)$ and any $\rho\in\mathsf{Rep}(G)(T)$,

$$(\mathcal{F}+\mathcal{G})^\gamma(\rho) = \sum_{\gamma_1+\gamma_2=\gamma}\mathcal{F}^{\gamma_1}(\rho)\cap\mathcal{G}^{\gamma_2}(\rho).$$

Indeed, the question is local on T for the Zariski topology, thus by definition of $\mathbb{STD}^\Gamma(G)$, we may assume that $P_{\mathcal{F}}\cap P_{\mathcal{G}}$ contains a maximal subtorus H of G_T. Then $\mathcal{F},\mathcal{G}$ lift to $f,g : \mathbb{D}_T(\Gamma)\to H$ and $(\mathcal{F}+\mathcal{G}) = \mathrm{Fil}(f+g)$. Let $V(\rho)_{\gamma_1,\gamma_2}$ be the subsheaf of $V(\rho|H)$ where $\mathbb{D}_T(\Gamma)$ acts by γ_1 through f and γ_2 through g. Then

$$\begin{aligned} \mathcal{F}^{\gamma_1}(\rho) &= \oplus_{\eta\geq\gamma_1}\oplus_{\eta'} V(\rho)_{\eta,\eta'} \\ \mathcal{G}^{\gamma_2}(\rho) &= \oplus_{\eta}\oplus_{\eta'\geq\gamma_2} V(\rho)_{\eta,\eta'} \\ (\mathcal{F}+\mathcal{G})^\gamma(\rho) &= \oplus_{\eta+\eta'\geq\gamma} V(\rho)_{\eta,\eta'} \end{aligned}$$

thus indeed $(\mathcal{F}+\mathcal{G})^{\gamma}(\rho)=\sum_{\gamma_1+\gamma_2=\gamma}\mathcal{F}^{\gamma_1}(\rho)\cap\mathcal{G}^{\gamma_2}(\rho)$.

COROLLARY 71. *Let $\varphi : G \to H$ be a morphism of reductive groups over S. Then for any S-scheme T and $t_1, t_2 \in \mathbb{C}^{\Gamma}(G)(T)$,*

$$\varphi(t_1+t_2) \preceq \varphi(t_1)+\varphi(t_2) \quad \textit{in} \quad \mathbb{C}^{\Gamma}(H)(T).$$

PROOF. We may assume that $T = s$ is a geometric point, and lift (t_1, t_2) to a pair of Γ-filtrations $(\mathcal{F}_1, \mathcal{F}_2) \in \mathbb{F}^{\Gamma}(G)(s)$ in osculatory relative position. Then $\varphi(\mathcal{F}_1)$ and $\varphi(\mathcal{F}_2)$ also are in standard relative position (cf. Remark 22) and the above formula shows that $\varphi(\mathcal{F}_1+\mathcal{F}_2)=\varphi(\mathcal{F}_1)+\varphi(\mathcal{F}_2)$ in $\mathbb{F}^{\Gamma}(H)(s)$. Thus

$$\begin{array}{rcl} \varphi\left(t(\mathcal{F}_1)+t(\mathcal{F}_2)\right) & = & \varphi\left(t(\mathcal{F}_1+\mathcal{F}_2)\right) \\ & = & t\left(\varphi(\mathcal{F}_1+\mathcal{F}_2)\right) \\ & = & t\left(\varphi(\mathcal{F}_1)+\varphi(\mathcal{F}_2)\right) \quad \preceq \quad t\left(\varphi(\mathcal{F}_1)\right)+t\left(\varphi(\mathcal{F}_2)\right) \end{array}$$

by proposition 24, i.e. $\varphi(t_1+t_2) \preceq \varphi(t_1)+\varphi(t_2)$ in $\mathbb{C}^{\Gamma}(H)(s)$. □

3.11.11. The morphism defined in section 2.3.4 has the following Tannakian description. Let P be a parabolic subgroup of G with unipotent radical U, and suppose that $P = P_{\mathcal{F}}$ for some Γ-filtration $\mathcal{F}$ on ω_S. For every $\rho \in \mathsf{Rep}(G)(S)$ and $\gamma \in \Gamma$, we may view $\mathrm{Gr}^{\gamma}_{\mathcal{F}}(\rho)=\mathcal{F}^{\gamma}(\rho)/\mathcal{F}^{\gamma}_{+}(\rho)$ as a representation of P/U. Then for every S-scheme T and every Γ-filtration $\mathcal{G}$ on ω_T such that P_T and $P_{\mathcal{G}}$ are in standard relative position (i.e. $P_T \cap P_{\mathcal{G}}$ is a smooth subscheme of G_T),

$$\mathrm{Gr}_P(\mathcal{G})\left(\mathrm{Gr}^{\gamma}_{\mathcal{F}}(\rho)\right)=\mathrm{Gr}^{\gamma}_{\mathcal{F}}(\mathcal{G},\rho)$$

where $\mathrm{Gr}^{\gamma}_{\mathcal{F}}(\mathcal{G},\rho)$ is the Γ-filtration on $\mathrm{Gr}^{\gamma}_{\mathcal{F}}(\rho)_T=\mathcal{F}^{\gamma}(\rho)_T/\mathcal{F}^{\gamma}_{+}(\rho)_T$ induced by the Γ-filtration $\mathcal{G}(\rho)$ on $V(\rho)_T$, so that for every $\theta \in \Gamma$,

$$\mathrm{Gr}^{\gamma}_{\mathcal{F}}(\mathcal{G},\rho)^{\theta}=\left(\mathcal{F}^{\gamma}(\rho)_T\cap\mathcal{G}^{\theta}(\rho)+\mathcal{F}^{\gamma}_{+}(\rho)_T\right)/\mathcal{F}^{\gamma}_{+}(\rho)_T.$$

This follows from the explicit description of Gr_P in the proof of proposition 26.

3.11.12. The functors $\mathbb{G}^{\Gamma}(-)$ and $\mathbb{F}^{\Gamma}(-)$ preserve closed immersions.

PROPOSITION 72. *Let $H \hookrightarrow G$ be a closed immersion of reductive group schemes over S. Then the induced morphisms*

$$\mathbb{G}^{\Gamma}(H)\to\mathbb{G}^{\Gamma}(G) \quad \textit{and} \quad \mathbb{F}^{\Gamma}(H)\to\mathbb{F}^{\Gamma}(G)$$

are finitely presented closed immersions.

PROOF. Plainly, $\mathbb{G}^{\Gamma}(H)\to\mathbb{G}^{\Gamma}(G)$ is a monomorphism. Let $x : T \to \mathbb{G}^{\Gamma}(G)$ be a morphism corresponding to $f : \mathbb{D}(\Gamma)_T \to G_T$. Put $Z = f^{-1}(H_T)$, a closed subgroup scheme of $Y = \mathbb{D}(\Gamma)_T$. For every morphism $a : T' \to T$, we have:

$$\begin{array}{rl} & x\circ a : T' \to \mathbb{G}^{\Gamma}(G) \text{ factors through } \mathbb{G}^{\Gamma}(H) \\ \iff & f_{T'} : \mathbb{D}(\Gamma)_{T'} \to G_{T'} \text{ factors through } H_{T'} \\ \iff & Z_{T'} = Y_{T'}. \end{array}$$

This last condition is represented by a closed subscheme of T by [**20**, VIII 6.3 & 6.4]. Thus $\mathbb{G}^{\Gamma}(H)\to\mathbb{G}^{\Gamma}(G)$ is relatively representable by closed immersions, i.e. itself a closed immersion. Since $\mathbb{G}^{\Gamma}(H)\to S$ and $\mathbb{G}^{\Gamma}(G)\to S$ are locally of finite presentation (by theorem 1), so is $\mathbb{G}^{\Gamma}(H)\hookrightarrow\mathbb{G}^{\Gamma}(G)$ by [**25**, 1.4.3.v], which therefore is a finitely presented closed immersion.

The second morphism $\mathbb{F}^{\Gamma}(H)\to\mathbb{F}^{\Gamma}(G)$ is a monomorphism: we have seen that Γ-filtrations on ω_H are uniquely determined by their value on the regular reprentation of H, which is a quotient of the restriction to H of the regular representation of G. Since $\mathbb{G}^\Gamma(H) \to \mathbb{G}^\Gamma(G) \to \mathbb{F}^\Gamma(G)$ is quasi-compact and $\mathbb{G}^\Gamma(H) \to \mathbb{F}^\Gamma(H)$ is surjective, $\mathbb{F}^\Gamma(H) \to \mathbb{F}^\Gamma(G)$ is quasi-compact [**25**, 1.1.3]. Since $\mathbb{F}^\Gamma(H) \to S$ and $\mathbb{F}^\Gamma(G) \to S$ are separated and locally of finite presentation, so is $\mathbb{F}^\Gamma(H) \to \mathbb{F}^\Gamma(G)$ by [**23**, 5.5.1.v] and [**25**, 1.4.3.v]. Since moreover $\mathbb{F}^\Gamma(H) \to S$ satisfies the valuative criterion of properness, so does $\mathbb{F}^\Gamma(H) \to \mathbb{F}^\Gamma(G)$, which thus is a proper morphism by [**24**, 7.3.8] and a (finitely presented) closed immersion by [**28**, 18.12.6]. □

3.12. Ranks and relative positions

Let G be a reductive group over S.

3.12.1. Recall from section 3.8 that for every S-scheme T and $\mathcal{F} \in \mathbb{F}^\Gamma(G)(T)$, the exact $\otimes$-functor $\mathrm{Gr}^\bullet_{\mathcal{F}} : \mathsf{Rep}^\circ(G)(T) \to \mathsf{Gr}^\Gamma\mathsf{LF}(T)$ yields a ring homomorphism

$$K_0(G_T) \to \mathcal{C}(T, \mathbb{Z}[\Gamma])$$

mapping the class of $\tau \in \mathsf{Rep}^\circ(G)(T)$ in $K_0(G_T)$ to the function

$$t \mapsto \sum_\gamma \dim_{k(t)} \left(\mathrm{Gr}^\gamma_{\mathcal{F}}(\tau) \otimes k(t)\right) \cdot e^\gamma$$

where e^γ is the basis element of $\mathbb{Z}[\Gamma]$ corresponding to $\gamma \in \Gamma$. This construction is functorial in T and invariant under the action of G on $\mathbb{F}^\Gamma(G)(T)$. It therefore induces a morphism of fpqc sheaves of commutative rings $(\mathsf{Sch}/S)^\circ \to \mathsf{Ring}$,

$$\kappa : \underline{K}_0(G) \to \underline{\mathrm{Mor}}\left(\mathbb{C}^\Gamma(G), \mathbb{Z}[\Gamma]_S\right)$$

where $\underline{K}_0(G)$ is the fpqc sheaf associated to the presheaf $T \mapsto K_0(G_T)$ while $\underline{\mathrm{Mor}}\left(\mathbb{C}^\Gamma(G), \mathbb{Z}[\Gamma]_S\right)$ is the fpqc sheaf of morphisms of S-schemes from the cone $\mathbb{C}^\Gamma(G)$ to the constant sheaf of rings $\mathbb{Z}[\Gamma]_S$.

3.12.2. Let now $(\mathcal{F}_1, \mathcal{F}_2) \in \mathbb{STD}^\Gamma(G)(T)$ be a pair of Γ-filtrations in standard relative position (cf. 2.3). Then the formula

$$\mathrm{Gr}^{\gamma_1,\gamma_2}_{\mathcal{F}_1,\mathcal{F}_2}(\tau) = \frac{\mathcal{F}_1^{\gamma_1}(\tau) \cap \mathcal{F}_2^{\gamma_2}(\tau)}{\mathcal{F}_{1,+}^{\gamma_1}(\tau) \cap \mathcal{F}_2^{\gamma_2}(\tau) + \mathcal{F}_1^{\gamma_1}(\tau) \cap \mathcal{F}_{2,+}^{\gamma_2}(\tau)}$$

also defines an exact $\otimes$-functor

$$\mathrm{Gr}^{\bullet,\bullet}_{\mathcal{F}_1,\mathcal{F}_2} : \mathsf{Rep}^\circ(G)(T) \longrightarrow \mathsf{Gr}^{\Gamma\times\Gamma}\mathsf{LF}(T).$$

Indeed, we have to show that $\mathrm{Gr}^{\gamma_1,\gamma_2}_{\mathcal{F}_1,\mathcal{F}_2}(\tau)$ is locally free of finite rank, exact in τ, and such that for every $\tau', \tau'' \in \mathsf{Rep}^\circ(G)(T)$ and $\gamma_1, \gamma_2 \in \Gamma$, the natural map

$$\oplus_{(\gamma_1',\gamma_2')+(\gamma_1'',\gamma_2'')=(\gamma_1,\gamma_2)} \mathrm{Gr}^{\gamma_1',\gamma_2'}_{\mathcal{F}_1,\mathcal{F}_2}(\tau') \otimes \mathrm{Gr}^{\gamma_1'',\gamma_2''}_{\mathcal{F}_1,\mathcal{F}_2}(\tau'') \to \mathrm{Gr}^{\gamma_1,\gamma_2}_{\mathcal{F}_1,\mathcal{F}_2}(\tau' \otimes \tau'')$$

is an isomorphism. All this is local in the fpqc topology on T. We may thus assume that $P_{\mathcal{F}_1} \cap P_{\mathcal{F}_2}$ contains a maximal torus H of G which is split, i.e. $H = \mathbb{D}_T(M)$ for some finitely generated free abelian group M, in which case $\mathcal{F}_1$ and $\mathcal{F}_2$ are split by morphisms $\mathcal{G}_1$ and $\mathcal{G}_2 : \mathbb{D}_T(\Gamma) \to \mathbb{D}_T(M)$. If $V(\tau) = \oplus_{m\in M} V(\tau)_m$ is the H-eigenspace decomposition of $\tau|_H$, we then have a canonical isomorphism

$$\mathrm{Gr}^{\gamma_1,\gamma_2}_{\mathcal{F}_1,\mathcal{F}_2}(\tau) \simeq \oplus_{m\in M:(m\circ\mathcal{G}_1, m\circ\mathcal{G}_2)=(\gamma_1,\gamma_2)} V(\tau)_m$$

and our claim easily follows. We thus obtain a ring homomorphism

$$K_0(G_T) \to \mathcal{C}(T, \mathbb{Z}[\Gamma\times\Gamma])$$

which maps the class of $\tau \in \mathsf{Rep}^{\circ}(G)(T)$ in $K_0(G_T)$ to the function

$$t \mapsto \sum_{\gamma_1,\gamma_2} \dim_{k(t)} \left(\mathrm{Gr}^{\gamma_1,\gamma_2}_{\mathcal{F}_1,\mathcal{F}_2}(\tau) \otimes k(t)\right) \cdot e^{\gamma_1} \otimes e^{\gamma_2}$$

where $e^{\gamma_1} \otimes e^{\gamma_2}$ is the basis element of $\mathbb{Z}[\Gamma \times \Gamma] = \mathbb{Z}[\Gamma] \otimes \mathbb{Z}[\Gamma]$ corresponding to the element (γ_1, γ_2) of $\Gamma \times \Gamma$.

3.12.3. The above construction is again functorial in T and invariant under the diagonal action of G on $\mathbb{STD}^{\Gamma}(G)$. It therefore induces a morphism of fpqc sheaves of commutative rings $(\mathsf{Sch}/S)^{\circ} \to \mathsf{Ring}$,

$$\kappa : \underline{K}_0(G) \to \underline{\mathrm{Mor}}\left(\mathbb{TSTD}^{\Gamma}(G), \mathbb{Z}[\Gamma \times \Gamma]_S\right).$$

3.12.4. If now $f : \mathbb{Z}[\Gamma \times \Gamma]_S \to X$ is a morphism of S-schemes, we denote by

$$\begin{array}{rccl} \langle -,- \rangle_f : & \underline{K}_0(G) & \to & \underline{\mathrm{Mor}}\left(\mathbb{STD}^{\Gamma}(G), X\right) \\ \langle -,- \rangle_f^{os} : & \underline{K}_0(G) & \to & \underline{\mathrm{Mor}}\left(\mathbb{C}^{\Gamma}(G)^2, X\right) \\ \langle -,- \rangle_f^{tr} : & \underline{K}_0(G) & \to & \underline{\mathrm{Mor}}\left(\mathbb{C}^{\Gamma}(G)^2, X\right) \end{array}$$

the morphisms of fpqc sheaves on S which are obtained by post-composition of κ with the obvious morphisms induced by f and, respectively: the quotient map

$$t_2 : \mathbb{STD}^{\Gamma}(G) \twoheadrightarrow \mathbb{TSTD}^{\Gamma}(G)$$

and the osculatory and transverse sections

$$os \quad \text{and} \quad tr : \mathbb{C}^{\Gamma}(G)^2 \hookrightarrow \mathbb{TSTD}^{\Gamma}(G)$$

of section 2.3. For $\tau \in \underline{K}_0(G)(S)$, we thus obtain morphisms of S-schemes

$$\begin{array}{rccc} \langle -,- \rangle_{f,\tau} : & \mathbb{STD}^{\Gamma}(G) & \to & X \\ \langle -,- \rangle_{f,\tau}^{os} : & \mathbb{C}^{\Gamma}(G)^2 & \to & X \\ \langle -,- \rangle_{f,\tau}^{tr} : & \mathbb{C}^{\Gamma}(G)^2 & \to & X \end{array}$$

By construction, for every S-scheme T and $(\mathcal{F}_1, \mathcal{F}_2) \in \mathbb{GEN}^{\Gamma}(G)(T)$,

$$\langle \mathcal{F}_1, \mathcal{F}_2 \rangle_{f,\tau} = \langle t(\mathcal{F}_1), t(\mathcal{F}_2) \rangle_{f,\tau}^{tr} \quad \text{in} \quad X(T).$$

3.12.5. We will only consider these constructions in the following situation: Γ is a subgroup of $\mathbb{R}$, X is the constant scheme $\mathbb{R}_S$, f is induced by the bilinear form $\Gamma \times \Gamma \ni (\gamma_1, \gamma_2) \mapsto \gamma_1\gamma_2 \in \mathbb{R}$ and τ is a genuine representation in $\mathsf{Rep}^{\circ}(G)(S)$. Then for any S-scheme T and $(\mathcal{F}_1, \mathcal{F}_2) \in \mathbb{STD}^{\Gamma}(G)(T)$, $\langle \mathcal{F}_1, \mathcal{F}_2 \rangle_\tau = \langle \mathcal{F}_1, \mathcal{F}_2 \rangle_{f,\tau}$ is the locally constant function $T \to \mathbb{R}$ given by

$$t \mapsto \sum_{\gamma_1,\gamma_2} \dim_{k(t)} \left(\mathrm{Gr}^{\gamma_1,\gamma_2}_{\mathcal{F}_1,\mathcal{F}_2}(\tau) \otimes k(t)\right) \cdot \gamma_1\gamma_2.$$

3.13. Appendix: pure subsheaves

Let X be a scheme.

LEMMA 73. *For $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ in $\mathsf{QCoh}(X)$, consider the following conditions:*

(1) *For every quasi-coherent sheaf $\mathcal{F}$ on X,*

$$0 \to \mathcal{A} \otimes \mathcal{F} \to \mathcal{B} \otimes \mathcal{F} \to \mathcal{C} \otimes \mathcal{F} \to 0 \qquad \textit{is exact in } \mathsf{QCoh}(X).$$

(2) *For every morphism* $f : Y \to X$,

$$0 \to f^*\mathcal{A} \to f^*\mathcal{B} \to f^*\mathcal{C} \to 0 \qquad \text{is exact in } \mathsf{QCoh}(Y).$$

(3) *For every morphism* $f : Y \to X$ *and quasi-coherent sheaf* $\mathcal{F}$ *on* Y,

$$0 \to f^*\mathcal{A} \otimes \mathcal{F} \to f^*\mathcal{B} \otimes \mathcal{F} \to f^*\mathcal{C} \otimes \mathcal{F} \to 0 \qquad \text{is exact in } \mathsf{QCoh}(Y).$$

Then $(1) \Leftarrow (2) \Leftrightarrow (3)$ *and* $(1) \Leftrightarrow (2) \Leftrightarrow (3)$ *if* X *is quasi-separated.*

Proof. Obviously $(3) \Rightarrow (1)$ and (2). Suppose (2) holds. Let $f : Y \to X$ be a morphism, $\mathcal{F}$ a quasi-coherent sheaf on Y, $g : Z \to Y$ the structural morphism of $Z = \operatorname{Spec}(\mathcal{O}_Y[\mathcal{F}])$ where $\mathcal{O}_Y[\mathcal{F}] = \mathcal{O}_Y \oplus \mathcal{F}$ is the quasi-coherent $\mathcal{O}_Y$-algebra defined by $\mathcal{F} \cdot \mathcal{F} = 0$. By assumption, $0 \to h^*\mathcal{A} \to h^*\mathcal{B} \to h^*\mathcal{C} \to 0$ is an exact sequence of quasi-coherent sheaves on Z, where $h = f \circ g$. Since g is affine,

$$0 \to g_*h^*\mathcal{A} \to g_*h^*\mathcal{B} \to g_*h^*\mathcal{C} \to 0$$

is an exact sequence of quasi-coherent sheaves on Y. But

$$g_*h^*\mathcal{X} = g_*g^*f^*\mathcal{X} = f^*\mathcal{X} \oplus f^*\mathcal{X} \otimes \mathcal{F}$$

for any $\mathcal{X}$ in $\mathsf{QCoh}(X)$, therefore

$$0 \to f^*\mathcal{A} \otimes \mathcal{F} \to f^*\mathcal{B} \otimes \mathcal{F} \to f^*\mathcal{C} \otimes \mathcal{F} \to 0$$

is exact and $(2) \Rightarrow (3)$. Suppose now that X is quasi-separated and (1) holds. Let $f : Y \to X$ be any morphism. Let $\{X_i\}$ and $\{Y_{i,j}\}$ be open coverings of X and Y by affine schemes such that $f(Y_{i,j}) \subset X_i$ and let $f_{i,j} : Y_{i,j} \to X_i$ be the induced morphism. Since $(f^*\mathcal{X})|_{Y_{i,j}} = f_{i,j}^*(\mathcal{X}|_{X_i})$ for every $\mathcal{X} \in \mathsf{QCoh}(X)$, we have to show that $0 \to f_{i,j}^*(\mathcal{A}_i) \to f_{i,j}^*(\mathcal{B}_i) \to f_{i,j}^*(\mathcal{C}_i) \to 0$ is exact on $Y_{i,j}$ for every i, j, with $\mathcal{X}_i = \mathcal{X}|_{X_i}$. Since $Y_{i,j}$ and X_i are affine, this amounts to showing that

$$0 \to \mathcal{A}_i \otimes \mathcal{O}_{i,j} \to \mathcal{B}_i \otimes \mathcal{O}_{i,j} \to \mathcal{C}_i \otimes \mathcal{O}_{i,j} \to 0$$

is exact on X_i for every i, j, for the quasi-coherent sheaf $\mathcal{O}_{i,j} = (f_{i,j})_*\mathcal{O}_{Y_{i,j}}$ on X_i. Since X is quasi-separated, the immersion $\iota_i : X_i \hookrightarrow X$ is quasi-compact and quasi-separated by [**25**, 1.2.2.i & 1.2.7.b], thus $\mathcal{F}_{i,j} = (\iota_i)_*\mathcal{O}_{i,j}$ is a quasi-coherent sheaf on X by [**25**, 1.7.4] and $0 \to \mathcal{A} \otimes \mathcal{F}_{i,j} \to \mathcal{B} \otimes \mathcal{F}_{i,j} \to \mathcal{C} \otimes \mathcal{F}_{i,j} \to 0$ is an exact sequence on X by assumption. Pulling back through the exact restriction functor $\iota_i^* : \mathsf{QCoh}(X) \to \mathsf{QCoh}(X_i)$ yields the desired result. □

Definition 74. We say that the sequence $0 \to \mathcal{A} \xrightarrow{\iota} \mathcal{B} \to \mathcal{C} \to 0$ is pure exact, or that ι is a pure monomorphism, or that $\iota(\mathcal{A})$ is a pure (quasi-coherent) subsheaf of $\mathcal{B}$ if the above condition (2) holds.

Lemma 75. *Let* $\mathcal{B}$ *be a quasi-coherent sheaf on* X. *Then*

$$\mathcal{P} : (\mathsf{Sch}/X)^\circ \to \mathsf{Set} \qquad T \mapsto \{\textit{pure quasi-coherent subsheaves } \mathcal{A} \textit{ of } \mathcal{B}_T\}$$

is an fpqc sheaf on Sch/X.

Proof. It is a functor: if $\mathcal{A} \in \mathcal{P}(T)$ and $\alpha : T' \to T$ is an X-morphism, the monomorphism $\alpha^*(\mathcal{A} \hookrightarrow \mathcal{B}_T)$ identifies $\alpha^*(\mathcal{A})$ with a quasi-coherent subsheaf of $\alpha^*(\mathcal{B}_T) = \mathcal{B}_{T'}$, which is pure since for any morphism $f' : Y \to T'$, if $f = \alpha \circ f'$, then $f'^* \circ \alpha^*(\mathcal{A} \hookrightarrow \mathcal{B}_T) = f^*(\mathcal{A} \hookrightarrow \mathcal{B}_T)$ is a monomorphism of quasi-coherent sheaves on Y since $\mathcal{A}$ is pure in $\mathcal{B}_T$. It is an fpqc sheaf: if $\{T_i \to T\}$ is an fpqc cover and $\mathcal{A}_i \in \mathcal{P}(T_i)$ have the same image $\mathcal{A}_{i,j} \in \mathcal{P}(T_i \times_T T_j)$, then the quasi-coherent subsheaves $\mathcal{A}_i$ of $\mathcal{B}_{T_i}$ glue to a quasi-coherent subsheaf $\mathcal{A}$ of $\mathcal{B}_T$ which is pure since for any $f : Y \to T$, $f^*(\mathcal{A} \hookrightarrow \mathcal{B}_T)$ is a monomorphism of quasi-coherent sheaves on Y as it becomes so in the fpqc cover $\{Y \times_T T_i \to Y\}$ of Y. $\square$

LEMMA 76. *Let $\mathcal{A}$ be a quasi-coherent subsheaf of $\mathcal{B}$.*

(1) *Suppose that locally on X for the fpqc topology, $\mathcal{A}$ is a direct factor of $\mathcal{B}$. Then $\mathcal{A}$ is a pure subsheaf of $\mathcal{B}$.*
(2) *Suppose that $\mathcal{A}$ is a pure subsheaf of $\mathcal{B}$ and $\mathcal{C} = \mathcal{B}/\mathcal{A}$ is finitely presented. Then locally on X for the Zariski topology, $\mathcal{A}$ is a direct factor of $\mathcal{B}$.*

PROOF. (1) A direct factor being obviously pure, this follows from the previous lemma. As for (2): the assumptions are local in the Zariski topology by the previous lemma, we may thus assume that $X = \mathrm{Spec}(R)$ for some ring R. Then $A = \Gamma(X, \mathcal{A})$ is a pure R-submodule of $B = \Gamma(X, \mathcal{B})$ in the sense of [**34**, Appendix to §7] by (2) $\Rightarrow$ (1) of lemma 73, and $C = B/A$ is a finitely presented R-module. Therefore A is a direct factor of B by [**34**, Theorem 7.14], i.e. $\mathcal{A}$ is a direct factor of $\mathcal{B}$. $\square$

CHAPTER 4

The vectorial Tits building $\mathbf{F}^{\Gamma}(G)$

Let $\mathcal{O}$ be a local ring, G a reductive group over $\mathrm{Spec}(\mathcal{O})$. We shall here take a closer look at the set $\mathbf{F}^{\Gamma}(G) = \mathbb{F}^{\Gamma}(G)(\mathcal{O})$ of sections of $\mathbb{F}^{\Gamma}(G)$ over $\mathrm{Spec}(\mathcal{O})$.

4.1. Combinatorial structures

4.1.1. We say that a morphism of posets $f : (\mathbf{X}, \leq) \to (\mathbf{Y}, \leq)$ is nice if

$$\forall x, y \in \mathbf{X} \times \mathbf{Y} \text{ with } f(x) \leq y, \text{ there is a unique } x' \in f^{-1}(y) \text{ with } x \leq x'.$$

We say that it is very nice if also

$$\forall x, y \in \mathbf{X} \times \mathbf{Y} \text{ with } f(x) \geq y, \text{ there is an } x' \in f^{-1}(y) \text{ with } x \geq x'.$$

4.1.2. We will define below an $\mathrm{Aut}(G)$-equivariant sequence of nice surjective morphisms of posets

$$\mathbf{SBP}(G) \xrightarrow{a} \mathbf{SP}(G) \xrightarrow{b} \mathbf{OPP}(G) \xrightarrow{p_1} \mathbf{P}(G) \xrightarrow{t} \mathbf{O}(G)$$

The group $\mathsf{G} = G(\mathcal{O})$ acts on it through $\mathrm{Int} : \mathsf{G} \to \mathrm{Aut}(G)$, and we will see that

$$\mathsf{G}\backslash\mathbf{SBP}(G) = \mathsf{G}\backslash\mathbf{SP}(G) = \mathsf{G}\backslash\mathbf{OPP}(G) = \mathsf{G}\backslash\mathbf{P}(G) = \mathbf{O}(G).$$

4.1.3. We first define our posets. We will use the following notations:

$$\begin{array}{rcl}
\mathbf{S}(G) & = & \{S : \text{maximal split torus of } G\} \\
\mathbf{B}(G) & = & \{B : \text{minimal parabolic subgroup of } G\} \\
\mathbf{P}(G) & = & \{P : \text{parabolic subgroup of } G\} \\
\mathbf{SP}(G) & = & \{(S, P) : Z_G(S) \subset P\} \\
\mathbf{SBP}(G) & = & \{(S, B, P) : Z_G(S) \subset B \subset P\} \\
\mathbf{OPP}(G) & = & \{(P, P') : \text{opposed parabolic subgroups of } G\}
\end{array}$$

Thus $\mathbf{P}(G) = \mathbb{P}(G)(\mathcal{O})$ and $\mathbf{OPP}(G) = \mathbb{OPP}(G)(\mathcal{O})$. In addition, we set

$$\mathbf{O}(G) = \text{image of } t : \mathbb{P}(G)(\mathcal{O}) \to \mathbb{O}(G)(\mathcal{O}).$$

We endow $\mathbf{P}(G)$ and $\mathbf{O}(G)$ with their natural partial orders and the remaining three sets $\mathbf{SBP}(G)$, $\mathbf{SP}(G)$ and $\mathbf{OPP}(G)$ with the following ones:

$$\begin{array}{rcl}
(S_1, B_1, P_1) \leq (S_2, B_2, P_2) & \iff & S_1 = S_2,\ B_1 = B_2 \text{ and } P_1 \subset P_2 \\
(S_1, P_1) \leq (S_2, P_2) & \iff & S_1 = S_2 \text{ and } P_1 \subset P_2 \\
(P_1, P_1') \leq (P_2, P_2') & \iff & P_1 \subset P_2 \text{ and } P_1' \subset P_2'
\end{array}$$

4.1.4. The morphism $t : \mathbf{P}(G) \to \mathbf{O}(G)$ maps P to its type $t(P)$. It is plainly a morphism of posets. It is surjective by definition of $\mathbf{O}(G)$, nice by [**21**, XXVI 3.8] and even very nice by [**21**, XXVI 5.5]. The group G acts trivially on $\mathbf{O}(G)$, and $\mathsf{G} \cdot P = t^{-1}t(P)$ by [**21**, XXVI 5.2], thus $\mathsf{G}\backslash\mathbf{P}(G) = \mathbf{O}(G)$.

4.1.5. The morphism $p_1 : \mathbf{OPP}(G) \to \mathbf{P}(G)$ maps (P, P') to P. It is plainly a morphism of posets, and it is surjective by [**21**, XXVI 2.3 & 4.3.2]. Consider now $(P, P') \in \mathbf{OPP}(G)$, $Q \in \mathbf{P}(G)$ and suppose first that $P \subset Q$. Since t is nice, there is a unique $Q' \in \mathbf{P}(G)$ with $P' \subset Q'$ and $t(Q') = \iota t(Q)$, where ι is the opposition involution of $\mathbf{O}(G)$. We have $(Q, Q') \in \mathbf{OPP}(G)$ by [**21**, XXVI 4.3.2 & 4.2.1], thus p_1 is nice. If $Q \subset P$, then $Q_L = Q \cap L$ is a parabolic subgroup of $L = P \cap P'$ and its Levi subgroups are the Levi subgroups of Q contained in L by [**21**, XXVI 1.20]. Since $p_1 : \mathbf{OPP}(L) \to \mathbf{P}(L)$ is surjective, there is a parabolic subgroup Q'_L of L opposed to Q_L. Then $Q'_L = Q' \cap L$ for a unique parabolic subgroup Q' of G contained in P', and $(Q, Q') \in \mathbf{OPP}(G)$ since $Q \cap Q' = Q_L \cap Q'_L$ is a Levi subgroup of Q_L and Q'_L, thus also of Q and Q'. Therefore p_1 is very nice. Finally, the stabilizer of P in G is $\mathsf{P} = P(\mathcal{O})$ by [**21**, XXVI 1.2], and $\mathsf{P} \cdot (P, P') = p_1^{-1}(P)$ by [**21**, XXVI 1.8 & 4.3.2], thus $\mathsf{G}\backslash\mathbf{OPP}(G) = \mathsf{G}\backslash\mathbf{P}(G)$.

4.1.6. The morphism $b : \mathbf{SP}(G) \to \mathbf{OPP}(G)$ maps (S, P) to $(P, \iota_S P)$, where $\iota_S P$ is defined in the next lemma, which also says that b is a morphism of posets.

Lemma 77. *For $S \in \mathbf{S}(G)$ and $P \in \mathbf{P}(G)$ with $Z_G(S) \subset P$, there exists a unique Levi subgroup L of P and a unique parabolic subgroup $\iota_S P$ of G opposed to P with $Z_G(S) \subset L, \iota_S P$. Moreover $L = P \cap \iota_S P$ and $P \mapsto \iota_S P$ preserves inclusions.*

Proof. By [**1**, XIV 3.20], there is a maximal torus T in $Z_G(S)$. It is also maximal in G and P. By [**21**, XXVI 1.6], there is a unique Levi subgroup L of P with $T \subset L$. We have to show that $Z_G(S) \subset L$. By [**21**, XXVI 6.11], this is equivalent to $R_{sp}(L) \subset S$, where $R_{sp}(L)$ is the split radical of L, i.e. the maximal split subtorus $R(L)_{sp}$ of the radical $R(L)$ of L. Since T is a maximal torus in L, $R(L)$ is contained in T, thus $R_{sp}(L)$ is contained in the maximal split subtorus T_{sp} of T, which obviously contains S and in fact equals S by maximality of S. This proves the existence and uniqueness of L. That of $\iota_S P$ follows from [**21**, XXVI 4.3.2] which also shows that $L = P \cap \iota_S P$. If $P \subset Q$, there is a unique $(Q, Q') \in \mathbf{OPP}(G)$ with $\iota_S P \subset Q'$ because p_1 is nice, and obviously $\iota_S Q = Q'$, thus $\iota_S P \subset \iota_S Q$. □

Starting with $(P, P') \in \mathbf{OPP}(G)$ put $L = P \cap P'$ and let S be a maximal split torus in G containing the split radical $R_{sp}(L)$ of L. Then $Z_G(S)$ is contained in $Z_G(R_{sp}(L))$ which equals L by [**21**, XXVI 6.11], thus $(S, P) \in \mathbf{SP}(G)$ and $b(S, P)$ equals (P, P'), i.e. b is surjective. It is obviously nice, although not *very* nice. The stabilizer of $b(S, P)$ in G is $\mathsf{L} = L(\mathcal{O})$ where $L = P \cap \iota_S P$, and $\mathsf{L} \cdot (S, P) = b^{-1} b(S, P)$ by [**21**, XXVI 6.16], thus $\mathsf{G}\backslash\mathbf{SP}(G) = \mathsf{G}\backslash\mathbf{OPP}(G)$. The opposition involution $\iota(P_1, P_2) = (P_2, P_1)$ of $\mathbf{OPP}(G)$ lifts to the involution $\iota(S, P) = (S, \iota_S P)$ of $\mathbf{SP}(G)$.

4.1.7. The morphism $a : \mathbf{SBP}(G) \to \mathbf{SP}(G)$ maps (S, B, P) to (S, P). It is plainly a nice morphism of poset, although not *very* nice. Fix $(S, P) \in \mathbf{SP}(G)$, let $L = P \cap \iota_S P$. Then [**21**, XXVI 1.20] sets up a bijection between: the set of minimal parabolic subgroup B of G with $Z_G(S) \subset B \subset P$ (the fiber $a^{-1}(S, P)$) and the set of minimal parabolic subgroups $B_L = B \cap L$ of L with $Z_G(S) \subset B_L$. The latter set is not empty by [**21**, XXVI 6.16], thus a is surjective. The stabilizer of (S, P) in G equals $\mathsf{N}_L(S) = N_L(S)(\mathcal{O})$ and $\mathsf{N}_L(S) \cdot (S, B, P) = a^{-1}(S, P)$ by [**21**, XXVI 7.2] applied to $Z_G(S) \subset L$, thus $\mathsf{G}\backslash\mathbf{SBP}(G) = \mathsf{G}\backslash\mathbf{SP}(G)$. The stabilizer of (S, B, P) in G is the stabilizer of (S, B), namely $\mathsf{Z}_G(S) = Z_G(S)(\mathcal{O})$ since $Z_G(S) = B \cap \iota_S B$.

4.1.8. By [**21**, XXVI 5.7], there is a smallest element $\circ$ in $\mathbf{O}(G)$. For

$$\mathbf{X} \quad \text{in} \quad \{\mathbf{SBP}(G), \mathbf{SP}(G), \mathbf{OPP}(G), \mathbf{P}(G)\},$$

the morphism $f : \mathbf{X} \to \mathbf{O}(G)$ is very nice. We've proved it already in the last two cases. Since f is nice, our assertion is equivalent to: $\mathbf{X}_{\min} = f^{-1}(\circ)$ where $\mathbf{X}_{\min}$ is the set of minimal elements in $\mathbf{X}$. This is obvious for $\mathbf{SBP}(G)$, and also for $\mathbf{SP}(G)$ since a is surjective. For any $x \in \mathbf{X}_{\min} = f^{-1}(\circ)$, there is then a unique section

$$\begin{tikzcd} (\mathbf{X}, \leq) \arrow[r, two heads, shift right, "f"'] & (\mathbf{O}(G), \leq) \arrow[l, shift right, "s_x"'] \end{tikzcd}$$

with $s_x(\circ) = x$, and these sections cover $\mathbf{X}$.

4.1.9. Let now $\Gamma = (\Gamma, +, \leq)$ be a non-trivial totally ordered commutative group and form the $\mathrm{Aut}(G)$-equivariant cartesian diagram of sets:

$$\begin{tikzcd} \mathbf{ACF}^\Gamma(G) \arrow[r, two heads, "a"] \arrow[d, "F"] & \mathbf{AF}^\Gamma(G) \arrow[r, two heads, "b"] \arrow[d, "F"] & \mathbf{G}^\Gamma(G) \arrow[r, two heads, "\mathrm{Fil}"] \arrow[d, "F"] & \mathbf{F}^\Gamma(G) \arrow[r, two heads, "t"] \arrow[d, "F"] & \mathbf{C}^\Gamma(G) \arrow[d, "F"] \\ \mathbf{SBP}(G) \arrow[r, two heads, "a"] & \mathbf{SP}(G) \arrow[r, two heads, "b"] & \mathbf{OPP}(G) \arrow[r, two heads, "p_1"] & \mathbf{P}(G) \arrow[r, two heads, "t"] & \mathbf{O}(G) \end{tikzcd}$$

where $\mathbf{C}^\Gamma(G)$ is the inverse image of $\mathbf{O}(G)$ under $F : \mathbb{C}^\Gamma(G)(\mathcal{O}) \to \mathbb{O}(G)(\mathcal{O})$. Of course we may and do identify $\mathbf{F}^\Gamma(G)$ with $\mathbb{F}^\Gamma(G)(\mathcal{O})$ and $\mathbf{G}^\Gamma(G)$ with $\mathbb{G}^\Gamma(G)(\mathcal{O})$, see section 2.2.6. With these identifications, we find:

$$\begin{aligned} \mathbf{AF}^\Gamma(G) &= \left\{(S, \mathcal{F}) \in \mathbf{S}(G) \times \mathbf{F}^\Gamma(G) \text{ with } Z_G(S) \subset P_{\mathcal{F}}\right\} \\ &= \left\{(S, \mathcal{G}) \in \mathbf{S}(G) \times \mathbf{G}^\Gamma(G) \text{ with } Z_G(S) \subset L_{\mathcal{G}}\right\} \\ &= \left\{(S, \mathcal{G}) : S \in \mathbf{S}(G), \mathcal{G} \in \mathbf{G}^\Gamma(S)\right\} \\ \mathbf{ACF}^\Gamma(G) &= \left\{(S, B, \mathcal{F}) : \mathbf{S}(G) \times \mathbf{B}(G) \times \mathbf{F}^\Gamma(G) \text{ with } Z_G(S) \subset B \subset P_{\mathcal{F}}\right\} \\ & \quad \left\{(S, B, \mathcal{G}) : S \in \mathbf{S}(G), \mathcal{G} \in \mathbf{G}^\Gamma(S) \text{ with } Z_G(S) \subset B \subset P_{\mathcal{G}}\right\}. \end{aligned}$$

The opposition involution ι of $\mathbf{G}^\Gamma(G)$ lifts to an involution of $\mathbf{AF}^\Gamma(G)$, given by

$$\iota(S, \mathcal{F}) = (S, \iota_S \mathcal{F}) \quad \text{or} \quad \iota(S, \mathcal{G}) = (S, \iota\mathcal{G}).$$

Here $\iota\mathcal{G} = \mathcal{G}^{-1}$ in $\mathbf{G}^\Gamma(S)$ and $(\mathrm{Fil}(\mathcal{G}), \mathrm{Fil}(\iota\mathcal{G})) = (\mathcal{F}, \iota_S \mathcal{F})$.

4.1.10. Fix $S \in \mathbf{S}(G)$. Let M be its group of characters, $R \subset M$ the roots of S in $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathfrak{g} = \mathfrak{g}_0 \oplus \oplus_{\alpha \in R} \mathfrak{g}_\alpha$ the corresponding decomposition of $\mathfrak{g}$. Put

$$W = (N_G(S)/Z_G(S))(\mathcal{O}) = N_G(S)(\mathcal{O})/Z_G(S)(\mathcal{O}).$$

By [**21**, XXVI 7.4], there exists a unique root datum $\mathscr{R} = (M, R, M^*, R^*)$ with Weyl group W and a W-equivariant bijection $B \leftrightarrow R_+$ between the set of all $B \in \mathbf{B}(G)$ with $Z_G(S) \subset B$ and the set of all systems of positive roots $R_+ \subset R$, given by

$$\mathrm{Lie}(B) = \mathfrak{g}_0 \oplus \oplus_{\alpha \in R_+} \mathfrak{g}_\alpha.$$

Fix one such B and let $\Delta \subset R_+$ be the corresponding set of simple roots. By [**21**, XXVI 7.7], there is an inclusion preserving bijection $P \leftrightarrow A$ between the set of all $P \in \mathbf{P}(G)$ with $B \subset P$ and the set of all subsets A of Δ, given by

$$\mathrm{Lie}(P) = \mathfrak{g}_0 \oplus \oplus_{\alpha \in R_A} \mathfrak{g}_\alpha$$

where $R_A = R_+ \coprod (\mathbb{Z}A \cap R_-)$ is the set of roots in $R = R_+ \coprod R_-$ which are either positive or in the group spanned by A. We write P_A for the parabolic associated to A. Since $f : \mathbf{SBP}(G) \to \mathbf{O}(G)$ is (very) nice, we obtain a poset bijection

$$f_{S,B} : (\{A \subset \Delta\}, \subset) \to (\mathbf{O}(G), \leq), \quad A \mapsto t(P_A).$$

Fix one such $P = P_A$. Then the fiber of $F : \mathbf{ACF}^\Gamma(G) \to \mathbf{SBP}(G)$ above (S, B, P) is the set of all $(S, B, \mathcal{G})$ with $\mathcal{G} \in \mathbf{G}^\Gamma(S) = \mathrm{Hom}(M, \Gamma)$ such that

$$\forall \alpha \in \Delta : \quad \begin{cases} \mathcal{G}(\alpha) = 0 & \text{if } \alpha \in A, \\ \mathcal{G}(\alpha) > 0 & \text{if } \alpha \notin A. \end{cases}$$

Since the elements of Δ are linearly independent and Γ is non-trivial, this fiber is not empty and $F : \mathbf{ACF}^\Gamma(G) \to \mathbf{SBP}(G)$ is surjective.

4.1.11. It follows that the five F's in our diagram are surjective. Their fibers are called facets, the type of a facet is its image in $\mathbf{O}(G)$, and all facets of the same type are canonically isomorphic. The facets of type $\circ$ are called chambers. For any $f' : \mathbf{X}' \to \mathbf{C}^\Gamma(G)$ over $f : \mathbf{X} \to \mathbf{O}(G)$ in our diagram, the closed facet of $x \in \mathbf{X}$ is $F^{-1}(\overline{x}) \subset \mathbf{X}'$ where $\overline{x} = \{y \geq x\}$. It is a disjoint union of finitely many facets. Since $x = \min FF^{-1}(\overline{x})$, closed facets have a well-defined type and those of the same type are canonically isomorphic. We equip the set of closed facets with the partial order given by inclusion, which is opposite to the partial order on $\mathbf{X}$. A closed chamber is a maximal closed facet, and the set of all closed chambers equals $\mathbf{X}_{\min} = f^{-1}(\circ)$. Since f is nice, every $x \in \mathbf{X}_{\min}$ defines compatible sections

$$\begin{tikzcd} \mathbf{X}' \arrow[r, two heads, "f'"'] \arrow[d, "F"'] & \mathbf{C}^\Gamma(G) \arrow[l, bend right, "s_x"'] \arrow[d, "F"] \\ \mathbf{X} \arrow[r, two heads, "f"'] & \mathbf{O}(G) \arrow[l, bend right, "s_x"'] \end{tikzcd}$$

and the closed chamber $F^{-1}(\overline{x})$ is the image of $s_x : \mathbf{C}^\Gamma(G) \to \mathbf{X}'$. Since f is very nice, any $x' \in \mathbf{X}'$ belongs to some closed chamber. Since $\mathsf{G}\backslash\mathbf{X}' = \mathbf{C}^\Gamma(G)$, any closed chamber is a fundamental domain for the action of G on $\mathbf{X}'$.

4.1.12. The facets which are minimal among the set of non-minimal facets are called panels. A panel $F^{-1}(x)$ bounds a chamber $F^{-1}(y)$ if $F^{-1}(x) \subset F^{-1}(\overline{y})$, i.e. $y \leq x$. Any panel bounds at least 3 chambers. Indeed, this means that a non-minimal parabolic subgroup P of G contains at least 3 minimal parabolic subgroups. To establish this, fix a Levi subgroup L of P – which exists by [**21**, XXVI 2.3] or the surjectivity of p_1. Then $Q \mapsto L \cap Q$ yields a bijection between the parabolic subgroups Q of G contained in P and the parabolic subgroups of L, by [**21**, XXVI 1.20]. Since P is non-minimal, L is not a minimal parabolic subgroup of itself. By [**21**, XXVI 5.11], it contains at least 3 such subgroups, and so does P.

4.1.13. The apartment attached to $S \in \mathbf{S}(G)$ is the subset $\mathbf{F}^\Gamma(S)$ of all $\mathcal{F}$'s in $\mathbf{F}^\Gamma(G)$ such that $Z_G(S) \subset P_{\mathcal{F}}$. It is canonically isomorphic to $\mathbf{G}^\Gamma(S)$ by the map which sends $\mathcal{G} : \mathbb{D}_{\mathcal{O}}(\Gamma) \to S$ to $\mathsf{Fil}(\mathcal{G})$. Our notations are thus consistent since

$$\mathbf{F}^\Gamma(S) = \mathbf{G}^\Gamma(S) = \mathbb{G}^\Gamma(S)(\mathcal{O}) = \mathbb{F}^\Gamma(S)(\mathcal{O}).$$

Since $F : \mathbf{AF}^\Gamma(G) \to \mathbf{SP}$ is surjective, $\mathbf{F}^\Gamma(S)$ is the disjoint union of the facets $F^{-1}(P)$ with $Z_G(S) \subset P$. Since $Z_G(S) = B \cap B'$ for some pair of opposed minimal parabolic subgroups of G, $\mathbf{F}^\Gamma(S)$ determines $Z_G(S) = \cap_{F^{-1}(P)\subset \mathbf{F}^\Gamma(S)} P$ and its split radical S. Thus $S \mapsto \mathbf{F}^\Gamma(S)$ is an $\mathrm{Aut}(G)$-equivariant bijection from $\mathbf{S}(G)$ onto the set $\mathbf{A}(G)$ of apartments in $\mathbf{F}^\Gamma(G)$. In particular,

$$\begin{aligned}\mathbf{AF}^\Gamma(G) &= \{(A,\mathcal{F}) : A \in \mathbf{A}(G), \mathcal{F} \in A\} \\ \mathbf{ACF}^\Gamma(G) &= \{(A,C,\mathcal{F}) : \mathcal{F} \in C = \text{closed chamber of } A \in \mathbf{A}(G)\}.\end{aligned}$$

Since $\mathbf{AF}^\Gamma(G) \to \mathbf{F}^\Gamma(G)$ is surjective, every $\mathcal{F} \in \mathbf{F}^\Gamma(G)$ belongs to some $A \in \mathbf{A}(G)$. The stabilizer of $\mathbf{F}^\Gamma(S)$ in $\mathsf{G} = G(\mathcal{O})$ equals $\mathsf{N}_G(S) = N_G(S)(\mathcal{O})$ and its pointwise stabilizer equals $\mathsf{Z}_G(S) = Z_G(S)(\mathcal{O})$. Thus $\mathsf{W}_G(S) = \mathsf{N}_G(S)/\mathsf{Z}_G(S)$ acts on $\mathbf{F}^\Gamma(S)$, and this gives the usual action of $\mathsf{W}_G(S)$ on $\mathbf{F}^\Gamma(S) = \mathbf{G}^\Gamma(S) = \mathrm{Hom}(\mathbb{D}_{\mathcal{O}}(\Gamma), S)$.

4.1.14. A panel bounds exactly two chambers in any apartment which contains it. Indeed, let $F^{-1}(Q)$ be a panel in $\mathbf{F}^\Gamma(S)$. Given [**21**, XXVI 1.20], we have to show that there are exactly two minimal parabolic subgroups of $L = Q \cap \iota_S Q$ containing $Z_G(S)$. By assumption, $\mathbf{O}(L) = \{\circ, t(L)\}$. Our claim then follows from 4.1.10.

4.1.15. For any $\mathcal{F}_1, \mathcal{F}_2 \in \mathbf{F}^\Gamma(G)$, there is an apartment $A \in \mathbf{A}(G)$ containing $\mathcal{F}_1$ and $\mathcal{F}_2$ if and only if $P_{\mathcal{F}_1}$ and $P_{\mathcal{F}_2}$ are in standard position [**21**, XXVI 4.5]. Indeed if $\mathcal{F}_1, \mathcal{F}_2 \in \mathbf{F}^\Gamma(S)$ for some $S \in \mathbf{S}(G)$, then $Z_G(S)$ contains a maximal torus T by [**1**, XIV 3.20], thus $T \subset Z_G(S) \subset P_{\mathcal{F}_1} \cap P_{\mathcal{F}_2}$. If conversely $T \subset P_{\mathcal{F}_1} \cap P_{\mathcal{F}_2}$ for some maximal torus T of G, then $\mathcal{F}_1, \mathcal{F}_2 \in \mathbf{F}^\Gamma(S)$ for any $S \in \mathbf{S}(G)$ containing the maximal split torus T_{sp} of T: if R_i is the split radical of the unique Levi subgroup L_i of $P_{\mathcal{F}_i}$ containing T [**21**, XXVI 1.6], then $R_i \subset T_{sp} \subset S$, therefore $Z_G(S) \subset Z_G(R_i) = L_i \subset P_{\mathcal{F}_i}$ by [**21**, XXVI 6.11]. We will denote by

$$\mathbf{Std}(G) = \mathbb{STD}(G)(\mathcal{O}) \quad \text{and} \quad \mathbf{Std}^\Gamma(G) = \mathbb{STD}^\Gamma(G)(\mathcal{O})$$

the corresponding subsets of $\mathbf{P}(G)^2$ and $\mathbf{F}^\Gamma(G)^2$, so that

$$\mathbf{Std}^\Gamma(G) = F^{-1}(\mathbf{Std}(G)) = \cup_{S \in \mathbf{S}(G)} \mathbf{F}^\Gamma(S) \times \mathbf{F}^\Gamma(S) \subset \mathbf{F}^\Gamma(G)^2.$$

For any $S \in \mathbf{S}(G)$, the map $+ : \mathbf{Std}^\Gamma(G) \to \mathbf{F}^\Gamma(G)$ of section 2.3.2 induces the natural commutative group structure on $\mathbf{F}^\Gamma(S) = \mathbf{G}^\Gamma(S) = \mathrm{Hom}(\mathbb{D}_{\mathcal{O}}(\Gamma), S)$.

4.1.16. For $P \in \mathbf{P}(G)$ with unipotent radical U and Levi L, we also define

$$\mathbf{Std}^\Gamma(P) = \left\{\mathcal{F} \in \mathbf{F}^\Gamma(G) : (P, P_{\mathcal{F}}) \in \mathbf{Std}(G)\right\} = \cup_{Z_G(S) \subset P} \mathbf{F}^\Gamma(S).$$

As explained in sections 2.3.3 and 2.3.4, the functorial map $\mathbf{F}^\Gamma(L) \to \mathbf{F}^\Gamma(G)$ lands in $\mathbf{Std}^\Gamma(P)$ and actually defines a section of a P-equivariant map

$$\mathrm{Gr}_P : \mathbf{Std}^\Gamma(P) \twoheadrightarrow \mathbf{F}^\Gamma(P/U)$$

which may be computed as follows: starting with $\mathcal{F} \in \mathbf{Std}^\Gamma(P)$, pick $S \in \mathbf{S}(G)$ such that $Z_G(S) \subset P \cap P_{\mathcal{F}}$, let $\mathcal{G} \in \mathbf{G}^\Gamma(S)$ be the corresponding splitting of $\mathcal{F}$ and let $\overline{\mathcal{G}}$ be the image of $\mathcal{G}$ in $\mathbf{G}^\Gamma(P/U)$. Then $\mathrm{Gr}_P(\mathcal{F}) = \mathrm{Fil}(\overline{\mathcal{G}})$ in $\mathbf{F}^\Gamma(P/U)$. Thus for $\mathcal{F} \in F^{-1}(P)$, $\mathcal{F} \in \mathbf{Std}^\Gamma(P)$ and $\mathrm{Gr}_P(\mathcal{F}) = \overline{\mathcal{F}}$ with $\overline{\mathcal{F}} \in \mathbf{G}^\Gamma(\overline{R}(P))$ as in 2.2.8.

Theorem. [**21**, XXVI 4.1.1] *If $\mathcal{O} = K$ is a field, then $\mathbf{Std}(G) = \mathbf{P}(G)^2$, thus also $\mathbf{Std}^\Gamma(G) = \mathbf{F}^\Gamma(G)^2$ and Gr_P is defined on the whole of $\mathbf{F}^\Gamma(G) = \mathbf{Std}^\Gamma(P)$:*

$$\mathrm{Gr}_P : \mathbf{F}^\Gamma(G) \twoheadrightarrow \mathbf{F}^\Gamma(P/U)$$

4.1.17. Suppose now that $\mathcal{O}$ is a Henselian local ring with residue field k.

PROPOSITION 78. *The specialization from $\mathcal{O}$ to k induces a map from the diagram of section 4.1.9 for G to the similar diagram for G_k. In the resulting commutative diagram, all the specialization maps $\mathbf{X}(G) \to \mathbf{X}(G_k)$ are surjective, all the squares involving two F's are cartesian, and $\mathbf{O}(G) \simeq \mathbf{O}(G_k)$, $\mathbf{C}^\Gamma(G) \simeq \mathbf{C}^\Gamma(G_k)$.*

PROOF. Since $\mathbb{G}^\Gamma$, $\mathbb{F}^\Gamma$, $\mathbb{C}^\Gamma$, $\mathbb{OPP}$, $\mathbb{P}$ and $\mathbb{O}$ are smooth over $\mathrm{Spec}(\mathcal{O})$, the specialization from $\mathcal{O}$ to k induces a map from the last two squares of our diagram for G to the last two squares of the analogous diagram for G_k, in which all specialization maps $\mathbf{X}(G) \to \mathbf{X}(G_k)$ are surjective by [**28**, 18.5.17]. Since $\mathbb{O}$ is finite étale over $\mathrm{Spec}(\mathcal{O})$, $\mathbf{O}(G) \to \mathbf{O}(G_k)$ is also injective by [**28**, 18.5.4-5], i.e. $\mathbf{O}(G) = \mathbf{O}(G_k)$. It follows that $\mathbf{P}(G) \twoheadrightarrow \mathbf{P}(G_k)$ induces $\mathbf{B}(G) \twoheadrightarrow \mathbf{B}(G_k)$. If S is a maximal split torus in G, then $Z_G(S)$ is a Levi subgroup of a minimal parabolic subgroup B of G, S is the maximal split subtorus of the radical R of $Z_G(S)$, thus R/S is an anisotropic torus, i.e. $\mathrm{Hom}(\mathbb{G}_{m,\mathcal{O}}, R/S) = 0$. Then by proposition 3, lemma 4 and [**28**, 18.5.4-5], also $\mathrm{Hom}(\mathbb{G}_{m,k}, R_k/S_k) = 0$, thus S_k is the maximal split subtorus of the radical R_k of the Levi subgroup $Z_G(S)_k = Z_{G_k}(S_k)$ of the minimal parabolic subgroup B_k of G_k, in particular S_k is a maximal split subtorus of G_k and the specialization map $\mathbf{S}(G) \to \mathbf{S}(G_k)$ is well-defined. It is surjective: starting with $\overline{S}$ in $\mathbf{S}(G_k)$, choose $\overline{B} \in \mathbf{B}(G_k)$ containing $Z_{G_k}(\overline{S})$, lift $\overline{B}$ to some $B \in \mathbf{B}(G)$, choose $S' \in \mathbf{S}(G)$ with $Z_G(S') \subset B$, write $\overline{S} = \mathrm{Int}(\overline{b})(S'_k)$ for some $b \in B(k)$, lift $\overline{b}$ to some $b \in B(\mathcal{O})$ using [**28**, 18.5.17] and set $S = \mathrm{Int}(b)(S')$. Then $S \in \mathbf{S}(G)$ and $S_k = \overline{S}$. The same argument shows that $\mathbf{SBP}(G) \to \mathbf{SBP}(G_k)$ and $\mathbf{SP}(G) \to \mathbf{SP}(G_k)$ are well-defined and surjective, from which follows that also $\mathbf{ACF}^\Gamma(G) \to \mathbf{ACF}^\Gamma(G_k)$ and $\mathbf{AF}^\Gamma(G) \to \mathbf{AF}^\Gamma(G_k)$ are well-defined. To establish all of the remaining claims, it is sufficient to show that $\mathbf{C}^\Gamma(G) \twoheadrightarrow \mathbf{C}^\Gamma(G_k)$ is also injective, which again follows from [**28**, 18.5.4-5] since $\mathbb{C}^\Gamma(G)$ is separated and étale over $\mathcal{O}$. Alternatively, fix (S, B) as above and let $s : \mathbf{C}^\Gamma(G) \hookrightarrow \mathbf{F}^\Gamma(G)$ and $s_k : \mathbf{C}^\Gamma(G_k) \hookrightarrow \mathbf{F}^\Gamma(G_k)$ be the corresponding sections. They are compatible with the specialization maps and there images are respectively contained in the apartments $\mathbf{F}^\Gamma(S)$ of $\mathbf{F}^\Gamma(G)$ and $\mathbf{F}^\Gamma(S_k)$ of $\mathbf{F}^\Gamma(G_k)$. Since $\mathbf{G}^\Gamma(S) \simeq \mathbf{G}^\Gamma(S_k)$, the specialization map $\mathbf{F}^\Gamma(G) \to \mathbf{F}^\Gamma(G_k)$ restricts to a bijection $\mathbf{F}^\Gamma(S) \simeq \mathbf{F}^\Gamma(S_k)$, therefore $\mathbf{C}^\Gamma(G) \twoheadrightarrow \mathbf{C}^\Gamma(G_k)$ is indeed injective. □

4.1.18. Suppose now that $\mathcal{O}$ is a valuation ring with fraction field K.

PROPOSITION 79. *The generization from $\mathcal{O}$ to K induces a map from the diagram of section 4.1.9 for G to the similar diagram for G_K. In the resulting commutative diagram, all the generization maps $\mathbf{X}(G) \to \mathbf{X}(G_K)$ are injective, they are bijective for $\mathbf{X} \in \{\mathbf{F}^\Gamma, \mathbf{C}^\Gamma, \mathbf{P}, \mathbf{O}\}$ and all the squares involving two F's are cartesian.*

PROOF. Since $\mathbb{G}^\Gamma$, $\mathbb{F}^\Gamma$, $\mathbb{C}^\Gamma$, $\mathbb{OPP}$, $\mathbb{P}$ and $\mathbb{O}$ are separated over $\mathrm{Spec}(\mathcal{O})$, the generization from $\mathcal{O}$ to K induces a map from the last two squares of our diagram for G to the last two squares of the analogous diagram for G_K, in which all generization maps $\mathbf{X}(G) \to \mathbf{X}(G_K)$ are injective. Since $\mathbb{O}$ and $\mathbb{P}$ are proper over $\mathrm{Spec}(\mathcal{O})$, the maps $\mathbf{P}(G) \to \mathbf{P}(G_K)$ and $\mathbf{O}(G) \to \mathbf{O}(G_K)$ are in fact bijective. It follows that $\mathbf{P}(G) \simeq \mathbf{P}(G_K)$ induces $\mathbf{B}(G) \simeq \mathbf{B}(G_K)$. If S is a maximal split torus in G, then $Z_G(S)$ is a Levi subgroup of a minimal parabolic subgroup B of G, S is the maximal split subtorus of the radical R of $Z_G(S)$, thus R/S is an anisotropic torus, i.e. $\mathrm{Hom}(\mathbb{G}_{m,\mathcal{O}}, R/S) = 0$. Then by proposition 3 and lemma 4, also $\mathrm{Hom}(\mathbb{G}_{m,K}, R_K/S_K) = 0$, thus S_K is the maximal split subtorus of the radical

R_K of the Levi subgroup $Z_G(S)_K = Z_{G_K}(S_K)$ of the minimal parabolic subgroup B_K of G_K, in particular S_K is a maximal split subtorus of G_K and the specialization map $\mathbf{S}(G) \to \mathbf{S}(G_K)$ is well-defined. It is injective by [**21**, XXII 5.8.3], so are $\mathbf{SBP}(G) \to \mathbf{SBP}(G_K)$ and $\mathbf{SP}(G) \to \mathbf{SP}(G_K)$, while $\mathbf{ACF}^\Gamma(G) \to \mathbf{ACF}^\Gamma(G_K)$ and $\mathbf{AF}^\Gamma(G) \to \mathbf{AF}^\Gamma(G_K)$ are well-defined. To establish the remaining claims, it is sufficient to show that $\mathbf{C}^\Gamma(G) \hookrightarrow \mathbf{C}^\Gamma(G_K)$ is also surjective, which again follows from lemma 4 since $\mathbb{C}^\Gamma(G)$ is a quasi-isotrivial twisted constant scheme over $\mathcal{O}$. Alternatively, fix (S,B) as above and let $s : \mathbf{C}^\Gamma(G) \hookrightarrow \mathbf{F}^\Gamma(G)$ and $s_K : \mathbf{C}^\Gamma(G_k) \hookrightarrow \mathbf{F}^\Gamma(G_k)$ be the corresponding sections. They are compatible with the generization maps and there images are respectively contained in the apartments $\mathbf{F}^\Gamma(S)$ of $\mathbf{F}^\Gamma(G)$ and $\mathbf{F}^\Gamma(S_K)$ of $\mathbf{F}^\Gamma(G_K)$. Since $\mathbf{G}^\Gamma(S) \simeq \mathbf{G}^\Gamma(S_K)$, the generization map $\mathbf{F}^\Gamma(G) \to \mathbf{F}^\Gamma(G_K)$ restricts to a bijection $\mathbf{F}^\Gamma(S) \simeq \mathbf{F}^\Gamma(S_K)$, therefore $\mathbf{C}^\Gamma(G) \hookrightarrow \mathbf{C}^\Gamma(G_K)$ is indeed surjective. □

REMARK 80. Under the identifications of theorem 34 (note that G is isotrivial over the valuation ring $\mathcal{O}$ by proposition 48), the inverse of $\mathbf{F}^\Gamma(G) \to \mathbf{F}^\Gamma(G_K)$ maps a Γ-filtration $\mathcal{F}_K$ on ω_K° to the Γ-filtration $\mathcal{F}$ on ω° defined by

$$\forall \tau \in \mathsf{Rep}^\circ(G)(\mathcal{O}),\, \gamma \in \Gamma : \quad \mathcal{F}^\gamma(\tau) = \mathcal{F}_K^\gamma(\tau) \cap V(\tau) \text{ in } V_K(\tau).$$

It is not at all obvious that this formula indeed defines a right exact functor!

4.1.19. Set $\mathbf{Gen}(G) = \mathbb{GEN}(G)(\mathcal{O})$, $\mathbf{Gen}^\Gamma(G) = \mathbb{GEN}^\Gamma(G)(\mathcal{O})$ and define

$$\begin{aligned} \mathbf{Gen}(Y) &= \{P \in \mathbf{P}(G) : \{P\} \times Y \subset \mathbf{Gen}(G)\} \\ \mathbf{Gen}^\Gamma(X) &= \left\{\mathcal{F} \in \mathbf{F}^\Gamma(G) : \{\mathcal{F}\} \times X \subset \mathbf{Gen}^\Gamma(G)\right\} \end{aligned}$$

for $Y \subset \mathbf{P}(G)$ and $X \subset \mathbf{F}^\Gamma(G)$. We say that Y (resp. X) is thin if

$$t(\mathbf{Gen}(Y)) = \mathbf{O}(G) \quad (\text{resp. } t(\mathbf{Gen}^\Gamma(X)) = \mathbf{C}^\Gamma(G)).$$

Plainly, $F^{-1}(\mathbf{Gen}(Y)) = \mathbf{Gen}^\Gamma(F^{-1}(Y))$ and $F(\mathbf{Gen}^\Gamma(X)) = \mathbf{Gen}(F(X))$, thus

$$\left(Y \text{ is thin} \Leftrightarrow F^{-1}(Y) \text{ is thin}\right) \quad \text{and} \quad (X \text{ is thin} \Leftrightarrow F(X) \text{ is thin}).$$

Moreover Y is thin $\Leftrightarrow \circ \in t(\mathbf{Gen}(Y))$ and a thin subset of a thin set is thin.

LEMMA 81. *Suppose that $\mathcal{O}$ is a strictly Henselian valuation ring with residue field k and fraction field K. Then any subset Y_K of $\mathbf{P}(G_K)$ (resp. X_K of $\mathbf{F}^\Gamma(G_K)$) whose image in $\mathbf{P}(G_k)$ is finite is a thin subset of $\mathbf{P}(G_K)$ (resp. $\mathbf{F}^\Gamma(G_K)$).*

PROOF. It is sufficient to treat the case of a subset Y_K of $\mathbf{P}(G_K)$. Let $Y \simeq Y_K$ be its pre-image in $\mathbf{P}(G)$ with finite image Y_k in $\mathbf{P}(G_k)$. For every $y \in Y_k$,

$$U_y = t^{-1}(\circ_k) \times \{y\} \cap \mathbb{GEN}(G_k)$$

is a non-empty open subscheme of the (geometrically) irreducible k-scheme $t^{-1}(\circ_k)$ by [**21**, XXVI 4.2.4.iii], thus $(\cap_{y\in Y_k} U_y)(k) = \mathbf{B}(G_k) \cap \mathbf{Gen}(Y_k)$ is not empty since k is algebraically closed and $t^{-1}(\circ_k)$ is of finite type over k. If $B \in \mathbf{B}(G)$ lifts $B_k \in \mathbf{B}(G_k) \cap \mathbf{Gen}(Y_k)$, then $\{B\} \times Y \subset \mathbf{Gen}(G)$ since $\mathbb{GEN}(G)$ is open in $\mathbb{P}(G)^2$, thus also $\{B_K\} \times Y_K \subset \mathbf{Gen}(G_K)$, i.e. $B_K \in \mathbf{Gen}(Y_K)$. Since also $B_K \in \mathbf{B}(G_K)$, Y_K is indeed a thin subset of $\mathbf{P}(G_K)$. □

4.2. Distances and angles

Suppose from now on that Γ is a subring of $\mathbb{R}$ with the induced total order on the underlying commutative group.

4.2.1. Recall from theorem 34 that for $\tau \in \mathsf{Rep}^\circ(G)(\mathcal{O})$, any $\mathcal{F} \in \mathbf{F}^\Gamma(G)$ defines a Γ-filtration $\mathcal{F}(\tau)$ on the (free) $\mathcal{O}$-module $V(\tau)$. For any $(\mathcal{F}_1, \mathcal{F}_2) \in \mathbf{Std}^\Gamma(G)$ and $\gamma_1, \gamma_2 \in \Gamma$, the $\mathcal{O}$-module

$$\mathrm{Gr}^{\gamma_1,\gamma_2}_{\mathcal{F}_1,\mathcal{F}_2}(\tau) = \frac{\mathcal{F}_1^{\gamma_1}(\tau) \cap \mathcal{F}_2^{\gamma_2}(\tau)}{\mathcal{F}_{1,+}^{\gamma_1}(\tau) \cap \mathcal{F}_2^{\gamma_2}(\tau) + \mathcal{F}_1^{\gamma_1}(\tau) \cap \mathcal{F}_{2,+}^{\gamma_2}(\tau)}$$

is free of finite rank: if $\mathcal{F}_i = \mathrm{Fil}(\mathcal{G}_i)$ with $\mathcal{G}_i \in \mathbf{G}^\Gamma(S) = \mathrm{Hom}(M, \Gamma)$ for some S in $\mathbf{S}(G)$ with $M = \mathrm{Hom}(S, \mathbb{G}_{m,\mathcal{O}})$, then $\mathcal{F}_i(\tau)^\gamma = \oplus_{\mathcal{G}_i(m) \geq \gamma} V(\tau)_m$ for any $i \in \{1, 2\}$ and $\gamma \in \Gamma$ where $V(\tau) = \oplus_{m \in M} V(\tau)_m$ is the eigenspace decomposition of $\tau|_S$, thus

$$\mathrm{Gr}^{\gamma_1,\gamma_2}_{\mathcal{F}_1,\mathcal{F}_2}(\tau) = \oplus_{m:\mathcal{G}_i(m)=\gamma_i} V(\tau)_m.$$

4.2.2. Since Γ is a subring of $\mathbb{R}$:

• Any apartment is endowed with a canonical structure of free Γ-module, and these structures are preserved by the action of $\mathrm{Aut}(G)$ on $\mathbf{F}^\Gamma(G)$. Indeed,

$$\mathbf{F}^\Gamma(S) = \mathbf{G}^\Gamma(S) = \mathrm{Hom}(M(S), \Gamma) \quad \text{with} \quad M(S) = \mathrm{Hom}(S, \mathbb{G}_{m,\mathcal{O}}).$$

• Any $\tau \in \mathsf{Rep}^\circ(G)(\mathcal{O})$ defines a G-invariant function

$$\langle -, - \rangle_\tau : \mathbf{Std}^\Gamma(G) \to \Gamma, \qquad \langle \mathcal{F}_1, \mathcal{F}_2 \rangle_\tau = \textstyle\sum_{\gamma_1,\gamma_2} \mathrm{rank}_{\mathcal{O}} \left(\mathrm{Gr}^{\gamma_1,\gamma_2}_{\mathcal{F}_1,\mathcal{F}_2}(\tau) \right) \cdot \gamma_1\gamma_2$$

whose restriction to $\mathbf{F}^\Gamma(S)$ is bilinear, symmetric and non-negative, given by

$$\langle \mathcal{F}_1, \mathcal{F}_2 \rangle_\tau = \textstyle\sum_{m \in M(S)} \mathrm{rank}_{\mathcal{O}} \left(V(\tau)_m \right) \cdot \mathcal{G}_1(m)\mathcal{G}_2(m)$$

if $\mathcal{F}_i \in \mathbf{F}^\Gamma(S)$ corresponds to $\mathcal{G}_i \in \mathrm{Hom}(M(S), \Gamma)$. Its kernel equals $\mathbf{G}^\Gamma(\ker(\tau|S))$, thus $\langle -, - \rangle_\tau$ is positive definite when τ is a faithful representation of G.

• Write $\|\mathcal{F}\|_\tau = \langle \mathcal{F}, \mathcal{F} \rangle_\tau^{1/2}$. Thus $\|-\|_\tau : \mathbf{F}^\Gamma(G) \to \mathbb{R}_+$ is a G-invariant function. It descends to a G-invariant function $\|-\|_\tau : \mathbf{C}^\Gamma(G) \to \mathbb{R}_+$ with $\|\mathcal{F}\|_\tau = \|t(\mathcal{F})\|_\tau$. We have the Cauchy-Schwartz inequality

$$\forall (\mathcal{F}_1, \mathcal{F}_2) \in \mathbf{Std}^\Gamma(G) : \qquad |\langle \mathcal{F}_1, \mathcal{F}_2 \rangle_\tau| \leq \|\mathcal{F}_1\|_\tau \, \|\mathcal{F}_2\|_\tau \, .$$

• We may thus also define a G-invariant angle

$$\measuredangle_\tau(-,-) : \mathbf{Std}^\Gamma(G) \to [0, \pi], \qquad \langle \mathcal{F}_1, \mathcal{F}_2 \rangle_\tau = \cos\left(\measuredangle_\tau(\mathcal{F}_1, \mathcal{F}_2)\right) \cdot \|\mathcal{F}_1\|_\tau \, \|\mathcal{F}_2\|_\tau$$

and a G-invariant function

$$d_\tau(-,-) : \mathbf{Std}^\Gamma(G) \to \mathbb{R}_+, \qquad d_\tau(\mathcal{F}_1, \mathcal{F}_2) = \sqrt{\|\mathcal{F}_1\|_\tau^2 + \|\mathcal{F}_2\|_\tau^2 - 2\,\langle \mathcal{F}_1, \mathcal{F}_2 \rangle_\tau}$$

inducing the distance $d_\tau(\mathcal{F}_1, \mathcal{F}_2) = \|\mathcal{F}_2 - \mathcal{F}_1\|_\tau$ on any apartment.

• If τ is faithful, then $d_\tau(\mathcal{F}_1, \mathcal{F}_2) = 0$ if and only if $\mathcal{F}_1 = \mathcal{F}_2$ and the map $\mathcal{G} \mapsto (\mathrm{Fil}(\mathcal{G}), \mathrm{Fil}(\iota\mathcal{G}))$ induces a G-equivariant bijection

$$\mathbf{G}^\Gamma(G) \simeq \left\{ (\mathcal{F}_1, \mathcal{F}_2) \in \mathbf{Std}^\Gamma(G) : \|\mathcal{F}_1\|_\tau = \|\mathcal{F}_2\|_\tau \text{ and } \measuredangle_\tau(\mathcal{F}_1, \mathcal{F}_2) = \pi \right\}.$$

4.2.3. For $(\mathcal{F}_1, \mathcal{F}_2) \in \mathbf{Std}^\Gamma(G)$, we also have the following formula

$$\langle \mathcal{F}_1, \mathcal{F}_2 \rangle_\tau = \textstyle\sum_\gamma \gamma \cdot \deg \mathrm{Gr}^\gamma_{\mathcal{F}_1}(\mathcal{F}_2, \tau) = \sum_\gamma \gamma \cdot \deg \mathrm{Gr}^\gamma_{\mathcal{F}_2}(\mathcal{F}_1, \tau)$$

where $\mathrm{Gr}^\gamma_{\mathcal{F}}(\mathcal{G}, \tau)$ is the filtration induced by $\mathcal{G}(\tau)$ on $\mathrm{Gr}^\gamma_{\mathcal{F}}(\tau) = \mathcal{F}^\gamma(\tau)/\mathcal{F}^\gamma_+(\tau)$, i.e.

$$\forall \theta \in \Gamma, \qquad \mathrm{Gr}^\gamma_{\mathcal{F}}(\mathcal{G}, \tau)^\theta = \mathcal{F}^\gamma(\tau) \cap \mathcal{G}^\theta(\tau) + \mathcal{F}^\gamma_+(\tau)/\mathcal{F}^\gamma_+(\tau)$$

and the degree of an $\mathbb{R}$-filtration $\mathcal{F}$ on a finite free $\mathcal{O}$-module V is given by

$$\deg(\mathcal{F}) = \textstyle\sum_\gamma \gamma \cdot \mathrm{rank}_{\mathcal{O}}(\mathrm{Gr}^\gamma_{\mathcal{F}}).$$

Choosing a splitting of $\mathcal{F}$, one checks easily that also

$$\deg(\mathcal{F}) = \deg(\det \mathcal{F})$$

where $\det \mathcal{F}$ is the $\mathbb{R}$-filtration on $\det V = \Lambda^r_{\mathcal{O}} V$, $r = \mathrm{rank}_{\mathcal{O}} V$ defined by

$$(\det \mathcal{F})^\gamma = \text{span of } \left\{ v_1 \wedge \cdots \wedge v_r : v_i \in \mathcal{F}^{\gamma_i}, \textstyle\sum \gamma_i = \gamma \right\}.$$

4.2.4. If Γ is a $\mathbb{Q}$-vector space, the decomposition

$$\mathbb{F}^\Gamma(G) = \mathbb{F}^\Gamma(G^{\mathrm{der}}) \times \mathbb{G}^\Gamma(Z(G))$$

of section 2.2.13 induces an analogous decomposition

$$\mathbf{F}^\Gamma(G) = \mathbf{F}^\Gamma(G^{\mathrm{der}}) \times \mathbf{G}^\Gamma(Z(G))$$

which is orthogonal in the following sense: for $\mathcal{F}_1, \mathcal{F}_2 \in \mathbf{Std}^\Gamma(G)$ and $\mathcal{F} \in \mathbf{F}^\Gamma(G)$,

$$\begin{aligned} \langle \mathcal{F}_1, \mathcal{F}_2 \rangle_\tau &= \langle \mathcal{F}_1^r, \mathcal{F}_2^r \rangle_\tau + \langle \mathcal{F}_1^c, \mathcal{F}_2^c \rangle_\tau \\ d_\tau(\mathcal{F}_1, \mathcal{F}_2)^2 &= d_\tau(\mathcal{F}_1^r, \mathcal{F}_2^r)^2 + d_\tau(\mathcal{F}_1^c, \mathcal{F}_2^c)^2 \\ \|\mathcal{F}\|_\tau^2 &= \|\mathcal{F}^r\|_\tau^2 + \|\mathcal{F}^c\|_\tau^2 \end{aligned}$$

where $\mathcal{F}^r \in \mathbf{F}^\Gamma(G^{\mathrm{der}}) \subset \mathbf{F}^\Gamma(G)$ and $\mathcal{F}^c \in \mathbf{G}^\Gamma(Z(G)) \subset \mathbf{F}^\Gamma(G)$ are the components of $\mathcal{F}$. We prove the first formula. Pick $S \in \mathbf{S}(G)$ with $\mathcal{F}_1, \mathcal{F}_2 \in \mathbf{F}^\Gamma(S)$. Then also $\mathcal{F}_i^r, \mathcal{F}_i^c \in \mathbf{F}^\Gamma(S)$ with $\mathcal{F}_i = \mathcal{F}_i^r + \mathcal{F}_i^c$ in the apartment $\mathbf{F}^\Gamma(S)$, thus

$$\langle \mathcal{F}_1, \mathcal{F}_2 \rangle_\tau = \langle \mathcal{F}_1^r, \mathcal{F}_2^r \rangle_\tau + \langle \mathcal{F}_1^c, \mathcal{F}_2^c \rangle_\tau + \langle \mathcal{F}_1^r, \mathcal{F}_2^c \rangle_\tau + \langle \mathcal{F}_1^c, \mathcal{F}_2^r \rangle_\tau$$

since $\langle -, - \rangle_\tau$ is a bilinear form on $\mathbf{F}^\Gamma(S)$. It is therefore sufficient to show that

$$\langle \mathcal{F}, \mathcal{G} \rangle_\tau = \langle \mathcal{G}, \mathcal{F} \rangle_\tau = \textstyle\sum_\gamma \gamma \cdot \deg \mathrm{Gr}_{\mathcal{G}}^\gamma(\mathcal{F}, \tau) = 0$$

for any $\mathcal{F} \in \mathbf{F}^\Gamma(G^{\mathrm{der}})$ and $\mathcal{G} \in \mathbf{G}^\Gamma(Z(G))$. Since $\mathcal{G} : \mathbb{D}_{\mathcal{O}}(\Gamma) \to Z(G)$ is central in G, $\tau = \oplus \tau_\gamma$ with $V(\tau_\gamma) = \mathcal{G}_\gamma(\tau)$ and $\mathrm{Gr}_{\mathcal{G}}^\gamma(\mathcal{F}, \tau) \simeq \mathcal{F}(\tau_\gamma)$ on $\mathrm{Gr}_{\mathcal{G}}^\gamma(\tau) \simeq \tau_\gamma$, thus

$$\deg \mathrm{Gr}_{\mathcal{G}}^\gamma(\mathcal{F}, \tau) = \deg \mathcal{F}(\tau_\gamma) = \deg\left(\det \mathcal{F}(\tau_\gamma)\right) = \deg\left(\mathcal{F}(\det \tau_\gamma)\right) = 0$$

because the restriction of $\det \tau_\gamma$ to G^{der} is trivial.

4.2.5. For $x, y \in \mathbf{O}(G)$, there is a single G-orbit of (P, Q)'s in $t^{-1}(x) \times t^{-1}(y)$ such that P and Q are in osculatory (resp. transverse) position [**21**, XXVI 5.3-5], and this orbit is contained in $\mathbf{Std}(G)$. Thus for any $x, y \in \mathbf{C}^\Gamma(G)$, there is a single G-orbit of $(\mathcal{F}_1, \mathcal{F}_2)$'s in $t^{-1}(x) \times t^{-1}(y)$ with the property that $P_{\mathcal{F}_1}$ and $P_{\mathcal{F}_2}$ are in osculatory (resp. transverse) position, and it is contained in $\mathbf{Std}^\Gamma(G)$. We set $\measuredangle_\tau^{os}(x, y) = \measuredangle_\tau(\mathcal{F}_1, \mathcal{F}_2)$ and $\langle x, y \rangle_\tau^{os} = \langle \mathcal{F}_1, \mathcal{F}_2 \rangle_\tau$ (resp. $\measuredangle_\tau^{tr}(x, y) = \measuredangle_\tau(\mathcal{F}_1, \mathcal{F}_2)$ and $\langle x, y \rangle_\tau^{tr} = \langle \mathcal{F}_1, \mathcal{F}_2 \rangle_\tau$), thus obtaining two other pairs of symmetric functions

$$\begin{aligned} \measuredangle_\tau^{os}(-,-) &: \mathbf{C}^\Gamma(G) \times \mathbf{C}^\Gamma(G) \to [0, \pi] \quad \text{and} \quad \langle -, - \rangle_\tau^{os} : \mathbf{C}^\Gamma(G) \times \mathbf{C}^\Gamma(G) \to \mathbb{R}, \\ \measuredangle_\tau^{tr}(-,-) &: \mathbf{C}^\Gamma(G) \times \mathbf{C}^\Gamma(G) \to [0, \pi] \quad \text{and} \quad \langle -, - \rangle_\tau^{tr} : \mathbf{C}^\Gamma(G) \times \mathbf{C}^\Gamma(G) \to \mathbb{R}. \end{aligned}$$

They are of course related by the formulas

$$\begin{aligned} \langle x, y \rangle_\tau^{os} &= \cos\left(\measuredangle_\tau^{os}(x, y)\right) \cdot \|x\|_\tau \|y\|_\tau, \\ \langle x, y \rangle_\tau^{tr} &= \cos\left(\measuredangle_\tau^{tr}(x, y)\right) \cdot \|x\|_\tau \|y\|_\tau. \end{aligned}$$

We also define yet another symmetric function

$$d_\tau : \mathbf{C}^\Gamma(G) \times \mathbf{C}^\Gamma(G) \to \mathbb{R}_+ \quad d_\tau(x, y) = \sqrt{\|x\|_\tau^2 + \|y\|_\tau^2 - 2\langle x, y \rangle_\tau^{os}}.$$

By construction, for $(\mathcal{F}_1, \mathcal{F}_2)$ in $\mathbf{Gen}^\Gamma(G) = \mathbb{GEN}^\Gamma(G)(\mathcal{O})$,

$$\measuredangle_\tau(\mathcal{F}_1, \mathcal{F}_2) = \measuredangle_\tau^{tr}(t(\mathcal{F}_1), t(\mathcal{F}_2)) \quad \text{and} \quad \langle \mathcal{F}_1, \mathcal{F}_2 \rangle_\tau = \langle \mathcal{F}_1, \mathcal{F}_2 \rangle_\tau^{tr}.$$

4.2.6. The above constructions are merely special cases of those of section 3.12. In particular, our functions are induced by morphisms of schemes over $\mathcal{O}$, and some of them (the "bilinear forms") still make sense for an arbitrary τ in the Grothendieck ring $K_0(G)$ of $\mathsf{Rep}^\circ(G)(\mathcal{O})$, or even for a global section of the sheafified version $\underline{K}_0(G)$ of that ring. However, the positivity of these forms requires some sort of effectiveness/faithfulness of the initial τ, which we have not tried to axiomatize. From now on until 4.2.11, we fix a faithful τ in $\mathsf{Rep}^\circ(G)(\mathcal{O})$.

4.2.7. Fix $(S, B) \in \mathbf{S}(G) \times \mathbf{B}(G)$ with $Z_G(S) \subset B$, let $B' = \iota_S B$ be the minimal parabolic subgroup of G containing $Z_G(S)$ opposed to B, and denote by

$$s, s' : \mathbf{C}^\Gamma(G) \hookrightarrow \mathbf{F}^\Gamma(G)$$

the corresponding sections of $t : \mathbf{F}^\Gamma(G) \twoheadrightarrow \mathbf{C}^\Gamma(G)$. Then for $x, y \in \mathbf{C}^\Gamma(G)$,

$$B \subset P_{s(x)} \cap P_{s(y)} \quad \text{and} \quad B' \subset P_{s'(x)} \cap P_{s'(y)},$$

thus $P_{s(x)}$ and $P_{s(y)}$ are in osculatory position while $P_{s(x)}$ and $P_{s'(y)}$ are in transverse position. It follows that for every $x, y \in \mathbf{C}^\Gamma(G)$,

$$\begin{array}{rclcrcl} \measuredangle_\tau^{os}(x,y) &=& \measuredangle_\tau(s(x), s(y)) & \text{and} & \langle x, y\rangle_\tau^{os} &=& \langle s(x), s(y)\rangle_\tau, \\ \measuredangle_\tau^{tr}(x,y) &=& \measuredangle_\tau(s(x), s'(y)) & \text{and} & \langle x, y\rangle_\tau^{tr} &=& \langle s(x), s'(y)\rangle_\tau. \end{array}$$

In particular, the "scalar products" are compatible with the monoid structure:

$$\begin{array}{rclcrcl} \langle x_1 + x_2, y\rangle_\tau^{os} &=& \langle x_1, y\rangle_\tau^{os} + \langle x_2, y\rangle_\tau^{os} & \text{and} & \langle x, y_1 + y_2\rangle_\tau^{os} &=& \langle x, y_1\rangle_\tau^{os} + \langle x, y_2\rangle_\tau^{os}, \\ \langle x_1 + x_2, y\rangle_\tau^{tr} &=& \langle x_1, y\rangle_\tau^{tr} + \langle x_2, y\rangle_\tau^{tr} & \text{and} & \langle x, y_1 + y_2\rangle_\tau^{tr} &=& \langle x, y_1\rangle_\tau^{tr} + \langle x, y_2\rangle_\tau^{tr}. \end{array}$$

Moreover, d_τ is a distance on $\mathbf{C}^\Gamma(G)$.

4.2.8. The following lemma is related to the angle rigidity axiom of [**29**, 4.1.2].

LEMMA 82. *For any $x, y \in \mathbf{C}^\Gamma(G)$, the set*

$$D_\tau(x,y) = \left\{ \measuredangle_\tau(\mathcal{F}_1, \mathcal{F}_2) : (\mathcal{F}_1, \mathcal{F}_2) \in \mathbf{Std}^\Gamma(G) \cap t^{-1}(x) \times t^{-1}(y) \right\}$$

is finite with

$$\min D_\tau(x,y) = \measuredangle_\tau^{os}(x,y) \quad \textit{and} \quad \max D_\tau(x,y) = \measuredangle_\tau^{tr}(x,y).$$

PROOF. Fix (S, B) and $s, s' : \mathbf{C}^\Gamma(G) \hookrightarrow \mathbf{F}^\Gamma(G)$ as above. Then any pair

$$(\mathcal{F}_1, \mathcal{F}_2) \in \mathbf{Std}^\Gamma(G) \cap t^{-1}(x) \times t^{-1}(y)$$

is G-conjugated to some pair in $\mathsf{W}_G(S) \cdot s(x) \times \mathsf{W}_G(S) \cdot s(y) \subset \mathbf{F}^\Gamma(S)^2$, thus

$$\begin{array}{rcl} D_\tau(x,y) &=& \left\{ \measuredangle_\tau(w_1 \cdot s(x), w_2 \cdot s(y)) : (w_1, w_2) \in \mathsf{W}_G(S)^2 \right\} \\ &=& \left\{ \measuredangle_\tau(s(x), w \cdot s(y)) : w \in \mathsf{W}_G(S) \right\} \end{array}$$

is finite. To establish our final claim, we have to show that

$$\langle s(x), s(y)\rangle_\tau \geq \langle s(x), w \cdot s(y)\rangle_\tau \geq \langle s(x), s'(y)\rangle_\tau$$

for every $w \in \mathsf{W}_G(S)$, which follows from [**7**, Proposition 18]. $\square$

COROLLARY 83. *The type map $t : \mathbf{F}^\Gamma(G) \twoheadrightarrow \mathbf{C}^\Gamma(G)$ is compatible with the d_τ's:*

$$\forall (\mathcal{F}_1, \mathcal{F}_2) \in \mathbf{Std}^\Gamma(G) : \qquad d_\tau(t(\mathcal{F}_1), t(\mathcal{F}_2)) \leq d_\tau(\mathcal{F}_1, \mathcal{F}_2).$$

4.2.9. Let us use the above notions to show that

PROPOSITION 84. *For a facet F, a chamber C and apartments A_1, A_2 in $\mathbf{F}^\Gamma(G)$ with $F \cup C \subset A_1 \cap A_2$, there exists $g \in \mathsf{G}$ with $gA_1 = A_2$ and $g \equiv 1$ on $\overline{F} \cup \overline{C}$.*

PROOF. In group theoretical terms, this means that for $P \in \mathbf{P}(G)$, $B \in \mathbf{B}(G)$ and $S_1, S_2 \in \mathbf{S}(G)$ with $Z_G(S_i) \subset B \cap P$, there is a $g \in \mathsf{G}$ such that $\mathrm{Int}(g)(S_1) = S_2$ and $g \in \mathsf{B} \cap \mathsf{P}$ with $\mathsf{B} = B(\mathcal{O})$, $\mathsf{P} = P(\mathcal{O})$. This does not depend upon Γ, and we may thus assume that $\Gamma = \mathbb{R}$. Since (S_1, B) and $(S_2, B) \in \mathbf{SP}(G)$ have the same image in $\mathbf{O}(G)$, there exists an element $g \in \mathsf{G}$ with $g(S_1, B) = (S_2, B)$, i.e. $\mathrm{Int}(g)(S_1) = S_2$ and $g \in \mathsf{B}$. We will show that also $g \in \mathsf{P}$, i.e. $g\mathcal{F} = \mathcal{F}$ for any $\mathcal{F} \in F^{-1}(P) \subset \mathbf{F}^{\mathbb{R}}(G)$. Note that $\mathcal{F}$, $g\mathcal{F}$ and the chamber $C = F^{-1}(B)$ are all contained in the apartment $\mathbf{F}^{\mathbb{R}}(S_2)$. Fix a faithful $\tau \in \mathsf{Rep}^\circ(G)(\mathcal{O})$. Then

$$\langle \mathcal{F}, \mathcal{F}' \rangle_\tau = \langle g\mathcal{F}, g\mathcal{F}' \rangle_\tau = \langle g\mathcal{F}, \mathcal{F}' \rangle_\tau$$

for all $\mathcal{F}' \in F^{-1}(B)$, thus $\mathcal{F} = g\mathcal{F}$ because $F^{-1}(B)$ is a non-empty open subset of the Euclidean space $\left(\mathbf{F}^{\mathbb{R}}(S_2), \langle -, - \rangle_\tau\right)$ by 4.1.10. □

4.2.10. Suppose for this and the next subsection that our local ring $\mathcal{O} = k$ is a field. Then every pair $(\mathcal{F}_1, \mathcal{F}_2) \in \mathbf{F}^\Gamma(G)$ is contained in some apartment since

THEOREM 85. [**21**, XXVI 4.1.1] $\mathbf{Std}(G) = \mathbf{P}(G)^2$ *and* $\mathbf{Std}^\Gamma(G) = \mathbf{F}^\Gamma(G)^2$.

COROLLARY 86. *For any apartments A_1, A_2 in $\mathbf{F}^\Gamma(G)$ and facets F_1, F_2 in $A_1 \cap A_2$, there exists $g \in \mathsf{G}$ mapping A_1 to A_2 with $g \equiv 1$ on $\overline{F}_1 \cup \overline{F}_2$.*

PROOF. Fix closed chambers $F_1 \subset \overline{C}_1 \subset A_1$ and $F_2 \subset \overline{C}_2 \subset A_2$ and choose an apartment A_3 containing C_1 and C_2. The previous proposition shows that there exists elements $g_1, g_2 \in \mathsf{G}$ such that $g_1 A_1 = A_3 = g_2 A_2$, $g_1 \equiv 1$ on $\overline{C}_1 \cup \overline{F}_2$ and $g_2 \equiv 1$ on $\overline{C}_2 \cup \overline{F}_1$. Then $g = g_2^{-1} g_1$ maps A_1 to A_2 and $g \equiv 1$ on $\overline{F}_1 \cup \overline{F}_2$. □

COROLLARY 87. *For a monomorphism $f : G_1 \to G_2$ of reductive groups over k, the induced map $f : \mathbf{F}^\Gamma(G_1) \to \mathbf{F}^\Gamma(G_2)$ is injective.*

PROOF. Fix a faithful $\tau \in \mathsf{Rep}^\circ(G_2)(k)$. Then $f^*\tau = \tau \circ f \in \mathsf{Rep}^\circ(G_1)(k)$ is also faithful and for every $\mathcal{F}, \mathcal{F}' \in \mathbf{F}^\Gamma(G_1)$,

$$\langle f(\mathcal{F}), f(\mathcal{F}') \rangle_\tau = \langle \mathcal{F}, \mathcal{F}' \rangle_{f^*\tau} \quad \text{and} \quad d_\tau(f(\mathcal{F}), f(\mathcal{F}')) = d_{f^*\tau}(\mathcal{F}, \mathcal{F}').$$

Therefore $f(\mathcal{F}) = f(\mathcal{F}')$ implies $\mathcal{F} = \mathcal{F}'$. □

COROLLARY 88. *Let P be a parabolic subgroup of G with unipotent radical U and Levi subgroup L. Then $\mathbf{F}^\Gamma(L)$ is a fundamental domain for the action of $U(k)$ on $\mathbf{F}^\Gamma(G)$. Let $r = r_{P,L} : \mathbf{F}^\Gamma(G) \twoheadrightarrow \mathbf{F}^\Gamma(L)$ be the corresponding retraction. Then*

$$\forall x, y \in \mathbf{F}^\Gamma(G) : \qquad d_\tau(rx, ry) \leq d_\tau(x, y).$$

COROLLARY 89. *The function $d_\tau : \mathbf{F}^\Gamma(G) \times \mathbf{F}^\Gamma(G) \to \mathbb{R}_+$ is a distance:*

$$\forall x, y, z \in \mathbf{F}^\Gamma(G) : \qquad d_\tau(x, y) \leq d_\tau(x, z) + d_\tau(z, y).$$

PROOF. Fix $S_0 \in \mathbf{S}(L)$. The $\mathsf{P} = P(k)$ and $\mathsf{L} = L(k)$ orbits of S_0 in $\mathbf{S}(G)$ are respectively equal to $\mathbf{S}(G, P) = \{S \in \mathbf{S}(G) : Z_G(S) \subset P\}$ and $\mathbf{S}(L)$. Since any $\mathcal{F} \in \mathbf{F}^\Gamma(G)$ belongs to $\mathbf{F}^\Gamma(S)$ for some $S \in \mathbf{S}(G, P)$, we find that with $\mathsf{U} = U(k)$,

$$\mathbf{F}^\Gamma(G) = \cup_{S \in \mathbf{S}(G,P)} \mathbf{F}^\Gamma(S) = \mathsf{P} \cdot \mathbf{F}^\Gamma(S_0) = \mathsf{U} \cdot \cup_{S \in \mathbf{S}(L)} \mathbf{F}^\Gamma(S) = \mathsf{U} \cdot \mathbf{F}^\Gamma(L).$$

Suppose that $\mathcal{F}, u\mathcal{F} \in \mathbf{F}^{\Gamma}(L)$ for some $u \in \mathsf{U}$, and choose an $S \in \mathbf{S}(L)$ with $\mathcal{F}, u\mathcal{F} \in \mathbf{F}^{\Gamma}(S)$. Since $Z_G(S) \subset L \subset P$, there is a $B \in \mathbf{B}(G)$ with $Z_G(S) \subset B \subset P$. Let $C = F^{-1}(B)$ be the corresponding (G-)chamber in $A = \mathbf{F}^{\Gamma}(S)$. Since $U \subset B$, $uC = C$ and $\mathcal{F}, C \in A \cap u^{-1}A$. Choose $g \in \mathsf{G}$ with $gu^{-1}A = A$, $g\mathcal{F} = \mathcal{F}$ and $gC = C$. Then g belongs to $\mathsf{B} = B(k)$, thus gu^{-1} belongs to $\mathsf{B} \cap \mathsf{N}_G(S) = \mathsf{Z}_G(S)$ which acts trivially on A. Therefore $u\mathcal{F} = gu^{-1}u\mathcal{F} = g\mathcal{F} = \mathcal{F}$ and $\mathbf{F}^{\Gamma}(L)$ is a fundamental domain for the action of U on $\mathbf{F}^{\Gamma}(G)$.

For $A \in \mathbf{A}(G)$ containing $F^{-1}(P)$, there is a unique $A_L \in \mathbf{A}(L) \cap \mathsf{U} \cdot \{A\}$ such that $r(x) = ux$ for any $x \in A$ and $u \in \mathsf{U}$ such that $uA = A_L$. Indeed, there is a $p = lu$ in $\mathsf{P} = \mathsf{LU}$ such that pA is an apartment of $\mathbf{F}^{\Gamma}(L)$, then $uA = l^{-1}pA \subset \mathbf{F}^{\Gamma}(L)$ and $r(x) = ux$ for every $x \in A$. Thus for $x, y \in A$, $d_\tau(rx, ry) = d_\tau(x, y)$.

For the remaining claims, we may assume that $\Gamma = \mathbb{R}$ and use induction on the semi-simple rank s of G. If $s = 0$ everything is obvious. If $s > 0$ but $G = L$, then r is the identity thus $d_\tau(rx, ry) = d_\tau(x, y)$ for every $x, y \in \mathbf{F}^{\mathbb{R}}(G)$. If $G \neq L$, choose an apartment A in $\mathbf{F}^{\mathbb{R}}(G)$ containing x and y, let $[x, y]$ be the corresponding segment of A, and write $[x, y] = \cup_{i=0}^{n-1}[x_i, x_{i+1}]$ for consecutive points $x_i \in [x, y]$ with $x_0 = x$, $x_n = y$ and $]x_i, x_{i+1}[$ contained in a facet $F_i \subset A$. Then there is an apartment containing $F^{-1}(P)$ and $\{x_i, x_{i+1}\} \subset \overline{F}_i$, thus $d_\tau(rx_i, rx_{i+1}) = d_\tau(x_i, x_{i+1})$ for every $i \in \{0, \cdots, n-1\}$. Since d_τ is a distance on $\mathbf{F}^{\mathbb{R}}(L)$ by our induction hypothesis,

$$d_\tau(rx, ry) \leq \textstyle\sum_{i=0}^{n-1} d_\tau(rx_i, rx_{i+1}) = \sum_{i=0}^{n-1} d_\tau(x_i, x_{i+1}) = d_\tau(x, y).$$

Finally for $x, y, z \in \mathbf{F}^{\mathbb{R}}(G)$, choose an apartment $\mathbf{F}^{\mathbb{R}}(S)$ containing x, y and a chamber $F^{-1}(B)$, let $r = r_{B, Z_G(S)}$ be the corresponding retraction. Then

$$d_\tau(x, y) = d_\tau(rx, ry) \leq d_\tau(rx, rz) + d_\tau(rz, ry) \leq d_\tau(x, z) + d_\tau(z, y).$$

This finishes the proof of corollaries 88 and 89. □

COROLLARY 90. *If $\Gamma = \mathbb{R}$, then $(\mathbf{F}^{\mathbb{R}}(G), d_\tau)$ is a complete CAT(0)-space.*

PROOF. Plainly, $(\mathbf{C}^{\mathbb{R}}(G), d_\tau)$ is a complete metric space. Let (x_n) be a Cauchy sequence in $\left(\mathbf{F}^{\mathbb{R}}(G), d_\tau\right)$. Then $t(x_n)$ is a Cauchy sequence in $(\mathbf{C}^{\mathbb{R}}(G), d_\tau)$ by corollary 83, it thus converges to some $y \in \mathbf{C}^{\mathbb{R}}(G)$. Now choose for each n a minimal pair $(S_n, B_n) \in \mathbf{SP}(G)$ corresponding to a section $s_n : \mathbf{C}^{\mathbb{R}}(G) \to \mathbf{F}^{\mathbb{R}}(G)$ passing through x_n, and let $y_n = s_n(y)$. Then $d_\tau(x_n, y_n) = d_\tau(t(x_n), y)$ converges to 0 and y_n is also a Cauchy sequence in $\mathbf{F}^{\mathbb{R}}(G)$. But $d_\tau(y_n, y_m)$ takes finitely many values by lemma 82, therefore y_n is stationary and x_n converges to its limit. Thus $(\mathbf{F}^{\mathbb{R}}(G), d_\tau)$ is complete, and a geodesic space by theorem 85. Finally, for any triple x, y, z in $\mathbf{F}^{\mathbb{R}}(G)$, choose a minimal pair $(S, B) \in \mathbf{SP}(G)$ such that $x, y \in \mathbf{F}^{\mathbb{R}}(S)$ and the middle point m of the segment $[x, y]$ of $\mathbf{F}^{\mathbb{R}}(S)$ belongs to $F^{-1}(B)$. Let $r = r_{B, Z_G(S)} : \mathbf{F}^{\mathbb{R}}(G) \to \mathbf{F}^{\mathbb{R}}(S)$ be the corresponding retraction and pick u in $\mathsf{U} = U(k)$ with $uz = r(z)$, where U is the unipotent radical of B. Then

$$\begin{aligned} d_\tau(z, m)^2 = d_\tau(uz, um)^2 = d_\tau(r(z), m)^2 &= \tfrac{1}{2} d_\tau(r(z), x)^2 + \tfrac{1}{2} d_\tau(r(z), y)^2 - \tfrac{1}{4} d_\tau(x, y)^2 \\ &\leq \tfrac{1}{2} d_\tau(z, x)^2 + \tfrac{1}{2} d_\tau(z, y)^2 - \tfrac{1}{4} d_\tau(x, y)^2 \end{aligned}$$

thus $\mathbf{F}^{\mathbb{R}}(G)$ is a CAT(0)-space by [**8**, II.1.9]. □

COROLLARY 91. *For any $\mathcal{F} \in \mathbf{F}^{\mathbb{R}}(G)$, the function $\mathcal{G} \mapsto \langle \mathcal{F}, \mathcal{G} \rangle_\tau$ from $\mathbf{F}^{\mathbb{R}}(G)$ to $\mathbb{R}$ is homogeneous, concave and $\|\mathcal{F}\|_\tau$-Lipschitzian.*

PROOF. Homogeneity means that $\langle \mathcal{F}, t\mathcal{G} \rangle_\tau = t \langle \mathcal{F}, \mathcal{G} \rangle_\tau$ for all $t \in \mathbb{R}_+$, which is obvious from the definitions. Concavity means that for any $\mathcal{G}_0$, $\mathcal{G}_1 \in \mathbf{F}^{\mathbb{R}}(G)$ and

$t \in [0,1]$, if $\mathcal{G}_t$ is the unique point at distance $td_\tau(\mathcal{G}_0, \mathcal{G}_1)$ from $\mathcal{G}_0$ on the geodesic segment $[\mathcal{G}_0, \mathcal{G}_1]$ of the uniquely geodesic space $(\mathbf{F}^{\mathbb{R}}(G), d_\tau)$ [**8**, II.1.4], then

$$\langle \mathcal{F}, \mathcal{G}_t \rangle_\tau \geq t \langle \mathcal{F}, \mathcal{G}_1 \rangle_\tau + (1-t) \langle \mathcal{F}, \mathcal{G}_0 \rangle_\tau .$$

Let $(0, f, g_0, g_1)$ be a comparison tetrahedron for $(0, \mathcal{F}, \mathcal{G}_0, \mathcal{G}_1)$ in the Euclidean space $\mathbb{R}^3$, by which we mean that the lengths of the edges containing 0 and the angles between them are the same for both tetrahedron. Then the lengths of the other three edges are also the same for both tetrahedron, since every triangle $(0, \mathcal{X}, \mathcal{Y})$ in $\mathbf{F}^{\mathbb{R}}(G)$ is flat by theorem 85. In particular, (f, g_0, g_1) is a comparison triangle for $(\mathcal{F}, \mathcal{G}_0, \mathcal{G}_1)$, thus $d_\tau(\mathcal{F}, \mathcal{G}_t) \leq d(f, g_t)$ where $g_t = tg_1 + (1-t)g_0$ in $\mathbb{R}^3$ by the previous corollary. Since $\|\mathcal{G}_t\| = \|g_t\|$ (because $(0, \mathcal{G}_0, \mathcal{G}_1)$ is flat), it follows that

$$\langle \mathcal{F}, \mathcal{G}_t \rangle_\tau \geq \langle f, g_t \rangle = t \langle f, g_1 \rangle + (1-t) \langle f, g_0 \rangle = t \langle \mathcal{F}, \mathcal{G}_1 \rangle_\tau + (1-t) \langle \mathcal{F}, \mathcal{G}_0 \rangle_\tau .$$

Similarly, we find that

$$|\langle \mathcal{F}, \mathcal{G}_1 \rangle_\tau - \langle \mathcal{F}, \mathcal{G}_0 \rangle_\tau| = |\langle f, g_1 \rangle - \langle f, g_0 \rangle| \leq \|f\| \, \|g_1 - g_0\| = \|\mathcal{F}\|_\tau \cdot d_\tau(\mathcal{G}_0, \mathcal{G}_1)$$

thus $\mathcal{G} \mapsto \langle \mathcal{F}, \mathcal{G} \rangle_\tau$ is indeed $\|\mathcal{F}\|_\tau$-Lipschitzian. □

COROLLARY 92. *For any* $\mathcal{F}, \mathcal{G}, \mathcal{H} \in \mathbf{F}^{\mathbb{R}}(G)$,

$$\langle \mathcal{F}, \mathcal{G} + \mathcal{H} \rangle_\tau \geq \langle \mathcal{F}, \mathcal{G} \rangle_\tau + \langle \mathcal{F}, \mathcal{H} \rangle_\tau .$$

PROOF. Apply the previous lemma to the middle point $\mathcal{G} + \mathcal{H}$ of $[2\mathcal{G}, 2\mathcal{H}]$. □

REMARK 93. We could pursue here with many further corollaries, but our knowledgeable readers will recognize that already with corollary 86, we have established that $\mathbf{F}^{\mathbb{R}}(G)$, together with its collections of apartments and facets (and the function d_τ for some choice of a faithful τ), is a (discrete) Euclidean building in the sense of [**40**, 6.1]. It is the vectorial (Tits) building defined in [**40**, 10.6]. But the construction given there singles out a pair $Z_G(S) \subset B$ and uses more of the finest results from [**5**]: $\mathbf{F}^{\mathbb{R}}(G)$ is the building associated to the saturated Tits system $(\mathsf{G}, \mathsf{B}, \mathsf{N}) = (G, B, N_G(S))(k)$. By contrast, we may retrieve some of the results of [**5**] using the strongly transitive and strongly type-preserving action of G on our globally constructed building $\mathbf{F}^{\mathbb{R}}(G)$, for instance the fact that $(\mathsf{G}, \mathsf{B}, \mathsf{N})$ is indeed a saturated Tits system [**40**, 8.6]. The main advantage of our construction is however that it is plainly functorial in G and k.

REMARK 94. Corollary 87 also immediately follows from proposition 72 together with [**2**, VIB 1.4.2].

4.2.11. If τ' is another faithful representation of G, the distances $d_{\tau'}$ and d_τ are equivalent. One checks it first on a fixed apartment A, thus obtaining constants $c, C > 0$ such that $cd_\tau(x,y) \leq d_{\tau'}(x,y) \leq Cd_\tau(x,y)$ for $x, y \in A$. Then this holds true for every $x, y \in \mathbf{F}^{\Gamma}(G)$, since any such pair is G-conjugated to one in A. We thus obtain a canonical metrizable G-invariant topology on $\mathbf{F}^{\Gamma}(G)$. The G-invariant functions of section 4.2.2 are continuous with respect to the canonical topology. The apartments and the "closed facets" of section 4.1.11 are topologically closed, being complete for the induced metrics. The canonical topology on $\mathbf{C}^{\Gamma}(G)$ is the quotient topology of the canonical topology on $\mathbf{G}^{\Gamma}(G)$, it is compatible with the monoid structure on $\mathbf{C}^{\Gamma}(G)$, the sections defined by the "closed chambers" are homeomorphisms and the functions defined in section 4.2.5 are continuous.

4.2.12. Suppose now that our local ring $\mathcal{O}$ is an integral domain with fraction field K and residue field k, giving rise to morphisms of cartesian squares

$$\begin{tikzcd}
\mathbf{F}^{\Gamma}(G_K) \arrow[dr, "t"] \arrow[dd, "F"'] & & \mathbf{F}^{\Gamma}(G) \arrow[ll] \arrow[rr] \arrow[dr, "t"] \arrow[dd, "F"'] & & \mathbf{F}^{\Gamma}(G_k) \arrow[dr, "t"] \arrow[dd, "F"'] & \\
& \mathbf{C}^{\Gamma}(G_K) \arrow[dd, "F"'] & & \mathbf{C}^{\Gamma}(G) \arrow[ll] \arrow[rr] \arrow[dd, "F"'] & & \mathbf{C}^{\Gamma}(G_k) \arrow[dd, "F"'] \\
\mathbf{P}(G_K) \arrow[dr, "t"] & & \mathbf{P}(G) \arrow[ll] \arrow[rr] \arrow[dr, "t"] & & \mathbf{P}(G_k) \arrow[dr, "t"] & \\
& \mathbf{O}(G_K) & & \mathbf{O}(G) \arrow[ll] \arrow[rr] & & \mathbf{O}(G_k)
\end{tikzcd}$$

We write $x \mapsto x_K$ for the generization maps, $x \mapsto x_k$ for the specialization maps.

PROPOSITION 95. *For any faithful $\tau \in \mathsf{Rep}^{\circ}(G)(\mathcal{O})$ and $x, y \in \mathbf{F}^{\Gamma}(G)$,*

$$\begin{array}{rcl}
\langle x_k, y_k \rangle_{\tau_k} & \geq & \langle x_K, y_K \rangle_{\tau_K} \\
\measuredangle_{\tau_k}(x_k, y_k) & \leq & \measuredangle_{\tau_K}(x_K, y_K) \\
d_{\tau_k}(x_k, y_k) & \leq & d_{\tau_K}(x_K, y_K) \\
\|x_k\|_{\tau_k} & = & \|x_K\|_{\tau_K}
\end{array}$$

PROOF. We may assume that $\Gamma = \mathbb{R}$. For $(x, y) \in \mathbf{Std}^{\mathbb{R}}(G)$, one checks easily that all of the above inequalities are in fact equalities. In particular,

$$\|x_k\|_{\tau_k} = \|x_K\|_{\tau_K}$$

for all $x \in \mathbf{F}^{\Gamma}(G)$. For an arbitrary pair (x, y) in $\mathbf{F}^{\mathbb{R}}(G)$, the facet decomposition of $\mathbf{F}^{\mathbb{R}}(G)$ induces a decomposition of the segment $[x, y] = \cup_{i=0}^{n-1}[x_i, x_{i+1}]$ as in the proof of corollary 88, with $(x_i, x_{i+1}) \in \mathbf{Std}^{\mathbb{R}}(G)$ for every i. Thus

$$d_{\tau_K}(x_K, y_K) = \sum_{i=0}^{n-1} d_{\tau_K}(x_{i,K}, x_{i+1,K}) = \sum_{i=0}^{n-1} d_{\tau_k}(x_{i,k}, x_{i+1,k}) \geq d_{\tau_k}(x_k, y_k)$$

and the other two inequalities easily follow. $\square$

REMARK 96. On the other hand for any $x, y \in \mathbf{C}^{\Gamma}(G)$,

$$\begin{array}{rcllrcl}
\langle x_k, y_k \rangle_{\tau_k}^{os} & = & \langle x_K, y_K \rangle_{\tau_K}^{os} & \text{and} & \measuredangle_{\tau_k}^{os}(x_k, y_k) & = & \measuredangle_{\tau_K}^{os}(x_K, y_K), \\
\langle x_k, y_k \rangle_{\tau_k}^{tr} & = & \langle x_K, y_K \rangle_{\tau_K}^{tr} & \text{and} & \measuredangle_{\tau_k}^{tr}(x_k, y_k) & = & \measuredangle_{\tau_K}^{tr}(x_K, y_K).
\end{array}$$

However, it does happen that $D_{\tau_k}(x_k, y_k) \neq D_{\tau_K}(x_K, y_K)$.

CHAPTER 5

Affine $\mathbf{F}(G)$-buildings

Let G be a reductive group over a field K. From now on, $\Gamma = (\mathbb{R}, +, \leq)$ and we drop it from our notations. We also fix a faithful finite dimensional representation τ of G and drop it from the notations of section 4.2.2. We use sans-serif fonts to denote the set of K-valued points of a K-scheme, as in $\mathsf{G} = G(K)$, $\mathsf{P} = P(K)$ etc...

5.1. The dominance order

5.1.1. Since $\Gamma = \mathbb{R}$ is divisible, the weak and strong partial dominance order on $\mathbb{C}^{\Gamma}(G)$ agree. We denote by $\leq$ the induced partial order on the commutative monoid $\mathsf{C}(G) = \mathbb{C}(G)(K)$ or its submonoid $\mathbf{C}(G) = t(\mathbf{F}(G))$. It is compatible with the monoid structure on $\mathbf{C}(G)$ and related to the decomposition

$$\mathbf{C}(G) = \mathbf{C}^r(G) \times \mathbf{G}(Z) \quad \text{with} \quad \mathbf{C}^r(G) = \mathbb{C}(G)^r(K) \text{ and } \mathbf{G}(Z) = \mathbb{C}(G)^c(K)$$

of section 2.2.13 as follows: for $x = (x^r, x^c)$ and $y = (y^r, y^c)$ in $\mathbf{C}^r(G) \times \mathbf{G}(Z)$,

$$x \leq y \quad \iff \quad x^r \leq y^r \quad \text{and} \quad x^c = y^c.$$

The poset $(\mathbf{C}(G), \leq)$ is a lattice and $\mathbf{G}(Z) \subset \mathbf{C}(G)$ is its subset of minimal elements.

5.1.2. Choose a minimal pair (S, B) in $\mathbf{SP}(G)$, giving rise to the relative based root data $\mathscr{R}_+ = (M, R, M^*, R^*; R_+)$ with Weyl group $\mathsf{W}_G(S) = W_G(S)(K)$, and to the partial dominance order $\leq$ on $\mathrm{Hom}^+(M, \mathbb{R})$ as defined in section 2.4.13. Let also $s : \mathbf{C}(G) \hookrightarrow \mathbf{F}(S)$ be the corresponding section of $t : \mathbf{G}(G) \twoheadrightarrow \mathbf{C}(G)$, whose image $C = s(\mathbf{C}(G))$ equals $\mathrm{Hom}^+(M, \mathbb{R})$ inside $\mathrm{Hom}(M, \mathbb{R}) = \mathbf{F}(S)$ by section 4.1.10. Let finally C^* be the dual cone of C in $\mathbf{F}(S)$ with respect to the scalar product $\langle -, - \rangle$ on $\mathbf{F}(S)$ which is attached to our chosen τ, so that

$$C^* = \{t \in \mathbf{F}(S) : \forall c \in C, \langle t, c \rangle \geq 0\}.$$

Then for every $x, y \in \mathbf{C}(G)$,

$$\begin{aligned}
x \leq y \quad &\iff \quad s(x) \leq s(y) \text{ in } \mathrm{Hom}^+(M, \mathbb{R}), \\
&\iff \quad s(x) \text{ belongs to the convex hull of } \mathsf{W}_G(S) \cdot s(y), \\
&\iff \quad s(y) - s(x) \text{ belongs to the dual cone } C^*, \\
&\iff \quad \forall z \in \mathbf{C}(G) : \quad \langle x, z \rangle^{os} \leq \langle y, z \rangle^{os}, \\
&\iff \quad \forall z \in \mathbf{C}(G) : \quad \langle x, z \rangle^{tr} \geq \langle y, z \rangle^{tr}.
\end{aligned}$$

The first equivalence follows from Proposition 31, the second from section 2.4.11, the third one from [**4**, 12.14] and the last two from the formulas of section 4.2.7. The equivalence of the first and third line on the right is actually a tautology, since in fact $C^* = \mathbb{R}_+ R_+^*$ in $\mathbf{F}(S) = \mathrm{Hom}(M, \mathbb{R})$. Indeed for $\alpha \in R$, let α_v be the unique element of $\mathbf{F}(S)$ such that $\langle x, \alpha_v \rangle = x(\alpha)$ for every $x \in \mathbf{F}(S)$. Then s_α is the orthogonal reflection of $\mathbf{F}(S)$ with respect to the hyperplane $\alpha_v^{\perp}$, thus $\alpha^* = \frac{\alpha_v}{\langle \alpha_v, \alpha_v \rangle}$ in $\mathbf{F}(S)$. Since $C = \{x \in \mathbf{F}(S) : \forall \alpha \in R_+,\ \langle x, \alpha_v \rangle \geq 0\}$, its dual cone C^* is spanned by the α_v's for $\alpha \in R_+$ and thus $C^* = \mathbb{R}_+ R_+^*$.

5.1.3. For every $x, y \in \mathbf{C}(G)$, we have

$$\begin{array}{rcl} x \leq y & \Longrightarrow & \|x\|^2 \leq \langle x, y \rangle^{os} \leq \|y\|^2 \\ x = y & \Longleftrightarrow & x \leq y \text{ and } \|x\| = \|y\|. \end{array}$$

Indeed the first implication follows from the equivalence

$$x \leq y \iff \forall z \in \mathbf{C}(G) : \quad \langle x, z \rangle^{os} \leq \langle y, z \rangle^{os}$$

with $z = x$ or y, and with s as above, it says that

$$x \leq y \Longrightarrow \|s(x)\|^2 \leq \|s(x)\| \, \|s(y)\| \cos \measuredangle(s(x), s(y)) \leq \|s(y)\|^2.$$

Thus if $x \leq y$ and $\|x\| = \|y\|$, $s(x) = s(y)$ and $x = y$.

5.1.4. The next proposition slightly refines proposition 24.

PROPOSITION 97. *For every $\mathcal{F}, \mathcal{G} \in \mathbf{F}(G)$, we have*

$$t(\mathcal{F} + \mathcal{G}) \leq t(\mathcal{F}) + t(\mathcal{G}) \quad \textit{in} \quad (\mathbf{C}(G), \leq)$$

with equality if and only if $\mathcal{F}, \mathcal{G} \in C$ for some closed chamber C of $\mathbf{F}(G)$.

PROOF. With notations as above, we may choose (S, B) such that $\mathcal{F}, \mathcal{G} \in \mathbf{F}(S)$ with $\mathcal{F} + \mathcal{G} \in C$, $C = s(\mathbf{C}(G))$. Set $\mathcal{F}' = s \circ t(\mathcal{F})$ and $\mathcal{G}' = s \circ t(\mathcal{G})$, so that

$$\mathcal{F} + \mathcal{G} = s\left(t(\mathcal{F} + \mathcal{G})\right) \quad \text{and} \quad \mathcal{F}' + \mathcal{G}' = s\left(t(\mathcal{F}) + t(\mathcal{G})\right).$$

The (acute) dual cone C^* defines a partial order $\leq$ on $\mathbf{F}(S)$, given by

$$x \leq y \iff y - x \in C^*.$$

Since $\mathcal{F}' \in (\mathsf{W}_G(S) \cdot \mathcal{F}) \cap C$ and $\mathcal{G}' \in (\mathsf{W}_G(S) \cdot \mathcal{G}) \cap C$, we have

$$\mathcal{F} \leq \mathcal{F}' \quad \text{and} \quad \mathcal{G} \leq \mathcal{G}'$$

by [**7**, VI §1 Proposition 18] (or lemma 29). Thus $\mathcal{F} + \mathcal{G} \leq \mathcal{F}' + \mathcal{G}'$ with equality if and only if $\mathcal{F} = \mathcal{F}'$ and $\mathcal{G} = \mathcal{G}'$, i.e. $\mathcal{F}$ and $\mathcal{G}$ belong to C. $\square$

5.1.5. The above inequality can also be established and somehow refined as follows. For every $z \in \mathbf{C}(G)$, there is an $\mathcal{H} \in \mathbf{F}(G)$ with $t(\mathcal{H}) = z$ such that $\mathcal{H}$ and $\mathcal{F} + \mathcal{G}$ are in (relative) transverse position, see 4.2.5. For any such $\mathcal{H}$, 4.2.7, lemma 82 and corollary 92 together imply that

$$\begin{array}{rcl} \langle t(\mathcal{F}) + t(\mathcal{G}), z \rangle^{tr} & = & \langle t(\mathcal{F}), z \rangle^{tr} + \langle t(\mathcal{G}), z \rangle^{tr} \\ & \leq & \langle \mathcal{F}, \mathcal{H} \rangle + \langle \mathcal{G}, \mathcal{H} \rangle \\ & \leq & \langle \mathcal{F} + \mathcal{G}, \mathcal{H} \rangle \\ & = & \langle t(\mathcal{F} + \mathcal{G}), z \rangle^{tr}. \end{array}$$

Thus indeed $t(\mathcal{F} + \mathcal{G}) \leq t(\mathcal{F}) + t(\mathcal{G})$ in $(\mathbf{C}(G), \leq)$.

5.2. Affine $\mathbf{F}(G)$-spaces and buildings

5.2.1. Affine $\mathbf{F}(G)$-spaces interact with the vectorial Tits building $\mathbf{F}(G)$ in the same way as affine spaces do with their underlying vector space.

DEFINITION 98. An affine $\mathbf{F}(G)$-space is a set $\mathbf{X}(G)$ equipped with:

- a left action $\mathsf{G} \times \mathbf{X}(G) \to \mathbf{X}(G)$, written $(g, x) \mapsto g \cdot x$ or gx,
- a G-equivariant *pull* map $\mathbf{X}(G) \times \mathbf{F}(G) \to \mathbf{X}(G)$, written $(x, \mathcal{F}) \mapsto x + \mathcal{F}$,
- a G-equivariant *apartment* map $\mathbf{S}(G) \to \mathcal{P}(\mathbf{X}(G))$, written $S \mapsto \mathbf{X}(S)$,

such that for (one or) every $S \in \mathbf{S}(G)$, the pull map sends $\mathbf{X}(S) \times \mathbf{F}(S)$ to $\mathbf{X}(S)$ and induces a structure of affine $\mathbf{G}(S)$-space (in the usual sense) on $\mathbf{X}(S)$.

5.2.2. The group $\mathsf{N}_G(S)$ thus acts on $\mathbf{X}(S)$ by affine morphisms, the vectorial part of this action equals $\nu_S^v : \mathsf{N}_G(S) \twoheadrightarrow \mathsf{W}_G(S) \subset \mathrm{Aut}(\mathbf{G}(S))$ and the kernel $\mathsf{Z}_G(S)$ of ν_S^v acts on $\mathbf{X}(S)$ by translations, through a $\mathsf{W}_G(S)$-equivariant morphism

$$\nu_{\mathbf{X},S} : \mathsf{Z}_G(S) \to \mathbf{G}(S).$$

For any other $S' \in \mathbf{S}(G)$, there is commutative diagram

$$\begin{tikzcd} \mathsf{Z}_G(S) \arrow[r, "\nu_{\mathbf{X},S}"] \arrow[d, "\simeq"'] & \mathbf{G}(S) \arrow[d, "\simeq"] \\ \mathsf{Z}_G(S') \arrow[r, "\nu_{\mathbf{X},S'}"] & \mathbf{G}(S') \end{tikzcd}$$

where the vertical maps are induced by $\mathrm{Int}(g)$ for any $g \in \mathsf{G}$ with $\mathrm{Int}(g)(S) = S'$. Set $\mathsf{W}_G = \varprojlim \mathsf{W}_G(S)$. The type of $\mathbf{X}(G)$ is the W_G-equivariant morphism

$$\nu_{\mathbf{X}} = \varprojlim \nu_{\mathbf{X},S} : \varprojlim \mathsf{Z}_G(S) \to \varprojlim \mathbf{G}(S)$$

which is obtained from these diagrams by taking the limits over all $S \in \mathbf{S}(G)$. We say that $\mathbf{X}(G)$ is discrete when the image of $\nu_{\mathbf{X}}$ is a discrete subgroup of the real vector space $\varprojlim \mathbf{G}(S)$. Equivalently: $\mathbf{X}(G)$ is discrete when the image of $\nu_{\mathbf{X},S}$ is a discrete subgroup of $\mathbf{G}(S)$ for one or every $S \in \mathbf{S}(G)$.

5.2.3. Affine $\mathbf{F}(G)$-buildings are affine $\mathbf{F}(G)$-spaces satisfying a long list of axioms, which shall be gradually introduced below. The following definition picks up an (hopefully minimal) subset of these axioms, from which all others will be derived in due time, along with various properties.

DEFINITION 99. An affine $\mathbf{F}(G)$-building is an affine $\mathbf{F}(G)$-space which satisfies the axioms $L(s)$, $R(s)$, $R(i)$, $\mathcal{C}^\circ$, NE, UN, CO and UG listed below.

5.2.4. Example. The Tits building $\mathbf{F}(G)$ itself, equipped with its left action of G, the addition map of section 2.3.2 and the apartment map of section 4.1.13 is a discrete affine $\mathbf{F}(G)$-space with trivial type $\nu_{\mathbf{F}} = 0$. We will see that it satisfies all of the required axioms, thus $\mathbf{F}(G) = (\mathbf{F}(G), +, \mathbf{F}(-))$ is an affine $\mathbf{F}(G)$-building.

5.2.5. Many apartments. An affine $\mathbf{F}(G)$-building $\mathbf{X}(G)$ satisfies

$L(s)$ For every $x \in \mathbf{X}(G)$ and $\mathcal{F} \in \mathbf{F}(G)$,

$$\mathbf{S}(x, \mathcal{F}) = \{S \in \mathbf{S}(G) : x \in \mathbf{X}(S) \text{ and } \mathcal{F} \in \mathbf{F}(S)\}$$

is not empty.

$R(s)$ For every $x, y \in \mathbf{X}(G)$,
$$\mathbf{S}(x, y) = \{S \in \mathbf{S}(G) : x, y \in \mathbf{X}(S)\}$$
is not empty.

$T(s)$ For every $x, y \in \mathbf{X}(G)$,
$$\mathbf{F}(x, y) = \{\mathcal{F} \in \mathbf{F}(G) : y = x + \mathcal{F}\}$$
is not empty.

Note that $R(s)$ implies $T(s)$ while $L(s)$ is equivalent to

$L'(s)$ For every $x \in \mathbf{X}(G)$ and every closed chamber C of $\mathbf{F}(G)$,
$$\mathbf{S}(x, C) = \{S \in \mathbf{S}(G) : x \in \mathbf{X}(S) \text{ and } C \subset \mathbf{F}(S)\}$$
is not empty.

This in turn implies that the pull map is well-behaved:

AC (*Action*) For every closed chamber C of $\mathbf{F}(G)$, the map
$$\mathbf{X}(G) \times C \hookrightarrow \mathbf{X}(G) \times \mathbf{F}(G) \xrightarrow{+} \mathbf{X}(G)$$
defines an action of the commutative monoid $(C, +)$ on $\mathbf{X}(G)$.

Thus for any $x \in \mathbf{X}(G)$ and $\mathcal{F}, \mathcal{G} \in \mathbf{F}(G)$, $x + 0 = x$ and
$$(x + \mathcal{F}) + \mathcal{G} = x + (\mathcal{F} + \mathcal{G}) = (x + \mathcal{G}) + \mathcal{F}$$
if $P_{\mathcal{F}}$ and $P_{\mathcal{G}}$ are in osculatory position. In particular,
$$(x + \lambda\mathcal{F}) + \mu\mathcal{F} = x + (\lambda + \mu)\mathcal{F}$$
for every $\lambda, \mu \geq 0$ and $\mathcal{F} \in \mathbf{F}(G)$.

5.2.6. Strong transitivity. An affine $\mathbf{F}(G)$-building $\mathbf{X}(G)$ satisfies

$L(i)$ For every $x \in \mathbf{X}(G)$ and $\mathcal{F} \in \mathbf{F}(G)$,
$$\mathsf{G}_{x,\mathcal{F}} = \{g \in \mathsf{G} : gx = x \quad \text{and} \quad g\mathcal{F} = \mathcal{F}\}$$
acts transitively on $\mathbf{S}(x, \mathcal{F})$.

$R(i)$ For every $x, y \in \mathbf{X}(G)$,
$$\mathsf{G}_{x,y} = \{g \in \mathsf{G} : gx = x \quad \text{and} \quad gy = y\}$$
acts transitively on $\mathbf{S}(x, y)$.

$T(i)$ For every $x, y \in \mathbf{X}(G)$, $\mathsf{G}_{x,y}$ acts transitively on $\mathbf{F}(x, y)$.

5.2.7. The labels of the L, R, or T-axioms reflect their equivalence with the surjectivity or injectivity of the relevant maps in the commutative diagram

$$\begin{tikzcd}
\mathsf{G}\backslash\left(\mathbf{X}(G) \times \mathbf{F}(G)\right) \arrow[r, "T"] & \mathsf{G}\backslash\left(\mathbf{X}(G) \times \mathbf{X}(G)\right) \\
\mathsf{N}_G(S)\backslash\left(\mathbf{X}(S) \times \mathbf{F}(S)\right) \arrow[r] \arrow[u, "L"] & \mathsf{N}_G(S)\backslash\left(\mathbf{X}(S) \times \mathbf{X}(S)\right) \arrow[u, "R"']
\end{tikzcd}$$

which is induced by the equivariant commutative diagram

$$\begin{tikzcd}
\mathbf{X}(G) \times \mathbf{F}(G) \arrow[r, "(x,\mathcal{F}) \mapsto (x, x+\mathcal{F})"] & \mathbf{X}(G) \times \mathbf{X}(G) \\
\mathbf{X}(S) \times \mathbf{F}(S) \arrow[r] \arrow[u, hook] & \mathbf{X}(S) \times \mathbf{X}(S) \arrow[u, hook]
\end{tikzcd}$$

The bottom map in each diagram is always bijective since $\mathbf{X}(S)$ is an affine $\mathbf{F}(S)$-space. Thus $R(i) \Rightarrow T(i)$, and $R(i) + L(s) + T(s)$ imply all of the above axioms.

5.2.8. The vectorial distance. It follows from the axioms already introduced that for an affine $\mathbf{F}(G)$-building $\mathbf{X}(G)$, there is a unique G-invariant map

$$\mathbf{d} : \mathbf{X}(G) \times \mathbf{X}(G) \to \mathbf{C}(G)$$

such that for every $x \in \mathbf{X}(G)$ and $\mathcal{F} \in \mathbf{F}(G)$,

$$\mathbf{d}(x, x + \mathcal{F}) = t(\mathcal{F}) \quad \text{in} \quad \mathbf{C}(G).$$

The following properties are easily established: for $x, y \in \mathbf{X}(G)$,

$$\mathbf{d}(y, x) = \mathbf{d}(x, y)^\iota \quad \text{and} \quad \mathbf{d}(x, y) = 0 \iff x = y.$$

Moreover for $x \in \mathbf{X}(G)$, $\mathcal{F} \in \mathbf{F}(G)$ and $0 \leq \lambda \leq \lambda'$,

$$\mathbf{d}(x + \lambda\mathcal{F}, x + \lambda'\mathcal{F}) = (\lambda' - \lambda) \cdot t(\mathcal{F}).$$

This *vectorial distance* $\mathbf{d}$ may also satisfy the following properties – and it does for affine $\mathbf{F}(G)$-buildings, by lemma 100 and proposition 101 below:

$\star$ For every $x, y \in \mathbf{X}(G)$ and $\mathcal{F}, \mathcal{G} \in \mathbf{F}(G)$,

$$\mathbf{d}(x + \mathcal{F}, y + \mathcal{G}) \leq \mathbf{d}(x, y) + \mathbf{d}(\mathcal{F}, \mathcal{G}) \quad \text{in} \quad \mathbf{C}(G).$$

TR (*Triangle inequality*) For every $x, y, z \in \mathbf{X}(G)$,

$$\mathbf{d}(x, z) \leq \mathbf{d}(x, y) + \mathbf{d}(y, z) \quad \text{in} \quad \mathbf{C}(G).$$

TR' For every $y \in \mathbf{X}(G)$ and $\mathcal{F}, \mathcal{G} \in \mathbf{F}(G)$,

$$\mathbf{d}(y + \mathcal{F}, y + \mathcal{G}) \leq \mathbf{d}(\mathcal{F}, \mathcal{G}) \quad \text{in} \quad \mathbf{C}(G).$$

NE (*Non expanding*) For every $x, y \in \mathbf{X}(G)$ and $\mathcal{F} \in \mathbf{F}(G)$,

$$\mathbf{d}(x + \mathcal{F}, y + \mathcal{F}) \leq \mathbf{d}(x, y) \quad \text{in} \quad \mathbf{C}(G).$$

$\mathcal{C}^\circ$ (*Continuity*) For every sequences (x_n), (y_n) and points x, y in $\mathbf{X}(G)$,

$$\left(\begin{array}{c} \mathbf{d}(x_n, x) \to 0 \\ \mathbf{d}(y_n, y) \to 0 \end{array} \text{ in } \mathbf{C}(G) \right) \Longrightarrow \left(\mathbf{d}(x_n, y_n) \to \mathbf{d}(x, y) \text{ in } \mathbf{C}(G) \right).$$

Note that $\star$ and TR' also involve the vectorial distance $\mathbf{d}$ for $\mathbf{F}(G)$ – it follows from 4.2.10 that the affine $\mathbf{F}(G)$-space $\mathbf{F}(G)$ indeed satisfies the required axioms for the existence of $\mathbf{d}$: $L(s) = R(s)$ is theorem 85 and $L(i) = R(i)$ is its corollary 86.

LEMMA 100. *The above properties of* $\mathbf{d}$ *are related as follows:*

$$\star \iff TR + TR' + NE \quad \textit{and} \quad TR \iff TR' \Longrightarrow \mathcal{C}^\circ.$$

PROOF. $(TR + TR' + NE \Rightarrow \star)$. For $x, y \in \mathbf{X}(G)$ and $\mathcal{F}, \mathcal{G} \in \mathbf{F}(G)$, we find

$$\mathbf{d}(x + \mathcal{F}, y + \mathcal{G}) \leq \mathbf{d}(x + \mathcal{F}, y + \mathcal{F}) + \mathbf{d}(y + \mathcal{F}, y + \mathcal{G}) \leq \mathbf{d}(x, y) + \mathbf{d}(\mathcal{F}, \mathcal{G})$$

using TR for the first inequality, NE and TR' for the second.

$(\star \Rightarrow TR + TR' + NE)$. Taking $x = y$ (resp. $\mathcal{F} = \mathcal{G}$) in $\star$ yields TR' (resp. NE). Taking $\mathcal{F} = 0$ and $\mathcal{G} \in \mathbf{F}(y, z)$ (using $T(s)$) yields TR.

$(TR \Rightarrow TR')$. For $x \in \mathbf{X}(G)$, $\mathcal{F}, \mathcal{G} \in \mathbf{F}(G)$ and $\lambda \in [0, 1]$, set

$$\mathcal{F}(\lambda) = (1 - \lambda)\mathcal{F} + \lambda\mathcal{G} \text{ in } \mathbf{F}(G) \quad \text{and} \quad x(\lambda) = x + \mathcal{F}(\lambda) \text{ in } \mathbf{X}(G).$$

Pick $S \in \mathbf{S}(G)$ with $\mathcal{F}, \mathcal{G} \in \mathbf{F}(S)$. There is a subdivision $0 = \lambda_0 < \cdots < \lambda_n = 1$ of $[0, 1]$ and for each $i \in \{1, \cdots, n\}$, a closed chamber C_i of $\mathbf{F}(S)$ such that $\mathcal{F}(\lambda) \in C_i$

for all $\lambda \in [\lambda_{i-1}, \lambda_i]$. By $L'(s)$, there is an $S_i \in \mathbf{S}(G)$ such that $x \in \mathbf{X}(S_i)$ and $C_i \subset \mathbf{F}(S_i)$, thus also $x(\lambda) \in \mathbf{X}(S_i)$ for every $\lambda \in [\lambda_{i-1}, \lambda_i]$. Then

$$\mathbf{d}(x+\mathcal{F}, x+\mathcal{G}) \leq \textstyle\sum_{i=1}^n \mathbf{d}(x(\lambda_{i-1}), x(\lambda_i)) = \sum_{i=1}^n \mathbf{d}(\mathcal{F}(\lambda_{i-1}), \mathcal{F}(\lambda_i)) = \mathbf{d}(\mathcal{F}, \mathcal{G})$$

using respectively TR in $\mathbf{X}(G)$ and trivial computations in $\mathbf{X}(S_i)$ and $\mathbf{F}(G)$.

($TR' \Rightarrow TR$). For $x, y, z \in \mathbf{X}(G)$, pick $\mathcal{F}, \mathcal{G} \in \mathbf{F}(G)$ with

$$x = y + \mathcal{F} \quad \text{and} \quad z = y + \mathcal{G}$$

using $T(s)$. Choose $S \in \mathbf{S}(G)$ such that $\mathcal{F}, \mathcal{G} \in \mathbf{F}(S)$, and set $\mathcal{F}' = \iota_S \mathcal{F}$. Then

$$\mathbf{d}(x, z) \leq \mathbf{d}(\mathcal{F}, \mathcal{G}) = t(\mathcal{F}' + \mathcal{G}) \leq t(\mathcal{F}') + t(\mathcal{G}) = \mathbf{d}(x, y) + \mathbf{d}(y, z)$$

using respectively TR' in $\mathbf{X}(G)$, proposition 97 and

$$t(\mathcal{F}') = t(\mathcal{F})^\iota = \mathbf{d}(y, x)^\iota = \mathbf{d}(x, y), \quad t(\mathcal{G}) = \mathbf{d}(y, z).$$

($TR \Rightarrow \mathcal{C}^\circ$). Suppose that $\mathbf{d}(x_n, x) \to 0$ and $\mathbf{d}(y_n, y) \to 0$ for sequences (x_n), (y_n) and points x, y in $\mathbf{X}(G)$. Then also $\mathbf{d}(x, x_n) \to 0$ and $\mathbf{d}(y, y_n) \to 0$ in $\mathbf{C}(G)$. Let c be a limit point of $\mathbf{d}(x_n, y_n)$ in the Alexandrov compactification $\mathbf{C}(G) \cup \{\infty\}$ of the locally compact space $\mathbf{C}(G)$. We have to show that $c = \mathbf{d}(x, y)$, for then $\mathbf{d}(x_n, y_n) \to \mathbf{d}(x, y)$ in $\mathbf{C}(G)$. By the triangle inequality TR,

$$\begin{array}{rcl} \mathbf{d}(x, y) & \leq & \mathbf{d}(x, x_n) + \mathbf{d}(x_n, y_n) + \mathbf{d}(y_n, y) \\ \text{and} \quad \mathbf{d}(x_n, y_n) & \leq & \mathbf{d}(x_n, x) + \mathbf{d}(x, y) + \mathbf{d}(y, y_n) \end{array}$$

for every $n \geq 0$, thus $c \in \mathbf{C}(G)$ and $\mathbf{d}(x, y) \leq c \leq \mathbf{d}(x, y)$, i.e. $c = \mathbf{d}(x, y)$. $\square$

5.2.9. The classical distance. These axioms imply that the composition

$$d : \mathbf{X}(G) \times \mathbf{X}(G) \to \mathbb{R}_+, \qquad x \mapsto \|\mathbf{d}(x)\|$$

of the vectorial distance $\mathbf{d}$ with the norm $\|-\| : \mathbf{C}(G) \to \mathbb{R}_+$ attached to our chosen τ is a genuine G-invariant distance on $\mathbf{X}(G)$. Its restriction to any apartment is Euclidean and $(\mathbf{X}(G), d)$ is a geodesic space: for $x, y \in \mathbf{X}(G)$ and any apartment $\mathbf{X}(S)$ containing x and y, the unique geodesic from x to y in $\mathbf{X}(S)$ is a geodesic from x to y in $\mathbf{X}(G)$. For any sequence (x_n) in $\mathbf{X}(G)$ and x in $\mathbf{X}(G)$, we have

$$x_n \to x \text{ in } (\mathbf{X}(G), d) \iff d(x_n, x) \to 0 \text{ in } \mathbb{R}_+ \iff \mathbf{d}(x_n, x) \to 0 \text{ in } \mathbf{C}(G).$$

The induced metrizable topology on $\mathbf{X}(G)$ thus does not depend upon τ (see also 4.2.11). We call it the canonical topology of $\mathbf{X}(G)$. The apartments are closed, being complete for the induced metric. The vectorial distance and pull map

$$\mathbf{d} : \mathbf{X}(G) \times \mathbf{X}(G) \to \mathbf{C}(G) \qquad \text{and} \qquad + : \mathbf{X}(G) \times \mathbf{F}(G) \to \mathbf{X}(G)$$

are continuous for the canonical topologies on $\mathbf{X}(G)$, $\mathbf{C}(G)$ and $\mathbf{F}(G)$ by $\mathcal{C}^\circ$ and $\star$.

5.2.10. The retractions. For an affine $\mathbf{F}(G)$-building, we also require:

UN (*Unipotent*) For every $x \in \mathbf{X}(G)$, $\mathcal{F} \in \mathbf{F}(G)$ and $u \in \mathsf{U}_\mathcal{F}$,

$$\lim_{s \to +\infty} \mathbf{d}(x + s\mathcal{F}, ux + s\mathcal{F}) = 0.$$

For $\mathcal{F} \in \mathbf{F}(G)$ and any Levi subgroup L of $P_\mathcal{F}$, we denote by $\mathcal{F}_L^\iota$ the unique filtration opposed to $\mathcal{F}$ with $P_\mathcal{F} \cap P_{\mathcal{F}_L^\iota} = L$. Thus $\mathcal{F}_L^\iota = \mathrm{Fil}(\iota\mathcal{G})$ where $\mathcal{G} \in \mathbf{G}(G)$ is the unique splitting of $\mathcal{F}$ with $L_\mathcal{G} = L$. We have $\mathcal{F}_L^\iota = \iota_S \mathcal{F}$ for any $S \in \mathbf{S}(L)$.

PROPOSITION 101. *Let $\mathbf{X}(G)$ be an affine $\mathbf{F}(G)$-space satisfying the axioms of sections 5.2.5 and 5.2.6, together with $\mathcal{C}^\circ$, NE and UN. Then it also satisfy TR. Moreover, for every parabolic subgroup P of G with Levi decomposition $P = U \rtimes L$,*

$$\mathbf{X}(L) = \cup_{S \in \mathbf{S}(L)} \mathbf{X}(S)$$

is a fundamental domain for the action of U on $\mathbf{X}(G)$ and the induced retraction

$$r_{P,L} : \mathbf{X}(G) \twoheadrightarrow \mathbf{X}(L)$$

is non-expanding for $\mathbf{d}$: for every $x, y \in \mathbf{X}(G)$,

$$\mathbf{d}(r_{P,L}(x), r_{P,L}(y)) \leq \mathbf{d}(x, y) \quad \text{in} \quad \mathbf{C}(G).$$

Finally, for any $\mathcal{F} \in F^{-1}(P)$, if $\mathcal{F}' = \mathcal{F}_L^\iota$, then for all $x \in \mathbf{X}(G)$,

$$r_{P,L}(x) = \lim_{s \to \infty} (x + s\mathcal{F}) + s\mathcal{F}' \quad \text{in} \quad (\mathbf{X}(G), d)\,.$$

PROOF. Fix P, L, $\mathcal{F}$ and $\mathcal{F}' = \mathcal{F}_L^\iota$ as above. For any $x \in \mathbf{X}(G)$, there is by $L(s)$ an $S' \in \mathbf{S}(G)$ such that $x \in \mathbf{X}(S')$ and $\mathcal{F} \in \mathbf{F}(S')$, i.e. $Z_G(S') \subset P_{\mathcal{F}}$. Let L' be the unique levi subgroup of $P_{\mathcal{F}}$ containing $Z_G(S')$ and let u be the unique element of U such that $\mathrm{Int}(u)(L') = L$. Then $S = \mathrm{Int}(u)(S')$ belongs to $\mathbf{S}(L)$ and ux belongs to $\mathbf{X}(S) \subset \mathbf{X}(L)$, thus $\mathsf{U} \cdot \mathbf{X}(L) = \mathbf{X}(G)$. For $s \geq 0$ and $x \in \mathbf{X}(G)$, set

$$r_s(x) = (x + s\mathcal{F}) + s\mathcal{F}' \quad \text{in} \quad \mathbf{X}(G).$$

Then $x \mapsto r_s(x)$ is non-expanding for $\mathbf{d}$ by NE and for any $u \in \mathsf{U}$,

$$\lim_{s \to \infty} \mathbf{d}(r_s(x), r_s(u \cdot x)) = 0 \quad \text{in} \quad \mathbf{C}(G)$$

by UN and NE. If x belongs to $\mathbf{X}(L)$, say $x \in \mathbf{X}(S)$ for some $S \in \mathbf{S}(L)$, then $\mathcal{F}, \mathcal{F}' \in \mathbf{F}(S)$ with $\mathcal{F} + \mathcal{F}' = 0$ in $\mathbf{F}(S)$, thus $r_s(x) = x$ for all $s \geq 0$ since $\mathbf{X}(S)$ is an affine $\mathbf{F}(S)$-space. If x and $u \cdot x$ belong to $\mathbf{X}(L)$, $\mathbf{d}(x, u \cdot x) = \mathbf{d}(r_s(x), r_s(u \cdot x))$ for all $s \geq 0$, thus $\mathbf{d}(x, u \cdot x) = 0$ and $x = u \cdot x$. In particular, $\mathbf{X}(L)$ is indeed a fundamental domain for the action of U on $\mathbf{X}(G)$. Let $r : \mathbf{X}(G) \twoheadrightarrow \mathbf{X}(L)$ be the corresponding retraction. For $x \in \mathbf{X}(G)$, pick $u \in \mathsf{U}$ such that $r(x) = u \cdot x$. Then

$$\mathbf{d}(r_s(x), r(x)) = \mathbf{d}(r_s(x), u \cdot x) = \mathbf{d}(r_s(x), r_s(u \cdot x)) \to 0.$$

Applying this to $x, y \in \mathbf{X}(G)$ and using $\mathcal{C}^\circ$, we find that

$$\lim_{s \to \infty} \mathbf{d}(r_s(x), r_s(y)) = \mathbf{d}(r(x), r(y)),$$

thus $\mathbf{d}(r(x), r(y)) \leq \mathbf{d}(x, y)$ since $\mathbf{d}(r_s(x), r_s(y)) \leq \mathbf{d}(x, y)$ for all $s \geq 0$.

Turning now to the proof of TR, first note that by proposition 97, the triangle inequality holds whenever x, y, z belong to $\mathbf{X}(S)$ for some $S \in \mathbf{S}(G)$. For a general triple x, y, z in $\mathbf{X}(G)$, choose $S \in \mathbf{S}(G)$ with $x, z \in \mathbf{X}(S)$ using $R(s)$, pick a minimal parabolic subgroup B of G with Levi subgroup $L = Z_G(S)$ and let $r : \mathbf{X}(G) \twoheadrightarrow \mathbf{X}(S)$ be the corresponding retraction. Then $r(x) = x$ and $r(z) = z$, thus indeed

$$\mathbf{d}(x, z) = \mathbf{d}(r(x), r(z)) \leq \mathbf{d}(r(x), r(y)) + \mathbf{d}(r(y), r(z)) \leq \mathbf{d}(x, y) + \mathbf{d}(y, z)$$

since the triangle inequality holds on $\mathbf{X}(S)$ and r is non-expanding for $\mathbf{d}$. □

COROLLARY 102. *The apartment map $S \mapsto \mathbf{X}(S)$ is then uniquely determined by the pull map $+ : \mathbf{X}(G) \times \mathbf{F}(G) \to \mathbf{X}(G)$: for every $S \in \mathbf{S}(G)$,*

$$\mathbf{X}(S) = \left\{ x \in \mathbf{X}(G) : \forall \mathcal{F}, \mathcal{F}' \in \mathbf{F}(S),\, (x + \mathcal{F}) + \mathcal{F}' = x + (\mathcal{F} + \mathcal{F}') \right\}.$$

PROOF. Let $\mathbf{X}'(S)$ be the right hand side. Plainly, $\mathbf{X}(S) \subset \mathbf{X}'(S)$. Conversely, pick a minimal parabolic subgroup B of G with Levi $Z_G(S)$, let $r : \mathbf{X}(G) \twoheadrightarrow \mathbf{X}(S)$ be the corresponding retraction, choose $\mathcal{F} \in F^{-1}(B)$ and set $\mathcal{F}' = \iota_S \mathcal{F}$. For x in $\mathbf{X}'(S)$, $(x + \lambda\mathcal{F}) + \lambda\mathcal{F}' = x$ for all $\lambda \geq 0$, thus $r(x) = x$ belongs to $\mathbf{X}(S)$. □

5.2.11. Standard geodesics. For $x \in \mathbf{X}(G)$ and $\mathcal{F} \in \mathbf{F}(G)$, the function

$$[0,1] \to \mathbf{X}(G) \quad (\text{resp. } \mathbb{R}_+ \to \mathbf{X}(G)) \qquad t \mapsto x + t\mathcal{F}$$

is a geodesic segment (resp. geodesic ray) in $(\mathbf{X}(G), d)$. We refer to these geodesics as the *standard* ones. Thus a geodesic (segment or ray) is standard precisely when it is contained in some apartment, and the set of all standard geodesics does not depend upon the choice of τ. If $(\mathbf{X}(G), d)$ is uniquely geodesic, then every geodesic segment is standard, but there might still be some non-standard geodesic rays.

5.2.12. Convexity. An affine $\mathbf{F}(G)$-building satisfies all of the above axioms, together with the following convexity axiom:

CO^+ For every pair of geodesics $x, y : [0,1] \to \mathbf{X}(G)$ in $(\mathbf{X}(G), d)$, the function

$$f : [0,1] \to \mathbf{C}(G), \qquad f(t) = \mathbf{d}(x(t), y(t))$$

is convex, i.e. for every λ and $t_1 \leq t_2$ in $[0,1]$,

$$f\left((1-\lambda)t_1 + \lambda t_2\right) \leq (1-\lambda) f(t_1) + \lambda f(t_2) \quad \text{in} \quad \mathbf{C}(G).$$

This implies that the metric space $(\mathbf{X}(G), d)$ itself is convex in the sense of [**8**, II.1.3]. In particular, it is uniquely geodesic and for every $x \in \mathbf{X}(G)$ and $\mathcal{F}, \mathcal{G} \in \mathbf{F}(G)$,

$$x + \mathcal{F} = x + \mathcal{G} \Longrightarrow \forall t \in [0,1] : \quad x + t\mathcal{F} = x + t\mathcal{G}.$$

PROPOSITION 103. *Let $\mathbf{X}(G)$ be an affine $\mathbf{F}(G)$-building. Let (P, P') be a pair of opposed parabolic subgroups of G with common Levi subgroup $L = P \cap P'$. Let*

$$r, r' : \mathbf{X}(G) \twoheadrightarrow \mathbf{X}(L)$$

be the corresponding retractions, as in proposition 101. Then

$$\mathbf{X}(L) = \{x \in \mathbf{X}(G) : r(x) = r'(x)\}.$$

PROOF. For $x \in \mathbf{X}(L)$, $r(x) = x = r'(x)$, thus x belongs to

$$\mathbf{X}'(L) = \{x \in \mathbf{X}(G) : r(x) = r'(x)\}.$$

Suppose conversely that $x \in \mathbf{X}'(L)$ and set $y = r(x) = r'(x)$. Pick a pair of opposed filtrations $(\mathcal{F}, \mathcal{F}')$ with $P_\mathcal{F} = P$, $P_{\mathcal{F}'} = P'$ and $\|\mathcal{F}\| = \|\mathcal{F}'\| = 1$. For $t \in \mathbb{R}$, set

$$X(t) = \begin{cases} x + |t|\,\mathcal{F} & \text{if } t \geq 0, \\ x + |t|\,\mathcal{F}' & \text{if } t \leq 0, \end{cases} \quad \text{and} \quad Y(t) = \begin{cases} y + |t|\,\mathcal{F} & \text{if } t \geq 0, \\ y + |t|\,\mathcal{F}' & \text{if } t \leq 0. \end{cases}$$

Plainly, $Y : \mathbb{R} \to \mathbf{X}(G)$ is a geodesic line and $d(X(t), Y(t)) \to 0$ when $|t| \to \infty$. Note that for any $0 \leq t_1, t_2 \leq t$, $d\left(Y(-t), Y(t)\right) = 2t$ is not greater than

$$d\left(Y(-t), X(-t)\right) + d\left(X(-t), X(-t_1)\right) + d\left(X(-t_1), X(t_2)\right) + d\left(X(t_2), X(t)\right) + d\left(X(t), Y(t)\right).$$

The second and fourth term sum to $2t - (t_1 + t_2)$, thus $t_1 + t_2 \leq d\left(X(-t_1), X(t_2)\right)$. Since also $d\left(X(-t_1), X(t_2)\right) \leq t_1 + t_2$, it follows that $X : \mathbb{R} \to \mathbf{X}(G)$ is a geodesic line as well. Since the metric d is convex, the function $t \mapsto d(X(t), Y(t))$ is convex. Since it is also bounded, it must be constant, thus actually trivial. In particular, $d(x, y) = d(X(0), Y(0)) = 0$, therefore $x = y$ belongs to $\mathbf{X}(L)$. □

DEFINITION 104. The enclosure of $x, z \in \mathbf{X}(G)$ is defined by

$$\begin{aligned}\Diamond(x,z) &= \{y \in \mathbf{X}(G) : \mathbf{d}(x,z) = \mathbf{d}(x,y) + \mathbf{d}(y,z)\} \\ &= \{y \in \mathbf{X}(G) : \mathbf{d}(x,z) \geq \mathbf{d}(x,y) + \mathbf{d}(y,z)\}\end{aligned}$$

COROLLARY 105. *For any $S \in \mathbf{S}(G)$ and $x, x' \in \mathbf{X}(S)$, let F and F' be the pair of opposed facets in $\mathbf{F}(S)$ such that $x' \in x + F$ and $x \in x' + F'$. Then*

$$\Diamond(x, x') = (x + \overline{F}) \cap (x' + \overline{F}').$$

In particular for any $x, z \in \mathbf{X}(G)$, the enclosure $\Diamond(x,z)$ is a closed and convex subset of $\mathbf{X}(G)$ which is contained in any apartment containing x and z.

PROOF. For $y \in \mathbf{X}(S)$, write $y = x + a$ and $x' = y + b$ with $a, b \in \mathbf{F}(S)$, so that

$$\mathbf{d}(x, x') = t(a+b), \quad \mathbf{d}(x,y) = t(a) \quad \text{and} \quad \mathbf{d}(y, x') = t(b).$$

Thus y belongs to $\Diamond(x, x')$ if and only if there exists a closed chamber C in $\mathbf{F}(S)$ containing a and b by proposition 97, which occurs precisely when a and b both belong to the closure $\overline{F}$ of the facet F of $\mathbf{F}(S)$ which contains $c = a + b$. Hence

$$\Diamond(x, x') \cap \mathbf{X}(S) = (x + \overline{F}) \cap (x' + \overline{F}').$$

In particular, the function $y \mapsto \mathbf{d}(x,y)$ is injective on $\Diamond(x,x') \cap \mathbf{X}(S)$. Now pick a pair of opposed minimal parabolic subgroups (B, B') of G with $B \cap B' = Z_G(S)$. Let $r, r' : \mathbf{X}(G) \twoheadrightarrow \mathbf{X}(S)$ be the corresponding retractions. For any $y \in \Diamond(x, x')$,

$$\mathbf{d}(x, r(y)) = \mathbf{d}(r(x), r(y)) \leq \mathbf{d}(x,y) \quad \text{and} \quad \mathbf{d}(r(y), x') = \mathbf{d}(r(y), r(x')) \leq \mathbf{d}(y, x')$$

since r is non-expanding for $\mathbf{d}$, therefore

$$\mathbf{d}(x,x') \leq \mathbf{d}(x, r(y)) + \mathbf{d}(r(y), x') \leq \mathbf{d}(x,y) + \mathbf{d}(y,x') = \mathbf{d}(x,x').$$

Thus $r(y)$ belongs to $\Diamond(x,x') \cap \mathbf{X}(S)$ and $\mathbf{d}(x, r(y)) = \mathbf{d}(x,y)$. Since the same conclusion holds for $r'(y)$, we obtain $r(y) = r'(y)$. Hence y belongs to $\mathbf{X}(S)$ and indeed $\Diamond(x,x') = (x + \overline{F}) \cap (x' + \overline{F}')$. The remaining assertions easily follow. □

5.2.13. Unique Geodesics. It may seem that the validity of the axiom CO^+ for a given affine $\mathbf{F}(G)$-space $\mathbf{X}(G)$ depends upon the chosen τ, but it does not. In fact, CO^+ is plainly equivalent to the conjonction of the following two axioms:

CO For any pair of standard geodesics $x, y : [0,1] \to \mathbf{X}(G)$ in $\mathbf{X}(G)$, the function

$$f : [0,1] \to \mathbf{C}(G), \qquad f(t) = \mathbf{d}(x(t), y(t))$$

is convex, i.e. for every λ and $t_1 \leq t_2$ in $[0,1]$,

$$f\left((1-\lambda)t_1 + \lambda t_2\right) \leq (1-\lambda) f(t_1) + \lambda f(t_2) \quad \text{in} \quad \mathbf{C}(G).$$

UG The metric space $(\mathbf{X}(G), d)$ is uniquely geodesic.

Now CO plainly does not depend upon the choice of τ, and UG also does not. Indeed, suppose that $\mathbf{X}(G)$ satisfies all of the above axioms (using τ in UG) and let τ' be another faithful representation of G, giving rise to a distance d' on $\mathbf{X}(G)$. We have to show that every geodesic segment $c : [0,1] \to \mathbf{X}(G)$ in $(\mathbf{X}(G), d')$ is standard, for then CO implies UG for $(\mathbf{X}(G), d')$. Now for all $t \in [0,1]$, we have

$$\begin{aligned} \mathbf{d}(c(0), c(1)) &\leq \mathbf{d}(c(0), c(t)) + \mathbf{d}(c(t), c(1)) \quad \text{in} \quad \mathbf{C}(G) \\ \text{and} \quad d'(c(0), c(1)) &= d'(c(0), c(t)) + d'(c(t), c(1)) \quad \text{in} \quad \mathbb{R}_+ \end{aligned}$$

from which easily follows that actually

$$\mathbf{d}(c(0), c(1)) = \mathbf{d}(c(0), c(t)) + \mathbf{d}(c(t), c(1)) \quad \text{in} \quad \mathbf{C}(G).$$

Thus $c(t)$ belongs to $\Diamond(c(0), c(1))$ for all $t \in [0,1]$ and c is standard, being indeed contained in any apartment which contains $c(0)$ and $c(1)$ by corollary 105.

5.2.14. By the usual dyadic, reparametrization and triangulation tricks, it is sufficient to test the inequalities in CO or CO^+ for $(t_1, t_2, \lambda) = (0, 1, \frac{1}{2})$, for pairs of geodesics issuing from the same point. Thus CO is equivalent to either one of

CO' For every $x \in \mathbf{X}(G)$, $\mathcal{F}, \mathcal{G} \in \mathbf{F}(G)$ and $\lambda \in [0,1]$,

$$\mathbf{d}(x + \lambda\mathcal{F}, x + \lambda\mathcal{G}) \leq \lambda \mathbf{d}(x + \mathcal{F}, x + \mathcal{G}) \quad \text{in} \quad \mathbf{C}(G).$$

CO'' For every $x \in \mathbf{X}(G)$ and $\mathcal{F}, \mathcal{G} \in \mathbf{F}(G)$,

$$\mathbf{d}(x + \tfrac{1}{2}\mathcal{F}, y + \tfrac{1}{2}\mathcal{G}) \leq \tfrac{1}{2}\mathbf{d}(x + \mathcal{F}, x + \mathcal{G}) \quad \text{in} \quad \mathbf{C}(G).$$

5.2.15. There is a unique, G-equivariant and continuous map

$$\mathbf{X}(G) \times \mathbf{X}(G) \times [0,1] \to \mathbf{X}(G), \quad (x, y, \lambda) \mapsto (1-\lambda)x + \lambda y$$

such that $(1-\lambda)x + \lambda y = x + \lambda\mathcal{F}$ for any $\mathcal{F} \in \mathbf{F}(x,y)$. We set

$$[x, y] = \{(1-\lambda)x + \lambda y : \lambda \in [0,1]\}$$

and call it the *segment* between x and y. It is contained in the enclosure $\Diamond(x, y)$, thus also contained in any apartment $\mathbf{X}(S)$ which contains x and y. In particular, the intersection of two apartments is a convex subset of both apartments. A *subdivision* of $[x, y]$ is a finite collection $x = x_0, \cdots, x_n = y$ of points in $[x, y]$ such that

$$x_i = (1-\lambda_i)x + \lambda_i y, \quad 0 \leq \lambda_0 \leq \cdots \leq \lambda_n = 1.$$

Thus $[x, y] = \cup_{i=1}^n [x_{i-1}, x_i]$ and

$$\mathbf{d}(x, y) = \textstyle\sum_{i=1}^n \mathbf{d}(x_{i-1}, x_i).$$

5.2.16. For an affine $\mathbf{F}(G)$-building $\mathbf{X}(G)$, we denote by

$$\mathbf{d}^r : \mathbf{X}(G) \times \mathbf{X}(G) \to \mathbf{C}^r(G) \quad \text{and} \quad \mathbf{d}^c : \mathbf{X}(G) \times \mathbf{X}(G) \to \mathbf{G}(Z)$$

the components of $\mathbf{d}$. These are G-invariant functions. For $x, y, z \in \mathbf{X}(G)$,

$$\mathbf{d}^r(x, z) \leq \mathbf{d}^r(x, y) + \mathbf{d}^r(y, z) \quad \text{and} \quad \mathbf{d}^c(x, z) = \mathbf{d}^c(x, y) + \mathbf{d}^c(y, z).$$

The function $g \mapsto \mathbf{d}^c(x, gx)$ thus does not depend upon x and defines a morphism

$$\nu^c_{\mathbf{X}} : \mathsf{G} \to \mathbf{G}(Z).$$

5.2.17. A morphism of affine $\mathbf{F}(G)$-spaces $f : \mathbf{X}(G) \to \mathbf{Y}(G)$ is a G-equivariant map between the underlying sets which is compatible with their structure maps:

$$f(\mathbf{X}(S)) \subset \mathbf{Y}(S) \quad \text{and} \quad f(x + \mathcal{F}) = f(x) + \mathcal{F}$$

for every $S \in \mathbf{S}(G)$, $x \in \mathbf{X}(G)$ and $\mathcal{F} \in \mathbf{F}(G)$. If $\mathbf{Y}(G)$ is an $\mathbf{F}(G)$-building, it is sufficent to require the second condition. A morphism of affine $\mathbf{F}(G)$-buildings is a morphism of the underlying affine $\mathbf{F}(G)$-spaces. Any such morphism is an automorphism: it is bijective on any appartment, thus globally bijective by $R(s)$. It is compatible with the $\mathbf{d}$-maps, and an isometry of the underlying metric spaces.

5.2.18. An automorphism θ of an affine $\mathbf{F}(G)$-building $\mathbf{X}(G)$ acts on the apartment $\mathbf{X}(S)$ by an $\mathsf{N}_G(S)$-equivariant translation, which is thus given by a vector θ_S in $\mathbf{G}(Z) = \mathbf{G}(S)^{\mathsf{W}_G(S)}$, where $Z = Z(G)$. The G-equivariance of θ then implies that $S \mapsto \theta_S$ is also G-equivariant, thus constant. It follows that

$$\mathrm{Aut}(\mathbf{X}(G)) = \mathbf{G}(Z)$$

with $\mathcal{G} \in \mathbf{G}(Z)$ acting on $\mathbf{X}(G)$ by $x \mapsto x + \mathcal{G}$.

5.2.19. For an affine $\mathbf{F}(G)$-building $\mathbf{X}(G)$, we define

$$\mathbf{X}^r(G) = \mathbf{X}(G)/\mathbf{G}(Z) \quad \text{and} \quad \mathbf{X}^e(G) = \mathbf{X}^r(G) \times \mathbf{G}(Z).$$

The group G acts: on the quotient $\mathbf{X}^r(G)$ of $\mathbf{X}(G)$, on $\mathbf{G}(Z)$ by translations through the morphism $\nu^c_{\mathbf{X}} : \mathsf{G} \to \mathbf{G}(Z)$, and on $\mathbf{X}^e(G)$ diagonally. Then, the formulas

$$\mathbf{X}^r(S) = \mathbf{X}(S)/\mathbf{G}(Z) \quad \text{and} \quad \mathbf{X}^e(S) = \mathbf{X}^r(S) \times \mathbf{G}(Z)$$

yield G-equivariant maps $\mathbf{X}^r : \mathbf{S}(G) \to \mathcal{P}(\mathbf{X}^r(G))$ and $\mathbf{X}^e : \mathbf{S}(G) \to \mathcal{P}(\mathbf{X}^e(G))$, the pull map on $\mathbf{X}(G)$ descends to a G-equivariant map $+ : \mathbf{X}^r(G) \times \mathbf{F}^r(G) \to \mathbf{X}^r(G)$, which together with the addition map on $\mathbf{G}(Z)$ yields a G-equivariant map

$$+ : \mathbf{X}^e(G) \times \mathbf{F}(G) \to \mathbf{X}^e(G) \qquad ([x], \theta) + \mathcal{F} = ([x] + \mathcal{F}^r, \theta + \mathcal{F}^c).$$

The resulting triple $\mathbf{X}^e(G)$ is yet another affine $\mathbf{F}(G)$-building, with $\nu_{\mathbf{X}^e} = \nu_{\mathbf{X}}$. In fact, any point $x_0 \in \mathbf{X}(G)$ defines an isomorphism of affine $\mathbf{F}(G)$-buildings

$$\mathbf{X}(G) \simeq \mathbf{X}^e(G) \qquad x \mapsto ([x], \mathbf{d}^c(x_0, x)).$$

Thus $\mathbf{X}^e(G)$ appears as a rigidified version of $\mathbf{X}(G)$: there are no non-trivial automorphisms of $\mathbf{X}^e(G)$ preserving the subspace $\mathbf{X}^r(G) \simeq \mathbf{X}^r(G) \times \{0\}$ of $\mathbf{X}^e(G)$. The decomposition $\mathbf{X}^e(G) = \mathbf{X}^r(G) \times \mathbf{G}(Z)$ is orthogonal in the following sense:

$$\forall (x, \theta), (x', \theta') \in \mathbf{X}^e(G) : \qquad d((x, \theta), (x', \theta'))^2 = d(x, x')^2 + d(\theta, \theta')^2.$$

This follows from the analogous result for $\mathbf{F}(G)$, see 4.2.4.

5.2.20. If $\mathbf{X}(G) = (\mathbf{X}(G), \mathbf{X}(-), +)$ is an affine $\mathbf{F}(G)$-space or building, then so is $\mathbf{X}_\lambda(G) = (\mathbf{X}(G), \mathbf{X}(-), +_\lambda)$ for any $\lambda > 0$ in $\mathbb{R}$, where $x +_\lambda \mathcal{F} = x + \lambda \mathcal{F}$. The types $\nu_{\mathbf{X}}$ of $\mathbf{X}(G)$ and $\nu_{\mathbf{X}_\lambda}$ of $\mathbf{X}_\lambda(G)$ are related by $\nu_{\mathbf{X}} = \lambda \cdot \nu_{\mathbf{X}_\lambda}$.

5.3. Further axioms

Let $\mathbf{X}(G)$ be an affine $\mathbf{F}(G)$-space.

5.3.1. The axiom $L(s)^+$. The following is a sharp strengthening of $L(s)$:

$L(s)^+$ For any $x \in \mathbf{X}(G)$ and $\mathcal{F}, \mathcal{G} \in \mathbf{F}(G)$, there exists $S \in \mathbf{S}(G)$ and $\epsilon > 0$ such that $\mathcal{F} \in \mathbf{F}(S)$ and $x + \lambda \mathcal{G} \in \mathbf{X}(S)$ for every $\lambda \in [0, \epsilon]$.

PROPOSITION 106. *If $\mathbf{X}(G)$ satisfies $L(s)^+$, $R(s)$, $R(i)$ and UN, then it is an affine $\mathbf{F}(G)$-building and $(\mathbf{X}(G), d)$ is a $CAT(0)$-space.*

Suppose that $\mathbf{X}(G)$ satisfies $L(s)^+$, $R(s)$, $R(i)$ and UN. Then it already satisfies all the axioms of section 5.2.5 and 5.2.6, giving rise to the vectorial distance $\mathbf{d}$ which is the subject of the remaining axioms. We do not yet know that $\mathbf{d}$ satisfies TR, thus $d = \|\mathbf{d}\|$ may not be a distance on $\mathbf{X}(G)$. But for any apartment $\mathbf{X}(S)$, the restriction of $\mathbf{d}$ to $\mathbf{X}(S)$ satisfies TR and d is a Euclidean distance on $\mathbf{X}(S)$.

LEMMA 107. *For $x \in \mathbf{X}(G)$ and $\mathcal{F}, \mathcal{G} \in \mathbf{F}(G)$, there exists $S \in \mathbf{S}(G)$, $\mathcal{G}^* \in \mathbf{F}(G)$ and $\epsilon > 0$ such that $x \in \mathbf{X}(S)$, $\mathcal{F}, \mathcal{G}^* \in \mathbf{F}(S)$ and*

$$\forall \lambda \in [0, \epsilon] : \quad x + \lambda \mathcal{G} = x + \lambda \mathcal{G}^* \in \mathbf{X}(S).$$

PROOF. By $L(s)^+$, there exists $S \in \mathbf{S}(G)$ and $\epsilon > 0$ such that $\mathcal{F} \in \mathbf{F}(S)$ and $x(\lambda) = x + \lambda \mathcal{G} \in \mathbf{X}(S)$ for $\lambda \in [0, \epsilon]$. For any $0 \leq \lambda \leq \lambda'$, $x(\lambda') = x(\lambda) + (\lambda' - \lambda)\mathcal{G}$ by AC, thus $\mathbf{d}(x(\lambda), x(\lambda')) = (\lambda' - \lambda) \cdot t(\mathcal{G})$ in $\mathbf{C}(G)$ and $d(x(\lambda), x(\lambda')) = (\lambda' - \lambda) \cdot \|\mathcal{G}\|$ in $\mathbb{R}_+$. In particular, $x(-) : [0, \epsilon] \to \mathbf{X}(S)$ is a geodesic segment in $\left(\mathbf{X}(S), d|_{\mathbf{X}(S)}\right)$. There is thus a unique $\mathcal{G}^* \in \mathbf{F}(S)$ such that $x(\lambda) = x + \lambda \mathcal{G}^*$ for $\lambda \in [0, \epsilon]$. □

LEMMA 108. *For any $x, y \in \mathbf{X}(G)$ and $\mathcal{G} \in \mathbf{F}(G)$, the function*

$$c : \mathbb{R}_+ \to \mathbf{C}(G), \qquad c(\lambda) = \mathbf{d}(y, x + \lambda \mathcal{G})$$

is continuous for the canonical topologies on $\mathbb{R}_+$ and $\mathbf{C}(G)$.

PROOF. Pick $\mathcal{F} \in \mathbf{F}(G)$ with $y = x + \mathcal{F}$ using $T(s)$. By the previous lemma, there exists $S \in \mathbf{S}(G)$, $\mathcal{G}^* \in \mathbf{F}(S)$ and $\epsilon > 0$ such that $x \in \mathbf{X}(S)$, $\mathcal{F} \in \mathbf{F}(S)$ and $x + \lambda \mathcal{G} = x + \lambda \mathcal{G}^*$ for $\lambda \in [0, \epsilon]$. Since $x, y \in \mathbf{X}(S)$ and $\mathcal{G}^* \in \mathbf{F}(S)$, the function $\lambda \mapsto \mathbf{d}(y, x + \lambda \mathcal{G}^*)$ is plainly continuous on $\mathbb{R}_+$, thus c is continuous on $[0, \epsilon]$. Changing x to $x + \lambda \mathcal{G}$, we find that c is right continous on $\mathbb{R}_+$. By $L(s)$, there is an $S' \in \mathbf{S}(G)$ with $x \in \mathbf{X}(S')$, $\mathcal{G} \in \mathbf{F}(S')$. Set $\mathcal{G}' = \iota_S \mathcal{G}$. Then for $\lambda' \geq \lambda \geq 0$,

$$\begin{array}{rrcll} & x(\lambda) & = & x(\lambda') + (\lambda' - \lambda) \cdot \mathcal{G}' & \text{in} \quad \mathbf{X}(G), \\ \text{thus} & c(\lambda) & = & \mathbf{d}\left(y, x(\lambda') + (\lambda' - \lambda) \cdot \mathcal{G}'\right) & \text{in} \quad \mathbf{C}(G). \end{array}$$

It follows that c is also left continuous on $\mathbb{R}_+$. □

LEMMA 109. *For any $x \in \mathbf{X}(G)$ and $\mathcal{F}, \mathcal{G} \in \mathbf{F}(G)$,*

$$x + \mathcal{F} = x + \mathcal{G} \quad \Longrightarrow \quad \forall \lambda \in [0, 1] : x + \lambda \mathcal{F} = x + \lambda \mathcal{G}.$$

PROOF. Suppose $x + \mathcal{F} = x + \mathcal{G}$, put $x(\lambda) = x + \lambda \mathcal{F}$, $y(\lambda) = x + \lambda \mathcal{G}$ and define

$$\lambda_0 = \inf\{1, \lambda \in [0, 1] \text{ such that } x(\lambda) \neq y(\lambda)\}.$$

Suppose that $\lambda_0 \in [0, 1[$. If $\lambda_0 \neq 0$, then since $x(\lambda) = y(\lambda)$ for all $\lambda \in [0, \lambda_0[$,

$$\mathbf{d}(x(\lambda_0), y(\lambda_0)) = \lim_{s \to s_0^-} \mathbf{d}(x(\lambda_0), y(\lambda)) = \lim_{s \to s_0^-} \mathbf{d}(x(\lambda_0), x(\lambda)) = 0$$

by the previous lemma, thus $x(\lambda_0) = y(\lambda_0)$. Changing $(x, \mathcal{F}, \mathcal{G})$ to

$$(x(\lambda_0), (1 - \lambda_0)\mathcal{F}, (1 - \lambda_0)\mathcal{G}),$$

we may assume that $\lambda_0 = 0$. By lemma 107, there exists $S \in \mathbf{S}(G)$, $\mathcal{G}^* \in \mathbf{F}(S)$ and $\epsilon > 0$ such that $x \in \mathbf{X}(S)$, $\mathcal{F} \in \mathbf{F}(S)$ and $y(\lambda) = x + \lambda \mathcal{G}^*$ for $\lambda \in [0, \epsilon]$. Since $x + \epsilon \mathcal{G} = x + \epsilon \mathcal{G}^*$ and $x + \mathcal{G} = x + \mathcal{F}$ in $\mathbf{X}(G)$, $t(\mathcal{G}^*) = t(\mathcal{G}) = t(\mathcal{F})$ in $\mathbf{C}(G)$ with $\mathcal{F}, \mathcal{G}^* \in \mathbf{F}(S)$, thus $\mathcal{G}^* = w\mathcal{F}$ for some $w \in \mathsf{W}_G(S)$. In the affine $\mathbf{F}(S)$-space $\mathbf{X}(S)$,

$$x(1) = x + \mathcal{F} = (x + \lambda \mathcal{G}^*) + (\mathcal{F} - \lambda \mathcal{G}^*) = y(\lambda) + (\mathcal{F} - \lambda w \mathcal{F})$$

for all $\lambda \in [0, \epsilon]$. Since $x(1) = y(1)$, we thus find that for $\lambda \in [0, \epsilon]$,

$$t((1 - \lambda)\mathcal{F}) = (1 - \lambda)t(\mathcal{G}) = \mathbf{d}(y(\lambda), y(1)) = \mathbf{d}(y(\lambda), x(1)) = t(\mathcal{F} - \lambda w \mathcal{F}).$$

Let C be a closed chamber in $\mathbf{F}(S)$ such that $\mathcal{F} - \lambda w \mathcal{F} \in C$ for all $\lambda \in [0, \epsilon]$ (shrinking ϵ if necessary). Since t is injective on C, $(1 - \lambda)\mathcal{F} = \mathcal{F} - \lambda w \mathcal{F}$ in $C \subset \mathbf{F}(S)$ for all $\lambda \in [0, \epsilon]$, thus $\mathcal{F} = w\mathcal{F} = \mathcal{G}^*$. But then $x(\lambda) = y(\lambda)$ for all $\lambda \in [0, \epsilon]$, a contradiction. Therefore $\lambda_0 = 1$, i.e. $x(\lambda) = y(\lambda)$ for all $\lambda \in [0, 1]$. □

Using $R(s)$ and the previous lemma, we may now define segments in $\mathbf{X}(G)$ and their subdivisions as in section 5.2.15, with $[x,y] \subset \mathbf{X}(S)$ if $x,y \in \mathbf{X}(S)$.

LEMMA 110. *For every $x,y \in \mathbf{X}(G)$ and $z \in \mathbf{X}(G)$ (resp. $\mathcal{F} \in \mathbf{F}(G)$), there exists a subdivision $x = x_0, \cdots, x_n = y$ of the segment $[x,y]$ and for $i \in \{1,\cdots,n\}$, an $S_i \in \mathbf{S}(G)$ such that $[x_{i-1}, x_i] \subset \mathbf{X}(S_i)$ and $z \in \mathbf{X}(S_i)$ (resp. $\mathcal{F} \in \mathbf{F}(S_i)$).*

PROOF. By $R(s)$, there is an $S \in \mathbf{S}(G)$ such that $x,y \in \mathbf{X}(S)$, so that

$$y = x + \mathcal{G}^+ \quad \text{and} \quad x = y + \mathcal{G}^- \quad \text{with} \quad \mathcal{G}^\pm \in \mathbf{F}(S), \quad \mathcal{G}^+ + \mathcal{G}^- = 0.$$

For $\lambda \in [0,1]$, set $x(\lambda) = x + \lambda\mathcal{G}^+$ and choose $\mathcal{F}_\lambda \in \mathbf{F}(G)$ such that $z = x(\lambda) + \mathcal{F}_\lambda$ using $T(s)$. By $L(s)^+$, there exists $\epsilon_\lambda > 0$ and $S^\pm_\lambda \in \mathbf{S}(G)$ such that $\mathcal{F}_\lambda \in \mathbf{F}(S^\pm_\lambda)$ (resp. $\mathcal{F} \in \mathbf{F}(S^\pm_\lambda)$) and $x(\lambda) + \mu\mathcal{G}^\pm \in \mathbf{X}(S^\pm_\lambda)$ for all $\mu \in [0,\epsilon_\lambda]$. Pick a finite set $\mathcal{S} \subset [0,1]$ such that $[0,1] \subset \cup_{\lambda\in\mathcal{S}}]\lambda - \epsilon_\lambda, \lambda + \epsilon_\lambda[$ and let $x = x_0, \cdots, x_n = y$ be the subdivision of $[x,y]$ defined by $[x,y] \cap \{x, y, x(\lambda), x(\lambda) \pm \epsilon_\lambda : \lambda \in \mathcal{S}\}$. Then for each $i \in \{1,\cdots,n\}$, there exists an $S_i \in \{S^\pm_\lambda : \lambda \in \mathcal{S}\}$ such that $[x_{i-1}, x_i] \in \mathbf{X}(S_i)$ and $z \in \mathbf{X}(S_i)$ (resp. $\mathcal{F} \in \mathbf{F}(S_i)$). □

LEMMA 111. *For a minimal parabolic subgroup $B = U \rtimes Z_G(S)$ of G, the apartment $\mathbf{X}(S)$ is a fundamental domain for the action of U on $\mathbf{X}(G)$ and the corresponding retraction $r : \mathbf{X}(G) \twoheadrightarrow \mathbf{X}(S)$ is non-expanding for $\mathbf{d}$.*

PROOF. First, $\mathbf{X}(G) = \mathsf{U} \cdot \mathbf{X}(S)$ by $L(s)$. For $x,y \in \mathbf{X}(S)$ and any $\mathcal{F} \in \mathbf{F}(S)$, $\mathbf{d}(x,y) = \mathbf{d}(x+\mathcal{F}, y+\mathcal{F})$ in $\mathbf{C}(G)$. If $y = ux$ with $u \in \mathsf{U}$ and $P_\mathcal{F} = B$, then also

$$\lim_{s\to\infty} \mathbf{d}(x + s\mathcal{F}, y + s\mathcal{F}) = 0$$

by UN, thus $\mathbf{d}(x,y) = 0$ and $x = y$, i.e. $\mathbf{X}(S)$ is a fundamental domain for the action of U on $\mathbf{X}(G)$. Let $r : \mathbf{X}(G) \twoheadrightarrow \mathbf{X}(S)$ be the corresponding retraction. For $x,y \in \mathbf{X}(G)$, there exists by the previous lemma a subdivision $x = x_0, \cdots, x_n = y$ of $[x,y]$ and for each $i \in \{1,\cdots,n\}$, an $S_i \in \mathbf{S}(G)$ such that $[x_{i-1}, x_i] \subset \mathbf{X}(S_i)$ and $Z_G(S_i) \subset B$. Then, there is a unique $u_i \in \mathsf{U}$ such that $\mathrm{Int}(u_i)(S_i) = S$, in which case also $u_i \cdot \mathbf{X}(S_i) = \mathbf{X}(S)$ and $r(z) = u_i z$ for all $z \in \mathbf{X}(S_i)$. We thus obtain

$$\begin{array}{rcl} \mathbf{d}(x,y) & = & \sum_{i=1}^n \mathbf{d}(x_{i-1}, x_i) \\ & = & \sum_{i=1}^n \mathbf{d}(u_i x_{i-1}, u_i x_i) \\ & = & \sum_{i=1}^n \mathbf{d}\left(r(x_{i-1}), r(x_i)\right) \\ & \geq & \mathbf{d}\left(r(x), r(y)\right) \end{array}$$

in $\mathbf{C}(G)$ by the known triangle inequality for $\mathbf{d}$ in $\mathbf{X}(S)$. □

LEMMA 112. *The vectorial distance $\mathbf{d}$ satisfies TR, NE, CO and $(\mathbf{X}(G), d)$ is a $CAT(0)$-metric space – thus $\mathbf{X}(G)$ also satisfies UG.*

PROOF. For $x,y,z \in \mathbf{X}(G)$, choose $S \in \mathbf{S}(G)$ with $x,z \in \mathbf{X}(S)$ using $R(s)$, pick a minimal parabolic subgroup B of G with Levi $Z_G(S)$ and let $r : \mathbf{X}(G) \twoheadrightarrow \mathbf{X}(S)$ be the corresponding retraction. Then

$$\mathbf{d}(x,z) \leq \mathbf{d}\left(x, r(y)\right) + \mathbf{d}\left(r(y), z\right) \leq \mathbf{d}(x,y) + \mathbf{d}(y,z)$$

by the triangle inequality in $\mathbf{X}(S)$ and the previous lemma. This proves TR.

For $x,y \in \mathbf{X}(G)$ and $\mathcal{F} \in \mathbf{F}(G)$, pick a subdivision $x = x_0, \cdots, x_n = y$ of $[x,y]$ and for each $i \in \{1,\cdots,n\}$, an $S_i \in \mathbf{S}(G)$ such that $[x_{i-1}, x_i] \in \mathbf{X}(S_i)$ and $\mathcal{F} \in \mathbf{F}(S_i)$, using lemma 110. Then

$$\mathbf{d}(x+\mathcal{F}, y+\mathcal{F}) \leq \sum_{i=1}^n \mathbf{d}(x_{i-1}+\mathcal{F}, x_i+\mathcal{F}) = \sum_{i=1}^n \mathbf{d}(x_{i-1}, x_i) = \mathbf{d}(x,y)$$

by the triangle inequality in $\mathbf{X}(G)$ that we have just proven and a trivial computation in the affine $\mathbf{F}(S_i)$-space $\mathbf{X}(S_i)$. This proves NE.

For $x \in \mathbf{X}(G)$ and $\mathcal{F}, \mathcal{G} \in \mathbf{F}(G)$, set $y = x + \mathcal{F}$, $z = x + \mathcal{G}$. By lemma 110, there is a subdivision $y = x_0, \cdots, x_n = z$ of the segment $[y, z]$ and for each $i \in \{1, \cdots, n\}$, an $S_i \in \mathbf{S}(G)$ such that $[x_{i-1}, x_i] \in \mathbf{X}(S_i)$ and $x \in \mathbf{X}(S_i)$. For $i \in \{0, \cdots, n\}$ and $\lambda \in [0, 1]$, set $x_i(\lambda) = (1 - \lambda)x + \lambda x_i$ in $[x, x_i]$. Then

$$\mathbf{d}(x_0(\lambda), x_n(\lambda)) \leq \textstyle\sum_{i=1}^n \mathbf{d}(x_{i-1}(\lambda), x_i(\lambda)) = \sum_{i=1}^n \lambda \mathbf{d}(x_{i-1}, x_i) = \lambda \mathbf{d}(y, z)$$

by the triangle inequality in $\mathbf{X}(G)$ and a trivial computation in the affine $\mathbf{F}(S_i)$-space $\mathbf{X}(S_i)$. This proves CO', from which CO follows.

To establish the $CAT(0)$-property, imagine a rigid comparison triangle $(\tilde{x}, \tilde{y}, \tilde{z})$ for (x, y, z), lying on a Euclidean 2-plane E. Add flex points $(\tilde{x}_1, \cdots, \tilde{x}_{n-1})$ on the segment $[\tilde{y}, \tilde{z}]$ corresponding to $(x_1, \cdots, x_{n-1})$, and push them (inward or outward) one by one, so that each $(\tilde{x}, \tilde{x}_{i-1}, \tilde{x}_i)$ becomes a comparison triangle for (x, x_{i-1}, x_i) (with $\tilde{x}_0 = \tilde{y}$ and $\tilde{x}_n = \tilde{z}$). If a last outward move occurs at the i-th step, then in the final configuration, the chord between $\tilde{x}_{i-1}$ and $\tilde{x}_{i+1}$ intersects the radius between $\tilde{x}$ and $\tilde{x}_i$ at some point $\tilde{y} = (1 - \nu)\tilde{x} + \nu \tilde{x}_i$, with $\nu \in [0, 1[$. For the corresponding point $y = (1 - \nu)x + \nu x_i$ on the segment $[x, x_i] \subset \mathbf{X}(S_i) \cap \mathbf{X}(S_{i+1})$, we would have:

$$\begin{aligned} d(x_{i-1}, y) + d(y, x_{i+1}) &= d(\tilde{x}_{i-1}, \tilde{y}) + d(\tilde{y}, \tilde{x}_{i+1}) \\ &< d(\tilde{x}_{i-1}, \tilde{x}_i) + d(\tilde{x}_i, \tilde{x}_{i+1}) \\ &= d(x_{i-1}, x_i) + d(x_i, x_{i+1}) \\ &= d(x_{i-1}, x_{i+1}) \end{aligned}$$

which contradicts the triangle inequality for d in $\mathbf{X}(G)$. It follows that there is no last outward move, i.e. no outward move at all. Thus for any $\lambda \in [0, 1]$, if $\tilde{x}(\lambda)$ is the point corresponding to $x(\lambda) = (1 - \lambda)y + \lambda z \in [y, z]$ on the articulated segment $[\tilde{y}, \tilde{z}]$ of our comparison triangle, the distance between $\tilde{x}$ and $\tilde{x}(\lambda)$ is not greater in the final configuration than it was initially. Since the final distance is the actual distance between x and $x(\lambda)$ in $(\mathbf{X}(G), d)$, this proves the required $CAT(0)$ inequality for x and the *standard* geodesic segment $x(-) : [0, 1] \to \mathbf{X}(G)$ from y to z. However, we still have to check that our metric space $(\mathbf{X}(G), d)$ is unically geodesic in the usual sense. Suppose therefore that $x'(-) : [0, 1] \to \mathbf{X}(G)$ is another geodesic segment between y and z. For $\lambda \in [0, 1]$, the $CAT(0)$-inequality that we have just established for the point $x'(\lambda)$ and the standard geodesic $x(-) : [0, 1] \to \mathbf{X}(G)$ implies that $x(\lambda) = x'(\lambda)$, thus indeed $x(-) = x'(-)$. □

5.3.2. Discrete Buildings. The following axiom is a strengthening of $R(i)$:

$R(i)^+$ For $S, S' \in \mathbf{S}(G)$, there is a $g \in \mathsf{G}$ with

$$\mathrm{Int}(g)(S) = S' \quad \text{and} \quad g \equiv \mathrm{Id} \text{ on } \mathbf{X}(S) \cap \mathbf{X}(S').$$

LEMMA 113. *A discrete affine $\mathbf{F}(G)$-building $\mathbf{X}(G)$ satisfies $R(i)^+$.*

PROOF. We may assume that $Z = \mathbf{X}(S) \cap \mathbf{X}(S') \neq \emptyset$. Then Z is a non-empty closed convex subset of the affine $\mathbf{F}(S)$-space $\mathbf{X}(S)$, therefore Z has non-empty interior as a subset of its affine span A in $\mathbf{X}(S)$. Let $\sim$ be the equivalence relation on Z defined by $x \sim y$ if and only if x and y have the same stabilizer in $\mathsf{N}_G(S)$. Since $\mathbf{X}(G)$ is discrete, there are countably many equivalence classes, thus one of them at least, say $E \subset Z$, has the property that the closure of E has a non-empty interior in A. Then A is also the affine span of $\overline{E}$ or E in $\mathbf{X}(S)$. Let $C \subset \mathsf{N}_G(S)$

be the common stabilizer of the points of E. Now for any $g_1, g_2 \in \mathsf{G}$ such that $\mathrm{Int}(g_i)(S) = S'$ and $g_1 x = g_2 x$ for some $x \in E$, $g_2 = g_1 c$ for some $c \in C$, thus $g_1 \equiv g_2$ on E, A and Z. Fix $x \in E$. Then for any $y \in Z$, there exists by $R(i)$ some $g_y \in \mathsf{G}$ such that $\mathrm{Int}(g_y)(S) = S'$ and $g_y x = x$, $g_y y = y$. For $y, z \in Z$, we have just seen that $g_y \equiv g_z$ on E, A and Z, thus $g_y z = g_z z = z$ and $g_y \equiv \mathrm{Id}$ on Z. □

LEMMA 114. *The metric of a discrete affine* $\mathbf{F}(G)$*-building is complete.*

PROOF. Suppose that $\mathbf{X}(G)$ is discrete and equip $\mathcal{C} = \mathsf{G}\backslash\mathbf{X}(G)$ with

$$\overline{d}(\alpha,\beta) = \inf D(\alpha,\beta) \quad \text{where} \quad D(\alpha,\beta) = \{d(x,y) : x \in \alpha, y \in \beta\}.$$

Fix $S \in \mathbf{S}(G)$. Then by $R(i)$ and $R(s)$, also $\mathcal{C} = \mathsf{N}_G(S)\backslash\mathbf{X}(S)$ and

$$D(\alpha,\beta) = \{d(a, n\cdot b) : n \in \mathsf{N}_G(S)\}$$

if (a,b) lifts (α,β) in $\mathbf{X}(S)$. Since $\mathsf{N}_G(S)\cdot b$ is discrete in the Euclidean space $\mathbf{X}(S)$ by assumption, if follows that there is a constant $\epsilon(\alpha,\beta) > 0$ such that

$$\forall (x,y) \in \alpha\times\beta : \ d(x,y) \le \overline{d}(\alpha,\beta) + \epsilon(\alpha,\beta) \Longrightarrow d(x,y) = \overline{d}(\alpha,\beta).$$

In particular, $\overline{d}$ is a distance on $\mathcal{C}$. Moreover, $(\mathcal{C},\overline{d})$ is complete: if (α_n) is a Cauchy sequence in $\mathcal{C}$, it lifts to a bounded sequence (a_n) in $\mathbf{X}(S)$, the latter has a subsequence $(a_{\varphi(n)})$ converging to some a in $\mathbf{X}(S)$, whose image in $\mathcal{C}$ is then a limit of (α_n). Let now (x_n) be a Cauchy sequence in $(\mathbf{X}(G), d)$. Its image (α_n) is a Cauchy sequence in $(\mathcal{C},\overline{d})$, which thus converges to some α in $\mathcal{C}$. For each n, lift α to some $y_n \in \mathbf{X}(G)$ with $d(x_n, y_n) = \overline{d}(\alpha_n, \alpha)$. Then (y_n) is also a Cauchy sequence in $\mathbf{X}(G)$, hence $d(y_n, y_m) \le \epsilon(\alpha,\alpha)$ for $n, m \gg 0$, which implies that (y_n) is actually stationary and (x_n) converges to its limit: $(\mathbf{X}(G), d)$ is complete. □

5.4. Walls and tight buildings

Let again $\mathbf{X}(G)$ be an affine $\mathbf{F}(G)$-space.

5.4.1. For $S \in \mathbf{S}(G)$, let $\Phi(S,G)$ be the set of roots of S in the Lie algebra

$$\mathrm{Lie}(G) = \mathfrak{g} = \mathfrak{g}_0 \oplus \oplus_{a\in\Phi(S,G)} \mathfrak{g}_a.$$

We denote by $U_a \subset G$ the root subgroup corresponding to some $a \in \Phi(G,S)$: if S_a denotes the neutral component of the kernel of $a : S \to \mathbb{G}_{m,K}$, then U_a is the unipotent radical of the unique parabolic subgroup of $Z_G(S_a)$ containing $Z_G(S)$ with Lie algebra $\mathfrak{g}_0 \oplus \oplus_{b\in\mathbb{N}a\cap\Phi(S,G)} \mathfrak{g}_b$. If $2a \in \Phi(S,G)$, then $U_{2a} \subset U_a$.

5.4.2. For any $u \in \mathsf{U}_a \setminus \{1\}$, there exists a unique triple $(u_1, u_2, m(u))$ with

$$u_1 u u_2 = m(u), \quad u_1, u_2 \in \mathsf{U}_{-a} \setminus \{1\} \quad \text{and} \quad m(u) \in \mathsf{N}_G(S).$$

Moreover, $\nu_S^v(m(u))$ is the symmetry $s_a \in \mathsf{W}_G(S)$ attached to a, given by

$$s_a : \mathbf{G}(S) \to \mathbf{G}(S), \quad s_a(x) = x - a(x)a^\vee$$

where $a^\vee : \mathbb{G}_{m,K} \to S$ is the coroot corresponding to a and $a(x) = a \circ x$ in

$$\mathbb{R} = \mathrm{Hom}(\mathbb{D}_K(\mathbb{R}), \mathbb{G}_{m,K}).$$

This follows from [**5**, §5] by [**9**, 6.1.2.2 & 6.1.3.c]. Considering the action of $m(u)$ on $\mathbf{X}(S)$, we thus obtain a unique affine hyperplane $\mathbf{X}(S,u)$ in $\mathbf{X}(S)$ which is preserved by $m(u)$. The underlying vector space is the fixed point set of s_a, namely

$$\mathbf{G}(S_a) = \{x \in \mathbf{G}(S) : s_a(x) = x\} = \{x \in \mathbf{G}(S) : a(x) = 0\}$$

and $m(u)$ acts on $\mathbf{X}(S,u)$ by $x \mapsto x + \nu_{\mathbf{X}}(S,u)$ for some $v_{\mathbf{X}}(S,u) \in \mathbf{G}(S_a)$.

EXAMPLE 115. For the affine $\mathbf{F}(G)$-space $\mathbf{X}(G) = \mathbf{F}(G)$, $\mathbf{X}(S,u) = \mathbf{G}(S_a)$ is the fixed point set of $m(u)$ acting on $\mathbf{G}(S) = \mathbf{F}(S)$ and $\nu_{\mathbf{X}}(S,u) = 0$.

5.4.3. Of course $m(u)$ fixes $\mathbf{X}(S,u)$ if and only if $\nu_{\mathbf{X}}(S,u) = 0$, and this happens when $m(u)$ already has finite order in $\mathsf{N}_G(S)$, which holds true for any $u \in \mathsf{U}_a \setminus \{1\}$ if $2a \notin \Phi(S,G)$. Indeed, set $\Phi'(S,G) = \{b \in \Phi(S,G) : 2b \notin \Phi(S,G)\}$. This is again a root system and $U_b \simeq \mathbb{G}_{a,K}^{n(b)}$ for some $n(b) \geq 1$ for all $b \in \Phi'(S,G)$. Choose a set of simple roots Δ' of $\Phi'(S,G)$ containing a and choose for each $b \in \Delta'$ a 1-dimensional K-subspace U'_b in $\mathsf{U}_b \simeq K^{n(b)}$, with $u \in \mathsf{U}'_a$. Then by [**5**, 7.2], there is a unique split reductive subgroup G' of G containing S with $\Phi(S,G') = \Phi'(S,G)$ such that the root subgroup U'_b of $b \in \Delta'$ in G' is the subgroup of U_b determined by U'_b, i.e. $\mathsf{U}'_b = U'_b(K)$. Then $(Z_{G'}(S_a), S, a)$ is an elementary system in the sense of [**21**, XX 1.3] by [**21**, XIX 3.9]. Let $f : S_{\mathcal{L}} \to Z_{G'}(S_a)$ be the corresponding morphism constructed in [**21**, XX 5.8] and let $X \neq 0$ be the unique element of $\mathcal{L} = \mathrm{Lie}(U'_a)$ with $f\left(\begin{smallmatrix} 1 & X \\ 0 & 1 \end{smallmatrix}\right) = u$. Since

$$\begin{pmatrix} 1 & 0 \\ -X^{-1} & 1 \end{pmatrix}\begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -X^{-1} & 1 \end{pmatrix} = \begin{pmatrix} 0 & X \\ -X^{-1} & 0 \end{pmatrix}$$

in $S_{\mathcal{L}}(K)$, we find that

$$m(u) = f\begin{pmatrix} 0 & X \\ -X^{-1} & 0 \end{pmatrix}, \quad m(u)^2 = f\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad m(u)^4 = 1.$$

On the other hand if $2a \in \Phi(G,S)$, then [**47**, 1.15] provides examples where $m(u)$ has infinite order. Note also that $m(u)$ fixes $\mathbf{X}(S,u)$ when there is a $z \in \mathsf{Z}_G(S)$ such that $zuz^{-1} = u^{-1}$: since $m(u^{-1}) = m(u)^{-1}$ and $m(zuz^{-1}) = zm(u)z^{-1}$,

$$\begin{array}{rcl} \mathbf{X}(S,u^{-1}) & = & \mathbf{X}(S,u) \\ \nu_{\mathbf{X}}(S,u^{-1}) & = & -\nu_{\mathbf{X}}(S,u) \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathbf{X}(S,zuz^{-1}) & = & \mathbf{X}(S,u) + \nu_{\mathbf{X},S}(z) \\ \nu_{\mathbf{X}}(S,zuz^{-1}) & = & \nu_{\mathbf{X}}(S,u) \end{array}$$

therefore $zuz^{-1} = u^{-1}$ implies $\nu_{\mathbf{X}}(S,u) = 0$. Note also that since

$$\left(m(u)u_2m(u)^{-1}\right)u_1u = m(u) \quad \text{and} \quad uu_2\left(m(u)^{-1}u_1m(u)\right) = m(u)$$

we find that $m(u_1) = m(u_2) = m(u)$, thus

$$\mathbf{X}(S,u_1) = \mathbf{X}(S,u_2) = \mathbf{X}(S,u) \quad \text{and} \quad \nu_{\mathbf{X}}(S,u_1) = \nu_{\mathbf{X}}(S,u_2) = \nu_{\mathbf{X}}(S,u).$$

5.4.4. For a subset $\Omega \neq \emptyset$ of $\mathbf{X}(S)$, we denote by G_Ω the pointwise stabilizer of Ω in G and by $\mathsf{G}_{S,\Omega}$ the subgroup of G spanned by $\mathsf{N}_G(S)_\Omega = \mathsf{G}_\Omega \cap \mathsf{N}_G(S)$ and

$$\left\{u \in \mathsf{U}_a \setminus \{1\} : a \in \Phi(G,S),\, \Omega \subset \mathbf{X}^+(S,u)\right\}$$

where for any $a \in \Phi(G,S)$ and $u \in \mathsf{U}_a \setminus \{1\}$,

$$\mathbf{X}^+(S,u) = \mathbf{X}(S,u) + \{\mathcal{F} \in \mathbf{F}(S) : a(\mathcal{F}) \geq 0\}.$$

When $\Omega = \{x\}$, we simply write $\mathsf{G}_x = \mathsf{G}_{\{x\}}$ and $\mathsf{G}_{S,x} = \mathsf{G}_{S,\{x\}}$. Thus

$$\mathsf{G}_\Omega = \cap_{x\in\Omega}\mathsf{G}_x \quad \text{and} \quad \mathsf{G}_{S,\Omega} \subset \cap_{x\in X}\mathsf{G}_{S,x}.$$

For $\emptyset \neq \Omega' \subset \Omega \subset \mathbf{X}(S)$, $\mathsf{G}_\Omega \subset \mathsf{G}_{\Omega'}$ and $\mathsf{G}_{S,\Omega} \subset \mathsf{G}_{S,\Omega'}$. Finally for $g \in \mathsf{G}$ and $S' = \mathrm{Int}(g)(S)$, $\Omega' = g \cdot \Omega$, one checks easily that

$$\mathrm{Int}(g)(\mathsf{G}_\Omega) = \mathsf{G}_{\Omega'} \quad \text{and} \quad \mathrm{Int}(g)(\mathsf{G}_{S,\Omega}) = \mathsf{G}_{S',\Omega'}.$$

5.4.5. Example. For the affine $\mathbf{F}(G)$-space $\mathbf{X}(G)=\mathbf{F}(G)$ and $u\in\mathsf{U}_a\setminus\{1\}$,

$$\mathbf{X}^+(S,u)=\{\mathcal{F}\in\mathbf{F}(S):a(\mathcal{F})\geq 0\}\,.$$

For $\mathcal{F}\in\mathbf{F}(S)$, $\mathsf{G}_{\mathcal{F}}=\mathsf{P}_{\mathcal{F}}$ and $\mathsf{G}_{S,\mathcal{F}}$ is therefore the group spanned by $\mathsf{N}_G(S)\cap\mathsf{P}_{\mathcal{F}}$ and the U_a's for $a\in\Phi(G,S)$, $a(\mathcal{F})\geq 0$. The U_a's with $a(\mathcal{F})>0$ span $\mathsf{U}_{\mathcal{F}}$ by [**5**, 3.11]. Moreover, the group $\mathsf{N}_G(S)\cap\mathsf{P}_{\mathcal{F}}=\mathsf{N}_L(S)$ and the U_a's with $a(\mathcal{F})=0$ together span $\mathsf{L}=L(K)$ where L is the Levi subgroup of $P_{\mathcal{F}}$ which contains $Z_G(S)$, by the Bruhat decomposition of L, see [**5**, 5.15]. Therefore

$$\mathsf{G}_{S,\mathcal{F}}=\mathsf{P}_{\mathcal{F}}=\mathsf{G}_{\mathcal{F}}.$$

5.4.6. We next consider the following axioms:

ST (*Stabilizers*) For some (or every) $S\in\mathbf{S}(G)$,

$$\forall x\in\mathbf{X}(S):\quad \mathsf{G}_{S,x}=\mathsf{G}_x.$$

ST^- For some (or every) $S\in\mathbf{S}(G)$,

$$\forall x\in\mathbf{X}(S):\quad \mathsf{G}_{S,x}\subset\mathsf{G}_x.$$

ST_1^- For some (or every) $S\in\mathbf{S}(G)$,

$$\forall\emptyset\neq\Omega\in\mathbf{X}(S):\quad \mathsf{G}_{S,\Omega}\subset\mathsf{G}_\Omega.$$

ST_2^- For some (or every) $S\in\mathbf{S}(G)$ and any $a\in\Phi(G,S)$, $u\in\mathsf{U}_a\setminus\{1\}$,

$$\exists x\in\mathbf{X}(S,u):\qquad ux=x.$$

UN^+ For $x\in\mathbf{X}(G)$, $\mathcal{F}\in\mathbf{F}(G)$ and $u\in\mathsf{U}_{\mathcal{F}}$,

$$\forall t\gg 0:\qquad u(x+t\mathcal{F})=x+t\mathcal{F}.$$

LEMMA 116. *These axioms are related as follows:*

$$\begin{aligned} & ST\Longrightarrow ST^-\iff ST_1^-\iff ST_2^- \\ \textit{and}\qquad & ST^-+L(s)\Longrightarrow UN^+\Longrightarrow UN. \end{aligned}$$

Under ST^-, for every $S\in\mathbf{S}(G)$, $a\in\Phi(G,S)$ and $u\in\mathsf{U}_a\setminus\{1\}$, $\nu_{\mathbf{X}}(S,u)=0$ and

$$\mathbf{X}^+(S,u)=\{x\in\mathbf{X}(S):ux=x\}=\{x\in\mathbf{X}(S):ux\in\mathbf{X}(S)\}\,.$$

PROOF. Plainly, $ST\Rightarrow ST^-$, $ST^-\Leftrightarrow ST_1^-$ and $UN^+\Rightarrow UN$. Fix $S\in\mathbf{S}(G)$, $a\in\Phi(G,S)$, $u\in\mathsf{U}_a\setminus\{1\}$. Since $x\in\mathbf{X}(S,u)$ implies $u\in\mathsf{G}_{S,x}$, $ST^-\Rightarrow ST_2^-$. On the other hand if u fixes some $x\in\mathbf{X}(S)$, it also fixes $x+\mathcal{F}$ for every $\mathcal{F}\in\mathbf{F}(S)$ with $a(\mathcal{F})\geq 0$ because $a(\mathcal{F})\geq 0\iff U_a\subset P_{\mathcal{F}}$, thus $ST_2^-\Rightarrow ST^-$. Under ST_2^-,

$$\mathbf{X}^+(S,u)\subset\{x\in\mathbf{X}(S):ux=x\}\subset\{x\in\mathbf{X}(S):ux\in\mathbf{X}(S)\}\,.$$

Applying this to $u_1,u_2\in\mathsf{U}_{-a}\setminus\{1\}$, we obtain: $u_1\equiv u_2\equiv\mathrm{Id}$ on $\mathbf{X}(S)\setminus\mathbf{X}^+(S,u)$. If x and ux belong to $\mathbf{X}(S)$ but x does not belong to $\mathbf{X}^+(S,u)$, then ux also does not belong to $\mathbf{X}^+(S,u)$, however $m(u)x=u_1uu_2x=u_1ux=ux$ does, a contradiction. This proves the required displayed equality, and $\nu_{\mathbf{X}}(S,u)=0$ since $m(u)=u_1uu_2$ with $u,u_1,u_2\equiv\mathrm{Id}$ on $\mathbf{X}(S,u)=\mathbf{X}(S,u_1)=\mathbf{X}(S,u_2)$. Suppose finally that ST^- and $L(s)$ hold. For $x\in\mathbf{X}(G)$, $\mathcal{F}\in\mathbf{F}(G)$ and any $u\in\mathsf{U}_{\mathcal{F}}$ with $u\neq 1$, pick $S\in\mathbf{S}(G)$ with $x\in\mathbf{X}(S)$ and $\mathcal{F}\in\mathbf{F}(S)$ using $L(s)$. Since $\mathsf{U}_{\mathcal{F}}$ is spanned by the U_a's with $a\in\Phi(G,S)$, $a(\mathcal{F})>0$ (as in Example 5.4.5), we may write $u=u_1\cdots u_n$ with $u_i\in\mathsf{U}_{a_i}\setminus\{1\}$ for some $a_i\in\Phi(G,S)$, $a_i(\mathcal{F})>0$. For any sufficiently large $t\geq 0$, $x+t\mathcal{F}\in\mathbf{X}(S)$ then belongs to $\mathbf{X}^+(S,u_i)$ for all $i\in\{1,\cdots,n\}$, thus $u_i(x+t\mathcal{F})=x+t\mathcal{F}$ by ST^- and $u(x+t\mathcal{F})=x+t\mathcal{F}$, which proves UN^+. $\square$

5.4.7. The next axiom is related to alcove-based retractions, see [**37**, 1.4].

HA (*Half-apartments*) For $S_1, S_2, S_3 \in \mathbf{S}(G)$, if $\mathbf{X}(S_i) \cap \mathbf{X}(S_j)$ contains an half-subspace of $\mathbf{X}(S_i)$ for every pair (i,j) in $\{1,2,3\}^2$, then

$$\mathbf{X}(S_1) \cap \mathbf{X}(S_2) \cap \mathbf{X}(S_3) \neq \emptyset.$$

LEMMA 117. *The axioms UN^+ and HA imply ST^-.*

PROOF. Fix $S \in \mathbf{S}(G)$, $a \in \Phi(G,S)$ and $u \in \mathsf{U}_a \setminus \{1\}$ and first note that

$$\{x \in \mathbf{X}(S) : ux \in \mathbf{X}(S)\} = \{x \in \mathbf{X}(S) : ux = x\}.$$

Indeed if x and ux belong to $\mathbf{X}(S)$, pick $\mathcal{F} \in \mathbf{F}(S)$ with $a(\mathcal{F}) > 0$. Then $U_a \subset U_{\mathcal{F}}$, thus $ux + t\mathcal{F} = u(x + t\mathcal{F}) = x + t\mathcal{F}$ for $t \gg 0$ by UN^+ and $ux = x$ since $\mathbf{X}(S)$ is an affine $\mathbf{F}(S)$-space. For every $t \in \mathbb{R}$, we define

$$\begin{aligned} \mathbf{X}(S,u,t) &= \mathbf{X}(S,u) + \{\mathcal{F} \in \mathbf{F}(S) : a(\mathcal{F}) = t\} \\ \mathbf{X}^+(S,u,t) &= \mathbf{X}(S,u,t) + \{\mathcal{F} \in \mathbf{F}(S) : a(\mathcal{F}) \geq 0\} \\ \mathbf{X}^-(S,u,t) &= \mathbf{X}(S,u,t) + \{\mathcal{F} \in \mathbf{F}(S) : a(\mathcal{F}) \leq 0\} \end{aligned}$$

If u fixes some $x \in \mathbf{X}(S,u,t)$, then also $u \equiv \mathrm{Id}$ on $\mathbf{X}^+(S,u,t)$ since

$$\forall \mathcal{F} \in \mathbf{F}(S) : \qquad a(\mathcal{F}) \geq 0 \iff U_a \subset P_{\mathcal{F}}.$$

By UN^+, u fixes some point in $\mathbf{X}(S)$, thus $u \equiv \mathrm{Id}$ on $\mathbf{X}^+(S,u,t)$ for $t \gg 0$. Let us now write $u_1 u u_2 = m(u)$ with $u_1, u_2 \in \mathsf{U}_{-a} \setminus \{1\}$. Then similarly for $i \in \{1,2\}$,

$$\{x \in \mathbf{X}(S) : u_i x \in \mathbf{X}(S)\} = \{x \in \mathbf{X}(S) : u_i x = x\}$$

and u_i fixes $\mathbf{X}^+(S,u_i,t) = \mathbf{X}^-(S,u,-t)$ for $t \gg 0$. Choose $T > 0$ such that

$$u \equiv \mathrm{Id} \text{ on } \mathbf{X}^+(S,u,T) \quad \text{and} \quad u_1 \equiv u_2 \equiv \mathrm{Id} \text{ on } \mathbf{X}^-(S,u,-T).$$

Then: $\mathbf{X}(S)$ and $u\mathbf{X}(S)$ contain the half-subspace $\mathbf{X}^+(S,u,T)$, $\mathbf{X}(S)$ and $u_1^{-1}\mathbf{X}(S)$ contain $\mathbf{X}^-(S,u,-T)$, while $u\mathbf{X}(S)$ and $u_1^{-1}\mathbf{X}(S)$ contain

$$u\mathbf{X}^-(S,u,-T) = uu_2\mathbf{X}^-(S,u,-T) = u_1^{-1}m(u)\mathbf{X}^-(S,u,-T) = u_1^{-1}\mathbf{X}^+(S,u,T).$$

Thus by HA, there is a point $x \in \mathbf{X}(S) \cap u\mathbf{X}(S) \cap u_1^{-1}\mathbf{X}(S)$. Any such point is fixed by u^{-1} and u_1, thus also by $m(u)u_2^{-1} = u_1 u$. In particular $u_2^{-1}(x) = m(u)^{-1}(x)$ also belongs to $\mathbf{X}(S)$, so that again x is fixed by u_2, as well as $m(u) = u_1 u u_2$. But then x belongs to $\mathbf{X}(S,u) = \{x \in \mathbf{X}(S) : m(u)(x) = x\}$ and it is fixed by u, which proves ST_2^-, from which ST follows by the previous lemma. $\square$

5.4.8. An affine $\mathbf{F}(G)$-building is *tight* if it satisfies ST. It then also satisfies the conclusion of lemma 116, and it is determined by its type. More precisely:

LEMMA 118. *Suppose that $\mathbf{X}(G)$ is a tight affine $\mathbf{F}(G)$-building and $\mathbf{Y}(G)$ is an affine $\mathbf{F}(G)$-building which satisfies ST^-. Then $\nu_{\mathbf{X}} = \nu_{\mathbf{Y}} \iff \mathbf{X}(G) \simeq \mathbf{Y}(G)$.*

PROOF. We have to show that $\nu_{\mathbf{X}} = \nu_{\mathbf{Y}}$ implies $\mathbf{X}^e(G) \simeq \mathbf{Y}^e(G)$. Suppose therefore that $\nu_{\mathbf{X}} = \nu_{\mathbf{Y}}$. Pick $S \in \mathbf{S}(G)$. By [**39**, 2.1.9], there is a finite subgroup of $\mathsf{N}_G(S)$ which maps surjectively onto $\mathsf{W}_G(S)$, and which thus has unique fixed points x_S in $\mathbf{X}^r(S)$ and y_S in $\mathbf{Y}^r(S)$. Let $\theta_S : \mathbf{X}^e(G) \to \mathbf{Y}^e(G)$ be the unique isomorphism of affine $\mathbf{F}(S)$-spaces mapping $(x_S, 0)$ to $(y_S, 0)$. Then θ_S is $\mathsf{N}_G(S)$-equivariant, and it is the unique $\mathsf{N}_G(S)$-equivariant isomorphism of affine $\mathbf{F}(S)$-spaces from $\mathbf{X}^e(S)$ to $\mathbf{Y}^e(S)$ mapping $\mathbf{X}^r(S)$ to $\mathbf{Y}^r(S)$. If $\mathrm{Int}(g)(S) = S'$, then $g \circ \theta_S = \theta_{S'} \circ g$. For $x \in \mathbf{X}^e(S) \cap \mathbf{X}^e(S')$, there is such a g in G_x by $R(i)$ for $\mathbf{X}(G)$. Thus g belongs to $\mathsf{G}_{S,x}$ by ST for $\mathbf{X}(G)$, which equals $\mathsf{G}_{S,\theta_S(x)}$ by definition. Then $g \in \mathsf{G}_{\theta_S(x)}$ by our assumption on $\mathbf{Y}(G)$, thus $\theta_{S'}(x) = \theta_{S'}(gx) = g\theta_S(x) = \theta_S(x)$. Our isomorphisms θ_S therefore glue to $\theta : \mathbf{X}^e(G) \to \mathbf{Y}^e(G)$, which is the desired isomorphism. □

Remark 119. A tight affine $\mathbf{F}(G)$-building $\mathbf{X}(G)$ can be retrieved from any apartment $\mathbf{X}(S)$ together with its $\mathsf{N}_G(S)$-action. It is the quotient of $\mathsf{G} \times \mathbf{X}(S)$ for the equivalence relation $\sim$ induced by $(g, x) \mapsto gx$, which indeed only depends upon the apartment: $(g, x) \sim (g', x')$ if and only if $g' = gkn$ and $x' = n^{-1}x$ for some $k \in \mathsf{G}_{S,x}$ and $n \in \mathsf{N}_G(S)$.

5.5. Metric properties

Let $\mathbf{X}(G)$ be an affine $\mathbf{F}(G)$-building. We shall here relate our mostly algebraic formalism to various notions pertaining to the non-canonical metric $d = d_\tau$: rays, tangent spaces and Busemann functions. For simplicity, we furthermore assume that $(\mathbf{X}(G), d)$ is a $CAT(0)$-space.

5.5.1. There is a G-equivariant commutative diagram

$$\begin{tikzcd}
\mathbf{X}(G)\times\mathbf{F}(G) \arrow[r, hook, "\alpha"] \arrow[rr, bend left, "\mathrm{Id}\times\iota"] \arrow[dr, bend right, "+"'] & \mathbf{RX}(G) \arrow[r, hook, "\beta"] \arrow[d, "\mathrm{ev}_1"] & \mathbf{X}(G)\times\mathcal{C}(\partial\mathbf{X}(G)) \\
& \mathbf{X}(G) &
\end{tikzcd}$$

where the various new sets and maps are defined as follows:

- $\mathbf{RX}(G)$ is the set of all functions $f : \mathbb{R}^+ \to \mathbf{X}(G)$ such that
$$\exists c_f \geq 0 \text{ s.t. } \forall t, u \in \mathbb{R}^+ : \qquad d(f(t), f(u)) = c_f \, |u - t| \, .$$
- $\partial\mathbf{X}(G)$ is the visual boundary $\{f \in \mathbf{RX}(G) : c_f = 1\} / \sim$ of $\mathbf{X}(G)$, where the equivalence relation $\sim$ is defined on the whole of $\mathbf{RX}(G)$ by
$$f \sim g \iff t \mapsto d\left(f(t), g(t)\right) \text{ is bounded.}$$
- $\mathcal{C}(\partial\mathbf{X}(G))$ is the cone $(\mathbb{R}^+ \times \partial\mathbf{X}(G)) / \approx$ where the equivalence relation $\approx$ just collapses $\{0\} \times \partial\mathbf{X}(G)$ to a single point $0 \in \mathcal{C}(\partial\mathbf{X}(G))$, so that
$$f \mapsto [f] = \begin{cases} (c_f, \text{class of } f(c_f^{-1} -)) & \text{if } c_f \neq 0, \\ 0 & \text{if } c_f = 0, \end{cases}$$
identifies the quotient $\mathbf{RX}(G)/\sim$ with the cone $\mathcal{C}(\partial\mathbf{X}(G))$.
- $\alpha(x, \mathcal{F})(t) = x + t\mathcal{F}$, $\beta(f) = (f(0), [f])$ and $\mathrm{ev}_1(f) = f(1)$.

By the axiom NE for $\mathbf{X}(G)$, $\beta \circ \alpha = \mathrm{Id}_{\mathbf{X}(G)} \times \iota$ for some G-equivariant map
$$\iota : \mathbf{F}(G) \hookrightarrow \mathcal{C}(\partial\mathbf{X}(G)).$$
The latter is injective: suppose that $x + t\mathcal{F} \sim y + t\mathcal{G}$ and pick $z \in \mathbf{X}(S)$ for some $S \in \mathbf{S}(G)$ such that $\mathcal{F}, \mathcal{G} \in \mathbf{F}(S)$. Then $z + t\mathcal{F} \sim x + t\mathcal{F} \sim y + t\mathcal{G} \sim z + t\mathcal{G}$, thus $\mathcal{F} = \mathcal{G}$ since $z + t\mathcal{F} \sim z + t\mathcal{G}$ in the affine $\mathbf{F}(S)$-space $\mathbf{X}(S)$. It follows that α is also injective. Finally β is injective by convexity of the CAT(0)-distance d, and it is also surjective when $(\mathbf{X}(G), d)$ is complete [**8**, II.8.2] (for instance in the discrete case, by lemma 114). By the axiom $L(s)$, α is bijective precisely when every geodesic ray in $\mathbf{X}(G)$ is standard, in which case ι and β are also bijective.

REMARK 120. The injectivity of ι implies that the apartment map $S \mapsto \mathbf{X}(S)$ is injective: $\mathbf{X}(S)$ determines $\mathcal{C}(\partial\mathbf{X}(S)) = \iota(\mathbf{F}(S))$, thus also $\mathbf{F}(S)$ and $S \in \mathbf{S}(G)$.

5.5.2. Fix $x \in \mathbf{X}(G)$ and $0 \neq \mathcal{F}, \mathcal{G} \in \mathbf{F}(G)$, set $y = x + \mathcal{F}$, $z = x + \mathcal{G}$. We may then define the following five different types of angles

$$0 \leq \measuredangle_x(\mathcal{F},\mathcal{G}) \leq \measuredangle_x(\overrightarrow{xy},\mathcal{G}) \leq \measuredangle_x^c(y,z) \leq \measuredangle(\overrightarrow{xy},\mathcal{G}) \leq \measuredangle^x(\mathcal{F},\mathcal{G}) \leq \pi.$$

First, $\measuredangle_x^c(y,z)$ is the angle at x in a comparison triangle for (x,y,z), so that

$$d(y,z) = \left(d(x,y)^2 + d(x,z)^2 - 2d(x,y)d(x,z)\cos\measuredangle_x^c(y,z)\right)^{1/2}.$$

More generally for every $(t,u) \in \mathbb{R}_+$, the distance $d(x+t\mathcal{F}, x+u\mathcal{G})$ equals

$$\left(t^2\left\|\mathcal{F}\right\|^2 + u^2\left\|\mathcal{G}\right\|^2 - 2tu\left\|\mathcal{F}\right\|\left\|\mathcal{G}\right\|\cos\measuredangle_x^c(x+t\mathcal{F}, x+u\mathcal{G})\right)^{1/2}.$$

By [**8**, II.3.1], the comparison angle function

$$(t,u) \in \mathbb{R}^2_> \mapsto \measuredangle_x^c(x+t\mathcal{F}, x+u\mathcal{G}) \in [0,\pi]$$

is non-decreasing in both variables. We define

$$\begin{array}{lclcl}
\measuredangle_x(\mathcal{F},\mathcal{G}) & = & \inf\left\{\measuredangle_x^c(x+t\mathcal{F},x+u\mathcal{G}) : t,u>0\right\} & = & \lim_{t,u\to 0}\measuredangle_x^c(x+t\mathcal{F},x+u\mathcal{G}) \\
\measuredangle_x(\overrightarrow{xy},\mathcal{G}) & = & \inf\left\{\measuredangle_x^c(y,x+u\mathcal{G}) : u>0\right\} & = & \lim_{u\to 0}\measuredangle_x^c(y,x+u\mathcal{G}) \\
\measuredangle(\overrightarrow{xy},\mathcal{G}) & = & \sup\left\{\measuredangle_x^c(y,x+u\mathcal{G}) : u>0\right\} & = & \lim_{u\to \infty}\measuredangle_x^c(y,x+u\mathcal{G}) \\
\measuredangle^x(\mathcal{F},\mathcal{G}) & = & \sup\left\{\measuredangle_x^c(x+t\mathcal{F},x+u\mathcal{G}) : t,u>0\right\} & = & \lim_{t,u\to \infty}\measuredangle_x^c(x+t\mathcal{F},x+u\mathcal{G})
\end{array}$$

We will also use the notations $\measuredangle_x(y,z) = \measuredangle_x(\mathcal{F},\mathcal{G}) = \measuredangle(\mathcal{F}_x,\mathcal{G}_x)$.

5.5.3. Let us immediately observe that:

LEMMA 121. *If $\mathcal{G}$ belongs to $\mathbf{G}(Z) \subset \mathbf{F}(G)$, then*

$$\measuredangle_x(\mathcal{F},\mathcal{G}) = \measuredangle_x(\overrightarrow{xy},\mathcal{G}) = \measuredangle_x^c(y,z) = \measuredangle(\overrightarrow{xy},\mathcal{G}) = \measuredangle^x(\mathcal{F},\mathcal{G}).$$

PROOF. Pick $S \in \mathbf{S}(G)$ with $x \in \mathbf{X}(S)$, $\mathcal{F} \in \mathbf{F}(S)$ using the axiom $L(s)$ for $\mathbf{X}(G)$. Then also $\mathcal{G} \in \mathbf{F}(S)$, thus everything stays in the flat Euclidean affine $\mathbf{F}(S)$-space $\mathbf{X}(S)$ on which all of our angles plainly agree. □

5.5.4. The smallest of these angles, also denoted by $\measuredangle(\mathcal{F}_x,\mathcal{G}_x)$, is the Alexandrov angle at x between the rays $x+t\mathcal{F}$ and $x+t\mathcal{G}$ [**8**, I.12]. It satisfies a triangle inequality: if $\mathcal{H} \in \mathbf{F}(G)$ is yet another nonzero filtration, then by [**8**, I.14],

$$\measuredangle_x(\mathcal{F},\mathcal{H}) \leq \measuredangle_x(\mathcal{F},\mathcal{G}) + \measuredangle_x(\mathcal{G},\mathcal{H}).$$

The tangent cone at x is the quotient $\mathbf{T}_x\mathbf{X}(G) = \mathbf{F}(G)/\sim_x$, where $\mathcal{F} \sim_x \mathcal{G}$ if and only if $\|\mathcal{F}\| = \|\mathcal{G}\|$ and $\measuredangle_x(\mathcal{F},\mathcal{G}) = 0$. This definition agrees with [**8**, II.3.18] by the axiom $R(s)$ for $\mathbf{X}(G)$. We denote by $\mathrm{loc}_x(\mathcal{F}) = \mathcal{F}_x$ the class of $\mathcal{F}$ in $\mathbf{T}_x\mathbf{X}(G)$. The norm $\|-\|$ and Alexandrov angle $\measuredangle_x(-,-)$ on $\mathbf{F}(G)$ descend to a norm and angle on $\mathbf{T}_x\mathbf{X}(G)$, thereby justifying our notation $\measuredangle(\mathcal{F}_x,\mathcal{G}_x) = \measuredangle_x(\mathcal{F},\mathcal{G})$. We also define a scalar product and a distance function on $\mathbf{T}_x\mathbf{X}(G)$ by the usual formulas:

$$\begin{array}{rcl}
\langle\mathcal{F}_x,\mathcal{G}_x\rangle & = & \|\mathcal{F}_x\|\,\|\mathcal{G}_x\|\cos\measuredangle(\mathcal{F}_x,\mathcal{G}_x) \\
d(\mathcal{F}_x,\mathcal{G}_x) & = & \sqrt{\|\mathcal{F}_x\|^2 + \|\mathcal{G}_x\|^2 - 2\langle\mathcal{F}_x,\mathcal{G}_x\rangle}.
\end{array}$$

By definition of the Alexandrov angle,

$$d(\mathcal{F}_x,\mathcal{G}_x) = \lim_{t\to 0}\tfrac{1}{t}d(x+t\mathcal{F},x+t\mathcal{G}).$$

These formulas for d respectively show that $d(\mathcal{F}_x, \mathcal{G}_x) = 0$ if and only if $\mathcal{F}_x = \mathcal{G}_x$, and that $d(\mathcal{F}_x, \mathcal{H}_x) \leq d(\mathcal{F}_x, \mathcal{G}_x) + d(\mathcal{G}_x, \mathcal{H}_x)$. Thus d is indeed a distance on $\mathbf{T}_x\mathbf{X}(G)$ and $\mathcal{F} \sim_x \mathcal{G}$ if and only if $\lim_{t\to 0} \frac{1}{t} d(x + t\mathcal{F}, x + t\mathcal{G}) = 0$.

5.5.5. By the very definition of $\mathbf{T}_x\mathbf{X}(G)$, there is a commutative diagram

$$\begin{tikzcd}
\mathbf{F}(G) \arrow[rr, two heads, "\mathcal{F}\mapsto x+\mathcal{F}"] \arrow[dr, two heads, "\mathrm{loc}_x"'] & & \mathbf{X}(G) \arrow[dl, two heads, "\mathrm{loc}_x^a"] \\
& \mathbf{T}_x\mathbf{X}(G) &
\end{tikzcd}$$

We may thus also define

$$\begin{aligned}
\measuredangle_x(y, z) &= \measuredangle\left(\mathrm{loc}_x^a(y), \mathrm{loc}_x^a(z)\right), \\
\langle y, z\rangle_x &= \left\langle \mathrm{loc}_x^a(y), \mathrm{loc}_x^a(z)\right\rangle, \\
d_x(y, z) &= d\left(\mathrm{loc}_x^a(y), \mathrm{loc}_x^a(z)\right).
\end{aligned}$$

5.5.6. Our second smallest angle actually equals the first one by [**8**, I.1.16]. Thus for $y = x + \mathcal{F}$, we have $\measuredangle_x(\overrightarrow{xy}, \mathcal{G}) = \measuredangle(\mathcal{F}_x, \mathcal{G}_x)$. Since

$$\lim_{t\to 0} \tfrac{1}{t}\left(d(y, x) - d(y, x + t\mathcal{G})\right) = \|\mathcal{F}\|\,\|\mathcal{G}\| \cos\measuredangle_x(\overrightarrow{xy}, \mathcal{G})$$

by definition of $\measuredangle_x(\overrightarrow{xy}, \mathcal{G})$, it follows that

$$\lim_{t\to 0} \tfrac{1}{t}\left(d(y, x) - d(y, x + t\mathcal{G})\right) = \left\langle \mathrm{loc}_x^a(y), \mathrm{loc}_x\mathcal{G}\right\rangle.$$

5.5.7. By definition of our largest angle $\measuredangle^x(\mathcal{F}, \mathcal{G})$, we have

$$\lim_{t\to\infty} \tfrac{1}{t} d(x + t\mathcal{F}, x + t\mathcal{G}) = \sqrt{\|\mathcal{F}\|^2 + \|\mathcal{G}\|^2 - 2\,\|\mathcal{F}\|\,\|\mathcal{G}\| \cos\measuredangle^x(\mathcal{F}, \mathcal{G})}.$$

For $z_1, z_2 \in \mathbf{X}(G)$, $x + t\mathcal{F} \sim z_1 + t\mathcal{F}$ and $x + t\mathcal{G} \sim z_2 + t\mathcal{G}$, thus also

$$\lim_{t\to\infty} \tfrac{1}{t} d(z_1 + t\mathcal{F}, z_2 + t\mathcal{G}) = \sqrt{\|\mathcal{F}\|^2 + \|\mathcal{G}\|^2 - 2\,\|\mathcal{F}\|\,\|\mathcal{G}\| \cos\measuredangle^x(\mathcal{F}, \mathcal{G})}.$$

In particular, $\measuredangle^x(\mathcal{F}, \mathcal{G})$ is independent of x. Taking $x \in \mathbf{X}(S)$ for some $S \in \mathbf{S}(G)$ with $\mathcal{F}, \mathcal{G} \in \mathbf{F}(S)$, we find that $\measuredangle^x(\mathcal{F}, \mathcal{G}) = \measuredangle(\mathcal{F}, \mathcal{G}) = \measuredangle_x(\mathcal{F}, \mathcal{G})$. Thus

$$d(\mathcal{F}, \mathcal{G}) = \lim_{t\to\infty} \tfrac{1}{t} d(z_1 + t\mathcal{F}, z_2 + t\mathcal{G})$$

for every $z_1, z_2 \in \mathbf{X}(G)$ and

$$\begin{aligned}
\measuredangle(\mathcal{F}, \mathcal{G}) &= \max\{\measuredangle(\mathcal{F}_x, \mathcal{G}_x) : x \in \mathbf{X}(G)\}, \\
\langle \mathcal{F}, \mathcal{G}\rangle &= \min\{\langle \mathcal{F}_x, \mathcal{G}_x\rangle : x \in \mathbf{X}(G)\}, \\
d(\mathcal{F}, \mathcal{G}) &= \max\{d(\mathcal{F}_x, \mathcal{G}_x) : x \in \mathbf{X}(G)\}.
\end{aligned}$$

5.5.8. Recall from [**8**, II.8.18-20] that for any $y, z \in \mathbf{X}(G)$, the function

$$t \mapsto d(y, z + t\mathcal{G}) - t\,\|\mathcal{G}\|$$

is non-increasing and bounded, the functions

$$y \mapsto d(y, z + t\mathcal{G}) - t\,\|\mathcal{G}\|$$

converge uniformly on bounded subsets of $\mathbf{X}(G)$ as $t \to \infty$ to

$$y \mapsto b_{z,\mathcal{G}}(y) = \lim_{t\to\infty}\left(d(y, z + t\mathcal{G}) - t\,\|\mathcal{G}\|\right),$$

and (for $\mathcal{G} \neq 0$) the Busemann function in two variables

$$(x, y) \mapsto b_{\mathcal{G}}(x, y) = b_{z,\mathcal{G}}(y) - b_{z,\mathcal{G}}(x)$$

does not depend upon z. Note that the proof of this last statement in *loc. cit.*, which only uses the "if" part of [**8**, II.8.19], does indeed not require the ambient CAT(0)-space to be complete. For any $\mathcal{G} \in \mathbf{F}(G)$ and $x, y \in \mathbf{X}(G)$, we set

$$\langle \overrightarrow{xy}, \mathcal{G} \rangle = \|\mathcal{G}\| \cdot \lim_{t\to\infty} \left(d(x, z + t\mathcal{G}) - d(y, z + t\mathcal{G}) \right)$$

which is thus well-defined, independent of z, and equal to $\|\mathcal{G}\| \cdot b_{\mathcal{G}}(y, x)$ if $\mathcal{G} \neq 0$. For $x, y, z \in \mathbf{X}(G)$, we have $\langle \overrightarrow{xz}, \mathcal{G} \rangle = \langle \overrightarrow{xy}, \mathcal{G} \rangle + \langle \overrightarrow{yz}, \mathcal{G} \rangle$, $\langle \overrightarrow{yx}, \mathcal{G} \rangle + \langle \overrightarrow{xy}, \mathcal{G} \rangle = 0$. Taking $z = x$ in the formula defining $\langle \overrightarrow{xy}, \mathcal{G} \rangle$, we find that

$$\langle \overrightarrow{xy}, \mathcal{G} \rangle = d(x, y) \cdot \|\mathcal{G}\| \cdot \cos \measuredangle(\overrightarrow{xy}, \mathcal{G})$$

by definition of our second largest angle $\measuredangle(\overrightarrow{xy}, \mathcal{G})$. The function

$$y \mapsto \langle \overrightarrow{xy}, \mathcal{G} \rangle = - \langle \overrightarrow{yx}, \mathcal{G} \rangle$$

is $\|\mathcal{G}\|$-Lipschitzian and concave (by convexity of d).

5.5.9. Returning to $y = x + \mathcal{F}$ and $z = x + \mathcal{G}$, we obtain

$$\langle \mathcal{F}, \mathcal{G} \rangle \leq \langle \overrightarrow{xy}, \mathcal{G} \rangle \leq \tfrac{1}{2} \left(d(x, y)^2 + d(x, z)^2 - d(y, z)^2 \right) \leq \langle \mathcal{F}_x, \mathcal{G}_x \rangle$$

with absolute values bounded by $\|\mathcal{F}\| \, \|\mathcal{G}\|$, as well as

$$d(\mathcal{F}_x, \mathcal{G}_x) \leq d(y, z) \leq \left(d(x, y)^2 + \|\mathcal{G}\|^2 - 2 \langle \overrightarrow{xy}, \mathcal{G} \rangle \right)^{1/2} \leq d(\mathcal{F}, \mathcal{G}).$$

In particular, the localization functions

$$\begin{array}{ccccc} (\mathbf{F}(G), d) & \to & (\mathbf{X}(G), d) & \to & (\mathbf{T}_x\mathbf{X}(G), d) \\ \mathcal{F} & \mapsto & x + \mathcal{F} = y & \mapsto & y_x = \mathcal{F}_x \end{array}$$

are non-expanding. For $S \in \mathbf{S}(G)$ with $x \in \mathbf{X}(S)$, they restrict to isometries

$$(\mathbf{F}(S), d) \simeq (\mathbf{X}(S), d) \simeq (\mathbf{T}_x\mathbf{X}(S), d)$$

where $\mathbf{T}_x\mathbf{X}(S) = \mathrm{loc}^a_x\mathbf{X}(S) = \mathrm{loc}_x\mathbf{F}(S)$. We refer to $\mathbf{T}_x\mathbf{X}(S)$ as the apartment of S in $\mathbf{T}_x\mathbf{X}(G)$. It is a (complete thus) closed subset of $\mathbf{T}_x\mathbf{X}(G)$.

5.5.10. Suppose that any two germs of geodesic segments in $\mathbf{X}(G)$ issuing from the same point are contained in some apartment of $\mathbf{X}(G)$. This is for instance the case when the strengthening $L(s)^+$ of $L(s)$ holds for $\mathbf{X}(G)$. Then:

(1) The axiom $R(s)$ holds for $\mathbf{T}_x\mathbf{X}(G)$, i.e. any two elements of $\mathbf{T}_x\mathbf{X}(G)$ belong to $\mathbf{T}_x\mathbf{X}(S)$ for some S in $\mathbf{S}(x) = \{S \in \mathbf{S}(G) : x \in \mathbf{X}(S)\}$;
(2) $\mathcal{F}_x = \mathcal{G}_x$ if and only if $x + t\mathcal{F} = x + t\mathcal{G}$ for all sufficiently small $t \geq 0$;
(3) $(\mathbf{T}_x\mathbf{X}(G), d)$ is a CAT(0)-space; and
(4) $v_2 \mapsto \langle v_1, v_2 \rangle$ is homogeneous, concave and $\|v_1\|$-Lipschitzian on $\mathbf{T}_x\mathbf{X}(G)$.

The first two properties are easy. To establish (3), first note that $(\mathbf{T}_x\mathbf{X}(G), d)$ is a geodesic space by (1), so it remains to establish the $CAT(0)$-inequality. Let thus v, v_0, v_1 be three points of $\mathbf{T}_x\mathbf{X}(G)$ and choose $S \in \mathbf{S}(x)$ such that v_0, v_1 belong to $\mathbf{T}_x\mathbf{X}(S)$. Lift v_i to $\mathcal{F}_i \in \mathbf{F}(S)$ and lift v to some $\mathcal{F} \in \mathbf{F}(G)$. For $u \in [0, 1]$, let $\mathcal{F}_u = (1 - u)\mathcal{F}_0 + u\mathcal{F}_1$ be the point at distance $ud(\mathcal{F}_0, \mathcal{F}_1)$ from $\mathcal{F}_0$ on the segment $[\mathcal{F}_0, \mathcal{F}_1]$ of $\mathbf{F}(S)$. Then $x_{u,t} = x + t\mathcal{F}_u$ is the point at distance $ud(x_{0,t}, x_{1,t})$ from $x_{0,t}$ on the segment $[x_{0,t}, x_{1,t}]$ of $\mathbf{X}(S)$ while $v_u = \mathrm{loc}_x\mathcal{F}_u$ is the point at distance

$ud(v_0, v_1)$ from v_0 on the segment $[v_0, v_1]$ of $\mathbf{T}_x\mathbf{X}(S)$. Set $x_t = x + t\mathcal{F}$. By the $CAT(0)$-inequality in $\mathbf{X}(G)$ applied to the triangle $(x_t, x_{0,t}, x_{1,t})$,

$$d(x_t, x_{u,t})^2 \leq (1-u)\cdot d(x_t, x_{0,t})^2 + u\cdot d(x_t, x_{1,t})^2 - u(1-u)\cdot d(x_{0,t}, x_{1,t})^2.$$

Dividing by t^2 and taking the limit as $t \to 0$ gives

$$d(v, v_u)^2 \leq (1-u)\cdot d(v, v_0)^2 + u\cdot d(v, v_1)^2 - u(1-u)\cdot d(v_0, v_1)^2$$

which is the $CAT(0)$-inequality for $\mathbf{T}_x\mathbf{X}(G)$. Given (3), the proof of (4) is entirely similar to that of corollary 91.

REMARK 122. By (2), the quotient $\mathbf{T}_x\mathbf{X}(G)$ of $\mathbf{F}(G)$ does not depend upon the chosen metric (i.e. chosen τ) for buildings satisfying the above condition on germs. Assuming instead that $(\mathbf{X}(G), d)$ is complete, [**8**, II.3.19] shows that the completion of $(\mathbf{T}_x\mathbf{X}(G), d)$ is always a CAT(0)-space.

5.5.11. If the axiom $L(s)^+$ holds for $\mathbf{X}(G)$, then $\mathcal{G} \mapsto \langle\overrightarrow{xy}, \mathcal{G}\rangle$ is $d(x,y)$-Lipschitzian. Indeed, for $\mathcal{G}_1, \mathcal{G}_2 \in \mathbf{F}(G)$, there is a subdivision $x = x_0, \cdots, x_n = y$ of the segment $[x, y]$ of $\mathbf{X}(G)$ and for each $i \in \{1, \cdots, n\}$, tori $S_{i,1}, S_{i,2} \in \mathbf{S}(G)$ such that $[x_{i-1}, x_i] \subset \mathbf{X}(S_{i,j})$ and $\mathcal{G}_j \in \mathbf{F}(S_{i,j})$ for $j \in \{1, 2\}$ by lemma 110. Then

$$\langle\overrightarrow{xy}, \mathcal{G}_1\rangle - \langle\overrightarrow{xy}, \mathcal{G}_2\rangle = \textstyle\sum_{i=0}^{n-1}\langle\overrightarrow{x_i x_{i+1}}, \mathcal{G}_1\rangle - \langle\overrightarrow{x_i x_{i+1}}, \mathcal{G}_2\rangle$$

with $\langle\overrightarrow{x_i x_{i+1}}, \mathcal{G}_j\rangle = \left\langle \mathrm{loc}^a_{x_i}(x_{i+1}), \mathrm{loc}_{x_i}(\mathcal{G}_j)\right\rangle$. Thus

$$|\langle\overrightarrow{xy}, \mathcal{G}_1\rangle - \langle\overrightarrow{xy}, \mathcal{G}_2\rangle| \leq \textstyle\sum_{i=0}^{n-1}\left\|\mathrm{loc}^a_{x_i}(x_{i+1})\right\|\cdot d\left(\mathrm{loc}_{x_i}(\mathcal{G}_1), \mathrm{loc}_{x_i}(\mathcal{G}_2)\right)$$

because $v_2 \mapsto \langle v_1, v_2\rangle$ is $\|v_1\|$-Lipschitzian on $\mathbf{T}_{x_i}\mathbf{X}(G)$. Since

$$\left\|\mathrm{loc}^a_{x_i}(x_{i+1})\right\| = d(x_i, x_{i+1}) \quad\text{and}\quad d\left(\mathrm{loc}_{x_i}(\mathcal{G}_1), \mathrm{loc}_{x_i}(\mathcal{G}_2)\right) \leq d(\mathcal{G}_1, \mathcal{G}_2)$$

we obtain the desired inequality:

$$|\langle\overrightarrow{xy}, \mathcal{G}_1\rangle - \langle\overrightarrow{xy}, \mathcal{G}_2\rangle| \leq \textstyle\sum_{i=0}^{n-1} d(x_i, x_{i+1})\cdot d(\mathcal{G}_1, \mathcal{G}_2) = d(x, y)\cdot d(\mathcal{G}_1, \mathcal{G}_2).$$

5.5.12. Convex projections. Let C be a closed convex subset of $\mathbf{X}(G)$ which is complete in the induced topology. Then for every $x \in \mathbf{X}(G)$, there is a unique point $p(x)$ in C such that $d(x, p(x)) = d(x, C) = \inf\{d(x, y) : y \in C\}$. We call

$$p : \mathbf{X}(G) \twoheadrightarrow C$$

the convex projection onto C. It is non-expanding, constant on the segment $[x, p(x)]$, the map $H : \mathbf{X}(G)\times[0,1] \to \mathbf{X}(G)$ associating to (x, t) the unique point at distance $td(x, p(x))$ from x on the segment $[x, p(x)]$ is a continuous homotopy from $\mathrm{Id}_{\mathbf{X}(G)}$ to p, and $\measuredangle_{p(x)}(x, y) \geq \frac{\pi}{2}$ for every $y \in C$ by [**8**, II.2.4], thus also

$$\langle x, y\rangle_{p(x)} \leq 0.$$

For any $\mathcal{F} \in \mathbf{F}(G)$ such that $p(x) + t\mathcal{F}$ belongs to C for all sufficiently small $t > 0$,

$$\left\langle\overrightarrow{p(x)x}, \mathcal{F}\right\rangle \leq \tfrac{1}{2}\left(d(x, C)^2 + \|\mathcal{F}\|^2 - d(x, p(x) + \mathcal{F})^2\right) \leq \left\langle \mathrm{loc}^a_{p(x)}(x), \mathrm{loc}_{p(x)}(\mathcal{F})\right\rangle \leq 0.$$

5.6. The affine $\mathbf{F}(P/U)$-space $\mathbf{T}_P^\infty\mathbf{X}(G)$

Let $\mathbf{X}(G)$ be an affine $\mathbf{F}(G)$-building. Fix a parabolic subgroup P of G and let U be the unipotent radical of P.

5.6.1. We have already seen that a Levi subgroup L of P determines:

(1) a parabolic subgroup P_L^ι of G opposed to P with $P\cap P_L^\iota = L$,
(2) a splitting map $F^{-1}(P)\hookrightarrow \mathbf{G}(L)$,
(3) an opposition map $F^{-1}(P)\ni\mathcal{F}\mapsto\mathcal{F}_L^\iota\in F^{-1}(P_L)$,
(4) a section $\mathbf{F}(P/U)\ni\mathcal{H}\mapsto\mathcal{H}_L\in\mathbf{F}(L)$ of $\mathrm{Gr}_P:\mathbf{F}(G)\twoheadrightarrow\mathbf{F}(P/U)$,
(5) a fundamental domain $\mathbf{X}(L)=\cup_{S\in\mathbf{S}(L)}\mathbf{X}(S)$ for the U-action on $\mathbf{X}(G)$,
(6) an L-equivariant, U-invariant retraction $\mathbf{X}(G)\ni x\mapsto x_L\in\mathbf{X}(L)$.

The splitting map takes $\mathcal{F}$ to its unique splitting $\mathcal{G}$ with $Z_G(\mathcal{G})=L$. We have $\mathcal{F}_L^\iota=\mathrm{Fil}(\iota\mathcal{G})$ and $P_{\mathcal{F}_L^\iota}=P_L^\iota$. The Gr_P-map and its section are discussed in 4.1.15, and Gr_P is defined everywhere by theorem 85. Finally (5) and (6) come from proposition 101, which also says that for any $\mathcal{F}$ in the facet $F^{-1}(P)$,

$$x_L=\lim_{t\to\infty}(x+t\mathcal{F})+t\mathcal{F}_L^\iota\quad\text{in }\mathbf{X}(G).$$

5.6.2. For any $x,y\in\mathbf{X}(G)$, the following conditions are equivalent:

(1) $\mathsf{U}\cdot x=\mathsf{U}\cdot y$,
(2) $\lim_{t\to\infty}\mathbf{d}(x+t\mathcal{F},y+t\mathcal{F})=0$ for some (or every) $\mathcal{F}\in F^{-1}(P)$,
(3) $\lim_{t\to\infty}d(x+t\mathcal{F},y+t\mathcal{F})=0$ for some (or every) $\mathcal{F}\in F^{-1}(P)$,
(4) $x_L=y_L$ for some (or every) Levi subgroup L of P

Indeed (1) $\Rightarrow$ (2) by the axiom UN, (2) $\Rightarrow$ (3) is trivial, (3) $\Rightarrow$ (4) because

$$(3)\overset{NE}{\Longrightarrow}\lim_{t\to\infty}d\left((x+t\mathcal{F})+t\mathcal{F}_L^\iota,(y+t\mathcal{F})+t\mathcal{F}_L^\iota\right)=0\Longrightarrow(4)$$

and (4) $\Rightarrow$ (1) is obvious. If $\mathbf{X}(G)$ satisfies UN^+, they are also equivalent to:
(5) $x+t\mathcal{F}=y+t\mathcal{F}$ for $t\gg 0$.
Indeed (1) $\Rightarrow$ (5) by UN^+ and plainly (5) $\Rightarrow$ (3).

5.6.3. For any $\overline{S}\in\mathbf{S}(P/U)$, there is a U-equivariant bijection between the set $\mathbf{S}(P,\overline{S})$ of all $S\in\mathbf{S}(G)$ with $Z_G(S)\subset P$ such that $P\to P/U$ induces an isomorphism from S to $\overline{S}$, and the set of all Levi subgroups L of P. It maps S to the unique Levi subgroup L_S containing $Z_G(S)$ and L to the unique lift S_L of $\overline{S}$ in $L\simeq P/U$, see lemma 77. In particular, $\mathbf{S}(P,\overline{S})$ is a U-torsor.

5.6.4. There is a structure of affine $\mathbf{F}(P/U)$-space on $\mathsf{U}\backslash\mathbf{X}(G)$,

$$\mathbf{T}_P^\infty\mathbf{X}(G)=\left(\mathsf{U}\backslash\mathbf{X}(G),+,\mathbf{T}_P^\infty\mathbf{X}(-)\right).$$

The $\mathsf{P/U}$-equivariant apartment map is defined by

$$\begin{aligned}\mathbf{T}_P^\infty\mathbf{X}(\overline{S}) &= \mathsf{U}\backslash\cup_{S\in\mathbf{S}(P,\overline{S})}\mathbf{X}(S)\\ &= \text{image of }\mathbf{X}(S)\text{ in }\mathsf{U}\backslash\mathbf{X}(G)\text{ for any }S\in\mathbf{S}(P,\overline{S}).\end{aligned}$$

The $\mathsf{P/U}$-equivariant pull map takes $x\in\mathsf{U}\backslash\mathbf{X}(G)$ and $\mathcal{H}\in\mathbf{F}(P/U)$ to

$$x+\mathcal{H}=\text{image of }x_L+\mathcal{H}_L\text{ in }\mathsf{U}\backslash\mathbf{X}(G)$$

where L is any Levi subgroup of P: if L' is another one, there is a unique $u\in\mathsf{U}$ such that $\mathrm{Int}(u)(L)=L'$. Then $x_{L'}=ux_L$, $\mathcal{H}_{L'}=u\mathcal{H}_L$, thus $x_{L'}+\mathcal{H}_{L'}=u(x_L+\mathcal{H}_L)$ and $x_L+\mathcal{H}_L$ have the same image in $\mathsf{U}\backslash\mathbf{X}(G)$. This defines an affine $\mathbf{F}(P/U)$-space: for any $\overline{S}\in\mathbf{S}(P/U)$, $x\mapsto x_L$ and $\mathcal{H}\mapsto\mathcal{H}_L$ yield bijections $\mathbf{T}_P^\infty\mathbf{X}(\overline{S})\to\mathbf{X}(S_L)$ and $\mathbf{F}(\overline{S})\to\mathbf{F}(S_L)$, thus $+$ indeed induces a structure of affine $\mathbf{F}(\overline{S})$-space on $\mathbf{T}_P^\infty\mathbf{X}(\overline{S})$.

5.6.5. There is a P/U-invariant distance on $\mathbf{T}_P^\infty\mathbf{X}(G)$, given by the formulas:

$$\begin{aligned} d(x,y) &= d(x_L,y_L) \\ &= \lim_{t\to\infty} d(x_0+t\mathcal{F},y_0+t\mathcal{F}) \\ &= \inf\left\{d(x',y') : (x',y')\in\mathbf{X}(G)^2,\ (x',y')\mapsto(x,y)\right\} \end{aligned}$$

In the first formula, L is any Levi subgroup of P. In the second formula, $\mathcal{F}\in\mathbf{F}(G)$ belongs to the facet $F^{-1}(P)$ and $(x_0,y_0)\in\mathbf{X}(G)^2$ lifts (x,y). The three formulas agree: writing d_i for the function defined by the i-th formula, first note that d_2 is well-defined (by NE), independent of the chosen lift (by UN), and not greater than $d(x_0,y_0)$ (by NE). Therefore $d_2(x,y)\le d_3(x,y)\le d_1(x,y)$. But

$$\begin{aligned} d_1(x,y) &= d\left(\lim_{t\to\infty}\left((x_0+t\mathcal{F})+t\mathcal{F}_L^\iota\right),\lim_{t\to\infty}\left((y_0+t\mathcal{F})+t\mathcal{F}_L^\iota\right)\right) \\ &= \lim_{t\to\infty} d\left(\left((x_0+t\mathcal{F})+t\mathcal{F}_L^\iota\right),\left((y_0+t\mathcal{F})+t\mathcal{F}_L^\iota\right)\right) \\ &\le \lim_{t\to\infty} d\left(x_0+t\mathcal{F},y_0+t\mathcal{F}\right)=d_2(x,y) \end{aligned}$$

by NE, so that indeed $d=d_1=d_2=d_3$. It is obviously a P/U-invariant distance, and it restricts to a Euclidean norm on any apartment $\mathbf{T}_P^\infty\mathbf{X}(\overline{S})$. The projection

$$\mathrm{Gr}_P^\infty:\mathbf{X}(G)\twoheadrightarrow\mathbf{T}_P^\infty\mathbf{X}(G)$$

is non-expanding and restricts to an isometry on $\mathbf{X}(L)$.

5.6.6. If $\mathbf{T}_P^\infty\mathbf{X}(G)$ satisfies $R(s)$ and $(\mathbf{X}(G),d)$ is a $CAT(0)$-metric space, then so is $(\mathbf{T}_P^\infty\mathbf{X}(G),d)$. Indeed, it is a geodesic space by $R(s)$, so it remains to establish the $CAT(0)$-inequality. Let thus v,v_0,v_1 be three points of $\mathbf{T}_P^\infty\mathbf{X}(G)$ and choose $\overline{S}\in\mathbf{S}(P/U)$ such that v_0,v_1 belong to $\mathbf{T}_P^\infty\mathbf{X}(\overline{S})$. Lift $\overline{S}$ to $S\in\mathbf{S}(P,\overline{S})$ and v_i to $x_i\in\mathbf{X}(S)$, and lift v to $x\in\mathbf{X}(G)$. For $u\in[0,1]$, let $x_u=(1-u)x_0+ux_1$ be the point at distance $ud(x_0,x_1)$ from x_0 on the segment $[x_0,x_1]$ of $\mathbf{X}(S)$. Fix $\mathcal{F}\in\mathbf{F}(S)$ with $P_\mathcal{F}=P$. Then for every $t\ge0$, $x_{u,t}=x_u+t\mathcal{F}$ is the point at distance $ud(x_{0,t},x_{1,t})$ from $x_{0,t}$ on the segment $[x_{0,t},x_{1,t}]$ of $\mathbf{X}(S)$ and $v_u=\mathrm{Gr}_P^\infty(x_u)$ is the point at distance $ud(v_0,v_1)$ from v_0 on the segment $[v_0,v_1]$ of $\mathbf{T}_P^\infty\mathbf{X}(\overline{S})$. Set $x_t=x+t\mathcal{F}$. By the $CAT(0)$-inequality in $\mathbf{X}(G)$,

$$d(x_t,x_{u,t})^2\le(1-u)\cdot d(x_t,x_{0,t})^2+u\cdot d(x_t,x_{1,t})^2-u(1-u)\cdot d(x_{0,t},x_{1,t})^2.$$

Taking the limit as $t\to\infty$ gives

$$d(v,v_u)^2\le(1-u)\cdot d(v,v_0)^2+u\cdot d(v,v_1)^2-u(1-u)\cdot d(v_0,v_1)^2$$

which is the $CAT(0)$-inequality for $\mathbf{T}_P^\infty\mathbf{X}(G)$.

Remark 123. Suppose that for any $x,y\in\mathbf{X}(G)$ and $\mathcal{F}\in\mathbf{F}(G)$ there is an $S\in\mathbf{S}(G)$ such that $x+t\mathcal{F}$ and $y+t\mathcal{F}$ belong to $\mathbf{X}(S)$ for $t\gg0$. Then also $\mathcal{F}\in\mathbf{F}(S)$ and the axiom $R(s)$ holds for $\mathbf{T}_P^\infty\mathbf{X}(G)$. If $\mathbf{X}(G)$ satisfies UN^+, this condition on $\mathbf{X}(G)$ is actually equivalent to the axiom $R(s)$ for $\mathbf{T}_P^\infty\mathbf{X}(G)$.

5.6.7. We shall always equip $\mathbf{F}(P/U)$ with the scalar product, distance, norm… which are induced by the representation $\mathrm{Gr}_\mathcal{F}^\bullet(\tau)$ of P/U. Here $\mathcal{F}$ is any filtration in the facet $F^{-1}(P)$, and we view $\mathrm{Gr}_\mathcal{F}^\bullet(\tau)=\oplus_\gamma\mathrm{Gr}_\mathcal{F}^\gamma(\tau)$ as a representation of P/U. If L is a Levi subgroup of P and $\mathcal{G}$ is the corresponding splitting of $\mathcal{F}$, the restriction of τ to L splits as $\tau|L=\oplus\tau_\gamma$ with $V(\tau_\gamma)=\mathcal{G}_\gamma(\tau)$ and the isomorphism $L\simeq P/U$ maps τ_γ to the representation $\mathrm{Gr}_\mathcal{F}^\gamma(\tau)$ of P/U, thus $\tau|L$ to $\mathrm{Gr}_\mathcal{F}^\bullet(\tau)$. In particular, $\mathrm{Gr}_\mathcal{F}^\bullet(\tau)$ is indeed a faithful representation of P/U and its isomorphism class $\mathrm{Gr}_P^\bullet(\tau)$ does not depend upon $\mathcal{F}$. It follows from this conventions that the isomorphism $\mathbf{F}(L) \simeq \mathbf{F}(P/U)$ is compatible with the scalar products, distances, norms... which are induced on $\mathbf{F}(L)$ and $\mathbf{F}(P/U)$ by the chosen faithful representation τ of G.

5.6.8. If $\mathbf{T}_P^\infty\mathbf{X}(G)$ satisfies $L(s)$, then $d(x, x+\mathcal{H}) = \|\mathcal{H}\|$ for $x \in \mathbf{T}_P^\infty\mathbf{X}(G)$ and $\mathcal{H} \in \mathbf{F}(P/U)$. Indeed, choose $\overline{S} \in \mathbf{S}(P/U)$ with $x \in \mathbf{T}_P^\infty\mathbf{X}(\overline{S})$ and $\mathcal{H} \in \mathbf{F}(\overline{S})$. Then $x_L \in \mathbf{X}(S_L)$ and $\mathcal{H}_L \in \mathbf{F}(S_L)$, thus also $x_L + \mathcal{H}_L \in \mathbf{X}(S_L)$. In particular, $(x+\mathcal{H})_L = x_L + \mathcal{H}_L$, thus $d(x, x+\mathcal{H}) = d(x_L, x_L + \mathcal{H}_L) = \|\mathcal{H}_L\| = \|\mathcal{H}\|$: if the affine $\mathbf{F}(P/U)$-space $\mathbf{T}_P^\infty\mathbf{X}(G)$ actually is an affine $\mathbf{F}(P/U)$-building, its "quotient" distance defined above agrees with its "building" distance defined in section 5.2.9.

5.6.9. Suppose again that $(\mathbf{X}(G), d)$ is a $CAT(0)$-space. The Busemann scalar product of section 5.5.8 on $\mathbf{X}(G)^2 \times \mathbf{F}(G)$, namely

$$\langle \overrightarrow{xy}, \mathcal{F}\rangle = \|\mathcal{F}\| \cdot \lim_{t\to\infty} \left(d(x, z+t\mathcal{F}) - d(y, z+t\mathcal{F})\right)$$

induces a function on $\mathbf{T}_P^\infty\mathbf{X}(G)^2 \times F^{-1}(P)$. Indeed for $u, v \in \mathsf{U}$ and $\mathcal{F} \in F^{-1}(P)$,

$$\begin{aligned}\lim_{t\to\infty} \left(d(ux, z+t\mathcal{F}) - d(vy, z+t\mathcal{F})\right) &= \lim_{t\to\infty} \left(d(x, u^{-1}z+t\mathcal{F}) - d(y, v^{-1}z+t\mathcal{F})\right)\\ &= \lim_{t\to\infty} \left(d(x, z+t\mathcal{F}) - d(y, z+t\mathcal{F})\right)\end{aligned}$$

by the triangle inequality and the axiom UN for $\mathbf{X}(G)$. If $\mathbf{T}_P^\infty\mathbf{X}(G)$ moreover satisfies $R(s)$, the resulting function depends only on the image $\overline{\mathcal{F}} = \mathrm{Gr}_P\mathcal{F}$ of $\mathcal{F}$ in

$$\mathbf{G}(\overline{R}(P)) = \mathbf{G}(Z(P/U)) \subset \mathbf{F}(P/U).$$

In fact, it is simply the corresponding Busemann scalar product

$$\left\langle \overrightarrow{xy}, \overline{\mathcal{F}}\right\rangle = \left\|\overline{\mathcal{F}}\right\| \cdot \lim_{t\to\infty} \left(d(x, z+t\overline{\mathcal{F}}) - d(y, z+t\overline{\mathcal{F}})\right)$$

on the CAT(0)-space $(\mathbf{T}_P^\infty\mathbf{X}(G), d)$. Indeed, pick $\overline{S} \in \mathbf{S}(P/U)$ with $x, z \in \mathbf{T}_P^\infty\mathbf{X}(\overline{S})$. Then $x_L, z_L \in \mathbf{X}(S_L)$, $\mathcal{F}$ equals $(\overline{\mathcal{F}})_L$ and belongs to $\mathbf{F}(S_L)$, $z_L + t\mathcal{F}$ belongs to $\mathbf{X}(S_L)$, therefore $(z+t\overline{\mathcal{F}})_L = (z_L + t(\overline{\mathcal{F}})_L)_L = (z_L + t\mathcal{F})_L = z_L + t\mathcal{F}$, and finally

$$d(x, z+t\overline{\mathcal{F}}) = d(x_L, z_L + t\mathcal{F})$$

for all $t \geq 0$, which proves our claim since also $\|\mathcal{F}\| = \left\|(\overline{\mathcal{F}})_L\right\| = \left\|\overline{\mathcal{F}}\right\|$. However, $z \mapsto z + t\overline{\mathcal{F}}$ is now a Clifford translation of $\mathbf{T}_P^\infty\mathbf{X}(G)$ [**8**, II 6.14]: we have just seen that it corresponds to $z_L \mapsto z_L + t\mathcal{F}$ on the isometric space $\mathbf{X}(L)$, and the latter map is non-expanding by NE with non-expanding inverse $z_L \mapsto z_L + t\mathcal{F}_L^\iota$. It is therefore an isometry, and a Clifford translation since $d(z_L, z_L + t\mathcal{F}) = t\,\|\mathcal{F}\|$ for all $z_L \in \mathbf{X}(L)$. If $\mathcal{F} \neq 0$, it now follows from [**8**, II 6.15] that the comparison angle

$$t \mapsto \measuredangle_x^c(y, x+t\overline{\mathcal{F}})$$

is constant. Taking $z = x$ in the defining formula for $\left\langle \overrightarrow{xy}, \overline{\mathcal{F}}\right\rangle$, we thus find that

$$\left\langle \overrightarrow{xy}, \overline{\mathcal{F}}\right\rangle = d(x,y) \cdot \|\mathcal{F}\| \cdot \cos\left(\lim_{t\to\infty} \measuredangle_x^c(y, x+t\overline{\mathcal{F}})\right)$$

equals

$$\left\langle \overrightarrow{xy}, \overline{\mathcal{F}}_x\right\rangle = d(x,y) \cdot \|\mathcal{F}\| \cdot \cos\left(\lim_{t\to 0} \measuredangle_x^c(y, x+t\overline{\mathcal{F}})\right).$$

Note that using [**8**, I.1.16] again, this is also equal to

$$\left\langle \mathrm{loc}_x^a(y), \overline{\mathcal{F}}_x\right\rangle = d(x,y) \cdot \|\mathcal{F}\| \cdot \cos\left(\lim_{t\to 0} \measuredangle_x^c(y_t, x+t\overline{\mathcal{F}})\right)$$

where $y_t = ty + (1-t)x$ is the point at distance $td(x,y)$ from x on the segment $[x,y]$ of the $CAT(0)$-space $\mathbf{T}_P^\infty \mathbf{X}(G)$. In any case, we obtain yet another series of formulas for the relevant Busemann function on $\mathbf{T}_P^\infty \mathbf{X}(G)$ or $\mathbf{X}(G)$.

REMARK 124. If the affine $\mathbf{F}(P/U)$-space $\mathbf{T}_{\overline{P}}^\infty \mathbf{X}(G)$ is an affine $\mathbf{F}(P/U)$-building, we may also directly apply lemma 121 to $\mathcal{G} = \overline{\mathcal{F}} = \mathrm{Gr}_P(\mathcal{F})$, thereby obtaining

$$\langle \overrightarrow{xy}, \mathcal{F} \rangle = \left\langle \mathrm{loc}^a_{\mathrm{Gr}_P^\infty(x)} \left(\mathrm{Gr}_P^\infty(y)\right), \mathrm{loc}_{\mathrm{Gr}_P^\infty(x)} \left(\mathrm{Gr}_P(\mathcal{F})\right) \right\rangle$$

where the second scalar product is in the tangent space $\mathbf{T}_{\mathrm{Gr}_P(x)}\left(\mathbf{T}_P^\infty \mathbf{X}(G)\right)$.

5.6.10. If $\mathbf{X}(L) + \mathbf{F}(L) \subset \mathbf{X}(L)$, then $\mathbf{X}(L)$ inherits from $\mathbf{X}(G)$ a structure of affine $\mathbf{F}(L)$-space, and the restriction of $\mathrm{Gr}_P^\infty : \mathbf{X}(G) \to \mathbf{T}_P^\infty \mathbf{X}(G)$ to $\mathbf{X}(L)$ is an isomorphism of affine $\mathbf{F}(L)$-spaces – viewing $\mathbf{T}_P^\infty \mathbf{X}(G)$ as an affine $\mathbf{F}(L)$-space through $\mathbf{F}(L) \simeq \mathbf{F}(P/U)$. Most of the above discussion then becomes much easier.

5.7. Example: $\mathbf{F}(G)$ as a tight affine $\mathbf{F}(G)$-building

5.7.1. Recall from example 5.2.4 that $(\mathbf{F}(G), +, \mathbf{F}(-))$ is a discrete affine $\mathbf{F}(G)$-space with trivial type. Under the identification $\mathbf{F}(G) = \mathbf{F}(\omega_K^\circ)$, the pull map may be computed as follows: for $\mathcal{F}_1, \mathcal{F}_2 \in \mathbf{F}(G)$, $\rho \in \mathsf{Rep}^\circ(G)(K)$ and $\gamma \in \mathbb{R}$,

$$(\mathcal{F}_1 + \mathcal{F}_2)^\gamma(\rho) = \sum_{\gamma_1+\gamma_2=\gamma} \mathcal{F}_1^{\gamma_1}(\rho) \cap \mathcal{F}_2^{\gamma_2}(\rho).$$

We have already mentioned that $\mathbf{F}(G)$ satisfies $L(s) = R(s)$ by theorem 85 and $L(i) = R(i)$ by corollary 86. Actually for $\mathcal{F}, \mathcal{G} \in \mathbf{F}(G)$, choosing S in $\mathbf{S}(G)$ with $\mathcal{F}, \mathcal{G} \in \mathbf{F}(S)$, we find using 4.1.10 that $\mathcal{F} + \eta\mathcal{G}$ belongs to a fixed closed chamber of $\mathbf{F}(S)$ for all sufficiently small $\eta \geq 0$, from which the stronger axiom $L(s)^+$ easily follows. We have also seen in example 5.4.5 that $\mathbf{F}(G)$ satisfies the axiom ST. It thus satisfies UN^+ and UN by lemma 116, and it is therefore a (tight) affine $\mathbf{F}(G)$-building by proposition 106. It also trivially satisfies the axiom HA, because every apartment contains the origin $0 \in \mathbf{F}(G)$. The latter is fixed by G, and it follows from lemma 118 that $\mathbf{F}(G)$ is, up to isomorphism, the unique affine $\mathbf{F}(G)$-building with a point fixed by G. Indeed, any such building has trivial type and satisfies ST_2^-, thus also ST^- by lemma 116.

5.7.2. The retractions of corollary 88 and proposition 101 agree, and so do the decompositions of sections 2.2.13 and 5.2.19 (with base point $0 \in \mathbf{F}(G)$).

5.7.3. The distance $d = d_\tau$ of section 4.2.10 is equal to the corresponding distance on the affine $\mathbf{F}(G)$-building $\mathbf{F}(G)$ defined in section 5.2.9. For $\mathcal{F}, \mathcal{G} \in \mathbf{F}(G)$ and $t \in [0,1]$, the unique point at distance $td(\mathcal{F}, \mathcal{G})$ from $\mathcal{F}$ on the segment $[\mathcal{F}, \mathcal{G}]$ in the CAT(0)-space $(\mathbf{F}(G), d)$ is equal to the sum $t\mathcal{G} + (1-t)\mathcal{F}$, as defined above, of the rescaled filtrations $t\mathcal{G}$ and $(1-t)\mathcal{F}$ of $\mathbf{F}(G)$: this is obvious in any apartment.

5.7.4. For a parabolic subgroup $P = U \rtimes L$ of G with unipotent radical U and Levi subgroup L, there is a commutative diagram

$$\begin{tikzcd}
& \mathbf{F}(L) \arrow[dl, bend right, "\simeq"', "\iota_L"] \arrow[dr, bend left, "\simeq"] \arrow[d, hook, shift left, "\iota_{L,G}"] & \\
& \mathbf{F}(G) \arrow[u, two heads, shift left, "r_{P,L}"] \arrow[dl, two heads, "\mathrm{Gr}_P"'] \arrow[dr, two heads, "\mathrm{Gr}_P^\infty"] & \\
\mathbf{F}(P/U) \arrow[rr, "\simeq"] & & \mathbf{T}_P^\infty \mathbf{F}(G)
\end{tikzcd}$$

where $\iota_L : \mathbf{F}(L) \simeq \mathbf{F}(P/U)$ and $\iota_{L,G} : \mathbf{F}(L) \hookrightarrow \mathbf{F}(G)$ are the L-equivariant maps functorially induced by the isomorphism $L \simeq P/U$ and the embedding $L \hookrightarrow G$, $r_{P,L} : \mathbf{F}(G) \twoheadrightarrow \mathbf{F}(L)$ is the U-invariant L-equivariant retraction of corollary 88, Gr_P is the P-equivariant morphism of section 2.3.4 (which is defined on the whole of $\mathbf{F}(G)$ by theorem 85) and $\mathrm{Gr}_P^\infty : \mathbf{F}(G) \twoheadrightarrow \mathbf{T}_P^\infty \mathbf{F}(G)$ is the P-equivariant quotient map onto $\mathbf{T}_P^\infty \mathbf{F}(G) = \mathsf{U} \backslash \mathbf{F}(G)$. Since Gr_P is P-equivariant (thus U-invariant) and $\mathrm{Gr}_P \circ \iota_{L,G} = \iota_L$, also $\mathrm{Gr}_P = \iota_L \circ r_{P,L}$. The right hand side triangles are plainly commutative, and this implies the existence of the bottom map bijection. One checks easily that it is an isomorphism of affine $\mathbf{F}(P/U)$-spaces. In particular: $\mathbf{T}_P^\infty \mathbf{F}(G)$ is an affine $\mathbf{F}(P/U)$-building, its "quotient" and "building" metric agree by 5.6.8, thus $\mathbf{F}(P/U) \to \mathbf{T}_P^\infty \mathbf{F}(G)$ is an isometry while $\mathrm{Gr}_P : \mathbf{F}(G) \twoheadrightarrow \mathbf{F}(P/U)$ is non-expanding. This gives the following formula: for any $\mathcal{F}, \mathcal{G}_1, \mathcal{G}_2 \in \mathbf{F}(G)$,

$$\lim_{t\to\infty} d\left(\mathcal{G}_1 + t\mathcal{F}, \mathcal{G}_2 + t\mathcal{F}\right) = d\left(\mathrm{Gr}_{\mathcal{F}}(\mathcal{G}_1), \mathrm{Gr}_{\mathcal{F}}(\mathcal{G}_2)\right) \leq d(\mathcal{G}_1, \mathcal{G}_2)$$

where $\mathrm{Gr}_{\mathcal{F}} = \mathrm{Gr}_{P_{\mathcal{F}}} : \mathbf{F}(G) \twoheadrightarrow \mathbf{F}(P_{\mathcal{F}}/U_{\mathcal{F}})$. Also,

$$\begin{array}{rcl}
\langle \mathrm{Gr}_P(\mathcal{G}_1), \mathrm{Gr}_P(\mathcal{G}_2) \rangle & \geq & \langle \mathcal{G}_1, \mathcal{G}_2 \rangle \\
\measuredangle\left(\mathrm{Gr}_P(\mathcal{G}_1), \mathrm{Gr}_P(\mathcal{G}_2)\right) & \leq & \measuredangle\left(\mathcal{G}_1, \mathcal{G}_2\right)
\end{array}$$

since Gr_P contracts the distances and preserves the norms.

REMARK 125. Here is a more direct proof of the fact that Gr_P is non-expanding: starting with $\mathcal{G}_1, \mathcal{G}_2 \in \mathbf{F}(G)$, cut the segment $[\mathcal{G}_1, \mathcal{G}_2]$ along its facet decomposition, going from $\mathcal{H}_0 = \mathcal{G}_1$ to $\mathcal{H}_n = \mathcal{G}_2$ with F constant on $]\mathcal{H}_{i-1}, \mathcal{H}_i[$. Then observe that Gr_P restrict to an isometry on any facet $F^{-1}(Q)$, $Q \in \mathbf{P}(G)$: it restricts to an isometry on any apartment containing $F^{-1}(P)$, and there is at least one such apartment which also contains $F^{-1}(Q)$ (along with its closure). Thus

$$\begin{array}{rcl}
d\left(\mathcal{G}_1, \mathcal{G}_2\right) & = & \sum_{i=1}^n d\left(\mathcal{H}_{i-1}, \mathcal{H}_i\right) \\
& = & \sum_{i=1}^n d\left(\mathrm{Gr}_P(\mathcal{H}_{i-1}), \mathrm{Gr}_P(\mathcal{H}_i)\right) \\
& \geq & d\left(\mathrm{Gr}_P(\mathcal{G}_1), \mathrm{Gr}_P(\mathcal{G}_2)\right)
\end{array}$$

by the triangle inequality in $\mathbf{F}(P/U)$. One can also probably establish the inequalities using the explicit formulas for the scalar products, but this involves playing around with three filtrations. In any case, these approaches do not yield an exact formula relating the distances on $\mathbf{F}(P/U)$ and $\mathbf{F}(G)$.

5.7.5. For $\mathcal{F}, \mathcal{G} \in \mathbf{F}(G)$, choose $S \in \mathbf{S}(G)$ with $\mathcal{F}, \mathcal{G} \in \mathbf{F}(S)$. Then by 4.1.10, there is a facet $F \subset \mathbf{F}(S)$ of $\mathbf{F}(G)$ such that $\mathcal{F} + t\mathcal{G}$ belongs to F for every sufficiently small $t > 0$. If $P \supset Z_G(S)$ is the corresponding parabolic subgroup, then $P \subset P_{\mathcal{F}}$

since $\mathcal{F}$ belongs to the closure of F, thus also $U_{\mathcal{F}} \subset U$ where U is the unipotent radical of P. In particular, $\mathcal{F} + tu\mathcal{G} = u(\mathcal{F} + t\mathcal{G}) = \mathcal{F} + t\mathcal{G}$ for every $u \in \mathsf{U}_{\mathcal{F}}$, thus

$$\mathrm{loc}_{\mathcal{F}} : \mathbf{F}(G) \twoheadrightarrow \mathbf{T}_{\mathcal{F}}\mathbf{F}(G) \qquad \mathcal{G} \mapsto \text{germ of } (t \mapsto \mathcal{F} + t\mathcal{G})$$

is $\mathsf{U}_{\mathcal{F}}$-invariant. Since $\mathrm{Gr}_{\mathcal{F}}$ induces a bijection $\mathsf{U}_{\mathcal{F}} \backslash \mathbf{F}(G) \simeq \mathbf{F}(P_{\mathcal{F}}/U_{\mathcal{F}})$, it follows that there is a canonical $\mathsf{P}_{\mathcal{F}}$-equivariant commutative diagram

$$\begin{tikzcd} & \mathbf{F}(G) \arrow[dl, "\mathrm{Gr}_{\mathcal{F}}"'] \arrow[dr, "\mathrm{loc}_{\mathcal{F}}"] & \\ \mathbf{F}(P_{\mathcal{F}}/U_{\mathcal{F}}) \arrow[rr, "\varphi"] & & \mathbf{T}_{\mathcal{F}}\mathbf{F}(G) \end{tikzcd}$$

For $S \in \mathbf{S}(G)$ with $\mathcal{F} \in \mathbf{F}(S)$, it restrict to a commutative diagram of isometries

$$\begin{tikzcd} & (\mathbf{F}(S), d) \arrow[dl, "\simeq"'] \arrow[dr, "\simeq"] & \\ (\mathbf{F}(\overline{S}), d) \arrow[rr, "\simeq"] & & (\mathbf{T}_{\mathcal{F}}\mathbf{F}(S), d) \end{tikzcd}$$

where $\overline{S}$ is the image of S in $P_{\mathcal{F}}/U_{\mathcal{F}}$. Since any two elements $x, y \in \mathbf{F}(P_{\mathcal{F}}/U_{\mathcal{F}})$ are contained in one such $\mathbf{F}(\overline{S})$, it follows that $\varphi : \mathbf{F}(P_{\mathcal{F}}/U_{\mathcal{F}}) \to \mathbf{T}_{\mathcal{F}}\mathbf{F}(G)$ is an isometry. It is therefore also compatible with the relevant norms, angles and scalar products. This gives the following explicit formulas:

$$d\left(\mathrm{Gr}_{\mathcal{F}}(\mathcal{G}_1), \mathrm{Gr}_{\mathcal{F}}(\mathcal{G}_2)\right) = \lim_{t \to 0} \tfrac{1}{t} d(\mathcal{G}_1 + t\mathcal{F}, \mathcal{G}_2 + t\mathcal{F}) \leq d(\mathcal{G}_1, \mathcal{G}_2)$$

and

$$\lim_{t \to 0} \tfrac{1}{t}\left(d(\mathcal{F} + \mathcal{G}_1, \mathcal{F}) - d(\mathcal{F} + \mathcal{G}_1, \mathcal{F} + t\mathcal{G}_2)\right) = \langle \mathrm{Gr}_{\mathcal{F}}(\mathcal{G}_1), \mathrm{Gr}_{\mathcal{F}}(\mathcal{G}_2) \rangle$$

for every $\mathcal{G}_1, \mathcal{G}_2 \in \mathbf{F}(G)$, with $\|\mathrm{Gr}_{\mathcal{F}}(\mathcal{G})\| = \|\mathcal{G}\|$ and

$$\langle \mathrm{Gr}_{\mathcal{F}}(\mathcal{G}_1), \mathrm{Gr}_{\mathcal{F}}(\mathcal{G}_2) \rangle = \textstyle\sum_\gamma \langle \mathrm{Gr}_{\mathcal{F}}^\gamma(\mathcal{G}_1, \tau), \mathrm{Gr}_{\mathcal{F}}^\gamma(\mathcal{G}_2, \tau) \rangle$$

Also: $\mathrm{loc}_{\mathcal{F}}(\mathcal{G}_1) = \mathrm{loc}_{\mathcal{F}}(\mathcal{G}_2)$ if and only if $\mathsf{U}_{\mathcal{F}} \cdot \mathcal{G}_1 = \mathsf{U}_{\mathcal{F}} \cdot \mathcal{G}_2$.

REMARK 126. The previous results yield a $\mathsf{P}_{\mathcal{F}}$-equivariant isometry between the tangent space $\mathbf{T}_{\mathcal{F}}\mathbf{F}(G)$ at $\mathcal{F}$ viewed as a point in the affine building and the "tangent space" $\mathbf{T}_{\mathcal{F}}^\infty\mathbf{F}(G)$ at $\mathcal{F}$ viewed as a boundary point. This reflects the homogeneity of the vectorial Tits building $\mathbf{F}(G)$: our isometry is induced by

$$(t\,\text{small}) \quad \mathcal{F} + t\mathcal{G} \mapsto t^{-1}(\mathcal{F} + t\mathcal{G}) = \mathcal{G} + t^{-1}\mathcal{F} \quad (t^{-1}\,\text{large})$$

5.7.6. There is also the localization map $\mathrm{loc}_{\mathcal{F}}^a : \mathbf{F}(G) \to \mathbf{T}_{\mathcal{F}}\mathbf{F}(G)$, which sends $\mathcal{G}$ to $\mathrm{loc}_{\mathcal{F}}^a(\mathcal{G}) = \mathrm{loc}_{\mathcal{F}}(\mathcal{H})$ if $\mathcal{G} = \mathcal{F} + \mathcal{H}$. Define $\mathrm{Gr}_{\mathcal{F}}^a = \varphi^{-1} \circ \mathrm{loc}_{\mathcal{F}}^a$, so that

$$\begin{tikzcd} & \mathbf{F}(G) \arrow[dl, "\mathrm{Gr}_{\mathcal{F}}^a"'] \arrow[dr, "\mathrm{loc}_{\mathcal{F}}^a"] & \\ \mathbf{F}(P_{\mathcal{F}}/U_{\mathcal{F}}) \arrow[rr, "\varphi"] & & \mathbf{T}_{\mathcal{F}}\mathbf{F}(G) \end{tikzcd}$$

One checks easily in $\mathbf{F}(S) \ni \mathcal{F}, \mathcal{G}$ that $\mathrm{Gr}_{\mathcal{F}}^a(\mathcal{G}) + \overline{\mathcal{F}} = \mathrm{Gr}_{\mathcal{F}}(\mathcal{G})$, where

$$\overline{\mathcal{F}} = \mathrm{Gr}_{\mathcal{F}}(\mathcal{F}) \in \mathbf{G}(Z(P_{\mathcal{F}}/U_{\mathcal{F}})) = \mathrm{Aut}\left(\mathbf{F}(P_{\mathcal{F}}/U_{\mathcal{F}})\right)$$

is the automorphism determined by $\mathcal{F}$ (see 2.2.6, 5.1 and 5.2.18). Thus

$$\begin{aligned}\left\langle \mathrm{Gr}^a_{\mathcal{F}}(\mathcal{G}), \mathrm{Gr}_{\mathcal{F}}(\mathcal{H})\right\rangle &= \left\langle \mathrm{Gr}_{\mathcal{F}}(\mathcal{G}), \mathrm{Gr}_{\mathcal{F}}(\mathcal{H})\right\rangle - \left\langle \overline{\mathcal{F}}, \mathrm{Gr}_{\mathcal{F}}(\mathcal{H})\right\rangle \\ &= \textstyle\sum_\gamma \left\langle \mathrm{Gr}^\gamma_{\mathcal{F}}(\mathcal{G},\tau), \mathrm{Gr}^\gamma_{\mathcal{F}}(\mathcal{H},\tau)\right\rangle - \gamma \deg(\mathrm{Gr}^\gamma_{\mathcal{F}}(\mathcal{H},\tau))\end{aligned}$$

This gives an explicit formula for

$$\lim_{t\to 0} \tfrac{1}{t}\left(d(\mathcal{G},\mathcal{F}) - d(\mathcal{G},\mathcal{F}+t\mathcal{H})\right) = \left\langle \mathrm{Gr}^a_{\mathcal{F}}(\mathcal{G}), \mathrm{Gr}_{\mathcal{F}}(\mathcal{H})\right\rangle.$$

5.7.7. It is finally very easy to compute the Busemann functions:

$$b_{0,\mathcal{G}}(\mathcal{F}) = \lim_{t\to\infty}\left(d(\mathcal{F},t\mathcal{G}) - t\left\|\mathcal{G}\right\|\right) = -\left\|\mathcal{F}\right\|\cos\measuredangle(\mathcal{F},\mathcal{G}).$$

It follows that for any $\mathcal{F},\mathcal{G},\mathcal{H}\in\mathbf{F}(G)$,

$$\left\langle \overrightarrow{\mathcal{F}\mathcal{G}}, \mathcal{H}\right\rangle = \left\langle \mathcal{G},\mathcal{H}\right\rangle - \left\langle \mathcal{F},\mathcal{H}\right\rangle.$$

Thus if C is a closed convex subset of $\mathbf{F}(G)$, $\mathcal{F}\in C$ is the convex projection of some $\mathcal{G}\in\mathbf{F}(G)$ and $\mathcal{H}\in\mathbf{F}(G)$ satisfies $\mathcal{F}+t\mathcal{H}\in C$ for all sufficiently $t>0$, then

$$\left\langle \mathcal{G},\mathcal{H}\right\rangle \leq \left\langle \mathcal{F},\mathcal{H}\right\rangle.$$

5.8. Example: a symmetric space

Let $K=\mathbb{R}$ and $G=GL(V)$, where V is an $\mathbb{R}$-vector space of dimension $n\in\mathbb{N}$.

5.8.1. By corollary 65, the tautological representation V of G identifies $\mathbf{G}(G)$ and $\mathbf{F}(G)$ with the sets $\mathbf{G}(V)$ and $\mathbf{F}(V)$ of all $\mathbb{R}$-graduations and $\mathbb{R}$-filtrations on V. Similarly, the action of G on the set $\mathbb{P}^1(V)$ of $\mathbb{R}$-lines in V identifies $\mathbf{S}(G)$ with

$$\mathbf{S}(V) = \left\{\mathcal{S}\subset\mathbb{P}^1(V) : V = \oplus_{L\in\mathcal{S}} L\right\}.$$

We denote by $\mathbf{F}(\mathcal{S})$ the apartment of $\mathbf{F}(V)$ corresponding to $\mathcal{S}\in\mathbf{S}(V)$. An $\mathbb{R}$-filtration $\mathcal{F}$ on V thus belongs to $\mathbf{F}(\mathcal{S})$ if and only if

$$\forall\gamma\in\mathbb{R}:\qquad \mathcal{F}^\gamma = \oplus_{L\in\mathcal{S},\,\mathcal{F}^\sharp(L)\geq\gamma} L \quad\text{where}\quad \mathcal{F}^\sharp(L) = \sup\{\lambda : L\subset\mathcal{F}^\lambda\}.$$

We also identify $\mathbf{C}(G)$ with $\mathbb{R}^n_\leq = \{\gamma_1\leq\cdots\leq\gamma_n : \gamma_i\in\mathbb{R}\}$ by the map which sends $t(\mathcal{F})$ to $\underline{t}(\mathcal{F}) = (t_i(\mathcal{F}))_{i=1}^n$, with $\sharp\{i : t_i(\mathcal{F})=\gamma\} = \dim_{\mathbb{R}}\mathrm{Gr}^\gamma_{\mathcal{F}}(V)$ for $\gamma\in\mathbb{R}$. The dominance order on $\mathbf{C}(G)$ defined in section 5.1 corresponds to

$$(\gamma_i)_{i=1}^n \leq (\gamma'_i)_{i=1}^n \iff \begin{cases}\sum_{j=1}^n\gamma_j = \sum_{j=1}^n\gamma'_j & \text{and}\\ \sum_{j=i}^n\gamma_j \leq \sum_{j=i}^n\gamma'_j & \text{for } 2\leq i\leq n.\end{cases}$$

The length $\|-\| : \mathbf{C}(G)\to\mathbb{R}_+$ attached to the tautological faithful representation V of G in 4.2.2 corresponds to the function $\|-\| : \mathbb{R}^n_\leq\to\mathbb{R}_+$ given by

$$\|\gamma_1\leq\cdots\leq\gamma_n\| = \sqrt{\gamma_1^2+\cdots+\gamma_n^2}.$$

5.8.2. The exponential $\exp:\mathbb{R}\to\mathbb{R}^\times$ defines an $\mathbb{R}$-valued section exp of the multiplicative group $\mathbb{D}(\mathbb{R})$, whose evaluation at the character $\gamma\in\mathbb{R}$ is given by

$$\gamma(\exp) = \exp(\gamma)\in\mathbb{R}^\times.$$

For $\mathcal{G}\in\mathbf{G}(V)$, we denote by $\mathcal{G}^\flat$ the endomorphism of V which acts by $\gamma\in\mathbb{R}$ on the direct summand $\mathcal{G}_\gamma$ of V. Viewing $\mathcal{G}$ as an $\mathbb{R}$-morphism $\mathbb{D}(\mathbb{R})\to G$, we thus have $\exp(\mathcal{G}^\flat) = \mathcal{G}(\exp)$ in $\mathsf{G}=G(\mathbb{R})$. The maps $\mathcal{G}\mapsto\mathcal{G}^\flat\mapsto\exp(\mathcal{G}^\flat)$ yield G-equivariant bijections between $\mathbf{G}(V)$, the set of diagonalizable endomorphisms of V and the set of diagonalizable elements of G with positive eigenvalues.

5.8.3. Let $\mathbf{B}(V)$ be the space of all Euclidean norms α on V, i.e. functions

$$\alpha : V \to \mathbb{R}_+$$

whose square α^2 is a positive definite quadratic form on V – thus $\mathbf{B}(V)$ may also be viewed as the space of all scalar products on V, or as the space of all ellipsoids in V. The group G acts transitively on $\mathbf{B}(V)$ by $(g \cdot \alpha)(v) = \alpha(g^{-1}v)$ and the stabilizer of α is the orthogonal group $\mathsf{O}(\alpha) \subset \mathsf{G}$, thus $\mathbf{B}(V) \simeq \mathsf{G}/\mathsf{O}(\alpha)$ is a smooth variety. We denote by $\mathbf{G}(V,\alpha) \subset \mathbf{G}(V)$ the set of all α-orthogonal $\mathbb{R}$-graduations on V, by $\mathbf{Sym}(V,\alpha) = \mathbf{G}(V,\alpha)^\flat$ the set of all α-symmetric endomorphisms of V, and by $\mathsf{G}(\alpha) = \exp(\mathbf{Sym}(V,\alpha))$ the set of all α-symmetric automorphisms of V with positive eigenvalues. Thus $\mathbf{Sym}(V,\alpha)$ is the tangent space of $\mathbf{B}(V)$ at α and the polar decomposition in G yields $\mathsf{G} = \mathsf{G}(\alpha) \cdot \mathsf{O}(\alpha)$. For $\mathcal{F} \in \mathbf{F}(V)$ and $\gamma \in \mathbb{R}$, we denote by $\mathcal{G}_\alpha(\mathcal{F})_\gamma$ the α-orthogonal complement of $\mathcal{F}_+^\gamma$ in $\mathcal{F}^\gamma$. Then $\mathcal{G}_\alpha(\mathcal{F})$ is the unique splitting of $\mathcal{F}$ in $\mathbf{G}(V,\alpha)$ and $\mathcal{G}_\alpha : \mathbf{F}(V) \to \mathbf{G}(V,\alpha)$ is an $\mathsf{O}(\alpha)$-equivariant section of $\mathrm{Fil} : \mathbf{G}(V) \twoheadrightarrow \mathbf{F}(V)$. We obtain a sequence of $\mathsf{O}(\alpha)$-equivariant bijections

$$\mathbf{F}(V) \xrightarrow{\mathcal{G}_\alpha} \mathbf{G}(V,\alpha) \xrightarrow{\flat} \mathbf{Sym}(V,\alpha) \xrightarrow{\exp} \mathsf{G}(\alpha) \xrightarrow{-\cdot\alpha} \mathbf{B}(V).$$

We set $g_\alpha(\mathcal{F}) = \exp(\mathcal{G}_\alpha(\mathcal{F})^\flat) = \mathcal{G}_\alpha(\mathcal{F})(\exp) \in \mathsf{G}(\alpha)$ and define

$$\alpha + \mathcal{F} = g_\alpha(\mathcal{F}) \cdot \alpha \quad \text{in} \quad \mathbf{B}(V).$$

Thus for any $\alpha \in \mathbf{B}(V)$, $\mathcal{F} \in \mathbf{F}(G)$ and $v \in V$,

$$(\alpha + \mathcal{F})(v) = \alpha\left(\textstyle\sum_\gamma e^{-\gamma} v_\gamma\right) : \quad v = \textstyle\sum_\gamma v_\gamma, \quad v_\gamma \in \mathcal{G}_\alpha(\mathcal{F})_\gamma.$$

For $\mathcal{S} \in \mathbf{S}(V)$, we denote by $\mathbf{B}(\mathcal{S})$ the set of α's in $\mathbf{B}(V)$ for which $V = \oplus_{L\in\mathcal{S}} L$ is an orthogonal decomposition. Thus for $\alpha \in \mathbf{B}(\mathcal{S})$ and $\mathcal{F} \in \mathbf{F}(\mathcal{S})$, we find that

$$(\alpha + \mathcal{F})^2(v) = \textstyle\sum_{L\in\mathcal{S}} (\alpha + \mathcal{F})^2(v_L) = \sum_{L\in\mathcal{S}} (e^{-\mathcal{F}^\sharp(L)}\alpha)^2(v_L)$$

where $v = \sum_{L\in\mathcal{S}} v_L$ with $v_L \in L$, therefore also $\alpha + \mathcal{F} \in \mathbf{B}(\mathcal{S})$.

5.8.4. The above formulas show that $\mathbf{B}(V) = (\mathbf{B}(V), \mathbf{B}(-), +)$ is an affine $\mathbf{F}(V)$-space. It is well-known that it satisfies $R(s)$, and $L(s)$ follows from the existence of the α-orthogonal splittings. Moreover for any $\alpha \in \mathbf{B}(V)$, the pull map

$$\mathbf{F}(V) \to \mathbf{B}(V), \qquad \mathcal{F} \mapsto \alpha + \mathcal{F}$$

is an $\mathsf{O}(\alpha)$-equivariant bijection. The Fischer-Courant theory tells us that the orbits of the diagonal action of G on $\mathbf{B}(V) \times \mathbf{B}(V)$ are classified by a G-equivariant map

$$\underline{\mathbf{d}} : \mathbf{B}(V) \times \mathbf{B}(V) \to \mathbb{R}^n_\leq$$

whose i-th component $\mathbf{d}_i : \mathbf{B}(V) \times \mathbf{B}(V) \to \mathbb{R}$ is given by

$$\mathbf{d}_i(\alpha,\beta) = -\log\left(\max\left\{\min\left\{\frac{\beta(x)}{\alpha(x)} : x \in W \setminus \{0\}\right\} : W \subset V, \dim_\mathbb{R} W = i\right\}\right).$$

Suppose that $\alpha, \beta \in \mathbf{B}(\mathcal{S}) \cap \mathbf{B}(\mathcal{S}')$ and choose $\mathbb{R}$-basis $e = (e_i)_{i=1}^n$ and $e' = (e'_i)_{i=1}^n$ of V such that $\mathcal{S} = \{\mathbb{R}e_i : i = 1, \cdots, n\}$, $\mathcal{S}' = \{\mathbb{R}e'_i : i = 1, \cdots, n\}$, e and e' are orthonormal for α, and $\beta(e_1) \geq \cdots \geq \beta(e_n)$, $\beta(e'_1) \geq \cdots \geq \beta(e'_n)$. Then necessarily

$$\forall i \in \{1, \ldots, n\} : \qquad \beta(e_i) = \exp\left(-\mathbf{d}_i(\alpha,\beta)\right) = \beta(e'_i)$$

The element $g \in \mathsf{G}$ mapping e to e' satisfies $g\mathcal{S} = \mathcal{S}'$, $g\alpha = \alpha$ and $g\beta = \beta$, which proves $R(i)$. The resulting vectorial distance $\mathbf{d}$ equals $\underline{\mathbf{d}}$ under the identification

$\mathbf{C}(G) \simeq \mathbb{R}^n_{\leq}$, i.e. for every $\alpha \in \mathbf{B}(V)$ and $\mathcal{F} \in \mathbf{F}(V)$, $\underline{\mathbf{d}}(\alpha, \alpha + \mathcal{F}) = \underline{t}(\mathcal{F})$. Indeed for any $X \in \mathbf{Sym}(V, \alpha)$ and $\beta = \exp(X) \cdot \alpha$, we have

$$\underline{\mathbf{d}}(\alpha, \beta) = (\gamma_1, \cdots, \gamma_n) \quad \text{in} \quad \mathbb{R}^n_{\leq}$$

where $\gamma_1 \leq \cdots \leq \gamma_n$ are the eigenvalues of X counted with multiplicities.

5.8.5. Define $\mathbf{d}^i(\alpha, \beta) = \sum_{j=0}^{i-1} \mathbf{d}_{n-j}(\alpha, \beta)$, so that

$$\begin{aligned} \mathbf{d}^i(\alpha, \beta) &= \max\left\{\mathbf{d}^i(\alpha|W, \beta|W) : W \subset V, \dim_{\mathbb{R}} W = i\right\} \\ &= \log \max \left\{ \frac{\Lambda^i(\alpha)(v)}{\Lambda^i(\beta)(v)} : v \in \Lambda^i(V) \setminus \{0\} \right\} \end{aligned}$$

where $\Lambda^i(\alpha)$ is the Euclidean norm on $\Lambda^i(V)$ induced by α. We have

$$\mathbf{d}^n(\alpha, \beta) = \log \left(\frac{\int_{\beta(v) \leq 1} dv}{\int_{\alpha(v) \leq 1} dv} \right)$$

for any Borel measure dv on V, therefore

$$\begin{aligned} \mathbf{d}^n(\alpha, \gamma) &= \mathbf{d}^n(\alpha, \beta) + \mathbf{d}^n(\beta, \gamma), \\ \mathbf{d}^n(\alpha, g\alpha) &= \log |\det(g)|, \\ \mathbf{d}^n(\alpha, \alpha + \mathcal{F}) &= \textstyle\sum_\gamma \gamma \dim_{\mathbb{R}} \mathrm{Gr}^\gamma_{\mathcal{F}}. \end{aligned}$$

In particular, if $\mathbf{d}^i(\alpha, \gamma) = \mathbf{d}^i(\alpha|W, \gamma|W)$ for some $W \subset V$, $\dim_{\mathbb{R}} W = i$, then

$$\mathbf{d}^i(\alpha, \gamma) = \mathbf{d}^i(\alpha|W, \beta|W) + \mathbf{d}^i(\beta|W, \gamma|W) \leq \mathbf{d}^i(\alpha, \beta) + \mathbf{d}^i(\beta, \gamma)$$

i.e. $\underline{\mathbf{d}}$ satisfies the triangle inequality TR.

5.8.6. We next show that for any $\alpha \in \mathbf{B}(V)$ and $\mathcal{F}, \mathcal{G} \in \mathbf{F}(V)$,

$$2 \cdot \underline{\mathbf{d}}(\alpha + \mathcal{F}, \alpha + \mathcal{G}) \leq \underline{\mathbf{d}}(\alpha + 2\mathcal{F}, \alpha + 2\mathcal{G}) \quad \text{in} \quad \mathbb{R}^n_{\leq}.$$

Put $f = g_\alpha(\mathcal{F})$, $g = g_\alpha(\mathcal{G})$. Then $f^2 = g_\alpha(2\mathcal{F})$, $g^2 = g_\alpha(2\mathcal{G})$ and we have to show

$$2 \cdot \underline{\mathbf{d}}(f\alpha, g\alpha) \leq \underline{\mathbf{d}}(f^2\alpha, g^2\alpha).$$

Let $h \mapsto h^*$ be the involution of G defined by α, so that $f^* = f$, $g^* = g$ and $f^{-2}g^2$ is conjugated to $gf^{-2}g = (gf^{-1})(gf^{-1})^*$. For $1 \leq i \leq n$ and $h \in \mathsf{G}$, write $\lambda_i(h)$ for the largest real eigenvalue of h acting on $\Lambda^i(V)$ and denote by $\langle -, - \rangle_{\alpha,i}$ the scalar product on $\Lambda^i(V)$ attached to its Euclidean norm $\Lambda^i(\alpha)$. Then

$$\begin{aligned} \exp\left(\mathbf{d}^i(f^2\alpha, g^2\alpha)\right) &= \max\left\{ \frac{\Lambda^i(\alpha)(f^{-2}v)}{\Lambda^i(\alpha)(g^{-2}v)} : v \in \Lambda^i(V) \setminus \{0\} \right\} \\ &= \max\left\{ \frac{\Lambda^i(\alpha)(f^{-2}g^2v)}{\Lambda^i(\alpha)(v)} : v \in \Lambda^i(V) \setminus \{0\} \right\} \\ &\geq \lambda_i(f^{-2}g^2) \quad = \quad \lambda_i(gf^{-2}g) \\ &= \max\left\{ \frac{\langle gf^{-2}gx, x \rangle_{\alpha,i}}{\langle x, x \rangle_{\alpha,i}} : x \in \Lambda^i(V) \setminus \{0\} \right\} \\ &= \max\left\{ \frac{\langle f^{-1}x, f^{-1}x \rangle_{\alpha,i}}{\langle g^{-1}x, g^{-1}x \rangle_{\alpha,i}} : x \in \Lambda^i(V) \setminus \{0\} \right\} \\ &= \exp\left(2\mathbf{d}^i(f\alpha, g\alpha)\right) \end{aligned}$$

with equality for $i = n$, which proves our claim. Thus $\underline{\mathbf{d}}$ satisfies CO'' and CO.

5.8.7. For $\alpha \in \mathbf{B}(V)$ and $\mathcal{F} \in \mathbf{F}(V)$, we denote by $\mathrm{Gr}_{\mathcal{F}}(\alpha)$ the Euclidean norm on $\mathrm{Gr}^{\bullet}_{\mathcal{F}}(V)$ induced by α through the isomorphism $V \simeq \mathrm{Gr}_{\mathcal{F}}(V)$ provided by the α-orthogonal splitting $\mathcal{G}_\alpha(\mathcal{F})$ of $\mathcal{F}$. We claim that for every $\alpha, \beta \in \mathbf{B}(V)$,

$$\lim_{t\to\infty} \underline{\mathbf{d}}(\alpha + t\mathcal{F}, \beta + t\mathcal{F}) = \underline{\mathbf{d}}(\mathrm{Gr}_{\mathcal{F}}(\alpha), \mathrm{Gr}_{\mathcal{F}}(\beta)) \quad \text{in} \quad \mathbb{R}^n_{\leq}.$$

Indeed, choosing an isomorphism $(\mathcal{G}_\alpha(\mathcal{F})_\gamma, \alpha|\mathcal{G}_\alpha(\mathcal{F})_\gamma) \simeq (\mathcal{G}_\beta(\mathcal{F})_\gamma, \beta|\mathcal{G}_\beta(\mathcal{F})_\gamma)$ for every $\gamma \in \mathbb{R}$, we obtain an element $g \in \mathsf{G}$ which fixes $\mathcal{F}$ and maps α to β. It then also maps $\alpha + t\mathcal{F} = g_\alpha(t\mathcal{F}) \cdot \alpha$ to $\beta + t\mathcal{F} = gg_\alpha(t\mathcal{F}) \cdot \alpha$, so that

$$\underline{\mathbf{d}}(\alpha + t\mathcal{F}, \beta + t\mathcal{F}) = \underline{\mathbf{d}}\left(\alpha, g_\alpha^{-1}(t\mathcal{F}) g g_\alpha(t\mathcal{F}) \cdot \alpha\right).$$

Let $L_\alpha(\mathcal{F})$ be the centralizer of $\mathcal{G}_\alpha(\mathcal{F})$, so that $P_{\mathcal{F}} = U_{\mathcal{F}} \rtimes L_\alpha(\mathcal{F})$. Write $g = u \cdot \ell$ with $u \in \mathsf{U}_{\mathcal{F}}$ and $\ell \in \mathsf{L}_\alpha(\mathcal{F})$, so that $g_\alpha^{-1}(t\mathcal{F}) g g_\alpha(t\mathcal{F}) = g_\alpha^{-1}(t\mathcal{F}) u g_\alpha(t\mathcal{F}) \cdot \ell$. Let then $\mathfrak{u}_{\mathcal{F}} = \oplus_{\gamma > 0} \mathfrak{u}_\gamma$ be the weight decomposition of $\mathfrak{u}_{\mathcal{F}} = \mathrm{Lie}(U_{\mathcal{F}})(\mathbb{R})$ induced by

$$\mathrm{ad} \circ \mathcal{G}_\alpha(\mathcal{F}) : \mathbb{D}(\mathbb{R}) \to G \to GL(\mathfrak{g})$$

where $\mathfrak{g} = \mathrm{Lie}(G)(\mathbb{R})$. Then $g_\alpha(t\mathcal{F})$ acts on $\mathfrak{u}_\gamma$ by $\exp(t\gamma)$, from which easily follows that $g_\alpha^{-1}(t\mathcal{F}) u g_\alpha(t\mathcal{F})$ converges to 1 in $\mathsf{U}_{\mathcal{F}}$ (for the real topology). It follows that

$$\lim_{t\to\infty} \underline{\mathbf{d}}(\alpha + t\mathcal{F}, \beta + t\mathcal{F}) = \underline{\mathbf{d}}\left(\alpha, \ell\alpha\right) = \underline{\mathbf{d}}\left(\mathrm{Gr}_{\mathcal{F}}(\alpha), \mathrm{Gr}_{\mathcal{F}}(\beta)\right).$$

Taking $\beta = u\alpha$ with $u \in \mathsf{U}_{\mathcal{F}}$, we obtain UN. On the other hand for any β, since

$$\mathbb{R}_+ \ni t \mapsto \underline{\mathbf{d}}(\alpha + t\mathcal{F}, \beta + t\mathcal{F}) \in \mathbb{R}^n_{\leq}$$

is convex and bounded, it is non-increasing, which proves NE.

5.8.8. Let $d = \|\underline{\mathbf{d}}\|$ be the G-invariant distance on $\mathbf{B}(V)$ attached to the faithful representation V of G, as in 5.2.9. We claim that the metric space $(\mathbf{B}(V), d)$ is $CAT(0)$. In particular, it is uniquely geodesic, thus $\mathbf{B}(V)$ also satisfies UG. To establish our claim, fix $\alpha \in \mathbf{B}(V)$, choose an α-orthonormal basis $(e_i)_{i=1}^n$ of V and use it to identify G with $GL(n, \mathbb{R})$, $\mathbf{Sym}(V, \alpha)$ with the vector space $S(n, \mathbb{R})$ of symmetric matrices in $M(n, \mathbb{R})$ and $\mathsf{G}(\alpha)$ with the open cone $P(n, \mathbb{R}) \subset S(n, \mathbb{R})$ of positive definite matrices. Let $\langle -, - \rangle$ be the scalar product on V attached to α and $g \mapsto g^*$ the corresponding involution of G. For $p \in \mathsf{G}(\alpha)$, $g \in \mathsf{G}$ and $v \in V$, set $g \cdot p = gpg^*$ and $\alpha_p(v) = \langle pv, v \rangle^{1/2}$. This defines an action of G on $\mathsf{G}(\alpha)$ and $p \mapsto \alpha_p$ is an isomorphism of differentiable manifold $\mathsf{G}(\alpha) \to \mathbf{B}(V)$ such that

$$\alpha_{g \cdot p} = (g^*)^{-1} \cdot \alpha_p \quad \text{and} \quad \alpha + \mathcal{F} = \alpha_{g_\alpha(\mathcal{F})^{-2}} \quad \text{in} \quad \mathbf{B}(V)$$

for any $g \in \mathsf{G}$, $p \in \mathsf{G}(\alpha)$ and $\mathcal{F} \in \mathbf{F}(V)$. In [**8**, II.10.31], $\mathsf{G}(\alpha)$ is equipped with a G-invariant Riemannian structure. Let d_α be the corresponding G-invariant Riemannian metric on $\mathsf{G}(\alpha)$ or $\mathbf{B}(V)$. For $X \in \mathbf{Sym}(V, \alpha)$ and $p = \exp(X) \in \mathsf{G}(\alpha)$,

$$d_\alpha^2(\alpha, \alpha_p) = \mathrm{Tr}(X^2)$$

by [**8**, II.10.42.(2)]. Thus for any $\mathcal{F} \in \mathbf{F}(V)$,

$$d_\alpha^2(\alpha, \alpha + \mathcal{F}) = 4\mathrm{Tr}\left(\left(\mathcal{G}_\alpha(\mathcal{F})^\flat\right)^2\right) = 4\left\|\mathcal{F}\right\|^2 = 4d^2(\alpha, \alpha + \mathcal{F})$$

since $\alpha + \mathcal{F} = \alpha_p$ with $p = \exp(-2\mathcal{G}_\alpha(\mathcal{F})^\flat)$ in $\mathsf{G}(\alpha)$. Therefore $d_\alpha(\alpha, \beta) = 2d(\alpha, \beta)$ for any $\beta \in \mathbf{B}(V)$ by $R(s)$ and $d_\alpha = 2d$ on $\mathbf{B}(V)$ since G acts transitively on $\mathbf{B}(V)$. Since the metric space $(\mathbf{B}(V), d_\alpha)$ is $CAT(0)$ by [**8**, II.10.39], so is $(\mathbf{B}(V), d)$.

5.8.9. We have thus established that $(\mathbf{B}(V), \mathbf{B}(-), +)$ is an affine $\mathbf{F}(V)$-building. If $S \in \mathbf{S}(G)$ corresponds to $\mathcal{S} \in \mathbf{S}(V)$, the type map $\nu_{\mathbf{B},S} : \mathsf{S} \to \mathbf{G}(S)$ maps $s \in \mathsf{S}$ to the unique morphism $\mathbb{D}_{\mathbb{R}}(\mathbb{R}) \to S$ whose composite with the character χ_L through which S acts on $L \in \mathcal{S}$ is the character $\log|\chi_L(s)| \in \mathbb{R}$ of $\mathbb{D}(\mathbb{R})$.

5.8.10. The computations of section 5.8.7 show that $\mathrm{Gr}_{\mathcal{F}}$ induces an isometry

$$\mathrm{Gr}_{\mathcal{F}} : \mathbf{T}^{\infty}_{P_{\mathcal{F}}}\mathbf{B}(V) \simeq \prod\nolimits_{\gamma} \mathbf{B}\left(\mathrm{Gr}^{\gamma}_{\mathcal{F}}(V)\right).$$

On the other hand, the tangent space $\mathbf{T}_{\alpha}\mathbf{B}(V)$ as defined in section 5.5.4 is equal to the corresponding tangent space $\mathbf{Sym}(V, \alpha)$ of the differential manifold $\mathbf{B}(V)$, its scalar product is given by $\langle X, Y\rangle = \mathrm{Tr}(XY)$ and the localization map

$$\mathrm{loc}_{\alpha} : \mathbf{F}(V) \twoheadrightarrow \mathbf{T}_{\alpha}\mathbf{B}(V)$$

maps $\mathcal{F} \in \mathbf{F}(V)$ to the α-symmetric endomorphism

$$\mathrm{loc}_{\alpha}(\mathcal{F}) = \left(\frac{d}{dt} g_{\alpha}(t\mathcal{F})\right)_{t=0} = \mathcal{G}_{\alpha}(\mathcal{F})^{\flat} \quad \text{in} \quad \mathbf{Sym}(V, \alpha).$$

CHAPTER 6

Bruhat-Tits buildings

We first keep the assumptions and notations of the previous chapter. Thus G will be a reductive group over a field K, $\mathsf{G} = G(K)$ and $\Gamma = (\mathbb{R}, +, \leq)$. In addition, we assume that K is equipped with a non-trivial, non-archimedean absolute value

$$|-| : K \to \mathbb{R}^+.$$

However, we will eventually return to the setting of chapter 4, with G a reductive group over the valuation ring $\mathcal{O}_K = \{x \in K : |x| \leq 1\}$ of K and $\Gamma = \mathbb{R}$. Note that then $\mathbf{P}(G) = \mathbf{P}(G_K)$, $\mathbf{F}(G) = \mathbf{F}(G_K)$ but $\mathbf{S}(G) \subsetneq \mathbf{S}(G_K)$ and $\mathbf{G}(G) \subsetneq \mathbf{G}(G_K)$.

6.1. The Bruhat-Tits building of $GL(V)$

6.1.1. Let $G = GL(V)$, where $V \neq 0$ is a K-vector space of dimension $n \in \mathbb{N}$. As in section 5.8, we thus have G-equivariant bijections

$$\begin{array}{rcccl} \mathbf{S}(G) & \simeq & \mathbf{S}(V) & = & \left\{\mathcal{S} \subset \mathbb{P}^1(V)(K) : V = \oplus_{L \in \mathcal{S}} L\right\}, \\ \mathbf{F}(G) & \simeq & \mathbf{F}(V) & = & \{\mathbb{R} - \text{filtrations on } V\}, \\ \mathbf{G}(G) & \simeq & \mathbf{G}(V) & = & \{\mathbb{R} - \text{graduations on } V\}. \end{array}$$

6.1.2. A K-norm (or simply: norm) on V is a function $\alpha : V \to \mathbb{R}^+$ such that

(1) $\alpha(v) = 0$ if and only if $v = 0$,
(2) $\alpha(\lambda v) = |\lambda|\, \alpha(v)$ for every $\lambda \in K$ and $v \in V$, and
(3) $\alpha(u + v) \leq \max\{\alpha(u), \alpha(v)\}$ for every $u, v \in V$.

The K-norm α is split by $\mathcal{S} \in \mathbf{S}(V)$ if and only if

$$\forall v \in V : \quad \alpha(v) = \max\{\alpha(v_L) : L \in \mathcal{S}\} \quad \text{where } v = \textstyle\sum_{L \in \mathcal{S}} v_L,\ v_L \in L.$$

It is splittable if it is split by $\mathcal{S}$ for some $\mathcal{S} \in \mathbf{S}(V)$. If K is locally compact, every K-norm on V is splittable by [**22**, Proposition 1.1].

6.1.3. We denote by $\mathbf{B}(V)$ the set of all splittable K-norms on V, by $\mathbf{B}(\mathcal{S})$ the subset of all K-norms split by $\mathcal{S}$. We let G act on $\mathbf{B}(V)$ by $(g \cdot \alpha)(v) = \alpha(g^{-1} v)$, and define the pull map $+ : \mathbf{B}(V) \times \mathbf{F}(V) \to \mathbf{B}(V)$ by

$$(\alpha + \mathcal{F})(v) = \min\left\{\max\left\{e^{-\gamma} \alpha(v_\gamma) : \gamma \in \mathbb{R}\right\} : v = \textstyle\sum_{\gamma \in \mathbb{R}} v_\gamma,\ v_\gamma \in \mathcal{F}^\gamma\right\}$$

where the sums $\sum_{\gamma \in \mathbb{R}} v_\gamma$ have finite support. We have to verify that this operation is well-defined. Note first that the axiom $L(s)$ follows from the second proof of [**11**, 1.5.ii]: for any $\alpha \in \mathbf{B}(V)$ and $\mathcal{F} \in \mathbf{F}(V)$, there is an $\mathcal{S} \in \mathbf{S}(V)$ with $\alpha \in \mathbf{B}(\mathcal{S})$ and $\mathcal{F} \in \mathbf{F}(\mathcal{S})$. Let us then identify $\mathbf{F}(\mathcal{S})$ with $\mathbb{R}^{\mathcal{S}}$ by $\mathcal{F} \mapsto \mathcal{F}^\sharp$ where

$$\mathcal{F}^\sharp(L) = \max\{\gamma \in \mathbb{R} : L \subset \mathcal{F}^\gamma\}, \qquad \mathcal{F}^\gamma = \oplus_{L : \mathcal{F}^\sharp(L) \geq \gamma} L.$$

Then for $v = \sum_{L \in \mathcal{S}} v_L$ in $V = \oplus_{L \in \mathcal{S}} L$, we find that

$$\inf\left\{\max\left\{e^{-\gamma} \alpha(v_\gamma) : \gamma \in \mathbb{R}\right\} \,\middle|\, {\scriptstyle v = \sum_\gamma v_\gamma \atop v_\gamma \in \mathcal{F}^\gamma}\right\} = \max\left\{e^{-\mathcal{F}^\sharp(L)} \alpha(v_L) : L \in \mathcal{S}\right\}.$$

Indeed for $v = \sum_\gamma v_\gamma$ with $v_\gamma = \sum_L v_{\gamma,L}$, $v_{\gamma,L} \in L$ and $v_{\gamma,L} = 0$ if $\gamma > \mathcal{F}^\sharp(L)$,

$$\begin{aligned} \max\left\{e^{-\gamma}\alpha(v_\gamma) : \gamma \in \mathbb{R}\right\} &= \max\left\{e^{-\gamma}\alpha(v_{\gamma,L}) : \gamma \in \mathbb{R},\, L \in \mathcal{S}\right\} \\ &\geq \max\left\{e^{-\mathcal{F}^\sharp(L)}\alpha(v_{\gamma,L}) : \gamma \in \mathbb{R},\, L \in \mathcal{S}\right\} \\ &\geq \max\left\{e^{-\mathcal{F}^\sharp(L)}\alpha(v_L) : L \in \mathcal{S}\right\} \end{aligned}$$

since $\alpha \in \mathbf{B}(\mathcal{S})$ (for the first equality) and $v_L = \sum_\gamma v_{\gamma,L}$ (for the last inequality), which provides the non-trivial required inequality in the displayed formula. Thus

$$(\alpha + \mathcal{F})(v) = \max\left\{e^{-\mathcal{F}^\sharp(L)}\alpha(v_L) : L \in \mathcal{S}\right\} \tag{6.1.1}$$

from which follows that $\alpha + \mathcal{F}$ is well-defined and again belongs to $\mathbf{B}(\mathcal{S})$.

6.1.4. The apartment and pull maps are plainly G-equivariant, and the above formula shows that the latter turns $\mathbf{B}(\mathcal{S})$ into an affine $\mathbf{F}(\mathcal{S})$-space, thus

$$\mathbf{B}(V) = (\mathbf{B}(V), +, \mathbf{B}(-))$$

is an affine $\mathbf{F}(G)$-space. If $S \in \mathbf{S}(G)$ corresponds to $\mathcal{S} \in \mathbf{S}(V)$, the type map

$$\nu_{\mathbf{B},S} : \mathsf{S} \to \mathbf{G}(S)$$

maps s to the unique $\mathcal{F} \in \mathbf{F}(S)$ with $\gamma_L(\mathcal{F}) = \log|\chi_L(s)|$ for all $L \in \mathcal{S}$, where $\chi_L : S \to \mathbb{G}_{m,k}$ is the character through which S acts on L.

6.1.5. In [**37**, §3], Parreau shows that the closely related set $\Delta = \mathbb{R}_+^\times \backslash \mathbf{B}(V)$ is an affine building in the sense of [**37**, 1.1] (see also [**11, 22**]). The axioms $R(s)$, $R(i)^+$, HA and $L(s)^+$ for $\mathbf{B}(V)$ respectively follow from the axioms $A3$, $A2$, $A5$ and proposition 1.8 for Δ in [**37**]. For $\alpha \in \mathbf{B}(V)$, $\mathcal{F} \in \mathbf{F}(V)$ and $u \in \mathsf{U}_\mathcal{F}$, pick $\mathcal{S} \in \mathbf{S}(V)$ such that $\alpha \in \mathbf{B}(\mathcal{S})$ and $\mathcal{F} \in \mathbf{F}(\mathcal{S})$ using $L(s)$. Write $\mathcal{S} = \{Kv_1, \cdots, Kv_n\}$ with $i \mapsto \gamma_i = \mathcal{F}^\sharp(Kv_i)$ non-increasing and identify $\mathbf{B}(\mathcal{S})$ with $\mathbb{R}^n$ by $\alpha \mapsto (\alpha_1, \cdots, \alpha_n)$ where $\alpha_i = -\log(\alpha(v_i))$. Then for $t \geq 0$, $t\mathcal{F} \in \mathbf{F}(\mathcal{S})$ acts on $\mathbf{B}(\mathcal{S}) \simeq \mathbb{R}^n$ by

$$(\alpha_1, \cdots, \alpha_n) \mapsto (\alpha_1 + t\gamma_1, \cdots, \alpha_n + t\gamma_n)$$

and the matrix $(u_{i,j})$ of $u \in \mathsf{U}_\mathcal{F}$ in the basis $(v_1, \cdots, v_n)$ of V satisfies $u_{i,i} = 1$ and $u_{i,j} \neq 0$ if and only if $\gamma_i > \gamma_j$ for $i \neq j$. Moreover, u fixes α if and only if $\alpha_j - \alpha_i \leq -\log|u_{i,j}|$ for all $1 \leq i, j \leq n$ by [**37**, 3.5]. Therefore u fixes $\alpha + t\mathcal{F}$ for all $t \gg 0$, i.e. $\mathbf{B}(V)$ satisfies UN^+. Thus by proposition 106, $\mathbf{B}(V)$ is an affine $\mathbf{F}(G)$-building whose underlying metric space is $CAT(0)$.

6.1.6. Let $Z \simeq \mathbb{G}_{m,K}$ be the center of G, so that $\mathbf{G}(Z) \simeq \mathbb{R}$ by the isomorphism which maps $\mathcal{G} : \mathbb{D}_K(\mathbb{R}) \to Z$ to the unique weight $\mathcal{G}^\sharp \in \mathbb{R}$ of the corresponding representation of $\mathbb{D}_K(\mathbb{R})$ on V. For $S \in \mathbf{S}(G)$ corresponding to $\mathcal{S} \in \mathbf{S}(V)$, the projection from $\mathbf{F}(S) = \mathbf{G}(S)$ to $\mathbf{G}(Z)$ then maps $\mathcal{F} \in \mathbf{F}(S)$ to the unique $\mathcal{G}$ with

$$\mathcal{G}^\sharp = \tfrac{1}{n} \textstyle\sum_{L \in \mathcal{S}} \mathcal{F}^\sharp(L).$$

It follows that the projection

$$\mathbf{d}^c : \mathbf{B}(V) \times \mathbf{B}(V) \to \mathbf{G}(Z)$$

of the distance $\mathbf{d} : \mathbf{B}(V) \times \mathbf{B}(V) \to \mathbf{C}(G)$ maps (α, β) to the unique $\mathcal{G}$ with

$$\mathcal{G}^\sharp = \tfrac{1}{n} \textstyle\sum_{i=1}^n \log\alpha(v_i) - \log\beta(v_i)$$

for any K-basis $(v_1, \cdots, v_n)$ of V such that $\alpha, \beta \in \mathbf{B}(\mathcal{S})$ with $\mathcal{S} = \{Kv_1, \cdots, Kv_n\}$. From [**37**, 3.2], we then deduce that the morphism

$$\nu_{\mathbf{B}}^c : \mathsf{G} \to \mathbf{G}(Z)$$

maps g to the unique $\mathcal{G}$ with $\mathcal{G}^\sharp = \frac{1}{n} \log |\det g|$. In particular, $|\det \mathsf{G}_\alpha| = 1$ for every $\alpha \in \mathbf{B}(V)$, and then [**37**, 3.5] implies ST: $\mathsf{G}_\alpha = \mathsf{G}_{S,\alpha}$ for all $\alpha \in \mathbf{B}(S)$, $S \in \mathbf{S}(G)$. Therefore $\mathbf{B}(V)$ is a tight affine $\mathbf{F}(G)$-building.

6.1.7. If the valuation of K is discrete, the map

$$\alpha \mapsto L = \{x \in V : \alpha(x) \leq 1\}$$

identifies the subset $\mathbf{B}^\circ(V) \subset \mathbf{B}(V)$ of K-norms α on V such that

$$\alpha(V \setminus \{0\}) = \left|K^\times\right|$$

with the set $\mathcal{L}(V)$ of $\mathcal{O}_K$-lattices in V. Then $\mathbf{B}^\circ(V)$ is stable under the action of G and of the subset $\mathbf{F}^{\log|K^\times|}(V) \subset \mathbf{F}(V)$ of $\log|K^\times|$-filtrations on V. It is then convenient to either normalize the valuation by requiring that $|K^\times| = e^{\mathbb{Z}}$, or to rescale the pull map as in 5.2.20. Then $\mathbf{F}^{\mathbb{Z}}(V)$ acts on $\mathbf{B}^\circ(V) \simeq \mathcal{L}(V)$ by

$$L + \mathcal{F} = \textstyle\sum_{i \in \mathbb{Z}} \pi^{-i} L \cap \mathcal{F}^i$$

for $L \in \mathcal{L}(V)$ and $\mathcal{F} \in \mathbf{F}^{\mathbb{Z}}(V)$, where $\pi \in \mathcal{O}_K$ is any uniformizer.

6.1.8. The space $\mathbf{B}(V)$ is known to be a realization of the Bruhat-Tits building of G; for a more general case, see [**11**].

6.2. The Bruhat-Tits building of G

6.2.1. For a general reductive group G over K, we have to make some assumption on the triple $(G, K, |-|)$: the existence of a *valuation* on the root datum $(\mathsf{Z}_G(S), (\mathsf{U}_a)_{a \in \Phi(G,S)})$ of $\mathsf{G} = G(K)$, in the sense of [**9**, 6.2.1]. Here S is a fixed element of $\mathbf{S}(G)$ and the notations are taken from section 5.4.1.

6.2.2. Let then $\mathbf{B}^r(G)$ and $\mathbf{B}^e(G) = \mathbf{B}^r(G) \times \mathbf{G}(Z)$ be respectively the reduced and extended Bruhat-Tits buildings of G, as defined in [**9**, §7] and [**10**, 4.2.16 & 5.1.29]. These two sets have compatible actions of G, they are covered by apartments $\mathbf{B}^r(S)$ and $\mathbf{B}^e(S) = \mathbf{B}^r(S) \times \mathbf{G}(Z)$ which are G-equivariantly parametrized by $\mathbf{S}(G)$, $\mathbf{B}^e(S)$ is an affine $\mathbf{F}(S)$-space on which $\mathsf{N}_G(S)$ acts by affine transformations with linear part $\nu_S^v : \mathsf{N}_G(S) \twoheadrightarrow \mathsf{W}_G(S)$ and the resulting action of $\mathsf{Z}_G(S)$ is given by a morphism $\nu_{\mathbf{B},S} : \mathsf{Z}_G(S) \to \mathbf{G}(S)$ which is uniquely characterized by the following property: for every morphism $\chi : Z_G(S) \to \mathbb{G}_{m,K}$, the induced morphism

$$\mathbf{G}(\chi|_S) \circ \nu_{\mathbf{B},S} : \mathsf{Z}_G(S) \longrightarrow \mathbf{G}(\mathbb{G}_{m,K})$$

maps z in $\mathsf{Z}_G(S)$ to $\log|\chi(z)|$ in $\mathbb{R} = \mathbf{G}(\mathbb{G}_{m,K})$. Similarly, the action of G on $\mathbf{G}(Z)$ is given by a morphism $\nu_{\mathbf{B}}^c : \mathsf{G} \to \mathbf{G}(Z)$ which is uniquely characterized by the following property: for every morphism $\chi : G \to \mathbb{G}_{m,K}$, the induced morphism

$$\mathbf{G}(\chi|Z) \circ \nu_{\mathbf{B}}^c : \mathsf{G} \to \mathbf{G}(\mathbb{G}_{m,K})$$

maps g in G to $\log|\chi(g)|$ in $\mathbb{R} = \mathbf{G}(\mathbb{G}_{m,K})$. There is a G-invariant distance

$$d : \mathbf{B}^e(G) \times \mathbf{B}^e(G) \to \mathbb{R}^+$$

inducing a Euclidean distance on each apartment, which turns $\mathbf{B}^e(G)$ into a CAT(0)-space. Finally, $\mathbf{B}^e(G)$ already satisfies our axiom $R(s)$ by [**9**, 7.4.18.i] as well as the following strengthening of ST and $R(i)^+$:

> For every subset $\Omega \neq \emptyset$ of $\mathbf{B}^e(S)$, the pointwise stabilizer $\mathsf{G}_\Omega \subset \mathsf{G}$ of Ω equals $\mathsf{G}_{S,\Omega}$ by [**9**, 7.4.4], and it acts transitively on the set of apartments containing Ω by [**9**, 7.4.9].

We denote by $+_S : \mathbf{B}^e(S) \times \mathbf{F}(S) \to \mathbf{B}^e(S)$ the given structure of affine $\mathbf{F}(S)$-space on $\mathbf{B}^e(S)$. These maps are compatible in the following sense:

$$g \cdot (x +_S \mathcal{F}) = (g \cdot x +_{g \cdot S} g \cdot \mathcal{F}).$$

6.2.3. Let us first prove $L(s)$: starting with $x \in \mathbf{B}^e(G)$ and $\mathcal{F} \in \mathbf{F}(G)$, choose a minimal parabolic subgroup $B' \subset P_{\mathcal{F}}$ with Levi $Z_G(S')$, pick $c \in \mathbf{B}^e(S')$ and form the sector $C' = c' +_{S'} F^{-1}(B')$ in $\mathbf{B}^e(S')$. By [**9**, 7.4.18.ii], there is another apartment $\mathbf{B}^e(S)$ containing x and a subsector C of C', which *a priori* is of the form $C = c +_S F^{-1}(B)$ for some minimal parabolic B with Levi $Z_G(S)$. Since $C \subset \mathbf{B}^e(S) \cap \mathbf{B}^e(S')$, there is a $g \in \mathsf{G}$ fixing C and mapping S to S'. Then $g \in \mathsf{B}$ since $\mathsf{G}_C = \mathsf{G}_{S,C} \subset \mathsf{B} \cap \mathsf{G}_c$, thus $\mathrm{Int}(g)(B) = B$ and $Z_G(S') \subset B$. Moreover

$$C = gC = g(c +_S F^{-1}(B)) = c +_{S'} F^{-1}(B)$$

thus actually $B = B'$ because

$$C = c +_{S'} F^{-1}(B) \subset c' +_{S'} F^{-1}(B') = C'$$

in the affine $\mathbf{F}(S')$-space $\mathbf{B}^e(S')$. Now $x \in \mathbf{B}^e(S)$ and also $\mathcal{F} \in \mathbf{F}(S)$ since

$$Z_G(S) \subset B = B' \subset P_{\mathcal{F}}.$$

This proves $L(s)$. Note also that for any $\mathcal{G} \in \mathbf{F}(S) \cap \mathbf{F}(S')$ in the closure of $F^{-1}(B)$,

$$c +_S \mathcal{G} = g(c +_S \mathcal{G}) = c +_{S'} \mathcal{G}$$

since g fixes c, $c +_S \mathcal{G}$ and $\mathcal{G}$. In particular, $c +_S t\mathcal{F} = c +_{S'} t\mathcal{F}$ for all $t \in \mathbb{R}^+$.

6.2.4. Suppose now that $x \in \mathbf{B}^e(S_1) \cap \mathbf{B}^e(S_2)$ and $\mathcal{F} \in \mathbf{F}(S_1) \cap \mathbf{F}(S_2)$ with $S_1, S_2 \in \mathbf{S}(G)$. We now show that $x +_{S_1} \mathcal{F} = x +_{S_2} \mathcal{F}$ in $\mathbf{B}^e(G)$. For $t \geq 0$, put

$$x_i(t) = x +_{S_i} t\mathcal{F} \quad \text{in} \quad \mathbf{B}^e(S_i).$$

The $CAT(0)$-property of d implies that $t \mapsto d(x_1(t), x_2(t))$ is a convex function, and it is therefore sufficient to show that it is also bounded. Let us choose minimal parabolic subgroups $Z_G(S_i) \subset B_i \subset P_{\mathcal{F}}$, and form the corresponding sectors

$$C_i = x +_{S_i} F^{-1}(B_i) \quad \text{in} \quad \mathbf{B}^e(S_i).$$

By [**9**, 7.4.18.iii], there is another apartment $\mathbf{B}^e(S)$ which contains subsectors

$$C_1' \subset C_1 \quad \text{and} \quad C_2' \subset C_2.$$

We have just seen that then $Z_G(S) \subset B_i$ (thus $\mathcal{F} \in \mathbf{F}(S)$) and $C_i' = y_i +_S F^{-1}(B_i)$ in $\mathbf{B}^e(S)$ for some y_i's in $\mathbf{B}^e(S)$, with moreover

$$y_i(t) = y_i +_S t\mathcal{F} = y_i +_{S_i} t\mathcal{F}$$

for $t \geq 0$. Then $t \mapsto d(y_1(t), y_2(t))$ and $t \mapsto d(x_i(t), y_i(t))$ are constant by elementary computations in $\mathbf{B}^e(S)$ and $\mathbf{B}^e(S_i)$ respectively, thus $t \mapsto d(x_1(t), x_2(t))$ is indeed bounded by the triangle inequality in $(\mathbf{B}^e(G), d)$.

6.2.5. We may at last define our pull map: for $x \in \mathbf{B}^e(G)$ and $\mathcal{F} \in \mathbf{F}(G)$, choose $S \in \mathbf{S}(G)$ with $x \in \mathbf{B}^e(S)$ and $\mathcal{F} \in \mathbf{F}(S)$ and set $x + \mathcal{F} = x +_S \mathcal{F}$ in $\mathbf{B}^e(G)$: this does not depend upon the chosen S. Our pull map is plainly G-equivariant, and induces the given structure of affine $\mathbf{F}(S)$-space on $\mathbf{B}^e(S)$. Therefore

$$\mathbf{B}^e(G) = (\mathbf{B}^e(G), +, \mathbf{B}^e(-))$$

is an affine $\mathbf{F}(S)$-space.

6.2.6. For $x \in \mathbf{B}^e(G)$ and $\mathcal{F}, \mathcal{G} \in \mathbf{F}(G)$, choose $S \in \mathbf{S}(G)$ with $x \in \mathbf{B}^e(S)$, $\mathcal{F} \in \mathbf{F}(S)$, let F be the "facet" in $\mathbf{B}^e(S)$ denoted by $\gamma(x, E)$ in [**5**, 7.2.4] with $E = \{t\mathcal{F} : t > 0\}$, let C be a "chamber" of $\mathbf{B}^e(S)$ containing F in its closure. Using [**9**, 7.4.18.ii] as above, we find that there is an apartment $\mathbf{B}^e(S')$ containing C with $\mathcal{G} \in \mathbf{F}(S')$. It then also contains F by [**9**, 7.4.8], which means that for some $\epsilon > 0$, it contains $x + \eta\mathcal{G}$ for every $\eta \in [0, \epsilon]$: this proves $L(s)^+$.

6.2.7. We already have the axioms $R(s)$, $R(i)^+$, $L(s)^+$ and ST, thus $\mathbf{B}^e(G)$ is a tight affine $\mathbf{F}(G)$-building by proposition 106 and lemma 116. Note that the axiom HA also holds for $\mathbf{B}^e(G)$ by [**37**, 1.4] and [**9**, 7.4.19]. If $G = GL(V)$, then $\mathbf{B}^e(G) \simeq \mathbf{B}(V)$ by lemma 118 (see also [**22, 11, 37**]).

6.2.8. The $CAT(0)$-distance d used above may be chosen to be one of our d_τ's, for some faithful representation τ of G. The affine $\mathbf{F}(G)$-space $\mathbf{B}^e(G)$ is discrete when $(K, |-|)$ is discrete, in which case $(\mathbf{B}^e(G), d)$ is a complete metric space by lemma 114 or [**9**, 2.5.12]. If $(K, |-|)$ is complete, then every geodesic ray or line in $\mathbf{B}^e(G)$ is contained in some apartment by [**39**, 2.3.8] and $R(s)$. Thus with the notations of section 5.5.1, $\mathbf{F}(G) \simeq \mathcal{C}(\partial \mathbf{B}^e(G))$ if $(K, |-|)$ is complete.

6.2.9. The Bruhat-Tits building $\mathbf{B}^e(G) = \mathbf{B}^e(G, |-|)$ depends upon the choice of the valuation $|-|$ on K, and so does its structure of affine $\mathbf{F}(G)$-building. However for $\nu > 0$, there is a $G(K)$-equivariant commutative diagram

$$\begin{array}{ccccc}
\mathbf{B}^e(G, |-|) & \times & \mathbf{F}(G_K) & \xrightarrow{+} & \mathbf{B}^e(G, |-|) \\
a \downarrow & & b \downarrow & & a \downarrow \\
\mathbf{B}^e(G, |-|^\nu) & \times & \mathbf{F}(G_K) & \xrightarrow{+} & \mathbf{B}^e(G, |-|^\nu)
\end{array}$$

where a is a canonical $G(K)$-equivariant map and $b(\mathcal{F}) = \nu\mathcal{F}$.

6.3. Functoriality for Bruhat-Tits buildings

6.3.1. Suppose for this section that the valuation ring of $(K, |-|)$, namely

$$\mathcal{O}_K = \{x \in K : |x| \leq 1\}$$

is Henselian. Then for every algebraic extension L of K, there is a unique absolute value $|-| : L \to \mathbb{R}^+$ on L which extends $|-| : K \to \mathbb{R}^+$, and its valuation ring

$$\mathcal{O}_L = \{x \in L : |x| \leq 1\}$$

is the integral closure of $\mathcal{O}_K$ in L, also Henselian. We say that L/K has a geometric property $\mathcal{P}$ over $\mathcal{O}_K$ if the corresponding morphism $\mathrm{Spec}(\mathcal{O}_L) \to \mathrm{Spec}(\mathcal{O}_K)$ does.

PROPOSITION 127. *Let G be a reductive group over $\mathcal{O}_K$.*

(1) *There is an extension L/K, finite étale and Galois over $\mathcal{O}_K$, splitting G.*

(2) *The Bruhat-Tits building $\mathbf{B}^e(G_K)$ exists and contains a canonical point*

$$\circ^e_{G,K} = \circ^e_G = (\circ^r_G, 0) \in \mathbf{B}^e(G_K) = \mathbf{B}^r(G_K) \times \mathbf{G}(Z(G_K))$$

with stabilizer $G(\mathcal{O}_K)$ in $G(K)$. The projection $\circ^r_G$ of $\circ^e_G$ is the unique fixed point of $G(\mathcal{O}_K)$ in $\mathbf{B}^r(G_K)$ if the residue field of $\mathcal{O}_K$ is neither $\mathbb{F}_2$ nor $\mathbb{F}_3$.

(3) *The apartments of $\mathbf{B}^e(G_K)$ containing $\circ^e_G$ are the $\mathbf{B}^e(S_K)$'s for $S \in \mathbf{S}(G)$.*

Proof. Let S be a maximal split torus of G and let T be a maximal torus of $Z_G(S)$ [**1**, XIV 3.20]. Then G and T are isotrivial by proposition 48, thus split by a finite étale cover of $\mathrm{Spec}(\mathcal{O}_K)$ which we may assume to be connected and Galois, i.e. of the form $\mathrm{Spec}(\mathcal{O}_L) \to \mathrm{Spec}(\mathcal{O}_K)$ where $\mathcal{O}_L$ is the normalization of $\mathcal{O}_K$ in a finite étale Galois extension L/K over $\mathcal{O}_K$ by [**28**, 18.10.12]. Since $\mathcal{O}_K$ is Henselian, $\mathcal{O}_L$ is also the valuation ring of $(L, |-|)$. Let (x_α) be a Chevalley system for $(G_{\mathcal{O}_L}, T_{\mathcal{O}_L})$, as defined in [**21**, XXIII 6.2], giving rise to a Chevalley valuation φ_L for G_L, as explained in [**9**, 6.2.3.b] and [**10**, 4.2.1], thus also to the reduced Bruhat-Tits building $\mathbf{B}^r(G_L)$ with its distinguished apartment $\mathbf{B}^r(T_L)$ and the distinguished point $\circ^r_G \equiv \varphi_L$ in $\mathbf{B}^r(T_L)$, as defined in [**9**, §7]. For $f = 0$, the group schemes $\mathfrak{G}^0_f \subset \mathfrak{G}_f \subset \hat{\mathfrak{G}}_f \subset \mathfrak{G}^\dagger_f$ constructed in [**10**, 4.3-6] are all equal to $G_{\mathcal{O}_L}$ [**10**, 4.6.22]. Thus by [**10**, 4.6.28], $G(\mathcal{O}_L)$ is the stabilizer of the distinguished point $\circ^e_G = (\circ^r_G, 0)$ of $\mathbf{B}^e(T_L) \subset \mathbf{B}^e(G_L)$ in $G(L)$, and $\circ^r_G$ is the unique fixed point of $G(\mathcal{O}_L)$ in $\mathbf{B}^r(G_L)$ by [**10**, 5.1.39] if the residue field of $\mathcal{O}_L$ is not equal to $\mathbb{F}_2$ or $\mathbb{F}_3$, which we can always assume.

The pair (G_K, K) satisfies the conditions of the pair denoted by $(H, K^\natural)$ in [**10**, 5.1.1]. The Galois group $\Sigma = \mathrm{Gal}(L/K)$ acts compatibly on $G(L)$ and $\mathbf{B}^e(G_L)$. It therefore fixes $\circ^e_G$, which thus belongs to $\mathbf{B}^e(T_L)^\Sigma = \mathbf{B}^e(S_K)$. Applying this to $Z_G(S)$ instead of G, we see that (G_K, K) also satisfies the assumption (DE) of [**10**, 5.1.5]. Then by [**10**, 5.1.20], the valuation φ_L descends to a valuation φ for G_K. The corresponding building $\mathbf{B}^e(G_K)$ is the fixed point set of Σ in $\mathbf{B}^e(G_L)$ by [**10**, 5.1.25]. The stabilizer of $\circ^e_G \in \mathbf{B}^e(G_K)$ in $G(K)$ equals $G(\mathcal{O}_K) = G(K) \cap G(\mathcal{O}_L)$ and again by [**10**, 5.1.39], $\circ^r_G$ is the unique fixed point of $G(\mathcal{O}_K)$ in $\mathbf{B}^r(G_K)$ if the residue field of $\mathcal{O}_K$ is not equal to $\mathbb{F}_2$ or $\mathbb{F}_3$. By construction, $\circ^e_G$ belongs to $\mathbf{B}^e(S_K)$. Therefore [**9**, 7.4.9] proves our last claim, since $G(\mathcal{O}_K)$ also acts transitively on $\mathbf{S}(G)$. □

6.3.2. We denote by $\mathbf{B}^e(G, K, |-|)$ the pointed affine $\mathbf{F}(G_K)$-building

$$\mathbf{B}^e(G, K, |-|) = (\mathbf{B}^e(G_K), \circ^e_G)$$

attached to a reductive group G over $\mathcal{O}_K$. It easily follows from [**10**, 5.1.41] that this construction is functorial in the Henselian pair $(K, |-|)$. More precisely, let HV be the category whose objects are pairs $(K, |-|)$ where K is a field and $|-| : K \to \mathbb{R}^+$ is a non-trivial, non-archimedean absolute value whose valuation ring $\mathcal{O}_K$ is Henselian. Then for every morphism $f : (K, |-|) \to (L, |-|)$ in HV and every reductive group G over $\mathcal{O}_K$, there is a canonical morphism $f : \mathbf{B}^e(G_K) \to \mathbf{B}^e(G_L)$ such that

$$f(\circ^e_G) = \circ^e_G, \quad f(gx) = f(g)f(x) \quad \text{and} \quad f(x + \mathcal{F}) = f(x) + f(\mathcal{F})$$

for every $x \in \mathbf{B}^e(G_K)$, $g \in G(K)$ and $\mathcal{F} \in \mathbf{F}(G_K)$. The first and last property already determine this morphism uniquely: by the axiom $T(s)$ for $\mathbf{B}^e(G_K)$, any element x of $\mathbf{B}^e(G_K)$ equals $\circ^e_G + \mathcal{F}$ for some for $\mathcal{F} \in \mathbf{F}(G_K)$.

Remark 128. The above functoriality amounts to saying that the mapping

$$\mathbf{B}^e(G_K) \ni \circ^e_G + \mathcal{F} \mapsto \circ^e_G + f(\mathcal{F}) \in \mathbf{B}^e(G_L)$$

is well-defined and equivariant with respect to $G(K) \to G(L)$. This indeed implies the equivariance with respect to $f : \mathbf{F}(G_K) \to \mathbf{F}(G_L)$ as follows. For $S \in \mathbf{S}(G)$ mapping into $S' \in \mathbf{S}(G_{\mathcal{O}_L})$, the above mapping restricts to a well-defined map $\mathbf{B}^e(S_K) \to \mathbf{B}^e(S'_L)$ which is equivariant with respect to $f : \mathbf{F}(S_K) \to \mathbf{F}(S'_K)$; by the axiom $L(s)$ for $\mathbf{B}^e(G_K)$ and proposition 79, any pair $(x, \mathcal{F})$ in $\mathbf{B}^e(G_K) \times \mathbf{F}(G_K)$ is conjugated by some $g \in G(K)$ to one in $\mathbf{B}^e(S_K) \times \mathbf{F}(S_K)$, thus

$$f\left(x + \mathcal{F}\right) = f(g^{-1}) f\left(gx + g\mathcal{F}\right) = f(g^{-1})\left(f(gx) + f(g\mathcal{F})\right) = f(x) + f(\mathcal{F}).$$

THEOREM 129. *The pointed affine $\mathbf{F}(G)$-building $\mathbf{B}^e(G, K, |-|)$ is also functorial in the reductive group G over $\mathcal{O}_K$: for every morphism $f : G \to H$ of reductive groups over $\mathcal{O}_K$, there is a unique morphism $f : \mathbf{B}^e(G_K) \to \mathbf{B}^e(H_K)$ such that*

$$f(\circ^e_G) = \circ^e_H, \quad f(gx) = f(g)f(x) \quad \text{and} \quad f(x + \mathcal{F}) = f(x) + f(\mathcal{F})$$

for every $x \in \mathbf{B}^e(G_K)$, $g \in G(K)$ and $\mathcal{F} \in \mathbf{F}(G_K)$.

This essentially follows from Landvogt's work in [**32**], which has no assumptions on the reductive groups over K but requires $(K, |-|)$ to be quasi-local, in particular discrete. The main difficulty there is the construction of base points with good properties, which is here trivialized by the given points $\circ^e_G$ and $\circ^e_H$. Note that again, the uniqueness of $f : \mathbf{B}^e(G_K) \to \mathbf{B}^e(H_K)$ follows from the first and last displayed requirements, and its existence amounts to showing that the mapping

$$\mathbf{B}^e(G_K) \ni \circ^e_G + \mathcal{F} \mapsto \circ^e_H + f(\mathcal{F}) \in \mathbf{B}^e(H_K)$$

is well-defined and equivariant with respect to $f : G(K) \to H(K)$. Given the identification $\mathbf{B}^e(GL(V)) \simeq \mathbf{B}(V)$, this theorem is closely related to the Tannakian theorem 132 below. We will prove the former as a corollary of the latter.

6.3.3. Assuming theorem 129, we may work out an analog of the discussion of section 5.7.4 for the pointed Bruhat-Tits building $\mathbf{B}^e(G, K)$. First, recall that $\mathbf{P}(G_K) = \mathbf{P}(G)$ since $\mathbb{P}(G)$ is projective over $\mathcal{O}_K$. Let thus $P \in \mathbf{P}(G)$ be a parabolic subgroup of G with unipotent radical U. For every Levi subgroup L of P, there is a canonical commutative diagram

$$\begin{tikzcd}
 & \mathbf{B}^e(L,K) = \mathbf{B}^e(L_K) \arrow[ddl, bend right, "\simeq"'] \arrow[ddr, bend left, "\simeq"] \arrow[d, hook, shift left, "\iota_{L,G}"] & \\
 & \mathbf{B}^e(G,K) = \mathbf{B}^e(G_K) \arrow[u, shift left, "r_{P,L}"] \arrow[dl, "\mathrm{Gr}_P"'] \arrow[dr, "\mathrm{Gr}_P^\infty"] & \\
\mathbf{B}^e(P/U,K) \arrow[rr, "\psi", "\simeq"'] & & \mathbf{T}_P^\infty \mathbf{B}^e(G_K)
\end{tikzcd}$$

where $\iota_L : \mathbf{B}^e(L, K) \simeq \mathbf{B}^e(P/U, K)$ and $\iota_{L,G} : \mathbf{B}^e(L, K) \hookrightarrow \mathbf{B}^e(G, K)$ are the $L(K)$-equivariant maps functorially induced by $L \simeq P/U$ and $L \hookrightarrow G$, $r_{P,L}$ is the $U(K)$-invariant, $L(K)$-equivariant retraction of proposition 101 onto the image $\cup_{S \in \mathbf{S}(L_K)} \mathbf{B}^e(S)$ of $\iota_{L,G}$, $\mathrm{Gr}_P = \iota_L \circ r_{P,L}$ is a $P(K)$-equivariant map, and the right hand side triangle comes from 5.6.4. Both Gr_P and Gr_P^∞ identify there codomain with $U(K) \backslash \mathbf{B}^e(G_K)$, which yields the existence and unicity of the $P(K)$-equivariant bijection $\psi : \mathbf{B}^e(P/U, K) \simeq \mathbf{T}_P^\infty \mathbf{B}^e(G_K)$ at the bottom of our diagram. Neither ψ nor Gr_P depends upon the choice of L: if L' is another Levi subgroup of P, there is a $u \in U(\mathcal{O}_K)$ such that $L' = uLu^{-1}$. The automorphism $\mathrm{Int}(u) : G \to G$ then induces by functoriality a commutative diagram

$$\begin{tikzcd}
\mathbf{B}^e(G,K) \arrow[r, two heads, "r_{P,L}"] \arrow[d, "\mathrm{Int}(u)"] & \cup_{S\in\mathbf{S}(L_K)}\mathbf{B}^e(S) \arrow[d, "\mathrm{Int}(u)"] & \mathbf{B}^e(L,K) \arrow[l, "\iota_{L,G}"'] \arrow[r, "\iota_L"] \arrow[d, "\mathrm{Int}(u)"] & \mathbf{B}^e(P/U,K) \arrow[d, "\mathrm{Id}"] \\
\mathbf{B}^e(G,K) \arrow[r, two heads, "r_{P,L'}"] & \cup_{S'\in\mathbf{S}(L'_K)}\mathbf{B}^e(S') & \mathbf{B}^e(L',K) \arrow[l, "\iota_{L',G}"'] \arrow[r, "\iota_{L'}"] & \mathbf{B}^e(P/U,K)
\end{tikzcd}$$

The first vertical map is also equal to the multiplication by u map on $\mathbf{B}^e(G,K)$:

$$\mathrm{Int}(u)(\circ_G^e + \mathcal{F}) = \circ_G^e + u\mathcal{F} = u(\circ_G^e + \mathcal{F})$$

for all $\mathcal{F} \in \mathbf{F}(G)$ since $u \in G(\mathcal{O}_K)$ fixes $\circ_G^e$. Thus Gr_P and ψ indeed do not depend upon the choice of L. One checks easily that ψ is an isomorphism of affine $\mathbf{F}(P_K/U_K)$-spaces. In particular: $\mathbf{T}_P^\infty\mathbf{B}^e(G_K)$ is an affine $\mathbf{F}(P_K/U_K)$-building, its "quotient" and "building" metric agree by 5.6.8, thus $\psi : \mathbf{B}^e(P/U,K) \to \mathbf{T}_P^\infty\mathbf{B}^e(G_K)$ is an isometry while $\mathrm{Gr}_P : \mathbf{B}^e(G,K) \twoheadrightarrow \mathbf{B}^e(P/U,K)$ is non-expanding when everyone is equipped with the metrics induced by a chosen faithful representation τ of G_K. This gives the following formula: for every $x,y \in \mathbf{B}^e(G_K)$ and $\mathcal{F} \in \mathbf{F}(G)$,

$$\lim_{t\to\infty} d_\tau(x+t\mathcal{F}, y+t\mathcal{F}) = d_{\mathrm{Gr}^\bullet_{\mathcal{F}}(\tau)}(\mathrm{Gr}_{\mathcal{F}}(x), \mathrm{Gr}_{\mathcal{F}}(y)) \leq d_\tau(x,y)$$

where $\mathrm{Gr}_{\mathcal{F}} = \mathrm{Gr}_{P_{\mathcal{F}}} : \mathbf{B}^e(G,K) \twoheadrightarrow \mathbf{B}^e(P_{\mathcal{F}}/U_{\mathcal{F}},K)$. Also:

$$\langle \overrightarrow{xy}, \mathcal{F}\rangle = \left\langle \overrightarrow{\mathrm{Gr}_{\mathcal{F}}(x)\mathrm{Gr}_{\mathcal{F}}(y)}, \overline{\mathcal{F}} \right\rangle$$

for every $x,y \in \mathbf{B}^e(G,K)$, with $\overline{\mathcal{F}} = \mathrm{Gr}_{\mathcal{F}}(\mathcal{F})$ in $\mathbf{G}(Z(P_{\mathcal{F}}/U_{\mathcal{F}})) = \mathbf{G}(\overline{R}(P_{\mathcal{F}}))$.

6.3.4. We may also establish some partial functoriality results when no base point is given, as in Landvogt's work. Fix a quasi-local (discrete, Henselian) pair $(K, |-|)$. For any reductive group G over K, there is a finite Galois extension L of K splitting G such that the reduced building $\mathbf{B}^r(G_L)$ contains a special point $\circ$ fixed by $\mathrm{Gal}(L/K)$. Indeed, let first L_1 be a finite Galois extension of K splitting G, and choose a facet F of $\mathbf{B}^r(G_{L_1})$ fixed by $\mathrm{Gal}(L_1/K)$, for instance one which intersects $\mathbf{B}^r(G_K)$. Then the barycenter $\circ$ of F is also fixed by $\mathrm{Gal}(L_1/K)$, and just like any barycenter of a facet of the Bruhat-Tits building of a split group, it becomes special over a sufficiently ramified extension L of L_1, which we may assume to be Galois over K. Write $\circ_G^e = (\circ, 0)$ for the corresponding $\mathrm{Gal}(L/K)$-invariant point of $\mathbf{B}^e(G_L)$ and let $G_\circ$ be the reductive group over $\mathcal{O}_L$ with generic fiber G_L such that $G_\circ(\mathcal{O}_L)$ is the stabilizer of $\circ_G^e$ in $G(L)$. Since $\circ_G^e$ is fixed by $\mathrm{Gal}(L/K)$, the Hopf $\mathcal{O}_L$-sub-algebra $\mathcal{A}(G_\circ)$ of $\mathcal{A}(G_L) = \mathcal{A}(G)_L$ is fixed by the action of $\mathrm{Gal}(L/K)$.

Let now τ be a finite dimensional K-representation of G, corresponding to a morphism $f : G \to H$, with $H = GL(V)$, $V = V(\tau)$. By [**44**, 1.5], every finitely generated $\mathcal{O}_L$-submodule M of V_L is contained in some $\mathcal{A}(G_\circ)$-sub-comodule F of V_L which is finitely generated (hence free) over $\mathcal{O}_L$. Since $\mathcal{A}(G_\circ)$ is flat over $\mathcal{O}_L$, there is a smallest such F, which we denote by $F(M)$. Since $\mathcal{A}(G_\circ)$ is stabilized by $\mathrm{Gal}(L/K)$, the map $M \mapsto F(M)$ is $\mathrm{Gal}(L/K)$-equivariant. Thus starting with a $\mathrm{Gal}(L/K)$-stable $\mathcal{O}_L$-lattice M of V_L, for instance the base change of an $\mathcal{O}_K$-lattice of V, we obtain an $\mathcal{O}_L$-model $f : G_\circ \to H_\circ$ of $f_L : G_L \to H_L$, with $H_\circ = GL(F(M))$, such that the point $\circ_H^e = (\circ, 0)$ corresponding to $H_\circ$ in $\mathbf{B}^e(H_L)$ is also fixed by $\mathrm{Gal}(L/K)$. Applying now the previous functoriality results to this

$\mathcal{O}_L$-morphism $f : G_\circ \to H_\circ$, we obtain: for every extension $(L', |-|)$ of $(L, |-|)$ in HV, there is a unique morphism $f : \mathbf{B}^e(G_{L'}) \to \mathbf{B}^e(H_{L'})$ such that

$$f(\circ_G^e) = \circ_H^e, \quad f(gx) = f(g)f(x) \quad \text{and} \quad f(x + \mathcal{F}) = f(x) + f(\mathcal{F})$$

for every $x \in \mathbf{B}^e(G_{L'})$, $g \in G(L')$ and $\mathcal{F} \in \mathbf{F}(G_{L'})$. Moreover, for every K-linear morphism $\sigma : (L', |-|) \to (L'', |-|)$ between two such extensions,

$$f(\sigma x) = \sigma f(x) \quad \text{in} \quad \mathbf{B}^e(G_{L''})$$

for every $x \in \mathbf{B}^e(G_{L'})$. Indeed if $x = \circ_G^e + \mathcal{F}$ with $\mathcal{F} \in \mathbf{F}(G_{L'})$, then

$$\begin{aligned} f(\sigma x) &= f(\sigma \circ_G^e + \sigma\mathcal{F}) = f(\circ_G^e + \sigma\mathcal{F}) = \circ_H^e + f(\sigma\mathcal{F}) \\ &= \sigma \circ_H^e + \sigma f(\mathcal{F}) = \sigma(\circ_H^e + f(\mathcal{F})) = \sigma f(x). \end{aligned}$$

6.4. A Tannakian formalism for Bruhat-Tits buildings

6.4.1. Let again $(K, |-|)$ be a field with a non-trivial, non-archimedean absolute value $|-| : K \to \mathbb{R}^+$, with valuation ring $\mathcal{O}_K$ and residue field k. We denote by $\mathsf{Norm}^\circ(K, |-|)$ the category whose objects are pairs (V, α) where V is a finite dimensional K-vector space and $\alpha : V \to \mathbb{R}^+$ is a splittable K-norm on V. A morphism $f : (V, \alpha) \to (V', \alpha')$ is a K-linear morphism $f : V \to V'$ such that $\alpha'(f(x)) \leq \alpha(x)$ for every $x \in V$. This defines an $\mathcal{O}_K$-linear rigid $\otimes$-category with neutral object $1_K = (K, |-|)$. The $\otimes$-products, inner homs and duals

$$\begin{aligned} (V_1, \alpha_1) \otimes (V_2, \alpha_2) &= (V_1 \otimes V_2, \alpha_1 \otimes \alpha_2) \\ \mathrm{Hom}\left((V_1, \alpha_1), (V_2, \alpha_2)\right) &= \left(\mathrm{Hom}\left(V_1, V_2\right), \mathrm{Hom}\left(\alpha_1, \alpha_2\right)\right) \\ (V, \alpha)^* &= (V^*, \alpha^*) \end{aligned}$$

are respectively given by : $\alpha_1 \otimes \alpha_2 = \mathrm{Hom}(\alpha_1^*, \alpha_2)$ under $V_1 \otimes V_2 = \mathrm{Hom}(V_1^*, V_2)$,

$$\begin{aligned} \mathrm{Hom}\left(\alpha_1, \alpha_2\right)(f) &= \sup\left\{ \frac{\alpha_2(f(x))}{\alpha_1(x)} : x \in V_1 \setminus \{0\} \right\}, \\ \alpha^*(f) &= \sup\left\{ \frac{|f(x)|}{\alpha(x)} : x \in V \setminus \{0\} \right\}. \end{aligned}$$

In addition, $\mathsf{Norm}^\circ(K, |-|)$ is an exact category in Quillen's sense: a short sequence

$$(V_1, \alpha_1) \xrightarrow{f_1} (V_2, \alpha_2) \xrightarrow{f_2} (V_3, \alpha_3)$$

is exact precisely when the underlying sequence of K-vector spaces is exact and

$$\alpha_1(x) = \alpha_2(f_1(x)), \qquad \alpha_3(z) = \inf\left\{\alpha_2(y) : y \in f_2^{-1}(z)\right\}$$

for every $x \in V_1$ and $z \in V_3$. For $\gamma \in \mathbb{R}$ and $(V, \alpha) \in \mathsf{Norm}^\circ(K, |-|)$, we set

$$\begin{aligned} B(\alpha, \gamma) &= \{x \in V : \alpha(x) < \exp(-\gamma)\}\} \\ \overline{B}(\alpha, \gamma) &= \{x \in V : \alpha(x) \leq \exp(-\gamma)\}\} \end{aligned}$$

These are $\mathcal{O}_K$-submodules of V and the functors $(V, \alpha) \mapsto B(\alpha, \gamma)$ are easily seen to be exact. However, $(V, \alpha) \mapsto \overline{B}(\alpha, \gamma)$ is *also* exact, because in fact every exact sequence in $\mathsf{Norm}^\circ(K)$ is split by [**11**, 1.5.ii + Appendix]! If M is an $\mathcal{O}_K$-lattice in V (by which we mean a finitely generated, thus free, $\mathcal{O}_K$-submodule spanning V), we denote by α_M the splittable K-norm on V with $\overline{B}(\alpha_M, 0) = M$ defined by

$$\alpha_M(x) = \inf\left\{|\lambda| : \lambda \in K,\, x \in \lambda M\right\} = \min\left\{|\lambda| : \lambda \in K,\, x \in \lambda M\right\}.$$

6.4.2. For $(K, |-|) \to (L, |-|)$, there is an exact $\mathcal{O}_K$-linear $\otimes$-functor

$$- \otimes L : \mathsf{Norm}^\circ(K, |-|) \to \mathsf{Norm}^\circ(L, |-|)$$

defined by $(V, \alpha) \otimes L = (V_L, \alpha_L)$ where $V_L = V \otimes L$ and

$$\begin{aligned} \alpha_L(v) &= \inf\left\{\max\{|x_k|\,\alpha(v_k)\} : v = \textstyle\sum v_k \otimes x_k,\ v_k \in V,\, x_k \in L\right\}, \\ &= \min\left\{\max\{|x_k|\,\alpha(v_k)\} : v = \textstyle\sum v_k \otimes x_k,\ v_k \in V,\, x_k \in L\right\}. \end{aligned}$$

For $(V, \alpha) \in \mathsf{Norm}^\circ(K, |-|)$, $\gamma \in \mathbb{R}$ and $x \in V$,

$$B(\alpha_L, \gamma) = B(\alpha, \gamma) \otimes \mathcal{O}_L, \quad \overline{B}(\alpha_L, \gamma) = \overline{B}(\alpha, \gamma) \otimes \mathcal{O}_L \quad \text{and} \quad \alpha = \alpha_L | V.$$

If M is an $\mathcal{O}_K$-lattice in V, then $\alpha_{M,L} = \alpha_{M \otimes \mathcal{O}_L}$.

6.4.3. We shall also consider the category $\mathsf{Norm}'(K)$ whose objects are triples (V, α, M) where (V, α) is an object of $\mathsf{Norm}^\circ(K)$ and M is an $\mathcal{O}_K$-lattice in V, with the obvious morphisms. It is again an $\mathcal{O}_K$-linear $\otimes$-category. The formula

$$\mathrm{loc}^\gamma(V, \alpha, M) = \text{image of } \overline{B}(\alpha, \gamma) \cap M \text{ in } M_k = M \otimes_{\mathcal{O}_K} k$$

defines an $\mathcal{O}_K$-linear $\otimes$-functor with values in $\mathsf{Fil}(k) = \mathsf{Fil}^{\mathbb{R}}\mathsf{LF}(k)$,

$$\mathrm{loc} : \mathsf{Norm}'(K) \to \mathsf{Fil}(k).$$

Indeed by the axiom $R(s)$ for $\mathbf{B}(V)$, there is an $\mathcal{O}_K$-basis $(e_1, \cdots, e_n)$ of M adapted to α, thus $\alpha(\sum x_i e_i) = \max\{|x_i|\, e^{-\gamma_i}\}$ where $\gamma_i = -\log \alpha(e_i)$ and

$$\mathrm{loc}^\gamma(V, \alpha, M) = \oplus_{\gamma_i \geq \gamma} k e_i$$

from which easily follows that loc is well-defined and compatible with $\otimes$-products.

6.4.4. For an extension $(K, |-|) \to (L, |-|)$ and a reductive group G over $\mathcal{O}_K$, we denote by $\mathbf{B}'(\omega_G^\circ, L, |-|)$ or simply $\mathbf{B}'(\omega_G^\circ, L)$ the set of all factorizations

$$\mathsf{Rep}^\circ(G)(\mathcal{O}_K) \xrightarrow{\alpha} \mathsf{Norm}^\circ(L, |-|) \xrightarrow{\text{forg}} \mathsf{Vect}(L)$$

of the fiber functor $\omega_{G,L}^\circ$ through an $\mathcal{O}_K$-linear $\otimes$-functor α. For $\tau \in \mathsf{Rep}^\circ(G)(\mathcal{O}_K)$ and $\alpha \in \mathbf{B}'(\omega_G^\circ, L)$, we denote by $\alpha(\tau)$ the corresponding L-norm on $V_L(\tau)$.

6.4.5. For $g \in G(L)$ and $\mathcal{F} \in \mathbf{F}(G_L)$, the following formulas

$$(g \cdot \alpha)(\tau) = \tau_L(g) \cdot \alpha(\tau) \quad \text{and} \quad (\alpha + \mathcal{F})(\tau) = \alpha(\tau) + \mathcal{F}(\tau)$$

respectively define an action of $G(L)$ on $\mathbf{B}'(\omega_G^\circ, L)$ and a $G(L)$-equivariant map

$$+ : \mathbf{B}'(\omega_G^\circ, L) \times \mathbf{F}(G_L) \to \mathbf{B}'(\omega_G^\circ, L).$$

6.4.6. We define the canonical L-norm $\alpha_{G,L}$ on $\omega_{G,L}^\circ$ by the formula

$$\alpha_{G,L}(\tau) = \alpha_{V_{\mathcal{O}_L}(\tau)} = \alpha_{V(\tau),L}.$$

By propositions 45 and 48, $G(\mathcal{O}_L)$ is the stabilizer of $\alpha_{G,L}$ in $G(L)$. We set

$$\mathbf{B}(\omega_G^\circ, L) \overset{\text{def}}{=} \alpha_{G,L} + \mathbf{F}(G_L).$$

This is a $G(\mathcal{O}_L)$-stable subset of $\mathbf{B}'(\omega_G^\circ, L)$ equipped with a $G(\mathcal{O}_L)$-equivariant map

$$\mathrm{can} : \mathbf{F}(G_L) \twoheadrightarrow \mathbf{B}(\omega_G^\circ, L), \qquad \mathrm{can}(\mathcal{F}) = \alpha_{G,L} + \mathcal{F}.$$

6.4.7. Any L-norm α on $\omega^{\circ}_{G,L}$ induces an $\mathcal{O}_K$-linear $\otimes$-functor

$$\alpha' : \mathsf{Rep}^{\circ}(G)(\mathcal{O}_K) \to \mathsf{Norm}'(L)$$

by the formula $\alpha'(\tau) = (V_L(\tau), \alpha(\tau), V_{\mathcal{O}_L}(\tau))$, thus also an $\mathcal{O}_K$-linear $\otimes$-functor

$$\mathrm{loc}(\alpha) : \mathsf{Rep}^{\circ}(G)(\mathcal{O}_K) \to \mathsf{Fil}(k_L), \quad \mathrm{loc}(\alpha) = \mathrm{loc} \circ \alpha'$$

where k_L is the residue field of $\mathcal{O}_L$. We may thus define

$$\mathbf{B}^{?}(\omega^{\circ}_G, L) = \{\alpha \in \mathbf{B}'(\omega^{\circ}_G, L) : \mathrm{loc}(\alpha) \text{ is exact}\}.$$

This is a $G(\mathcal{O}_L)$-stable subset of $\mathbf{B}'(\omega^{\circ}_G, L)$ equipped with a $G(\mathcal{O}_L)$-equivariant map

$$\mathrm{loc} : \mathbf{B}^{?}(\omega^{\circ}_G, L) \to \mathbf{F}(G_{k_L}).$$

6.4.8. All of the above constructions are functorial in G, $(K, |-|)$ and $(L, |-|)$, using pre- or post-composition with the obvious exact $\otimes$-functors

$$\begin{array}{rclcrcl}
\mathsf{Rep}^{\circ}(G_2)(\mathcal{O}_K) & \longrightarrow & \mathsf{Rep}^{\circ}(G_1)(\mathcal{O}_K) & & G_1 & \to & G_2 \\
\mathsf{Rep}^{\circ}(G)(\mathcal{O}_{K_1}) & \longrightarrow & \mathsf{Rep}^{\circ}(G)(\mathcal{O}_{K_2}) & \text{for} & (K_1, |-|_1) & \to & (K_2, |-|_2) \\
\mathsf{Norm}^{\circ}(L_1, |-|_1) & \longrightarrow & \mathsf{Norm}^{\circ}(L_2, |-|_2) & & (L_1, |-|_1) & \to & (L_2, |-|_2)
\end{array}$$

LEMMA 130. *For any reductive group G over $\mathcal{O}_K$, we have*

$$\mathbf{B}(\omega^{\circ}_G, L) \subset \mathbf{B}^{?}(\omega^{\circ}_G, L) \subset \mathbf{B}'(\omega^{\circ}_G, L)$$

and the composition $\mathrm{loc} \circ \mathrm{can} : \mathbf{F}(G_L) \to \mathbf{F}(G_{k_L})$ *is the reduction map*

$$\mathbf{F}(G_L) \xleftarrow{\simeq} \mathbf{F}(G_{\mathcal{O}_L}) \xrightarrow{\mathrm{red}} \mathbf{F}(G_{k_L}).$$

For any $S \in \mathbf{S}(G_{\mathcal{O}_L})$, the functorial map $\mathbf{B}'(\omega^{\circ}_S, L) \to \mathbf{B}'(\omega^{\circ}_G, L)$ is injective.

PROOF. By proposition 79, any $\mathcal{F} \in \mathbf{F}(G_L)$ belongs to $\mathbf{F}(S_L)$ for some S in $\mathbf{S}(G_{\mathcal{O}_L})$. Pre-composing with $\mathsf{Rep}^{\circ}(G)(\mathcal{O}_K) \to \mathsf{Rep}^{\circ}(S)(\mathcal{O}_L)$ yields the vertical maps of the commutative diagram

$$\begin{tikzcd}
\mathbf{F}(S_L) \arrow[r, two heads, "\mathrm{can}"] \arrow[d, hook] & \mathbf{B}(\omega^{\circ}_S, L) \arrow[r, hook] \arrow[d] & \mathbf{B}'(\omega^{\circ}_S, L) \arrow[d] & \mathbf{B}^{?}(\omega^{\circ}_S, L) \arrow[l, hook'] \arrow[r, "\mathrm{loc}"] \arrow[d] & \mathbf{F}(S_{k_L}) \arrow[d, hook] \\
\mathbf{F}(G_L) \arrow[r, two heads, "\mathrm{can}"] & \mathbf{B}(\omega^{\circ}_G, L) \arrow[r, hook] & \mathbf{B}'(\omega^{\circ}_G, L) & \mathbf{B}^{?}(\omega^{\circ}_G, L) \arrow[l, hook'] \arrow[r, "\mathrm{loc}"] & \mathbf{F}(G_{k_L})
\end{tikzcd}$$

which reduces us to the case $K = L$, $G = S$ treated below. □

LEMMA 131. *Suppose that $G = S$ is a split torus. Then all maps in*

$$\mathbf{F}(S_L) \xrightarrow{\mathrm{can}} \mathbf{B}(\omega^{\circ}_S, L) \hookrightarrow \mathbf{B}'(\omega^{\circ}_S, L) \hookleftarrow \mathbf{B}^{?}(\omega^{\circ}_S, L) \xrightarrow{\mathrm{loc}} \mathbf{F}(S_{k_L})$$

are isomorphisms of pointed affine $\mathbf{G}(S)$-spaces. Moreover, $S(L)$ acts on

$$\mathbf{B}(\omega^{\circ}_S, L) = \mathbf{B}^{?}(\omega^{\circ}_S, L) = \mathbf{B}'(\omega^{\circ}_S, L)$$

by translations through the morphism

$$\nu_{\mathbf{B},S} : S(L) \to \mathbf{G}(S)$$

which maps $s \in S(L)$ to the unique morphism $\nu_{\mathbf{B},S}(s) : \mathbb{D}_{\mathcal{O}_K}(\mathbb{R}) \to S$ whose composition with any character χ of S is the character $\log|\chi(s)| \in \mathbb{R}$ *of* $\mathbb{D}_{\mathcal{O}_K}(\mathbb{R})$.

PROOF. Put $M = \mathrm{Hom}(S, \mathbb{G}_{m,\mathcal{O}_K})$ and let ρ_m be the representation of S on $\mathcal{O}_K$ given by the character $m \in M$. For $\tau \in \mathsf{Rep}^\circ(S)(\mathcal{O}_K)$, let $\tau = \oplus \tau_m$ be the weight decompositions of τ. Recall from section 3.10.6 that the formulas

$$\mathcal{F}^\gamma(\tau) = \oplus_{\mathcal{F}^\sharp(m)\geq\gamma} V(\tau_m), \quad \mathcal{F}^\sharp(m) = \sup\{\gamma : \mathcal{F}^\gamma(\rho_m) \neq 0\}$$

yield isomorphisms between $\mathbf{F}(S) = \mathbf{G}(S)$ and $\mathrm{Hom}(M, \mathbb{R})$. Similarly, the formulas

$$\alpha(\tau)(x) = \max\left\{e^{-\alpha^\sharp(m)}\alpha_{V_{\mathcal{O}_L}(\tau_m)}(x_m) : m \in M\right\}, \quad \alpha^\sharp(m) = -\log \alpha(\rho_m)(1_{\mathcal{O}_K})$$

where $x = \sum x_m$ is the decomposition of x in $V_L(\tau) = \oplus V_L(\tau_m)$ yield isomorphisms between $\mathbf{B}'(\omega_S^\circ, L)$ and $\mathrm{Hom}(M, \mathbb{R})$. One then checks easily that

$$\alpha_{S,L}^\sharp = 0, \quad (\alpha + \mathcal{F})^\sharp = \alpha^\sharp + \mathcal{F}^\sharp \quad \text{and} \quad s \cdot \alpha = \alpha + \nu_{\mathbf{B},S}(s)$$

as well as $\mathrm{loc}^\gamma(\alpha)(\tau) = \oplus_{\alpha^\sharp(m)\geq\lambda} V_{k_L}(\tau_m)$, from which the lemma follows. $\square$

6.4.9. For $S \in \mathbf{S}(G_{\mathcal{O}_L})$, we identify $\mathbf{B}(\omega_S^\circ, L)$ with its image in $\mathbf{B}(\omega_G^\circ, L)$ and call it the apartment attached to S. The pull map on $\mathbf{B}'(\omega_G^\circ, L)$ thus induces a structure of affine $\mathbf{F}(S_L)$-space on $\mathbf{B}(\omega_S^\circ, L)$, and the action of $G(L)$ on $\mathbf{B}'(\omega_G^\circ, L)$ restricts to an action of $S(L)$ on $\mathbf{B}(\omega_S^\circ, L)$, by translations through the above morphism $\nu_{\mathbf{B},S} : S(L) \to \mathbf{G}(S_L)$.

6.4.10. We now restrict our attention to Henselian fields, so that $\mathbf{B}^e(G, L, |-|)$ is also well-defined, functorial in $(L, |-|)$, and equal to $\circ_G^e + \mathbf{F}(G_L)$ by $T(s)$. Given the functorial properties of $\mathbf{B}(\omega_G^\circ, L)$, theorem 129 immediately follows from:

THEOREM 132. *The formula* $\circ_G^e + \mathcal{F} \mapsto \alpha_{G,L} + \mathcal{F}$ *defines a functorial bijection*

$$\boldsymbol{\alpha} : \mathbf{B}^e(G, L, |-|) \to \mathbf{B}(\omega_G^\circ, L, |-|)$$

such that for every $x \in \mathbf{B}^e(G_L)$, $g \in G(L)$ *and* $\mathcal{F} \in \mathbf{F}(G_L)$,

$$\boldsymbol{\alpha}(\circ_G^e) = \alpha_G, \quad \boldsymbol{\alpha}(g \cdot x) = g \cdot \boldsymbol{\alpha}(x) \quad \textit{and} \quad \boldsymbol{\alpha}(x + \mathcal{F}) = \boldsymbol{\alpha}(x) + \mathcal{F}.$$

PROOF. Fix an extension $(L, |-|) \to (L', |-|)$ such that $G' = G_{\mathcal{O}_{L'}}$ splits and consider the following diagram, where $\mathcal{F} \in \mathbf{F}(G_L)$ and $\mathcal{F}' \in \mathbf{F}(G_{L'}) = \mathbf{F}(G'_{L'})$:

$$\begin{tikzcd}
\circ_G^e + \mathcal{F} \arrow[d, dotted, "?"'] & \mathbf{B}^e(G, L) \arrow[rr] \arrow[d, dotted, "\boldsymbol{\alpha}"'] \arrow[dr, dotted, "\boldsymbol{\beta}"] & & \mathbf{B}^e(G', L') \arrow[d, dotted, "\boldsymbol{\alpha}'"] \arrow[dl, dotted, "\boldsymbol{\beta}'"'] & \circ_{G'}^e + \mathcal{F}' \arrow[d, dotted, "?"] \\
\alpha_{G,L} + \mathcal{F} & \mathbf{B}'(\omega_G^\circ, L) \arrow[r, "-\otimes L'"] & \mathbf{B}'(\omega_G^\circ, L') & \mathbf{B}'(\omega_{G'}^\circ, L') \arrow[l, "\mathrm{Res}"'] & \alpha_{G',L'} + \mathcal{F}'
\end{tikzcd}$$

The bottom maps are respectively induced by post and pre-composition with

$$-\otimes L' : \mathsf{Norm}^\circ(L) \to \mathsf{Norm}^\circ(L') \quad \text{and} \quad -\otimes \mathcal{O}_{L'} : \mathsf{Rep}^\circ(G)(\mathcal{O}_K) \to \mathsf{Rep}^\circ(G')(\mathcal{O}_{L'}).$$

If $\boldsymbol{\alpha}'$ is well-defined and equivariant with respect to the operations of $G(L')$ and $\mathbf{F}(G_{L'})$, so is $\boldsymbol{\beta}'$. Then $\boldsymbol{\beta}$ is well-defined and equivariant with respect to the operations of $G(L)$ and $\mathbf{F}(G_L)$. But $\mathbf{B}'(\omega_G^\circ, L) \to \mathbf{B}'(\omega_G^\circ, L')$ is injective, thus $\boldsymbol{\alpha}$ is also well-defined and equivariant with respect to the operations of $G(L)$ and $\mathbf{F}(G_L)$. Its image equals $\mathbf{B}(\omega_G^\circ, L)$ by definition, which is thus stable under the operations of $G(L)$ and $\mathbf{F}(G_L)$ on $\mathbf{B}'(\omega_G^\circ, L)$. Since $\mathrm{loc}(\alpha_{G,L} + \mathcal{F}) = \mathcal{F}_{k_L}$ for every $\mathcal{F} \in \mathbf{F}(G_L)$, the restriction of $\boldsymbol{\alpha}$ to any apartment $\mathbf{B}^e(S_L) = \circ_G^e + \mathbf{F}(S_L)$ for $S \in \mathbf{S}(G_{\mathcal{O}_L})$ is injective. Since any pair of points in $\mathbf{B}^e(G, L)$ is $G(L)$-conjugated to one in such an apartment by the axiom $R(s)$ for $\mathbf{B}^e(G, L)$, $\boldsymbol{\alpha} : \mathbf{B}^e(G, L) \to \mathbf{B}(\omega_G^\circ, L)$ is a bijection. This reduces us to the case where G is split over $\mathcal{O}_K$ and $K = L$.

Suppose that $\circ_G^e + \mathcal{F}_1 = \circ_G^e + \mathcal{F}_2 = x$ in $\mathbf{B}^e(G_K)$ for some $\mathcal{F}_1, \mathcal{F}_2 \in \mathbf{F}(G_K)$, choose $S_i \in \mathbf{S}(G_K)$ such that $\mathcal{F}_i \in \mathbf{F}(S_i)$ and $\circ_G^e \in \mathbf{B}^e(S_i)$ using $L(s)$ for $\mathbf{B}^e(G_K)$, and then choose $g \in G(K)$ fixing $\circ_G^e$ and x such that $\mathrm{Int}(g)(S_1) = S_2$ using $R(i)$ for $\mathbf{B}^e(G_K)$. Then $S_i \in \mathbf{S}(G)$ and $g \in G(\mathcal{O}_K)$ by proposition 127, moreover $g\mathcal{F}_1 = \mathcal{F}_2$ since $\mathbf{B}^e(S_2)$ is an affine $\mathbf{F}(S_2)$-space. Thus $g(\alpha_G + \mathcal{F}_1) = \alpha_G + \mathcal{F}_2$ in $\mathbf{B}'(\omega_G^\circ, K)$, since $G(\mathcal{O}_K)$ fixes α_G. But g fixes the point $x = \circ_G^e + \mathcal{F}_1$ of $\mathbf{B}^e(S_1)$, thus g fixes $\alpha_G + \mathcal{F}_1$ in $\mathbf{B}'(\omega_G^\circ, K)$ by lemma 133 below, therefore $\alpha_G + \mathcal{F}_1 = \alpha_G + \mathcal{F}_2$ and our map $\boldsymbol{\alpha} : \mathbf{B}^e(G, K) \to \mathbf{B}'(\omega_G^\circ, K)$ is indeed well-defined.

It is plainly $G(\mathcal{O}_K)$-equivariant. For any $S \in \mathbf{S}(G)$, the $G(K)$-equivariant map $\boldsymbol{\alpha}_S$ of lemma 133 below coincides with $\boldsymbol{\alpha}$ on $\mathbf{B}^e(S_K)$, thus $\boldsymbol{\alpha}$ equals $\boldsymbol{\alpha}_S$ everywhere since every point of $\mathbf{B}^e(G_K)$ is conjugated to one in $\mathbf{B}^e(S_K)$ by some element in $G(\mathcal{O}_K)$. Therefore $\boldsymbol{\alpha}$ is $G(K)$-equivariant. Since every pair in $\mathbf{B}^e(G_K) \times \mathbf{F}(G_K)$ is conjugated to one in $\mathbf{B}^e(S_K) \times \mathbf{F}(S_K)$ by some element in $G(K)$, our $\boldsymbol{\alpha}$ is also compatible with the operations of $\mathbf{F}(G_K)$. □

LEMMA 133. *Suppose that G is split over $\mathcal{O}_K$ and let $(K, |-|) \to (L, |-|)$ be any extension in* HV. *Then for any $S \in \mathbf{S}(G)$, there is a unique map*

$$\boldsymbol{\alpha}_S : \mathbf{B}^e(S_L) \to \mathbf{B}'(\omega_S^\circ, L)$$

such that for all $x \in \mathbf{B}^e(S_L)$ and $\mathcal{F} \in \mathbf{F}(S_L)$,

$$\boldsymbol{\alpha}_S(\circ_G^e) = \alpha_{G,L} \quad \textit{and} \quad \boldsymbol{\alpha}_S(x + \mathcal{F}) = \boldsymbol{\alpha}_S(x) + \mathcal{F}$$

Moreover, it extends uniquely to a $G(L)$-equivariant map

$$\boldsymbol{\alpha}_S : \mathbf{B}^e(G_L) \to \mathbf{B}'(\omega_G^\circ, L).$$

PROOF. The uniqueness of both maps is obvious. Since $\mathbf{B}^e(S_L)$ and $\mathbf{B}(\omega_S^\circ, L)$ are affine $\mathbf{G}(S_L)$-spaces on which $S(L)$ acts by translations through the same morphism $\nu_{\mathbf{B},S} : S(L) \to \mathbf{G}(S_L)$, the unique isomorphism of affine $\mathbf{G}(S_L)$-spaces

$$\boldsymbol{\alpha}_S : \mathbf{B}^e(S_L) \to \mathbf{B}(\omega_S^\circ, L)$$

mapping $\circ_G^e \in \mathbf{B}^e(S_L)$ to $\alpha_{G,L} \in \mathbf{B}(\omega_S^\circ, L)$ is $S(L)$-equivariant. Since $G(\mathcal{O}_L)$ fixes $\circ_G^e \in \mathbf{B}^e(G_L)$ and $\alpha_{G,L} \in \mathbf{B}'(\omega_G^\circ, L)$, the induced embedding

$$\boldsymbol{\alpha}_S : \mathbf{B}^e(S_L) \to \mathbf{B}'(\omega_G^\circ, L)$$

is also equivariant for the actions of $N_G(S)(L) = N_G(S)(\mathcal{O}_L) \cdot S(L)$. To extend the latter map to a $G(L)$-equivariant morphism on the whole tight building $\mathbf{B}^e(G_L)$, it remains to establish the following claim – see remark 119:

> For every $x \in \mathbf{B}^e(S_L)$, the $G(L)$-stabilizer of $x \in \mathbf{B}^e(G_L)$ is contained in the $G(L)$-stabilizer of $\boldsymbol{\alpha}_S(x) \in \mathbf{B}'(\omega_G^\circ, L)$.

This is true for $x = \circ_G^e$, where both stabilizers equal $G(\mathcal{O}_L)$. This is therefore also true for any x in $S(L) \cdot \circ_G^e = \circ_G^e + \nu_{\mathbf{B},S}(S(L))$ since $\boldsymbol{\alpha}_S$ is $S(L)$-equivariant. To clarify the proof, note that the base change maps from K to L identify

$$\begin{array}{rclccc} F &=& \mathbf{F}(S_K) & \text{with} & \mathbf{F}(S_L) & \subset & \mathbf{F}(G_L) \\ A &=& \mathbf{B}^e(S_K) & \text{with} & \mathbf{B}^e(S_L) & \subset & \mathbf{B}^e(G_L) \\ B &=& \mathbf{B}(\omega_S^\circ, K) & \text{with} & \mathbf{B}(\omega_S^\circ, L) & \subset & \mathbf{B}'(\omega_G^\circ, L) \end{array}$$

and the isomorphism of affine F-space $\boldsymbol{\alpha}_S : A \to B$ also does not depend upon L. What depends upon L is the subset $\Lambda(L) = \circ_G^e + \nu_{\mathbf{B},S}(\mathbf{S}(L))$ of A on which we know the validity of our claim. So let us fix x and $\alpha = \boldsymbol{\alpha}_S(x)$ as above, as well as some $g \in G(L)$ such that $gx = x$. By lemma 134 below, there is an extension

$(L,|-|) \to (L',|-|)$ in HV such that $\log|L'^{\times}| = \mathbb{R}$. Then $\Lambda(L') = A$, thus $g\alpha = \alpha$ in $\mathbf{B}'(\omega_G^\circ, L')$ since $gx = x$ in $\mathbf{B}^e(G_{L'})$. But $\mathbf{B}'(\omega_G^\circ, L) \to \mathbf{B}'(\omega_G^\circ, L')$ is injective and $G(L)$-equivariant, thus also $g\alpha = \alpha$ in $\mathbf{B}'(\omega_G^\circ, L)$, which proves our claim. $\square$

LEMMA 134. *Let L be a field with a non-archimedean absolute value $|-|$. There is an extension $(L',|-|)$ of $(L,|-|)$ with L' algebraically closed and $\log|L'^{\times}| = \mathbb{R}$.*

PROOF. By [**6**, VI, §8, Proposition 9], we may assume that L is algebraically closed. Then $\log|L^{\times}|$ is a divisible subgroup of $\mathbb{R}$, i.e. a $\mathbb{Q}$-vector space. Let $(\delta_i)_{i\in I}$ be a $\mathbb{Q}$-basis of $\mathbb{R}/\log|L^{\times}|$ and lift each δ_i to $d_i \in \mathbb{R}$. Let $(t_i)_{i\in I}$ be independent variables and let L' be an algebraic closure of the purely transcendental extension $M = K((t_i)_{i\in I})$ of K. By Zorn's lemma and [**6**, VI, §10, Proposition 1], there is a unique extension of $|-|$ to a non-archimedean absolute value on M such that $\log|t_i| = d_i$ for every $i \in I$. The latter again extends to L', and then $\log|L'^{\times}|$ equals $\mathbb{R}$, being a divisible subgroup of $\mathbb{R}$ which contains $\log|L^{\times}|$ and all d_i's. $\square$

6.4.11. The theorem implies various properties of $\mathbf{B}(\omega_G^\circ, K)$, for instance: $\mathbf{B}(\omega_G^\circ, K)$ is a tight affine $\mathbf{F}(G_K)$-building. For an extension $(K,|-|) \to (L,|-|)$, the map $\mathbf{B}(\omega_{G_{\mathcal{O}_L}}^\circ, L) \to \mathbf{B}(\omega_G^\circ, L)$ is an isomorphism of affine $\mathbf{F}(G_L)$-buildings. For a closed immersion $G_1 \hookrightarrow G_2$, the map $\mathbf{B}(\omega_{G_1}^\circ, K) \to \mathbf{B}(\omega_{G_2}^\circ, K)$ is injective. For a central isogeny $G_1 \twoheadrightarrow G_2$, the map $\mathbf{B}(\omega_{G_1}^\circ, K) \to \mathbf{B}(\omega_{G_2}^\circ, K)$ is an isomorphism. Thus $\mathbf{B}(\omega_G^\circ, K)$ has canonical decompositions analogous to those of section 2.2.13. This last property also follows from 5.2.19.

6.4.12. Fix a faithful representation τ in $\mathsf{Rep}^\circ(G)(\mathcal{O}_K)$ and drop it from the notations for the induced distances, angles, scalar products... For $x, y \in \mathbf{B}^e(G,K)$,

$$d(x,y) = d\left(\boldsymbol{\alpha}(x), \boldsymbol{\alpha}(y)\right) = d\left(\boldsymbol{\alpha}(x)(\tau), \boldsymbol{\alpha}(y)(\tau)\right)$$

where the last distance is computed in the space of K-norms on $V_K(\tau)$.

6.4.13. Fix $\mathcal{F}_1, \mathcal{F}_2 \in \mathbf{F}(G_K)$. Suppose that for some $\epsilon > 0$,

$$\forall t \in [0,\epsilon]: \qquad \alpha_G + t\mathcal{F}_1 = \alpha_G + t\mathcal{F}_2 \quad \text{in} \quad \mathbf{B}(\omega_G^\circ, K).$$

Then the reductions $\mathcal{F}_{1,k}$ and $\mathcal{F}_{2,k}$ are equal in $\mathbf{F}(G_k)$ by lemma 130. Suppose conversely that $\mathcal{F}_{1,k} = \mathcal{F}_{2,k}$, and choose an apartment $\mathbf{B}^e(S)$ in $\mathbf{B}^e(G_K)$ containing the germs of $t \mapsto \circ_G^e + t\mathcal{F}_i$ for $i \in \{1,2\}$ – in particular, S belongs to $\mathbf{S}(G)$ since $\circ_G^e$ belongs to $\mathbf{B}^e(S)$. Then there are unique $\mathcal{F}_i^*$ in $\mathbf{F}(S)$ such that, for some $\epsilon > 0$, $\circ_G^e + t\mathcal{F}_i = \circ_G^e + t\mathcal{F}_i^*$ in $\mathbf{B}^e(G_K)$ for all $t \in [0,\epsilon]$. But then also $\alpha_G + t\mathcal{F}_i = \alpha_G + t\mathcal{F}_i^*$ in $\mathbf{B}(\omega_G^\circ, K)$, thus $\mathcal{F}_{i,k} = \mathcal{F}_{i,k}^*$ in $\mathbf{F}(G_k)$, therefore $\mathcal{F}_{1,k}^* = \mathcal{F}_{2,k}^*$ and $\mathcal{F}_1^* = \mathcal{F}_2^*$ since the reduction map is injective on $\mathbf{F}(S)$, thus again $\alpha_G + t\mathcal{F}_1 = \alpha_G + t\mathcal{F}_2$ for all $t \in [0,\epsilon]$. This yields canonical identifications

$$\begin{tikzcd}
& \mathbf{F}(G_K) \arrow[dl, bend right, "\mathrm{loc}_{\circ_G^e}"'] \arrow[d, "\mathrm{loc}_{\alpha_G}"] \arrow[dr, bend left, "\mathrm{red}"] & \\
\mathbf{T}_{\circ_G^e}\mathbf{B}^e(G_K) \arrow[r, "\simeq"] \arrow[rr, bend right, "\simeq", "\kappa"'] & \mathbf{T}_{\alpha_G}\mathbf{B}(\omega_G^\circ, K) \arrow[r, "\simeq"] & \mathbf{F}(G_k)
\end{tikzcd}$$

between the localization maps of 5.5.4 and the reduction map on $\mathbf{F}(G_K)$. By restriction to an apartment $\mathbf{F}(S_K)$ with $S \in \mathbf{S}(G)$, one checks that the isomorphism

$$\kappa : \mathbf{T}_{\circ^e_G}\mathbf{B}^e(G_K) \xrightarrow{\simeq} \mathbf{F}(G_k)$$

is compatible with the distances, scalar products etc... attached to our chosen τ as in 5.5.4 and 4.2.12, and also that κ fits in a commutative diagram

$$\begin{tikzcd} \mathbf{B}^e(G,K) \arrow[r, "\boldsymbol{\alpha}"] \arrow[d, "\mathrm{loc}^a_{\circ^e_G}"'] & \mathbf{B}(\omega^\circ_G, K) \arrow[d, "\mathrm{loc}"] \\ \mathbf{T}_{\circ^e_G}\mathbf{B}^e(G_K) \arrow[r, "\kappa"] & \mathbf{F}(G_k) \end{tikzcd}$$

Thus for every $\mathcal{F}, \mathcal{G} \in \mathbf{F}(G)$ and $x, y \in \mathbf{B}^e(G_K)$,

$$\measuredangle_\circ(\mathcal{F},\mathcal{G}) = \measuredangle(\mathcal{F}_k, \mathcal{G}_k) \quad \text{and} \quad \measuredangle_\circ(x,y) = \measuredangle(\mathrm{loc}\circ\boldsymbol{\alpha}(x), \mathrm{loc}\circ\boldsymbol{\alpha}(y))$$

where we have abbreviated $\circ^e_G = \circ$. In particular,

$$\begin{aligned} \lim_{t\to 0} \tfrac{1}{t} d(\circ + t\mathcal{F}, \circ + t\mathcal{G}) &= d(\mathcal{F}_k, \mathcal{G}_k) \\ \lim_{t\to 0} \tfrac{1}{t}\left(d(x, \circ + t\mathcal{F}) - d(x,\circ)\right) &= \langle \mathrm{loc}\circ\boldsymbol{\alpha}(x), \mathcal{F}_k\rangle \end{aligned}$$

As for the vector valued distance $\mathbf{d} : \mathbf{B}^e(G_K) \times \mathbf{B}^e(G_K) \to \mathbf{C}(G_K)$, we have

$$\mathbf{d}(\circ, x) = t(\mathrm{loc}\circ\boldsymbol{\alpha}(x)) \quad \text{in} \quad \mathbf{C}(G_K) = \mathbf{C}(G_k).$$

6.4.14. For a parabolic subgroup P of G with unipotent radical U, the Gr_P-map of section 6.3.3 induces an analogous $P(K)$-equivariant map

$$\mathrm{Gr}_P : \mathbf{B}(\omega^\circ_G, K) \to \mathbf{B}(\omega^\circ_{P/U}, K).$$

For $\mathcal{F} \in \mathbf{F}(G)$, set $\mathrm{Gr}_\mathcal{F} = \mathrm{Gr}_{P_\mathcal{F}}$. For $\rho \in \mathsf{Rep}^\circ(G)(\mathcal{O}_K)$, $\gamma \in \mathbb{R}$ and $\alpha \in \mathbf{B}'(\omega^\circ_G, K)$, let $\mathrm{Gr}^\gamma_\mathcal{F}(\alpha,\rho)$ be the K-norm on $\mathrm{Gr}^\gamma_\mathcal{F}(\rho)_K$ induced by $\alpha(\rho)$ on $V_K(\rho)$, i.e.

$$\mathrm{Gr}^\gamma_\mathcal{F}(\alpha,\rho)(\overline{x}) = \inf\left\{\alpha(\rho)(x) : x \in \mathcal{F}^\gamma_K(\rho),\, x \equiv \overline{x} \bmod \mathcal{F}^\gamma_{+,K}(\rho)\right\}$$

for every $\overline{x}$ in $\mathrm{Gr}^\gamma_\mathcal{F}(\rho)_K = \mathcal{F}^\gamma_K(\rho)/\mathcal{F}^\gamma_{+,K}(\rho)$. By the axiom $L(s)$ for the $\mathbf{F}(V_K(\rho))$-building $\mathbf{B}(V_K(\rho))$ of splittable K-norms on $V_K(\rho)$, $\mathrm{Gr}^\gamma_\mathcal{F}(\alpha,\rho)$ is a splittable K-norm on $\mathrm{Gr}^\gamma_\mathcal{F}(\rho)_K$. Viewing $\mathrm{Gr}^\gamma_\mathcal{F}(\rho)$ as a representation of $P_\mathcal{F}/U_\mathcal{F}$, we have:

$$\forall \alpha \in \mathbf{B}(\omega^\circ_G, K) : \qquad \mathrm{Gr}_\mathcal{F}(\alpha)\left(\mathrm{Gr}^\gamma_\mathcal{F}(\rho)\right) = \mathrm{Gr}^\gamma_\mathcal{F}(\alpha,\rho).$$

Indeed, both sides only depend upon the $U_\mathcal{F}(K)$-orbit of α, and we may thus assume that α belongs to the image of $\mathbf{B}(\omega^\circ_L, K) \to \mathbf{B}(\omega^\circ_G, K)$ for some fixed Levi subgroup L of $P_\mathcal{F}$, i.e. $\alpha = \alpha_G + \mathcal{H}$ for some $\mathcal{H} \in \mathbf{F}(L)$. Then $r_{P_\mathcal{F},L}(\alpha) = \alpha$, thus $\mathrm{Gr}_\mathcal{F}(\alpha) = \alpha_{P/U} + \overline{\mathcal{H}}$ where $\overline{\mathcal{H}}$ is the image of $\mathcal{H}$ in $\mathbf{F}(P_\mathcal{F}/U_\mathcal{F})$. On the other hand, the chosen L gives a splitting $\mathcal{G} \in \mathbf{G}(Z(L))$ of $\mathcal{F}$, thus also a splitting $\rho|_L = \oplus_\gamma \rho_\gamma$ with $\rho_\gamma \in \mathsf{Rep}^\circ(L)(\mathcal{O})$, $V(\rho_\gamma) = \mathcal{G}_\gamma(\rho)$. Since α is the image of $\alpha_L + \mathcal{H}$ in $\mathbf{B}^e(\omega^\circ_G, K)$, this splitting is adapted to α: $\alpha(\rho) = \oplus\alpha_\gamma(\rho)$ where $\alpha_\gamma(\rho) = \alpha_L(\rho_\gamma) + \mathcal{H}(\rho_\gamma)$. Thus $\mathrm{Gr}^\gamma_\mathcal{F}(\alpha,\rho) \simeq \alpha_\gamma(\rho)$ under the K-linear isomorphism $V_K(\rho_\gamma) \simeq \mathrm{Gr}^\gamma_\mathcal{F}(\rho)_K$ induced by the $(L \to P_\mathcal{F}/U_\mathcal{F})$-equivariant isomorphism $\rho_\gamma \simeq \mathrm{Gr}^\gamma_\mathcal{F}(\rho)$. It follows that indeed

$$\mathrm{Gr}^\gamma_\mathcal{F}(\alpha,\rho) = \alpha_{P/U}(\mathrm{Gr}^\gamma_\mathcal{F}(\rho)) + \overline{\mathcal{H}}(\mathrm{Gr}^\gamma_\mathcal{F}(\rho)) = \mathrm{Gr}_\mathcal{F}(\alpha)(\mathrm{Gr}^\gamma_\mathcal{F}(\rho)).$$

For the distances attached to our chosen τ, we thus obtain

$$\lim_{t\to\infty} d_\tau(x + t\mathcal{F}, y + t\mathcal{F}) = d\left(\mathrm{Gr}^\bullet_\mathcal{F}(\boldsymbol{\alpha}(x),\tau), \mathrm{Gr}^\bullet_\mathcal{F}(\boldsymbol{\alpha}(y),\tau)\right)$$

for every $x, y \in \mathbf{B}^e(G_K)$, $\mathcal{F} \in \mathbf{F}(G)$.

6.4.15. Combining the previous two computations, we also obtain a formula for the Busemann scalar product on $\mathbf{B}^e(G_K)$. Recall from section 6.3.3 (and 5.6.9) that for any $x, y \in \mathbf{B}^e(G_K)$ and $\mathcal{F} \in \mathbf{F}(G)$, we have

$$\langle \overrightarrow{xy}, \mathcal{F} \rangle = \left\langle \overrightarrow{\mathrm{Gr}_{\mathcal{F}}(x)\mathrm{Gr}_{\mathcal{F}}(y)}, \overline{\mathcal{F}} \right\rangle = \left\langle \mathrm{loc}^a_{\mathrm{Gr}_{\mathcal{F}}(x)} \left(\mathrm{Gr}_{\mathcal{F}}(y)\right), \mathrm{loc}_{\mathrm{Gr}_{\mathcal{F}}(x)} \left(\overline{\mathcal{F}}\right) \right\rangle$$

where the second and third scalar product are respectively the Busemann scalar product on $\mathbf{B}^e(P_K/U_K)$ and the scalar product on its tangent space at $\mathrm{Gr}_{\mathcal{F}}(x)$, with $(P, U) = (P_{\mathcal{F}}, U_{\mathcal{F}})$. For $x = \circ^e_G$, $\mathrm{Gr}_{\mathcal{F}}(x) = \circ^e_{P/U}$ and we thus obtain

$$\langle \overrightarrow{xy}, \mathcal{F} \rangle = \left\langle \mathrm{loc}\left(\mathrm{Gr}_{\mathcal{F}}(\boldsymbol{\alpha}(y))\right), \overline{\mathcal{F}}_k \right\rangle$$

with the scalar product of $\mathbf{F}(P_k/U_k)$ attached to the faithful representation

$$\mathrm{Gr}^{\bullet}_{\mathcal{F}}(\tau) = \oplus_\gamma \mathrm{Gr}^{\gamma}_{\mathcal{F}}(\tau)$$

of P/U. Since $\overline{\mathcal{F}}(\mathrm{Gr}^\gamma_{\mathcal{F}}(\tau))$ is the $\mathbb{R}$-filtration with a single jump at γ,

$$\langle \overrightarrow{xy}, \mathcal{F} \rangle = \textstyle\sum_\gamma \gamma \cdot \deg\left(\mathrm{loc}\left(\mathrm{Gr}_{\mathcal{F}}(\boldsymbol{\alpha}(y))\right)\left(\mathrm{Gr}^\gamma_{\mathcal{F}}(\tau)\right)\right).$$

By definition of the morphism $\mathrm{loc} : \mathbf{B}(\omega^\circ_G, K) \to \mathbf{F}(G_k)$,

$$\mathrm{loc}\left(\mathrm{Gr}_{\mathcal{F}}(\boldsymbol{\alpha}(y))\right)\left(\mathrm{Gr}^\gamma_{\mathcal{F}}(\tau)\right) = \mathrm{loc}\left(\mathrm{Gr}^\gamma_{\mathcal{F}}(\tau)_K, \mathrm{Gr}^\gamma_{\mathcal{F}}(\boldsymbol{\alpha}(y), \tau), \mathrm{Gr}^\gamma_{\mathcal{F}}(\tau)\right).$$

The degree of this filtration is the degree of its determinant. Since the functors

$$\mathrm{loc} : \mathsf{Norm}'(K) \to \mathsf{Fil}(k) \quad \text{and} \quad \mathrm{Gr}^{\bullet}_{\mathcal{F}}\left(\boldsymbol{\alpha}(y)\right)' : \mathsf{Rep}^\circ(P/U)(\mathcal{O}_K) \to \mathsf{Norm}'(K)$$

are exact $\otimes$-functors, they both commute with the determinant. The degrees which occur in the last displayed formula for $\langle \overrightarrow{xy}, \mathcal{F} \rangle$ are therefore given by

$$\deg\left(\mathrm{loc}\left(\Lambda^\gamma_{\mathcal{F}}(\tau)_K, \Lambda^\gamma_{\mathcal{F}}(\boldsymbol{\alpha}(y), \tau), \Lambda^\gamma_{\mathcal{F}}(\tau)\right)\right)$$

where $\Lambda^\gamma_{\mathcal{F}}(\tau) = \det\left(\mathrm{Gr}^\gamma_{\mathcal{F}}(\tau)\right)$ is a rank one representation of P/U and

$$\Lambda^\gamma_{\mathcal{F}}(\boldsymbol{\alpha}(y), \tau) = \det\left(\mathrm{Gr}^\gamma_{\mathcal{F}}(\boldsymbol{\alpha}(y), \tau)\right) = \mathrm{Gr}^{\bullet}_{\mathcal{F}}\left(\boldsymbol{\alpha}(y)\right)\left(\Lambda^\gamma_{\mathcal{F}}(\tau)\right)$$

is a K-norm on $\Lambda^\gamma_{\mathcal{F}}(\tau)_K$. For a rank one object (V, α, L) in $\mathsf{Norm}'(K)$, the degree of $\mathrm{loc}(V, \alpha, L)$ is simply the largest $\gamma \in \mathbb{R}$ such that $L \subset \overline{B}(\alpha, \gamma)$. Equivalently,

$$\deg\left(\mathrm{loc}(V, \alpha, L)\right) = -\log\left(\sup\left\{\alpha(\ell) : \ell \in L\right\}\right) = -\log\left(\alpha(\ell_0)\right)$$

where $L = \mathcal{O}_K \cdot \ell_0$. Thus, still assuming that $x = \circ^e_G$, we finally obtain

$$\begin{aligned} \langle \overrightarrow{xy}, \mathcal{F} \rangle &= -\textstyle\sum_\gamma \gamma \cdot \log\left(\sup\left\{\Lambda^\gamma_{\mathcal{F}}\left(\boldsymbol{\alpha}(y), \tau\right) | \Lambda^\gamma_{\mathcal{F}}(\tau)\right\}\right) \\ &= -\textstyle\sum_\gamma \gamma \cdot \log\left(\Lambda^\gamma_{\mathcal{F}}\left(\boldsymbol{\alpha}(y), \tau\right)\left(e^\gamma_1 \wedge \cdots \wedge e^\gamma_{r_\gamma}\right)\right) \end{aligned}$$

where $(e^\gamma_1, \cdots, e^\gamma_{r_\gamma})$ is an $\mathcal{O}_K$-basis of $\mathrm{Gr}^\gamma_{\mathcal{F}}(\tau)$. For a general x in $\mathbf{B}^e(G_K)$, we find:

$$\langle \overrightarrow{xy}, \mathcal{F} \rangle = \textstyle\sum_\gamma \gamma \cdot \log\left(\dfrac{\Lambda^\gamma_{\mathcal{F}}\left(\boldsymbol{\alpha}(x), \tau\right)}{\Lambda^\gamma_{\mathcal{F}}\left(\boldsymbol{\alpha}(y), \tau\right)}\left(e^\gamma_1 \wedge \cdots \wedge e^\gamma_{r_\gamma}\right)\right).$$

Note that if we are given some $\mathcal{G} \in \mathbf{F}(P/U)$ with $\mathrm{Gr}_{\mathcal{F}}(y) = \mathrm{Gr}_{\mathcal{F}}(x) + \mathcal{G}$, then simply

$$\langle \overrightarrow{xy}, \mathcal{F} \rangle = \left\langle \mathcal{G}, \overline{\mathcal{F}} \right\rangle = \textstyle\sum_\gamma \gamma \cdot \deg\left(\mathcal{G}\left(\mathrm{Gr}^\gamma_{\mathcal{F}}(\tau)\right)\right).$$

6.4.16. For every $\nu > 0$, there is a $G(K)$-equivariant commutative diagram

$$\begin{array}{ccccc}\mathbf{B}(\omega_G^\circ, K, |-|) & \times & \mathbf{F}(G_K) & \xrightarrow{+} & \mathbf{B}(\omega_G^\circ, K, |-|) \\ a\downarrow & & b\downarrow & & a\downarrow \\ \mathbf{B}(\omega_G^\circ, K, |-|^\nu) & \times & \mathbf{F}(G_K) & \xrightarrow{+} & \mathbf{B}(\omega_G^\circ, K, |-|^\nu)\end{array}$$

where $a(\alpha) = \alpha^\nu$ and $b(\mathcal{F}) = \nu\mathcal{F}$. It is compatible with the analogous diagram of section 6.2.9 via the relevant $\boldsymbol{\alpha}$-maps.

6.4.17. For $x \in \mathbf{B}^e(G_K)$, the K-norm $\boldsymbol{\alpha}(x) \in \mathbf{B}(\omega_G^\circ, K)$ is exact and extends to a K-norm on ω_G' as in 3.6.4. Thus by proposition 49, it yields a K-norm $\boldsymbol{\alpha}(x)(\rho)$ on $V_K(\rho)$ for any representation ρ of G on a flat $\mathcal{O}_K$-module $V(\rho)$. We set

$$\boldsymbol{\alpha}_{\mathrm{ad}}(x) = \boldsymbol{\alpha}(x)(\rho_{\mathrm{ad}}), \quad \boldsymbol{\alpha}_{\mathrm{reg}}(x) = \boldsymbol{\alpha}(x)(\rho_{\mathrm{reg}}) \quad \text{and} \quad \boldsymbol{\alpha}_{\mathrm{adj}}(x) = \boldsymbol{\alpha}(x)(\rho_{\mathrm{adj}}).$$

PROPOSITION 135. *Suppose that $(K, |-|)$ is discrete, say $|K^\times| = q^{\mathbb{Z}}$ with $q > 1$. Let $(\mathfrak{g}_{x,r})_{r\in\mathbb{R}}$ be the Moy-Prasad filtration attached to x on $\mathfrak{g}_K = \mathrm{Lie}(G_K)$. Then*

$$\forall x \in \mathbb{R}: \qquad \mathfrak{g}_{x,r} = \left\{v \in \mathfrak{g}_K : \boldsymbol{\alpha}_{\mathrm{ad}}(x)(v) \le q^{-r}\right\}.$$

PROOF. Given the definition of $\mathfrak{g}_{x,r}$ (by étale descent from the quasi-split case) and proposition 127, we may assume that G splits over $\mathcal{O}_K$. Changing $|-|$ to $|-|^\nu$ with $\nu = \frac{1}{\log q}$, we may also assume that $q = e$. Fix $S \in \mathbf{S}(G)$ with x in $\mathbf{B}^e(S_K)$ and write $x = \circ_G^e + \mathcal{F}$ for some $\mathcal{F} \in \mathbf{F}(S_K)$, so that also $\boldsymbol{\alpha}(x) = \alpha_G + \mathcal{F}$. Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \oplus_{\beta\in\Phi(G,S)}\mathfrak{g}_\beta$ be the weight decomposition of $\mathfrak{g}$ and $\mathcal{F}^\sharp : M \to \mathbb{R}$ the morphism corresponding to $\mathcal{F}$, where $M = \mathrm{Hom}(S, \mathbb{G}_{m,\mathcal{O}_K})$. Then for every $r \in \mathbb{R}$,

$$\overline{B}\left(\boldsymbol{\alpha}_{\mathrm{ad}}(x), r\right) = \mathfrak{g}_{0,r} \oplus \oplus_{\beta\in\Phi(G,S)}\mathfrak{g}_{\beta,r}$$

where $\mathfrak{g}_{\beta,r} = \overline{B}\left(\alpha_{\mathfrak{g}_\beta}, r - \mathcal{F}^\sharp(\beta)\right)$ for $\beta \in \Phi(G,S) \cup \{0\}$. For $r = 0$, this is the Lie algebra $\mathfrak{g}_x$ of the group scheme $\mathfrak{G}_x$ over $\mathcal{O}_K$ attached to x in [**10**]. Comparing now this formula with the definition of $\mathfrak{g}_{x,r}$ in [**3**, 2.1.3] proves our claim. □

Let G_K^{an} be the analytic Berkovich space attached to G_K. In [**38**, 2.2], the authors construct a canonical map $\vartheta : \mathbf{B}^e(G_K) \to G_K^{\mathrm{an}}$, thus attaching to every $x \in \mathbf{B}^e(G_K)$ a multiplicative K-semi-norm $\vartheta(x)$ on $\mathcal{A}(G_K)$.

PROPOSITION 136. *For every $x \in \mathbf{B}^e(G_K)$, $\boldsymbol{\alpha}_{\mathrm{adj}}(x) = \vartheta(x)$. In particular, the K-norm $\boldsymbol{\alpha}_{\mathrm{adj}}(x)$ on $\mathcal{A}(G_K)$ is multiplicative and $\vartheta(x)$ is a norm.*

PROOF. Equip G_K^{an} with the action of $G(K)$ induced by ρ_{adj}. Then $x \mapsto \vartheta(x)$ is $G(K)$-equivariant and compatible with extensions $(K, |-|) \to (L, |-|)$ in the sense that $\vartheta(x) = \vartheta(x_L)|\mathcal{A}(G_K)$ for every $x \in \mathbf{B}^e(G_K)$ [**38**, Proposition 2.8]. The map $x \mapsto \boldsymbol{\alpha}_{\mathrm{adj}}(x)$ has the same properties. We may thus assume that G splits over $\mathcal{O}_K$, and again choosing L with $\log|L^\times| = \mathbb{R}$, we merely have to show that $\vartheta(\circ_G^e) = \boldsymbol{\alpha}_{\mathrm{adj}}(\circ_G^e) = \alpha_{\mathcal{A}(G)}$. By definition: $\{\vartheta(x)\}$ is the Shilov boundary of a K-affinoid subgroup G_x of G_K^{an}. For $x = \circ_G^e$, G_x is the affinoid group G^{an} attached to G, and its Shilov boundary is the gauge norm attached to $\mathcal{A}(G)$, i.e. $\alpha_{\mathcal{A}(G)}$. □

Since the multiplication on $\mathcal{A}(G)$ is a morphism $\rho_{\mathrm{reg}} \otimes \rho_{\mathrm{reg}} \to \rho_{\mathrm{reg}}$ in $\mathsf{Rep}'(G)(\mathcal{O}_K)$, the K-norm $\boldsymbol{\alpha}_{\mathrm{reg}}(x)$ on $\mathcal{A}(G_K)$ is sub-multiplicative. Since for $\tau \in \mathsf{Rep}^\circ(G)(\mathcal{O}_K)$, the co-module map $V(\tau) \to V(\tau) \otimes \mathcal{A}(G)$ is a pure monomorphism $\tau \hookrightarrow \tau_0 \otimes \rho_{\mathrm{reg}}$ in $\mathsf{Rep}'(G)(\mathcal{O}_K)$, $\boldsymbol{\alpha}(x)(\tau)$ is the restriction of $\alpha_{V(\tau_0)} \otimes \boldsymbol{\alpha}_{\mathrm{reg}}(x)$ to $V_K(\tau)$, thus $\boldsymbol{\alpha}_{\mathrm{reg}}(x)$ determines $\boldsymbol{\alpha}(x)$ and $\boldsymbol{\alpha}_{\mathrm{reg}}$ is a $G(K)$-equivariant embedding of $\mathbf{B}^e(G_K)$ into the space of sub-multiplicative K-norms on $\mathcal{A}(G)$ (equipped with the regular action).

6.4.18. Some final remarks:

(1) We have not given an intrinsic characterization of the subset $\mathbf{B}(\omega_G^\circ, K)$ of $\mathbf{B}'(\omega_G^\circ, K)$. We expect that $\mathbf{B}(\omega_G^\circ, K) = \mathbf{B}^?(\omega_G^\circ, K)$, or perhaps even that $\mathbf{B}(\omega_G^\circ, K)$ is equal to the $G(K)$-stable subset of exact norms in $\mathbf{B}'(\omega_G^\circ, K)$.

(2) Suppose that $\mathcal{O}$ is a valuation ring of height > 1 with fraction field K. Then $\Gamma = K^\times/\mathcal{O}^\times$ is a totally ordered commutative group which can not be embedded into $\mathbb{R}$. Let G be a reductive group over $\mathcal{O}$. Replacing $\mathbb{R}$ with Γ in the above constructions, it might be possible to define a "Bruhat-Tits" building $\mathbf{B}(\omega_G^\circ, K)$ with compatible actions of $G(K)$ and $\mathbf{F}^\Gamma(G_K)$, made of factorizations of the fiber functor $\omega_{G,K}^\circ : \mathsf{Rep}^\circ(G)(\mathcal{O}) \to \mathsf{Vect}(K)$ through a suitable category of "Γ-norms". The type maps should be the tautological morphisms $\nu : S(K) \to \mathbf{G}^\Gamma(S)$ mapping $s \in S(K)$ to the unique morphism $\nu(s) : \mathbb{D}_K(\Gamma) \to S$ whose composite with a character χ of S is the image of $\chi(s)$ in $\Gamma = K^\times/\mathcal{O}^\times$.

(3) There might also be a similar Tannakian formalism for the symmetric spaces of reductive groups over $\mathbb{R}$, with factorizations of fiber functors through a category of Euclidean spaces, using compact forms of the adjoint groups as base point.

Nomenclature

$\\|-\\|_\tau$	Length on $\mathbf{F}^\Gamma(G)$ or $\mathbf{C}^\Gamma(G)$ defined page 83.
$\measuredangle_\tau(-,-)$	Angle on $\mathbf{Std}^\Gamma(G)$ defined page 83.
$\measuredangle_\tau^{os}(-,-)$	Osculatory angle on $\mathbf{C}^\Gamma(G)$ defined page 84.
$\measuredangle_\tau^{tr}(-,-)$	Transverse angle on $\mathbf{C}^\Gamma(G)$ defined page 84.
$\measuredangle_x(\mathcal{F},\mathcal{G})$	Alexandrov angle at x between $x+t\mathcal{F}$ and $x+t\mathcal{G}$, page 109.
$\measuredangle_x^c(y,z)$	Angle at x in a comparison Euclidean triangle for (x,y,z).
$\measuredangle(\overrightarrow{xy},\mathcal{G})$	Busemann angle at x between y and $x+t\mathcal{G}$, page 109.
$\langle -,- \rangle_\tau$	Scalar product on $\mathbf{Std}^\Gamma(G)$ defined page 83.
$\langle -,- \rangle_\tau^{os}$	Osculatory scalar product on $\mathbf{C}^\Gamma(G)$ defined page 84.
$\langle -,- \rangle_\tau^{tr}$	Transverse scalar product on $\mathbf{C}^\Gamma(G)$ defined page 84.
$\leq$	Weak dominance partial order on $\mathbb{C}^\Gamma(G)$, page 20.
$\preceq$	Strong dominance partial order on $\mathbb{C}^\Gamma(G)$, page 20.
$\langle \overrightarrow{xy},\mathcal{G} \rangle$	Busemann scalar product between $[x,y]$ and $x+t\mathcal{G}$, defined page 111.
$1_G^\natural$	Counit $1_G^\natural : \mathcal{A}(G) \to \mathcal{O}_S$ of $\mathcal{A}(G)$.
1_G	Unit section $1_G : S \to G$ of a group scheme G over S.
1_S	Trivial representation of G on $\mathcal{O}_S$.
$\mathcal{A}(G)$	Hopf algebra of G.
$\mathbf{ACF}^\Gamma(G)$	Set of all triples $(S,B,\mathcal{F})$ with $S \in \mathbf{S}(G)$ and $\mathcal{F} \in F^{-1}(B) \subset \mathbf{F}^\Gamma(S)$, page 78.
ad	Morphism $\mathrm{ad} : G \to G^{\mathrm{ad}}$.
$\mathbf{AF}^\Gamma(G)$	Set of all pairs $(S,\mathcal{F})$ with $S \in \mathbf{S}(G)$ and $\mathcal{F} \in \mathbf{F}^\Gamma(S)$, page 78.
$\boldsymbol{\alpha}$	Functorial isomorphism $\mathbf{B}^e(G,L) \to \mathbf{B}(\omega_G^\circ, L)$ defined page 135.
$\alpha_{G,L}$	Canonical L-norm on $\omega_{G,L}^\circ$ defined page 133.
$\mathrm{Aut}^\otimes(\mathcal{F})$	Sheaf of tensor automorphisms of a fiber functor $\mathcal{X}$ preserving a Γ-filtration $\mathcal{F}$ on $\mathcal{X}$, page 38.
$\mathrm{Aut}^\otimes(\mathcal{G})$	Sheaf of tensor automorphisms of a fiber functor $\mathcal{X}$ preserving a Γ-graduation $\mathcal{G}$ on $\mathcal{X}$, page 38.
$\mathrm{Aut}^\otimes(\omega)$	Sheaf of tensor automorphisms of ω, page 38.
$\mathrm{Aut}^\otimes(\omega^\circ)$	Sheaf of tensor automorphisms of ω°, page 42.
$\mathrm{Aut}^\otimes(V)$	Sheaf of tensor automorphisms of V, page 38.
$\mathrm{Aut}^\otimes(V^\circ)$	Sheaf of tensor automorphisms of V°, page 42.
$B(\alpha,\gamma)$	Open ball of radius $\exp(-\gamma)$ for α, page 132.
$\overline{B}(\alpha,\gamma)$	Closed ball of radius $\exp(-\gamma)$ for α, page 132.
$\mathbf{B}(G)$	Set of all minimal parabolic subgroups of G, page 76.
$\mathbf{B}(\omega_G^\circ, L)$	Space of all good L-norms on $\omega_{G,L}^\circ$, defined page 133.
$\mathbf{B}'(\omega_G^\circ, L)$	Space of all L-norms on $\omega_{G,L}^\circ$, defined page 133.
$\mathbf{B}^?(\omega_G^\circ, L)$	Space of all nice L-norms on $\omega_{G,L}^\circ$, defined page 134.
$\mathbf{B}^e(G)$	Extended Bruhat-Tits building of G, page 126.

$\mathbf{B}^e(G, K)$	Pointed extended Bruhat-Tits building for G over $\mathcal{O}_K$, page 129.
$\mathbf{B}^r(G)$	Reduced Bruhat-Tits building of G, page 126.
$\mathcal{C}(\partial\mathbf{X}(G))$	Cone on the visual boundary of $\mathbf{X}(G)$, page 108.
$\mathbb{C}^\Gamma(G)$	Scheme of types of Γ-graduations of Γ-filtrations on G, page 16.
$\mathbf{C}^\Gamma(G)$	Cone of types of Γ-filtrations or Γ-graduations on G, page 78.
$\mathbb{C}^\Gamma(G)^c$	Central part of $\mathbb{C}^\Gamma(G)$, page 21.
$\mathbb{C}^\Gamma(G)^r$	Reduced part of $\mathbb{C}^\Gamma(G)$, page 21.
c_ρ	Comodule structure morphism $V(\rho) \to V(\rho) \otimes \mathcal{A}(G)$ of ρ.
can	Map $\mathbf{F}(G_L) \twoheadrightarrow \mathbf{B}(\omega_G^\circ, L)$, defined page 133.
C°	Category opposed to C.
$\mathbf{d}$	Vectorial distance on an affine $\mathbf{F}(G)$-building, defined page 94.
d_τ	Distance on $\mathbf{Std}^\Gamma(G)$ defined page 83.
$\mathbb{D}_S(M)$	Diagonalizable group scheme over S with character group M.
$\partial\mathbf{X}(G)$	Visual boundary of $\mathbf{X}(G)$, page 108.
$\mathbb{DYN}(G)$	Dynkin scheme of G, defined page 13.
e	2.71828182846...
F	Facet morphism on $\mathbb{G}^\Gamma(G)$, $\mathbb{F}^\Gamma(G)$) or $\mathbb{C}^\Gamma(G)$, page 16.
$\mathbb{F}^\Gamma(G)$	Scheme of Γ filtrations on G, page 16.
$\mathbf{F}^\Gamma(G)$	Set of all Γ-filtrations on G, page 78.
$\mathbb{F}^\Gamma(G)^c$	Central part of $\mathbb{F}^\Gamma(G)$, page 21.
$\mathbb{F}^\Gamma(G)^r$	Reduced part of $\mathbb{F}^\Gamma(G)$, page 21.
$\mathbb{F}^\Gamma(\omega)$	Sheaf of Γ-filtrations on ω, page 38.
$\mathbb{F}^\Gamma(\omega^\circ)$	Sheaf of Γ-filtrations on ω°, page 42.
$\mathbb{F}^\Gamma(V)$	Sheaf of Γ-filtrations on V, page 38.
$\mathbb{F}^\Gamma(V^\circ)$	Sheaf of Γ-filtrations on V°, page 42.
$\mathcal{F}_L^\iota$	Filtration opposed to $\mathcal{F}$ with respect to a Levi L of $P_\mathcal{F}$, page 95.
Fil	Functor $\mathrm{Fil} : \mathsf{Gr}^\Gamma\mathsf{LF} \to \mathsf{Fil}^\Gamma\mathsf{LF}$, page 39.
Fil	Functor $\mathrm{Fil} : \mathsf{Gr}^\Gamma\mathsf{QCoh} \to \mathsf{Fil}^\Gamma\mathsf{QCoh}$, page 35.
Fil	Morphism $\mathrm{Fil} : \mathbb{G}^\Gamma(G) \to \mathbb{F}^\Gamma(G)$ defined on page 16.
Fil	Morphism $\mathrm{Fil} : \mathbb{G}^\Gamma(X) \to \mathbb{F}^\Gamma(X)$ for $X = V$ or ω, page 38.
Fil	Morphism $\mathrm{Fil} : \mathbb{G}^\Gamma(X) \to \mathbb{F}^\Gamma(X)$ for $X = V^\circ$ or ω°, page 42.
$\mathrm{Fil}(\mathcal{G})$	Γ-filtration induced by a Γ-graduation $\mathcal{G}$, page 35.
$\mathsf{Fil}^\Gamma\mathsf{LF}$	Category of Γ-filtered finite locally free sheaves on schemes, page 39.
$\mathsf{Fil}^\Gamma\mathsf{LF}(X)$	Category of Γ-filtered finite locally free sheaves on X, page 39.
$\mathsf{Fil}^\Gamma\mathsf{QCoh}$	Category of Γ-filtered quasi-coherent sheaves on schemes, page 35.
$\mathsf{Fil}^\Gamma\mathsf{QCoh}(X)$	Category of Γ-filtered quasi-coherent sheaves on X, page 35.
$\overline{\mathcal{F}}$	Morphism $\mathbb{D}(\Gamma) \to \overline{R}(P_\mathcal{F})$ attached to a Γ-filtration $\mathcal{F}$, page 17.
G	$\mathsf{G} = G(\mathcal{O})$ for a group scheme G over a local ring $\mathcal{O}$.
G^{ab}	Abelianization $G^{\mathrm{ab}} = G/G^{\mathrm{der}}$ of G.
G^{ad}	Adjoint group $G^{\mathrm{ad}} = G/Z(G)$ of G.
G^{der}	Derived group of G.
$\mathbb{G}^\Gamma(G)$	Scheme of Γ-graduations on G, defined page 12.
$\mathbf{G}^\Gamma(G)$	Set of all Γ-graduations on G, page 78.
$\mathbb{G}^\Gamma(G)^c$	Central part of $\mathbb{G}^\Gamma(G)$, page 21.
$\mathbb{G}^\Gamma(G)^r$	Reduced part of $\mathbb{G}^\Gamma(G)$, page 21.
$\mathbb{G}^\Gamma(\omega)$	Sheaf of Γ-graduations on ω, page 38.
$\mathbb{G}^\Gamma(\omega^\circ)$	Sheaf of Γ-graduations on ω°, page 42.
$\mathbb{G}^\Gamma(V)$	Sheaf of Γ-graduations on V, page 38.

$\mathbb{G}^{\Gamma}(V^{\circ})$	Sheaf of Γ-graduations on V°, page 42.
G^{ss}	Semi-simplification $G^{\mathrm{ss}} = G/R(G)$ of G.
$\mathbb{G}_a$	Additive group over $\mathrm{Spec}(\mathbb{Z})$, $\mathbb{G}_a(R) = R$.
$\mathbb{G}_m$	Multiplicative group over $\mathrm{Spec}(\mathbb{Z})$, $\mathbb{G}_m(R) = R^{\times}$.
G_{Ω}	Pointwise stabilizer of Ω in G, page 105.
$\mathsf{G}_{S,\Omega}$	Apartment based avatar of G_{Ω}, page 105.
ΓR^*	Γ-subgroup spanned by the coroots, page 26.
$(\Gamma R^*)_{sat}$	Saturation of ΓR^*, page 26.
$\Gamma_+ R^*_+$	Γ_+-cone spanned by the positive coroots, page 26.
$(\Gamma_+ R^*_+)_{sat}$	Saturation of $\Gamma_+ R^*_+$, page 26.
$\Gamma(X, \mathcal{F})$	Sections of a sheaf $\mathcal{F}$ over X.
$\Gamma(X/S)$	Sections of a morphism $X \to S$.
Γ_+	$\Gamma_+ = \{\gamma \in \Gamma : \gamma \geq 0\}$.
$\mathbb{GEN}(G)$	Scheme of pairs of parabolic subgroups of G in generic relative position, page 22.
$\mathbb{GEN}^{\Gamma}(G)$	Scheme of pairs of Γ-filtrations on G in generic relative position, page 22.
Gr	Functor $\mathrm{Gr} : \mathsf{Fil}^{\Gamma}\mathsf{LF} \to \mathsf{Gr}^{\Gamma}\mathsf{LF}$, page 39.
Gr	Functor $\mathrm{Gr} : \mathsf{Fil}^{\Gamma}\mathsf{QCoh} \to \mathsf{Gr}^{\Gamma}\mathsf{QCoh}$, page 35.
$\mathrm{Gr}^{\bullet}$	Morphism $\mathrm{Gr}^{\bullet} : \mathrm{Aut}^{\otimes}(\mathcal{F}) \to \mathrm{Aut}^{\otimes}(\mathrm{Gr}^{\bullet}_{\mathcal{F}})$, page 38.
Grr	Growling sound indicative of frustration with useless generalities.
$\mathsf{Gr}^{\Gamma}\mathsf{LF}$	Category of Γ-graded finite locally free sheaves on schemes, page 39.
$\mathsf{Gr}^{\Gamma}\mathsf{LF}(X)$	Category of Γ-graded finite locally free sheaves on X, page 39.
$\mathsf{Gr}^{\Gamma}\mathsf{QCoh}$	Category of Γ-graded quasi-coherent sheaves on schemes, page 35.
$\mathsf{Gr}^{\Gamma}\mathsf{QCoh}(X)$	Category of Γ-graded quasi-coherent sheaves on X, page 35.
$\mathrm{Gr}_{\mathcal{F}}(\mathcal{M})$	Γ-graded quasi-coherent sheaf associated with a Γ-filtration $\mathcal{F}$ on a quasi-coherent sheaf $\mathcal{M}$, page 35.
$\mathrm{Gr}^{\gamma}_{\mathcal{F}}(\tau)$	Graded piece of the Γ-filtration $\mathcal{F}(\tau)$ on $V(\tau)$.
Gr_P	$P(K)$-equivariant map $\mathbf{B}^e(G, K) \twoheadrightarrow \mathbf{B}^e(P/U, K)$, page 130.
Gr_P	P-equivariant map $\mathbf{F}(G) \twoheadrightarrow \mathbf{F}(P/U)$, page 117.
$\mathrm{Gr}_P(\alpha)$	K-norm on $\omega^{\circ}_{P/U}$ induced by a K-norm α on ω°_G, page 138.
$\mathrm{Gr}_P(\mathcal{F})$	Γ-filtration on P/U induced by a Γ-filtration $\mathcal{F}$ on G in standard relative position with a parabolic subgroup P of G, page 25.
Gr^{∞}_P	Projection $\mathbf{X}(G) \twoheadrightarrow \mathbf{T}^{\infty}_P \mathbf{X}(G)$, page 114.
Group	Category of groups.
$S - \mathsf{Group}$	Category of group schemes over S.
H°	Neutral component of a group scheme H.
$\underline{\mathrm{Hom}}$	Sheafified version of Hom.
$\mathrm{Hom}^+(M, \Gamma)$	Dominant morphisms in $\mathrm{Hom}(M, \Gamma)$.
HV	Category of Henselian valued fields, page 129.
$\mathcal{I}(G)$	Augmentation ideal of $\mathcal{A}(G)$.
$\mathrm{Int}(g)$	Inner automorphism $h \mapsto ghg^{-1}$.
ι	Opposition involution of $\mathbb{G}^{\Gamma}(G)$ or $\mathbb{C}^{\Gamma}(G)$, defined page 17.
ι	Opposition involution of $\mathbb{OPP}(G)$ or $\mathbb{O}(G)$, defined page 14.
ι_S	Opposition involution on the apartment of S in $\mathbf{F}^{\Gamma}(G)$, page 78.
ι_S	Opposition involution on the apartment of S in $\mathbf{P}(G)$, page 77.
$K_0(G)$	Grothendieck ring of $\mathsf{Rep}^{\circ}(G)(S)$, page 55.
LF	Category of finite locally free sheaves on schemes, page 39.

$\mathsf{LF}(X)$	Category of finite locally free sheaves on X, page 39.
$\mathrm{Lie}(G)$	Lie algebra of G.
loc	Functor $\mathsf{Norm}'(K) \to \mathsf{Fil}(k)$, defined page 133.
loc	Map $\mathbf{B}^?(\omega_G^\circ, L) \to \mathbf{F}(G_{k_L})$, defined page 134.
loc_x	Projection $\mathbf{F}(G) \twoheadrightarrow \mathbf{T}_x\mathbf{X}(G)$, page 109.
loc_x^a	Localization $\mathbf{X}(G) \twoheadrightarrow \mathbf{T}_x\mathbf{X}(G)$, page 110.
M_d	Dominant weights.
$\mu_G^\natural$	Comultiplication $\mu_G^\natural : \mathcal{A}(G) \to \mathcal{A}(G) \otimes \mathcal{A}(G)$ of $\mathcal{A}(G)$.
$N_G(x)$	Normalizer of x in G.
$\mathsf{Norm}^\circ(K)$	Category of splittable normed finite K-vector spaces, page 132.
$\mathsf{Norm}'(K)$	Category of normed K-spaces with a lattice, defined page 133.
$\nu_{\mathbf{X}}$	Type morphism of an $\mathbf{F}(G)$-building $\mathbf{X}(G)$ defined page 92.
$\nu_{\mathbf{X},S}$	Morphism $\mathsf{Z}_G(S) \to \mathbf{G}(S)$ defined page 92.
$\circ$	Smallest element of $\mathbf{O}(G)$, page 78.
$\mathbb{O}(G)$	Scheme of types of parabolic subgroups of G, defined page 13.
$\mathbf{O}(G)$	Set of all types of parabolic subgroups of G, page 76.
$\circ_G^e$	Canonical point of $\mathbf{B}^e(G_K)$ attached to G over $\mathcal{O}_K$, page 129.
$\circ_G^r$	Canonical point of $\mathbf{B}^r(G_K)$ attached to G over $\mathcal{O}_K$, page 129.
ω_X	Fiber functor $\omega_X : \mathsf{Rep}(G)(S) \to \mathsf{QCoh}(X)$, page 36.
ω_X°	Fiber functor $\omega_X^\circ : \mathsf{Rep}^\circ(G)(S) \to \mathsf{LF}(X)$, page 41.
$\mathbb{OPP}(G)$	Scheme of pairs of opposed parabolic subgroups of G, page 14.
$\mathbf{OPP}(G)$	Set of all pairs of opposed parabolic subgroups of G, page 76.
os	Osculatory section $os : \mathbb{O}(G) \to \mathbb{TSTD}(G)$, defined page 22.
$\mathbb{P}(G)$	Scheme of parabolic subgroups of G, defined page 13.
$\mathbf{P}(G)$	Set of all parabolic subgroups of G, page 76.
$P_{\mathcal{F}}$	Parabolic subgroup of G fixing $\mathcal{F}$, page 17.
P_u	Universal parabolic subgroup of $G_{\mathbb{P}(G)}$.
$\Phi(S, G)$	Roots of S in $\mathrm{Lie}(G)$.
π	3.14159265359...
QCoh	Category of quasi-coherent sheaves on schemes, page 35.
$\mathsf{QCoh}(X)$	Category of quasi-coherent sheaves on X, page 35.
$R(G)$	Radical of G.
$R^u(P)$	Unipotent radical of P.
$R_{\mathbb{O}(G)}$	A torus over $\mathbb{O}(G)$ defined on page 13.
$R_{\mathbb{OPP}(G)}$	Radical of the universal Levi subgroup of $G_{\mathbb{OPP}(G)}$.
$R_{\mathbb{P}(G)}$	Radical $\overline{R}(P_u)$ of P_u/U_u, a torus over $\mathbb{P}(G)$.
$r_{P,L}$	Retraction $r_{P,L} : \mathbf{F}^\Gamma(G) \twoheadrightarrow \mathbf{F}^\Gamma(L)$, defined page 86.
$r_{P,L}$	Retraction $r_{P,L} : \mathbf{X}(G) \twoheadrightarrow \mathbf{X}(L)$, defined page 96.
$\mathsf{Rep}(G)$	Fibered category of algebraic representations of G on quasi-coherent sheaves, page 36.
$\mathsf{Rep}(G)(X)$	Category of algebraic representations of G on quasi-coherent sheaves over X, page 36.
$\mathsf{Rep}^\circ(G)$	Fibered category of algebraic representations of G on finite locally free sheaves, page 41.
$\mathsf{Rep}^\circ(G)(X)$	Category of algebraic representations of G on finite locally free sheaves over X, page 41.
$\mathsf{Rep}'(G)(S)$	Full sub-category of $\mathsf{Rep}(G)(S)$ defined on page 47.
ρ^n	Adjoint representation of G on $\mathcal{I}(G)/\mathcal{I}(G)^{n+1}$, page 52.

$\rho^\vee$	Dual of ρ.
ρ_0	Trivial representation of G on $V(\rho)$.
ρ_{ad}	Adjoint representation of G on $\mathrm{Lie}(G)$.
ρ_{adj}	Adjoint representation of G on $\mathcal{A}(G)$, page 52.
$\rho^\circ_{\mathrm{adj}}$	Adjoint representation of G on $\mathcal{I}(G)$, page 52.
ρ_n	Dual of ρ^n, page 52.
ρ_{reg}	Regular representation of G on $\mathcal{A}(G)$, page 45.
Ring	Category of commutative rings.
$\overline{R}(P)$	Radical of P/U, where U is the unipotent radical of P.
$\mathbf{RX}(G)$	Rays in $\mathbf{X}(G)$, page 108.
$\mathbf{S}(G)$	Set of all maximal split tori of G, page 76.
$\mathbf{SBP}(G)$	Set of triples (S, B, P) in $\mathbf{S}(G) \times \mathbf{B}(G) \times \mathbf{P}(G)$ with $Z_G(S) \subset B \subset P$, page 76.
Sch	Category of schemes.
Sch/S	Category of schemes over S.
Set	Category of sets.
$\sim_{\mathrm{Par}}$	Par-equivalence on $\mathbb{G}^\Gamma(G)$, defined page 16.
$\mathbf{SP}(G)$	Set of all pairs (S, P) in $\mathbf{S}(G) \times \mathbf{P}(G)$ with $Z_G(S) \subset P$, page 76.
$\mathbb{STD}(G)$	Scheme of pairs of parabolic subgroups of G in standard relative position, defined page 22.
$\mathbf{Std}(G)$	Set of all pairs of parabolic subgroups of G in standard relative position, page 80.
$\mathbb{STD}(Z)$	Pull-back of $\mathbb{STD}(G) \hookrightarrow \mathbb{P}(G)^2$ through $Z \to \mathbb{P}(G)^2$.
$\mathbb{STD}^\Gamma(G)$	Scheme of pairs of Γ-filtrations on G in standard relative position, defined page 22.
$\mathbf{Std}^\Gamma(G)$	Set of all pairs of Γ-filtrations on G in standard relative position, page 80.
t	Type morphism $t : \mathbb{P}(G) \to \mathbb{O}(G)$, defined page 13.
t	Type morphism $t : \mathbb{F}^\Gamma(G) \to \mathbb{C}^\Gamma(G)$, defined page 16.
t_2	Type morphism $t_2 : \mathbb{STD}(G) \to \mathbb{TSTD}(G)$, defined page 22.
t_2	Type morphism $t_2 : \mathbb{STD}^\Gamma(G) \to \mathbb{TSTD}^\Gamma(G)$, defined page 22.
$\mathbf{T}^\infty_P\mathbf{X}(G)$	Quotient of $\mathbf{X}(G)$ by the unipotent radical U of P, page 112.
$\mathbf{T}_x\mathbf{X}(G)$	Tangent space at x in $\mathbf{X}(G)$, defined page 109.
tr	Transverse section $tr : \mathbb{O}(G) \to \mathbb{TSTD}(G)$, defined page 22.
$\mathbb{TSDT}(G)$	Scheme of types of pairs of parabolic subgroups of G in standard relative position, defined page 22.
$\mathbb{TSDT}^\Gamma(G)$	Scheme of types of pairs of Γ-filtrations on G in standard relative position, defined page 22.
U_a	Root subgroup of G for $a \in \Phi(G, S)$, page 104.
$U_{\mathcal{F}}$	Unipotent radical of $P_{\mathcal{F}}$.
U_u	Unipotent radical of P_u.
V	Fiber functor $V : \mathsf{Rep}(G) \to \mathsf{QCoh}/S$, page 36.
V°	Fiber functor $V^\circ : \mathsf{Rep}^\circ(G) \to \mathsf{LF}/S$, page 41.
$W_G(S)$	Weyl group of S in G, $W_G(S) = N_G(S)/Z_G(S)$.
$X(\rho)$	Filtered set of subrepresentations of ρ on finite locally free subsheaves of $V(\rho)$, page 47.
X_T	Pull-back or base change of some X over S through $T \to S$.
$Z(G)$	Center of G.

$Z_G(x)$	Centralizer of x in G.

Bibliography

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