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ON THE LEXICOGRAPHIC DEGREE OF TWO-BRIDGE KNOTS

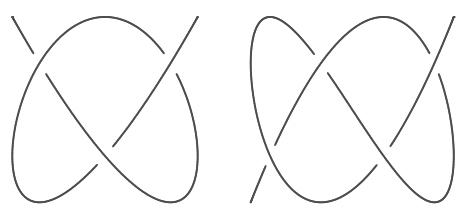
ERWAN BRUGALLÉ, PIERRE-VINCENT KOSELEFF, AND DANIEL PECKER

Abstract. We study the degree of polynomial representations of knots. We obtain the lexicographic degree for two-bridge torus knots and generalized twist knots. The proof uses the braid theoretical method developed by Orevkov to study real plane curves, combined with previous results from [KP10] and [BKP14]. We also give a sharp lower bound for the lexicographic degree of any knot, using real polynomial curves properties.

1. Introduction

It is known that every knot in $\mathbf{S}^3$ can be represented as the closure of the image of a polynomial embedding $\mathbb{R} \to \mathbb{R}^3 \subset \mathbf{S}_3$, see [Sh92, Va90]. In these early papers on the subject, only a few specific examples were given. The two-bridge torus knots form the first infinite family of knots for which a polynomial representation was explicitly given, see [RS96, KP08, KP11, KP10].

Example 1.1 (Harmonic knots ([KP11, KP12]))**.** The harmonic knot $\mathrm{H}(a, b, c)$ is the polynomial knot parametrized by $(T_a(t), T_b(t), T_c(t))$, where T_n is the classical Chebyshev polynomial $T_n(\cos t) = \cos nt$ and a, b, c are pairwise coprime integers. The two knot diagrams depicted in Figure 1 are the *xy-diagrams* of the knots



$3_1 = \mathrm{H}(3, 4, 5)$ $\quad$ $4_1 = \mathrm{H}(3, 5, 7)$

Figure 1. Two harmonic knots

$\mathrm{H}(3, 4, 5)$ and $\mathrm{H}(3, 5, 7)$, corresponding to the projection $(x, y, z) \mapsto (x, y)$.

The *multidegree* of a polynomial map $\gamma : \mathbb{R} \to \mathbb{R}^n, t \mapsto (P_i(t))$ is the n-tuple $(\deg(P_i))$. The *lexicographic degree* of a knot K is the minimal multidegree, for the lexicographic order, of a polynomial knot whose closure in $\mathbf{S}^3$ is isotopic to K. The unknot has lexicographic degree $(-\infty, -\infty, 1)$, and one sees easily that the lexicographic degree of any other knot is (a, b, c) with $3 \leq a < b < c$. Given a knot, it is in general a difficult problem to determine its lexicographic degree.

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In particular, the corresponding diagram might not have the minimal number of crossings.

In this paper we investigate the case of *two-bridge* knots, that are the knots for which $a = 3$, see [Cr04, KP10, BKP14]. We focus in particular on two-bridge torus knots and generalized twist knots, that we respectively denote by $C(m)$ with m odd, and $C(m, n)$ with mn positive and even (see Figure 2). Note that $C(-m)$ and $C(-m, -n)$ are the mirror images of $C(m)$ and $C(m, n)$.

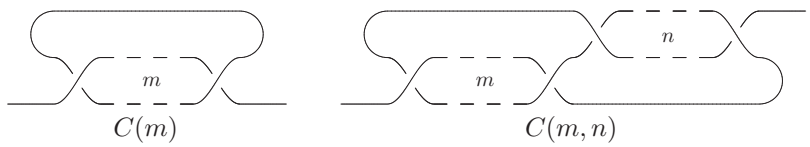


FIGURE 2. Two-bridge torus knots and generalized twist knots

Figure 3 shows typical examples, $C(5)$ and $C(4, 3)$.

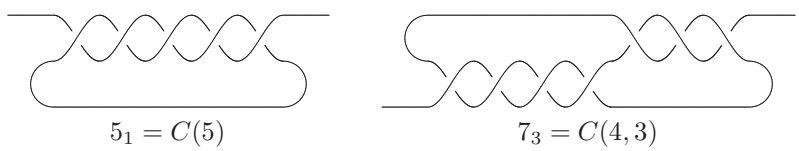


FIGURE 3. The knots 5_1 and 7_3

Mishra claimed in [Mi00] that the lexicographic degree of the torus knot $C(m)$, $m \not\equiv 2 \pmod 3$, is $(3, m + 1, c)$. Later on, it was proved in [KP08] that this is false for every integer $m > 5$. At that stage, the lexicographic degree of only a few knots was known, namely the trefoil, the figure-eight knot, the torus knot 5_1 [Mi00], and the torus knot 7_1 [KP08].

For every two-bridge knot of crossing number N, a polynomial representation of degree $(3, b, c)$, with $b + c = 3N$ and $b \geq N + 1$ is constructed in [KP10]. From that construction an upper bound for the lexicographic degree of two-bridge knots is immediately deduced. The main result of this paper is that this upper bound is sharp for two-bridge torus knots and for generalized twist knots.

Theorem 3.9. *Let K be the two-bridge knot $C(m)$, or $C(m, n)$ with $mn > 0$, and let N be its crossing number. Then the lexicographic degree of K is*

$$\left(3, \left[\tfrac{3N-1}{2}\right], \left[\tfrac{3N}{2}\right] + 1\right).$$

The proof goes by the study of the plane curve defined by the xy-projection of a polynomial knot, and uses Orevkov's braid theoretical approach to study real plane curves (see [Ore99] or Section 2). Like in the case of knots, we define the lexicographic degree of a long knot diagram as the minimal multidegree, for the lexicographic order, of a polynomial knot whose xy-diagram is isotopic in $\mathbb{R}^2$ to D.

The main idea in the proof of Theorem 3.9 is as follows. Suppose that D_1 is a trigonal long knot diagram with lexicographic degree $(3, b, c)$. If D_2 is a trigonal diagram obtained from D_1 by an isotopy through trigonal diagrams which never increases the number of crossings, then one may expect that it has lexicographic degree $(3, b', c')$ with $b' \leq b$. If this was the case, then Theorem 3.9 would follow from the study of the alternating diagrams $C(m)$ or $C(m, n)$, combined with the following theorem.

Theorem A. ([BKP14]) *Let K be the two-bridge knot $C(m)$, or $C(m, n)$ with $mn > 0$, and let D be a trigonal diagram of K. Then it is possible to transform D into an alternating trigonal diagram, so that all intermediate diagrams remain trigonal, and the number of crossings never increases.*

Unfortunately, one cannot ensure that the lexicographic degree of a trigonal knot diagram does not increase during an isotopy, even if this isotopy never increases the number of crossings. For this to be true, one needs to enlarge the set of objects we are studying, namely we have to consider *real pseudoholomorphic curves* (see Corollary 3.7). Note that a real algebraic curve is real pseudoholomorphic, however the converse is not true in general.

We conclude the paper by providing a lower bound for the lexicographic degree of every non-trivial knot. We also give the lexicographic degree of some infinite families of knots, showing in particular that this lower bound is sharp.

Theorem 4.3. *Let K be a knot of crossing number $N > 0$. Then the lexicographic degree of K is at least $(3, N+1, 2N-1)$. For every integer $N \not\equiv 2 \pmod 3, N \geq 3$, there exists a knot of crossing number N and lexicographic degree $(3, N+1, 2N-1)$.*

The paper is organized as follows. We recall Orevkov's braid theoretical method in Section 2. To avoid technical difficulties coming from an excess of generality, we restrict ourselves to the case of real curves of bidegree $(3, b)$, whereas the whole section generalizes to all bidegrees. We prove Theorem 3.9 in Section 3. To keep the exposition as light as possible we keep working with real algebraic curves as long as pseudoholomorphic curves are not needed. In particular we prove the lower bound of Theorem 3.9 in the particular case of the diagrams $C(m)$ and $C(m, n)$ for real algebraic curves (Proposition 3.1). We introduce real pseudoholomorphic curves only in Section 3.2 to deduce the general lower bound of Theorem 3.9 from Proposition 3.1 (Proposition 3.8). In Section 4, we first obtain a sharp lower bound for alternating diagrams. The proof is elementary, it is based on polynomial plane curves properties. As a consequence, we deduce Theorem 4.3 and an upper bound for the crossing number of alternating knots of polynomial degree d.

2. The braid of a real algebraic curve in $\mathbb{C}^2$

Here we recall basic facts about the braid theoretical approach developed by Orevkov to study real algebraic curves in $\mathbb{C}^2$.

2.1. **Link associated to a real algebraic curve.** Given $C \subset \mathbb{C}^2$ a real algebraic curve, we denote by $\mathbb{R}C$ the *real part* of C, i.e. $\mathbb{R}C = C \cap \mathbb{R}^2$. Let us fix an orientation preserving diffeomorphism $\Phi : \mathbb{R}^4 \to \{(x, y) \in \mathbb{C}^2 \mid \operatorname{Im}(x) > 0\}$, and let us denote by B_r the image by Φ of the 4-ball of radius r, and by $S_r = \partial B_r$ the

image by Φ of the 3-sphere of radius r. If $C \subset \mathbb{C}^2$ is a real algebraic curve, then all links $S_r \cap C$ (resp. all surfaces $B_r \cap C$) are isotopic if r is large enough.

Definition 2.1. *The link $L_C = S_r \cap C$, for r large enough, is called the link associated to the real algebraic curve C.*

We denote $U_C = B_r \cap C$, and we orient L_C as the boundary of U_C. We will use the following standard proposition in the proof of Theorem 3.9. The linking number of two oriented links L_1 and L_2 is denoted by $\mathrm{lk}\,(L_1, L_2)$; the algebraic intersection number of two smooth surfaces U_1 and U_2 in B_r intersecting transversally and in $B_r \setminus S_r$ is denoted by $U_1 \cdot U_2$.

Proposition 2.2. *If L_1 and L_2 are two sublinks of L_C which bound two components U_1 and U_2 of N_C, then*

$$\mathrm{lk}\,(L_1, L_2) = U_1 \cdot U_2.$$

It turns out that in some situations, one can read directly on the real part $\mathbb{R}C$ an expression of the oriented link L_C as a closed braid. For the sake of simplicity, we restrict ourselves here to the very special case of trigonal rational curves. For the general case, we refer to [Ore99]. In what follows, we are interested in curves in $\mathbb{C}^2$ up to isotopies respecting the fibration of $\mathbb{C}^2$ by vertical lines.

Definition 2.3. *An isotopy on a curve $C \subset \mathbb{C}^2$ is called a $\mathcal{L}$-*isotopy *if it commutes with the projection $\pi : \mathbb{C}^2 \to \mathbb{C}$, $(x, y) \mapsto x$.*

Let us consider a polynomial map

$$\begin{array}{rccc} \gamma : & \mathbb{C} & \longrightarrow & \mathbb{C}^2 \\ & t & \longmapsto & (P(t), Q(t)) \end{array}$$

where $P(t)$ and $Q(t)$ are two real polynomials of degrees 3 and $b \geq 1$. Replacing $P(t)$ with $-P(t)$ if necessary, we may assume that $P(t)$ is positive for t large enough. We may also assume that $b = 3k - 1$ or $b = 3k - 2$. If $b = 3k$, then we can replace the map γ by the map $t \to (P(t), Q(t) - \alpha P^k(t))$. In the target space $\mathbb{C}^2$, this corresponds to performing the real change of coordinates $y = y - \alpha x^k$ and does not affect the topology of the real algebraic curve $C = \gamma(\mathbb{C})$ in $\mathbb{C}^2$.

We suppose in addition that the map γ is generic enough so that:

— γ is an immersion;
— if $\gamma(t_1) = \gamma(t_2)$ for $t_1 \neq t_2$, then $\gamma'(t_1) \neq \gamma'(t_2)$ and $\gamma^{-1}(\gamma(t_1)) = \{t_1, t_2\}$.

In other words, the only singularities of the embedded algebraic curve $C \subset \mathbb{C}^2$ are nodes. Since C is real, it has three kinds of nodes:

(1) the intersection of two real branches of C (i.e. $t_1, t_2 \in \mathbb{R}$); such a real node is called a crossing, see Figure 4a;
(2) the intersection of two complex conjugated branches of C (i.e. $t_2 = \overline{t_1}$); such a real node is called solitary node, see Figure 4b;
(3) the intersection of two branches of C which are neither real nor complex conjugated; all such nodes lie in $\mathbb{C}^2 \setminus \mathbb{R}^2$ and come in pairs of complex conjugated nodes.

We denote by N the number of crossings of C, by α the number of solitary nodes of C and by 2β its number of nodes lying in $\mathbb{C}^2 \setminus \mathbb{R}^2$. Hence the real part $\mathbb{R}C$ is the union of $\gamma(\mathbb{R})$ together with the α solitary nodes of C.

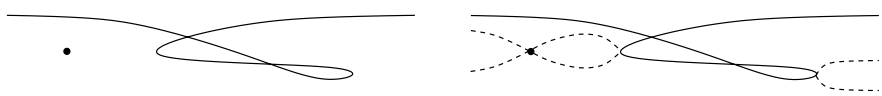
Lemma 2.4. *Given a map $\gamma : \mathbb{C} \to \mathbb{C}^2$ as above, we have $b - 1 = N + \alpha + 2\beta$.*

a) A crossing b) A solitary node

FIGURE 4.

Proof. The Newton polygon T of the curve $\gamma(C)$ is the triangle with vertices $(0,0),(0,3)$ and $(b,0)$, and the number of nodes of $\gamma(C)$ is the number of points in $\mathbb{Z}^2$ contained in the interior of T. □

Example 2.5. Let $1/\sqrt{3} < \nu < 1$. Using elementary elimination theory, one sees that the real part of the parametrized curve $\gamma : t \mapsto (T_3(t), T_4(t+\nu))$ is depicted, up to $\mathcal{L}$-isotopy, in Figure 5a. It has two crossings and one solitary node.



a) $\mathbb{R}C$ b) A

FIGURE 5.

Now let us consider the set $A = \pi^{-1}(\mathbb{R}) \cap C$. It consists in the real part $\mathbb{R}C$ together with some pairs of complex conjugated arcs, meeting at critical points of the map $\pi_{|\mathbb{R}C}$ (i.e. at points of $\mathbb{R}C$ which intersect a vertical line non-transversely).

Example 2.6. The set A in the case of the parametrization of Example 2.5 is depicted in Figure 5b. The dotted curves correspond to complex conjugate arcs.

The set A is not a manifold because of critical points of the map $\pi_{|\mathbb{R}C}$. However there is a canonical way to perturb A to a 1-dimensional manifold and to recover the link L_C. Let R_1, R_2 and ε be some positive real numbers, and let

$$V_1 = \pi^{-1}\left(\{x \in \mathbb{C} \mid |x| \leq R_1 \text{ and } \mathrm{Im}\,(x) > \varepsilon\}\right) \quad \text{and} \quad V_2 = \{y \in \mathbb{C} \mid |y| \leq R_2\}.$$

Then for R_1 and R_2 large enough, and ε small enough, the link $C \cap \partial\,(V_1 \times V_2)$ in $\partial\,(V_1 \times V_2) \simeq S^3$ is precisely the link L_C, and the surface $C \cap (V_1 \times V_2)$ is precisely the surface U_C in $V_1 \times V_2 \simeq B^4$. Moreover the link $C \cap \partial\,(V_1 \times V_2)$ appears naturally as the closure of a braid b_C, that we describe in the next section.

2.2. **The link L_C as a closed braid.** Recall that the *group of braids with* 3*-strings* is defined as

$$B_3 = \langle \sigma_1, \sigma_2 | \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle.$$

This terminology comes from the fact that there exists a natural geometric interpretation of σ_1, σ_2, and of the relation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$ as depicted in Figure 6. Note that a string of a braid is implicitly oriented from left to right. In particular if an oriented link L is represented as the closure of a braid b, then the linking number of two sublinks L_1 and L_2 of L is half the sum of all exponents of b corresponding to the crossings of L_1 and L_2.

a) σ_1 b) σ_2 c) $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$

FIGURE 6.

Let us associate an $\mathcal{L}$-*scheme* to the real curve C. Denote by $l_1 < l_2 < \ldots < l_s$ the real vertical lines of $\mathbb{C}^2$ which are not transversal to C (the order on the l_i's is the one induced by the canonical orientation of $\mathbb{R}$). Recall that the degree of the polynomial $Q(t)$ is $b = 3k-1$ or $b = 3k-2$.

Definition 2.7. *The $\mathcal{L}$-scheme $\mathcal{L}_C$ realized by C is given by the sequence $c_1c_2\ldots c_{s+1}$ where c_i with $1 \leq i \leq s$ is given by*

- *$c_i = \supset_j$ (resp. $c_i = \subset_j$) if $1 \leq i \leq s$ and l_i has an ordinary tangency point with C which is a local maximum (resp. minimum) of the map $\pi|_{\mathbb{R}C}$;*
- *$c_i = \times_j$ (resp. $c_i = \bullet_j$) if $i \leq s$ and l_i passes through a crossing (resp. solitary node) of $\mathbb{R}C$;*

(in each case $j = 1$ if the transverse intersection point of $\mathbb{R}l_i$ and $\mathbb{R}C$ is below the second point of $\mathbb{R}l_i \cap \mathbb{R}C$, and $j = 2$ otherwise)

and c_{s+1} is given by

- *$c_{s+1} = \downarrow$ (resp. $\uparrow$) if $b = 3k-1$, and $Q(t)$ is positive (resp. negative) for t large enough;*
- *$c_{s+1} = \vee$ (resp. $\wedge$) if $b = 3k-2$, and $Q(t)$ is positive (resp. negative) for t large enough.*

Example 2.8. The $\mathcal{L}$-scheme realized by the parametrization of Example 2.5 is

$$\bullet_1 \subset_1 \times_2 \times_1 \supset_1 \vee.$$

Then replace the $\mathcal{L}$-scheme $c_1c_2\ldots c_{s+1}$ by $c'_0c_1c_2\ldots c'_{s+1}$ where

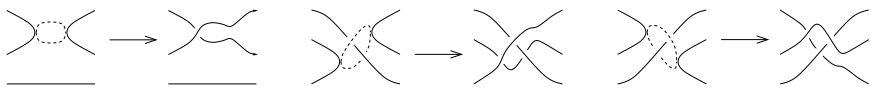
- $c'_0 = \supset_2$ (resp. $\supset_1$) and $c'_{s+1} = \subset_1$ if $c_{s+1} = \downarrow$ and k is even (resp. odd);
- $c'_0 = \supset_1$ (resp. $\supset_2$) and $c'_{s+1} = \subset_2$ if $c_{s+1} = \uparrow$ and k is even (resp. odd);
- $c'_0 = \supset_2$ (resp. $\supset_1$) and $c'_{s+1} = \subset_1\supset_1\subset_1$ if $c_{s+1} = \vee$ and k is even (resp. odd);
- $c'_0 = \supset_1$ (resp. $\supset_2$) and $c'_{s+1} = \subset_2\supset_2\subset_2$ if $c_{s+1} = \wedge$ and k is even (resp. odd).

Finally replace each $\bullet_j$ with $\subset_j\supset_j$ and each $\times_j$ with $\supset_j\subset_j$ in $c'_0c_1c_2\ldots c'_{s+1}$, and do the following final substitutions:

- replace each $\supset_j\subset_j$ by σ_j^{-1} (see Figure 7a);
- replace each $\supset_1\subset_2$ by $\sigma_1^{-1}\sigma_2^{-1}\sigma_1$ (see Figure 7b);
- replace each $\supset_2\subset_1$ by $\sigma_2^{-1}\sigma_1^{-1}\sigma_2$ (see Figure 7c).

Then we obtain a braid $b_{\mathbb{R}}$. We define the braid associated to the $\mathcal{L}$-scheme $\mathcal{L}_C$, denoted b_C, as the braid $b_{\mathbb{R}}(\sigma_1\sigma_2\sigma_1)^k$.

Example 2.9. After the two replacements described above, we obtain in the case of Example 2.5



a) $\supset_j \subset_j \to \sigma_j^{-1}$ b) $\supset_1 \subset_2 \to \sigma_1^{-1}\sigma_2^{-1}\sigma_1$ c) $\supset_2 \subset_1 \to \sigma_2^{-1}\sigma_1^{-1}\sigma_2$

FIGURE 7.

$$\overbrace{\supset_2}^{\sigma_2^{-1}}\ \overbrace{\subset_1 \bullet_1 \supset_1}^{\sigma_1^{-1}\sigma_2\sigma_1^{-1}}\ \overbrace{\subset_1}^{\sigma_2^{-1}}\ \overbrace{\supset_2 \times_2 \subset_2}^{\sigma_2^{-1}}\ \overbrace{\supset_1 \times_1 \subset_1}^{\sigma_1^{-1}}\ \overbrace{\supset_1}^{\sigma_1^{-2}}\ \overbrace{\subset_1 \vee \supset_1 \subset_1}^{\sigma_1^{-2}}\ (\sigma_1\sigma_2\sigma_1)^2$$

which finally gives the braid

$$b_C = \sigma_2^{-1}\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-3}(\sigma_1\sigma_2\sigma_1)^2$$

which is the trivial braid, see Figure 8.

The braid b_C of the parametrization given in Example 2.5

FIGURE 8.

Proposition 2.10. *The closure of the braid b_C is the oriented link L_C associated to C.*

Proof. The only part which is not already contained in [Ore99, Section 3] is the determination of the braid $C \cap \pi^{-1}(\gamma)$ where γ is the arc in $\mathbb{C}$ parametrized by $u \in [0;\pi] \to \varepsilon + R_1 \exp(2i\pi u)$ (recall that ε and R_1 have been introduced at the end of Section 2.1). But this braid is dictated by the monomials of an equation of C in $\mathbb{C}^2$ which are dominating at infinity. Up to a real multiplicative constant, these monomials are precisely

— $y^3 - \lambda x^{3k-1}$ if $c_{s+1} = \downarrow$,
— $y^3 + \lambda x^{3k-1}$ if $c_{s+1} = \uparrow$,
— $y^3 - \lambda x^{3k-2}$ if $c_{s+1} = \vee$,
— $y^3 + \lambda x^{3k-2}$ if $c_{s+1} = \wedge$,

where λ is some positive real number. Now the result follows from straightforward local computations as in [Ore99, Section 3]. □

3. PROOF OF THEOREM 3.9

Here we apply the method described in the previous section to the study of the lexicographic degree of two-bridge knots. Given a long knot diagram D in $\mathbb{R}^2$, we denote by $\overline{D}$ its projection to $\mathbb{R}^2$ (i.e. we forget about the sign of the crossings).

3.1. **The case of alternating diagrams.** We first consider the special case of alternating diagrams, from which we will deduce the general case in Section 3.2.

Proposition 3.1. *Let D be the alternating trigonal diagram $C(m)$ with m odd, or $C(m,n)$ with mn even and positive. If $\gamma : \mathbb{R} \to \mathbb{R}^2$ is a polynomial map of bidegree $(3,b)$ such that $\gamma(\mathbb{R})$ is $\mathcal{L}$-isotopic to $\overline{D}$, then $b \geq \left[\frac{3N-1}{2}\right]$, where N is the crossing number of the knot.*

Proof. Up to a change of coordinates in $\mathbb{C}^2$, we may suppose that $b = 3k-1$ or $b = 3k-2$. By genericity argument, we may also suppose that $C = \gamma(\mathbb{C})$ is a nodal curve in $\mathbb{C}^2$. Finally by symmetry we may further assume that m is even when $m+n$ is odd.

In the case of $C(m)$, the $\mathcal{L}$-scheme of C contains the patterns $\subset_2 (\times_1)^m \supset_2$. In the case of $C(m,n)$, the $\mathcal{L}$-scheme of C contains the patterns $\subset_2 (\times_1)^m(\times_2)^n \supset_1$ (see Figure 9). By the Riemann–Hurwitz formula the map $\pi \circ \gamma$ has exactly two

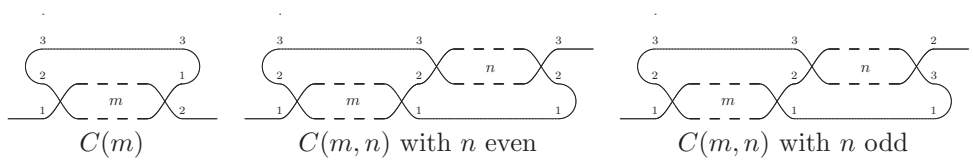


FIGURE 9. Diagrams for $C(m)$ or $C(m,n)$ knots

ramification points, which correspond precisely to the patterns $\subset$ and $\supset$ in the $\mathcal{L}$-scheme of C. In particular, these two ramification points are real. Hence, still by the Riemann–Hurwitz formula, the surface U_C is made of three embedded disks U_1, U_2, and U_3. Each disk U_i bounds the subknot L_i of the link L_C. The positivity of self-intersections of a complex algebraic curve and Proposition 2.2 implies that $\mathrm{lk}\,(L_i, L_j) = U_i \cdot U_j \geq 0$. On the other hand U_i and U_j intersect transversally at complex conjugated nodes, thus we have

$$\sum_{1 \leq i < j \leq 3} U_i \cdot U_j = \beta,$$

and therefore

$$\mathrm{lk}\,(L_i, L_j) \leq \beta, \quad 1 \leq i < j \leq 3. \tag{1}$$

Moreover Lemma 2.4 states that

$$N + \alpha + 2\beta = b - 1. \tag{2}$$

Let us label the components L_1, L_2, and L_3 as depicted in Figure 9, and let us evaluate the quantity $2\mathrm{lk}\,(L_1, L_3)$ from the algorithm described in Section 2.2. Clearly, no crossing point of $\mathbb{R}C$ contributes to $2\mathrm{lk}\,(L_1, L_3)$, and the factor $(\sigma_1\sigma_2\sigma_1)^k$ contributes k to $2\mathrm{lk}\,(L_1, L_3)$. As in Section 2.2, we denote by $c_1 \dots c_{s+1}$ the $\mathcal{L}$-scheme realized by C, and by $c'_0 c_1 \dots c_s c'_{s+1}$ the result of replacing c_{s+1} by c'_0 and c'_{s+1}. We have $N = n + m$ and $s = N + \alpha + 2$. Let $p \geq 1$ such that $c_p = \subset_2$. Then we have $c_{p+N+1} = \supset_2$ and $c_q = \bullet_j$ for $1 \leq q \leq p-1$ and $N + p + 2 \leq q \leq s$. We can estimate the contribution of $b_{\mathbb{R}}$ to $2\mathrm{lk}\,(L_1, L_3)$ as follows:

— in the case of $C(m)$ or $C(m,n)$ with N odd:
each pattern c_q with $p+N+1 \leq q \leq s$ contributes -1; each pattern $c_q c_{q+1}$ with $1 \leq q \leq p-1$ contributes at least -1; the pattern $c'_0 c_1$ contributes at least 0; the pattern c'_{s+1} contributes 0 if $b = 3k-1$, and -1 if $b = 3k-2$.

— in the case of $C(m,n)$ with N even:
each pattern $c_q c_{q+1}$ with $1 \leq q \leq p-1$ or $p+N+1 \leq q \leq s-1$ contributes at least -1, as well as the pattern $c'_0 c_1$; the pattern $c_s c'_{s+1}$ contributes at least -1 if $b = 3k-1$, and at least 0 if $b = 3k-2$.

Altogether, writing $d = 3k-1-\varepsilon$, we obtain

$$2\mathrm{lk}\,(L_1, L_3) \geq k - \alpha - 1 - \varepsilon$$

if N is odd, and

$$2\mathrm{lk}\,(L_1, L_3) \geq k - \alpha - 2 + \varepsilon$$

if N is even. These last inequalities together with identities (1) and (2) give

$$b - k \geq N - \varepsilon$$

if N is odd, and

$$b - k \geq N - 1 + \varepsilon$$

if N is even. Hence we obtain

$$b \geq \left[\tfrac{3N-1}{2}\right]$$

which proves the proposition. □

Remark 3.2. The proof of Proposition 3.1 shows that if $b = \left[\frac{3N-1}{2}\right]$ then $\beta = 0$ and $\alpha = \frac{N-3}{2}$ or $\alpha = \frac{N}{2} - 2$ depending on the parity of N. Moreover the mutual position of the solitary nodes of C is imposed: the $\mathcal{L}$-scheme realized by C does not contain the pattern $(\bullet_j)^2$ (i.e. the pattern corresponding to two successive solitary nodes must be $\bullet_j \bullet_{j\pm 1}$).

Remark 3.3. Let us consider the harmonic knot $\mathrm{H}(3,7,11)$. It is proved in [KP08] that its crossing number is 6. Then its lexicographic degree is $(3,7,c)$, which is smaller than the upper bound given in Proposition 3.1. This is the knot 6_3 which is neither a torus knot nor a generalized twist knot ([KP10]). In Section 4 we shall see that the lexicographic degree of 6_3 is $(3,7,11)$.

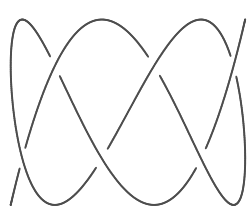


FIGURE 10. The harmonic knot $\mathrm{H}(3,7,11) = 6_3$

3.2. **Reduction to the case of alternating diagrams: real pseudoholomorphic curves.** As explained in the introduction it might not be true that the lexicographic degree of a trigonal long knot diagram does not increases along an isotopy through trigonal diagrams which never increases the number of crossings. For this to be true we have to consider *real pseudoholomorphic curves* (see Corollary 3.7).

We consider $\mathbb{C}^2$ equipped with the standard Fubini–Study symplectic form

$$\omega(u+iv, u'+iv') = u \wedge v + u' \wedge v'.$$

Recall that an *almost complex structure* J on $\mathbb{C}^2$ is a smooth section $J : \mathbb{C}^2 \to End_{\mathbb{R}}(T\mathbb{C}^2)$ such that $J_p^2 = -Id_{T_p\mathbb{C}^2}$ for any $p \in \mathbb{C}^2$, and that J is said to be *tamed* by ω if $\omega(v, Jv) > 0$ for any non-null vector $v \in T\mathbb{C}^2$. Such an almost complex structure is called *real* if the standard complex conjugation *conj* on $\mathbb{C}^2$ is J-antiholomorphic (i.e. $conj \circ J = J^{-1} \circ conj$). For example, the standard complex structure J_0 on $\mathbb{C}^2$ is real and tamed by ω. An almost complex structure is called *admissible* if it is real, tamed by ω, and coincide with J_0 outside a compact set in $\mathbb{C}^2$.

The next definition follows the lines of [OS02, Section 1] and [MS04, Chapter 2]. We also denote by J_0 (resp. *conj*) the standard complex structure (resp. complex conjugation) on $\mathbb{C}$.

Definition 3.4. *Let J be an admissible almost complex structure on $\mathbb{C}^2$. A* real (rational) J-holomorphic curve *is a map $\gamma : \mathbb{C} \to \mathbb{C}^2$ which is polynomial outside a compact set of $\mathbb{C}$, and such that*

$$d\gamma' \circ J_0 = J \circ d\gamma' \quad \text{and} \quad \gamma \circ conj = conj \circ \gamma.$$

A real pseudoholomorphic curve *$\gamma : \mathbb{C} \to \mathbb{C}^2$ is a map which is J-holomorphic for some admissible almost complex structure J on $\mathbb{C}^2$.*

Example 3.5. Every polynomial real map $\gamma : \mathbb{R} \to \mathbb{R}^2$ is a real J_0-holomorphic curve. However not every real pseudoholomorphic curve is J_0-holomorphic.

By definition, the image $C = \gamma(\mathbb{C})$ of any real pseudoholomorphic curve γ coincide with a real algebraic curve C_0 outside a compact set of $\mathbb{C}^2$. We say that γ is of *bidegree* $(3, b)$ if the curve C_0 has for Newton polygon the triangle with vertices $(0,0), (0,3)$, and $(0,b)$. As in the algebraic case, we may assume for our purposes that if a real pseudoholomorphic curve is of bidegree $(3, b)$, then $b = 3k-1$ or $b = 3k-2$. Note that nodal pseudoholomorphic curves satisfy the adjunction formula (see [MS04, Chapter 2]), in particular Lemma 2.4 still holds for pseudoholomorphic curves.

The braid theoretical approach described in Section 3 extends word by word to the case of nodal real pseudoholomorphic curves of bidegree $(3, b)$ (see for example [Ore99, OS02]). Indeed, in these latter two sections we only use the behavior at infinity of real algebraic curves and the three following facts:

(1) any intersection of two complex algebraic curves is positive;
(2) there exists a unique vertical line passing through a given point p of $\mathbb{C}^2$, which is real if p is real;
(3) the map π is a real holomorphic map.

Let us fix an admissible almost complex structure J on $\mathbb{C}^2$. The word by word extension of Sections 2.1 and 2.2 to J-holomorphic curves follows from the three following facts:

(1') any intersection of two J-holomorphic curves is positive (see [MS04, Appendix E]);

(2') there is an analogue of the pencil of vertical lines in $\mathbb{C}^2$: consider the standard embedding

$$\begin{array}{ccc} \mathbb{C}^2 & \longrightarrow & \mathbb{C}P^2 \\ (x,y) & \longmapsto & [x:y:1] \end{array};$$

then given a point $p \in \mathbb{C}^2$ there exists a unique J-holomorphic line in $\mathbb{C}P^2$ passing through p and $[0:1:0]$ (see [Gro85]); by uniqueness, this line is real if p is real; this produces a real pencil of "vertical" J-holomorphic lines on $\mathbb{C}^2$;

(3') since J is standard outside a compact set of $\mathbb{C}^2$, the line $\{y = b\}$ is J-holomorphic for $|b|$ large enough; replace the map π by the map $\pi_b : \mathbb{C}^2 \to \{y = b\}$ which associates to each point p of $\mathbb{C}^2$ the (unique) intersection point of $\{y = b\}$ with the J-holomorphic vertical line passing through p; the map π_b is clearly a real J-holomorphic map.

The main advantage in dealing with real pseudoholomorphic curves rather than real algebraic curves is that we have the following proposition.

Proposition 3.6 (Orevkov, [Ore99])**.** *Let $\gamma : \mathbb{C} \to \mathbb{C}^2$ be a real nodal pseudoholomorphic curve of bidegree $(3, b)$ realizing an $\mathcal{L}$-scheme $\mathcal{L}_\gamma$, and let $\mathcal{L}'_\gamma$ be the $\mathcal{L}$-scheme obtained from $\mathcal{L}_\gamma$ by one of the following elementary operations :*

$$\times_j \supset_{j\pm1} \leftrightarrow \times_{j\pm1} \supset_j \qquad \subset_{j\pm1} \times_j \leftrightarrow \subset_j \times_{j\pm1} \qquad \times_j \times_j \to \emptyset$$

$$\times_j \supset_j \leftrightarrow \supset_j \bullet_j \qquad \subset_j \times_j \leftrightarrow \bullet_j \subset_j \qquad \times_{j+1} \times_j \times_{j+1} \leftrightarrow \times_j \times_{j+1} \times_j$$

Then there exists a real nodal pseudoholomorphic curve of bidegree $(3, b)$ realizing $\mathcal{L}'_\gamma$.

Corollary 3.7. *Let D_1 and D_2 be two trigonal long knot diagrams such that D_2 is obtained from D_1 by an isotopy of trigonal diagrams which never increase the number of crossings. If there exists a real nodal pseudoholomorphic curve $\gamma_1 : \mathbb{C} \to \mathbb{C}^2$ of bidegree $(3, b)$ such that $\gamma_1(\mathbb{R})$ is $\mathcal{L}$-isotopic to $\overline{D}_1$, then there also exists a real nodal pseudoholomorphic curve $\gamma_2 : \mathbb{C} \to \mathbb{C}^2$ of bidegree $(3, b)$ such that $\gamma_2(\mathbb{R})$ is $\mathcal{L}$-isotopic to $\overline{D}_2$.*

Proposition 3.8. *Let K be the two-bridge knot $C(m)$ with m odd, or $C(m, n)$ with mn positive and even, and let N be its crossing number. Let D be any trigonal diagram of K. If there exists a real nodal pseudoholomorphic curve $\gamma : \mathbb{C} \to \mathbb{C}^2$ of bidegree $(3, b)$ such that $\gamma(\mathbb{R})$ is $\mathcal{L}$-isotopic to $\overline{D}$, then $b \geq \left[\frac{3N-1}{2}\right]$.*

Proof. According to Corollary 3.7 and Theorem A, it is enough to prove the proposition when D is alternating, which has been done in Proposition 3.1 (recall that real pseudoholomorphic curves satisfy Lemma 2.4, so the proof of Proposition 3.1 extends word by word to this case). □

Theorem 3.9. *Let K be the two-bridge knot $C(m)$ with m odd, or $C(m, n)$ with mn positive and even, and let N be its crossing number. Then the lexicographic degree of K is*

$$\left(3, \left[\tfrac{3N-1}{2}\right], \left[\tfrac{3N}{2}\right] + 1\right).$$

Proof. Let K be any two-bridge knot. It is proved in [KP10] that there exists a polynomial curve $\gamma : \mathbb{R} \to \mathbb{R}^3$ of degree $(3, b, c)$ such that $N < b < c$, $b + c = 3N$, and $\gamma(\mathbb{R})$ is the knot K. In particular we have $b \leq \left[\frac{3N-1}{2}\right]$ and $c \geq \left[\frac{3N}{2}\right] + 1$. If K is $C(m)$ or $C(m, n)$, then by Proposition 3.8 we have $b \geq \left[\frac{3N-1}{2}\right]$. This implies that $c \leq \left[\frac{3N}{2}\right] + 1$, and the theorem is proved. □

Remark 3.10. It is shown in [KP11] that the harmonic knot $\mathrm{H}(3, 3n + 2, 3n + 1)$ is the torus knot $\mathrm{T}(2, 2n + 1) = C(2n + 1)$. By Theorem 3.9, this is an explicit polynomial parametrization of minimal lexicographic degree $(3, 3n + 1, 3n + 2)$. In this example, the number of crossings of the diagram is greater than the crossing number of the knot. In [KP09] it is proved that there exists a polynomial representation of $\mathrm{T}(2, 2n + 1)$ which is of minimal degree, and such that the number of crossings is minimal.

4. A lower bound

Theorem 4.3 provides a general lower bound for the lexicographic degree of every non-trivial knot, furthermore this lower bound is sharp.

Proposition 4.1. *Let $\gamma : \mathbb{R} \to \mathbb{R}^3$ be a polynomial map of degree (a, b, c) whose image is a smooth knot. Suppose that a and b are coprime, and that the xy-diagram of $\gamma(t)$ is alternating with $(a - 1)(b - 1)/2$ crossings. Then $c \geq ab - a - b$.*

The proof of Proposition 4.1 uses the following lemma of Frobenius type.

Lemma 4.2. *Let a and b be coprime positive integers. The number of integers of the form $n = \alpha a + \beta b, n \leq ab - a - b - 1,\ (\alpha, \beta) \in \mathbb{Z}^2_{\geq 0}$, is equal to $(a - 1)(b - 1)/2$.*

Proof. Let $N = (a-1)(b-1)/2$. First, let us show that the integer $2N-1 = ab-a-b$ is not of the form $\alpha a + \beta b$, with $(\alpha, \beta) \in \mathbb{Z}^2_{\geq 0}$.

If we had $ab - a - b = \alpha a + \beta b$, then $\alpha \equiv -1 \pmod{b}$, and $\alpha \geq b - 1$. Hence we deduce that $\alpha a + \beta b \geq ab - a + \beta b > 2N - 1$, contrary to our hypothesis

Now consider the sets $E = \{(\alpha, \beta) \in \mathbb{Z}^2_{\geq 0}, \alpha a + \beta b \leq 2N - 2\}$ and $Q = \{(\alpha, \beta) \in \mathbb{Z}^2_{\geq 0},\ \alpha \leq b - 2,\ \beta \leq a - 2\}$. The set Q has $2N$ elements, and contains E. Denoting $s(\alpha, \beta) = \alpha a + \beta b,\ \alpha' = b - 2 - \alpha,$ and $\beta' = a - 2 - \beta$, we have $s(\alpha, \beta) + s(\alpha', \beta') = 2(2N - 1)$, and then $(\alpha', \beta') \in Q - E$ if and only if $(\alpha, \beta) \in E$.

Consequently, half the elements of Q are in E, and then E has N elements. Since a and b are coprime, the mapping s is an injection from E to $\mathbb{Z}_{\geq 0}$, which concludes the proof. □

Proof of Proposition 4.1. The $N = (a - 1)(b - 1)/2$ crossing points of the xy-projection of $\gamma(t) = (x(t), y(t), z(t))$ correspond to parameter pairs (s_i, t_i) such that

$$x(t_i) = x(s_i), \quad y(t_i) = y(s_i), \quad z(t_i) > z(s_i).$$

Consider V the vector subspace of $\mathbb{R}[t]$ generated by the polynomials $x(t)^\alpha y(t)^\beta$, $\alpha a + \beta b \leq 2N - 2$. By Lemma 4.2, the dimension of V is N. Let us define the linear mapping $\varphi : V \to \mathbb{R}^N$ by $\varphi(h) = \Big(h(t_1), \ldots, h(t_N)\Big)$. If a polynomial h is in the kernel of φ, then $h(t_1) = h(s_1) = \cdots = h(t_N) = h(s_N) = 0$, and the polynomial h has $2N$ distinct roots. Since $\deg(h) \leq 2N - 2$, we deduce that $h = 0$.

Consequently the mapping φ is an isomorphism and there exists $h \in V$ such that $h(t_i) = h(s_i) = (z(t_i) + z(s_i))/2$.

Let us define $\tilde{z}(t) = z(t) - h(t)$. The images of γ and $\tilde{\gamma}(t) = \big(x(t), y(t), \tilde{z}(t)\big)$ are isotopic and realize the same diagram. By construction, at each crossing we have $\tilde{z}(t_i) > 0 > \tilde{z}(s_i)$.

Let us write $\{t_1, \dots, t_N, s_1, \dots, s_N\} = \{\tau_1, \dots, \tau_{2N}\}$ where $\tau_j < \tau_{j+1}$. Since the diagram realized by $\tilde{\gamma}(t)$ is alternating, we may assume that $(-1)^i \tilde{z}(\tau_i) > 0$. Consequently the polynomial $\tilde{z}(t)$ has at least one root in the interval (τ_j, τ_{j+1}), and $\deg(\tilde{z}) \geq 2N-1$. Since $\deg(h) \leq 2N-2$, we conclude that $\deg(z) \geq 2N-1$. □

As a consequence we deduce:

Theorem 4.3. *Let K be a knot of crossing number $N \neq 0$. Then the lexicographic degree of K is at least $(3, N+1, 2N-1)$. For every integer $N \not\equiv 2 \pmod 3$, $N \geq 3$, there exists a knot of crossing number N and lexicographic degree $(3, N+1, 2N-1)$*

Proof. Let (a, b, c) be the lexicographic degree of K. Since K is nontrivial, we have $a \geq 3$. Suppose that $a = 3$. Then K is a two-bridge knot, and b is not divisible by 3. Moreover by Bézout's theorem we have $N \leq (a-1)(b-1)/2 = b-1$, that is $b \geq N+1$.

Let $\gamma(t) : \mathbb{R} \to \mathbb{R}^3$ be a polynomial parametrization of K of degree $(3, b, c)$. If $b = N+1$, then the xy-diagram of $\gamma(t)$ has the minimal number of crossings. Since K is a two-bridge knot, it is an alternating knot, and the xy-diagram of $\gamma(t)$ is alternating. By Proposition 4.1, we conclude that $c \geq 2N-1$.

On the other hand, when $N \not\equiv 2 \pmod 3$, the harmonic knot $\mathrm{H}(3, N+1, 2N-1)$ has crossing number N (see [KP08]), hence its lexicographic degree is $(3, N+1, 2N-1)$. □

We also obtain the lexicographic degree of an infinite family of three-bridge knots.

Corollary 4.4. *Let* H *be the harmonic knot* $\mathrm{H} = \mathrm{H}(5, b, 4b-5)$*, with* $b \not\equiv 0 \pmod 5$*. The knot* H *is a three-bridge knot of lexicographic degree* $(5, b, 4b-5)$*.*

Proof. The xy-projection of the knot H has $N = 2(b-1)$ crossings and is alternating (see [KP11]). If H was two-bridged, then it would have a minimal diagram which would be alternating and two-bridged. By Tait's flyping conjecture, this is not the case since the xy-projection of H is alternating and three-bridged.

Let $\big(x(t), y(t), z(t)\big)$ be a polynomial knot isotopic to H. Since H is not two-bridged, we have $\deg(x) \geq 5$. If $\deg(x) = 5$, then by Bézout's theorem we have $N \leq 2\big(\deg(y) - 1\big)$, that is $\deg(y) \geq b$. If $\deg(y) = b$, then by Proposition 4.1 we obtain $\deg(z) \geq 5b - 5 - b = 4b - 5$. □

Remark 4.5. *The lexicographic degree of the harmonic knot* $\mathrm{H}(a, b, ab-a-b)$ *is not always* $(a, b, ab-a-b)$*. For example, if $a = 4$ this knot is two-bridged and has a polynomial parametrization of degree* $(3, b', c')$*, which is lexicographically smaller than* $(4, b, 3b-4)$*.*

Proposition 4.1 can also be used to study the maximal degree of a polynomial knot, that is the maximum degree of its components.

Corollary 4.6. *Let K be a polynomial knot of degree d and crossing number N. We have*

$$N \leq \frac{(d-2)(d-3)}{2}.$$

Moreover, if $d > 5$ and if K is alternating, then we have:

$$N \leq \frac{(d-1)(d-4)}{2}.$$

Proof. Using an affine transformation, we can suppose that the lexicographic degree of K is $\leq (d-2, d-1, d)$. By Bézout's theorem we see that the number of crossings of the xy-projection is at most $(d-2)(d-3)/2$, which proves the first assertion. If we had $N = (d-2)(d-3)/2$, then the lexicographic degree of K would be $(d-2, d-1, d)$. By Proposition 4.1, we have $d \geq (d-2)(d-1) - (d-2) - (d-1) = d^2 - 5d + 5$, which is impossible, since $d > 5$. Consequently, we have $N \leq (d-2)(d-3)/2 - 1 = (d-1)(d-4)/2$. □

Remark 4.7. *Since knots of crossing number smaller than 8 are alternating, we deduce that the only knot of degree 4 is the trivial knot, the only nontrivial knot of degree 5 is the trefoil, and the crossing number of a knot of degree 6 is at most 5.*

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