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On stabilizability conditions for discrete-time switched linear systems

Mirko Fiacchini[1], Antoine Girard[2], Marc Jungers[3]

***Abstract*— In this paper we consider the stabilizability property for discrete-time switched linear systems. Novel conditions, in LMI form, are presented that permit to combine generality with computational affordability. The relations and implications between different conditions, new ones and taken from literature, for stabilizability are analyzed to infer and compare their conservatism and their complexity.**

I. Introduction

Switched systems are characterized by dynamics that may change along the time among a finite number of possible dynamical behaviors, see [7]. Each behavior is determined by a mode and the active one is selected by means of a function of time, referred to as switching law. The interest that such kind of systems rose in the last decades relies in their capability of modelling complex real systems, as embedded or networked, and also for the theoretical issues involved. Their dynamical properties, in fact, are often not intuitive nor trivial and the problems related to analysis and control design may result rather challenging, also for linear switched system, see [8].

The problem of stability or stabilizability, depending on the assumption on the switching law, of linear switched system attracted many research efforts, see the overview [9]. Conditions for stability, that is when the switching law is considered an exogenous signal, have been proposed: the joint spectral radius approach [4]; the polyhedral Lyapunov functions [1], [10] and the path-dependent switched Lyapunov ones [6].

In this paper we are considering the problem of stabilizability of switched systems, namely the condition under which a switching law can be designed for the system to be asymptotically stable. Sufficient conditions have been provided in literature, mainly based on *min-switching* policies, introduced in [11], developed in [5], [7] and leading to Lyapunov-Metzler inequalities [3]. A necessary and sufficient condition, based on set theory, for the stabilizability of discrete-time switched linear systems has appeared recently in [2]. Nevertheless, this condition might result to be often computationally unaffordable, as it requires to check whether some particular set is contained in the union of others. On the other hand, such computational complexity appears to be inherent the problem itself, then avoidable only at the price of introducing some conservatism.

The first main objective of this paper is to propose new conditions for stabilizability of discrete-time switched linear systems which could conjugate computational affordability with generality. Moreover, we provide geometrical and numerical insights on different stabilizability conditions to quantify their conservatism and the relations between them and with the necessary and sufficient ones. We proved the implications between the conditions, which permit to get a clear picture of their relations, their conservatism and their complexity.

The paper is organized as follows: Section II presents the problem of stabilizability of switched systems. The Section III provides the analysis of the Lyapunov-Metzler approach and proposes a generalization. In Section IV a novel condition, in LMI form, is given and its relations with other ones are analyzed. A numerical example is presented in Section V and Section VI draws some conclusions.

Notation: Given $n \in \mathbb{N}$, define $\mathbb{N}_n = \{x \in \mathbb{N} : 1 \leq x \leq n\}$. Given $\alpha \in \mathbb{R}^n$, α_i denote its i-th element; given $\pi \in \mathbb{R}^{n\times m}$, π_{ij} is the entry of i-th row and j-th column. Given $\Omega \subseteq \mathbb{R}^n$ define the interior of Ω as $\mathrm{int}(\Omega)$. Given $P \in \mathbb{R}^{n\times n}$ with $P > 0$ denote with $\mathscr{E}(P) = \left\{x \in \mathbb{R}^n : \ x^T P x \leq 1\right\}$, the related ellipsoid. The i-th element of a finite set of matrices is denoted, with slight abuse, as A_i.

II. Problem statement

We consider the problem of stabilizability of autonomous discrete-time switched linear system of the form

$$x_{k+1} = A_{\sigma(k)} x_k, \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state at time $k \in \mathbb{N}$ and $\sigma : \mathbb{N} \to \mathbb{N}_q$ is the switching law and $\{A_i\}_{i\in\mathbb{N}_q}$, with $A_i \in \mathbb{R}^{n\times n}$ for all $i \in \mathbb{N}_q$. The following assumption, not necessary, is supposed to hold throughout the paper for simplicity.

Assumption 1: All the matrices A_i, with $i \in \mathbb{N}_q$, are invertible and non-Schur.

The following notations are employed in the paper:

- $\mathscr{I} = \mathbb{N}_q$: finite set of switching modes.
- $\mathscr{I}^k = \prod_{j=1}^k \mathscr{I}$: all the possible sequences of modes of length k.
- $\mathscr{I}^{[1:N]} = \bigcup_{k=1}^N \mathscr{I}^k$: all the possible sequences of modes of length from 1 to N.
- $\bar{N} = \sum_{k=1}^N q^k$: given $N \in \mathbb{N}$, number of elements $i \in \mathscr{I}^{[1:N]}$. Analogous for $\bar{M}$.

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- given $i=(i_1,\dots,i_k)$ such that $i\in\mathscr{I}^{[1:N]}$ and a set Ω, define:

$$\begin{aligned}&\mathbb{A}_i=\textstyle\prod_{j=1}^k A_{i_j}=A_{i_k}\cdots A_{i_1},\\&\Omega_i=\Omega_i(\Omega)=\{x\in\mathbb{R}^n:\ \mathbb{A}_i x\in\Omega\},\\&\mathscr{B}_i=\{x\in\mathbb{R}^n:\ x^T\mathbb{A}_i^T\mathbb{A}_i x\le 1\}.\end{aligned}$$

  and then $\mathscr{B}_i=\Omega_i(\mathscr{B})$ with $\mathscr{B}=\{x\in\mathbb{R}^n:\ x^Tx\le 1\}$. The dependence of Ω_i on Ω is omitted when clear from the context.
- $\mathscr{M}_N$: set of Metzler matrices of dimension N, i.e. matrices $\pi\in\mathbb{R}^{N\times N}$ whose elements are nonnegative and $\sum_{j=1}^N\pi_{ji}=1$ for all $i\in\mathbb{N}_N$.

We recall hereafter the main results proposed in [2] on the stabilizability of switched linear systems (1). These results are based on the Algorithm 1 in [2] that basically consists in computing the successive pre-images of a C*-set $\Omega\subseteq\mathbb{R}^n$ with respect to all the possible modes. A C*-set $\Omega\subseteq\mathbb{R}^n$ is a compact, star-convex set containing the origin in its interior. The stabilizability of the system (1) is equivalent to the fact that the algorithm ends with a finite number of steps. The Theorem 1 in [2], recalled below, provides a set-theory based necessary and sufficient condition for stabilizability of the system (1).

Theorem 2 ([2]): Let Ω be a C*-set. The switched system (1) is stabilizable if and only if there exists $N\in\mathbb{N}$ such that

$$\Omega\subseteq\operatorname{int}\Big(\bigcup_{i\in\mathscr{I}^{[1:N]}}\Omega_i\Big). \tag{2}$$

Since the stabilizability property is not dependent on the choice of the initial C*-set Ω, focussing on the case $\Omega=\mathscr{B}$ and ellipsoidal pre-images entails no loss of generality, see [2]. Then condition (2) can be replaced by

$$\mathscr{B}\subseteq\operatorname{int}\Big(\bigcup_{i\in\mathscr{I}^{[1:N]}}\mathscr{B}_i\Big), \tag{3}$$

for what concerns stabilizability, although the value N might depend on the choice of Ω.

The set inclusions (2) or (3) are the stop conditions of the algorithm and then must be numerically tested at every step. The main computational issue is that determining numerically if a C*-set Ω is included into the interior of the union of some C*-sets is very complex in general, also in the case of ellipsoidal sets. On the other hand, the condition given by Theorem 2 provides an exact characterization of the complexity inherent to the problem of stabilizing a switched linear system.

The objective of this paper is to consider alternative conditions for stabilizability, taken from the literature and novel ones, to provide geometrical and numerical insights and analyze their conservatism by comparison with the necessary and sufficient one given in Theorem 2.

III. Lyapunov-Metzler BMI conditions

The condition we are considering first is related to the Lyapunov-Metzler inequality, proposed in [3]. The condition is sufficient and characterized by a set of BMI inequalities involving the Metzler matrices. The result is recalled below.

Theorem 3 ([3]): If there exist $P_i>0$, with $i\in\mathscr{I}$, and $\pi\in\mathscr{M}_q$ such that

$$A_i^T\left(\sum_{j=1}^q\pi_{ji}P_j\right)A_i-P_i<0,\qquad\forall i\in\mathscr{I}, \tag{4}$$

holds, then the switched system (1) is stabilizable.

As proved in the paper [3], the satisfaction of condition (4) implies that the homogeneous function induced by the set $\bigcup_{i\in\mathscr{I}}\mathscr{E}(P_i)$ is a control Lyapunov function for the system.

Our first objective is the geometrical meaning of the Lyapunov-Metzler condition and which is the relation with the necessary and sufficient condition for stabilizability given in Theorem 2.

A. Lyapunov-Metzler conditions for $q=2$

Consider, thus, $q=2$ for the moment. Given $i\in\mathscr{I}$, we analyze the geometrical meaning of the Lyapunov-Metzler condition, that is

$$A_i^T(\alpha_iP_1+\beta_iP_2)A_i-P_i<0,\quad\forall i\in\{1,2\}, \tag{5}$$

with $\alpha_i+\beta_i=1$ and $\alpha_i,\beta_i\ge 0$. Define

$$\begin{aligned}&\mathscr{E}_1=\mathscr{E}(P_1)=\left\{x\in\mathbb{R}^n:\ x^TP_1x\le 1\right\},\\&\mathscr{E}_2=\mathscr{E}(P_2)=\mathscr{E}\left\{x\in\mathbb{R}^n:\ x^TP_2x\le 1\right\},\\&\Theta=\bigcup_{\substack{\alpha+\beta=1\\ \alpha,\beta\ge 0}}\mathscr{E}(\alpha P_1+\beta P_2)=\{x\in\mathbb{R}^n:\exists\alpha,\beta\ge 0\\&\qquad\qquad\text{s.t. }\alpha+\beta=1,\ x^T(\alpha P_1+\beta P_2)x\le 1\}.\end{aligned} \tag{6}$$

The condition (5) implies that the image of $\mathscr{E}_i$ through the linear map given by A_i is contained in the interior of one element of the uncountable sets $\mathscr{E}(\alpha_iP_1+\beta_iP_2)$. In fact, if (5) holds then for all $x\in\mathscr{E}_i$, i.e. with $x^TP_ix\le 1$, we have that

$$x^TA_i^T(\alpha_iP_1+\beta_iP_2)A_i^Tx<x^TP_ix\le 1$$

which implies $x\in\mathscr{E}(\alpha_iP_1+\beta_iP_2)$. Thus (5) implies $A_i\mathscr{E}_i\subseteq\mathscr{E}(\alpha_iP_1+\beta_iP_2)$.

Now, we are interested in the particular sets (of sets) $\mathscr{E}(\alpha P_1+\beta P_2)$. It can proved that the set Θ is exactly the union of $\mathscr{E}_1$ and $\mathscr{E}_2$. This equivalence is proved below for a generic number of ellipsoids.

Lemma 4: Given $P_i>0$, $i\in\mathbb{N}_m$, the set defined by

$$\Gamma=\bigcup_{\alpha_i\ge 0}^{\alpha_1+\cdots+\alpha_m=1}\mathscr{E}\left(\sum_{i\in\mathbb{N}_m}\alpha_iP_i\right) \tag{7}$$

is such that

$$\Gamma=\bigcup_{i\in\mathbb{N}_m}\mathscr{E}(P_i). \tag{8}$$

Proof: The equality (8) is satisfied if and only if the following conditions

$$\bigcup_{i\in\mathbb{N}_m}\mathscr{E}(P_i)\subseteq\Gamma\subseteq\bigcup_{i\in\mathbb{N}_m}\mathscr{E}(P_i) \tag{9}$$

hold. The first inclusion in (9) is trivially proved by noticing that for $\alpha_j=1$ and the other coefficients $\alpha_k=0$ for all $j\ne k$, then $\mathscr{E}\left(\sum_{i\in\mathbb{N}_m}\alpha_iP_i\right)=\mathscr{E}(P_j)$.

Consider the second condition in (9) and suppose that $x \in \Gamma$. Then, from the definition of Γ, we have that there exist $\alpha_i^* \in [0,1]$ such that $\sum_{i\in\mathbb{N}_m} \alpha_i^* = 1$ and

$$\sum_{i\in\mathbb{N}_m} \alpha_i^* x^T P_i x \leq 1. \tag{10}$$

All the terms $x^T P_i x$ being non-negative, it yields

$$\min_{i\in\mathbb{N}_m} x^T P_i x \leq \sum_{i\in\mathbb{N}_m} \alpha_i^* x^T P_i x \leq 1. \tag{11}$$

That leads to the existence of $i^* \in \mathbb{N}_m$ such that $x \in \mathscr{E}(P_{i^*})$ and finally $x \in \bigcup_{i\in\mathbb{N}_m} \mathscr{E}(P_i)$. ■

The set Γ is defined as the union of a set of ellipsoids parameterized with respect to α_i and $\sum_{i\in\mathbb{N}_m} \alpha_i = 1$. Such an equality can be relaxed in the inequality $\sum_{i\in\mathbb{N}_m} \alpha_i \geq 1$, as shown in the following.

Corollary 5: Given $P_i > 0$, $i \in \mathbb{N}_m$, the set Γ defined in (8) verifies

$$\Gamma = \bigcup_{\alpha_i \geq 0}^{\alpha_1+\cdots+\alpha_m \geq 1} \mathscr{E}\left(\sum_{i\in\mathbb{N}_m} \alpha_i P_i\right)$$

Proof: The result follows directly from the fact that, if $s = \sum_{i\in\mathbb{N}_m} \alpha_i \geq 1$ then

$$\sum_{i\in\mathbb{N}_m} \alpha_i P_i = s \sum_{i\in\mathbb{N}_m} \frac{\alpha_i}{s} P_i \geq \sum_{i\in\mathbb{N}_m} \tilde{\alpha}_i P_i, \tag{12}$$

with $\tilde{\alpha}_i = \frac{\alpha_i}{s}$ and thus $\sum_{i\in\mathbb{N}_m} \tilde{\alpha}_i = 1$. Then $x \in \mathscr{E}(\sum_{i\in\mathbb{N}_m} \alpha_i P_i)$ with $\sum_{i\in\mathbb{N}_m} \alpha_i \geq 1$ implies $x \in \mathscr{E}(\sum_{i\in\mathbb{N}_m} \tilde{\alpha}_i P_i)$ with $\sum_{i\in\mathbb{N}_m} \tilde{\alpha}_i = 1$, which means that $x \in \Gamma$. ■

Using the results above with $m = 2$, we can now prove the equivalence between the conditions of the type (5) and the set inclusion, for $i \in \{1,2\}$. The idea is to prove that all the ellipsoids contained in the union of $\mathscr{E}_1$ and $\mathscr{E}_2$ are contained in one of the ellipsoids parameterized by all the α and β such that $\alpha + \beta \geq 1$.

Lemma 6: Given $P, P_1, P_2 > 0$ and the sets defined in (6), the inclusion

$$\mathscr{E}(P) \subseteq \mathrm{int}(\Theta), \tag{13}$$

holds if and only if there exist $\alpha, \beta \geq 0$ such that $\alpha + \beta = 1$ and

$$\alpha P_1 + \beta P_2 < P. \tag{14}$$

Proof: We assume without loss of generality that $\mathscr{E}(P) \nsubseteq \mathscr{E}_1$ (the case $\mathscr{E}(P) \nsubseteq \mathscr{E}_2$ is symmetric). Sufficiency consists in proving that (14) implies (13). Notice that (14) is equivalent to

$$x^T(\alpha P_1 + \beta P_2)x < x^T P x,$$

for all $x \in \mathbb{R}^n$ and if $x \in \mathscr{E}$ then $x \in \mathrm{int}(\Theta)$, from the definitions (6), and (13) holds.

For necessity, let us assume that $\mathscr{E}(P) \subseteq \mathrm{int}(\mathscr{E}_1 \cup \mathscr{E}_2)$. Then, for all $x \in \mathscr{E}(P)$, such that $x \notin \mathrm{int}(\mathscr{E}_1)$, it must hold $x \in \mathrm{int}(\mathscr{E}_2)$. This can be reformulated as follows:

$$\forall x \in \mathbb{R}^n,\ \left(x^T P x \leq 1 \text{ and } x^T P_1 x \geq 1\right) \implies x^T P_2 x < 1.$$

Let $y \in \mathbb{R}^n$, $y \neq 0$, such that $y^T(P_1 - P)y \geq 0$. Let $x = y/\sqrt{y^T P y}$, then $x^T P x = 1$ and

$$x^T P_1 x = \frac{y^T P_1 y}{y^T P y} \geq 1.$$

Then, it follows that we must have $x^T P_2 x < 1$ and therefore $y^T(P - P_2)y > 0$. Summarizing, we have shown

$$\forall y \in \mathbb{R}^n,\ y \neq 0,\ \left(y^T(P_1 - P)y \geq 0\right) \implies y^T(P - P_2)y > 0.$$

Since $\mathscr{E}(P) \nsubseteq \mathscr{E}_1$, it follows that there exists y_0 such that $y_0^T(P_1 - P)y_0 > 0$. Then, it follows from the S-lemma that there exists $\beta \geq 0$ such that

$$P - P_2 > \beta(P_1 - P).$$

Then, it follows that $\alpha_1 P_1 + \alpha_2 P_2 < P$ with $\alpha_1 = \beta/(1+\beta)$ and $\alpha_2 = 1/(1+\beta)$. ■

The results stated in Lemma 6 is not true for Θ given by more than two ellipsoids, see for instance the counter-example given in Example 13. In general, the LMI condition analogous to (14) is only sufficient for the set inclusion.

Using the Lemma 6, we can prove that the Lyapunov-Metzler condition is equivalent to the set theory one with $N = 1$.

Theorem 7: If $q = 2$, the Lyapunov-Metzler condition (5) holds if and only if (2) holds with $N = 1$ and $\Omega = \Theta$.

Proof: To prove sufficiency suppose that (2) holds with $N = 1$ and $\Omega = \mathscr{E}_1 \cup \mathscr{E}_2$, see Lemma 4. This means that $\mathscr{I} = \{1,2\}$ and then $\mathscr{E}_1 \cup \mathscr{E}_2 \subseteq \mathrm{int}(\Omega_1(\mathscr{E}_1 \cup \mathscr{E}_2) \cup \Omega_2(\mathscr{E}_1 \cup \mathscr{E}_2))$. This is equivalent, by Lemma 6, to the existence of $\alpha_1, \alpha_2 \in [0,1]$ such that $\alpha_i A_i^T P_1 A_i + (1-\alpha_i) A_1^T P_2 A_i < P_i$ for $i \in \{1,2\}$, that is to satisfaction of (5).

For necessity, assume that (5) is satisfied. Consider $i = 1$, the case $i = 2$ is analogous. Condition (5) results in $A_1^T(\pi_{11}P_1 + \pi_{21}P_2)A_1 < P_1$ that is equivalent, from Lemma 4, to

$$\begin{aligned}\mathscr{E}_1 &\subseteq \mathrm{int}\big(\mathscr{E}(A_1^T P_1 A_1) \cup \mathscr{E}(A_1^T P_2 A_1)\big)\\ &\subseteq \mathrm{int}\big(\Omega_1(\mathscr{E}_1) \cup \Omega_1(\mathscr{E}_2)\big) \subseteq \mathrm{int}\big(\Omega_1(\Omega)\big) \subseteq \mathrm{int}\big(\Omega_1 \cup \Omega_2\big),\end{aligned}$$

where the third inequality follows from

$$\Omega_1(\mathscr{E}_i) = \{x : A_1 x \in \mathscr{E}_i\} \subseteq \{x : A_1 x \in \mathscr{E}_1 \cup \mathscr{E}_2\} = \Omega_1(\Omega).$$

Then, $\Omega = \mathscr{E}_1 \cup \mathscr{E}_2 \subseteq \mathrm{int}\big(\Omega_1 \cup \Omega_2\big)$. ■

Thus, when the system has two modes, the Lyapunov-Metzler condition and the set-inclusion one with $N = 1$ are equivalent. This equivalence is lost, in general, for higher number of modes or bigger N.

B. Lyapunov-Metzler conditions for $q > 2$

An interesting issue is whether the Lyapunov-Metzler condition is necessary and sufficient also for a generic number of modes q. From the results presented in the following sections, one can infer that the Lyapunov-Metzler condition (5) is only sufficient for stabilizability, in general. See the implications in Figure 2 below, some of which are proved in the paper.

A direct generalization of the Lyapunov-Metzler condition can be given, by removing the unnecessary link between the number of ellipsoids (and matrices P_i) and the system modes.

Proposition 8: If there exist $M \in \mathbb{N}$ and $P_i > 0$, with $i \in \mathscr{I}^{[1:M]}$, and $\pi \in \mathscr{M}_{\tilde{M}}$ such that

$$\mathbb{A}_i^T \left(\sum_{j \in \mathscr{I}^{[1:M]}} \pi_{ji} P_j \right) \mathbb{A}_i - P_i < 0, \qquad \forall i \in \mathscr{I}^{[1:M]}, \tag{15}$$

holds, then the switched system (1) is stabilizable.

Proof: The proof is analogous to that one of the classical Lyapunov-Metzler condition, see [3]. ■

Proposition 8 extends the Lyapunov-Metzler condition providing a more general one. An interesting issue, that we are considering in our current research, is the relation with the necessary and sufficient condition for stabilizability, as well as with other ones.

Remark 9: The condition (15) can be interpreted in terms of the classical Lyapunov-Metzler condition (4) by considering the switched system obtained by defining one fictitious mode for every matrix $\mathbb{A}_i$ with $i \in \mathscr{I}^{[1:M]}$. Thus, testing the generalized Lyapunov-Metzler condition is equivalent to checking the classical one for a system whose modes are related to every possible sequence of the original system (1), with of length M or less.

IV. LMI SUFFICIENT CONDITION

The main drawback of the necessary and sufficient set-inclusion condition for stabilizability is, as already stated, its inherent complexity. On the other hand, the Lyapunov-Metzler-based approach leads to a more affordable BMI sufficient condition. Nevertheless, the complexity could be still computationally prohibitive. Our next aim is to formulate an alternative condition that could be checked efficiently, a convex one.

Theorem 10: The switched system (1) is stabilizable if there exist $N \in \mathbb{N}$ and $\alpha \in \mathbb{R}^{\tilde{N}}$ such that $\alpha \geq 0$, $\sum_{i \in \mathscr{I}^{[1:N]}} \alpha_i \geq 1$ and

$$\sum_{i \in \mathscr{I}^{[1:N]}} \alpha_i \mathbb{A}_i^T \mathbb{A}_i < I. \tag{16}$$

Proof: The result follows directly from the fact that (16) implies (3), as a consequence of Lemma 4. ■

We can also give an interpretation of the previous result in terms of Lyapunov functions and derive a natural controller synthesis technique. Let us assume that (16) holds, then there exists $\rho \in [0,1)$ such that

$$\sum_{i \in \mathscr{I}^{[1:N]}} \alpha_i \mathbb{A}_i^T \mathbb{A}_i \leq \rho^2 I.$$

Also, for all $x \in \mathbb{R}^n$, it holds

$$\min_{i \in \mathscr{I}^{[1:N]}} (x^T \mathbb{A}_i^T \mathbb{A}_i x) \leq \rho^2 x^T x. \tag{17}$$

We can now describe the stabilizing control strategy. The controller does not necessarily select at each time step $k \in \mathbb{N}$ which input should be applied. This is done only at given instant $\{k_p\}_{p \in \mathbb{N}}$ with $k_0 = 0$, and $k_p < k_{p+1} \leq k_p + N$, for all $p \in \mathbb{N}$. At time k_p, the controller selects the sequence of inputs to be applied up to step $k_{p+1} - 1$. The instant k_{p+1} is also determined by the controller at time k_p. More precisely, the controller acts as follows for all $p \in \mathbb{N}$, let

$$i_p = \arg \min_{i \in \mathscr{I}^{[1:N]}} (x_{k_p}^T \mathbb{A}_i^T \mathbb{A}_i x_{k_p}). \tag{18}$$

Then, the next instant k_{p+1} is given by

$$k_{p+1} = k_p + l(i_p), \tag{19}$$

with $l(i_p)$ length of i_p, and the controller applies the sequence of inputs

$$\sigma_{k_p + j - 1} = i_{p,j}, \quad \forall j \in \{1, \ldots, l(i_p)\}. \tag{20}$$

Theorem 11: Let us assume that (16) holds, and consider the control strategy given by (18), (19), (20). Then, for all $x_0 \in \mathbb{R}^n$, for all $k \in \mathbb{N}$,

$$\|x_k\| \leq \rho^{k/N-1} L^{N-1} \|x_0\| \tag{21}$$

where $L \geq \|A_i\|$, for all $i \in \mathscr{I}$. Then, the controlled switched system is globally asymptotically stable.

Proof: Using the proposed control strategy, we have $x_{k_{p+1}} = \mathbb{A}_{i_p} x_{k_p}$ for all $p \in \mathbb{N}$. Then, it follows from (17) and (18) that $\|x_{k_{p+1}}\| \leq \rho \|x_{k_p}\|$ and thus for all $p \in \mathbb{N}$, $\|x_{k_p}\| \leq \rho^p \|x_{k_0}\|$. Moreover, since $k_{p+1} - k_p \leq N$ and $L > 1$ from Assumption 1, we have for all $p \in \mathbb{N}$:

$$\|x_k\| \leq L^{k-k_p} \|x_{k_p}\| \leq \rho^p L^{N-1} \|x_0\|, \ \forall k \in \{k_p, \ldots, k_{p+1} - 1\}. \tag{22}$$

Now let $k \in \mathbb{N}$, and let $p \in \mathbb{N}$ be such that $k \in \{k_p, \ldots, k_{p+1} - 1\}$ then necessarily $p \geq \lfloor k/N \rfloor \geq k/N - 1$. Then (21) follows from (22). ■

One particular case in which the LMI condition is guaranteed to have a solution follows.

Corollary 12: If there exist $N \in \mathbb{N}$ and $i_1, i_2 \in \mathscr{I}^{[1:N]}$ such that $\mathscr{B} \subseteq \mathrm{int}(\mathscr{B}_{i_1} \cup \mathscr{B}_{i_2})$ then there is $\alpha \in [0,1]$ such that

$$\alpha \mathbb{A}_{i_1}^T \mathbb{A}_{i_1} + (1-\alpha) \mathbb{A}_{i_2}^T \mathbb{A}_{i_2} < I.$$

Proof: The property is a consequence of Lemma 6. ■

The condition presented in Theorem 10 is just sufficient unless there exists, among the $\mathscr{E}_i$, two ellipsoids containing $\mathscr{B}$ in their union, see Corollary 12. This is proved by the following counter-example.

Example 13: The aim of this illustrative example is to show a case for which the inclusion condition (3) is satisfied with $N = 1$, but there is not a finite value of $\hat{N} \in \mathbb{N}$ for which the condition (16). Consider the three modes given by the matrices

$$A_1 = AR(0), \quad A_2 = AR\left(\frac{2\pi}{3}\right), \quad A_3 = AR\left(\frac{-2\pi}{3}\right),$$

where

$$A = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}, \quad R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}, \tag{23}$$

with $a = 0.6$. Set $\Omega = \mathscr{B}$. By geometric inspection of Figure 1 (left), condition (3) holds at the first step, i.e. for $N = 1$. On the other hand, A_i are such that $\det(A_i^T A_i) = \alpha^2 \alpha^{-2} = 1$ and $\mathrm{trace}(A_i^T A_i) = \alpha^2 + \alpha^{-2} = 3.1378$ while the determinant

and trace of the matrix defining $\mathscr{B}$ are 1 and 2, respectively. Notice that $\alpha^2+\alpha^{-2}>2$ for every α different from 1 or -1 and $\alpha^2+\alpha^{-2}=2$ otherwise.

For every N and every $\mathscr{B}_i$ with $i\in\mathscr{I}^{[1:N]}$, the related $\mathbb{A}_i$ is such that $\det(\mathbb{A}_i^T\mathbb{A}_i)=1$ and $\mathrm{trace}(\mathbb{A}_i^T\mathbb{A}_i)\geq 2$. Notice that, for all the matrices $Q>0$ in $\mathbb{R}^{2\times 2}$ such that $\det(Q)=1$, then $\mathrm{trace}(Q)\geq 2$ and $\mathrm{trace}(Q)=2$ if and only if $Q=I$, since the determinant is the product of the eigenvalues and the trace its sum. Thus, for every subset of the ellipsoids $\mathscr{B}_i$, determined by a subset of indices $K\subseteq\mathscr{I}^{[1:N]}$, we have that

$$\sum_{i\in K}\alpha_i\mathbb{A}_i^T\mathbb{A}_i<I,$$

cannot hold, since either $\mathrm{trace}(\mathbb{A}_i^T\mathbb{A}_i)>2$ or $\mathbb{A}_i^T\mathbb{A}_i=I$.

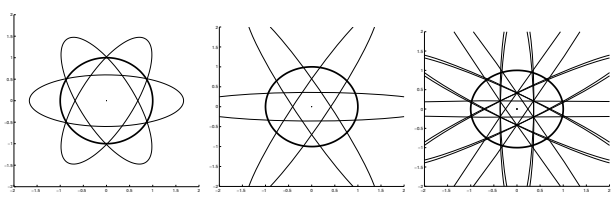

Fig. 1. $\mathscr{B}_1$ $\mathscr{B}_2$ $\mathscr{B}_3$ and Ω (left). $\mathscr{B}_{1i}$, with $i\in\mathbb{N}_3$, and Ω (center). $\mathscr{B}_{1ij}$, with $i,j\in\mathbb{N}_3$, and Ω (right).

Thus the LMI condition (16) is sufficient but not necessary.

In what follows we provide a relation with the generalized Lyapunov-Metzler condition (15). Recall that the Lyapunov-Metzler condition regards nonconvex sets and sequences of length one (possibly of extended systems) whereas the LMI one concerns quadratic Lyapunov functions and switching control sequences. It can be proved that the LMI sufficient condition (16) holds if and only if the generalized Lyapunov-Metzler one can be satisfied.

Theorem 14: There exist $M\in\mathbb{N}$, $P_i>0$, with $i\in\mathscr{I}^{[1:M]}$, and $\pi\in\mathscr{M}_{\bar{M}}$ such that (15) holds if and only if there exists $N\in\mathbb{N}$ and $\alpha\in\mathbb{R}^{\bar{N}}$ such that $\alpha\geq 0$, $\sum_{i\in\mathscr{I}^{[1:N]}}\alpha_i\geq\ 1$ and (16) holds.

Proof: First we prove that satisfaction of (15) implies the existence of N such that (16) holds. Suppose that for appropriate P_i, with $i\in\mathscr{I}^{[1:M]}$, and $\pi\in\mathscr{M}_{\bar{M}}$, (15) holds or, equivalently, that there exists $\lambda\in[0,1)$ such that

$$\mathbb{A}_m^T\Big(\sum_{j\in\mathscr{I}^{[1:M]}}\pi_{jm}P_j\Big)\mathbb{A}_m\leq\lambda P_m,\quad\forall m\in\mathscr{I}^{[1:M]}.$$

Let us choose an arbitrary $m\in\mathscr{I}^{[1:M]}$. We have

$$\mathbb{A}_m^T\Big(\sum_{j\in\mathscr{I}^{[1:M]}}\pi_{jm}\mathbb{A}_j^T\Big(\sum_{k\in[1:M]}\pi_{kj}P_k\Big)\mathbb{A}_j\Big)\mathbb{A}_m$$
$$\leq\lambda\mathbb{A}_m^T\Big(\sum_{j\in\mathscr{I}^{[1:M]}}\pi_{jm}P_j\Big)\mathbb{A}_m\leq\lambda^2P_m,$$

which is equivalent to

$$\mathbb{A}_m^T\Big(\sum_{j\in\mathscr{I}^{[1:M]}}\pi_{jm}\sum_{k\in\mathscr{I}^{[1:M]}}\pi_{kj}\Big(\mathbb{A}_j^TP_k\mathbb{A}_j\Big)\Big)\mathbb{A}_m\leq\lambda^2P_m.$$

From $\lambda<1$ and Assumption 1, we have that, for every $m\in\mathscr{I}^{[1:M]}$ there exists $s(m)=s\in\mathbb{N}$ such that $\lambda^sP_m\leq\mathbb{A}_m^T\mathbb{A}_m$ and then

$$\mathbb{A}_m^T\Big(\sum_{i_1\in\mathscr{I}^{[1:M]}}\pi_{i_1m}\ldots\sum_{i_s\in[1:M]}\pi_{i_si_{s-1}}\Big(\mathbb{A}_i^TP_{i_s}\mathbb{A}_i\Big)\Big)A_m<\lambda^sP_m\leq\mathbb{A}_m^T\mathbb{A}_m,$$

with $i=(i_1,\ldots,i_{s-1})\in\mathscr{I}^{[1:M(s-1)]}$. Since there is no loss of generality, assume that $I\leq P_k$ for all $k\in\mathscr{I}^{[1:M]}$ and then

$$\mathbb{A}_m^T\Big(\sum_{i_1\in\mathscr{I}^{[1:M]}}\pi_{i_1m}\ldots\sum_{i_{s-1}\in\mathscr{I}^{[1:M]}}\pi_{i_{s-1}i_{s-2}}\Big(\mathbb{A}_i^T\mathbb{A}_i\Big)\Big)\mathbb{A}_m$$
$$\leq\mathbb{A}_m^T\Big(\sum_{i_1\in\mathscr{I}^{[1:M]}}\pi_{i_1m}\ldots\sum_{i_s\in\mathscr{I}^{[1:M]}}\pi_{i_si_{s-1}}\Big(\mathbb{A}_i^TP_{i_s}\mathbb{A}_i\Big)\Big)\mathbb{A}_m<\mathbb{A}_m^T\mathbb{A}_m,$$

that implies

$$\sum_{i_1\in\mathscr{I}^{[1:M]}}\pi_{i_1m}\ldots\sum_{i_{s-1}\in\mathscr{I}^{[1:M]}}\pi_{i_{s-1}i_{s-2}}\Big(\mathbb{A}_i^T\mathbb{A}_i\Big)<I,\qquad(24)$$

from Assumption 1. Denoting for every $i\in\mathscr{I}^{[1:M(s-1)]}$ the parameter $\alpha_i=\pi_{i_1m}\pi_{i_2i_1}\ldots\pi_{i_{s-1}i_{s-2}}$ it can be proved that $0\leq\alpha_i\leq 1$ and $\sum_{i\in\mathscr{I}^{[1:M(s-1)]}}\alpha_i=1$ and (24) is equivalent to (16).

We prove now that the satisfaction of the LMI condition (16) with appropriate N and α implies that the generalized Lyapunov-Metzler one is satisfied for with adequate M. From (16) one have

$$\mathbb{A}_i^T\Big(\sum_{j\in\mathscr{I}^{[1:N]}}\alpha_j\mathbb{A}_j^T\mathbb{A}_j\Big)\mathbb{A}_i<\mathbb{A}_i^T\mathbb{A}_i,\quad\forall i\in\mathscr{I}^{[1:N]},$$

which is equivalent to (15) with $P_j=\mathbb{A}_j^T\mathbb{A}_j$ and $\pi_{ji}=\alpha_j$, for all $i,j\in\mathscr{I}^{[1:N]}$ and $M=N$. ■

Remark 15: Notice that, in the first part of proof of Theorem 14, there is a dependence on the index $m\in\mathscr{I}^{[1:M]}$. In reality, for every other possible index, the stabilizability result would be same. The only difference would be the length s, that depends on m, and the values of the parameters α_i, that should be written as dependent on m.

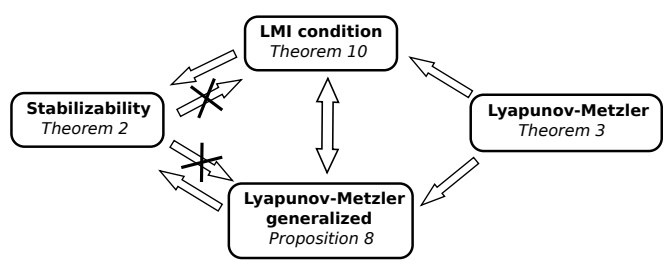

Fig. 2. Implications diagram of stabilizability conditions.

The implications between the stabilizability conditions are summarized in the diagram in Figure 2. It is remarkable that the LMI condition concerns a convex problem, thus more affordable than the BMI one related to Lyapunov-Metzler inequalities, but it is less conservative.

V. Numerical Example

Consider the system (1) with $q = 2$, $n = 2$, $x_0 = (-3,3)^T$ and

$$A_1 = 1.01R\left(\frac{\pi}{5}\right), \qquad A_2 = \begin{bmatrix} -0.6 & -2 \\ 0 & -1.2 \end{bmatrix}.$$

Matrices A_1 and A_2 are not Schur. With $N = 1,\ldots,6$ the LMI (16) is unfeasible, but there is a solution for $N = 7$. By applying the min-switching strategy (18)-(20), one obtains a stabilizing switching law, which concatenates elements of $\mathscr{I}^{[1,7]}$, respectively of lengths $\{7,6,5,7,7,\ldots\}$. The time-varying length of the switching subsequences is a consequence of the state dependency of the min-switching strategy. The switching law is depicted on Figure 3 (left). Each resulting subsequence of $\mathscr{I}^{[1,7]}$ is alternatively depicted by $\circ$ or $\star$ to emphasize its length. The induced state trajectory is drawn in Figure 3 (right) and converges to the origin.

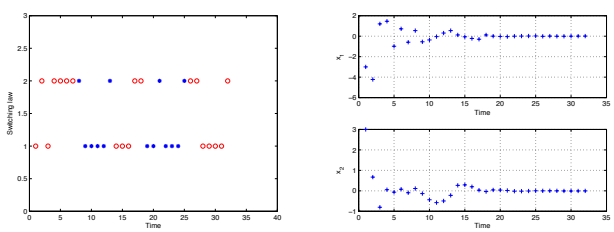

Fig. 3. Switching law (left). State trajectory in time (right).

The Lyapunov function associated with the min-switching strategy (18)-(20) is shown in Figure 4. As for the switching law, the Lyapunov function values issued from the same subsequence are alternatively depicted by $\circ$ or $\star$. In addition, the last Lyapunov function value of each subsequence is given by a big $\square$, leading to a decreasing function as expected. Nevertheless, the Lyapunov function may increase during the first samples of each subsequence.

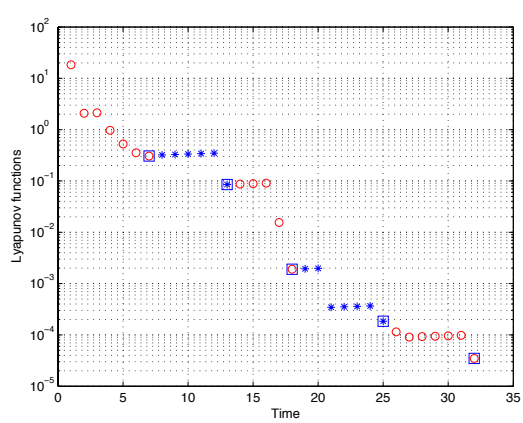

Fig. 4. Lyapunov function in time.

The method in paper [2] leads to satisfy the stop condition (3) after 5 iterations, as depicted in Figure 5 and 6 (left). The resulting Lyapunov function is depicted in Figure 6 (right), which does not increase and strictly decreases every each 5 steps. This is a main difference with the previous method. Of course the stabilizing switching law given in this framework and depicted in Figure 6 (right) is different.

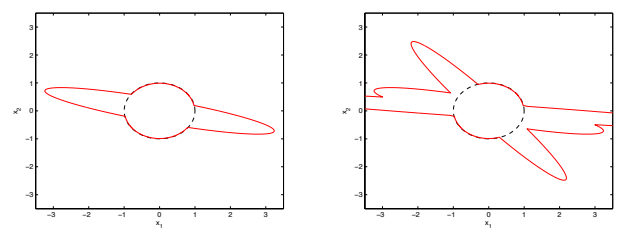

Fig. 5. Step 1 (left). Step 2 (right).

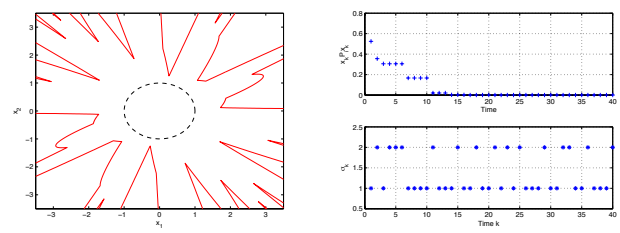

Fig. 6. Step 5 (left). Lyapunov function and switching law (right).

In order to strengthen the value of our approach, it should be noticed that no solution has been numerically found for the Lyapunov-Metzler inequalities (4). That has been numerically checked by fixing the Metzler matrix (fully characterized only by two parameters and a grid of the parameters with step of 0.01) which yields the Lyapunov-Metzler inequalities to LMI form.

VI. Conclusion

In this paper we provide a characterization of the relations and implications of different conditions, new and known ones, for stabilizability of switched linear systems. A comparison in terms of conservatism and complexity is presented. Extensions to novel conditions and new computational methods for testing stabilizability are the objectives of our current and future work on this topic.

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