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# A HALF-SPACE THEOREM FOR GRAPHS OF CONSTANT MEAN CURVATURE $0 < H < \frac{1}{2}$ IN $\mathbb{H}^2 \times \mathbb{R}$

L. MAZET AND G. A. WANDERLEY

ABSTRACT. We study a half-space problem related to graphs in  $\mathbb{H}^2 \times \mathbb{R}$ , where  $\mathbb{H}^2$  is the hyperbolic plane, having constant mean curvature  $H$  defined over unbounded domains in  $\mathbb{H}^2$ .

## 1. INTRODUCTION

The half-space theorem by Hoffman and Meeks [10] states that if a properly immersed minimal surface  $S$  in  $\mathbb{R}^3$  lies on one side of some plane  $P$ , then  $S$  is a plane parallel to  $P$ . As a consequence, they proved the strong half-space theorem which says that two properly immersed minimal surfaces in  $\mathbb{R}^3$  that do not intersect must be parallel planes.

These theorems have been generalized to some other ambient simply connected homogeneous manifolds with dimension 3. For example, we have half-space theorems with respect to horospheres in  $\mathbb{H}^3$  [17], vertical minimal planes in  $Nil_3$  and  $Sol_3$  [3],[4] and entire minimal graph in  $Nil_3$  [4]. It is known that there is no half-space theorem for horizontal slices in  $\mathbb{H}^2 \times \mathbb{R}$ , since rotational minimal surfaces (catenoids) are contained in a slab [14]-[15], but one has half-space theorems for constant mean curvature  $\frac{1}{2}$  surfaces in  $\mathbb{H}^2 \times \mathbb{R}$  [9].

In [11], the first author proved a general half-space theorem for constant mean curvature surfaces. Under certain hypothesis, he proved that in a Riemannian 3-manifold of bounded geometry, a constant mean curvature  $H$  surface on one side of a parabolic constant mean curvature  $H$  surface  $\Sigma$  is an equidistant surface to  $\Sigma$ .

In Euclidian spaces of dimension higher than 4, there is no strong half-space theorem, since there exist rotational proper minimal hypersurfaces contained in a slab.

In [13], Menezes proves a half-space theorem for some complete vertical minimal graphs, more precisely, she looks at some particular graphs  $\Sigma \subset M \times \mathbb{R}$  over a domain  $D \subset M$ , where  $M$  is a Hadamard surface with bounded curvature, these graphs are called ideal Scherk graphs and their existence was proved by Collin and Rosenberg in [2] for  $\mathbb{H}^2$  and by Galvez and Rosenberg in [5] in the general case.

**Theorem 1.** *Let  $M$  denote a Hadamard surface with bounded curvature and let  $\Sigma = \text{Graph}(u)$  be an ideal Scherk graph over an admissible polygonal domain  $D \subset M$ . If  $S$  is a properly immersed minimal surface contained in  $D \times \mathbb{R}$  and disjoint from  $\Sigma$ , then  $S$  is a translate of  $\Sigma$ .*

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In this paper, we are interested in the case where the graph has constant mean curvature. More precisely, we consider graphs over  $\mathbb{H}^2$  with constant mean curvature  $0 < H < \frac{1}{2}$ . In that case, we prove a result similar to the one of Menezes. We notice that the value  $H = \frac{1}{2}$  is critical in this setting (see [12, 16] for the  $H = \frac{1}{2}$  case).

The graphs that we will work with are graphs of functions  $u$  defined in a domain  $D \subset \mathbb{H}^2$  whose boundary  $\partial D$  is composed of complete arcs  $\{A_i\}$  and  $\{B_j\}$ , such that the curvatures of the arcs with respect to the domain are  $\kappa(A_i) = 2H$  and  $\kappa(B_j) = -2H$ . These graphs will have constant mean curvature and  $u$  will assume the value  $+\infty$  on each  $A_i$  and  $-\infty$  on each  $B_j$ . These domains  $D$  will be called *Scherk type domains* and the functions  $u$  *Scherk type solutions*. The existence of these graphs is assured by A. Folha and S. Melo in [6]. There, the authors give necessary and sufficient conditions on the geometry of the domain  $D$  to prove the existence of such a solution. In this context, we prove the following result.

**Theorem 2.** *Let  $D \subset \mathbb{H}^2$  be a Scherk type domain and  $u$  be a Scherk type solution over  $D$  (for some value  $0 < H < \frac{1}{2}$ ). Denote by  $\Sigma = \text{Graph}(u)$ . If  $S$  is a properly immersed CMC  $H$  surface contained in  $D \times \mathbb{R}$  and above  $\Sigma$ , then  $S$  is a vertical translate of  $\Sigma$ .*

The original idea of Hoffman and Meeks is to use the 1-parameter family of catenoids as *a priori* barriers to control minimal surfaces on one side of a plane (here *a priori* means that the choice of catenoids is independent of the particular minimal surface you want to control). In more general situations, it is not easy to construct such a continuous family of barriers so some authors use a discrete family (see for example [4, 18]). Menezes works also with such a discrete family. In our case, it does not seem possible to construct such a family in an easy way. Our approach is based on the existence of only one barrier whose construction depends on the particular surface  $S$ .

This paper is organized as follows. In Section 2, we will give a brief presentation of the Scherk type graphs and the result of Folha and Melo. Section 3 contains the proof of Theorem 2, so one of the main step is the existence of the barriers which uses the Perron method. We also prove a uniqueness result for the constant mean curvature equation.

## 2. CONSTANT MEAN CURVATURE SCHERK TYPE GRAPHS

In this section we present the theorem by A. Folha and S. Melo in [6] that assure the existence of constant mean curvature graphs which take the boundary value  $+\infty$  on certain arcs  $A_i$  and  $-\infty$  on arcs  $B_j$ . All along this section  $H$  will be a real constant in  $(0, \frac{1}{2})$ .

First, let us fix some notations. Let  $\mathbb{H}^2$  be the hyperbolic plane, and  $\mathbb{H}^2 \times \mathbb{R}$  be endowed with the product metric. Let  $D$  be a simply connected domain in  $\mathbb{H}^2$  and  $u : D \rightarrow \mathbb{R}$  a function. Denote by

$$\Sigma = \text{Graph}(u) = \{(x, u(x)), x \in D\}.$$

The upward unit normal to  $\Sigma$  is given by

$$(1) \quad N = \frac{1}{W}(\partial_t - \nabla u),$$

where

$$(2) \quad W = \sqrt{1 + |\nabla u|^2}.$$

The graph  $\Sigma$  has mean curvature  $H$  if  $u$  satisfies the equation

$$(3) \quad \mathcal{L}u := \operatorname{div} \frac{\nabla u}{W} - 2H = 0,$$

where the divergence and the gradient are taken with respect to the metric on  $\mathbb{H}^2$ . Let us now give some definitions.

**Definition 1.** We say that the boundary of a domain  $D$  in  $\mathbb{H}^2$  is a *2H-polygon* if its boundary is made of a finite number of complete arcs with constant curvature  $2H$  and the cluster points of  $D$  in  $\partial_\infty \mathbb{H}^2$  are the end-points of these arcs. The arcs are called the edges of  $D$  and the cluster points are the vertices of  $D$ .

We notice that a complete curve in  $\mathbb{H}^2$  with constant curvature  $2H$  is proper.

If  $\Omega$  is a domain whose boundary is a *2H-polygon*, we will denote by  $A_i$  (resp.  $B_i$ ) the arcs of the boundary whose curvature is  $2H$  (resp.  $-2H$ ) with respect to the inward pointing unit normal.

**Definition 2.** We say that a domain  $D$  in  $\mathbb{H}^2$  is a *Scherk type domain* if its boundary is a *2H-polygon* and if each vertex is the end point of one arc  $A_i$  and one arc  $B_j$ .

Such a domain  $D$  is drawn in Figure 1.

**Definition 3.** Let  $\Omega$  be a Scherk type domain. We say that  $P$  is an *admissible inscribed polygon* if  $P \subset \Omega$  is a domain whose boundary is a *2H-polygon* and its vertices are among the ones of  $\Omega$ .

Let  $D$  be a Scherk type domain, in [6], Folha and Melo study the following Dirichlet problem

$$(4) \quad \begin{cases} \mathcal{L}(u) = 0 & \text{in } D \\ u = +\infty & \text{on } A_i \\ u = -\infty & \text{on } B_i \end{cases}$$

In order to state the result of Folha and Melo, let us introduce some notations. Let  $P$  be an admissible inscribed polygon in  $D$  and let  $\{d_i\}_{i \in I}$  denote the vertices of  $P$ . Consider the set

$$\Theta = \{(\mathcal{H}_i)_{i \in I} \mid \mathcal{H}_i \text{ is a horodisk at } d_i \text{ and } \mathcal{H}_i \cap \mathcal{H}_j = \emptyset \text{ if } i \neq j\}.$$

we notice that, by choosing sufficiently small horodisks,  $\Theta$  is not empty.

Let  $(\mathcal{H}_i)_{i \in I}$  be in  $\Theta$  such that the following is true : each arc  $A_i$  and  $B_j$  meets exactly two of these horodisks. Denote by  $\tilde{A}_i$  the compact arc of  $A_i$  which is the part of  $A_i$  outside these two horodisks. Let  $|A_i|$  denote the length of  $\tilde{A}_i$ . We introduce the same notations of the  $B_j$ . For each arc  $\eta_j \in \partial P$ , we also define  $\tilde{\eta}_j$  and  $|\eta_j|$  in the same way.

We define

$$\alpha(\partial P) = \sum_{A_i \in \partial P} |A_i|, \quad \beta(\partial P) = \sum_{B_i \in \partial P} |B_i| \quad \text{and} \quad l(\partial P) = \sum_j |\eta_j|,$$

where  $\partial P = \cup_j \eta_j$ . We remark that a Scherk type domain has finite area. So we can introduce  $\mathcal{A}(D)$  the area of  $D$  and  $\mathcal{A}(P)$  the area of  $P$ .

With these definitions we can state the main theorem of [6].

**Theorem 3.** *Let  $D$  be a Scherk type domain. Then there exists a solution  $u$  for the Dirichlet problem (4) in  $D$  if and only if for some choice of the horodisks (in  $\Theta$ ) at the vertices,*

$$\alpha(\partial D) = \beta(\partial D) + 2HA(D)$$

*and for any admissible inscribed polygons  $P \neq D$ ,*

$$2\alpha(\partial P) < l(\partial P) + 2HA(\Omega) \quad \text{and} \quad 2\beta(\partial P) < l(\partial P) - 2HA(\Omega).$$

It could seem that the conditions depend on the choice of the horodisks in  $\Theta$ , actually they are independent of that choice if the horodisks are small enough. The details and the proof of this theorem can be found in [6].

### 3. THE MAIN RESULT

In this section we will prove the following result

**Theorem 2.** *Let  $D \subset \mathbb{H}^2$  be a Scherk type domain and  $u$  be a Scherk type solution over  $D$  (for some value  $0 < H < \frac{1}{2}$ ). Denote by  $\Sigma = \text{Graph}(u)$ . If  $S$  is a properly immersed CMC  $H$  surface contained in  $D \times \mathbb{R}$  and above  $\Sigma$ , then  $S$  is a vertical translate of  $\Sigma$ .*

The proof of the theorem consists in constructing barriers to control the surface  $S$ . Before starting the proof, let us give some notations and preliminary results that we will use.

So we fix a value of  $H \in (0, \frac{1}{2})$ , a Scherk type domain  $D$  and a Scherk type solution  $u$ . Let  $y \in D$  and  $B_y$  and  $B'_y$  be open balls centered in  $y$  such that  $B_y \subsetneq B'_y \subsetneq D$  (see Figure 1). The following result consists in constructing a first barrier to control  $S$ .

**Lemma 1.** *There exists a constant  $\epsilon > 0$  such that for all  $t \in [0, \epsilon)$  there exists  $v \in C^2(B'_y \setminus B_y) \cap C^0(\partial(B'_y \setminus B_y))$  such that  $v$  solves (3) and  $v = u$  in  $\partial B'_y$  and  $v = u + t$  over  $\partial B_y$ .*

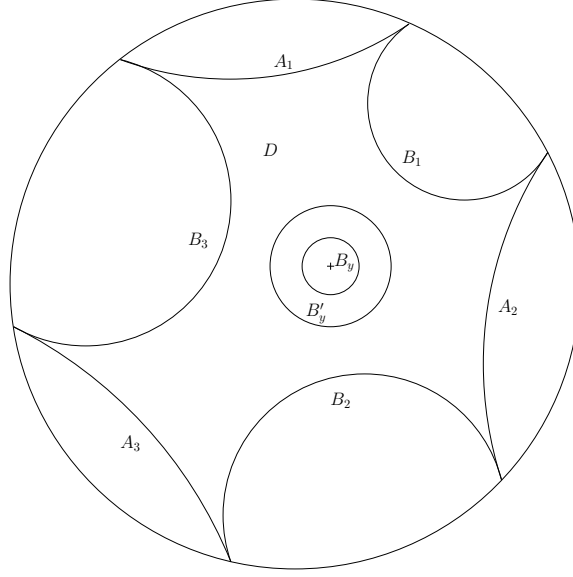
*Proof.* Consider the operator  $F : C^{2,\alpha}(\overline{B'_y \setminus B_y}) \times C^{2,\alpha}(\partial(B'_y \setminus B_y)) \longrightarrow C^{0,\alpha}(\overline{B'_y \setminus B_y}) \times C^{2,\alpha}(\partial(B'_y \setminus B_y))$  given by

$$F(v, \phi) = (\mathcal{L}v, v - \phi).$$

Observe that

$$F(u, u) = 0.$$

Moreover, consider the operator


 FIGURE 1. The Scherk type domain  $D$  and the balls  $B_y$  and  $B'_y$ 

$$T := D_1 F(u, u) : C^{2,\alpha}(\overline{B'_y \setminus B_y}) \longrightarrow C^{0,\alpha}(\overline{B'_y \setminus B_y}) \times C^{2,\alpha}(\partial(B'_y \setminus B_y))$$

$$h \longmapsto \lim_{t \rightarrow 0} \frac{F(u + th, u) - F(u, u)}{t}$$

We have that

$$T(h) = \left( \operatorname{Div} \left( \frac{\nabla h - \frac{\nabla u}{W} \langle \frac{\nabla u}{W}, \nabla h \rangle}{W} \right), h \right).$$

Observe that  $T$  is a linear operator, of the form  $T = (T_1, T_2)$  where  $T_1$  is an elliptic operator of the form

$$T_1(v) = a^{ij}(x) D_{ij} v + b^i(x) D_i v; \quad a^{ij} = a^{ji}.$$

Moreover, since  $|\nabla u| \leq C$ , we have that  $\frac{|\nabla u|}{W} \leq C' < 1$ , this implies that  $T_1$  is uniformly elliptic. We also have that the coefficients of  $T_1$  belong to  $C^{0,\alpha}(\overline{B'_y \setminus B_y})$ . It follows by Theorem 6.14 in [7] that if  $g \in C^{0,\alpha}(\overline{B'_y \setminus B_y})$  and  $\phi \in C^{2,\alpha}(\partial(B'_y \setminus B_y))$ , then there exists a unique  $w \in C^{2,\alpha}(\overline{B'_y \setminus B_y})$  such that  $T_1(w) = g$  in  $B'_y \setminus B_y$  and  $w = \phi$  in  $\partial(B'_y \setminus B_y)$ . Then,  $T_1$  is invertible.

We conclude that  $T$  is invertible. It follows by the implicit function theorem that for all  $\phi$  close to  $u$  there exists a solution of  $\mathcal{L}v = 0$  in  $B'_y \setminus \overline{B_y}$  with  $v = \phi$  in  $\partial(B'_y \setminus B_y)$ . In other words, it exists  $\epsilon > 0$  such that for all  $t \in [0, \epsilon)$  there exists  $v$  such that  $v$  solves (3) and  $v = u$  over  $\partial B'_y$  and  $v = u + t$  over  $\partial B_y$ .  $\square$

Let  $S$  be as in Theorem 2. Give  $p \in D$ , define  $g(p)$  by

$$g(p) = \inf_{t \in \mathbb{R}} \{(p, t) \in S\} \in \mathbb{R} \cup \{+\infty\}.$$

Observe that  $g$  is a lower semicontinuous functions and  $g \geq u$ . From now on, we will assume that  $g > u$  (the case where  $g(p) = u(p)$  for a point  $p \in D$  will be considered in the proof of the theorem). Then for  $\epsilon' > 0$  sufficiently small, we have that

$$(5) \quad g > u + \epsilon' \quad \text{on } \partial B_y.$$

Now, let  $\epsilon$  be as in Lemma 1, fix  $\epsilon' < \epsilon$  where  $\epsilon'$  satisfies (5) and  $v$  given by Lemma 1 associated to  $\epsilon'$ . We will construct a second barrier to control the surface  $S$ . More precisely, we will prove the existence of a function  $\beta$  that satisfies

$$(6) \quad \mathcal{L}\beta = 0 \quad \text{in } D \setminus \overline{B_y},$$

$$(7) \quad \beta = u + \epsilon' \quad \text{in } \partial B_y.$$

**Proposition 1.** *There is a solution  $\beta \in C^2(D \setminus \overline{B_y}) \cap C^0(D \setminus B_y)$  for the Dirichlet problem (6)-(7) such that  $\max(u, v) \leq \beta \leq \min(u + \epsilon', g)$ .*

*Proof.* To prove this proposition we will use the Perron method. Let us recall the framework of this method. A function  $w \in C^0(D \setminus B_y)$  is called a *subsolution* for  $\mathcal{L}$  if for any compact subdomain  $U \subset D \setminus B_y$  and any solution  $h$  of (3) with  $w \leq h$  on the boundary of  $U$ , we have  $w \leq h$  on  $U$ .

First, observe that  $u$  is a subsolution for (3). Moreover, if  $w'$  and  $w$  are subsolutions, the continuous function  $\max(w', w)$  is also a subsolution.

Let  $\Delta \subset D \setminus B_y$  be a geodesic disk of small radius such that  $\kappa(\partial\Delta) \geq 2H$ . Theorem 3.2 in [8] implies that the Dirichlet problem for Equation (3) can be solved in  $\Delta$ . So, for any such disk  $\Delta$  and subsolution  $w$ , we can define a continuous function  $M_\Delta(w)$  as

$$M_\Delta(w)(x) = \begin{cases} w(x) & \text{if } x \in D \setminus \Delta \\ \nu(x) & \text{if } x \in \Delta \end{cases}$$

where  $\nu$  is the solution of  $\mathcal{L}\nu = 0$  in  $\Delta$ , with  $\nu = w$  in  $\partial\Delta$ .

Also, define  $u_+ = \min(u + \epsilon', g)$  and  $u_- = \max(u, v)$ . Denote by  $\Gamma$  the set of all subsolutions  $w$  such that  $w \leq u_+$  on  $D \setminus B_y$ ,

**Claim 1.** *If  $w \in \Gamma$ , then  $M_\Delta(w) \in \Gamma$ .*

*Proof of Claim 1.* First we have to prove that  $M_\Delta(w)$  is a subsolution. So, take a arbitrary compact subdomain  $U \subset D \setminus B_y$ , and let  $h$  a solution of (3) in  $U$  with  $M_\Delta(w) \leq h$  on  $\partial U$ . Since  $w = M_\Delta(w)$  in  $U \setminus \Delta$  we have that  $M_\Delta(w) \leq h$  in  $U \setminus \Delta$ .

Moreover,  $M_\Delta(w)$  is a solution of (3) in  $\Delta$ . Then, by the maximum principle, we have that  $M_\Delta(w) \leq h$  in  $U \cap \Delta$ . So,  $M_\Delta(w) \leq h$  in  $U$ . Since  $U$  is arbitrary, it follows that  $M_\Delta(w)$  is a subsolution in  $D \setminus B_y$ .

Now we have to prove that  $M_\Delta(w) \leq u_+$ . Observe that in  $D \setminus \Delta$ ,  $M_\Delta(w) = w$ , since  $w \in \Gamma$ , then  $w \leq u_+$ , and so,  $M_\Delta(w) \leq u_+$  in  $D \setminus \Delta$ .

On the other hand,  $M_\Delta(w) = \nu$  in  $\Delta$ , where  $\nu$  is a solution of  $\mathcal{L}(\nu) = 0$  in  $\Delta$  and  $\nu = w$  on  $\partial\Delta$ . Thus  $\nu \leq u_+ = \min(u + \epsilon', g)$  on  $\partial\Delta$ . It follows by the maximum principle that  $\nu \leq u + \epsilon'$  in  $\Delta$ . So, we have to prove that  $\nu \leq g$  in  $\Delta$ .

Suppose that there exists  $q \in \Delta$  such that  $(\nu - g)(q) < 0$ . Then, there exists  $p \in \Delta$  such that  $(\nu - g)(p) = \min(\nu - g) = C < 0$ .

Now, observe that the graph of  $g$  in  $\Delta$  is a piece of the surface  $S$ , let us denote it by  $S_g$ . Since  $g \geq \nu + C$ , then the graph  $\Sigma_{\nu+C}$  of  $\nu + C$  is a CMC  $2H$  surface which is below the surface  $S_g$ . Moreover,  $(p, g(p))$  is a point of contact of  $\Sigma_{\nu+C}$  and  $S_g$ , and by the maximum principle,  $S_g = \Sigma_{\nu+C}$ . It follows that  $g = \nu + C$  in  $\Delta$ , since  $\nu \leq g$  on  $\partial\Delta$  then  $C \geq 0$ , and this contradicts  $C < 0$ . Then  $\nu \leq g$  in  $\Delta$ .  $\square$

We define our solution by the following formula

$$\beta(q) = \sup_{w \in \Gamma} w(q).$$

Observe that  $u_-$  is a subsolution, since  $u$  and  $v$  are subsolutions. Also,  $u \leq u_+ = \min(u + \epsilon', g)$ . Moreover, in the proof of Claim 1 we see that  $v \leq u_+$ . Then  $u_- = \max(u, v) \leq u_+$ . We conclude that  $u_- \in \Gamma$ , then  $\Gamma$  is non empty, and  $u_+$  is an upper bound for any  $w$  in  $\Gamma$ , thus  $\beta$  is well defined.

The method of Perron claims that  $\beta$  is a solution of Equation (3).

**Claim 2.** *The function  $\beta$  is a solution of (3) in  $D \setminus \overline{B_y}$ .*

*Proof of Claim 2.* Let  $p \in D \setminus B_y$  and  $\Delta \subset D \setminus B_y$  a geodesic disk of small radius centered at  $p$  as above. By definition of  $\beta$  there exists a sequence of subsolutions  $(w_n)$  such that  $w_n(p) \rightarrow \beta(p)$ . Then, consider the sequence of subsolutions  $M_\Delta(w_n)$ , we have that  $M_\Delta(w_n)(p) \rightarrow \beta(p)$ . Also, we have that  $M_\Delta(w_n)$  is a bounded sequence of solutions of (3) in  $\Delta$ , so, by considering a subsequence if necessary, we can assume that it converges to a solution  $\bar{w}$  on  $\Delta$  with  $\beta \geq \bar{w}$  and  $\bar{w}(p) = \beta(p)$ . Let us prove that  $\beta = \bar{w}$  on  $\Delta$ , then  $\beta$  will be a solution of (3).

We have that  $\beta \geq \bar{w}$ . Suppose that there is a point  $q \in \Delta$  where  $\beta(q) > \bar{w}(q)$ . So, there is a subsolution  $s$  such that  $s(q) > \bar{w}(q)$ . Now consider the sequence of subsolutions  $M_\Delta(\max(s, w_n))$ . We have that  $M_\Delta(\max(s, w_n))$  is a sequence of solutions of (3) in  $\Delta$ . Thus, considering a subsequence, it converges to a solution  $\bar{s} \geq \bar{w}$  of (3) in  $\Delta$ .

So, we have  $\bar{w}$  and  $\bar{s}$  solutions of (3) in  $\Delta$ , with  $\bar{w}(p) = \beta(p) = \bar{s}(p)$ , thus by the maximum principle we have that  $\bar{w} = \bar{s}$  in  $\Delta$ .

But, since  $M_\Delta(\max(s, w_n)) \geq s$ , we have that  $\bar{s} \geq s$ . This implies that  $\bar{s}(q) \geq s(q) > \bar{w}(q)$ , which contradicts  $\bar{w} = \bar{s}$  in  $\Delta$ .  $\square$

Until now we have a function  $\beta \in C^2(D \setminus \overline{B_y})$  solution of (3) in  $D \setminus \overline{B_y}$  such that  $u \leq \beta \leq \min(u + \epsilon', g)$ . But we don't have any information about  $\beta$  on the boundary  $\partial B_y$ . So, the next step is prove that  $\beta$  is continuous up to  $\partial B_y$  and  $\beta = u + \epsilon'$  in  $\partial B_y$ .

**Claim 3.** *The function  $\beta$  is continuous up to the boundary  $\partial B_y$  and takes the value  $u + \epsilon'$  on it.*



*Proof of Claim 3.* We have that  $\beta(q) = \sup_{w \in \Gamma} w(q)$ , then,  $\beta(q) \leq u_+(q) \leq u + \epsilon'$ . On the other hand,  $u_-(q) = \max(u, v) \in \Gamma$ , then  $u_-(q) \leq \beta(q)$ . Moreover, in  $\partial B_y$  we have that  $u_-(q) = u + \epsilon'$ . Thus, let  $p \in \partial B_y$ , and  $\{x_n\}$  a sequence such that  $x_n \rightarrow p$ . We have

$$u_-(x_n) \leq \beta(x_n) \leq u_+(x_n),$$

since

$$\lim_{x_n \rightarrow p} u_-(x_n) = \lim_{x_n \rightarrow p} u_+(x_n) = u(p) + \epsilon',$$

we have that

$$\lim_{x_n \rightarrow p} \beta(x_n) = u(p) + \epsilon'.$$

Then we define  $\beta(p)$  to be continuous on  $D \setminus B_y$  and  $\beta = u + \epsilon'$  in  $\partial B_y$ . □

This concludes the proof of Proposition 1. □

We have proved the existence of a function  $\beta$  defined on  $D \setminus B_y$  such that  $\max(u, v) \leq \beta \leq \min(u + \epsilon', g)$  and  $\beta \in C^2(D \setminus \overline{B_y}) \cap C^0(D \setminus B_y)$ . Now we observe that, since  $v \leq \beta \leq u + \epsilon'$  we have by Theorem 1.1 in [19] that  $\nabla \beta$  is uniformly bounded close to  $\partial B_y$ . This implies that  $\beta$  is a solution of a linear uniformly elliptic equation. It follows by Corollary 8.36 in [7] that  $\beta \in C^{1,\alpha}(D \setminus \overline{B_y})$ , and then  $\beta \in C^1(D \setminus \overline{B_y})$ .

In the next result we will prove a uniqueness result for CMC graphs in  $\mathbb{H}^2 \times \mathbb{R}$  defined over unbounded domain in  $\mathbb{H}^2$  whose the existence was proved by A. Folha and S. Melo in [6].

**Proposition 2.** *Let  $\beta \in C^1(D \setminus \overline{B_y})$  solution of the Dirichlet problem (6)-(7) such that  $u \leq \beta \leq u + \epsilon'$ . Then,  $\beta = u + \epsilon'$ .*

*Proof.* Let us first analyze the boundary  $\partial D$ . As in section 2, let  $A_1, B_1, \dots, A_k, B_k$  be the edges of the  $\partial D$  with  $u(A_i) = +\infty = \beta(A_i)$  and  $u(B_i) = -\infty = \beta(B_i)$ .

For each  $i$ , let  $\mathcal{H}_i(n)$  be a horodisk asymptotic to the vertex  $d_i$  of  $D$  such that  $\mathcal{H}_i(n+1) \subset \mathcal{H}_i(n)$  and  $\cap_n \mathcal{H}_i(n) = \emptyset$ . For each side  $A_i$ , let us denote by  $A_i^n$  the compact subarc of  $A_i$  which is the part of  $A_i$  outside the two horodisks, and by  $|A_i^n|$  the length of  $A_i^n$ . Analogously, we define  $B_i^n$  for each side  $B_i$ . Denote by  $C_i^n$  the compact arc of  $\partial \mathcal{H}_i(n)$  contained in the domain  $D$  and let  $P^n$  be the subdomain of  $D$  bounded by the closed curve formed by the arcs  $A_i^n$ ,  $B_i^n$  and  $C_i^n$  and let us denote  $\Delta^n = \partial(P^n \setminus \overline{B_y})$ .

We have by the theorem of Stokes that

$$0 = \int_{P^n \setminus B_y} \text{Div} \left( \frac{\nabla u}{W_u} - \frac{\nabla \beta}{W_\beta} \right) = \int_{\Delta^n} \left( \frac{\nabla u}{W_u} - \frac{\nabla \beta}{W_\beta} \right) \cdot \eta$$

where  $W_u^2 = 1 + |\nabla u|^2$  and  $W_\beta^2 = 1 + |\nabla \beta|^2$ .

Thus

$$0 = \int_{A_i^n} \left( \frac{\nabla u}{W_u} - \frac{\nabla \beta}{W_\beta} \right) \cdot \eta + \int_{B_i^n} \left( \frac{\nabla u}{W_u} - \frac{\nabla \beta}{W_\beta} \right) \cdot \eta \\ + \int_{C_i^n} \left( \frac{\nabla u}{W_u} - \frac{\nabla \beta}{W_\beta} \right) \cdot \eta + \int_{\partial B_y} \left( \frac{\nabla u}{W_u} - \frac{\nabla \beta}{W_\beta} \right) \cdot \eta.$$

By theorem 5.1 in [6], these integrals can be estimated. We have

$$0 < |A_i^n| - |A_i^n| + |B_i^n| - |B_i^n| + 2|C_i^n| + \int_{\partial B_y} \left( \frac{\nabla u}{W_u} - \frac{\nabla \beta}{W_\beta} \right) \cdot \eta.$$

Then

$$\int_{\partial B_y} \left( \frac{\nabla u}{W_u} - \frac{\nabla \beta}{W_\beta} \right) \cdot \eta > -2|C_i^n|.$$

Now, we have that  $\beta \leq u + \epsilon'$ , suppose that  $\beta < u + \epsilon'$ , it follows by Lemmas 1 and 2 in [1] that  $\int_{\partial B_y} \left( \frac{\nabla u}{W_u} - \frac{\nabla \beta}{W_\beta} \right) \cdot \eta < 0$ . This yields a contradiction, since  $|C_i^n| \rightarrow 0$  when  $n \rightarrow \infty$ , so we conclude that  $\beta = u + \epsilon'$ .  $\square$

Now we are able to prove our main theorem.

*Proof of Theorem 2.* We know that  $S$  is a properly immersed CMC surface contained in  $D \times \mathbb{R}$  above  $\Sigma$ . Then, let  $y \in D$ ,  $B_y \subset D$  and  $\epsilon'$  as above. We have three cases to analyze

1) Suppose that there exists  $p \in D$  such that  $g(p) = u(p)$  (is this the case we had let aside before Proposition 1). In this case, by the maximum principle, we conclude that  $u = \beta$ .

2) Suppose that  $g > u$  and  $\inf(g - u) = 0$ . In this case, by the Proposition 1 there exists  $\beta$  solution of (6)-(7) defined over  $D \setminus B_y$  such that  $u \leq \beta \leq g$ . Moreover, Proposition 2 assures that  $\beta = u + \epsilon'$ . This yields a contradiction, since we assume that  $\inf(g - u) = 0$ .

3) Finally, suppose that  $g > u$  and  $\inf(g - u) = \alpha > 0$ . Then, pushing up  $\Sigma$  by a vertical translations, that is, by considering  $u + \alpha$  instead of  $u$ , we have now that  $g \geq u + \alpha$  and  $\inf(g - \alpha - u) = 0$ , this case reduces to cases (1) and (2) and we conclude that  $\beta = u + \alpha$ , where  $\alpha$  is a constant.  $\square$

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