Cyclicity and invariant subspaces in the Dirichlet spaces

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CYCLICITY AND INVARIANT SUBSPACES IN DIRICHLET SPACES

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ABSTRACT. Let μ be a positive finite measure on the unit circle and $\mathcal{D}(\mu)$ the associated Dirichlet space. The generalized Brown-Shields conjecture asserts that an outer function $f \in \mathcal{D}(\mu)$ is cyclic if and only if $c_\mu(Z(f)) = 0$, where c_μ is the capacity associated with $\mathcal{D}(\mu)$ and $Z(f)$ is the zero set of f. In this paper we prove that this conjecture is true for measures with countable support. We also give in this case a complete and explicit characterization of invariant subspaces.

1. INTRODUCTION

The Dirichlet space $\mathcal{D}(\mu)$, associated with μ, consists of holomorphic functions on the unit disc whose derivatives are square integrable when weighted against the Poisson integral of μ. In this paper we study cyclic vectors and invariant subspaces of the shift operator on $\mathcal{D}(\mu)$. The corresponding problem for the Hardy space H^2 was solved by Beurling in [2]: the cyclic vectors are precisely the outer functions and the invariant subspaces are generated by inner functions. Brown–Shields in [4] studied cyclicity in the classical Dirichlet space $\mathcal{D}$. They proved that the set of zeros of cyclic functions in the Dirichlet space has zero logarithmic capacity and this led them to ask whether any outer function with this property is cyclic, see also [6, 7, 11, 16] on the study of cyclic vectors. A series of results was obtained by Richter and Richter–Sundberg in [12, 13, 14, 15, 16] for Dirichlet spaces $\mathcal{D}(\mu)$ and especially for the description of their invariant subspaces. More recently, Guillot in [10] obtained a precise characterization of cyclic vectors for Dirichlet spaces associated with finitely atomic measures. We refer the reader to [8] on these problems. In this work, we focus our attention in the study of the cyclic vectors and the invariant subspaces for the shift operator acting on the Dirichlet space $\mathcal{D}(\mu)$ associated with the measures with countable support.

We now introduce the necessary notation. Let H^2 be the classical *Hardy space* of the open unit disc $\mathbb{D}$. If μ is a positive Borel measure on $\mathbb{T}$, the *Dirichlet space* $\mathcal{D}(\mu)$ is the set of all functions $f \in H^2$ such that

$$\mathcal{D}_\mu(f) = \frac{1}{\pi}\int_{\mathbb{D}} |f'(z)|^2 P_\mu(z) dA(z) < \infty$$

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where $dA(re^{it}) = (1/\pi)rdrdt$ denotes the normalized area measure on $\mathbb{D}$ and P_μ is the Poisson integral of μ :

$$P_\mu(z) = \int_{\mathbb{T}} \frac{1-|z|^2}{|1-\bar{\zeta}z|^2} d\mu(\zeta).$$

The space $\mathcal{D}(\mu)$ is endowed with the norm

$$\|f\|_\mu^2 := \|f\|_{H^2}^2 + \mathcal{D}_\mu(f).$$

The classical Dirichlet space $\mathcal{D}$ is precisely $\mathcal{D}(m)$ where m denotes the normalized Lebesgue measure.

Given $f \in \mathcal{D}(\mu)$, we denote by $[f]_{\mathcal{D}(\mu)}$ the smallest invariant subspace of $\mathcal{D}(\mu)$ containing f; namely,

$$[f]_{\mathcal{D}(\mu)} := \overline{\{pf : p \text{ is a polynomial}\}}.$$

We say that f is *cyclic* for $\mathcal{D}(\mu)$ if $[f]_{\mathcal{D}(\mu)} = \mathcal{D}(\mu)$. Denote by S the shift operator on $\mathcal{D}(\mu)$, that is the multiplication by z on $\mathcal{D}(\mu)$. A closed subspace $\mathcal{M}$ of $\mathcal{D}(\mu)$ is called invariant if $S\mathcal{M} \subset \mathcal{M}$. The lattice of all closed invariant subspaces of the shift operator will be denoted by $\mathrm{Lat}(S, \mathcal{D}(\mu))$.

In this paper we are interested in a characterization of the cyclic functions of $\mathcal{D}(\mu)$ and in a description of $\mathrm{Lat}(S, \mathcal{D}(\mu))$. If $d\mu(e^{it}) = 0$ then $\mathcal{D}(0) = H^2$, and, in this case, Beurling's theorem asserts that all closed invariant subspaces are given by ΘH^2, where Θ is an inner function. As a consequence, a function $f \in H^2$ is cyclic for H^2 if and only if f is outer. In order to extend Beurling's theorem to the classical Dirichlet space, Richter in [13] was led to introduce Dirichlet type spaces. First, he proved that every cyclic, analytic 2-isometry is unitarily equivalent to the shift operator on some $\mathcal{D}(\mu)$. Recall that a bounded operator on a Hilbert space is called 2-isometry if

$$T^{*2}T^2 - 2T^*T - I = 0$$

and it is called *analytic* if $\bigcap_{n\geq 0} T^n\mathcal{H} = \{0\}$. T his result allowed him to prove that every invariant subspace for $\mathcal{D}$ is of the form $\phi\mathcal{D}(|\phi|^2 dm)$, where ϕ is an extremal function for $\mathcal{D}$, that is

$$\|\phi\|_\mu = 1 \quad \text{and} \quad \langle \phi, z^n\phi\rangle_{\mathbb{D}(\mu)} = 0, \quad n \geq 1.$$

This characterization does not allow to describe the cyclic functions for $\mathcal{D}$. Brown and Shields showed in [4] that if f is cyclic for $\mathcal{D}$ then f is outer and the zero set of its radial limit

$$\mathcal{Z}_{\mathbb{T}}(f) = \{\zeta \in \mathbb{T} : \lim_{r\to 1-} f(r\zeta) = 0\}$$

is of logarithmic capacity zero. Brown and Shields further conjectured that the converse is also true. This problem remains open, some partial results of this conjecture can be found in [8, 11, 16].

Richter and Sundberg in [16] extended the characterization of invariant subspaces to all Dirichlet spaces. Indeed, they proved that

$$\mathrm{Lat}(S, \mathcal{D}(\mu)) = \left\{\phi\mathcal{D}(|\phi|^2 d\mu) : \phi \text{ is an extremal function for } \mathcal{D}(\mu)\right\}.$$

As before, the description of cyclic functions remains an open problem. To state a general Brown-Shields conjecture we will introduce a notion of capacity associated with these spaces.

The harmonic Dirichlet space, $\mathcal{D}^h(\mu)$, associated with μ is given by

$$\mathcal{D}^h(\mu) := \left\{f \in L^2(\mathbb{T}) : \|f\|_2^2 + \mathcal{D}_\mu(f) < \infty\right\},$$

where $\mathcal{D}_\mu(f) = \int_{\mathbb{D}} |\nabla P[f]|^2 P_\mu dA$. Note that $\mathcal{D}^h(\mu)$ is a Dirichlet space in the sense of Beurling–Deny, and following [3], the c_μ-capacity of an open subset $U \subset \mathbb{T}$ is defined by

$$c_\mu(U) := \inf\left\{\|u\|_\mu^2 : u \in \mathcal{D}^h(\mu), \;\; u \geq 0 \text{ and } u \geq 1 \text{ a.e. on } U\right\}.$$

As usual we define the c_μ-capacity of any subset $F \subset \mathbb{T}$ by

$$c_\mu(F) = \inf\{c_\mu(U) : U \;\; \text{open}, \;\; F \subset U\}.$$

The capacity c_μ is the Choquet capacity [5, 10] and so for every borelian subset E of $\mathbb{T}$ we have

$$c_\mu(E) = \sup\{c_\mu(K) : K \text{ compact }, K \subset E\}.$$

In the case $\mu = m$, it is well known that c_m is comparable to the logarithmic capacity. For more details see [1, Theorem 2.5.5].

We say that a property holds c_μ-quasi-everywhere (c_μ-q.e.) if it holds everywhere outside a set of c_μ -capacity 0. So, c_μ-q.e. implies a.e. Note that for every function $f \in \mathcal{D}(\mu)$, the radial limits of f exist q.e., see [10]. Recall also that c_μ satisfies a weak-type inequality. Namely:

$$c_\mu(\{\zeta \in \mathbb{T} : |f(\zeta)| \geq t \quad c_\mu\text{-q.e.}\}) \leq \frac{\|f\|_\mu^2}{t^2}, \qquad f \in \mathcal{D}^h(\mu).$$

As consequence, the invariant subspace $\mathcal{M}_\mu(E)$ defined by

$$\mathcal{M}_\mu(E) := \{g \in \mathcal{D}(\mu) : g|E = 0 \; c_\mu\text{-q.e}\}.$$

is closed in $\mathcal{D}(\mu)$. Using these facts it is easy to verify that if f is cyclic in $\mathcal{D}(\mu)$ then f is outer and $c_\mu(\mathcal{Z}_\mathbb{T}(f)) = 0$, where The generalized Brown-Shields conjecture claims that the converse is also true.

For $f \in \mathcal{D}(\mu)$, denote by $\mathcal{Z}(f)$ the zeros set of f on the disc and set

$$\underline{\mathcal{Z}}(f) := \{\zeta \in \mathbb{T} : \liminf_{z\to\zeta} |f(z)| = 0\}.$$

In this paper we will prove the following two results:

Theorem 1. *Let μ be a positive finite measure on $\mathbb{T}$. Let $f \in \mathcal{D}(\mu)$ be such that* $\operatorname{supp}\mu \cap \underline{\mathcal{Z}}(f)$ *is countable. The following assertions are equivalent.*

(1) f is cyclic for $\mathcal{D}(\mu)$.
(2) f is an outer function and $c_\mu(\mathcal{Z}_{\mathbb{T}}(f)) = 0$.

This theorem asserts, in particular, that the generalized Brown–Shields conjecture is true if the support of μ is countable. We also obtain, in this case, an explicit characterization of invariant subspaces of $\mathcal{D}(\mu)$.
Let $\mathcal{M} \in \mathrm{Lat}(S, \mathcal{D}(\mu))$, we denote by $\Theta_{\mathcal{M}}$ the greatest common inner divisor of the inner parts of the non-zero functions of $\mathcal{M}$ and

$$\mathcal{Z}_{\mathbb{T}}(\mathcal{M}) = \bigcap_{f\in\mathcal{M}} \mathcal{Z}_{\mathbb{T}}(f).$$

We have the following characterization which completes the Richter–Sundberg result [16] in the case when $\operatorname{supp}\mu$ is countable.

Theorem 2. *Let μ be a positive finite measure on $\mathbb{T}$ such that $\operatorname{supp}\mu$ is countable. Let $\mathcal{M} \in \mathrm{Lat}(S, \mathcal{D}(\mu))$ and let $\Theta_{\mathcal{M}}$ be the greatest common inner divisor of $\mathcal{M}$. Then,*

$$\mathcal{M} = \Theta_{\mathcal{M}} H^2 \cap \mathcal{M}_\mu(E),$$

where $E = \{\lambda \in \operatorname{supp}\mu \,:\, c_\mu(\{\lambda\}) > 0$ and $\lambda \in \mathcal{Z}_{\mathbb{T}}(\mathcal{M})\}$.

Note that by [9, Corollary 5.2]

$$c_\mu(\{\lambda\}) > 0 \iff \int_0^1 \frac{dr}{(1-r)P_\mu(r\lambda)} < +\infty.$$

The plan of the paper is the following. The next section gives a background on Dirichlet spaces. In Section 3, we give a description of invariant subspaces generated by polynomials. In Sections 4 we collect some results on closed ideals of $\mathcal{D}(\mu) \cap H^\infty$. Sections 5 and 6 are devoted to the proof of Theorems 1 and 2.

2. Background on the Dirichlet type spaces

In this section we recall some results from the Richter–Sundberg papers [14, 15, 16] about Dirichlet type spaces which will be used in the proofs of our Theorems, see also [8].

Every function $f \in H^2$ has non-tangential limits almost everywhere on the unit circle $\mathbb{T} = \partial\mathbb{D}$. We denote by $f(\zeta)$ the non-tangential limit of f at $\zeta \in \mathbb{T}$ if it exists.

Let μ be a positive finite measure on the unit circle, the associated Dirichlet space $\mathcal{D}(\mu)$ is the set of all analytic functions $f \in H^2$, such that

$$\mathcal{D}_\mu(f) := \int_{\mathbb{T}} \mathcal{D}_\xi(f) d\mu(\xi) < \infty,$$

where $\mathcal{D}_\xi(f)$ is the *local Dirichlet integral* of f at $\xi \in \mathbb{T}$ given by

$$\mathcal{D}_\xi(f) := \int_{\mathbb{T}} \frac{|f(e^{it}) - f(\xi)|^2}{|e^{it} - \xi|^2} \frac{dt}{2\pi}.$$

Here $f(\lambda)$ ($\lambda = e^{it}$or ζ) is the radial limit of f at λ, that is $f(\lambda) = \lim_{r\to 1-} f(r\lambda)$.

Recall that a function I is inner if it is a bounded holomorphic function on $\mathbb{D}$ such that $|I| = 1$ a.e. on $\mathbb{T}$. The function O is called outer if it is of the form

$$O(z) = \exp \frac{1}{2\pi} \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log \varphi(\zeta) |d\zeta|, \qquad z \in \mathbb{D}.$$

where φ is a positive function such that $\log \varphi \in L^1(\mathbb{T})$. Note that $|f| = \varphi$ a.e. on $\mathbb{T}$.

Theorem 2.1. *Let $f \in H^2$ and $f = BS_\nu f_o$ with B a Blaschke product, ν a singular measure associated with S_ν a singular inner factor of f, f_o an outer function and let $\zeta \in \mathbb{T}$ such that $f_o(\zeta)$ exists. Then*

$$\mathcal{D}_\zeta(f) = \sum_{\gamma \in \mathcal{Z}(B)} P_{\delta_\gamma}(\zeta) |f_o(\zeta)|^2 + 2 \int_{\mathbb{T}} \frac{d\nu(\lambda)}{|\lambda - \zeta|^2} |f_o(\zeta)|^2 + \int_{\mathbb{T}} \frac{|f(\lambda)|^2 - |f(\zeta)|^2 - 2|f(\zeta)|^2 \log |f(\lambda)/f(\zeta)|}{|\lambda - \zeta|^2} \frac{|d\lambda|}{2\pi}, \quad (1)$$

where δ_γ is the dirac measure on γ.

Proof. See [14, Theorem 3.1].

□

Theorem 2.2. *Let $f, g \in \mathcal{D}(\mu)$. If $|f(z)| \le |g(z)|$ on $\mathbb{D}$, then $[f]_{\mathcal{D}(\mu)} \subset [g]_{\mathcal{D}(\mu)}$.*

Proof. See [14, Theorem 4.1].

□

If f and g are outer functions, then we define the outer function function $f \wedge g$ by $\min\{|f(e^{it})|, |g(e^{it})|\}$.

Theorem 2.3. *Let $f, g \in \mathcal{D}(\mu)$ be outer functions and let h be the outer function given by $|h| := |f| \wedge |g|$ a.e on $\mathbb{T}$. Then $h \in \mathcal{D}(\mu)$ and $[h]_{\mathcal{D}(\mu)} = [f]_{\mathcal{D}(\mu)} \cap [g]_{\mathcal{D}(\mu)}$. If further $fg \in \mathcal{D}(\mu)$, then*

$$[fg]_{\mathcal{D}(\mu)} = [f]_{\mathcal{D}(\mu)} \cap [g]_{\mathcal{D}(\mu)}.$$

Proof. See [15, Lemma 2.2 and Theorem 4.5].

□

Theorem 2.4. *Let $\mathcal{M} \in \mathrm{Lat}(S, \mathcal{D}(\mu))$. Then there exists a multiplier ϕ of $\mathcal{D}(\mu)$ such that $\mathcal{M} = [\phi]_{\mathcal{D}(\mu)}$.*

Proof. See [15, Theorem 3.5] and [16, Theorem 3.2].

□

Theorem 2.5. *Let $\mathcal{M} \in \mathrm{Lat}(S, \mathcal{D}(\mu)$, and let $\Theta_{\mathcal{M}}$ the greatest common inner divisor of $\mathcal{M}$. Then, there is an outer $f \in \mathcal{D}(\mu) \cap H^\infty$ such that*

$$\mathcal{M} = \Theta_{\mathcal{M}} H^2 \cap [f]_{\mathcal{D}(\mu)}.$$

In fact f can be chosen so that f and $\Theta_{\mathcal{M}} f$ are multipliers of $\mathcal{D}(\mu)$.

Proof. See [15, Theorem 5.3]. □

3. Invariant subspaces generated by polynomials

In this section we characterize closed invariant subspaces generated by polynomials. Let $\zeta \in \mathbb{T}$. We will say that ζ is a bounded point evaluation of $\mathcal{D}(\mu)$ if there exists a constant $M > 0$ such that for every polynomial p

$$|p(\zeta)| \leq M\|p\|_\mu.$$

It means that the functional $p \to p(\zeta)$ extends to a continuous functional on $\mathcal{D}(\mu)$. Since the polynomials are dense in $\mathcal{D}(\mu)$, this extension is unique and will be denoted by $L_\zeta(f) = f(\zeta)$.
Note also that by [10], c_μ is comparable to c^a_μ, where

$$c^a_\mu(E) = \inf\big\{\|f\|^2_\mu : \ f \in \mathcal{D}(\mu) \text{ and } \ |f| \geq 1 \text{ a.e. on a neighborhood of } \ E\big\}.$$

Since

$$\|f\|_\mu \geq \|f_o\|_\mu \geq \|f_o \wedge 1\|_\mu,$$

where f_o is the outer part of f, then

$$c^a_\mu(E) = \inf\big\{\|f\|^2_\mu : \ f \in \mathcal{D}(\mu) \text{ is an outer function, } |f| = 1 \text{ a.e. on a neighborhood of } E\big\}.$$

Using this fact, we have the following lemma.

Lemma 3.1. *Let $\zeta \in \mathbb{T}$. The following properties are equivalent:*

(1) ζ is a bounded point evaluation of $\mathcal{D}(\mu)$.
(2) The polynomial $z - \zeta$ is not cyclic for $\mathcal{D}(\mu)$.
(3) $c_\mu(\zeta) > 0$.

Proof. (1) $\Longrightarrow$ (3): Suppose that ζ is a bounded point evaluation for $\mathcal{D}(\mu)$. We will prove that $c^a_\mu(\zeta) > 0$. Let $f \in \mathcal{D}(\mu)$ be an outer function such that $|f| = 1$ a.e. on an open arc I centered at ζ. Then, there is $\lambda \in \mathbb{T}$ such that

$$f(z) = \lambda \exp \frac{1}{2\pi} \int_{\mathbb{T}\setminus I} \frac{\eta + z}{\eta - z} \log|f(\eta)| d\eta|, \qquad z \in \mathbb{D}.$$

Hence f is analytic in a neighborhood of ζ. Since L_ζ is bounded, we have $|L_\zeta(f)| = |f(\zeta)| = 1$. So $\|f\|_\mu \geq 1/\|L_\zeta\|$, and consequently, $c^a_\mu(\zeta) \geq 1/\|L_\zeta\|$. This proves that (1) implies (3).

(3) $\Longrightarrow$ (2): Suppose that $c_\mu(\zeta) > 0$. Then $\mathcal{M}(\zeta)$ is a proper closed invariant subspace of $\mathcal{D}(\mu)$ and $z - \zeta$ is not cyclic.

(2) $\Longrightarrow$ (1): Suppose now that $z - \zeta$ is not cyclic for $\mathcal{D}(\mu)$. The space $\mathcal{D}(\mu)/[(z-\zeta)]_{\mathcal{D}(\mu)}$ is of dimension one. Let π be the canonical surjection defined by

$$\begin{array}{rccc} \pi : & \mathcal{D}(\mu) & \to & \mathcal{D}(\mu)/[(z-\zeta)]_{\mathcal{D}(\mu)} \\ & f & \to & \pi(f) = f + [(z-\zeta)]_{\mathcal{D}(\mu)} \end{array}$$

and let ρ be the isomorphism from $\mathcal{D}(\mu)/[(z-\zeta)]_{\mathcal{D}(\mu)}$ to $\mathbb{C}$ satisfying $\rho(\pi(1))=1$. It is easy to verify that $L_\zeta = \rho\circ\pi$, which proves that L_ζ is continuous.

□

Lemma 3.2. *Let $\zeta\in\mathbb{T}$ and let n be an integer $n\geq 1$. We have*

(1) If $c_\mu(\zeta)=0$ then $[(z-\zeta)^n]_{\mathcal{D}(\mu)}=\mathcal{D}(\mu)$.
(2) If $c_\mu(\zeta)>0$ then $[(z-\zeta)^n]_{\mathcal{D}(\mu)}=\mathcal{M}_\mu(\zeta)$.

Proof. By Theorem 2.3, we have $[(z-\zeta)^n]_{\mathcal{D}(\mu)}=[(z-\zeta)]_{\mathcal{D}(\mu)}$. The two assertions come from Lemma 3.1.

□

To state the characterization of closed invariant subspaces generated by polynomials, we need some notations.
To any $\Lambda=\{(z_1,n_1),(z_2,n_2),\dots,(z_k,n_k)\}$, where $z_j\in\mathbb{D}$ and $n_j\in\mathbb{N}^*$, we associate a polynomial $p_\Lambda=\Pi_{j=1}^k(z-z_j)^{n_j}$. Let $E\subset\mathbb{T}$, we define

$$\mathcal{M}_\mu(\Lambda,E)=\{f\in\mathcal{D}(\mu):\ f\in p_\Lambda H^2 \text{ and } f_{|E}=0\}.$$

For a polynomial p, let $\Lambda_p=\{(z,n)\in\mathbb{D}\times\mathbb{N}^* : z \text{ is a zero of } p \text{ of order } n\}$. We have

Theorem 3.3. *Let p be a polynomial. Then*

$$[p]_{\mathcal{D}(\mu)}=\mathcal{M}_\mu(\Lambda_p,E),$$

where $E=\{\zeta\in\mathbb{T}:\ p(\zeta)=0$ and $c_\mu(\zeta)>0\}$.

Proof. The proof is based on classical arguments and on the Lemma 3.2.

□

4. Closed ideals of $\mathcal{D}(\mu)\cap H^\infty$

First, we will state the following lemmas which will be used in the sequel.

Lemma 4.1. *Let $f\in\mathcal{D}(\mu)\cap H^\infty$ and $g\in H^\infty$ be outer functions such that $\|f\|_\infty\leq 1$ and $\|g\|_\infty\leq 1$. Let h be the outer function given by*

$$|h(\zeta)|=\begin{cases}|f(\zeta)|, & \zeta\in V,\\ |g(\zeta)|, & \zeta\notin V,\end{cases}$$

where V is a closed neighborhood of $\operatorname{supp}(\mu)$. *Then $h\in\mathcal{D}(\mu)$ and*

$$\mathcal{D}_\mu(h)\leq\mathcal{D}_\mu(f)+\frac{\mu(\mathbb{T})}{\operatorname{dist}(\mathbb{T}\setminus V,\operatorname{supp}(\mu))^2}\Big(\|g\|_2^2+2\|\log 1/|g|\|_{L^1(\mathbb{T})}\Big). \tag{2}$$

Proof. Note that $\|h\|_2^2 \leq \|f\|_2^2 + \|g\|_2^2$, so $h \in H^2$. By (1), for $\zeta \in \operatorname{supp}\mu$, let $\delta = \operatorname{dist}(\mathbb{T} \setminus V, \operatorname{supp}(\mu))$, we have

$$\begin{aligned}
\mathcal{D}_\zeta(h) &= \int_{\lambda\in V} + \int_{\lambda\in\mathbb{T}\setminus V} \frac{|h(\lambda)|^2 - |h(\zeta)|^2 - 2|h(\zeta)|^2 \log|h(\lambda)/h(\zeta)|}{|\lambda-\zeta|^2} \frac{|d\lambda|}{2\pi} \\
&\leq \mathcal{D}_\zeta(f) + \frac{1}{\delta^2}\int_{\lambda\in\mathbb{T}\setminus V} \Big(|g(\lambda)|^2 + 2|f(\zeta)|^2 \log\frac{1}{|g(\lambda)|} - |f(\zeta)|^2 \log\frac{e}{|f(\zeta)|^2}\Big)\frac{|d\lambda|}{2\pi} \\
&\leq \mathcal{D}_\zeta(f) + \frac{1}{\delta^2}\Big(||g||_2^2 + 2|f(\zeta)|^2 \int_{\mathbb{T}} \log\frac{1}{|g(\lambda)|}\frac{|d\lambda|}{2\pi}\Big).
\end{aligned}$$

It's clear that (2) follows from this inequality and thus $h \in \mathcal{D}(\mu)$.

□

Lemma 4.2. *Let $V = (e^{ia}, e^{ib})$, and let $f \in \mathcal{D}(\mu) \cap H^\infty$ be an outer function. Let f_V be the outer function defined by*

$$|f_V(\zeta)| = \begin{cases} |(\zeta - e^{ia})(e^{ib} - \zeta)||f(\zeta)|, & \zeta \in V, \\ |(\zeta - e^{ia})(e^{ib} - \zeta)|, & \zeta \in \mathbb{T}\setminus V. \end{cases}$$

Then $f_V \in \mathcal{D}(\mu) \cap H^\infty$.

Proof. Set $u(z) = (z - e^{ia})(e^{ib} - z)$, again by (1), for $\zeta \in \operatorname{supp}\mu$, we have

$$\begin{aligned}
\mathcal{D}_\zeta(f_V) &= \int_{\lambda\in V} + \int_{\lambda\in\mathbb{T}\setminus V} \frac{|u(\lambda)|^2 - |u(\zeta)|^2 - 2|u(\zeta)|^2 \log|u(\lambda)/u(\zeta)|}{|\lambda-\zeta|^2} \frac{|d\lambda|}{2\pi} \\
&\leq \mathcal{D}_\zeta(uf) + D_\xi(u).
\end{aligned}$$

Clearly, $f_V \in \mathcal{D}(\mu) \cap H^\infty$.

□

In order to state our result, we introduce some notions. Let $f \in \mathcal{D}(\mu)$ and let $\mathcal{M} \in \operatorname{Lat}(S, \mathcal{D}(\mu))$. Set

$$\underline{\mathcal{Z}}(f) := \{z \in \overline{\mathbb{D}} : \liminf_{w\to z} |f(w)| = 0\}$$

and

$$\underline{\mathcal{Z}}(\mathcal{M}) := \bigcap_{f\in\mathcal{M}} \underline{\mathcal{Z}}(f).$$

Recall that $\mathcal{D}(\mu)\cap H^\infty$ is a Banach algebra endowed with the pointwise multiplication and equipped with the norm

$$\|f\|_{\infty,\mu} = \|f\|_\infty + \mathcal{D}_\mu(f)^{1/2}.$$

Theorem 4.3. *Let $\mathcal{J}$ be a closed ideal of $\mathcal{D}(\mu)\cap H^\infty$. Let*

$$\pi : \mathcal{D}(\mu)\cap H^\infty \longrightarrow \mathcal{D}(\mu)\cap H^\infty/J,$$

be the canonical surjection. Then

$$\sigma(\pi(u)) = \underline{\mathcal{Z}}(\mathcal{J})$$

where $u : z \to z$ is the identity map and $\sigma(\pi(u))$ is the spectrum of $\pi(u)$.

Proof. First, we prove that $\underline{\mathcal{Z}}(\mathcal{J}) \subset \sigma(\pi(u))$. Let $\lambda \notin \sigma(\pi(u))$, then there exists $f \in \mathcal{D}(\mu) \cap H^\infty$ and $g \in \mathcal{J}$ such that $(\lambda - z)f(z) = 1 - g(z)$. Which gives obviously that $\lim_{z\to\lambda} g(z) = 1$ and $\lambda \notin \underline{\mathcal{Z}}(\mathcal{J})$.

Conversely, Let $\lambda \notin \underline{\mathcal{Z}}(\mathcal{J})$. Hence there exists $g \in \mathcal{J}$ and $c > 0$ such that $c \leq |g(z)|$ on a neighborhood of λ, $V_\lambda = \{z \in \mathbb{D} : |z - \lambda| \leq \delta/2\}$.

Note that if $w \in C^2(\mathbb{T})$ and $\log|w| \in L^1(\mathbb{T})$ then the outer function f_w defined by $|f_w| = w$, a.e. on $\mathbb{T}$, belongs to $C^2(\overline{\mathbb{D}}) \cap \mathrm{Hol}(\mathbb{D}) \subset \mathcal{D}(\mu)$. Now consider the smooth function $w \in C^2(\mathbb{T})$ $w \geq c > 0$ defined by

$$w(\zeta) = \begin{cases} c & \text{if} \quad |\zeta - \lambda| \leq \delta/2, \\ 1 & \text{if} \quad |\zeta - \lambda| \geq \delta. \end{cases}$$

Suppose that $\|g\|_\infty \leq 1$. Write $g = g_i g_o$ the inner–outer factorization of g. Consider the outer function

$$|h| = |g_o| \wedge |f_w| = |g_o| \wedge |w|,$$

by Theorem 2.3, $h \in \mathcal{D}(\mu)$. Now consider the outer function

$$|\widetilde{h}(e^{it})| = (|g_o| \vee |f_w|)(e^{it}) = \max\{|g_o(e^{it})|, w(e^{it})\}.$$

Clearly,

$$g f_w = g_i(g_o \wedge f_w) \times (g_o \vee f_w) = g_i h \times \widetilde{h} \quad \text{ on } \mathbb{D}$$

and $\widetilde{h} \in \mathcal{D}(\mu) \cap H^\infty$ is invertible. So $g_i h \in \mathcal{J}$. Let

$$\psi = \frac{1}{(g_i h)(\lambda)} \frac{g_i h - g_i h(\lambda)}{u - \lambda}.$$

Note that the function $g_o \wedge f_w = f_w$ on $\{\zeta \in \mathbb{T} : |\zeta - \lambda| \leq \delta/2\}$ is C^∞ on V_λ and

$$h(z) = \exp \int_{|\zeta-\lambda|\leq\delta/2} \frac{\zeta + z}{\zeta - z} \log c \frac{|d\zeta|}{2\pi} \times \exp \int_{|\zeta-\lambda|\geq\delta/2} \frac{\zeta + z}{\zeta - z} \log|h(\zeta)| \frac{|d\zeta|}{2\pi},$$

the two functions are clearly C^∞ on V_λ and hence ψ is C^2 on V_λ. Thus $\psi \in \mathcal{D}(\mu) \cap H^\infty$. So $\pi(H)(\lambda - \pi(u)) = \pi(1)$ and we get $\lambda \notin \sigma(\pi(u))$ which finishes the proof.

□

Lemma 4.4. *Let $\mathcal{J}$ be a closed ideal of $\mathcal{D}(\mu) \cap H^\infty$ such that $\mathcal{J}$ contains an outer function, and $\underline{Z}(\mathcal{J}) = \{\zeta_0\} \subset \mathbb{T}$. Then $(z - \zeta_0) \in \mathcal{J}$.*

Proof. Consider again the canonical surjection

$$\pi : \mathcal{D}(\mu) \cap H^\infty \longrightarrow \mathcal{D}(\mu) \cap H^\infty / \mathcal{J},$$

and let $u : z \to z$ be the identity map. Given $\lambda \in \mathbb{D}$ and $f \in \mathcal{J}$, we define

$$L_\lambda(f)(z) = \begin{cases} \dfrac{f(z) - f(\lambda)}{z - \lambda}, & z \in \mathbb{D} \setminus \{\lambda\}, \\ f'(\lambda), & z = \lambda. \end{cases}$$

Since $L_\lambda f - L_0 f = \lambda L_\lambda L_0 f$, It is obvious that $L_\lambda(f) \in \mathcal{D}(\mu) \cap H^\infty$. So

$$\pi(L_\lambda(f))(\lambda\pi(1) - \pi(u)) = f(\lambda)\pi(1), \qquad \lambda \in \mathbb{D}.$$

Let

$$\Psi(\lambda) = (\lambda\pi(1) - \pi(u))^{-1}\pi(1), \qquad \lambda \in \mathbb{C}\backslash\{1\}.$$

This function is well defined and analytic in $\mathbb{C}\backslash\{1\}$ by Theorem 4.3. Let $f \in \mathcal{J}$ be an outer function. So

$$(\lambda\pi(1) - \pi(u))^{-1} = (1/f(\lambda))\pi(L_\lambda(f)), \qquad \lambda \in \mathbb{D}.$$

Since f is outer, we have

$$\|\Psi(\lambda)\| \leq \begin{cases} \dfrac{c_1}{(|\lambda|-1)^2} & \text{if} \quad |\lambda| > 1. \\ c_2 \exp \dfrac{\varepsilon}{1-|\lambda|} & \text{if} \quad |\lambda| < 1, \end{cases}$$

where c_1, c_2 are positive constant depending on f. Then by Phragmén-Lindelöf principal, we get that Ψ is a polynomial of $1/(1-\lambda)$ of degree at most 2. Let $g(z) = (\zeta_0 - z)^2$, we have

$$\pi(g) - g(\lambda)\pi(1) = (\lambda\pi(1) - \pi(u))\pi(L_\lambda(g))$$

so

$$\pi(g)(\lambda\pi(1) - \pi(u))^{-1} = \pi(L_\lambda(g)) + g(\lambda)(\lambda - \pi(u))^{-1}$$

and $\pi(g)(\lambda\pi(1) - \pi(u))^{-1}$ is bounded. It follows that $\pi(g) = 0$ and $(\zeta_0 - z)^2 \in \mathcal{J}$.

Let a functional ℓ in the dual space of $\mathcal{D}(\mu) \cap H^\infty$ be such that ℓ is orthogonal to $\mathcal{J}$. We have

$$\langle \ell, z^n(\zeta_0 - z)^2 \rangle = 0, \qquad n \geq 0.$$

Let $\widehat{\ell}(n) = \langle \ell, z^n \rangle$, the last equality implies that

$$\zeta_0^2\widehat{\ell}(n) - 2\zeta_0\widehat{\ell}(n+1) + \widehat{\ell}(n+2) = 0 \qquad n \geq 0.$$

So $\widehat{\ell}(n) = (a + bn)\zeta_0^n$ for some constants $a, b \in \mathbb{C}$. In the other hand

$$|\widehat{\ell}(n)| \leq \|\ell\| \, \|z^n\|_{\mathcal{D}(\mu)\cap H^\infty} = O(\sqrt{n}).$$

Hence $b = 0$ and $\langle \ell, (\zeta_0 - z) \rangle = 0$. Which gives $z - \zeta - 0 \in \mathcal{J}$ and completes the proof.

□

Remark. The preceding result can be extended to closed ideals such that there greatest common inner divisor is 1.

5. Cyclicity in $\mathcal{D}(\mu)$

Lemma 5.1. *Let $f \in \mathcal{D}(\mu) \cap H^\infty$ be an outer function. If $\underline{\mathcal{Z}}(f) \cap \operatorname{supp}(\mu) = \emptyset$ then f is cyclic for $\mathcal{D}(\mu)$.*

Proof. Suppose that $\|f\|_\infty \le 1$. Since $\underline{\mathcal{Z}}(f) \cap \operatorname{supp}\mu = \emptyset$, there exists a closed neighborhood V of $\operatorname{supp}\mu$ such that $\underline{\mathcal{Z}}(f) \cap V = \emptyset$. Consider the following outer functions

$$|h(\zeta)| = \begin{cases} 1, & \zeta \in V \\ |f(\zeta)|, & \zeta \notin V \end{cases}$$

and

$$|g(\zeta)| = \begin{cases} |f(\zeta)|, & \zeta \in V \\ 1, & \zeta \notin V. \end{cases}$$

By Lemma 4.1, $h, g \in \mathcal{D}(\mu) \cap H^\infty$ and by Theorem 2.3

$$[f]_{\mathcal{D}(\mu)} = [hg]_{\mathcal{D}(\mu)} = [h]_{\mathcal{D}(\mu)} \cap [g]_{\mathcal{D}(\mu)}.$$

Note that $\underline{\mathcal{Z}}(g) = \underline{\mathcal{Z}}(f) \cap V = \emptyset$. Hence $|g(z)| \ge c > 0$ on $\mathbb{D}$, and then g is cyclic for $\mathcal{D}(\mu) \cap H^\infty$. So

$$[f]_{\mathcal{D}(\mu)} = [h]_{\mathcal{D}(\mu)}. \tag{3}$$

Let $\delta > 0$, and consider the following outer function

$$|h_\delta(\zeta)| = \begin{cases} 1, & \zeta \in V \\ 1/(|f(\zeta)| + \delta), & \zeta \notin V \end{cases}$$

We have $\lim_{\delta\to 0} |h_\delta h(\zeta)| = 1$ a.e on $\mathbb{T}$. Since $\underline{\mathcal{Z}}(f) \cap \operatorname{supp}(\mu) = \emptyset$, By Lemma 4.1

$$\begin{aligned} \mathcal{D}_\mu(h_\delta h(\zeta)) &\le \frac{1}{\operatorname{dist}(\operatorname{supp}\mu, \mathbb{T}\setminus V)^2}\Big(\mu(\mathbb{T}) + 2\Big\|\log\frac{|f|+\delta}{|f|}\Big\|_{L^1(\mathbb{T})}\Big) \\ &\le \frac{1}{\operatorname{dist}(\operatorname{supp}\mu, \mathbb{T}\setminus V)^2}\Big(\mu(\mathbb{T}) + 4\| \,|\log|f|| \,\|_{L^1(\mathbb{T})}\Big). \end{aligned}$$

So

$$\liminf_{\delta\to 0} D_\mu(h_\delta h) < \infty.$$

Since h_δ and h are outer, $|h_\delta h(z)| \le |h(z)|/\delta$ for $z \in \mathbb{D}$, by Theorem 2.2, we have $h_\delta h \in [h]_{\mathcal{D}(\mu)\cap H^\infty}$ for all $\delta > 0$. So h is cyclic for $\mathcal{D}(\mu)$ and by (3) f is also cyclic for $\mathcal{D}(\mu)$. □

Lemma 5.2. *Let $f \in \mathcal{D}(\mu) \cap H^\infty$ be outer function. Then*

$$\underline{\mathcal{Z}}([f]_{\mathcal{D}(\mu)}) \subset \underline{\mathcal{Z}}(f) \cap \operatorname{supp}\mu.$$

Proof. Let $\zeta \in \underline{\mathcal{Z}}(f)\setminus\operatorname{supp}(\mu)$, There exists a neighborhood V_ζ of ζ such that $V_\zeta \cap \operatorname{supp}(\mu) = \emptyset$. Write

$$f = f_1 \times f_2, \qquad f_1, f_2 \in \mathcal{D}(\mu) \cap H^\infty$$

where $\underline{\mathcal{Z}}(f_2) \subset \mathbb{T} \setminus V_\zeta$ and $\underline{\mathcal{Z}}(f_1) \subset V_\zeta$, so $\underline{\mathcal{Z}}(f_1) \cap \operatorname{supp}(\mu) = \emptyset$. By Proposition 5.1, f_1 is cyclic for $\mathcal{D}(\mu)$. So

$$[f]_{\mathcal{D}(\mu)} = [f_1 f_2]_{\mathcal{D}(\mu)} = [f_1]_{\mathcal{D}(\mu)} \cap [f_2]_{\mathcal{D}(\mu)} = [f_2]_{\mathcal{D}(\mu)}.$$

Hence $\zeta \notin \underline{\mathcal{Z}}([f]_{\mathcal{D}(\mu)})$. □

Proof Theorem 1. If f is cyclic for $\mathcal{D}(\mu)$, then it's clear that f is an outer function and $c_\mu(\mathcal{Z}(f)) = \emptyset$.

Conversely, if $c_\mu(\mathcal{Z}_{\mathbb{T}}(f)) = 0$. By Lemma 5.2 we have

$$\underline{\mathcal{Z}}([f]_{\mathcal{D}(\mu)}) \subset \operatorname{supp}\mu \cap \underline{\mathcal{Z}}(f).$$

If $\underline{\mathcal{Z}}([f]_{\mathcal{D}(\mu)}) = \emptyset$. then by Lemma 5.2 and Lemma 5.1 f is cyclic and the theorem is proved.

Suppose now that

$$\underline{\mathcal{Z}}([f]_{\mathcal{D}(\mu)}) \neq \emptyset.$$

Since $\underline{\mathcal{Z}}([f]_{\mathcal{D}(\mu)})$ is a countable set, $\underline{\mathcal{Z}}([f]_{\mathcal{D}(\mu)})$ have an isolated point, noted by ζ_0. Let $V = (e^{ia}, e^{ib})$ be a neighborhood of ζ_0 such that $V \cap \underline{\mathcal{Z}}([f]_{\mathcal{D}(\mu)}) = \{\zeta_0\}$. Consider the outer functions $h = f_V$ and $g = f_{\mathbb{T}\setminus V}$:

$$|h(\zeta)| = \begin{cases} |(\zeta - e^{ia})(e^{ib} - \zeta)||f(\zeta)|, & \zeta \in V \\ |(\zeta - e^{ia})(e^{ib} - \zeta)|, & \zeta \notin V \end{cases}$$

and

$$|g(\zeta)| = \begin{cases} |(\zeta - e^{ia})(e^{ib} - \zeta)|, & \zeta \in V \\ |(\zeta - e^{ia})(e^{ib} - \zeta)||f(\zeta)|, & \zeta \notin V. \end{cases}$$

By Lemma 4.2, $h, g \in \mathcal{D}(\mu) \cap H^\infty$ and $hg \in [f]_{\mathcal{D}(\mu)}$. Define the closed division ideal $\mathcal{J}$ by

$$\mathcal{J} := \{\psi \in \mathcal{D}(\mu) \cap H^\infty : \psi g \in [f]_{\mathcal{D}(\mu)}\}.$$

Since $[f]_{\mathcal{D}(\mu)} \cap H^\infty \subset \mathcal{J}$ and $h \in \mathcal{J}$,

$$\underline{\mathcal{Z}}(\mathcal{J}) \subset \underline{\mathcal{Z}}([f]_{\mathcal{D}(\mu)}) \cap \underline{\mathcal{Z}}(h) \subset \{\zeta_0\}.$$

Hence by Lemma 4.4, $z - \zeta_0 \in \mathcal{J}$ and then $(z - \zeta_0)g \in [f]_{\mathcal{D}(\mu)}$. Since by Theorem 2.3, we have $[(z - \zeta_0)g]_{\mathcal{D}(\mu)} = [(z - \zeta_0)]_{\mathcal{D}(\mu)} \cap [g]_{\mathcal{D}(\mu)}$. We get

$$[(z - \zeta_0)]_{\mathcal{D}(\mu)} \cap [g]_{\mathcal{D}(\mu)} \subset [f]_{\mathcal{D}(\mu)}.$$

Now we distinguish 2 cases.

- If $c_\mu(\{\zeta_0\}) = 0$, then $z - \zeta_0$ is cyclic and hence $g \in [f]_{\mathcal{D}(\mu)}$.
- If $c_\mu(\{\zeta_0\}) > 0$, since $c_\mu(\mathcal{Z}_{\mathbb{T}}(f)) = 0$, then $\zeta_0 \notin \mathcal{Z}_{\mathbb{T}}(f)$ and so $h(\zeta_0) \neq 0$. Now write

$$g = \underbrace{\frac{h}{h(\zeta_0)} g}_{\in [f]_{\mathcal{D}(\mu)}} + \underbrace{\frac{h(\zeta_0) - h}{h(\zeta_0)} g}_{\in [f]_{\mathcal{D}(\mu)}} \in [f]_{\mathcal{D}(\mu)}.$$

In the two cases $g \in [f]_{\mathcal{D}(\mu)}$ which gives a contradiction since and $\zeta_0 \in \underline{\mathcal{Z}}[f]$ and $g(\zeta_0) \neq 0$. So $\underline{\mathcal{Z}}([f]_{\mathcal{D}(\mu)}) = \emptyset$ and By Lemma 4.3, we obtain that f is cyclic for $\mathcal{D}(\mu)$.

□

6. Invariant subspaces of $\mathcal{D}(\mu)$

This section is devoted to the proof of Theorem 2. Let $f \in \mathcal{D}(\mu)$ be an outer function. Note that the outer function $h = f \wedge 1$ belongs to $\mathcal{D}(\mu) \cap H^\infty$ and $\underline{\mathcal{Z}}(h) = \underline{\mathcal{Z}}(f)$. Without loss of generality we can suppose that $f \in \mathcal{D}(\mu) \cap H^\infty$.

By Theorem 2.5, there exists an inner function Θ and an outer function $f \in \mathcal{D}(\mu) \cap H^\infty$ such that

$$\mathcal{M} = \Theta H^2 \cap [f]_{\mathcal{D}(\mu)}.$$

Let $\mathcal{M}_\mu(E)$ be the closed invariant subspace of $\mathcal{D}(\mu)$ given by

$$\mathcal{M}_\mu(E) = \{\psi \in \mathcal{D}(\mu) : \psi | E = 0\},$$

where $E = \{\lambda \in \mathcal{Z}(f) \cap \operatorname{supp} \mu : c_\mu(\{\lambda\}) > 0\}$.

We need to show that

$$\mathcal{M}_\mu(E) = [f]_{\mathcal{D}(\mu)}.$$

Let $\psi \in [f]_{\mathcal{D}(\mu)}$, and let $\lambda \in E$, since $c_\mu(\lambda) > 0$, $\psi(\lambda)$ exists and $\psi(\lambda) = 0$. So $\psi \in \mathcal{M}_\mu(E)$ and we have $[f]_{\mathcal{D}(\mu)} \subset \mathcal{M}_\mu(E)$.

Now we show the opposite inclusion. By Theorem 2.4, there exists an outer function $\phi \in \mathcal{M}_\mu(E) \cap H^\infty$ such that $\mathcal{M}_\mu(E) = [\phi]_{\mathcal{D}(\mu)}$. Consider the closed division ideal

$$\mathcal{J} = \{\psi \in \mathcal{D}(\mu) \cap H^\infty : \psi\phi \in [f]_{\mathcal{D}(\mu)}\}.$$

We have $f \in \mathcal{J}$ and by Lemma 5.2,

$$\underline{\mathcal{Z}}(\mathcal{J}) \subset \underline{\mathcal{Z}}(f) \cap \operatorname{supp} \mu.$$

• If $\underline{\mathcal{Z}}(\mathcal{J}) = \emptyset$, then by Theorem 4.3, we get $\mathcal{J} = \mathcal{D}(\mu) \cap H^\infty$.

• If $\underline{\mathcal{Z}}(\mathcal{J}) \neq \emptyset$. Since $\operatorname{supp} \mu$ is countable, the set $\underline{\mathcal{Z}}(J)$ have an isolated point ζ_0. Let $V = (e^{ia}, e^{ib})$ be a neighborhood of ζ_0 such that $V \cap \underline{\mathcal{Z}}(\mathcal{J}) = \{\zeta_0\}$. Consider again the outer functions $h = f_V$ and $g = f_{\mathbb{T}\setminus V}$. By Lemma 4.2, $g, h \in \mathcal{D}(\mu) \cap H^\infty$. Now let

$$\widetilde{\mathcal{J}} := \{\psi \in \mathcal{D}(\mu) \cap H^\infty : \psi g \in \mathcal{J}\}.$$

We have $h \in \widetilde{\mathcal{J}}$ and so $\underline{\mathcal{Z}}(\widetilde{\mathcal{J}}) \subset \{\zeta_0\}$. Since h is an outer function, by Lemma 4.4, $(z - \zeta_0) \in \widetilde{\mathcal{J}}$. Thus

$$(z - \zeta_0) g\phi \in [f]_{\mathcal{D}(\mu)}.$$

As before we distinguish 2 cases.

- If $c_\mu(\{\zeta_0\}) = 0$, then $z - \zeta_0$ is cyclic . Hence $g\phi \in [f]_{\mathcal{D}(\mu)}$.

- If $c_\mu(\{\zeta_0\}) > 0$, we distinguish again 2 cases.
  - if $f(\zeta_0) = 0$ then $\zeta_0 \in E$, $\mathcal{M}_\mu(\{\zeta_0\}) = \mathcal{M}_\mu(E)$ and

$$\begin{aligned}[(z-\zeta_0)\phi]_{\mathcal{D}(\mu)} &= [(z-\zeta_0)]_{\mathcal{D}(\mu)} \cap [\phi]_{\mathcal{D}(\mu)} \\ &= \mathcal{M}_\mu(E) \\ &= [\phi]_{\mathcal{D}(\mu)}.\end{aligned}$$

  So

$$\begin{aligned}[g\phi]_{\mathcal{D}(\mu)} &= [g]_{\mathcal{D}(\mu)} \cap [\phi]_{\mathcal{D}(\mu)} \\ &= [g]_{\mathcal{D}(\mu)} \cap [(z-\zeta_0)\phi]_{\mathcal{D}(\mu)} \\ &= [(z-\zeta_0)g\phi]_{\mathcal{D}(\mu)} \\ &\subset [f]_{\mathcal{D}(\mu)}.\end{aligned}$$

  - if $f(\zeta_0) \neq 0$ then $h(\zeta_0) = f_V(\zeta_0) \neq 0$ and $h \in \widetilde{\mathcal{J}}$. As before

$$g\phi = \frac{h}{h(\zeta_0)} g\phi + \frac{h(\zeta_0) - h}{h(\zeta_0)} g\phi \in [f]_{\mathcal{D}(\mu)}.$$

In all cases we have $g\phi \in [f]_{\mathcal{D}(\mu)}$. We get $g \in \mathcal{J}$, but this is absurd, since $g(\zeta_0) \neq 0$. So $\underline{\mathcal{Z}}(\mathcal{J}) = \emptyset$. By Theorem 4.3, we get $\mathcal{J} = \mathcal{D}(\mu) \cap H^\infty$.

So $\mathcal{D}(\mu) \cap H^\infty \subset \mathcal{J}$ and

$$\mathcal{M}_\mu(E) \subset [f]_{\mathcal{D}(\mu)}.$$

Now the proof is complete.

□

7. Final remarks

- Note that if μ is a positive finite measure such that $\operatorname{supp}\mu \subset E_1 \cup E_2$, where E_1 and E_2 are disjoint closed subsets of $\mathbb{T}$, then $\mathcal{D}(\mu) = \mathcal{D}(\mu_{|E_1}) \cap \mathcal{D}(\mu_{|E_2})$. In this case every closed invariant subspace $\mathcal{M}$ of $\mathcal{D}(\mu)$ can be written as $\mathcal{M} = \mathcal{M}_1 \cap \mathcal{M}_2$, where $\mathcal{M}_i$ is an invariant subspace of $\mathcal{D}(\mu_{|E_i})$ $(i = 1, 2)$.

- A closed set E is union of a countable set and perfect set. Using the same argument as in the proof of Theorem 1, one can prove that Brown-Shields conjecture is true for $\mathcal{D}(\mu)$ if and only if Brown-Shields conjecture is true for $\mathcal{D}(\mu_{|\mathcal{P}(\operatorname{supp}\mu)})$ where $\mathcal{P}(\operatorname{supp}\mu)$ is the perfect core of the support of μ.

References

[1] D. Adams, L. Hedberg, Function spaces and potential theory, Springer, Berlin 1996.
[2] A. Beurling, On two problems concerning linear operators in Hilbert space, Acta Math. 81 (1949), 239–255.
[3] A. Beurling, J. Deny, Dirichlet spaces, Proc. Nat. Acad. Sci. USA 45 (1959) 208–215.
[4] L. Brown, A. Shields. Cyclic vectors in the Dirichlet space, Trans. Amer. Math. Soc. 285 (1984), 269-304.
[5] L. Carleson, Selected Problems on Exceptional sets. Van Nostrand Mathematical Studies, No. 13. Princeton, NJ: Van Nostrand.

[6] O. El-Fallah, K. Kellay, T. Ransford. Cyclicity in the Dirichlet space, Ark. Mat. 44 (2006), 61-86.
[7] O. El-Fallah, K. Kellay, T. Ransford. On the Brown-Shields conjecture for cyclicity in the Dirichlet space. Adv. Math., 222 (2009) (6): 2196-2214.
[8] O. El-Fallah, K. Kellay, J. Mashreghi, T. Ransford. A primer on the Dirichlet space. Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge, 2014.
[9] O. El-Fallah, Y. Elmadani, K. Kellay, Kernel estimates and capacity in the Dirichlet spaces. http://arxiv.org/abs/1411.1036
[10] D. Guillot, Fine boundary behavior and invariant subspaces of harmonically weighted Dirichlet spaces. Complex Anal. Oper. Theory. To appear
[11] H. Hedenmalm, A. Shields. Invariant subspaces in Banach spaces of analytic functions, Michigan Math. J. 37 (1990), 91-104.
[12] S. Richter, A representation theorem for cyclic analytic two-isometries, Trans. Am. Math. Soc., 328 (1994) 1–26.
[13] S. Richter, Invariant subspaces of the Dirichlet shift, J. Reine Angew. Math., 386 (1988) 205-220.
[14] S. Richter and C. Sundberg, A formula for the local Dirichlet integral, Michigan Math. J., 38 (1991) 355-379.
[15] S. Richter, C. Sundberg, Multipliers and invariant subspaces in the Dirichlet space, J. Operator Theory 28 (1992) 167-186.
[16] S. Richter and S. Sundberg, Invariant subspaces of the Dirichlet shift and pseudocontinuations, Trans. Am. Math. Soc., 341 51984) 863-879.

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