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Structure of W_4 -immersion free graphs*

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Abstract

We study the structure of graphs that do not contain the wheel on 5 vertices W_4 as an immersion, and show that these graphs can be constructed via 1, 2, and 3-edge-sums from subcubic graphs and graphs of bounded treewidth.

1 Introduction

All graphs in this paper are undirected and may have multiple edges. See the monograph by Diestel [4] for undefined notation and terminology. A recurrent theme in structural graph theory is the study of graphs which exclude a fixed pattern. The notion of appearing as a pattern gives rise to various graph containment relations. Maybe the most famous example is the minor relation that has been widely studied, in particular with the fundamental results of Kuratowski and Wagner who proved that planar graphs are exactly those graphs that contain neither K_5 nor $K_{3,3}$ as a (topological) minor. Another famous such example is Wagner's theorem, which describes the structure of K_5 -minor free graphs.

The structure of graphs that exclude a fixed graph as a minor was studied in the seminal Graph Minor series of papers by Robertson and Seymour [11]. However, while the structure of graphs that exclude a fixed graph H as a minor has been extensively studied, the structure of graphs excluding a fixed graph H as a topological minor or as an immersion has not received as much attention. While a general structure theorem for topological minor free graphs was very recently provided by Grohe and Marx [9], finding an exact characterization of the graphs that exclude K_5 as a topological minor remains a notorious open problem. Similarly, Wollan gave a structure theorem for graphs excluding complete graphs as immersions [16]. A graph G contains a graph H

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as an immersion if H can be obtained from G by a sequence of vertex deletions, edge deletions and replacing edge-disjoint paths with single edges. Observe that if a graph G contains a graph H as a topological minor, then G also contains H as an immersion, as vertex-disjoint paths are also edge-disjoint. In 2011, DeVos et al. [3] proved that if the minimum degree of a graph G is at least $200t$ then G contains the complete graph on t vertices as an immersion. In [6] Ferrara et al. provided a lower bound on the minimum degree of any graph G in order to ensure that a given graph H is contained in G as an immersion.

In this paper, we prove a structural characterization of the graphs that exclude W_4 as an immersion and show that they can be constructed from graphs that are either subcubic or have treewidth bounded by a constant. We denote by W_4 the wheel with 4 spokes, i.e., the graph obtained from a cycle on 4 vertices by adding a universal vertex. The structure of graphs that exclude W_4 and W_5 as a topological minor has been studied by Farr [5]. For the case of W_4 , he proved that these graphs can be constructed via clique-sums of order at most 3 from subcubic graphs. However, this characterization only applies to simple graphs. More recently, Robinson and Farr studied the structure of graphs that do not contain W_5 [14] and W_6 [15] as a topological minor.

We conclude this section with some elementary definitions that will be required later.

Definition 1 *An immersion of H in G is a function α with domain $V(H) \cup E(H)$, such that:*

- $\alpha(v) \in V(G)$ for all $v \in V(H)$, and $\alpha(u) \neq \alpha(v)$ for all distinct $u, v \in V(H)$;
- for each edge e of H , $\alpha(e)$ is a path of G with ends $\alpha(u), \alpha(v)$;
- for all distinct $e, f \in E(H)$, $E(\alpha(e) \cap \alpha(f)) = \emptyset$.

Definition 2 *Let G, G_1 , and G_2 be graphs. Let $t \geq 1$ be a positive integer. The graph G is a t -edge-sum of G_1 and G_2 if the following holds. There exist vertices $v_i \in V(G_i)$ such that $|E_{G_i}(v_i)| = t$ for $i \in [2]$ and a bijection $\pi : E_{G_1}(v_1) \rightarrow E_{G_2}(v_2)$ such that G is obtained from $(G_1 - v_1) \cup (G_2 - v_2)$ by adding an edge xy for every pair of edges e_1 and e_2 such that $e_1 = xv_1$, $e_2 = yv_2$, and $v_2 = \pi(v_1)$. We say that the edge-sum is internal if both G_1 and G_2 contain at least 2 vertices and denote the internal t -edge-sum of G_1 and G_2 by $G_1 \hat{\oplus}_t G_2$.*

The (*elementary*) *wall* of height r is the graph W_r with vertex set $V(W_r) = \{(i, j) \mid i \in [r + 1], j \in [2r + 2]\}$ in which we make two vertices (i, j) and (i', j') adjacent if and only if either $i = i'$ and $j' \in \{j - 1, j + 1\}$ or $j' = j$ and $i' = i + (-1)^{i+j}$, and then remove all vertices of degree 1. The vertices of this vertex set are called *original* vertices of the wall. A *subdivided wall* of height r is the graph obtained from W_r after replacing some of its vertices by internally vertex-disjoint paths.

It is well known that large treewidth ensures the existence of a large wall as a topological minor. Recently, Chekuri and Chuzhoy proved that polynomial treewidth suffices to ensure the existence of a wall as a topological minor, solving a long standing open problem.

Theorem 1 [1] *Let G be a graph and r a positive integer. If G does not contain W_r as a topological minor, then G has treewidth $O(r^{98})$.*

2 Structure of graphs excluding W_4 as an immersion

In this section, we prove the main result of our paper, namely we provide a structure theorem for graphs that exclude W_4 as an immersion. We first show that the property of containing W_4 as an immersion is closed under 3-edge sums.

Theorem 2 *Let G , G_1 , and G_2 be graphs such that $G = G_1 \hat{\oplus}_t G_2$, with $t \in [3]$. Then, G contains W_4 as an immersion if and only if G_1 or G_2 does as well.*

We now provide a technical lemma that will be crucial for the proof of Theorem 4.

Lemma 1 *There exists a function f such that for every integer $r \geq 60000$ and every graph G that does not contain W_4 as an immersion, has no internal 3 edge-cut, and has a vertex u with $d(u) \geq 4$, if $\text{tw}(G) \geq f(r)$, then there exist sets $Z = \{z_1, \dots, z_r\}$, S_1, \dots, S_r , and X , that satisfy the following properties:*

- (i) $z_i \in S_i, \forall i \in \{1, \dots, r\}$;
- (ii) $z_i \in S_j, \forall i \neq j \in \{1, \dots, r\}$;
- (iii) $u \in \bigcap_{i \in \{1, \dots, r\}} S_i$;
- (iv) $\partial(S_i) \leq 6$;
- (v) $G[S_i]$ is connected, $\forall i \in \{1, \dots, r\}$;
- (vi) $X \cap S_i = \emptyset, \forall i \in \{1, \dots, r\}$;
- (vii) For every $Z' \subseteq Z$ such that $|Z'| \geq 7$, there is a 7-flow from Z' to X ;

Lemma 1 essentially states that large treewidth yields a large number of vertex disjoint cycles that are highly connected to each other, and an additional disjoint set that is highly connected to these cycles. However, this, together with the assumption that W_4 does not immerse in G , implies that there cannot be a large flow between a vertex of degree at least 4 and one of the cycles. We will combine this fact with the notion of important separators to obtain Lemma 2. Please refer to e.g., [2, 10] for a definition of (X, Y) -important separator.

Theorem 3 [2, 10] *Let $X, Y \subseteq V(G)$ be two sets of vertices in graph G , let $k \geq 0$ be an integer, and let S_k be the set of all (X, Y) -important separators of size at most k . Then $|S_k| \leq 4^k$ and S_k can be constructed in time $|S_k| \cdot n^{O(1)}$.*

Theorem 3 states that the number of important separators of a certain size is bounded. The next lemma uses this fact together with Lemma 1.

Lemma 2 *Let G be a graph such that G does not contain W_4 as an immersion, has no internal 3 edge-cut and has a vertex u with $d(u) \geq 4$. Then the treewidth of G is upper bounded by a constant.*

We are now ready to prove the main theorem of our paper.

Theorem 4 *Let G be a graph that does not contain W_4 as an immersion. Then the prime graphs of a decomposition of G via i -edge-sums, $i \in [3]$, are either subcubic graphs, or have treewidth bounded from above by a constant.*

We conclude this section by noting that Theorem 4 is that it is in a sense tight: indeed, both the fact that we decompose along edge-sums of order at most 3 and the requirement that a unique vertex of degree at least 4 is sufficient to enforce small treewidth are necessary. The fact that decomposing along internal 3 edge sums is necessary can be seen from the fact that there are internally 3 connected graphs that have vertices of degree at least 4 and yet do not contain W_4 as an immersion, e.g. a cycle where every edge is doubled.

References

- [1] Chekuri, C. and Chuzhoy, J.: Polynomial Bounds for the Grid-Minor Theorem. *ArXiv e-prints*, abs/1305.6577 (2013)
- [2] Chen, J., Liu, Y., and Lu S.: An improved parameterized algorithm for the minimum node multiway cut problem. In: Proceedings of WADS 2007, LNCS, vol. 4619, pp. 495–506, Springer, Berlin (2007)
- [3] DeVos, M., Dvořák, Z., Fox, J., McDonald, J., Mohar, B. and Scheide, D.: Minimum degree condition forcing complete graph immersion. *ArXiv e-prints*, abs/1101.2630 (2011)
- [4] Diestel R.: *Graph theory*, volume 173 of *Graduate Texts in Mathematics*. Springer-Verlag, Berlin, electronic edition (2005)
- [5] Farr, G.: The subgraph homeomorphism problem for small wheels. *Discrete Mathematics*, 71(2):129–142 (1988)
- [6] Ferrara, M., Gould, R. J, Tansey, G. and Whalen, T.: On H-immersions. *Journal of Graph Theory*, 57(3):245–254 (2008)
- [7] Giannopoulou, A. C., Kaminski, M. and Thilikos, D. M.: Forbidding Kuratowski graphs as immersions. *Arxiv e-prints*, abs/1207.5329 (2012)
- [8] Giannopoulou, A. C., Salem, I. and Zoros, D.: Effective computation of immersion obstructions for unions of graph classes. In: Proceedings of SWAT 2012, LNCS, vol. 7357, pp. 165–176, Springer, Berlin (2012)
- [9] Grohe, M. and Marx, D.: Structure theorem and isomorphism test for graphs with excluded topological subgraphs. In: Proceedings of STOC 2012, pp. 173–192, ACM, New York (2012)
- [10] Marx, D.: Parameterized graph separation problems. *Theoretical Computer Science*, 351(3):394–406 (2006)
- [11] Robertson, N. and Seymour, P. D.: Graph Minors. XXI. Excluding a non-planar graph. *Journal of Combinatorial Theory, Series B*, 89(1):43–76 (2003)
- [12] Robertson, N. and Seymour, P. D.: Graph Minors. XX. Wagner’s conjecture. *Journal of Combinatorial Theory, Series B*, 92(2):325–357 (2004)
- [13] Robertson, N. and Seymour, P. D.: Graph Minors. XXIII. Nash-Williams’ immersion conjecture. *Journal of Combinatorial Theory, Series B*, 100(2):181–205 (2010)
- [14] Robinson, R. and Farr, G.: Structure and recognition of graphs with no 6-wheel subdivision. *Algorithmica*, 55(4):703–728 (2009)
- [15] Robinson, R. and Farr, G.: Graphs with no 7-wheel subdivision. *Arxiv e-prints*, abs/1311.0573 (2013)
- [16] Wollan, P.: The structure of graphs not admitting a fixed immersion. *Arxiv e-prints*, abs/1302.3867 (2013)