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The Parameterized Complexity of Graph Cyclability*

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Abstract. The cyclability of a graph is the maximum integer k for which every k vertices lie on a cycle. The algorithmic version of the problem, given a graph G and a non-negative integer k , decide whether the cyclability of G is at least k , is NP-hard. We prove that this problem, parameterized by k , is co-W[1]-hard. We give an FPT algorithm for planar graphs that runs in time $2^{2^{O(k^2 \log k)}} \cdot n^2$. Our algorithm is based on a series of graph theoretical results on cyclic linkages in planar graphs.

1 Introduction

In the opening paragraph of his book *Extremal Graph Theory* Béla Bollobás notes: “Perhaps the most basic property a graph may possess is that of being connected. At a more refined level, there are various functions that may be said to measure the connectedness of a connected graph.” Indeed, connectivity is one of the fundamental properties considered in graph theory and studying different variants of connectivity gives a better understanding of this property. Many such alternative connectivity measures have been studied in graph theory but very little is known about their algorithmic properties. The main goal of this paper is to focus on one of such parameters – *cyclability* – from an algorithmic point of view. Cyclability can be thought of as a quantitative measure of Hamiltonicity, or as a natural “tuning” parameter between connectivity and Hamiltonicity.

Cyclability. For a positive integer k , a graph is k -cyclable if every k vertices lie on a common cycle; we assume that any graph is 1-cyclable. Respectively, the *cyclability* of a graph G is the maximum integer k for which G is k -cyclable. Cyclability is well studied in the graph theory literature. Dirac proved that

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cyclability of a k -connected graph is at least k , for $k \geq 2$ [8]. Watkins and Mesner [32] characterized the extremal graphs for the theorem of Dirac. There is a variant of cyclability restricted only to a set of vertices of a graph. Generalizing the theorem of Dirac, Flandrin et al. [18] proved that if a set of vertices S in a graph G is k -connected, then there is a cycle in G through any k vertices of S . (A set of vertices S is k -connected in G if a pair of vertices in S cannot be separated by removing at most $k - 1$ vertices of G .) Another avenue of research is lower-bounds on cyclability of graphs in restricted families. For example, every 3-connected claw-free graph has cyclability at least 6 [26] and every 3-connected cubic planar graph has cyclability at least 23 [3].

Clearly, a graph G is Hamiltonian if and only if its cyclability equals $|V(G)|$. Therefore, we can think of cyclability as a quantitative measure of Hamiltonicity. A graph G is *hypohamiltonian* if it is not Hamiltonian but all graphs obtained from G by deleting one vertex are. Clearly, a graph G is hypohamiltonian if and only if its cyclability equals $|V(G)| - 1$. Hypohamiltonian graphs appear in combinatorial optimization and are used to define facets of the traveling salesman polytope [22]. Curiously, the computational complexity of deciding whether a graph is hypohamiltonian seems to be open.

To our knowledge no algorithmic study of cyclability has been done so far. In this paper we initiate this study. For this, we consider the following problem.

CYCLABILITY

Input: A graph G and a non-negative integer k .

Question: Is every k -vertex set S in G cyclable, i.e., is there a cycle C in G such that $S \subseteq V(C)$?

CYCLABILITY with $k = |V(G)|$ is HAMILTONICITY and HAMILTONICITY is NP-complete for planar cubic graphs [21]. Hence, we have the following.

Proposition 1. *CYCLABILITY is NP-hard for cubic planar graphs.*

Parameterized complexity. A parameterized problem has as instances pairs (I, k) where I is the main part and k is the parameterized part. Parameterized Complexity settles the question of whether a parameterized problem is solvable by an algorithm (we call it *FPT-algorithm*) of time complexity $f(k) \cdot |I|^{O(1)}$ where $f(k)$ is a function that does not depend on n . If such an algorithm exists, we say that the parameterized problem belongs in the class FPT. In a series of fundamental papers (see [14,15,12,13]), Downey and Fellows invented a series of complexity classes, namely the classes such as $W[1] \subseteq W[2] \subseteq \dots \subseteq W[SAT] \subseteq W[P] \subseteq XP$ and proposed special types of reductions such that hardness for some of the above classes makes it rather impossible that a problem belongs in FPT (we stress that $FPT \subseteq W[1]$). We mention that XP is the class of parameterized problems such that for every k there is an algorithm that solves that problem in time $O(|I|^{f(k)})$, for some function f (that does not depend on $|I|$). For more on parameterized complexity, we refer the reader to [11], [19], or [24].

Our results. In this paper we deal with the parameterized complexity of CYCLABILITY parameterized by k . It is easy to see that CYCLABILITY is in XP. For a graph G , we can check all possible subsets X of $V(G)$ of size k . For each subset

X , we consider $k!$ orderings of its vertices, and for each sequence of k vertices x_1, \dots, x_k of X , we use the celebrated result of Robertson and Seymour [29], to check whether there are k disjoint paths that join x_{i-1} and x_i for $i \in \{1, \dots, k\}$ assuming that $x_0 = x_k$. We return a YES if and only if there is an ordering that has the required disjoint paths for each set X .

Is it possible that CYCLABILITY is FPT when parameterized by k ? By showing that CYCLABILITY is co-W[1]-hard (even for split graphs⁵), we show that this is rather unlikely. However, we prove that the problem is FPT on planar graphs:

Theorem 1. *It is W[1]-hard to decide for a split graph G and a positive integer k , whether G has k vertices such that there is no cycle in G that contains these k vertices, when the problem is parameterized by k .*

Theorem 2. *The CYCLABILITY problem, when parameterized by k , is in FPT when its input graphs are restricted to be planar graphs. Moreover, the corresponding FPT-algorithm runs in $2^{2^{O(k^2 \log k)}} \cdot n^2$ steps.*

Our techniques. Theorem 3 is proved in the appendix and the proof is a reduction from the CLIQUE problem.

The two key ingredients in the proof of Theorem 2 are a new two-step version of the *irrelevant vertex technique*, a new combinatorial concept of *cyclic linkages* and a strong notion of *vitality* on them (vital linkages played an important role in the Graph Minors series, in [30] and [27]). The proof of Theorem 2 is presented in Section 3 (with references to the appendix). Below we give a rough sketch of our method.

We work with a variant of CYCLABILITY in which some vertices (initially all) are colored. We only require that every k colored vertices lie on a common cycle. If the treewidth of the input graph G is “small” (bounded by an appropriate function of k), we employ a dynamic programming routine to solve the problem. Otherwise, there exists a cycle in a plane embedding of G such that the graph H in the interior of that cycle is “bidimensional” (contains a large subdivided wall) but still of bounded treewidth. This structure permits to distinguish in H a sequence \mathcal{C} of, sufficiently many, concentric cycles that are all traversed by some, sufficiently many, paths of H . Our first aim is to check whether the distribution of the colored vertices in these cycles yields some “big uncolored area” of H . In this case we declare some “central” vertex of this area *problem-irrelevant* in the sense that its removal creates an equivalent instance of the problem. If such an area does not exist, then R is “uniformly” distributed inside the cycle sequence \mathcal{C} . Our next step is to set up a sequence of instances of the problem, each corresponding to the graph “cropped” by the interior of the cycles of \mathcal{C} , where all vertices of a sufficiently big “annulus” in it are now uncolored. As the graphs of these instances are subgraphs of H and therefore they have bounded treewidth, we can get an answer for all of them by performing a sequence of dynamic programming calls (each taking a linear number of steps). At this point, we prove that if one of these instances is a NO-instance then we just report that the initial instance is a NO-instance and we stop. Otherwise, we

⁵ A graph G is *split* if $V(G)$ can be partitioned into a clique and an independent set.

pick a colored vertex inside the most “central” cycle of \mathcal{C} and we prove that this vertex is *color-irrelevant*, i.e., an equivalent instance is created when this vertex is not any more colored. In any case, the algorithm produces either a solution or some “simpler” equivalent instance that either contains a vertex less or a colored vertex less. This permits a linear number of recursive calls of the same procedure. To prove that these two last critical steps work correctly, we have to introduce several combinatorial tools. One of them is the notion of strongly vital linkages, a variant of the notion of vital linkages introduced in [30], which we apply to terminals traversed by cycles instead of terminals linked by paths, as it has been done in [30]. This notion of vitality permits a significant restriction of the expansion of the cycles that certify that sets of k vertices are cyclable and is able to justify both critical steps of our algorithm. The proofs of the combinatorial results that support our algorithm are presented in Section 4 and we believe that they have independent combinatorial importance.

Structure of the paper. The paper is organized as follows. In Section 2 we give a set of definitions that are necessary for the presentation of our algorithm. The main steps of the algorithm are presented in Section 3 and the combinatorial results (along with the necessary definitions) are presented in Section 4. Section 5 is devoted to the co-W[1]-hardness of CYCLABILITY for general graphs. We conclude with some discussion and open questions in Section 7.

2 Definitions and preliminary results

For any graph G , $V(G)$ (respectively $E(G)$) denotes the *set of vertices* (respectively *edges*) of G . A graph G' is a *subgraph* of a graph G if $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$, and we denote this by $G' \subseteq G$. If S is a set of vertices or a set of edges of a graph G , the graph $G \setminus S$ is the graph obtained from G after the removal of the elements of S . Given two graphs G_1 and G_2 , we define $G_1 \cap G_2 = (V(G_1) \cap V(G_2), E(G_1) \cap E(G_2))$ and $G_1 \cup G_2 = (V(G_1) \cup V(G_2), E(G_1) \cup E(G_2))$.

For every vertex $v \in V(G)$ the *neighborhood* of v in G , denoted by $N_G(v)$, is the subset of vertices that are adjacent to v , and its size is called the *degree* of v in G , denoted by $\deg_G(v)$. The maximum degree $\Delta(G)$ of a graph G is the maximum value taken by \deg_G over $V(G)$. A *cycle* of G is a subgraph of G that is connected and all its vertices have degree 2. We call a set of vertices $S \subseteq V(G)$ *cyclable* if for some cycle C of G , it holds that $S \subseteq V(C)$.

Treewidth. A *tree decomposition* of a graph G is a pair $\mathcal{D} = (\mathcal{X}, T)$ in which T is a tree and $\mathcal{X} = \{X_i \mid i \in V(T)\}$ is a family of subsets of $V(G)$ such that:

- $\bigcup_{i \in V(T)} X_i = V(G)$
- for each edge $e = \{u, v\} \in E(G)$ there exists an $i \in V(T)$ such that both u and v belong to X_i
- for all $v \in V$, the set of nodes $\{i \in V(T) \mid v \in X_i\}$ forms a connected subtree of T .

The *width* of a tree decomposition is $\max\{|X_i| \mid i \in V(T)\} - 1$. The *treewidth* of a graph G (denoted by $\mathbf{tw}(G)$) is the minimum width over all possible tree decompositions of G .

Concentric cycles. Let G be a graph embedded in the sphere \mathbb{S}_0 and let $\mathcal{D} = \{D_1, \dots, D_r\}$, be a sequence of closed disks in \mathbb{S}_0 . We call \mathcal{D} *concentric* if $D_1 \subseteq D_2 \subseteq \dots \subseteq D_r$ and no point belongs in the boundary of two disks in \mathcal{D} . We call a sequence $\mathcal{C} = \{C_1, \dots, C_r\}$, $r \geq 2$, of cycles of G *concentric* if there exists a concentric sequence of closed disks $\mathcal{D} = \{D_1, \dots, D_r\}$, such that C_i is the boundary of D_i , $i \in \{1, \dots, r\}$. For $i \in \{1, \dots, r\}$, we set $\bar{C}_i = D_i$, $\dot{C}_i = \bar{C}_i \setminus C_i$, and $\hat{C}_i = G \cap D_i$ (notice that \bar{C}_i and \dot{C}_i are sets while \hat{C}_i is a subgraph of G). Given $i, j, i \leq j - 1$, we denote by $\hat{A}_{i,j}$ the graph $\hat{C}_j \setminus \hat{C}_i$. Finally, given a $q \geq 1$, we say that a set $R \subseteq V(G)$ is q -dense in \mathcal{C} if, for every $i \in \{1, \dots, r - q + 1\}$, $V(\hat{A}_{i,i+q-1}) \cap R \neq \emptyset$.

Railed annulus. Let r and q be integers such that $r \geq 2$ and $q \geq 1$ and let G be a graph embedded in the \mathbb{S}_0 . A (r, q) -*railed annulus* in G is a pair $(\mathcal{C}, \mathcal{W})$ such that $\mathcal{C} = \{C_1, C_2, \dots, C_r\}$ is a sequence of r concentric cycles that are all met by a sequence \mathcal{W} of q paths P_1, P_2, \dots, P_q (called *rails*) in such a way that $\mathbf{UW} \subseteq A_{1,r}$ and the intersection of a cycle and a rail is always connected, that is, it is a (possibly trivial) path.

Walls and subdivided walls. Let $k \geq 1$. A *wall of height k* is the graph obtained from a $((k + 1) \times (2 \cdot k + 2))$ -grid with vertices (x, y) , $x \in \{1, \dots, 2 \cdot k + 4\}$, $y \in \{1, \dots, k + 1\}$, after the removal of the “vertical” edges $\{(x, y), (x, y + 1)\}$ for odd $x + y$, and then the removal of all vertices of degree 1. We denote such a wall by W_k . A *subdivided wall of height k* is a wall obtained from W_k after replacing some of its edges by paths without common internal vertices. We call such a path an *edge-path* of W . The *perimeter* P_W of a subdivided wall W of height k is the cycle defined by its boundary. Let $C_2 = P_W$ and let C_1 be any cycle of W that has no common vertices with P_W . Notice that $\mathcal{C} = \{C_1, C_2\}$ is a sequence of concentric cycles in G . We define the *compass K_W of W in G* as the graph \hat{C}_2 .

Layers of a wall. Let W be a subdivided wall of height $h \geq 2$. The *layers* of W are recursively defined as follows. The first layer, J_1 , of W is its perimeter. For $i \in \{2, \dots, \lfloor \frac{h}{2} \rfloor\}$, the i -th layer, J_i , of W is the perimeter of the subwall W' obtained from W by removing its perimeter and repetitively removing occurring vertices of degree 1. We denote the *layer set* of W by $\mathcal{J}_W = \{J_1, \dots, J_{\lfloor \frac{h}{2} \rfloor}\}$

Given a graph G we denote by $\mathbf{gw}(G)$ the maximum h for which G contains a subdivided wall of height h as a subgraph. The next lemma follows easily combining results in [20], [23], and [28].

Lemma 1. *If G is a planar graph, then $\mathbf{tw}(G) \leq 9 \cdot \mathbf{gw}(G) + 1$.*

3 The algorithm

This section is devoted to the proof of Theorem 2. We consider the following slightly more general problem.

PLANAR ANNOTATED CYCLABILITY

Input: A plane graph G , a set $R \subseteq V(G)$, and a non-negative integer k .
Question: Does there exist, for every set S of k vertices in R , a cycle C of G such that $S \subseteq V(C)$?

In this section, for simplicity, we denote PLANAR ANNOTATED CYCLABILITY by Π . Theorem 2 follows directly from the following lemma.

Lemma 2. *There is an algorithm that solves Π in $2^{2^{O(k^2 \log k)}} \cdot n^2$ steps.*

Problem/color-irrelevant vertices. Let (G, k, R) be an instance of Π . We call a vertex $v \in V(G) \setminus R$ *problem-irrelevant* if (G, k, R) is a YES-instance if and only if $(G \setminus v, k, R)$ is a YES-instance. We call a vertex $v \in R$ *color-irrelevant* when (G, k, R) is a YES-instance if and only if $v \in R$ and $(G, k, R \setminus \{v\})$ is a YES-instance.

Before we present the algorithm of Lemma 2, we need to introduce three algorithms that are used in it as subroutines.

Algorithm DP $(G, R, k, q, \mathcal{D})$

Input: A graph G , a vertex set $R \subseteq V(G)$, two non-negative integers k and q , where $k \leq q$, and a tree decomposition \mathcal{D} of G of width q .

Output: An answer whether (G, R, k) is a YES-instance of Π or not.

Running time: $2^{2^{O(q \cdot \log q)}} \cdot n$.

Algorithm **DP** is based on dynamic programming on tree decompositions of graphs. The technical details are omitted in this extended abstract.

Algorithm Compass (G, q)

Input: A planar graph G and a non-negative integer q .

Output: Either a tree decomposition of G of width at most $18q$ or a subdivided wall W of G of height q and a tree decomposition \mathcal{D} of the compass K_W of W of width at most $18q$.

Running time: $2^q \cdot n$.

We describe algorithm **Compass** in Subsection 3.1.

Algorithm concentric_cycles (G, R, k, q, W)

Input: A planar graph G , a set $R \subseteq V(G)$, a non-negative integer k , and a subdivided wall W of G of height at least $392k^2 + 40k$.

Output: Either a problem-irrelevant vertex v or a sequence $\mathcal{C} = \{C_1, C_2, \dots, C_{98k+2}\}$ of concentric cycles of G , with the following properties:

- (1) $C_1 \cap R \neq \emptyset$.
- (2) The set R is $32k$ -dense in \mathcal{C} .
- (3) There exists a sequence \mathcal{W} of $2k + 1$ paths in K_W such that $(\mathcal{C}, \mathcal{W})$ is a $(98k + 2, 2k + 1)$ -railed annulus.

Running time: $O(n)$.

We describe Algorithm **concentric_cycles** in Subsection 3.2. We now use the above three algorithms to describe the main algorithm of this paper that is the following.

Algorithm Planar-Annotated-Cyclability(G, R, k)

Input: A planar graph G , a set $R \subseteq V(G)$, and a non-negative integer k .

Output: An answer whether (G, R, k) is a YES-instance of Π , or not.

Running time: $2^{2^{O(k^2 \log k)}} \cdot n^2$.

[Step 1.] Let $r = 98k^2 + 2k$, $y = 16k$, and $q = 2y + 4r$. If **Compass**(G, q) returns a tree decomposition of G of width $w = 18q$, then return **DP**(G, R, k, w) and stop. Otherwise, the algorithm **Compass**(G, q) returns a subdivided wall W of G of height q and a tree decomposition \mathcal{D} of the compass K_W of W of width at most w .

[Step 2.] If the algorithm **concentric_cycles**(G, R, k, q, W) returns a problem-irrelevant vertex v , then return **Planar-Annotated-Cyclability**($G \setminus v, R \setminus v, k$) and stop. Otherwise, it returns a sequence $\mathcal{C} = \{C_1, C_2, \dots, C_r\}$ of concentric cycles of G with the properties (1)–(3).

[Step 3.] For every $i \in \{1, \dots, r - 98k - 2\}$ let w_i be a vertex in $\hat{A}_{i+k, i+33k} \cap R$ (this vertex exists as, from property (2), R is $32k$ -dense in \mathcal{C}), let $R_i = (R \cap V(\hat{C}_i)) \cup \{w_i\}$, and let \mathcal{D}_i be a tree decomposition of \hat{C}_i of width at most $w - 1$ – this tree decomposition can be constructed in linear time from \mathcal{D} as each \hat{C}_i is a subgraph of K_W .

[Step 4.] If, for some $i \in \{1, \dots, r - 98k - 2\}$, the algorithm **DP**($\hat{C}_i, R_i, k, q, \mathcal{D}_i$) returns a negative answer, then return a negative answer and stop. Otherwise return **Planar-Annotated-Cyclability**($G, R \setminus v, k$) where v is some vertex of \hat{C}_1 that belongs in R (the choice of v is possible due to property (1)).

Proof of Lemma 2. The only non-trivial step in the above algorithm is Step 4. Its correctness follows from Lemma 6, presented in Subsection 3.3.

We now proceed with the analysis of the running time of the algorithm. Observe first that the call of **Compass**(G, q) in Step 1 takes $2^{k^{O(1)}} \cdot n$ steps and, in case, a tree decomposition is returned, the **DP** requires $2^{2^{O(k^2 \log k)}} \cdot n$ steps. For Step 2, the algorithm **concentric_cycles** takes $O(n)$ steps and if it returns a problem-irrelevant vertex, then the whole algorithm is applied again for a graph with one vertex less. Suppose now that Step 2 returns a sequence \mathcal{C} of concentric cycles of G with the properties (1)–(3). Then the algorithm **DP** is called $O(k^2)$ times and this takes in total $2^{2^{O(k^2 \log k)}} \cdot n$ steps. After that, the algorithm either concludes to a negative answer or is called again with one vertex less in the set R . In both cases where the algorithm is called again we have that the quantity $|V(G)| + |R|$ is becoming smaller. This means that the recursive calls of the algorithm cannot be more than $2n$. Therefore the total running time is bounded by $2^{2^{O(k^2 \log k)}} \cdot n^2$ as required. \square

3.1 The algorithm Compass

Before we start the description of algorithm **Compass** we present a result that follows by Proposition 1, the algorithms in [25] and [4], and the fact that finding a subdivision of a planar k -vertex graph H that has maximum degree 3 in a

graph G can be done, using dynamic programming, in $2^{O(k \cdot \log k)} \cdot n$ steps (see also [1]).

Lemma 3. *There exists an algorithm A_1 that, given a graph G and an integer h , outputs either a tree decomposition of G of width at most $9h$ or a subdivided wall of G of height h . This algorithm runs in $2^{h^{O(1)}} \cdot n$ steps.*

*Description of algorithm **Compass*** Let $q' = 9q$. We use the routine A_2 that receives as input a subdivided wall W of G with height equal to some even number h and outputs a subdivided wall W' of G such that with W' has height $h/2$ and $|V(K_{W'})| \leq |V(G)|/4$. A_2 uses the fact that, in W , there are 4 vertex-disjoint subdivided subwalls of W of height $h/2$. Among them, A_2 outputs the one with the minimum number of vertices and this can be done in $O(n)$ steps. The algorithm **Compass** uses as subroutines the routine A_2 and the algorithm A_1 of Lemma 3.1.

Algorithm **Compass**(G, q)

[Step 1.] if $A_1(G, 2q)$ outputs a tree decomposition \mathcal{D} of G with width at most $2q'$ then return \mathcal{D} ,
otherwise it outputs a subdivided wall W of G of height $2q$
[Step 2.] Let $W' = A_2(W)$
if $A_1(K_{W'}, 2q)$ outputs a tree decomposition \mathcal{D} of $K_{W'}$ with width at most $2q'$ then return W' and \mathcal{D} ,
otherwise $W \leftarrow W'$ and go to Step 2.

Notice that, if A terminates after the first execution of step 1, then it outputs a tree decomposition of G of width at most $2q'$. Otherwise, the output is a subdivided wall W' of height k in G and a tree decomposition of $K_{W'}$ of width at most $2q'$ (notice that as long as this is not the case, the algorithm keeps returning to step 2). The application of routine A_2 ensures that the number of vertices of every new K_W is at least four times smaller than the one of the previous one. Therefore, the i -th call of the algorithm A_1 requires $\mathcal{O}(2^{h^{O(1)}} \cdot \frac{n}{2^{2(i-1)}})$ steps. As $\sum_{i=0}^{\infty} \frac{1}{2^{2i}} = O(1)$, algorithm **Compass** has the same running time as algorithm A_1 .

3.2 The Algorithm concentric_cycles

We require first the following two lemmata, The first one is strongly based on the combinatorial Lemma 11 that is the main result of Section 4.

Lemma 4. *Let (G, R, k) be an instance of Π and let $\mathcal{C} = \{C_1, \dots, C_r\}$ be a sequence of concentric cycles in G such that $V(\hat{C}_r) \cap R = \emptyset$. If $r \geq 16 \cdot k$, then all vertices in \hat{C}_1 are problem-irrelevant.*

Proof. We observe that for every vertex $v \in V(G)$, if $(G \setminus v, R, k) \in \Pi$ then $(G, R, k) \in \Pi$ because $G \setminus v$ is a subgraph of G and thus every cycle that exists in $G \setminus v$ also exists in G .

Assume now that $(G, R, k) \in \Pi$, let $v \in \hat{C}_1$, and let $S \subseteq V(G)$, $|S| \leq k$. We will prove that there exists a cycle in $G \setminus v$ containing all vertices of S . As

$(G, R, k) \in \Pi$, there is a cyclic linkage $\mathcal{L} = (C, S)$ in G . If $v \notin V(C)$, then $C \subseteq G \setminus v$ and we are done. If $v \in V(C)$, let $\mathcal{L}' = (C', S)$ be a \mathcal{C} -weakly cheap cyclic linkage in graph $H = G[V(C) \cup (\cup_{i=1}^r V(C_i))]$, and assume that $v \in V(C')$. Then C' meets all cycles of \mathcal{C} and its penetration in \mathcal{C} is more than $16 \cdot |S|$, which contradicts to Lemma 11. Thus, $v \notin V(C')$ which means that there exists a cyclic linkage with terminals S that does not meet v . As S was arbitrarily chosen, vertex v is problem-irrelevant. \square

Lemma 5. *Let y, r, q, z be positive integers such that $y+1 \leq z \leq r$, G be a graph embedded on \mathbb{S}_0 and let $R \subseteq V(G)$ be the set of annotated vertices of G . Given a subdivided wall W in G of height $h = 2 \cdot \max\{y, \lceil \frac{q}{8} \rceil\} + 4r$, then either G contains a sequence $\mathcal{C}' = \{C'_1, C'_2, \dots, C'_y\}$ of concentric cycles such that $V(\hat{C}'_y) \cap R = \emptyset$ or a sequence $\mathcal{C} = \{C_1, C_2, \dots, C_r\}$ of concentric cycles such that:*

1. $C_1 \cap R \neq \emptyset$.
2. R is z -dense in \mathcal{C} .
3. There exists a collection \mathcal{W} of q paths in K_W , such that $(\mathcal{C}, \mathcal{W})$ is a (r, q) -railed annulus in G .

Moreover, a sequence \mathcal{C}' or \mathcal{C} of concentric cycles as above can be constructed in $O(n)$ steps.

Proof. Let $p = \max\{y, \lceil \frac{q}{8} \rceil\}$. We are given a subdivided wall W of height $h = 2 \cdot p + 4r$. We define $\mathcal{C} = \{C_1, \dots, C_r\}$ such that $C_i = J_{\frac{h}{2}-p-2i+2}$, $i \in \{1, \dots, r\}$. Notice that there is a collection \mathcal{W} of $8p$ vertex disjoint paths in W such that $(\mathcal{C}, \mathcal{W})$ is a (r, q) -railed annulus. If $(\bar{C}_1) \cap R = \emptyset$, then $\mathcal{C}' = \{J_{\frac{h}{2}}, \dots, J_{\frac{h}{2}+y-1}\}$ is a sequence of concentric cycles where $(\bar{J}_{\frac{h}{2}+y-1}) \subseteq \text{int}(C_1)$ and we are done. Otherwise, we have that \mathcal{C} satisfies property 1. Suppose now that Property 2 does not hold for \mathcal{C} . Then there exists some $i \in \{1, \dots, r\}$ such that $A_{i, i+z-1} \cap R = \emptyset$. Notice that $A_{i, i+z-1}$ contains $2z-1 > 2y$ layers of W which are crossed by at least $2y$ of the paths in \mathcal{W} (these paths certainly exist as $2y < 8p$). This implies the existence of a wall of height $2y$ in $A_{i, i+z-1}$ which, in turn contains a sequence $\mathcal{C}' = \{C'_1, \dots, C'_y\}$ of concentric cycles. As $(\bar{C}'_y) \subseteq A_{i, i+z-1}$ we have that $V(\hat{C}'_y) \cap R = \emptyset$ and we are done. It remains to verify property 3 for \mathcal{C} . This follows directly by including in \mathcal{W}' any $q \leq 8p$ of the disjoint paths of \mathcal{W} . Then $(\mathcal{C}, \mathcal{W}')$ is the required (r, q) -railed annulus. It is easy to verify that all steps of this proof can be turned to an algorithm that runs in linear, on n , number of steps.

Description of algorithm `concentric_cycles` This algorithm first applies the algorithm of Lemma 5 for $y = 16k$, $r = 98k^2 + 2k$, $q = 2k + 1$, and $z = 32k$. If the output is a sequence $\mathcal{C}' = \{C'_1, C'_2, \dots, C'_y\}$ of concentric cycles such that $V(\hat{C}'_y) \cap R = \emptyset$ then, the algorithm returns a vertex w of \hat{C}'_1 . As $V(\hat{C}'_y) \cap R = \emptyset$, Lemma 4, implies that w is problem-irrelevant. If the output is a sequence \mathcal{C} the it remains to observe that conditions 1–3 match the specifications of algorithm `concentric_cycles`.

3.3 Correctness of algorithm `Planar_Annotated_Cyclability`

As mentioned in the proof of Lemma 2, the main step – [step 4] – of algorithm `Planar_Annotated_Cyclability` is based in Lemma 6 bellow.

Lemma 6. *Let (G, R, k) be an instance of problem Π and let $b = 98k + 2$ and $r = 98k^2 + 2k$. Let also $(\mathcal{C}, \mathcal{W})$ be an $(r, 2k + 1)$ -railed annulus in G , where $\mathcal{C} = \{C_1, \dots, C_r\}$ is a sequence of concentric cycles such that \hat{C}_1 contains some vertex $v \in R$ and that, R is $32k$ -dense in \mathcal{C} . For every $i \in \{1, \dots, r - b\}$ let $R_i = (R \cap V(\hat{C}_i)) \cup \{w_i\}$, where $w_i \in V(\hat{A}_{i+k+1, 33k+i+1}) \cap R$. If (\hat{C}_{i+b}, R_i, k) is a NO-instance of Π , for some $i \in \{1, \dots, r - b\}$, then (G, R, k) is a NO-instance of Π . Otherwise vertex v is color-irrelevant.*

We first prove the following lemma, which reflects the use of the rails of a railed annulus and is crucial for the proof of Lemma 6.

Lemma 7. *Let G be a graph embedded on the sphere \mathbb{S}_0 , r, k two positive integers, and $(\mathcal{C}, \mathcal{W})$ be an $(r, 2 \cdot k + 1)$ -railed annulus of G with $\mathcal{C} = \{C_1, \dots, C_r\}$ being its sequence of concentric cycles, $\mathcal{W} = \{W_1, \dots, W_{2k+1}\}$ its rails and $r \geq 16 \cdot k$. Let also $S \subseteq V(G)$ such that $S \cap \hat{C}_r = \emptyset$ and $|S| = k$. Then for every two vertices $u, v \in V(C_1)$, if there exists a cyclic linkage $\mathcal{L} = (\mathcal{C}, S)$, with penetration $k + 1 \leq p_{\mathcal{C}}(\mathcal{L}) \leq r - 1$, in G , then there exists a path $P_{u,v}$ with ends u and v that meets all vertices of S .*

Proof. Let $f_{\mathcal{L}}$ be the path-coloring of the cyclic linkage \mathcal{L} . As W_i is a path with endpoints $w'_i \in V(C_1)$ and $w''_i \in V(C_r)$, we define the ordering $\{w'_i, \dots, w''_i\}$ of $V(W_i)$ and call it the natural ordering of W_i . Furthermore, for every $W_i \in \mathcal{W}$, let $m_{\mathcal{L}}(W_i) = f_{\mathcal{L}}(P)$ if P is the first path (with respect to the natural ordering of W_i) of $\mathcal{P}(\mathcal{L})$ that W_i meets and $m_{\mathcal{L}}(W_i) = 0$ if W_i does not meet C .

Let $C_j \in \mathcal{C}$. We pick an arbitrary vertex $v_1^j \in V(C_j)$ and order $V(C_j)$ starting from v_1^j and continuing in clockwise order. Let $\{v_1^j, \dots, v_{|V(C_j)|}^j\}$ be such an ordering of the vertices of C_j . We assign to each vertex of $v_i^j \in C_j$ a “color” from the set $\{0, \dots, k\}$ as follows: $c_{\mathcal{L}}(v_i^j) = 0$ if $v_i^j \notin V(C_j) \cap V(C)$ and $c_{\mathcal{L}}(v_i^j) = f_{\mathcal{L}}(P)$ if $v_i^j \in V(C_j) \cap V(P)$, where $P \in \mathcal{P}(\mathcal{L})$.

For the rest of the proof, if P_0 is a path, $P_0(v, w)$ is the subpath of P_0 with endpoints v and w . We examine two cases:

1. At least $k + 1$ paths of \mathcal{W} (i.e. rails of the railed annulus) meet C . Then, as $|\mathcal{P}(\mathcal{L})| = k$, there exist two rails $W_i, W_j \in \mathcal{W}$ and a path $P \in \mathcal{P}(\mathcal{L})$ such that $m_{\mathcal{L}}(W_i) = m_{\mathcal{L}}(W_j) = f_{\mathcal{L}}(P)$. Let $V(C_1) \cap V(W_i)$ be the vertices of path $Q_{1,i}$ and $V(C_1) \cap V(W_j)$ the vertices of path $Q_{1,j}$. Then, we let $x \in V(C_1)$ be the endpoint of $Q_{1,i}$ that is not w'_i and $y \in V(C_1)$ be the endpoint of $Q_{1,j}$ that is not w'_j (notice that x and y can coincide with u and v). Let also x' be the vertex of $V(P) \cap V(W_i)$ with the least index in the natural ordering of W_i and y' be the vertex of $V(P) \cap V(W_j)$ with the least index in the natural ordering of W_j . We observe that there exist two vertex disjoint paths P_1 and P_2 with endpoints either v, x and u, y or v, y and u, x , respectively. We define path $P_{u,v} = (C \setminus P(x', y')) \cup W_i(x, x') \cup W_j(y, y') \cup P_1 \cup P_2$. Path $P_{u,v}$ has the desired properties.
2. There exist $k' = k + 1$ paths, say $\mathcal{W}' = \{W_1, \dots, W_{k'}\}$, of \mathcal{W} that do not meet C . As the penetration of C is at least $k + 1$, for every $j \in \{r - k, \dots, r\}$, $V(C_j \cap C) \neq \emptyset$. For every $i \in \{1, \dots, k'\}$ and every $j \in \{r - k, \dots, r\}$ we

assign to the vertex w_i^j of $V(W_i \cap C_j)$ with the least index in the natural ordering of W_i , a “color” from the set $\{1, \dots, k\}$ as follows: $c_{\mathcal{L}}(w_i^j) = c_{\mathcal{L}}(v)$ if there exists a $v \in V(C)$ and a subpath $C_j(w_i^j, v)$ (starting from w_i^j and following C_j in counter-clockwise order) such that it does not contain any other vertices of $V(C)$ as internal vertices. For every $W_i \in \mathcal{W}'$, we assign to W_i a set of colors, $\chi_i = \bigcup_{j=1}^{k+1} c_{\mathcal{L}}(w_i^j)$. Let \mathcal{P} be the set of all maximal paths of C_r without internal vertices in C . Certainly, any $W_i \in \mathcal{W}'$ intersects exactly one path of \mathcal{P} . We define the equivalence relation \sim on the set of rails \mathcal{W}' as follows: $W_i \sim W_{i'}$ if and only if W_i and $W_{i'}$ intersect the same path of \mathcal{P} . We distinguish two subcases:

- The number of equivalence classes of \sim is k' . Then, there exist two rails $W_i, W_{i'} \in \mathcal{W}'$ and $j, j' \in \{r-k, \dots, r\}$ such that $c_{\mathcal{L}}(w_i^j) = c_{\mathcal{L}}(w_{i'}^{j'}) = c_{\mathcal{L}}(P)$ for some path $P \in \mathcal{P}(\mathcal{L})$.
- The number of equivalence classes of \sim is strictly less than k' . Then, there exist two rails $W_i, W_{i'} \in \mathcal{W}'$ such that $c_{\mathcal{L}}(w_i^j) = c_{\mathcal{L}}(w_{i'}^{j'})$ for every $j \in \{r-k, \dots, r\}$. Therefore, there exist $j, j' \in \{r-k, \dots, r\}$ with $j \neq j'$ such that $c_{\mathcal{L}}(w_i^j) = c_{\mathcal{L}}(w_{i'}^{j'}) = c_{\mathcal{L}}(P)$ for some path $P \in \mathcal{P}(\mathcal{L})$ (this holds because $|\{r-k, \dots, r\}| = k+1$).

For both subcases, as $c_{\mathcal{L}}(w_i^j) = c_{\mathcal{L}}(P)$, there exist a $v_j \in V(P)$ and a subpath $C_j(w_i^j, v_j)$ of C_j and, similarly, as $c_{\mathcal{L}}(w_{i'}^{j'}) = c_{\mathcal{L}}(P)$, there exist a $v_{j'} \in V(P)$ and a subpath $C_{j'}(w_{i'}^{j'}, v_{j'})$ of $C_{j'}$. These two subpaths do not contain any other vertices of C apart from v_j and $v_{j'}$, respectively. Moreover, let x be the vertex of $V(W_i \cap C_1)$ of the least index in the natural ordering of W_i and y the vertex of $V(W_{i'} \cap C_1)$ of the least index in the natural ordering of $W_{i'}$. As in case 1, observe that there exist two vertex disjoint paths P_1 and P_2 with endpoints either v, x and u, y or v, y and u, x , respectively. We define path $P_{u,v} = (C \setminus P(v_j, v_{j'})) \cup C_j(w_i^j, v_j) \cup C_{j'}(w_{i'}^{j'}, v_{j'}) \cup W_i(w_i^j, x) \cup W_{i'}(w_{i'}^{j'}, y) \cup P_1 \cup P_2$. Path $P_{u,v}$ has the desired properties \square

Proof (of Lemma 6). We first prove that if (\hat{C}_{i+b}, R_i, k) is a YES-instance of Π for every $i \in \{1, \dots, r-b\}$, then (G, R, k) is a YES-instance of Π iff $(G, R \setminus v, k)$ is a YES-instance of Π .

For the non-trivial direction, we assume that $(G, R \setminus v, k)$ is a YES-instance of Π and we have to prove that (G, R, k) is also a YES-instance of Π . Let $S \subseteq R$ with $|S| \leq k$. We have to prove that S is cyclable in G . We examine two cases:

1. $v \notin S$. As $(G, R \setminus v, k)$ is a YES-instance of Π , clearly there exists a cyclic linkage $\mathcal{L} = (C, S)$ in G , i.e., S is cyclable in G .
2. $v \in S$. As $r \geq k(98k+1)$ and $S \leq k$, there exists i such that $A_{i, i+98k} \cap S = \emptyset$. We distinguish two sub-cases:

Subcase 1. $S \subseteq \bar{C}_{i+98k+1}$. Then, as $(\hat{C}_{i+98k+1}, R_{i+98k+1}, k)$ is a YES-instance of Π , then S is cyclable in $\hat{C}_{i+98k+1}$ and therefore also in G .

Subcase 2. There is a partition $\{S_1, S_2\}$ of S into two non-empty sets, such that $S_1 \subset \hat{C}_i$ and $S_1 \cap \bar{C}_{i+98k+1} = \emptyset$. As R is $32k$ -dense in \mathcal{C} , there exists a vertex $v_1 \in S \cap A_{i+k+1, i+33k+1}$ and a vertex $v_2 \in S \cap A_{50+k+1, i+82k+1}$.

For $i \in \{1, 2\}$, let $S'_i = S_i \cup \{v_i\}$ and observe that $|S_i| \leq k$. Let $\mathcal{C}_1 = \{C_{i+49k}, \dots, C_i\}$ and $\mathcal{C}_2 = \{C_{i+49k}, \dots, C_{98k}\}$. As $(\hat{C}_{i+98k+1}, R_{98k+1}, k)$ is a YES-instance of Π , S'_1 is cyclable in $\hat{C}_{i+98k+1}$. Also, $(G, R \setminus v, k)$ is a YES-instance, S'_2 is cyclable in G . For each $i \in \{1, 2\}$, there exists a cyclic linkage $\mathcal{L}_i = (C_i, S'_i)$ that has penetration at least $k+1$ in \mathcal{C}_i . We may assume that \mathcal{L}_i is \mathcal{C}_i -cheap. Then, By Lemma 4, the penetration of \mathcal{L}_i in \mathcal{C}_i is at most $49k$. Let $\mathcal{L}'_i = (C_i, S_i), i \in \{1, 2\}$. For notational convenience we rename \mathcal{C}_1 and \mathcal{C}_2 where $\mathcal{C}_1 = \{C_1^1, \dots, C_{49k+1}^1\}$ and $\mathcal{C}_2 = \{C_1^2, \dots, C_{49k+1}^2\}$ (notice that $C_{49k+1}^1 = C_1^2$). Let x, y be two distinct vertices in C_{i+49k} . For $i \in \{1, 2\}$, we apply Lemma 7, for $r = 49k+1, k, \mathcal{C}_i, \mathcal{W}$, and x and y and obtain two paths $P_i, i \in \{1, 2\}$, such that $S_i \subseteq V(P_i)$ and whose endpoints are x and y . Clearly, $P_1 \cup P_2$ is a cycle whose vertex set contains S as a subset. Therefore S is cyclable in G , as required. \square

4 Vital cyclic linkages

Tight sequences. A sequence $\mathcal{C} = \{C_1, \dots, C_r\}$ of concentric cycles of G is *tight* in G , if

- C_1 is *surface minimal*, i.e., there is no closed disk D of \mathbb{S} that is properly contained in \bar{C}_1 and whose boundary is a cycle of G ;
- for every $i \in \{1, \dots, r-1\}$, there is no closed disk D such that $\bar{C}_i \subset D \subset \bar{C}_{i+1}$ and such that the boundary of D is a cycle of G .

Graph Linkages. Let G be a graph. A *graph linkage* in G is a pair $\mathcal{L} = (H, T)$ such that H is a subgraph of G without isolated vertices and T is a subset of the vertices of H , called *terminals* of \mathcal{L} , such that every vertex of H with degree different than 2 is contained in T . Set $\mathcal{P}(\mathcal{L})$, which we call *path set* of the graph linkage \mathcal{L} , contains all paths of H whose endpoints are in T and do not have any other vertex in T . The *pattern* of L is the graph

$$(T, \{\{s, t\} \mid \mathcal{P}(\mathcal{L}) \text{ contains a path from } s \text{ to } t \text{ in } H\}).$$

Two graph linkages of G are *equivalent* if they have the same pattern and are *isomorphic* if their patterns are isomorphic. A graph linkage $\mathcal{L} = (H, T)$ is called *weakly vital* (reps. *strongly vital*) in G if $V(H) = V(G)$ and there is no other equivalent (resp. isomorphic) graph linkage that is different from \mathcal{L} . Clearly, if a graph linkage \mathcal{L} is strongly vital then it is also weakly vital. We call a graph linkage \mathcal{L} *linkage* if its pattern has maximum degree 1 (i.e., it consists of a collection of paths). We call a graph linkage \mathcal{L} *cyclic linkage* if its pattern is a cycle.

CGL-configurations. Let G be a graph embedded on the sphere \mathbb{S}_0 . Then we say that a pair $\mathcal{Q} = (\mathcal{C}, \mathcal{L})$ is a *CGL-configuration* of *depth* r if $\mathcal{C} = \{C_1, \dots, C_r\}$ is a sequence of concentric cycles in G , $\mathcal{L} = (H, T)$ is a graph linkage in G , and $T \cap V(\hat{C}_r) = \emptyset$, i.e., all vertices in the terminals of \mathcal{L} are outside \hat{C}_r . The *penetration* of \mathcal{L} in \mathcal{C} , $p_{\mathcal{C}}(\mathcal{L})$, is the number of cycles of \mathcal{C} that are intersected

by the paths of \mathcal{L} (when $\mathcal{L} = (C, S)$ is cyclic we will sometimes refer to the penetration of \mathcal{L} as the penetration of cycle C). We say that \mathcal{Q} is *touch-free* if for every path $P \in \mathcal{L}$, the number of connected components of $P \cap C_r$ is not 1.

Cheap graph linkages. Let G be a graph embedded on the sphere \mathbb{S}_0 , let $\mathcal{C} = \{C_1, \dots, C_r\}$ be a sequence of cycles in G , and let $\mathcal{L} = (H, T)$ be a graph linkage where $T \subseteq V(G \setminus \hat{C}_r)$ (notice that $(\mathcal{C}, \mathcal{L})$ is a CGL-configuration). We define the function c that matches graph linkages of G to positive integers such that

$$c(\mathcal{L}) = |E(\mathcal{L}) \setminus \bigcup_{i \in \{1, \dots, r\}} E(C_i)|.$$

A graph linkage \mathcal{L} of G is \mathcal{C} -*strongly cheap* (resp. \mathcal{C} -*weakly cheap*), if $T(\mathcal{L}) \cap \hat{C}_r = \emptyset$ and there is no other isomorphic (resp. equivalent) graph linkage \mathcal{L}' such $c(\mathcal{L}) > c(\mathcal{L}')$. Obviously if \mathcal{L} is \mathcal{C} -strongly cheap then it is also \mathcal{C} -weakly cheap.

The proof of the next lemma is based on a suitable adaptation of the results in [2] about weakly vital linkages to strongly vital cyclic linkages.

Tilted grids. Let G be a graph. A *tilted grid* of G is a pair $\mathcal{U} = (\mathcal{X}, \mathcal{Z})$ where $\mathcal{X} = \{X_1, \dots, X_r\}$ and $\mathcal{Z} = \{Z_1, \dots, Z_r\}$ are both sequences of $r \geq 2$ vertex-disjoint paths of G such that

- for each $i, j \in \{1, \dots, r\}$ $I_{i,j} = X_i \cap Z_j$ is a (possibly edgeless) path of G ,
- for $i \in \{1, \dots, r\}$ the subpaths $I_{i,1}, I_{i,2}, \dots, I_{i,r}$ appear in this order in X_i .
- for $j \in \{1, \dots, r\}$ the subpaths $I_{1,j}, I_{2,j}, \dots, I_{r,j}$ appear in this order in Z_j .
- $E(I_{1,1}) = E(I_{1,r}) = E(I_{r,1}) = E(I_{r,r}) = \emptyset$,
- the graph $G_{\mathcal{U}}^*$ taken from the graph $G_{\mathcal{U}} = (\bigcup_{i \in \{1, \dots, r\}} X_i) \cup (\bigcup_{i \in \{1, \dots, r\}} Z_i)$ after contracting all edges in $\bigcup_{(i,j) \in \{1, \dots, r\}^2} I_{i,j}$ is isomorphic to the $(r \times r)$ -grid.

Tidy tilted grids. Given a plane graph G and a graph linkage $\mathcal{L} = (H, T)$ of G we say that a tilted grid $\mathcal{U} = (\mathcal{X}, \mathcal{Z})$ of G is an \mathcal{L} -*tidy tilted grid* of G if $T \cap D_{\mathcal{U}} = \emptyset$ and $D_{\mathcal{U}} \cap L = \mathbf{U}\mathcal{Z}$ where $D_{\mathcal{U}}$ is the closed interior of the perimeter of $G_{\mathcal{U}}$.

From graph linkages to linkages. Let G be a graph and let $\mathcal{L} = (H, T)$ be a graph linkage of G . We denote by $G_{\mathcal{L}}$ the graph obtained by subdividing all edges incident to terminals and then removing the terminals. Similarly, we define $\mathcal{L}^* = (H^*, T^*)$ so that H^* is the graph obtained by subdividing all edges incident to terminals, removing the terminals, and considering as terminals the subdivision vertices. Notice that \mathcal{L}^* is a linkage of $G_{\mathcal{L}}$. Notice that if \mathcal{L} is strongly vital then \mathcal{L}^* is not necessarily strongly vital. However, if \mathcal{L} is weakly vital, then so is \mathcal{L}^* .

The following proposition follows combining Lemmata 5, and 6, and Observation 3 of [2].

Proposition 2. *Let G be a graph embedded on the sphere \mathbb{S}_0 and let $\mathcal{Q} = (\mathcal{C}, \mathcal{L})$ be a touch-free CGL-configuration of G , where \mathcal{C} is tight in G and \mathcal{L} is a \mathcal{C} -weakly cheap linkage whose penetration in \mathcal{C} is at least r . Then G contains some \mathcal{L} -tidy tilted grid in G of capacity at least $r/(4 \cdot |\mathcal{P}(\mathcal{L})|)$.*

Lemma 8. *Let G be a graph embedded on the sphere \mathbb{S}_0 . If G contains a strongly vital cyclic linkage $\mathcal{L} = (C, T)$, then G does not contain a \mathcal{L} -tidy tilted grid of capacity 4.*

Proof. Assume that $\mathcal{L} = (C, T)$ is a strongly vital cyclic linkage in G and that Γ is a \mathcal{L} -tidy tilted grid of capacity 4 in G . Let also Γ' be the (4×4) -grid that we get after contracting all edges of Γ whose ends have both degree 2 and G' the graph resulting from G after these contractions. Let $V(\Gamma') = \{v_{ij} \mid i, j \in \{1, \dots, 4\}\}$ and $E(\Gamma') = \{\{v_{ij}, v_{i'j'}\} \mid |i - i'| + |j - j'| = 1\}$. Observe that Γ' is also a \mathcal{L} -tidy tilted grid of capacity 4 in G' and that, if \mathcal{L} is not strongly vital in G' then it is also not strongly vital in G . Let $H = \Gamma \cup C$ and H' be the contraction of H that we get after contracting all edges of H whose ends have both degree 2. It is not hard to confirm that for every possible H' , its corresponding contraction H^* is isomorphic to H^* , where $H^* = \Gamma' \cup P_1 \cup P_2 \cup P_3 \cup P_4$, where for every $i \in \{1, 2, 3, 4\}$, P_i is a path of length 2 such that P_1 connects v_{11} with v_{12} , P_2 connects v_{13} with v_{14} , P_3 connects v_{41} with v_{44} and P_4 connects v_{42} with v_{43} (i.e. for every cyclic linkage $\mathcal{L} = (C, T)$ if we contract all edges of $H = \Gamma \cup C$ whose ends have degree 2, we get a graph isomorphic to H^* which is a (4×4) -grid in addition to some paths that are subgraphs of C).

It remains to show that there exists a cyclic linkage $\mathcal{L}' = (C', T)$ in G' , where C' is different from C .

Lemma 9. *Let G be a graph embedded on the sphere \mathbb{S}_0 that is the union of $r \geq 2$ concentric cycles $\mathcal{C} = \{C_1, \dots, C_r\}$ and one more cycle C of G . Assume that \mathcal{C} is tight in G , $T \cap V(\hat{C}_r) = \emptyset$ and the cyclic linkage $\mathcal{L} = (C, T)$ is strongly vital in G . Then $r \leq 16 \cdot |T| - 1$.*

Proof (of Lemma 9). Let $\sigma : \mathcal{P}(\mathcal{L}) \rightarrow T$ such that that σ is a bijection that maps each path of $\mathcal{P}(\mathcal{L})$ to some of its endpoints. For every $i \in \{1, \dots, r\}$, we define $\mathcal{Q}^{(i)} = (C^{(i)}, \mathcal{L}^{(i)})$ where $C^{(i)} = \{C_1, \dots, C_i\}$ and $\mathcal{L}^{(i)} = (C, T^{(i)})$ where $T^{(i)} = T \setminus \{\sigma(P) \mid P \cap \hat{C}_i = \emptyset\}$. Notice that $|T^{(i)}| \leq |T| - (i - 1)$. In the trivial case where every $\mathcal{Q}^{(i)}$ is not touch-free we derive easily that $r = |T|$ and we are done. Otherwise, let $\mathcal{Q}' = (C', \mathcal{L}')$ be the touch-free CGL-configuration in $\{\mathcal{Q}^{(1)}, \dots, \mathcal{Q}^{(r)}\}$ of the highest index, say i . Certainly, $C' = C^{(i)}$ and \mathcal{Q}' is tight in G . Moreover, \mathcal{L}' is strongly vital in G . From Lemma 8, G does not contain an \mathcal{L}' -tidy tilted grid of capacity 4. Clearly, $G_{\mathcal{L}'}$ as well does not contain an \mathcal{L}'^* -tidy tilted grid of capacity 4. Recall now that, as \mathcal{L}' is strongly vital in G , it is also weakly vital in G and therefore \mathcal{L}'^* is weakly vital in $G_{\mathcal{L}'}$. Notice also that $\mathcal{Q}'^* = (C', \mathcal{L}'^*)$ is a CGL-configuration of $G_{\mathcal{L}'}$ where C' is tight in $G_{\mathcal{L}'}$. As \mathcal{L}'^* is weakly vital in $G_{\mathcal{L}'}$, then, by its uniqueness, \mathcal{L}'^* is C' -weakly cheap. Recall that the penetration of \mathcal{L}' in C' is $r - (i - 1)$ and so is the penetration of \mathcal{L}'^* in C' . As \mathcal{Q}' , and therefore \mathcal{Q}'^* as well, is touch-free we can apply Proposition 2 and obtain that $G_{\mathcal{L}'}$ contains some \mathcal{L}'^* -tidy tilted grid of capacity at least $(r - (i - 1))/(4 \cdot |\mathcal{P}(\mathcal{L}'^*)|)$. We conclude that $(r - (i - 1))/(4 \cdot |\mathcal{P}(\mathcal{L})|) < 4$, therefore $r \leq 16 \cdot |T| - 1$ as required. \square

A corollary of Lemma 9 with independent combinatorial interest is the following.

Corollary 1. *If a plane graph G contains a strongly vital cyclic linkage $\mathcal{L} = (C, T)$, then $\mathbf{tw}(G) = O(|T|^{3/2})$.*

Notice that, according to what is claimed in [2], we cannot restate the above corollary for weakly vital linkages, unless we change the bound to be an exponential one. That way, the fact that treewidth is (unavoidably, due to [2]) exponential to the number of terminals for (weakly) vital linkages is caused by the fact that the ordering of the terminals is predetermined.

Lemma 10. *Let G be a graph embedded on the sphere \mathbb{S}_0 that is the union of r concentric cycles $\mathcal{C} = \{C_1, \dots, C_r\}$ and a hamiltonian cycle C of G . Let also $T \cap V(\hat{C}_r) = \emptyset$. If $\mathcal{L} = (C, T)$ is \mathcal{C} -strongly cheap then \mathcal{L} is a strongly vital cyclic linkage in G .*

Proof. Assume that \mathcal{L} is not strongly vital in G , i.e., there is an other, isomorphic to $\mathcal{L} = (C, T)$, cyclic linkage $\mathcal{L}' = (C', T')$ in G . Then there exists an edge $e \in E(C') \cap E(C)$. But, as $E(G) = E(C) \cup \bigcup_{i=1}^r E(C_i)$, we get that $e \in \bigcup_{i=1}^r E(C_i)$. Thus, $|E(C') \cap \bigcup_{i=1}^r E(C_i)| < |E(C) \cap \bigcup_{i=1}^r E(C_i)|$ and, by the definition of cheap graph linkages, $c(\mathcal{L}) > c(\mathcal{L}')$, which contradicts to the hypothesis that \mathcal{L} is \mathcal{C} -strongly cheap. Therefore, $\mathcal{L} = (C, T)$ is a strongly vital cyclic linkage in G , as claimed. \square

We are now able to prove the main combinatorial result of this paper.

Lemma 11. *Let G be a plane graph with some sequence of concentric cycles $\mathcal{C} = \{C_1, \dots, C_r\}$. Let also $\mathcal{L} = (C, T)$ be a cyclic linkage of G where $T \cap V(\hat{C}_r) = \emptyset$. If \mathcal{L} is \mathcal{C} -strongly cheap then the penetration of \mathcal{L} in \mathcal{C} is at most $r \leq 16 \cdot |T| - 1$.*

Proof. Suppose that some path in $\mathcal{P}(\mathcal{L})$ intersects $16 \cdot |T|$ cycles in the set $\mathcal{C}^* = \{C_{r-16 \cdot |T|+1}, \dots, C_r\}$. Let G' be the graph obtained by $C \cup \mathbf{UC}^*$ after dissolving all vertices not in T that have degree 2 and let $\mathcal{L}' = (C', T)$ be the linkage of G' obtained from \mathcal{L} if we dissolve the same vertices in the paths of \mathcal{L} . Similarly, by dissolving vertices of degree 2 in the cycles of \mathcal{C}^* we obtain a new sequence of concentric cycles that, for notational convenience, we denote by $\mathcal{C}' = \{C_1, \dots, C_{r'}\}$, where $r' = 16 \cdot |T|$. \mathcal{L}' is \mathcal{C}' -strongly cheap because \mathcal{L} is \mathcal{C} -strongly cheap. Notice that C' is a Hamiltonian cycle of G' and, from Lemma 10, \mathcal{L}' is a strongly vital cyclic linkage of G' . We also assume that \mathcal{C}' is tight (otherwise replace it by a tight one and observe that, by its uniqueness, \mathcal{L}' will be cheap to this new one as well). As \mathcal{L}' is \mathcal{C}' -strongly cheap and \mathcal{C}' is tight, from Lemma 9, $r' \leq 16 \cdot |T| - 1$, a contradiction. \square

5 Hardness of the Cyclability Problem

In this section, we examine the hardness of CYCLABILITY in general graphs. We prove the following theorem:

Theorem 3. *It is W[1]-hard to decide for a split graph G and a positive integer k , whether G has k vertices such that there is no cycle in G that contains these k vertices, when the problem is parameterized by k .*

We first introduce some further notation.

For a set of edges S of a graph G , $G[S]$ is the *subgraph induced by S* , i.e., S is the set of edges of $G[S]$ and the vertices of $G[S]$ are the vertices of G incident to edges from S . We denote by $\delta(G) = \min\{\deg_G(v) \mid v \in V(G)\}$ the *minimum degree* of G . For a set of vertices U , $N_G(U) = \cup_{v \in U} N_G(v) \setminus U$. A cycle C in a graph G is *Hamiltonian* if $V(C) = V(G)$. Respectively, a graph H is *Hamiltonian* if it has a Hamiltonian cycle. A *matching* is a set of pairwise non-adjacent edges. A vertex v is *saturated* in a matching M if v is incident to an edge of M . By $x_1 \dots x_p$ we denote the path with the vertices x_1, \dots, x_p and the edges $\{x_1, x_2\}, \dots, \{x_{p-1}, x_p\}$, and we use $x_1 \dots x_p x_1$ to denote the cycle with the vertices x_1, \dots, x_p and the edges $\{x_1, x_2\}, \dots, \{x_{p-1}, x_p\}, \{x_p, x_1\}$. For a path $P = x_1 \dots x_p$ and a vertex y , yP (Py resp.) is the path $yx_1 \dots x_p$ ($x_1 \dots x_p y$ resp.). If $P_1 = x_1 \dots x_p$ and $P_2 = y_1 \dots y_q$ are paths such that $V(P_1) \cap V(P_2) = \{x_p\} = \{y_1\}$, then $P_1 + P_2$ is the *concatenation* of P_1 and P_2 , i.e., the path $x_1 \dots x_{p-1} y_1 \dots y_q$.

We need some auxiliary results.

The following lemma is due to Erdős [17]. Define the function $f(n, \delta)$ by

$$f(n, \delta) = \begin{cases} \binom{n-\delta}{2} + \delta^2 & \text{if } n \geq 6\delta - 2, \\ \binom{(n+1)/2}{2} + \binom{(n-1)/2}{2} & \text{if } n \leq 6\delta - 3 \text{ and } n \text{ is odd,} \\ \binom{(n+2)/2}{2} + \binom{(n-2)/2}{2} & \text{if } n \leq 6\delta - 4 \text{ and } n \text{ is even.} \end{cases}$$

Lemma 12 ([17]). *Let G be a graph with $n \geq 3$ vertices. If $\delta(G) \geq n/2$ or $|E(G)| > f(n, \delta(G))$, then G is Hamiltonian.*

Lemma 13. *Let $k \geq 75$ be an odd integer and let H be a graph such that*

- i) $(k-2)(k-3)/2 < |E(H)| \leq k(k-1)/2 + 1$,
- ii) $\delta(H) \geq (k-1)/2$,
- iii) *there is a set $S \subseteq E(H)$ such that $|S| > (k-2)(k-3)/2$ and $G[S]$ has at most $k+2$ vertices.*

Then H is Hamiltonian.

Proof. Let H be an n -vertex graph that satisfies i)–iii). Let $S \subseteq E(H)$ be a set such that $|S| > (k-2)(k-3)/2$ and $G[S]$ has at most $k+2$ vertices. Let also $U = V(H) \setminus V(G[S])$. Denote by R the set of edges of G incident to vertices of U . Since $|S| > (k-2)(k-3)/2$ and $|E(H)| \leq k(k-1)/2 + 1$, $|R| \leq 2k-3$. Because $\delta(H) \geq (k-1)/2$, $|R| \geq |U|\delta(H)/2 \geq |U|(k-1)/4$. We have that $|U| \leq 7$, i.e., H has at most $k+9$ vertices. Then because $k \geq 75$, we obtain that $n \geq 6\delta(G) - 3$,

$$\binom{(n+1)/2}{2} + \binom{(n-1)/2}{2} \leq \frac{(k-2)(k-3)}{2} < |E(H)|$$

and

$$\binom{(n+2)/2}{2} + \binom{(n-2)/2}{2} \leq \frac{(k-2)(k-3)}{2} < |E(H)|.$$

We have that $|E(H)| > f(n, \delta(H))$, and by Lemma 12, H is Hamiltonian. \square

We are now in the position to prove Theorem 3:

Proof (of Theorem 3). We reduce the CLIQUE problem. Recall that CLIQUE asks for a graph G and a positive integer k , whether G has a clique of size k . This problem is well known to be W[1]-complete [16] when parameterized by k . Notice that CLIQUE remains W[1]-complete when restricted to the instances where k is odd. To see it, it is sufficient to observe that if the graph G' is obtained from a graph G by adding a vertex adjacent to all the vertices of G , then G has a clique of size k if and only if G' has a clique of size $k + 1$. Hence, any instance of CLIQUE can be reduced to the instance with an odd value of the parameter. Clearly, the problem is still W[1]-hard if the parameter $k \geq c$ for any constant c .

Let (G, k) be an instance of CLIQUE where $k \geq 75$ is odd. We construct the graph G'_k as follows.

- For each vertex $x \in V(G)$, construct $s = (k - 1)/2$ vertices v_x^i for $i \in \{1, \dots, s\}$ and form a clique of size ns from all these vertices by joining them by edges pairwise.
- Construct a vertex w and edges $\{w, v_x^i\}$ for $x \in V(G)$, $i \in \{1, \dots, s\}$.
- For each edge $\{x, y\} \in E(G)$, construct a vertex u_{xy} and edges $\{u_{xy}, v_x^i\}, \{u_{xy}, v_y^i\}$ for $i \in \{1, \dots, s\}$; we assume that $u_{xy} = u_{yx}$.

Let $k' = k(k - 1)/2 + 1$. It is straightforward to see that G' is a split graph. We show that G has a clique of size k if and only if there are k' vertices in G'_k such that there is no cycle in G'_k that contains these k' vertices.

Suppose that G has a clique X of size k . Let $Y = \{u_{xy} \in V(G') \mid x, y \in X, x \neq y\}$ and $Z = Y \cup \{w\}$. Because $|X| = k$, $|Z| = k(k - 1)/2 + 1 = k'$. Observe that Y is an independent set in G'_k and $|Y| = |N_{G'}(Y)|$. Hence, for any cycle C in G'_k such that $Y \subseteq V(C)$, $V(C) \subseteq Y \cup N_{G'_k}(Y)$. Because $w \notin Y \cup N_{G'_k}(Y)$, w does not belong to any cycle that contains the vertices of Y . We have that no cycle in G'_k contains Z of size k' .

Now we show that if G has no cliques of size k , then for any $Z \subseteq V(G'_k)$ of size k' , there is a cycle C in G'_k such that $Z \subseteq V(C)$. We use the following claim.

Claim. *Suppose that G has no cliques of size k . Then for any non-empty $Z \subseteq \{u_{xy} \mid x, y \in V(G)\}$ of size at most $k(k - 1)/2 + 1$, there is a cycle C in G'_k such that $Z \subseteq V(C) \subseteq Z \cup N_G(Z)$ and C has an edge $\{v_x^i, v_y^j\}$ for some $x, y \in V(G)$ and $i, j \in \{1, \dots, s\}$.*

Proof (of Claim). For a set $Z \subseteq \{u_{xy} \mid x, y \in V(G)\}$, we denote by $S(Z)$ the set of edges $\{\{x, y\} \in E(G) \mid u_{xy} \in Z\}$, and $H(Z) = G[S(Z)]$.

If $Z = \{u_{xy}\}$, then the triangle $u_{xy}v_x^1v_x^2u_{xy}$ is a required cycle, and the claim holds. Let $r = |Z| \geq 2$ and assume inductively that the claim is fulfilled for smaller sets.

Suppose that $H(Z)$ has a vertex x with $\deg_{H(Z)}(x) \leq (k-3)/2$. Let $N_{H(Z)}(x) = \{y_1, \dots, y_t\}$. Notice that $t \leq (k - 3)/2 = s - 1$. Denote by Z' the set obtained from Z by the deletion of $u_{xy_1}, \dots, u_{xy_t}$, and let $H' = H(Z')$. If $Z' = \emptyset$, then the cycle $C = v_x^1u_{xy_1}v_x^2 \dots v_x^t u_{xy_t} v_x^{t+1} v_x^1$ satisfies the conditions and the claim holds. Suppose that $Z' \neq \emptyset$. Then, by induction, there is a cycle C' in G'_k such that $Z \subseteq V(C') \subseteq Z \cup N_G(Z)$ and C' has an edge $\{v_a^i, v_b^j\}$ for some $a, b \in V(G)$

and $i, j \in \{1, \dots, s\}$. We consider the path $P = v_x^1 u_{xy_1} v_x^2 \dots v_x^t u_{xy_t} v_x^{t+1}$. Then we delete $\{v_a^i, v_b^j\}$ and replace it by the path $v_a^i P v_b^j$. Denote the obtained cycle by C . It is straightforward to verify that $Z \subseteq V(C) \subseteq Z \cup N_G(Z)$ and $\{v_a^i, v_x^1\} \in E(C)$, i.e., the claim is fulfilled.

From now we assume that $\delta(H(Z)) \geq (k-1)/2$. We consider three cases.

Case 1. $r \leq (k-2)(k-3)/2$.

Consider the graph G'_{k-2} . We show that this graph has a matching M of size r such that every vertex of Z is saturated in M . By the Hall's theorem (see, e.g., [7]), it is sufficient to show that for any $Z' \subseteq Z$, $|Z'| \leq |N_{G'_{k-2}}(Z')|$. Let p be the smallest positive integer such that $|Z'| \leq p(p-1)/2$. By the definition of G'_{k-2} , $|N_{G'_{k-2}}(Z')| \geq p(k-3)/2$. Because $p \leq k-2$, $|Z'| \leq p(p-1)/2 \leq p(k-3)/2 \leq |N_{G'_{k-2}}(Z')|$.

Let M be a matching in G'_{k-2} of size r such that every vertex of Z is saturated in M . Clearly, M is a matching in G'_k that saturates Z as well. Let x_1, \dots, x_q be the vertices of G such that for $i \in \{1, \dots, q\}$, $\{v_{x_i}^1, \dots, v_{x_i}^s\}$ contains saturated in M vertices. Because $v_{x_i}^1, \dots, v_{x_i}^s$ have the same neighborhoods, we assume without loss of generality that for $i \in \{1, \dots, q\}$, $v_{x_i}^1, \dots, v_{x_i}^{t_i}$ are saturated. Observe that since M is a matching in G'_{k-2} , $t_i \leq s-1$. For $i \in \{1, \dots, q\}$ and $j \in \{1, \dots, t_i\}$, denote by u_i^j the vertex of Z such that $\{v_{x_i}^j, u_i^j\} \in M$. We define the path $P_i = v_{x_i}^1 u_i^1 v_{x_i}^2 \dots u_i^{t_i} v_{x_i}^{t_i+1}$ for $i \in \{1, \dots, q\}$. Because all the vertices $v_{x_i}^j$ are pairwise adjacent, by adding the edges $\{v_{x_1}^{t_1+1}, v_{x_2}^1\}, \dots, \{v_{x_{q-1}}^{t_{q-1}+1}, v_{x_q}^1\}, \{v_{x_q}^{s_q+1}, v_{x_1}^1\}$, we obtain from the the paths P_1, \dots, P_q a cycle. Denote it by C . We have that $Z \subseteq V(C) \subseteq Z \cup N_G(Z)$ and $\{v_{x_1}^{t_1+1}, v_{x_2}^1\} \in E(C)$, and we conclude that the claim holds.

Case 2. $(k-2)(k-3)/2 < r$ and for any $S \subseteq E(H(Z))$ such that $|S| > (k-2)(k-3)/2$, $H(Z)[S]$ has at least $k+3$ vertices.

We use the same approach as in Case 1 and show that G'_{k-2} has a matching M of size r such that every vertex of Z is saturated in M . We have to show that for any $Z' \subseteq Z$, $|Z'| \leq |N_{G'_{k-2}}(Z')|$. If $|Z'| \leq (k-2)(k-3)/2$, we use exactly the same arguments as in Case 1. Suppose that $|Z'| > (k-2)(k-3)/2$. Then $|S(Z')| = |Z'| > (k-2)(k-3)/2$. Hence, $H(Z)[S(Z')]$ has at least $k+3$ vertices. It implies that $|N_{G'_{k-2}}(Z')| \geq (k+3)(k-3)/2$. Because $k \geq 75$ and $|Z'| \leq r \leq k(k-1)/2+1$, $|N_{G'_{k-2}}(Z')| \geq (k+3)(k-3)/2 \geq k(k-1)/2+1 \geq |Z'|$. Given a matching M that saturates Z , we construct a cycle that contains Z in exactly the same way as in Case 1 and prove that the claim holds.

Case 3. $(k-2)(k-3)/2 < r$ and there is $S \subseteq E(H(Z))$ such that $|S| > (k-2)(k-3)/2$ and $H(Z)[S]$ has at most $k+2$ vertices.

By Lemma 13, $H(Z)$ is Hamiltonian. Let $p = |V(H(Z))|$ and denote by $R = x_1 \dots x_p x_1$ a Hamiltonian cycle in $H(Z)$. Let $U = \{u_{x_1 x_2}, \dots, u_{x_{p-1} x_1}\}$ and let $Z' = Z \setminus U$.

We again consider G'_{k-2} . We show that this graph has a matching M of size $|Z'|$ such that every vertex of Z' is saturated in M . We have to prove that for any $Z'' \subseteq Z'$, $|Z''| \leq |N_{G'_{k-2}}(Z'')|$. If $|Z''| \leq (k-2)(k-3)/2$, we use exactly the same arguments as in Case 1. Suppose that $|Z''| > (k-2)(k-3)/2$. Let q be

the smallest positive integer such that $|Z''| \leq q(q-1)/2$. Clearly, $q > k-2$. We consider the following three cases depending on the value of q .

Case a). $q = k-1$. Then $H(Z'')$ has at least $k-1$ vertices and at least $(k-2)(k-3)/2 + 1$ edges. Because $|Z| \leq k(k-1)/2 + 1$, $H(Z)$ has at most $2k-3$ edges that are not edges of $H(Z'')$. Because $\delta(H(Z)) \geq (k-1)/2$ and $k \geq 75$, $H(Z)$ has at most 4 vertices that are not adjacent to the edges of $H(Z'')$. Then at most 8 edges of the Hamiltonian cycle R in $H(Z)$ do not join vertices of $H(Z'')$ with each other. We obtain that at least $k-9$ edges of R join vertices of $H(Z'')$ with each other.

Suppose that $H(Z'')$ has $k-1$ vertices. Then $|Z''| \leq (k-1)(k-2)/2 - (k-9) \leq (k^2 - 5k + 20)/2$. Because $H(Z'')$ has $k-1$ vertices, $|N_{G'_{k-2}}(Z'')| = (k-1)(k-3)/2$. Since $k \geq 75$, $|Z''| \leq |N_{G'_{k-2}}(Z'')|$.

Suppose that $H(Z'')$ has k vertices. If $H(Z)$ has a vertex x that is not adjacent to the edges of $H(Z'')$, then at least $(k-1)/2$ vertices of Z that correspond to the edges incident to x are not in Z'' . Then $|Z''| \leq |Z| - (k-1)/2 - (k-9) \leq (k^2 - 4k + 21)/2$. Because $|N_{G'_{k-2}}(Z'')| = k(k-3)/2$ and $k \geq 75$, $|Z''| \leq |N_{G'_{k-2}}(Z'')|$. If $H(Z)$ has no vertex that is not adjacent to the edges of $H(Z'')$, then the edges of R join vertices of $H(Z'')$ with each other. We have that $|Z''| \leq k(k-1)/2 - k = k(k-3)/2$ and $|Z''| \leq |N_{G'_{k-2}}(Z'')|$.

Finally, if $H(Z'')$ has at least $k+1$ vertices, then $|N_{G'_{k-2}}(Z'')| \geq (k+1)(k-3)/2 \geq (k-1)(k-2)/2 \geq |Z''|$.

Case b). $q = k$. Then $H(Z'')$ has at least k vertices and at least $(k-1)(k-2)/2 + 1$ edges. Because $|Z| \leq k(k-1)/2 + 1$, $H(Z)$ has at most $k-1$ edges that are not edges of $H(Z'')$. Because $\delta(H(Z)) \geq (k-1)/2$ and $k \geq 75$, $H(Z)$ has at most 2 vertices that are not adjacent to the edges of $H(Z'')$. Then at most 4 edges of the Hamiltonian cycle R in $H(Z)$ do not join vertices of $H(Z'')$ with each other. We obtain that at least $k-4$ edges of R join vertices of $H(Z'')$ with each other.

Suppose that $H(Z'')$ has k vertices. If $H(Z)$ has a vertex x that is not adjacent to the edges of $H(Z'')$, then at least $(k-1)/2$ vertices of Z that correspond to the edges incident to x are not in Z'' . Then $|Z''| \leq |Z| - (k-1)/2 - (k-4) \leq (k^2 - 4k + 11)/2$. Because $|N_{G'_{k-2}}(Z'')| = k(k-3)/2$ and $k \geq 75$, $|Z''| \leq |N_{G'_{k-2}}(Z'')|$. If $H(Z)$ has no vertex that is not adjacent to the edges of $H(Z'')$, then the edges of R join vertices of $H(Z'')$ with each other. We have that $|Z''| \leq k(k-1)/2 - k = k(k-3)/2$ and $|Z''| \leq |N_{G'_{k-2}}(Z'')|$.

Suppose that $H(Z'')$ has at least $k+1$ vertices. Then R has at least $k+1$ edges and $|Z'| \leq |Z| - (k+1) \leq k(k-3)/2$. As $|N_{G'_{k-2}}(Z'')| \geq (k+1)(k-3)/2$, $|Z''| \leq |N_{G'_{k-2}}(Z'')|$.

Case c). $q \geq k+1$. Then $H(Z'')$ has at least $k+1$ vertices. We have that R has at least $k+1$ edges and $|Z'| \leq |Z| - (k+1) \leq k(k-3)/2$. Because $|N_{G'_{k-2}}(Z'')| \geq (k+1)(k-3)/2$, $|Z''| \leq |N_{G'_{k-2}}(Z'')|$.

We conclude that for any $Z'' \subseteq Z'$, $|Z''| \leq |N_{G'_{k-2}}(Z'')|$. Hence, G'_{k-2} has a matching M of size r such that every vertex of Z' is saturated in M .

Clearly, M is a matching in G'_k as well. Recall that $R = x_1 \dots x_p x_1$ is a Hamiltonian cycle in $H(Z)$ and $U = \{u_{x_1 x_2}, \dots, u_{x_{p-1} x_1}\}$. For $i \in \{1, \dots, p\}$, let t_i be the number of vertices in $\{v_{x_i}^1, \dots, v_{x_i}^s\}$ that are saturated in M . Because M is a matching in G'_{k-1} , $t_i \leq s - 1$.

We prove that there is $j \in \{1, \dots, p\}$ such that $t_j < s - 1$. Let q be the smallest positive integer such that $|Z| \leq q(q-1)/2$. The graph $H(Z)$ has at least q vertices. Suppose first that it has exactly q vertices. Then $p = q$ and $Z' = Z \setminus U$ has at most $p(p-1)/2 - p = p(p-3)/2$ vertices. Also $|N_{G'_{k-2}}(Z)| = p(k-3)/2$. If $p < k$, at least one vertex in $N_{G'_{k-2}}(Z)$ is not saturated and the statement holds. Let $p = k$. Then because G has no cliques of size k , $|Z| < k(k-1)/2$ and $|Z'| < k(k-3)/2$. We have that $|Z'| < |N_{G'_{k-2}}(Z)|$ and at least one vertex in $N_{G'_{k-2}}(Z)$ is not saturated. If $p \geq k+1$, then $|Z| = k(k-1)/2 + 1$. We have that $|Z'| \leq k(k-3)/2$ and $|N_{G'_{k-2}}(Z)| \geq (k+1)(k-3)/2$. Hence, there is a non-saturated vertex in $N_{G'_{k-2}}(Z)$. Suppose now that $H(Z)$ has at least $q+1$ vertices. Then $p \geq q+1$ and $|Z'| \leq q(q-1)/2 - (q+1) = q(q-3)/2 - 1$. As $|N_{G'_{k-2}}(Z)| \geq (q+1)(k-3)/2$, $|Z'| < |N_{G'_{k-2}}(Z)|$ if $q \leq k$. If $q \geq k+1$, then $|Z'| \leq |Z| - (k+2) \leq (k(k-1)/2 + 1) - (k+2) \leq k(k-3)/2 - 1$. Because $|N_{G'_{k-2}}(Z)| \geq (k+2)(k-3)/2$, we again have a non-saturated vertex in $N_{G'_{k-2}}(Z)$. We considered all cases and conclude that at least one vertex of $N_{G'_{k-2}}(Z)$ is not saturated in M . Hence, there is $j \in \{1, \dots, p\}$ such that $t_j < s - 1$. Without loss of generality we assume that $j = p$.

Because $v_{x_1}^1, \dots, v_{x_i}^s$ have the same neighborhoods, we assume without loss of generality that for $i \in \{1, \dots, p\}$, $v_{x_i}^1, \dots, v_{x_i}^{t_i}$ are saturated. For $i \in \{1, \dots, q\}$ and $j \in \{1, \dots, t_i\}$, denote by u_i^j the vertex of Z' such that $\{v_{x_i}^j, u_i^j\} \in M$. Notice that it can happen that $t_i = 0$ and we have no such saturated vertices. We define the path $P_i = v_{x_i}^1 u_i^1 v_{x_i}^2 \dots u_i^{s_i} v_{x_i}^{t_i+1}$ if $t_i \geq 1$ and let $P_i = v_{x_i}^1$ if $t_i = 0$ for $i \in \{1, \dots, p\}$. Let $P = P_1 + v_{x_1}^{t_1+1} u_{x_1 x_2} v_{x_2}^1 + \dots + v_{x_{p-1}}^{t_{p-1}+1} u_{x_{p-1} x_p} v_{x_p}^1 + P_p$ and then form the cycle C from P by joining the end-vertices of P by $v_{x_p}^{t_p+1} v_{x_p}^{t_p+2} u_{x_p x_1} v_{x_1}^1$ using the fact that $t_p \leq s - 2$. We have that $Z \subseteq V(C) \subseteq Z \cup N_G(Z)$ and $v_{x_p}^{t_p+1} v_{x_p}^{t_p+2} \in E(C)$. It concludes Case 3 and the proof of the claim. \square

Let $Z \subseteq V(G'_k)$ be a set of size k' . Let $Z' = Z \cap \{u_{xy} | \{x, y\} \in E(G)\}$. If $Z' = \emptyset$, then Z is a clique and there is a cycle C in G'_k such that $Z \subseteq V(C)$. Suppose that $Z' \neq \emptyset$. By Claim, there is a cycle C' in G'_k such that $Z' \subseteq V(C') \subseteq Z' \cup N_G(Z')$ and C' has an edge $\{v_x^i, v_y^j\}$ for some $x, y \in V(G')$ and $i, j \in \{1, \dots, s\}$. Let $\{u_1, \dots, u_p\} = Z \setminus V(C')$. Notice that these vertices are pairwise adjacent and adjacent to v_x^i, v_y^j . We construct the cycle C from C' by replacing $\{v_x^i, v_y^j\}$ by the path $v_x^i u_1 \dots u_p v_y^j$. It remains to observe that $Z \subseteq V(C) \subseteq Z \cup N_G(Z)$. \square

6 Dynamic Programming for Planar Cyclability

The purpose of this section is to present a dynamic programming-based algorithm for the problem PLANAR ANNOTATED CYCLABILITY:

Algorithm DP(G, R, k, q, \mathcal{D})

Input: A graph G , a vertex set $R \subseteq V(G)$, two non-negative integers k and q , where $k \leq q$, and a tree decomposition \mathcal{D} of G of width q .

Output: An answer whether (G, R, k) is a YES-instance of PLANAR ANNOTATED CYCLABILITY problem, or not.

Running time: $2^{2^{O(q \cdot \log q)}} \cdot n$.

We observe that the question of PLANAR ANNOTATED CYCLABILITY can be expressed in monadic second-order logic (MSOL). It is sufficient to notice that an instance (G, R, k) is a YES-instance of PLANAR ANNOTATED CYCLABILITY if and only if for any (not necessarily distinct) $v_1, \dots, v_k \in R$, there are sets $X \subseteq V(G)$ and $S \subseteq E(G)$ such that $v_1, \dots, v_k \in X$ and $C = (X, S)$ is a cycle. The property of $C = (X, S)$ being a cycle is equivalent to asking whether

- i) for any $x \in X$, there are two distinct $e_1, e_2 \in S$ such that x is incident to e_1 and e_2 ,
- ii) for any $x \in X$ and any three pairwise distinct $e_1, e_2, e_3 \in S$, e_1 is not incident to x or e_2 is not incident to x or e_3 is not incident to x , and
- iii) for any $Z_1, Z_2 \subseteq X$ such that $Z_1 \cap Z_2 = \emptyset$, $Z_1 \neq \emptyset$, $Z_2 \neq \emptyset$ and $Z_1 \cup Z_2 = X$, there is $\{x, y\} \in S$ such that $x \in Z_1$ and $y \in Z_2$.

By the celebrated Courcelle's theorem (see, e.g., [6,5]), any problem that can be expressed in MSOL can be solved in linear time for graphs of bounded treewidth.

As we saw, PLANAR ANNOTATED CYCLABILITY can be solved in $f(q, k) \cdot n$ steps if the treewidth of an input graph is at most q . Next, we give a dynamic programming algorithm in order to achieve better running time than the one that Courcelle's theorem provides:

Nice tree decompositions. Let G be a graph. A tree decomposition $\mathcal{D} = (T, \mathcal{X})$ of G is called a *nice tree decomposition* of G if T is rooted to some leaf r and:

1. for any leaf $l \in V(T)$ where $l \neq r$, $X_l = \emptyset$ (we call X_l leaf node of \mathcal{D} , except from X_r which we call *root* node)
2. the root and any non-leaf $t \in V(T)$ have one or two children
3. if t has two children t_1 and t_2 , then $X_t = X_{t_1} = X_{t_2}$ and X_t is called *join* node
4. if t has one child t' , then
 - either $X_t = X_{t'} \cup \{v\}$ (we call X_t *insert* node and v *insert* vertex)
 - or $X_t = X_{t'} \cup \{v\}$ (we call X_t *forget* node and v *forget* vertex)

Pairings. Let S be a set and let $|S| = n$. A *pairing* of S , is a graph H (we allow the existence of loops) such that $V(H) = S$ and $\deg_H(x) \leq 2, \forall x \in S$. We denote by $\mathcal{P}(S)$ the set of all pairings of S .

Let (G, R, k) be an instance of PLANAR ANNOTATED CYCLABILITY and set $V \setminus R = B$. Let also (T, \mathcal{X}, r) be a nice tree decomposition of G , where r is the

root of T . For every $x \in V(T)$ let T_t be the subtree of T rooted at t (the vertices of T_t are x and its descendants in T). Then for every $t \in V(T)$, we define

$$G_t = G \left[\bigcup_{t' \in V(T_t)} X_{t'} \right]$$

It is clear that the restriction of a cycle, C of G , in G_t is a collection \mathcal{Q} of vertex disjoint paths in G_t with ends in X_t . Intuitively, this collection of paths indicates “the way that C passes through X_t ”. Each collection of paths in \mathcal{Q} of G_t can be mapped to a unique pairing $P_{\mathcal{Q}} \in \mathcal{P}(X_t)$.

Let S be a set and $i \in \mathbb{N}$. We define $S^{(i)} = \{A \subseteq S \mid |A| \leq i\}$ and $S^{[i]} = \{A \subseteq S \mid |A| = i\}$.

7 Discussion

Notice that we have no proof that CYCLABILITY is in NP. The definition of the problem classifies it directly in Π_2^P . This prompts us to conjecture the following:

Conjecture 1. CYCLABILITY is Π_2^P -complete.

Moreover, while we have proved that CYCLABILITY is co-W[1]-hard, we have no evidence on which level of the parameterized complexity hierarchy it belongs (lower than the XP class). We find it an intriguing question whether there is some $i \geq 1$ for which CYCLABILITY is W[i]-complete (or co-W[i]-complete).

Clearly, a challenging question is whether the, double exponential, parametric dependence of our FPT-algorithm can be improved. We believe that this is not possible and we suspect that this issue might be related with Conjecture 1.

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