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# Square Roots of Minor Closed Graph Classes

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## Abstract

Let  $\mathcal{G}$  be a graph class. The *square root* of  $\mathcal{G}$  contains all graphs whose square belong in  $\mathcal{G}$ . We prove that if  $\mathcal{G}$  is non-trivial and minor closed, then all graphs in its square root have carving-width bounded by some constant depending only on  $\mathcal{G}$ . As a consequence, every square root of such a graph class has a linear time recognition algorithm.

*Keywords:* square roots of graphs, branch-width, carving-width, graph minors.

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## 1. Introduction

Let  $\mathcal{G}$  be a graph class. The *square root* of  $\mathcal{G}$  is defined as the graph class

$$\sqrt{\mathcal{G}} = \{G \mid G^2 \in \mathcal{G}\},$$

where the *square*  $G^2$  of a graph  $G$  is the graph obtained from  $G$  after adding edges between all pairs of vertices that share a common neighbor.

In [3], Harary, Karp, and Tutte provided a complete characterization of the graphs in  $\sqrt{\mathcal{P}}$  where  $\mathcal{P}$  is the class of all planar graphs. Notice that planar graphs are *minor closed*, i.e. a minor of every graph in  $\mathcal{P}$  also belongs in  $\mathcal{P}$ .

According to the characterization of [3], all graphs in  $\sqrt{\mathcal{P}}$  are outerplanar and have bounded degree. This implies that graphs in  $\sqrt{\mathcal{P}}$  have a very specific “tree-like” structure and it is a natural question whether this is the case for more general graph classes. In this paper we extend this result, in the sense that the same tree-like property holds for every minor closed graph class that is non-trivial (i.e. that does not contain all graphs). In

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fact, we prove (in Section 3) that, in this case, the correct “tree-likeness” property is given by the parameter of carving-width, introduced by Seymour and Thomas in [10]. As a consequence, we prove in Section 4 that the square root of any non-trivial minor closed graph class has a linear time recognition algorithm. This extends the algorithmic results of [5] where a linear time algorithm was given for recognizing the square roots of planar graphs.

## 2. Definitions

We next give some definitions that are necessary in order to formally define carving width. This will permit us to give the formal statement of our combinatorial result.

*Boundaries in graphs and hypergraphs.* In this paper we deal with graphs and hypergraphs. For a (hyper)graph  $G$  we denote by  $V(G)$  its vertex set and by  $E(G)$  the set of its (hyper)edges. If  $S \subseteq V(G)$  (resp.  $F \subseteq E(G)$ ) we denote  $\bar{S} = V(G) \setminus S$  (resp.  $\bar{F} = E(G) \setminus F$ ).

Given a vertex set  $S \subseteq V(G)$ , we define by  $E_G(S)$  as the set of hyperedges containing vertices in  $S$ . For simplicity we also denote  $E_G(v) = E_G(\{v\})$ . We also define  $\Delta(G) = \max\{|E_G(v)| \mid v \in V(G)\}$ . Given a set  $S \subseteq V(G)$ , we define

$$\partial_G(S) = E_G(S) \cap E_G(\bar{S}).$$

Notice that  $\partial_G$  is a symmetric function, i.e. for every  $S \subseteq V(G)$ , it holds that  $\partial_G(S) = \partial_G(\bar{S})$ . Also given a set  $F \subseteq E(G)$ , we set

$$\partial_G^*(F) = \left( \bigcup_{f \in F} f \right) \cap \left( \bigcup_{f \in \bar{F}} f \right)$$

Given a hypergraph  $G$  we define its *dual* as the hypergraph

$$G^* = (E(G), \{E_G(v) \mid v \in V(G)\}).$$

Notice that the hypergraphs  $G$  and  $G^*$  have the same incidence graph with the roles of their two parts reversed. Given a set  $S \subseteq V(G)$  (resp.  $F \subseteq E(G)$ ) we denote by  $S^*$  (resp.  $F^*$ ) their dual hyperedges (resp. vertices) in  $G^*$ .

Using duality, we also define  $\Delta^*(G) = \Delta(G^*)$ . Clearly, for a simple graph  $G$ ,  $\Delta^*(G) = 2$ . Moreover, the above definitions imply that for every  $F \subseteq E(G)$ ,  $(\partial_G^*(F))^* = \partial_{G^*}(F^*)$ .

A graph  $H$  is a *minor* of a graph  $G$ , and we write  $H \leq G$ , if  $H$  can be obtained by some subgraph of  $G$  after contracting edges (the contracting an

edge  $e = \{x, y\}$  is the operation that introduces removes  $x$  and  $y$  from  $G$  and introduces a new vertex  $v_e$  that is made adjacent with all the neighbors of  $x$  and  $y$  in  $G$ , except from  $x$  and  $y$ ). A graph class  $\mathcal{G}$  is *minor closed* if every minor of a graph in  $\mathcal{G}$  is also a graph in  $\mathcal{G}$ .

*Carving-width.* Given a tree  $T$  we denote the set of its leaves by  $L(T)$  and we call it *ternary* if all vertices in  $V(T) \setminus L(T)$  have degree 3. A *carving decomposition* of a hypergraph  $G$  is a pair  $(T, \rho)$ , where  $T$  is a ternary tree and  $\rho$  is a bijection from  $V(G)$  to  $L(T)$ . The *bridge function*  $\beta : E(T) \rightarrow 2^{E(G)}$  of a carving decomposition maps every edge  $e$  of  $T$  to the set  $\partial_G(\rho^{-1}(L(T')))$  where  $T'$  is one of the two connected components of  $T \setminus e$ . The *width* of  $(T, \rho)$  is equal to  $\max_{e \in E(T)} |\beta(e)|$  and the *carving-width* of  $G$ ,  $\mathbf{cw}(G)$ , is the minimum width over all carving decompositions of  $G$ . The following observation is a direct consequence of the definitions.

**Observation 1.** *For every hypergraph  $G$ , it holds that  $\Delta(G) \leq \mathbf{cw}(G)$ .*

The main combinatorial result of this paper is the following.

**Theorem 1.** *For every non-trivial minor closed graph class  $\mathcal{G}$  there is a constant  $c_{\mathcal{G}}$  such that all graphs in  $\mathcal{G}$  have carving-width at most  $c_{\mathcal{G}}$ .*

The proof of Theorem 1 uses the parameter of branch-width defined in [7].

*Branch-width.* A *branch decomposition* of a graph  $G$  is a pair  $(T, \tau)$ , where  $T$  is a ternary tree and  $\tau$  is a bijection from  $E(G)$  to  $L(T)$ . The *boundary function*  $\omega : E(T) \rightarrow 2^{V(G)}$  of a branch decomposition maps every edge  $e$  of  $T$  to the set  $\partial_G^*(\tau^{-1}(L(T')))$  where  $T'$  is one of the two connected components of  $T - \{e\}$ . The *width* of  $(T, \tau)$  is equal to  $\max_{e \in E(T)} |\omega(e)|$  and the *branch-width* of  $G$ ,  $\mathbf{bw}(G)$ , is the minimum width over all branch decompositions of  $G$ .

The following observation is a direct consequence of the duality between the functions  $\partial_G$  and  $\partial_G^*$ .

**Observation 2.** *For every hypergraph  $G$  it holds that  $\mathbf{bw}(G) = \mathbf{cw}(G^*)$ .*

### 3. Walls and squares

**Walls.** A *wall of height  $k$* ,  $k \geq 1$ , is obtained from a  $((k+1) \times (2k+2))$ -grid with vertices  $(x, y)$ ,  $x \in \{0, \dots, 2k+1\}$ ,  $y \in \{0, \dots, k\}$ , after removing the “vertical” edges  $\{(x, y), (x, y+1)\}$  for odd  $x+y$ . We denote such a wall by

$W_k$ . A *subdivided wall of height  $k$*  is obtained by the wall  $W_k$  with some edges of  $W_k$  replaced by paths without common internal vertices. If, in such a subdivided wall, all edges have been subdivided at least once, then we say that it is *properly subdivided*.

The following result follows from the results in [6] and [7].

**Proposition 1 ([6] and [7]).** *There is a function  $g : \mathbb{N} \rightarrow \mathbb{N}$  such that every graph  $G$  with branchwidth at least  $g(k)$  contains the  $(k \times k)$ -grid as a minor.*

A direct consequence of Proposition 1 is the following.

**Lemma 1.** *There exists a function  $f$  such that every graph  $G$  with branchwidth at least  $f(k)$  contains a properly subdivided wall of height  $k$  as a subgraph.*

**Proof.** Let  $W'_k$  be the graph taken from a  $k$ -wall if we subdivide each edge once. Notice that  $W'_k$  is a subgraph of the  $((4k + 3) \times (4k + 3))$ -grid. The lemma follows from Proposition 1 and the fact that minor relation and topological minor relation are identical when the host graph has maximum degree 3 (notice that  $\Delta(W'_k) \leq 3$ ).  $\square$

Let  $H$  be a  $((k + 1) \times (2k + 2))$ -grid where  $k$  is a positive odd integer. Let also  $W_k$  a spanning subgraph that is a  $k$ -wall as explained above. Let also  $G$  be a subdivision of  $W_k$  where the non-subdivision vertices are denote as coordinates of  $H$  (we call these vertices *original*). In particular, we denote the vertex of  $H$  with coordinates  $i, j$  by  $a^{i,j}$  (see Figure 1). For  $i \in \{0, \dots, k - 1\}$  and  $j \in \{0, \dots, 2k + 1\}$  where  $i + j$  is even, we denote by  $P_{i,j}^{(v)}$  the path that has replaced the edge  $\{a^{i,j}, a^{i+1,j}\}$  in  $W_k$  and we call these paths *vertical*. For  $i \in \{0, \dots, k\}$  and  $j \in \{0, \dots, 2k\}$ , we denote by  $P_{i,j}^{(h)}$  the path that has replaced the edge  $\{a^{i,j}, a^{i,j+1}\}$  and we call these paths *horizontal*. We denote by  $L_i, i \in \{0, \dots, k\}$  the path  $P_{i,0}^h \oplus \dots \oplus P_{i,2k}^h$ , we direct it from  $a^{i,0}$  to  $a^{i,2k}$  and we call it  *$i$ -th line* of  $G$ . We also define, for  $l \in \{0, \dots, k\}$ , the  *$l$ -th meander* of  $G$  as the path

$$\begin{aligned} M_l &= P_{0,2l}^{(v)} \oplus P_{1,2l}^{(h)} \oplus P_{1,2l+1}^{(v)} \oplus P_{2,2l}^{(h)} \oplus P_{2,2l}^{(v)} \oplus \dots \\ &\quad \oplus P_{k-2,2l+1}^{(v)} \oplus P_{k-1,2l}^{(h)} \oplus P_{k-1,2l}^{(v)}. \end{aligned}$$

Finally, for every original vertex  $a^{i,j}$  where  $0 \leq j \leq 2k$  ( $1 \leq j \leq 2k + 1$ ), we define **right** $_G(a^{i,j})$  (**left** $_G(a^{i,j})$ ) as the vertex of the directed path  $L_i$  that appears right after (before)  $a^{i,j}$ .

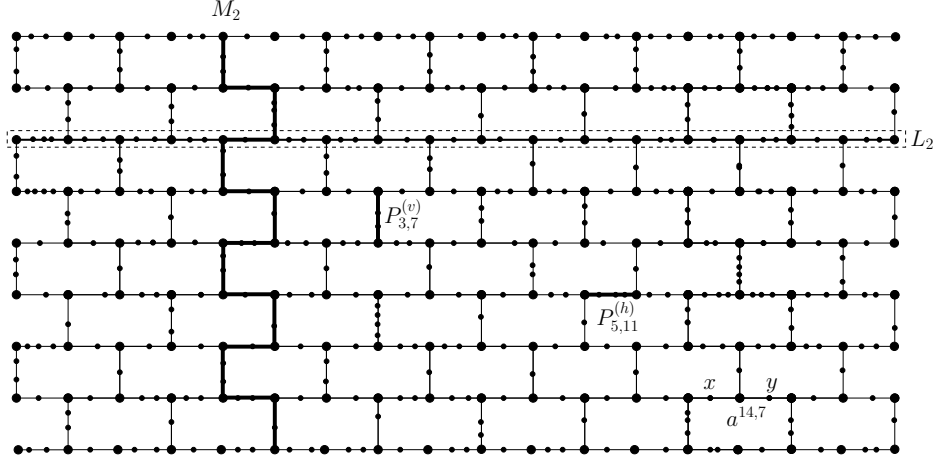


Figure 1: A properly subdivided wall  $G$  of height 8. The big (resp. small) vertices are the original (subdivision) vertices. The paths  $P_{3,7}^{(v)}$ ,  $P_{5,11}^{(h)}$ ,  $L_2$  and the 2-meander  $M_2$  are depicted in  $G$ . Notice that  $x = \text{left}_G(a^{14,7})$  and  $y = \text{right}_G(a^{14,7})$ .

We denote by  $K_r$  the complete graph on  $r$  vertices. Also we use the notation  $K_{r,q}$  for the complete bipartite graph with parts of size  $r$  and  $q$ .

**Lemma 2.** *Let  $k$  be a positive odd integer. If  $G$  is a properly subdivided wall of height  $k$ , then  $K_{k+2} \leq G^2$ .*

**Proof.** For each  $i \in \{0, \dots, k\}$  we define the path  $P_i$  of  $G^2$  as follows: If  $L_i = (x_0, x_1, \dots, x_m)$  is the  $i$ -th horizontal path of  $G$ , then  $\bar{L}_i = (x_0, x_2, x_4, \dots, x_{m'})$  where  $m' = m$  or  $m' = m - 1$  depending on whether  $m$  is even or not. We set  $S = \bigcup_{i \in \{0, \dots, k\}} V(\bar{L}_i)$ .

Our next step is to define the paths  $Q_j$  of  $G^2$ ,  $l \in \{0, \dots, k\}$ . Let  $M_l$  be the  $l$ -th meander of  $G$ . Following the definition of a meander, we denote

- $P_{i,2l}^{(v)} = (y_1^{i,2l}, \dots, y_{w_{i,2l}}^{i,2l})$ , for even  $i \in \{0, \dots, k-1\}$  (assuming that  $y_1^{i,2l} = a^{i,2l}$  and  $y_{w_{i,2l}}^{i,2l} = a^{i+1,2l}$ ),
- $P_{i,2l+1}^{(v)} = (y_1^{i,2l+1}, \dots, y_{w_{i,2l+1}}^{i,2l+1})$ , for odd  $i \in \{0, \dots, k-2\}$  (assuming that  $y_1^{i,2l+1} = a^{i,2l+1}$  and  $y_{w_{i,2l+1}}^{i,2l+1} = a^{i+1,2l+1}$ ),
- $P_{i,2l}^{(h)} = (z_1^{i,2l}, \dots, z_{m_{i,2l}}^{i,2l})$ ,  $i \in \{1, \dots, k-1\}$  (assuming that  $z_1^{i,2l} = a^{i,2l}$  and  $z_{m_{i,2l}}^{i,2l} = a^{i,2l+1}$ ).

For each even  $i \in \{0, \dots, k-1\}$ , we set  $\bar{P}_{i,2l}^{(v)} = (\bar{y}_1^{i,2l}, y_2^{i,2l}, \dots, y_{w_i,2l-1}^{i,2l}, \bar{y}_{w_i,2l}^{i,2l})$  where

$$\bar{y}_1^{i,2l} = \begin{cases} y_1^{i,2l} & \text{if } y_1^{i,2l} \notin S \\ \mathbf{right}_G(y_1^{i,2l}) & \text{if } y_1^{i,2l} \in S \end{cases}$$

$$\bar{y}_{w_i,2l}^{i,2l} = \begin{cases} y_{w_i,2l}^{i,2l} & \text{if } y_{w_i,2l}^{i,2l} \notin S \\ \mathbf{right}_G(y_{w_i,2l}^{i,2l}) & \text{if } y_{w_i,2l}^{i,2l} \in S. \end{cases}$$

and for each odd  $i \in \{0, \dots, k-1\}$ , we set  $\bar{P}_{i,2l+1}^{(v)} = (\bar{y}_1^{i,2l+1}, y_2^{i,2l+1}, \dots, y_{w_i,2l-1}^{i,2l+1}, \bar{y}_{w_i,2l+1}^{i,2l+1})$  where

$$\bar{y}_1^{i,2l+1} = \begin{cases} y_1^{i,2l+1} & \text{if } y_1^{i,2l+1} \notin S \\ \mathbf{left}_G(y_1^{i,2l+1}) & \text{if } y_1^{i,2l+1} \in S \end{cases}$$

$$\bar{y}_{w_i,2l+1}^{i,2l+1} = \begin{cases} y_{w_i,2l+1}^{i,2l+1} & \text{if } y_{w_i,2l+1}^{i,2l+1} \notin S \\ \mathbf{left}_G(y_{w_i,2l+1}^{i,2l+1}) & \text{if } y_{w_i,2l+1}^{i,2l+1} \in S. \end{cases}$$

Observe that for each  $i \in \{1, \dots, k-1\}$  and  $l \in \{0, \dots, k\}$  the vertices in  $P_{i,2l}^{(h)} \setminus S$  induce a path in  $G^2$  which we denote by  $\bar{P}_{i,2l}^{(h)}$ .

Notice, that for every odd  $i \in \{1, \dots, k-2\}$  it holds that

$$V(\bar{P}_{i-1,2l}^{(v)}) \cap V(\bar{P}_{i,2l}^{(h)}) = \{\bar{y}_{w_{i-1},2l}^{i-1,2l}\} \quad (1)$$

$$V(\bar{P}_{i,2l}^{(h)}) \cap V(P_{i,2l+1}^{(v)}) = \{\bar{y}_1^{i,2l+1}\}. \quad (2)$$

Also, for every even  $i \in \{2, \dots, k-1\}$  is also holds that

$$V(\bar{P}_{i-1,2l+1}^{(v)}) \cap V(\bar{P}_{i,2l}^{(h)}) = \{\bar{y}_{w_{i-1},2l+1}^{i-1,2l+1}\} \quad (3)$$

$$V(\bar{P}_{i,2l}^{(h)}) \cap V(\bar{P}_{i,2l}^{(v)}) = \{\bar{y}_1^{i,2l}\}. \quad (4)$$

From (1)–(4) we obtain that

$$\bar{M}_j = \bar{P}_{0,2l}^{(v)} \oplus \bar{P}_{1,2l}^{(h)} \oplus \bar{P}_{1,2l+1}^{(v)} \oplus \bar{P}_{2,2l}^{(h)} \oplus \bar{P}_{2,2l}^{(v)} \oplus \dots \oplus \bar{P}_{k-2,2l+1}^{(v)} \oplus \bar{P}_{k-1,2l}^{(h)} \oplus \bar{P}_{k-1,2l}^{(v)}$$

is a path in  $G^2$  for each  $l \in \{0, \dots, k\}$ .

As paths in  $\bar{\mathcal{M}} = \{\bar{M}_l \mid l \in \{0, \dots, k\}\}$  avoid all vertices of  $S$ , we have that for each  $i, l \in \{0, \dots, k\}$   $V(\bar{L}_i) \cap V(\bar{M}_l) = \emptyset$ . Moreover, by construction, any two paths in  $\bar{\mathcal{M}}$  are vertex disjoint. Recall also that the same holds for the paths in  $\bar{\mathcal{L}} = \{\bar{L}_i \mid i \in \{0, \dots, k-1\}\}$ .

We claim that for each  $i, l \in \{0, \dots, k\}$  there is an edge  $e_{i,l}$  in  $G^2$  with one endpoint in  $\bar{L}_i$  and the other in  $\bar{M}_l$ . Indeed it is easy to see that one can take

$$e_{i,l} = \begin{cases} \{a^{i,2l}, \mathbf{left}(a^{i,2l})\} & \text{if } (i, l) \in \{(0, k), (k, k)\} \\ \{a^{i,2l}, \mathbf{right}(a^{i,2l})\} & \text{otherwise} \end{cases}$$

Now we are ready to prove that  $G^2$  contains  $K_{k+2}$  as a minor. For this, we first remove from  $G^2$  all edges that are neither edges of paths in  $\bar{\mathcal{M}} \cup \bar{\mathcal{L}}$  nor edges in  $\mathcal{E} = \{e_{i,l} \mid (i, l) \in \{0, \dots, k\}^2\}$ . Then we contract all edges of the remaining graph except from those in  $\mathcal{E}$  and we end up to a graph isomorphic to  $K_{k+1, k+1}$  that, in turn, can be further contracted to  $K_{k+2}$ .  $\square$

**Extended carving-decompositions.** Let us relax the definition of a carving decomposition in the case of graphs in the following way: let  $G$  be a graph,  $T$  be a ternary tree rooted on some, say  $r$ , of its leaves, and a surjection  $\sigma : L(T) \setminus \{r\} \rightarrow V(G)$ . As  $T$  is rooted each edge of  $T$  can be seen as a directed edge pointing towards the root. Let  $f = (x, y)$  be such an edge and let  $T_x$  and  $T_y$  be the two connected components of  $T \setminus f$  such that  $r \in V(T_y)$ . Let also  $V_x = \sigma(L(T_x))$  and  $V_y = \sigma(L(T_y))$ . We define the **bridge set** of  $f$  as  $\beta(f) = \{\{a, b\} \in E(G) \mid a \in V_x \text{ and } b \in V_y \setminus V_x\}$ . We call each such pair  $(T, \sigma)$  an *extended carving decomposition* of  $G$  and we define its *width* as the maximum  $|\beta(f)|$  over all  $f \in E(T)$ .

**Lemma 3.** *If  $G$  has an extended carving decomposition of width at most  $k$ , then it also has a carving decomposition of width at most  $k$ .*

**Proof.** Let  $(T, \sigma)$  be such an extended carving decomposition. We apply on  $(T, \sigma)$  the following transformation until this is not possible any more: if for some  $v$ , in  $\sigma^{-1}(v)$  contains at least two vertices  $a$  and  $b$  then replace  $\sigma$  by  $\sigma \setminus \{(a, v)\}$  and replace  $T$  by the tree that is obtained if we remove  $a$  and then we dissolve the resulting vertex of degree 2. It is easy to see that each new pair  $(T, \sigma)$  has width at most  $k$ . Also when the above procedure finishes,  $\sigma^{-1}$  is a bijection from  $V(G)$  to  $L(T) \setminus \{r\}$ . If now we remove  $r$  from  $T$  and dissolve the resulting vertex of degree 2, we have a branch decomposition  $(T', \sigma^{-1})$  of  $G$  with width at most  $k$ .  $\square$

**Lemma 4.** *For every graph  $G$ ,  $\mathbf{cw}(G) \leq \Delta(G) \cdot \mathbf{bw}(G)$ .*



**Proof.** We set  $\delta = \Delta(G)$ . Let  $(T, \tau)$  be a branch decomposition of  $G$  with width at most  $k$ . We now define an extended carving decomposition  $(T', \sigma)$  of  $G$  as follows: Take  $T$  and for each  $v \in L(T)$  we define two new vertices  $v_1$  and  $v_2$  and we make them adjacent to  $v$ . For each such  $v$  where  $\tau^{-1}(v) = \{z, w\}$ , we set  $\sigma(v_1) = z$  and  $\sigma(v_2) = w$ . In the resulting tree we pick arbitrarily an edge, we subdivide it and we make the subdivision vertex adjacent with new vertex  $r$ . In the resulting graph  $T'$ ,  $r$  will be the root. Let  $f = (x, y)$  be an edge of  $T'$ . We set  $X = \sigma(L(T_x)) \setminus \sigma(L(T_y))$ ,  $Y = \sigma(L(T_y)) \setminus \sigma(L(T_x))$  and  $S = \sigma(L(T_x)) \cap \sigma(L(T_y))$ . Notice that  $\omega(f)$ , as defined by the branch decomposition  $(T, \mu)$ , is exactly the set  $S$ , therefore  $S$  is a separator of  $G$  and  $|S| \leq k$ . We conclude that each edge in  $\omega(f)$  has one endpoint in  $S$  and the other in  $Y$ . As each vertex in  $S$  has degree  $\leq \Delta(G)$ , we obtain that  $\omega(f) \leq k \cdot d$ . Therefore,  $(T', \sigma)$  is an extended branch decomposition of  $G$  with width  $\leq k \cdot d$ . The lemma now follows from Lemma 3.  $\square$

We are now ready to prove the main result of this paper.

**Proof of Theorem 1.** As  $\mathcal{G}$  is non-trivial class we choose a graph  $H \notin \mathcal{G}$  and we assume that  $H$  has  $h$  vertices. As  $\mathcal{G}$  is minor closed,  $\mathcal{G}$  excludes  $K_h$  as a minor. Therefore if  $G \in \sqrt{\mathcal{G}}$ , then  $G^2$  does not contain  $K_h$  as a minor. From Lemma 1,  $G$  does not contain as a subgraph a properly subdivided wall of height  $\geq h - 2$ . From Lemma 1,  $\mathbf{bw}(G) < f(h - 2)$ . Notice also that  $\Delta(G) \leq h - 1$ . Then, from Lemma 4,  $\mathbf{cw}(G) < (h - 1) \cdot f(h - 2)$  and the Theorem follows if we set  $c_{\mathcal{G}} = (h - 1) \cdot f(h - 2)$ .  $\square$

In the rest of this section we will also bound the branch-width of the squares of the graphs in  $\mathcal{G}$ . This will be useful for the algorithmic consequences of our results in Section 4.

**Lemma 5.** *For every graph  $G$  it holds that  $\mathbf{cw}(G^2) \leq 2 \cdot (\mathbf{cw}(G))^2 - \mathbf{cw}(G)$ .*

**Proof.** Let  $(T, \rho)$  be a carving decomposition of  $G$  of width at most  $k$ . We will prove that  $(T, \rho)$  is a carving decomposition of  $G^2$  of width at most  $2k^2 - k$ . We use the notation  $\beta$  and  $\beta'$  in order to distinguish the bridge sets for  $G$  and  $G^2$  respectively. Let  $f \in E(T)$ . As  $\Delta(G) \leq \mathbf{cw}(G)$ , it is enough to prove that  $\beta'(f) \leq 2 \cdot k \cdot (\Delta(G) - 1) + k$ . To prove this, we first define  $S = \rho^{-1}(T')$  where  $T'$  is one of the connected components of  $T - \{e\}$ . We define  $L = S \cap (\bigcup_{e \in \beta(f)} e)$  and  $R = \bar{S} \cap (\bigcup_{e \in \beta(f)} e)$ . Finally, for each  $e = \{x, y\} \in \beta(f)$  where  $x \in L$  and  $y \in R$ , we define the sets

$$\begin{aligned} Q_R(e) &= \{\{x, z\} \mid z \in N_G(y) \cap \bar{S}\} \\ Q_L(e) &= \{\{z, y\} \mid z \in N_G(x) \cap S\} \end{aligned}$$

We claim that each edge  $e' = \{v, u\} \in \beta'(f) \setminus \beta(f)$  belongs in  $Q_R(e) \cup Q_L(e)$  for some  $e \in \beta(f)$ . For this, suppose that  $e' = \{v, u\}$  where  $v \in S$  and  $u \in \bar{S}$ . As  $e'$  is not an edge in  $G$  and there is a vertex  $z \in V(G)$  such that  $\{v, z\}, \{z, u\} \in E(G)$ . If  $z \in S$  then  $\{z, u\} \in \beta(e)$  and therefore  $e' \in Q_L(e)$  for  $e = \{z, u\}$ . If  $z \in \bar{S}$  then  $\{v, z\} \in \beta(e)$  and therefore  $e' \in Q_R(e)$  for  $e = \{v, z\}$ . In both cases, the claim holds.

Notice now that for each  $e = \{x, y\} \in \beta(f)$ ,  $|Q_R(e) \cup Q_L(e)| = |Q_R(e)| + |Q_L(e)|$ . As  $|Q_R(e)| \leq |N_G(y)| - 1 \leq \Delta(G) - 1$  and  $|Q_L(e)| \leq |N_G(x)| - 1 \leq \Delta(G) - 1$ , we conclude that  $|Q_R(e) \cup Q_L(e)| \leq 2 \cdot (\Delta(G) - 1)$ . Therefore  $|\beta'(f) \setminus \beta(f)| \leq 2 \cdot k \cdot (\Delta(G) - 1)$  and the claim follows.  $\square$

**Lemma 6.** *Let  $\mathcal{G}$  be a class that is minor closed and let  $G \in \sqrt{\mathcal{G}}$ . Then  $\mathbf{bw}(G^2) \leq 4 \cdot c_{\mathcal{G}}^2 - 2 \cdot c_{\mathcal{G}}$ .*

**Proof.** Let  $G^2$  be a graph in  $\mathcal{G}$ . From Theorem 1,  $\mathbf{cw}(G) \leq c_{\mathcal{G}}$  and, from Lemma 5,  $\mathbf{cw}(G^2) \leq 2 \cdot c_{\mathcal{G}}^2 - c_{\mathcal{G}}$ . As  $G^2$  is a graph, we have that  $\Delta^*(G^2) = 2$ . This together with Observation 2 and Lemma 4, implies that  $\mathbf{bw}(G^2) \leq 4 \cdot c_{\mathcal{G}}^2 - 2 \cdot c_{\mathcal{G}}$ .  $\square$

#### 4. Linear time recognition of $\sqrt{\mathcal{G}}$

In this section we study the complexity of the following problem when  $\mathcal{G}$  is a minor closed graph class.

<p style="text-align: center;">SQUARE ROOT OF <math>\mathcal{G}</math>  <i>Instance:</i> A graph <math>G</math>.  <i>Question:</i> is <math>G</math> a member of <math>\sqrt{\mathcal{G}}</math>?</p>
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Before we deal with the above problem, we briefly expose some known facts on the recognition of minor closed graph classes. Let  $\mathcal{G}$  be such a class. We denote by  $\mathbf{obs}(\mathcal{G})$  the set of minor minimal graph that do not belong in  $\mathcal{G}$ . Notice that any two graphs in  $\mathbf{obs}(\mathcal{G})$  are incomparable with respect to the minor relation. In their Graph Minors series, Robertson and Seymour [9] proved that each such set is finite, and therefore the size of its graphs is bounded by some constant  $c'_{\mathcal{G}}$ . Therefore, to check whether an input graph  $G$  is a member of some minor-closed  $\mathcal{G}$ , it suffices to check whether some graph on  $\mathbf{obs}(\mathcal{G})$  is a minor of  $G$ . If this is the case, return NO as answer, otherwise return YES. In the same series of papers, Robertson and Seymour proved that to check whether a fixed size graph  $H$  is a minor

of an  $n$ -vertex graph  $G$  can be done in  $O(n^3)$  steps. Moreover, when the input graph  $G$  has branchwidth at most  $k$ , this check can be done in time  $f(k) \cdot n$ , that is linear in  $n$  (see e.g. [8, 1, 4]). We use this fact to prove our main algorithmic result.

**Theorem 2.** *For every non-trivial graph class  $\mathcal{G}$  that is minor closed, there exists an algorithm deciding  $\sqrt{\mathcal{G}}$  in linear time.*

**Proof.** Let  $c_{\mathcal{G}}$  be the constant of Theorem 1 and let  $G$  be an input graph to the problem of the recognition of  $\sqrt{\mathcal{G}}$ . If  $\Delta(G) > c_{\mathcal{G}}$ , then, from Observation 1,  $\mathbf{cw}(G) > c_{\mathcal{G}}$  and the answer to the problem is NO. Assuming that  $\Delta(G) \leq c_{\mathcal{G}}$ , the computation of  $G^2$  can be done in linear time. From Lemma 6, if  $G \in \sqrt{\mathcal{G}}$ , then the branchwidth of  $G^2$  is at most  $c'_{\mathcal{G}} = 4 \cdot c_{\mathcal{G}}^2 - 2 \cdot c_{\mathcal{G}}$ . We now check whether  $\mathbf{bw}(G^2) \leq c'_{\mathcal{G}}$ , e.g. using the algorithm of [2]. If the answer is negative, then we safely return NO. Otherwise, it remains to check whether  $G \in \mathcal{G}$  for a graph of bounded branchwidth and this can be done in linear time as commented above.  $\square$

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