



HAL
open science

Exponential mixing properties for time inhomogeneous diffusion processes with killing

Pierre del Moral, Denis Villemonais

► **To cite this version:**

Pierre del Moral, Denis Villemonais. Exponential mixing properties for time inhomogeneous diffusion processes with killing. 2014. hal-01083297v1

HAL Id: hal-01083297

<https://hal.science/hal-01083297v1>

Preprint submitted on 17 Nov 2014 (v1), last revised 19 Mar 2016 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Exponential mixing properties for time inhomogeneous diffusion processes with killing

Pierre Del Moral*, Denis Villemonais †

December 8, 2014

Abstract

We consider an elliptic and time-inhomogeneous diffusion process with time-periodic coefficients evolving in a bounded domain of \mathbb{R}^d with a smooth boundary. The process is killed when it hits the boundary of the domain (hard killing) or after an exponential time (soft killing) associated with some bounded rate function. The branching particle interpretation of the non absorbed diffusion again behaves as a set of interacting particles evolving in an absorbing medium. Between absorption times, the particles evolve independently one from each other according to the diffusion semigroup; when a particle is absorbed, another selected particle splits into two offsprings. This article is concerned with the stability properties of these non absorbed processes. Under some classical ellipticity properties on the diffusion process and some mild regularity properties of the hard obstacle boundaries, we prove a uniform exponential strong mixing property of the process conditioned to not be killed. We also provide uniform estimates w.r.t. the time horizon for the interacting particle interpretation of these non-absorbed processes, yielding what seems to be the first results of this type for this class of non-homogenous diffusion processes evolving in soft and hard obstacles.

Keywords: process with absorption; uniform mixing property; time-inhomogeneous diffusion process.

2010 Mathematics Subject Classification. Primary: 60J25; 37A25; 60B10; 60F99. Secondary: 60J80; 60G10; 92D25.

1 Introduction

Let D be a bounded open subset of \mathbb{R}^d ($d \geq 1$) whose boundary ∂D is of class C^2 and consider the stochastic differential equation

$$dZ_t = \sigma(t, Z_t)dB_t + b(t, Z_t)dt, \quad Z_0 \in D, \quad (1)$$

*School of Maths and Stats, UNSW Sydney, Australia

†TOSCA project-team, INRIA Nancy – Grand Est, IECL – UMR 7502, Université de Lorraine B.P. 70239, 54506 Vandoeuvre-lès-Nancy Cedex, France

where B is a standard d dimensional Brownian motion. We assume that the functions

$$\sigma : \begin{array}{l} [0, +\infty[\times \mathbb{R}^d \rightarrow \mathbb{R}^d \times \mathbb{R}^d \\ (t, x) \mapsto \sigma(t, x) \end{array} \quad \text{and } b : \begin{array}{l} [0, +\infty[\times \mathbb{R}^d \rightarrow \mathbb{R}^d \\ (t, x) \mapsto b(t, x) \end{array}$$

are bounded continuous on $[0, +\infty[\times \mathbb{R}^d$. Moreover, we assume that they are time-periodic and Lipschitz in $x \in D$ uniformly in $t \in [0, +\infty[$. This means that there exists two constants $\Pi \geq 0$ and $C_0 > 0$ such that, for all $x, y \in D$ and $t \geq 0$,

$$\begin{aligned} \sigma(t + \Pi, x) &= \sigma(t, x) \text{ and } b(t + \Pi, x) = b(t, x), \\ \|\sigma(t, x) - \sigma(t, y)\| + |b(t, x) - b(t, y)| &\leq C_0|x - y|. \end{aligned}$$

In particular, there exists a solution to the stochastic differential equation 1 (see [13, Theorem 3.10, Chapter 5]). Moreover, the solution is path-wise unique up to time $\tau_D = \inf\{t \geq 0, Z_t \notin D\}$ (see [13, Theorem 3.7, Chapter 5]).

For all $s > 0$ and any probability distribution μ on D , we denote by $(Z_{s,t}^\mu)_{t \geq s}$ the unique solution to this stochastic differential equation starting at time $s > 0$ with distribution μ , killed when it hits the boundary and killed with a rate $\kappa(t, Z_{s,t}^x) \geq 0$, where

$$\kappa : [0, +\infty[\times D \rightarrow \mathbb{R}_+$$

is a uniformly bounded non-negative measurable function. By “*the process is killed*”, we mean that the process is sent to a cemetery point $\partial \notin D$, so that the killed process is càdlàg almost surely. If there exists $x \in D$ such that $\mu = \delta_x$, we set $(Z_{s,t}^x)_{t \geq s} = (Z_{s,t}^{\delta_x})_{t \geq s}$. When the process is killed by hitting the boundary, we say that it undergoes a *hard* killing; when the process is killed strictly before reaching the boundary (because of the killing rate κ), we say that it undergoes a *soft* killing. We denote the killing time by $\tau_\partial = \inf\{t \geq s, Z_{s,t}^x = \partial\}$.

Let $(Q_{s,t})_{0 \leq s \leq t}$ be the semi-group of the diffusion process with killing. It is defined, for all $0 \leq s \leq t$, $x \in D$ and for any bounded measurable function $f : \mathbb{R}^d \cup \{\partial\} \mapsto \mathbb{R}$ which vanishes outside D , by

$$Q_{s,t}f(x) = \mathbb{E}(f(Z_{s,t}^x)\mathbf{1}_{t < \tau_\partial}) = \mathbb{E}(f(Z_{s,t}^x)).$$

We emphasize that, for any probability measure μ on D , the law of $Z_{s,t}^\mu$ is given by the probability measure $\mu Q_{s,t}$, defined by

$$\mu Q_{s,t}(f) = \int_D Q_{s,t}f(x) d\mu(x).$$

In this paper, we focus on the long time behaviour of the distribution of $Z_{s,t}^\mu$ conditioned to not be killed when it is observed, that is to the event $\{t < \tau_\partial\}$. This distribution is given by

$$\mathbb{P}(Z_{s,t}^\mu \in \cdot \mid t < \tau_\partial) = \frac{\mu Q_{s,t}(\cdot)}{\mu Q_{s,t}(\mathbf{1})}. \quad (2)$$

When there is no hard killing (*i.e.* $D = \mathbb{R}^d$) and the solutions to the stochastic differential equation (1) without killing satisfy some proper mixing properties, it is known (see for instance Del Moral and Miclo [9, 10]) that the conditional distribution of the process with soft killing ($\|\kappa\|_\infty < \infty$) entails the following exponential mixing property : there exist two constants $C > 0$ and $\gamma > 0$ such that, for any probability measure μ_1 and μ_2 on D ,

$$\left\| \frac{\mu_1 Q_{s,t}}{\mu_1 Q_{s,t}(\mathbf{1})} - \frac{\mu_2 Q_{s,t}}{\mu_2 Q_{s,t}(\mathbf{1})} \right\|_{TV} \leq C e^{-\gamma t}, \quad (3)$$

where $\|\cdot\|_{TV}$ denotes the total variation norm between measures on \mathbb{R}^d . When there is only hard killing (*i.e.* $\kappa = 0$) and when the coefficients of the stochastic differential equation (1) are time-independent and of class C^1 , it is known that the conditioned distribution (2), which only depends on the difference $t - s$ and on μ , converges when t goes to $+\infty$ to a limiting distribution on D , called the Yaglom limit (see for instance Pinsky [20] and Gong, Qian and Zhao [16]). The stability analysis of the discrete time version of these particle absorption processes with hard and soft obstacles is also developed by Del Moral and Doucet in [5] and by Del Moral and Guionnet in [6]. In a recent result, Champagnat and Villemonais [3] also provide a general necessary and sufficient criterion for the existence of a Yaglom limit and the mixing property 3 for time homogeneous processes.

Our main result, stated in Section 2, is a sufficient criterion ensuring that the mixing property (3) holds for time inhomogeneous diffusion processes with both soft and hard killings. The main tools of the proof, developed in Section 4, are a tightness result for conditional distributions (see Villemonais [24]) and a coupling between diffusion processes (see Priola and Wang in [21]).

In Section 3, we prove an interesting consequence of our main result. Namely, we show that our criterion also implies that the approximation method developed in [25] converges uniformly in time to the conditional distribution of the stochastic differential equation (1).

2 Main result

Let $\phi_D : D \mapsto \mathbb{R}_+$ denotes the Euclidean distance to the boundary ∂D :

$$\phi_D(x) = d(x, \partial D) = \inf_{z \in \partial D} |x - z|,$$

$\|\cdot\|$ being the Euclidean norm. According to [11, Chapter 5, Section 4], we fix $a > 0$ such that ϕ_D is of class C_b^2 on the boundary's neighbourhood $D^a \subset D$ defined by

$$D^a = \{x \in D \text{ such that } \phi_D(x) < a\}.$$

Assumption (H). We assume that

1. there exists a constant $c_0 > 0$ such that

$$c_0|y| \leq |\sigma(t,x)y|, \forall (t,x,y) \in [0, +\infty[\times D \times \mathbb{R}^d.$$

2. there exist two measurable functions $f : [0, +\infty[\times D^a \rightarrow \mathbb{R}_+$ and $g : [0, +\infty[\times D^a \rightarrow \mathbb{R}$ such that $\forall (t,x) \in [0, +\infty[\times D^a$,

$$\sum_{k,l} \frac{\partial \phi_D}{\partial x_k}(z) \frac{\partial \phi_D}{\partial x_l}(z) [\sigma \sigma^*]_{kl}(t,x) = f(t,z) + g_i(t,x),$$

and such that

- (a) f is of class C^1 in time and of class C^2 in space, and the successive derivatives of f are uniformly bounded,
- (b) there exists a positive constant $k_g > 0$ such that, for all $(t,x) \in [0, +\infty[\times D^a$,

$$|g(t,x)| \leq k_g \phi_D(x),$$

The first point of Assumption (H) is a classical ellipticity assumption. The second point means that $\sum \partial_k \phi_D \partial_l \phi_D [\sigma \sigma^*]_{kl}$ can be approximated by a smooth function near the boundary, the error term being bounded by $k_g \phi_D$.

Theorem 1. *Assume that assumption (H) is satisfied. Then there exist two constants $C > 0$ and $\gamma > 0$ such that*

$$\sup_{\mu_1, \mu_2 \in \mathcal{M}_1(D)} \left\| \frac{\mu_1 Q_{0,T}}{\mu_1 Q_{0,T} \mathbf{1}} - \frac{\mu_2 Q_{0,T}}{\mu_2 Q_{0,T} \mathbf{1}} \right\|_{TV} \leq C e^{-\gamma T}.$$

Before turning to the proof of Theorem 1 in Section 4, we present in the next section an interesting consequence of Theorem 1 for an approximation method based on a Fleming-Viot type interacting particle system. Note that the approximation method described in the next section is itself used in the proof of Theorem 1, through the tightness results obtained in [24].

3 Uniform convergence of a Fleming-Viot type approximation method to the conditional distribution

We present in this section an interesting consequence of the mixing property for the approximation method developed in [25]: assuming that Assumption (H) holds, we prove that the approximation converges uniformly in time. The particle approximation method has been introduced by Burdzy, Holyst, Ingberman and March [1] for standard Brownian motions, and studied later by Burdzy, Holyst and March [2] and by Grigorescu and Kang [18] for Brownian motions, by Del Moral and Miclo [8, 10] and by Rousset [22] for jump-diffusion processes with smooth killings, Ferrari and Marić [14] for Markov processes

in countable state spaces, in [23] for diffusion processes and in [25] for general Markov processes. The discrete time version of these interacting particle models on general measurable spaces, including approximations of non absorbed trajectories in terms of genealogical trees is developed in [5, 6, 7]. For a more detailed discussion, including applications of these discrete generation particle techniques in advanced signal processing, statistical machine learning, and quantum physics, we also refer the reader to the recent monograph [4], and the references therein.

The approximation method is based on a sequence of Fleming-Viot type interacting particle systems whose associated sequence of empirical distributions converges to the conditioned distribution (2) when the number of particles tends to infinity. More precisely, fix $N \geq 2$ and let us define the Fleming-Viot type interacting particle system with N particles. The system of N particles $(X_{s,t}^{1,N}, \dots, X_{s,t}^{N,N})_{t \geq s}$ starts from an initial state $(X_{s,s}^{1,N}, \dots, X_{s,s}^{N,N}) \in D^N$, then:

- The particles evolve as N independent copies of $\mathcal{Z}_{s,\cdot}^{X_{s,s}^{i,N}}$ until one of them, say $X^{i_1,N}$, is killed. The first killing time is denoted by τ_1^N . We emphasize that under our hypotheses, the particle killed at time τ_1^N is unique [25].
- At time τ_1^N , the particle $X^{i_1,N}$ jumps on the position of an other particle, chosen uniformly among the $N - 1$ remaining ones. After this operation, the position $X_{s,\tau_1^N}^{i,N}$ is in D , for all $i \in \{1, \dots, N\}$.
- Then the particles evolve as N independent copies of $\mathcal{Z}_{\tau_1^N, \cdot}^{X_{s,\tau_1^N}^{i,N}}$ until one of them, say $X^{i_2,N}$, is killed. This second killing time is denoted by τ_2^N . Once again, the killed particle is uniquely determined.
- At time τ_2^N , the particle $X^{i_2,N}$ jumps on the position of an other particle, chosen uniformly among the $N - 1$ remaining ones.
- Then the particles evolve as independent copies of $\mathcal{Z}_{\tau_2^N, \cdot}^{X_{s,\tau_2^N}^{i,N}}$ and so on.

We denote by $0 < \tau_1^N < \tau_2^N < \dots < \tau_n^N < \dots$ the sequence of killing/jump times of the process. By [25], Assumption (H) implies that

$$\lim_{n \rightarrow \infty} \tau_n^N = +\infty, \text{ almost surely.}$$

In particular, the above algorithm defines a Markov process $(X_{s,t}^{1,N}, \dots, X_{s,t}^{N,N})_{t \geq s}$. For all $N \geq 2$ and all $0 \leq s \leq t$, we denote by $\mu_{s,t}^N$ the empirical distribution of $(X_{s,t}^{1,N}, \dots, X_{s,t}^{N,N})$, which means that

$$\mu_{s,t}^N(\cdot) = \frac{1}{N} \sum_{i=1}^N \delta_{X_{s,t}^{i,N}}(\cdot) \in \mathcal{M}_1(D),$$

where $\mathcal{M}_1(D)$ denotes the set of probability measures on D . This branching type particle model with fixed population size is closely related to the class of Moran and Fleming-Viot processes arising in measure valued super-processes [8, 12, 15]. The main difference with these super-processes comes from the fact that the occupation measures of the system converge to a deterministic limit measure valued process, as the size of the population tends to ∞ . These particle absorption models can also be interpreted as an extended version of the Nanbu type mean field particle model developed by C. Graham and S. Méléard [17] in the context of spatially homogeneous Boltzmann equations. The next results provide an uniform estimate w.r.t. the time horizon.

Theorem 2. *Assume that Hypothesis (H) holds and that the family of empirical distributions $(\mu_{s,s}^N)_{s \geq 0, N \geq 2}$ of the initial distributions of the interacting particle system described above is uniformly tight. Then*

$$\lim_{N \rightarrow \infty} \sup_{s \geq 0} \sup_{t \in [s, +\infty[} \sup_{f \in \mathcal{B}_1(D)} \mathbb{E} \left| \mu_{s,t}^N(f) - \frac{\mu_0^N Q_{s,t}(f)}{\mu_0^N Q_{s,t}(\mathbf{1})} \right| = 0.$$

Proof. Fix $\epsilon > 0$. Our aim is to prove that there exists $N_\epsilon \geq 2$ such that, for all $N \geq N_\epsilon$ and all measurable function $f : D \rightarrow \mathbb{R}$ satisfying $\|f\|_\infty \leq 1$, we have

$$\sup_{s,t \in [0, +\infty[} \mathbb{E} \left| \mu_{s,s+t}^N(f) - \frac{\mu_{s,s}^N Q_{s,s+t}(f)}{\mu_{s,s}^N Q_{s,s+t}(\mathbf{1})} \right| \leq \epsilon. \quad (4)$$

Let γ be the constant of Theorem 1 and fix $t_0 \geq 1$ such that $2e^{-\gamma(t_0-1)} \leq \epsilon/6$. In a first step, we prove that (4) holds for $t \leq t_0$. In a second step we prove that it holds for $t \geq t_0$.

Step 1: Inequality (4) holds for $t \leq t_0$.

Since the sequence of initial distributions is assumed to be uniformly tight (w.r.t. the time parameter and the size of the system), there exists $\alpha_1 = \alpha_1(\epsilon) > 0$ such that, $\forall N \geq 2$,

$$\mathbb{E}(\mu_{s,s}^N(D^{\alpha_1})) \leq \frac{\epsilon}{8}.$$

Now, since the coefficients of the SDE (1) and the killing rate κ are uniformly bounded, the probability for the process \mathcal{Z}_s^x , starting from $x \in (D^{\alpha_1})^c$ to be killed after time $s+t$ is uniformly bounded below by a positive constant. In other words, the constant $\beta = \beta(\epsilon)$ defined below is positive :

$$\beta \stackrel{def}{=} \inf \{Q_{s,s+t_0} \mathbf{1}_D(x), s \in [0, +\infty[, x \in (D^{\alpha_1})^c\} > 0.$$

For all $t \in [0, t_0]$, we have

$$\mathbb{E} \left| \mu_{s,s+t}^N(f) - \frac{\mu_{s,s}^N Q_{s,s+t}(f)}{\mu_{s,s}^N Q_{s,s+t}(\mathbf{1}_D)} \right| = \mathbb{E} \left[\mathbb{E} \left(\left| \mu_{s,s+t}^N(f) - \frac{\mu_{s,s}^N Q_{s,s+t}(f)}{\mu_{s,s}^N Q_{s,s+t}(\mathbf{1}_D)} \right| \mid \mu_{s,s}^N \right) \right],$$

where, by [25, Theorem 2.2],

$$\mathbb{E} \left(\left| \mu_{s,s+t}^N(f) - \frac{\mu_{s,s}^N Q_{s,s+t}(f)}{\mu_{s,s}^N Q_{s,s+t}(\mathbf{1}_D)} \right| \mid \mu_{s,s}^N \right) \leq \frac{2(1+\sqrt{2})}{\sqrt{N} \mu_{s,s}^N Q_{s,s+t}(\mathbf{1}_D)}.$$

Since $\|f\|_\infty \leq 1$, we have $\left| \mu_{s,s+t}^N(f) - \frac{\mu_{s,s}^N Q_{s,s+t}(f)}{\mu_{s,s}^N Q_{s,s+t}(\mathbf{1}_D)} \right| \leq 2$ almost surely and we deduce from the previous inequality that

$$\begin{aligned} \mathbb{E} \left| \mu_{s,s+t}^N(f) - \frac{\mu_{s,s}^N Q_{s,s+t}(f)}{\mu_{s,s}^N Q_{s,s+t}(\mathbf{1}_D)} \right| &\leq \frac{\epsilon}{2} + 2 \mathbb{P} \left(\frac{2(1+\sqrt{2})}{\sqrt{N} \mu_{s,s}^N Q_{s,s+t}(\mathbf{1}_D)} \geq \frac{\epsilon}{2} \right) \\ &\leq \frac{\epsilon}{2} + 2 \mathbb{P} \left(\mu_{s,s}^N Q_{s,s+t}(\mathbf{1}_D) \leq \frac{4(1+\sqrt{2})}{\epsilon \sqrt{N}} \right) \end{aligned}$$

But $\mu_{s,s}^N Q_{s,s+t}(\mathbf{1}_D) \geq \mu_{s,s}^N ((D^{\alpha_1})^c) \beta_\epsilon$ almost surely, thus

$$\begin{aligned} \mathbb{E} \left| \mu_{s,s+t}^N(f) - \frac{\mu_{s,s}^N Q_{s,s+t}(f)}{\mu_{s,s}^N Q_{s,s+t}(\mathbf{1}_D)} \right| &\leq \frac{\epsilon}{2} + 2 \mathbb{P} \left(\mu_{s,s}^N ((D^{\alpha_1})^c) \leq \frac{4(1+\sqrt{2})}{\epsilon \sqrt{N} \beta_\epsilon} \right) \\ &\leq \frac{\epsilon}{2} + 2 \mathbb{P} \left(\mu_{s,s}^N (D^{\alpha_1}) \geq 1 - \frac{4(1+\sqrt{2})}{\epsilon \sqrt{N} \beta_\epsilon} \right) \\ &\leq \frac{\epsilon}{2} + \frac{2}{1 - \frac{4(1+\sqrt{2})}{\epsilon \sqrt{N} \beta_\epsilon}} \mathbb{E} (\mu_{s,s}^N (D^{\alpha_1})) \\ &\leq \frac{\epsilon}{2} + 2 \frac{1}{1 - \frac{4(1+\sqrt{2})}{\epsilon \sqrt{N} \beta_\epsilon}} \frac{\epsilon}{8}, \end{aligned}$$

where we used Markov's inequality. Finally, there exists $N_1 = N_1(\epsilon) \geq 2$ such that, $\forall N \geq N_1$,

$$\sup_{s \geq 0} \sup_{t \in [0, t_0]} \mathbb{E} \left| \mu_{s,s+t}^N(f) - \frac{\mu_{s,s}^N Q_{s,s+t}(f)}{\mu_{s,s}^N Q_{s,s+t}(\mathbf{1}_D)} \right| \leq \epsilon.$$

Step 2: Inequality (4) holds for $t \geq t_0$.

Fix now $t \geq t_0$. We have

$$\begin{aligned} \mathbb{E} \left| \mu_{s,s+t}^N(f) - \frac{\mu_{s,s}^N Q_{s,s+t}(f)}{\mu_{s,s}^N Q_{s,s+t}(\mathbf{1}_D)} \right| &\leq \mathbb{E} \left| \mu_{s,s+t}^N(f) - \frac{\mu_{s,s+1+t-t_0}^N Q_{s+1+t-t_0,s+t}(f)}{\mu_{s,s+1+t-t_0}^N Q_{s+1+t-t_0,s+t}(\mathbf{1}_D)} \right| \\ &\quad + \mathbb{E} \left| \frac{\mu_{s,s+1+t-t_0}^N Q_{s+1+t-t_0,s+t}(f)}{\mu_{s,s+1+t-t_0}^N Q_{s+1+t-t_0,s+t}(\mathbf{1}_D)} - \frac{\mu_{s,s}^N Q_{s,s+t}(f)}{\mu_{s,s}^N Q_{s,s+t}(\mathbf{1}_D)} \right|. \end{aligned}$$

By [24, Theorem 3.1], there exists $\alpha_2 = \alpha_2(\epsilon) > 0$ and $N_2 = N_2(\epsilon) \geq 2$ such that for all $s \in [0, +\infty[$, $t \geq t_0$ and $N \geq N_2$,

$$\mathbb{E} (\mu_{s,s+1+t-t_0}^N (D^{\alpha_2})) \leq \epsilon.$$

Since α_1 and α_2 can be chosen arbitrarily small, one can assume without loss of generality that $\alpha_1 = \alpha_2$. Now, using step 1 for the particle system with initial distribution $\mu_{s,s+1+t-t_0}^N$ and the Markov property for the particle system, we deduce that, for all $N \geq 2$,

$$\mathbb{E} \left| \mu_{s,s+t}^N(f) - \frac{\mu_{s,s+1+t-t_0}^N Q_{s+1+t-t_0,s+t}(f)}{\mu_{s,s+1+t-t_0}^N Q_{s+1+t-t_0,s+t}(\mathbf{1}_D)} \right| \leq \frac{\epsilon}{2} + 2 \frac{1}{1 - \frac{4(1+\sqrt{2})}{\epsilon\sqrt{N}\beta_\epsilon}} \frac{\epsilon}{8}.$$

By Theorem 1, we also have

$$\mathbb{E} \left| \frac{\mu_{s,s+1+t-t_0}^N Q_{s+1+t-t_0,s+t}(f)}{\mu_{s+1+t-t_0}^N Q_{s+1+t-t_0,s+t}(\mathbf{1}_D)} - \frac{\mu_{s,s} Q_{s,s+t}(f)}{\mu_{s,s}^N Q_{s,s+t}(\mathbf{1}_D)} \right| \leq 2e^{-\gamma(t_0-1)} = \epsilon/6.$$

We deduce from the two previous inequality that there exists $N_3 = N_3(\epsilon) \geq N_2$ such that, $\forall N \geq N_3$,

$$\mathbb{E} \left| \mu_{s,s+t}^N(f) - \frac{\mu_{s,s}^N Q_{s,s+t}(f)}{\mu_{s,s}^N Q_{s,s+t}(\mathbf{1}_D)} \right| \leq \epsilon.$$

Conclusion.

Setting $N_\epsilon = N_1 \vee N_3$, we have proved (4), which concludes the proof of Theorem 2. \square

4 Proof of the main result

In Subsection 4.1, we present a coupling construction for multi-dimensional time-inhomogeneous diffusion processes. In Subsection 4.2, we derive from this coupling two intermediate results which are key steps for the proof of Theorem 1. The first result (Lemma 4) concerns the existence of a path $(x_{s,t})_{0 \leq s \leq t}$ and a constant $r_0 > 0$ such that

$$\inf_{x \in B(x_{s,t}, r_0)} Q_{s,t} \mathbf{1}_D(x) \geq \frac{1}{2} \|Q_{s,t} \mathbf{1}_D\|_\infty, \forall 0 \leq s + \Pi + 1 \leq t,$$

where we recall that Π is the time-period of the coefficients σ and b . The second result (Lemma 5) states the existence of a constant $\beta > 0$ and a family of probability measures $(\nu_s^{x_1, x_2})_{s \geq 0, (x_1, x_2) \in D \times D}$ such that, for all $s \geq 0$, all $(x_1, x_2) \in D \times D$ and any non-negative measurable function f ,

$$\frac{Q_{s,s+1} f(x_i)}{Q_{s,s+1} \mathbf{1}_D(x_i)} \geq \beta \nu_s^{x_1, x_2}(f), \quad i = 1, 2.$$

We conclude the proof of Theorem 1 in Subsection 4.3, showing that Lemma 4 and Lemma 5 imply the strong mixing property.

4.1 Coupling

In the following proposition, we state the existence of a coupling for multi-dimensional time-inhomogeneous diffusion processes. The result also provides a bound for the coupling probability that will be very useful in the next subsections.

Proposition 3. *For all $s \geq 0$ and all $(y^1, y^2) \in D \times D$, there exists a diffusion process $(Y_{s,t}^1, Y_{s,t}^2)_{t \geq s}$ such that*

1. $(Y_{s,t}^1)_{t \geq s}$ has the same law as $(\mathcal{Z}_{s,t}^{y^1})_{t \geq s}$,
2. $(Y_{s,t}^2)_{t \geq s}$ has the same law as $(\mathcal{Z}_{s,t}^{y^2})_{t \geq s}$;
3. $Y_{s,t}^1$ and $Y_{s,t}^2$ are equal almost surely after the coupling time

$$T_c^s = \inf\{t \geq 0, Y_{s,t}^1 = Y_{s,t}^2\},$$

where $\inf \emptyset = +\infty$ by convention.

4. There exists a constant $c > 0$ which doesn't depend on s, t such that

$$\mathbb{P}(t < \tau_{\partial}^1 \vee \tau_{\partial}^2 \text{ and } T_c^s > t \wedge \tau_{\partial}^1 \wedge \tau_{\partial}^2) \leq \frac{c|y_1 - y_2|}{\sqrt{1 \wedge (t - s)}},$$

where τ_{∂}^1 and τ_{∂}^2 denote the killing times of Y^1 and Y^2 respectively.

The proof of Proposition 3 is given in [21] for time-homogeneous diffusion processes, though a careful check of the arguments shows that the authors do not use the time-homogeneity of the coefficients to derive the existence and other properties of the coupling. We do not write the proof in details, but we recall the idea behind the coupling construction. The proof of the 4th statement of Proposition 3 requires fine estimates and calculus which are plainly detailed in [21].

By Assumption (H), there exists $\lambda_0 > 0$ such that $\sigma\sigma^* - \lambda_0 I$ is definite positive for all t, x . Let $\sigma_0 := \sqrt{\sigma\sigma^* - \lambda_0 I}$ be the unique symmetric definite positive matrix function such that $\sigma_0^2 = \sigma\sigma^* - \lambda_0 I$. Without loss of generality, one can choose λ_0 small enough so that σ_0 is uniformly positive definite. We define

$$u(x, y) = \frac{k(|x - y|)(x - y)}{(k(|x - y|) + 1)|x - y|} \text{ and } C_t(x, y) = \lambda_0 (I - 2u(x, y)u(x, y)^*) + \sigma_0(t, x)\sigma_0(t, y)^*,$$

where $k(r) = (k_0 r^2 / 2 \vee r)^{\frac{1}{4}}$. Before the coupling time, the coupling process is generated by

$$L_t(x, y) = \frac{1}{2} \sum_{i,j=1}^d \left\{ [\sigma(t, x)\sigma(t, x)^*]_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} + [\sigma(t, y)\sigma(t, y)^*]_{i,j} \frac{\partial^2}{\partial y_i \partial y_j} \right. \\ \left. + 2[C_t(x, y)]_{i,j} \frac{\partial^2}{\partial x_i \partial y_j} \right\} + \sum_{i=1}^d \left\{ b_i(t, x) \frac{\partial}{\partial x_i} + b_i(t, y) \frac{\partial}{\partial y_i} \right\}.$$

The coefficients of L_t are continuous and bounded over \mathbb{R}^d , then, for all $s \geq 0$ and $x = (y^1, y^2) \in D \times D$, there exists a not necessarily unique process $(X_{s,t}^x)_{t \geq 0}$ with values in \mathbb{R}^{2d} to the martingale problem associated with $(L_t)_{t \geq s}$ (see [19, Theorem 2.2, Chapter IV]). We define Y^1 and Y^2 as the two marginal components of X^x , so that $X_{s,t}^x = (Y_{s,t}^1, Y_{s,t}^2)$ almost surely. We consider the coupling time T_c^s of $X_{s,\cdot}^x$, which is defined by

$$T_c^s = \inf\{t \geq s, \text{ such that } Y_{s,t}^1 = Y_{s,t}^2\}.$$

We define Y^1 and Y^2 as follows:

$$Y_t^i = \begin{cases} Y_t^i, & t \leq T_c^s, \\ Y_t^1, & t > T_c^s. \end{cases}$$

Moreover, each marginal process Y^i , $i = 1, 2$, is killed either when it hits the boundary ∂D or with a rate κ .

4.2 Intermediate results

In this section, we prove Lemmas 4 and 5, which are essential part of the proof of Theorem 1. We recall that Π denotes the time-period of the coefficients of the SDE 1. Note also that Lemma 4 is the only part of the paper which makes use of the periodicity assumption.

Lemma 4. *For any $0 \leq s < t$, let us denote by $x_{s,t} \in D$ the point at which $Q_{s,t} \mathbf{1}_D$ is maximal. There exists a positive constant $r_0 > 0$ (independent of s and t) such that*

$$\inf_{x \in B(x_{s,t}, r_0)} Q_{s,t} \mathbf{1}_D(x) \geq \frac{1}{2} \|Q_{s,t} \mathbf{1}_D\|_\infty, \quad \forall 0 \leq s \leq s + \Pi + 1 \leq t,$$

where $B(x_{s,t}, r_0)$ denotes the ball of radius r_0 centred on $x_{s,t}$.

Proof of Lemma 4. Fix $s \geq 0$ and let $(Y_{s,\cdot}^1, Y_{s,\cdot}^2)$ be the coupling of Proposition 3, starting from to points y_1 and y_2 in D . From the properties (1) and (2) of the proposition, we deduce that, for any measurable bounded function f which vanishes outside D , we have

$$|Q_{s,s+\Pi} f(y^1) - Q_{s,s+\Pi} f(y^2)| \leq \mathbb{E} |f(Y_{s,s+\Pi}^1) - f(Y_{s,s+\Pi}^2)|$$

where $f(Y_{s,s+\Pi}^1) = f(Y_{s,s+\Pi}^2) = 0$ if $s + \Pi \geq \tau_\partial^1 \vee \tau_\partial^2$ and, by property (3) of Proposition 3, $Y_{s,s+\Pi}^1 = Y_{s,s+\Pi}^2$ if $T_c^s \leq s + \Pi \wedge \tau_\partial^1 \wedge \tau_\partial^2$. Thus we have

$$\begin{aligned} |Q_{s,s+\Pi} f(y^1) - Q_{s,s+\Pi} f(y^2)| &\leq \|f\|_\infty \mathbb{P}(s + \Pi < \tau_\partial^1 \vee \tau_\partial^2 \text{ and } T_c^s > (s + \Pi) \wedge \tau_\partial^1 \wedge \tau_\partial^2) \\ &\leq \|f\|_\infty \frac{c|y_1 - y_2|}{\sqrt{1 \wedge \Pi}}, \end{aligned} \quad (5)$$

by the fourth property of Proposition 3.

By Proposition [24, Theorem 4.1] with $\epsilon = 1/2$ and $t_0 = 1$, there exists $\alpha_0 > 0$ such that, for all $0 \leq s \leq s + \Pi + 1 \leq t$,

$$Q_{s+\Pi,t}\mathbf{1}_D(x) \leq 2Q_{s+\Pi,t}\mathbf{1}_{(D^{\alpha_0})^c}(x).$$

We emphasize that α_0 does not depend on s,t . Now, since the coefficients of the SDE (1) and the killing rate κ are assumed to be uniformly bounded on D , there exists a positive constant c_{α_0} , such that, for any $t \geq 0$,

$$\inf_{x \in (D^{\alpha_0})^c} Q_{t,t+\Pi}\mathbf{1}_D(x) \geq c_{\alpha_0} > 0.$$

In particular, we have

$$\mathbf{1}_{(D^{\alpha_0})^c} \leq \frac{Q_{t,t+\Pi}\mathbf{1}_D}{c_{\alpha_0}}.$$

We deduce that, for all $0 \leq s \leq s + \Pi + 1 \leq t$,

$$Q_{s+\Pi,t}\mathbf{1}_D \leq 2Q_{s+\Pi,t} \frac{Q_{t,t+\Pi}\mathbf{1}_D}{c_{\alpha_0}},$$

so that

$$\|Q_{s+\Pi,t}(\mathbf{1}_D)\|_{\infty} \leq \frac{2}{c_{\alpha_0}} \|Q_{s+\Pi,t+\Pi}\mathbf{1}_D\|_{\infty} = \frac{2}{c_{\alpha_0}} \|Q_{s,t}\mathbf{1}_D\|_{\infty},$$

by the time-periodicity assumption on the coefficients of the SDE (1). Applying inequality (5) to $f = Q_{s+\Pi,t}(\mathbf{1}_D)$ and using the semi-group property of $(Q_{s,t})_{s \leq t}$, we deduce that, for all $s \leq s + \Pi + 1 \leq t$,

$$|Q_{s,t}\mathbf{1}_D(y^1) - Q_{s,t}\mathbf{1}_D(y^2)| \leq \frac{2c|y^1 - y^2|}{c_{\alpha_0}\sqrt{1 \wedge \Pi}} \|Q_{s,t}\mathbf{1}_D\|_{\infty}.$$

For any $0 \leq s \leq s + \Pi + 1 \leq t$, let $x_{s,t}$ be such that $Q_{s,t}\mathbf{1}_D(x_{s,t}) = \|Q_{s,t}\mathbf{1}_D\|_{\infty}$. We have by the previous inequality,

$$Q_{s,t}f(y) \geq \|Q_{s,t}\mathbf{1}_D\|_{\infty} - \frac{2c|x_{s,t} - y|}{c_{\alpha_0}\sqrt{1 \wedge \Pi}} \|Q_{s,t}\mathbf{1}_D\|_{\infty}, \quad \forall y \in D.$$

Choosing $r_0 = \frac{c_{\alpha_0}}{4c}\sqrt{1 \wedge \Pi}$, one obtains Lemma 4. □

Lemma 5. *There exist a constant $\beta > 0$ and a family of probability measures denoted by $(\nu_s^{x_1, x_2})_{s \geq 0, (x_1, x_2) \in D \times D}$ such that, for all $s \geq 0$, for all $(x_1, x_2) \in D \times D$, $i \in \{1, 2\}$ and for any non-negative measurable function f ,*

$$\frac{Q_{s,s+1}f(x_i)}{Q_{s,s+1}\mathbf{1}_D(x_i)} \geq \beta \nu_s^{x_1, x_2}(f).$$

Moreover, for any $r_1 > 0$, we have for all $x \in D$

$$\inf_{s \geq 0, (x_1, x_2) \in D \times D} \nu_s^{x_1, x_2}(B(x, r_1) \cap D) > 0.$$

Proof of Lemma 5. Let us first prove that there exist a constant $\rho_0 > 0$ and a fixed point $x_0 \in D$ such that, for any $(y_1, y_2) \in B(x_0, \rho_0) \times B(x_0, \rho_0)$ and any $s \geq 0$, there exists a probability measure $\mu_s^{y_1, y_2}$ which fulfills

$$\mathbb{E} \left(f(\mathcal{Z}_{s+\frac{2}{3}, s+1}^{y_i}) \right) \geq \frac{1}{2} \mu_s^{y_1, y_2}(f), \quad (6)$$

for any $i \in \{1, 2\}$. Fix $x_0 \in D$ and $s \geq 0$. Let ρ_0 be a positive constant which will be fixed later in the proof. Let y_1, y_2 be two elements of $B(x_0, \rho_0)$ and let $(Y_{s+\frac{2}{3}, \cdot}^1, Y_{s+\frac{2}{3}, \cdot}^2)$ be the coupling of Proposition 3 starting from $(y_1, y_2) \in D \times D$ at time $s + \frac{2}{3}$. We define the event \mathcal{E} by

$$\mathcal{E} = \{s + 1 \geq \tau_\partial^1 \vee \tau_\partial^2 \text{ or } T_c^s \leq (s + 1) \wedge \tau_\partial^1 \wedge \tau_\partial^2\},$$

where T_c^s is the coupling time of Proposition 3, and τ_∂^1 and τ_∂^2 the killing times of Y^1 and Y^2 respectively. By definition of the killing time, $s + 1 \geq \tau_\partial^1 \vee \tau_\partial^2$ implies $Y_{s+\frac{2}{3}, s+1}^1 = Y_{s+\frac{2}{3}, s+1}^2 = \partial$. Moreover, by the coupling property (3) of Proposition 3, $T_c^s \leq (s + 1) \wedge \tau_\partial^1 \wedge \tau_\partial^2$ implies $Y_{s+\frac{2}{3}, s+1}^1 = Y_{s+\frac{2}{3}, s+1}^2$. Finally,

$$\mathcal{E} \subset \left\{ Y_{s+\frac{2}{3}, s+1}^1 = Y_{s+\frac{2}{3}, s+1}^2 \right\},$$

so that

$$\mathbb{E} \left(f(Y_{s+\frac{2}{3}, s+1}^2) | \mathcal{E} \right) = \mathbb{E} \left(f(Y_{s+\frac{2}{3}, s+1}^1) | \mathcal{E} \right).$$

We have then (the first equality being a consequence of Proposition 3 (1)), for any measurable function f which vanishes outside D ,

$$\begin{aligned} \mathbb{E} \left(f(\mathcal{Z}_{s+\frac{2}{3}, s+1}^{y_i}) \right) &= \mathbb{E} \left(f(Y_{s+\frac{2}{3}, s+1}^i) \right) \\ &\geq \mathbb{E} \left(f(Y_{s+\frac{2}{3}, s+1}^i) | \mathcal{E} \right) \mathbb{P}(\mathcal{E}) \\ &\geq \mathbb{E} \left(f(Y_{s+\frac{2}{3}, s+1}^1) | \mathcal{E} \right) \mathbb{P}(\mathcal{E}). \end{aligned}$$

But Proposition 3 (4) implies

$$\mathbb{P}(\mathcal{E}) \geq 1 - 3c|y_1 - y_2| \geq 1 - 6c\rho_0$$

so that

$$\mathbb{E} \left(f(\mathcal{Z}_{s+\frac{2}{3}, s+1}^{y_i}) \right) \geq \mathbb{E} \left(\mathbf{1}_D(Y_{s+\frac{2}{3}, s+1}^1) | \mathcal{E} \right) (1 - 6c\rho_0) \mu_s^{y_1, y_2}(f), \quad \forall i = 1, 2, \quad (7)$$

where the probability measure $\mu_s^{y_1, y_2}$ on D is defined by

$$\mu_s^{y_1, y_2}(f) = \frac{\mathbb{E} \left(f(Y_{s+\frac{2}{3}, s+1}^1) | \mathcal{E} \right)}{\mathbb{E} \left(\mathbf{1}_D(Y_{s+\frac{2}{3}, s+1}^1) | \mathcal{E} \right)}.$$

It only remains to find a lower bound for $\mathbb{E}\left(\mathbf{1}_D(Y_{s+\frac{2}{3},s+1}^1)|\mathcal{E}\right)$ to conclude that (6) holds for a well chosen ρ_0 . We have

$$\begin{aligned}\mathbb{E}\left(\mathbf{1}_D(Y_{s+\frac{2}{3},s+1}^1)|\mathcal{E}\right) &= \frac{1}{\mathbb{P}(\mathcal{E})}\mathbb{E}\left(\mathbf{1}_D(Y_{s+\frac{2}{3},s+1}^1)\right) - \frac{1-\mathbb{P}(\mathcal{E})}{\mathbb{P}(\mathcal{E})}\mathbb{E}\left(\mathbf{1}_D(Y_{s+\frac{2}{3},s+1}^1)|\mathcal{E}^c\right) \\ &\geq \frac{1}{\mathbb{P}(\mathcal{E})}\mathbb{E}\left(\mathbf{1}_D(\mathcal{Z}_{s+\frac{2}{3},s+1}^{y_1})\right) - \frac{1-\mathbb{P}(\mathcal{E})}{\mathbb{P}(\mathcal{E})} \\ &\geq \mathbb{E}\left(\mathbf{1}_D(\mathcal{Z}_{s+\frac{2}{3},s+1}^{y_1})\right) - \frac{6c\rho_0}{1-6c\rho_0}.\end{aligned}$$

Now, for an arbitrarily fixed point $x_0 \in D$, one can choose $\rho_0 > 0$ small enough so that $d(\partial D, B(x_0, \rho_0)) > 0$. Then, since the coefficients of the SDE (1) and the killing rate κ are uniformly bounded on D , we have

$$\epsilon_0 \stackrel{def}{=} \inf_{s \geq 0} \inf_{y_1 \in B(x_0, \rho_0)} \mathbb{E}\left(\mathbf{1}_D(\mathcal{Z}_{s+\frac{2}{3},s+1}^{y_1})\right) > 0.$$

Finally, we deduce from (7) that

$$\mathbb{E}\left(f(\mathcal{Z}_{s+\frac{2}{3},s+1}^{y_i})\right) \geq \left(\epsilon_0 - \frac{6c\rho_0}{1-6c\rho_0}\right)(1-6c\rho_0)\mu_s^{y_1, y_2}(f),$$

for any non-negative measurable function which vanishes outside D . In particular, choosing ρ_0 small enough, we deduce that, for all $i \in \{1, 2\}$,

$$\mathbb{E}\left(f(\mathcal{Z}_{s+\frac{2}{3},s+1}^{y_i})\right) \geq \frac{\epsilon_0}{2}\mu_s^{y_1, y_2}(f). \quad (8)$$

Let us now conclude the proof of Lemma 5. By [24, Theorem 4.1] with $\mu = \delta_x$, there exists a constant $\alpha_1 > 0$ such that, for all $s \geq 0$ and all $x \in D$,

$$\begin{aligned}Q_{s, s+\frac{1}{3}}\mathbf{1}_{(D^{\alpha_1})^c}(x) &\geq \frac{1}{2}Q_{s, s+\frac{1}{3}}\mathbf{1}_D(x) \\ &\geq \frac{1}{2}Q_{s, s+1}\mathbf{1}_D(x).\end{aligned} \quad (9)$$

Since the coefficients of the SDE (1) and the killing rate κ are uniformly bounded, we have

$$\epsilon_1 \stackrel{def}{=} \inf_{s \geq 0, x \in (D^{\alpha_1})^c} Q_{s+\frac{1}{3}, s+\frac{2}{3}}\mathbf{1}_{B(x_0, \rho_0)}(x) > 0.$$

In particular, we deduce from (9) that, for all $x \in D$,

$$\begin{aligned}Q_{s, s+\frac{2}{3}}\mathbf{1}_{B(x_0, \rho_0)}(x) &= Q_{s, s+\frac{1}{3}}\left(Q_{s+\frac{1}{3}, s+\frac{2}{3}}\mathbf{1}_{B(x_0, \rho_0)}\right)(x) \\ &\geq \epsilon_1 Q_{s, s+\frac{1}{3}}\mathbf{1}_{(D^{\alpha_1})^c}(x) \\ &\geq \frac{\epsilon_1}{2}Q_{s, s+1}\mathbf{1}_D(x)\end{aligned} \quad (10)$$

Finally, we have, for all $x_1, x_2 \in D \times D$,

$$\begin{aligned}
Q_{s,s+1}f(x_i) &\geq \int_{B(x_0,\rho_0)} Q_{s+\frac{2}{3},s+1}f(y_1) \left[\delta_{x_1} Q_{s,s+\frac{2}{3}} \right] (dy_1) \\
&\geq \frac{1}{Q_{s,s+\frac{2}{3}} \mathbf{1}_{B(x_0,\rho_0)}(x_2)} \times \\
&\quad \int_{B(x_0,\rho_0)} \int_{B(x_0,\rho_0)} Q_{s+\frac{2}{3},s+1}f(y_1) \left[\delta_{x_1} Q_{s,s+\frac{2}{3}} \otimes \delta_{x_2} Q_{s,s+\frac{2}{3}} \right] (dy_1, dy_2) \\
&\geq \frac{\epsilon_0}{2Q_{s,s+\frac{2}{3}} \mathbf{1}_{B(x_0,\rho_0)}(x_2)} \times \\
&\quad \int_{B(x_0,\rho_0)} \int_{B(x_0,\rho_0)} \mu_s^{y_1, y_2}(f) \left[\delta_{x_1} Q_{s,s+\frac{2}{3}} \otimes \delta_{x_2} Q_{s,s+\frac{2}{3}} \right] (dy_1, dy_2) \text{ by (8)} \\
&\geq \epsilon_0 \frac{Q_{s,s+\frac{2}{3}} \mathbf{1}_{B(x_0,\rho_0)}(x_1)}{2} \nu_s^{x_1, x_2}(f),
\end{aligned}$$

where $\nu_s^{x_1, x_2}$ is the probability measure on D defined by

$$\nu_s^{x_1, x_2}(f) = \frac{\int_{B(x_0,\rho_0)} \int_{B(x_0,\rho_0)} \mu_s^{y_1, y_2}(f) \left[\delta_{x_1} Q_{s,s+\frac{2}{3}} \otimes \delta_{x_2} Q_{s,s+\frac{2}{3}} \right] (dy_1, dy_2)}{Q_{s,s+\frac{2}{3}} \mathbf{1}_{B(x_0,\rho_0)}(x_1) Q_{s,s+\frac{2}{3}} \mathbf{1}_{B(x_0,\rho_0)}(x_2)}.$$

This and Inequality (10) allow us to conclude the proof of the first part of Lemma 5.

Fix $r_1 > 0$ and let us prove the second part of the lemma. We have, for all $(y_1, y_2) \in (B(x_0, \rho_0))^2$ and all $x \in D$,

$$\begin{aligned}
\mu_s^{y_1, y_2}(B(x, r_1)) &= \frac{\mathbb{E} \left(\mathbf{1}_{B(x, r_1)}(Y_{s+\frac{2}{3}, s+1}^1) | \mathcal{E} \right)}{\mathbb{E} \left(\mathbf{1}_D(Y_{s+\frac{2}{3}, s+1}^1) | \mathcal{E} \right)} \\
&\geq \mathbb{E} \left(\mathbf{1}_{B(x, r_1)}(Y_{s+\frac{2}{3}, s+1}^1) | \mathcal{E} \right) \mathbb{P}(\mathcal{E}) \\
&\geq \mathbb{E} \left(\mathbf{1}_{B(x, r_1)}(Y_{s+\frac{2}{3}, s+1}^1) \right) - (1 - \mathbb{P}(\mathcal{E})) \\
&\geq \delta_{y_1} Q_{s+\frac{2}{3}, s+1}(B(x, r_1)) - 6c\rho_0.
\end{aligned}$$

We emphasize that, because of the boundedness and the regularity of D , $B(x, r_1) \cap D$ contains a ball of minimal volume uniformly over $x \in D$. Then, since the coefficients of the SDE (1) and the killing rate κ are assumed to be uniformly bounded, we have

$$\epsilon_2 \stackrel{def}{=} \inf_{s \geq 0, x \in D, y_1 \in B(x, \rho_0)} Q_{s+\frac{2}{3}, s+1} \mathbf{1}_{B(x, r_1)}(y_1) > 0,$$

where we recall that ρ_0 is chosen small enough so that $d(\partial D, B(x_0, \rho_0)) > 0$. We deduce that

$$\mu_s^{y_1, y_2}(B(x, r_1)) \geq \epsilon_2/2.$$

Finally, by definition of $\nu_s^{x,y}$, we deduce that

$$\nu_s^{x,y}(B(x,r_1)) \geq \epsilon_2/2, \forall x \in D.$$

This concludes the proof of Lemma 5. \square

4.3 Conclusion of the proof of Theorem 1

In this section, we conclude the proof of Theorem 1. Let us define, for all $0 \leq s \leq t \leq T$ the linear operator $R_{s,t}^T$ by

$$R_{s,t}^T f(x) = \frac{Q_{s,t}(fQ_{t,T}\mathbf{1}_D)(x)}{Q_{s,T}\mathbf{1}_D(x)},$$

for all $x \in D$ and any bounded measurable function f . Let us remark that the value $R_{s,t}^T f(x)$ is the expectation of $f(\mathcal{Z}_{s,t}^x)$ conditioned to $T < \tau_\partial$. Indeed we have

$$\begin{aligned} \mathbb{E}(f(\mathcal{Z}_{s,t}^x) \mid T < \tau_\partial) &= \frac{\mathbb{E}\left(f(\mathcal{Z}_{s,t}^x)\mathbf{1}_{\mathcal{Z}_{s,T}^x \in D}\right)}{\mathbb{E}(T < \tau_\partial)} \\ &= \frac{\mathbb{E}\left(f(\mathcal{Z}_{s,t}^x)\mathbb{E}\left(\mathbf{1}_{\mathcal{Z}_{s,T}^x \in D} \mid \mathcal{Z}_{s,t}^x\right)\right)}{Q_{s,T}\mathbf{1}_D(x)} \\ &= \frac{\mathbb{E}\left(f(\mathcal{Z}_{s,t}^x)\mathbb{E}\left(\mathbf{1}_{\mathcal{Z}_{s,T}^x \in D} \mid \mathcal{Z}_{s,t}^x\right)\right)}{Q_{s,T}\mathbf{1}_D(x)}, \end{aligned}$$

by the Markov property. Finally, since $\mathbb{E}\left(\mathbf{1}_{\mathcal{Z}_{s,T}^x \in D} \mid \mathcal{Z}_{s,t}^x\right) = Q_{t,T}\mathbf{1}_D(\mathcal{Z}_{s,t}^x)$, we get the announced result.

For any $T > 0$, the family $(R_{s,t}^T)_{0 \leq s \leq t \leq T}$ is a semi-group. Indeed, we have for all $0 \leq u \leq s \leq t \leq T$

$$R_{u,s}^T(R_{s,t}^T f)(x) = \frac{Q_{u,s}(R_{s,t}^T f Q_{s,T}\mathbf{1}_D)(x)}{Q_{u,T}\mathbf{1}_D(x)},$$

where, for all $y \in D$,

$$R_{s,t}^T f(y) Q_{s,T}\mathbf{1}_D(y) = Q_{s,t}(fQ_{t,T}\mathbf{1}_D)(y),$$

then

$$\begin{aligned} R_{u,s}^T R_{s,t}^T f(x) &= \frac{Q_{u,s}(Q_{s,t}(fQ_{t,T}\mathbf{1}_D))(x)}{Q_{u,T}\mathbf{1}_D(x)} \\ &= \frac{Q_{u,t}(fQ_{t,T}\mathbf{1}_D)(x)}{Q_{u,T}\mathbf{1}_D(x)} = R_{u,t}^T f(x), \end{aligned}$$

where we have used that $(Q_{s,t})_{s \leq t}$ is a semigroup.

In order to prove the exponential mixing property of Theorem 1, we need the following lemma, whose proof is postponed to the end of this subsection.

Lemma 6. *There exists a constant $\beta' > 0$ and a family of probability measures $(\eta_s^{x_1, x_2})_{s, x_1, x_2}$ such that, for all $0 \leq s \leq T - \Pi - 2$, we have*

$$R_{s, s+1}^T f(x_i) \geq \beta' \eta_s^{x_1, x_2}(f), \quad i = 1, 2,$$

for all $(x_1, x_2) \in D \times D$ and any non-negative measurable function f .

For any orthogonal probability measures μ_1, μ_2 on D , we have

$$\begin{aligned} \|\mu_1 R_{s, s+1}^T - \mu_2 R_{s, s+1}^T\|_{TV} &= \sup_{f \in \mathcal{B}_1(D)} |\mu_1 R_{s, s+1}^T(f) - \mu_2 R_{s, s+1}^T(f)| \\ &\leq \sup_{f \in \mathcal{B}_1(D)} \int_{D \times D} |R_{s, s+1}^T f(x) - R_{s, s+1}^T f(y)| (\mu_1 \otimes \mu_2)(dx, dy), \end{aligned}$$

where $\mathcal{B}_1(D)$ denotes the set of measurable functions f such that $\|f\|_\infty \leq 1$, and $\|\cdot\|_{TV}$ the total variation norm for signed measures. For any $x, y \in D \times D$ and any $f \in \mathcal{B}_1(D)$, we have by Lemma 6, for all $s \leq T - \Pi - 2$,

$$\begin{aligned} |R_{s, s+1}^T f(x) - R_{s, s+1}^T f(y)| &= |(R_{s, s+1}^T f(x) - \beta' \eta_s^{x, y}(f)) - (R_{s, s+1}^T f(y) - \beta' \eta_s^{x, y}(f))| \\ &\leq 2(1 - \beta'). \end{aligned}$$

Since μ_1 and μ_2 are assumed to be orthogonal probability measures, we have $\|\mu_1 - \mu_2\|_{TV} = 2$, so that

$$\|\mu_1 R_{s, s+1}^T - \mu_2 R_{s, s+1}^T\|_{TV} \leq (1 - \beta') \|\mu_1 - \mu_2\|_{TV}.$$

If μ_1 and μ_2 are two different but not orthogonal probability measures, one can apply the previous result to the orthogonal probability measures $\frac{(\mu_1 - \mu_2)_+}{(\mu_1 - \mu_2)_+(D)}$ and $\frac{(\mu_1 - \mu_2)_-}{(\mu_1 - \mu_2)_-(D)}$. Then

$$\begin{aligned} &\left\| \frac{(\mu_1 - \mu_2)_+}{(\mu_1 - \mu_2)_+(D)} R_{s, s+1}^T - \frac{(\mu_1 - \mu_2)_-}{(\mu_1 - \mu_2)_-(D)} R_{s, s+1}^T \right\|_{TV} \\ &\leq (1 - \beta') \left\| \frac{(\mu_1 - \mu_2)_+}{(\mu_1 - \mu_2)_+(D)} - \frac{(\mu_1 - \mu_2)_-}{(\mu_1 - \mu_2)_-(D)} \right\|_{TV}. \end{aligned}$$

But $(\mu_1 - \mu_2)_+(D) = (\mu_1 - \mu_2)_-(D)$ since $\mu_1(D) = \mu_2(D) = 1$, then, multiplying the obtained inequality by $(\mu_1 - \mu_2)_+(D)$, we deduce that

$$\|(\mu_1 - \mu_2)_+ R_{s, s+1}^T - (\mu_1 - \mu_2)_- R_{s, s+1}^T\|_{TV} \leq (1 - \beta') \|(\mu_1 - \mu_2)_+ - (\mu_1 - \mu_2)_-\|_{TV}.$$

But $(\mu_1 - \mu_2)_+ - (\mu_1 - \mu_2)_- = \mu_1 - \mu_2$, so that

$$\|\mu_1 R_{s, s+1}^T - \mu_2 R_{s, s+1}^T\|_{TV} \leq (1 - \beta') \|\mu_1 - \mu_2\|_{TV}.$$

In particular, using the semigroup property of $(R_{s,t}^T)_{s,t}$, we deduce that

$$\begin{aligned} & \|\delta_x R_{0,T-\Pi-2}^T - \delta_y R_{0,T-\Pi-2}^T\|_{TV} \\ &= \delta_x R_{0,T-\Pi-3} R_{T-\Pi-3,T-\Pi-2}^T - \delta_y R_{0,T-\Pi-3} R_{T-\Pi-3,T-\Pi-2}^T \|_{TV} \\ &\leq (1 - \beta') \|\delta_x R_{0,T-\Pi-3}^T - \delta_y R_{0,T-\Pi-3}^T\|_{TV} \leq 2(1 - \beta')^{[T-\Pi-2]}, \end{aligned}$$

where $[T - \Pi - 2]$ denotes the integer part of $T - \Pi - 2$. Theorem 1 is thus proved for any pair of probability measures (δ_x, δ_y) , with $(x, y) \in D \times D$, for a good choice of C and γ which are now assumed to be fixed.

Let μ be a probability measure on D and $y \in D$. We have

$$\begin{aligned} \left| \mu Q_{0,T}(f) - \mu Q_{0,T}(\mathbf{1}_D) \frac{\delta_y Q_{0,T}(f)}{\delta_y Q_{0,T}(\mathbf{1}_D)} \right| &= \left| \int_D Q_{0,T} f(x) - \delta_x Q_{0,T}(\mathbf{1}_D) \frac{\delta_y Q_{0,T}(f)}{\delta_y Q_{0,T}(\mathbf{1}_D)} d\mu(x) \right| \\ &\leq \int_D C e^{-\gamma T} \delta_x Q_{0,T}(\mathbf{1}_D) d\mu(x), \end{aligned}$$

by Theorem 1 for Dirac initial measures that we just proved. Dividing by $\mu Q_{0,T}(\mathbf{1}_D) = \int_D \delta_x Q_{0,T}(\mathbf{1}_D) d\mu(x)$, we deduce that

$$\left| \frac{\mu Q_{0,T}(f)}{\mu Q_{0,T}(\mathbf{1}_D)} - \frac{\delta_y Q_{0,T}(f)}{\delta_y Q_{0,T}(\mathbf{1}_D)} \right| \leq C e^{-\gamma T},$$

for any $f \in \mathcal{B}_1$. The same procedure, replacing δ_y by any probability measure, leads us to Theorem 1.

Proof of Lemma 6. By Lemma 5, there exist $\beta > 0$ and a family of probability measures denoted by $(\nu_s^{x_1, x_2})_{s \geq 0, (x_1, x_2) \in D \times D}$ such that, for any $(x_1, x_2) \in D \times D$ and any $s \geq 0$, we have for all $i \in \{1, 2\}$

$$Q_{s,s+1} f(x_i) \geq Q_{s,s+1} \mathbf{1}_D(x_i) \beta \nu_s^{x_1, x_2}(f),$$

for any non-negative measurable function f . Then we have

$$\begin{aligned} R_{s,s+1}^T f(x_i) &= \frac{Q_{s,s+1}(f Q_{s+1,T} \mathbf{1}_D)(x_i)}{Q_{s,T} \mathbf{1}_D(x_i)} \\ &\geq \frac{\beta \nu_s^{x_1, x_2}(f Q_{s+1,T} \mathbf{1}_D) Q_{s,s+1} \mathbf{1}_D(x_i)}{Q_{s,T} \mathbf{1}_D(x_i)}. \end{aligned}$$

Since $s + 1 + \Pi + 1 \leq T$ by assumption, we deduce from Lemma 4 that there exist $x_{s+1,T} \in D$ and $r_0 > 0$ such that

$$\inf_{x \in B(x_{s+1,T}, r_0)} Q_{s+1,T} \mathbf{1}_D(x) \geq \frac{1}{2} \|Q_{s+1,T} \mathbf{1}_D\|_\infty. \quad (11)$$

Now, we define the probability measure $\eta_s^{x_1, x_2}$ by

$$\eta_s^{x_1, x_2}(A) \stackrel{\text{def}}{=} \frac{\nu_s^{x_1, x_2}(A \cap B(x_s, T, r_0))}{\nu_s^{x_1, x_2}(B(x_s, T, r_0))}, \forall A \subset D.$$

By the second part of Lemma 5, $\nu_s^{x_1, x_2}(B(x_s, T, r_0))$ is uniformly bounded below by a constant $\epsilon > 0$ which only depend on r_0 . We deduce that

$$\begin{aligned} R_{s, s+1}^T f(x_1) &\geq \frac{\frac{\epsilon}{2} \eta_s^{x_1, x_2}(f) Q_{s, s+1} \mathbf{1}_D(x_1) \|Q_{s+1, T} \mathbf{1}_D\|_\infty}{Q_{s, T} \mathbf{1}_D(x_1)} \\ &\geq \frac{\epsilon}{2} \eta_s^{x_1, x_2}(f), \end{aligned}$$

by the Markov property. This concludes the proof of Lemma 6. □

References

- [1] K. Burdzy, R. Holyst, D. Ingeman, and P. March. Configurational transition in a Fleming-Viot-type model and probabilistic interpretation of laplacian eigenfunctions. *J. Phys. A*, 29(29):2633–2642, 1996.
- [2] K. Burdzy, R. Holyst, and P. March. A Fleming-Viot particle representation of the Dirichlet Laplacian. *Comm. Math. Phys.*, 214(3):679–703, 2000.
- [3] N. Champagnat and D. Villemonais. Exponential convergence to quasi-stationary distribution and Q-process. *ArXiv e-prints*, Apr. 2014.
- [4] P. Del Moral. *Mean field simulation for Monte Carlo integration*. Chapman & Hall/CRC Monographs on Statistics & Applied Probability, 2013.
- [5] P. Del Moral and A. Doucet. Particle motions in absorbing medium with hard and soft obstacles. *Stochastic Analysis and Applications*, 22(5):1175–1204, 2004.
- [6] P. Del Moral and A. Guionnet. On the stability of interacting processes with applications to filtering and genetic algorithms. *Annales de l'Institut Henri Poincaré*, 37(2):155–194, 2001.
- [7] P. Del Moral and L. Miclo. Branching and interacting particle systems approximations of feynman-kac formulae with applications to non-linear filtering. *Séminaire de Probabilités XXXIV, Lecture Notes in Mathematics, Springer*, 1729:1–145, 2000.
- [8] P. Del Moral and L. Miclo. A moran particle system approximation of feynman-kac formulae. *Stochastic Processes and their Applications*, 86(2):193–216, 2000.
- [9] P. Del Moral and L. Miclo. On the stability of nonlinear semigroup of feynman-kac type. *Annales de la Faculté des Sciences de Toulouse*, 11(2):135–175, 2002.

- [10] P. Del Moral and L. Miclo. Particle approximations of Lyapunov exponents connected to Schrödinger operators and Feynman-Kac semigroups. *ESAIM Probab. Stat.*, 7:171–208, 2003.
- [11] M. C. Delfour and J.-P. Zolésio. *Shapes and geometries*, volume 4 of *Advances in Design and Control*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2001. Analysis, differential calculus, and optimization.
- [12] S. Ethier and S. Krone. Comparing Fleming-Viot and Dawson-Watanabe processes. *Stochastic Processes and their Applications*, 60(2):171–190, 1995.
- [13] S. N. Ethier and T. G. Kurtz. *Markov processes*. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons Inc., New York, 1986. Characterization and convergence.
- [14] P. A. Ferrari and N. Marić. Quasi stationary distributions and Fleming-Viot processes in countable spaces. *Electron. J. Probab.*, 12:no. 24, 684–702 (electronic), 2007.
- [15] W. H. Fleming and M. Viot. Some measure-valued Markov processes in population genetics theory. *Indiana University Mathematics Journal*, 28(5):817–843, 1979.
- [16] G. L. Gong, M. P. Qian, and Z. X. Zhao. Killed diffusions and their conditioning. *Probab. Theory Related Fields*, 80(1):151–167, 1988.
- [17] C. Graham and S. Méléard. Stochastic particle approximations for generalized Boltzmann models and convergence estimates. *Ann. Probab.*, 25(1):115–132, 1997.
- [18] I. Grigorescu and M. Kang. Hydrodynamic limit for a Fleming-Viot type system. *Stochastic Process. Appl.*, 110(1):111–143, 2004.
- [19] N. Ikeda and S. Watanabe. *Stochastic differential equations and diffusion processes*. North-Holland Mathematical Library, 1989.
- [20] R. G. Pinsky. On the convergence of diffusion processes conditioned to remain in a bounded region for large time to limiting positive recurrent diffusion processes. *Ann. Probab.*, 13(2):363–378, 1985.
- [21] E. Priola and F.-Y. Wang. Gradient estimates for diffusion semigroups with singular coefficients. *J. Funct. Anal.*, 236(1):244–264, 2006.
- [22] M. Rousset. On the control of an interacting particle estimation of Schrödinger ground states. *SIAM J. Math. Anal.*, 38(3):824–844 (electronic), 2006.
- [23] D. Villemonais. Interacting particle systems and Yaglom limit approximation of diffusions with unbounded drift. *Electronic Journal of Probability*, 16:1663–1692, 2011.

- [24] D. Villemonais. Uniform tightness for time-inhomogeneous particle systems and for conditional distributions of time-inhomogeneous diffusion processes. *Markov Process. Related Fields*, 19(3):543–562, 2013.
- [25] D. Villemonais. General approximation method for the distribution of markov processes conditioned not to be killed. *ESAIM: Probability and Statistics*, eFirst, 7 2014.