Kernel estimate and capacity in the Dirichlet spaces

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ABSTRACT. Let μ be a positive finite measure on the unit circle. The associated Dirichlet space $\mathcal{D}(\mu)$ consists of holomorphic functions on the unit disc whose derivatives are square integrable when weighted against the Poisson integral of μ. First, we give an estimate of the norm of the reproducing kernel k^μ of $\mathcal{D}(\mu)$. Next, we study the notion of μ-capacity associated with $\mathcal{D}(\mu)$, in the sense of Beurling–Deny. Namely, we give an estimate of μ-capacity of arcs in terms of the norm of k^μ. We also provide a new condition on closed sets to be μ-polar. Our method is based on sharp estimates of norms of some outer test functions which allow us to transfer these problems to an estimate of the reproducing kernel of an appropriate weighted Sobolev space.

1. INTRODUCTION

Let H^2 denote the classical *Hardy space* of analytic functions on the unit disc $\mathbb{D}$ having square summable Taylor coefficients at the origin. Every function $f \in H^2$ has non-tangential limits almost everywhere on the unit circle $\mathbb{T} = \partial\mathbb{D}$. We denote by $f(\zeta)$ the non-tangential limit of f at $\zeta \in \mathbb{T}$ if it exists.
Let μ be a positive finite measure on $\mathbb{T}$, the *Dirichlet type space* $\mathcal{D}(\mu)$ is the set of analytic functions $f \in H^2$, such that

$$\mathcal{D}_\mu(f) := \frac{1}{2\pi}\int_{\mathbb{T}}\int_{\mathbb{T}} \frac{|f(\zeta)-f(\xi)|^2}{|\zeta-\xi|^2}|d\zeta|d\mu(\xi) < \infty.$$

The space $\mathcal{D}(\mu)$ is endowed with the norm

$$\|f\|_\mu^2 := \|f\|_{H^2}^2 + \mathcal{D}_\mu(f).$$

If $d\mu(e^{it}) = 0$, then $\mathcal{D}(\mu) = H^2$ and if $d\mu(e^{it}) = dm(t) = dt/2\pi$, the normalized arc measure on $\mathbb{T}$, then $\mathcal{D}(\mu)$ is the classical *Dirichlet space* $\mathcal{D}$.
These spaces were introduced by Richter in [12] by examining the 2-*isometries* on the Hilbert spaces. A bounded operator T in a Hilbert space $\mathcal{H}$ is called 2-*isometry* if

$$T^{*2}T^2 - 2T^*T - I = 0,$$

is said to be *cyclic* if there exists $x \in \mathcal{H}$ such that $\text{span}\{T^n x, n \geq 0\}$ is dense in $\mathcal{H}$ and is called *analytic* if $\bigcap_{n\geq 0} T^n\mathcal{H} = \{0\}$. Richter in [12] proved that every cyclic, analytic

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2–isometry can be represented as a multiplication by z on a Dirichlet type space $\mathcal{D}(\mu)$ for some positive finite measure μ. As consequence [11, 12] Richter gave an analogue of Beurling's theorem for the Dirichlet space.

1.1. **Reproducing kernels.** The *reproducing kernel* plays an important role in the study of Hilbert spaces of analytic functions. In particular, it allows to determine the rate of growth of functions near the boundary and its tangential behavior; their properties are closely related to embedding theorems, sampling and interpolation sets, and other topics. Let $P[\mu]$ be the Poisson integral of the positive finite measure μ on $\mathbb{T}$

$$P[\mu](z) = \int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta - z|^2} d\mu(\zeta), \qquad z \in \mathbb{D}.$$

In the following theorem, we provide an asymptotic estimate of the reproducing kernel k^μ of $\mathcal{D}(\mu)$ on the diagonal.

Theorem 1. *Let μ be a finite positive measure on $\mathbb{T}$. We have*

$$k^\mu(z,z) \asymp 1 + \int_0^{|z|} \frac{dr}{(1-r)P[\mu](rz/|z|) + (1-r)^2},$$

where the implied constants are absolute.

Let us recall that Shimorin [15] proved that all Dirichlet type spaces have complete Nevanlinna–Pick reproducing kernels. As an important consequence (see [2, §9.4] and [14, Theorem 1]), each sequence $\mathcal{Z} = \{z_n\} \subset \mathbb{D}$ satisfying Shapiro–Shields condition $\sum_{z\in\mathcal{Z}} 1/k^\mu(z,z) < \infty$ is a zero set of $\mathcal{D}(\mu)$. Theorem 1 allows to give examples of zero sets of $\mathcal{D}(\mu)$.

To prove the lower estimates of $k^\mu(z,z)$, we establish a sharp norm estimates of some outer functions which peak near z. This allows us to transfer our problem to an estimation of the norm of the kernel of an appropriate weighted Sobolev space. In fact and roughly speaking, Theorem 1 says that $k^\mu(z,z) \asymp K_\varphi(1-|z|, 1-|z|)$ where K_φ is the reproducing kernel of the weighted Sobolev space defined by

$$W^2(\varphi) := \big\{ f \in C\big((0,2\pi]\big) : \ f(x) = f(1) + \int_x^{2\pi} g(t)dt, \ g \in L^2((0,2\pi), \varphi dt)\big\},$$

where $\varphi(t) = tP[\mu]((1-t)z/|z|) + t^2$.

1.2. **Capacity.** The Dirichlet type space is closely related to some notions of potential theory. Let $\mathcal{D}^h(\mu)$ be the harmonic version of $\mathcal{D}(\mu)$ given by

$$\mathcal{D}^h(\mu) := \Big\{ f \in L^2(\mathbb{T}) : \ \|f\|_\mu^2 := \|f\|_{L^2(\mathbb{T})}^2 + \mathcal{D}_\mu(f) < \infty \Big\}.$$

$\mathcal{D}^h(\mu)$ is a Dirichlet space in the sense of Beurling–Deny [3]. Some aspects of the potential theory associated to the general Dirichlet spaces were studied in several papers (see for

instance [8]). In this paper we will focus on the notion of capacity. We recall at first, the definition of capacity in the sense of Beurling–Deny. Let U be an open subset of the unit circle. The c_μ-capacity of U is defined by

$$c_\mu(U) := \inf\left\{\|u\|_\mu^2 : u \in \mathcal{D}^h(\mu),\ \ u \geq 0 \text{ and } u \geq 1 \text{ a.e. on } U\right\}. \tag{1}$$

As usual we define the c_μ-capacity of any subset $F \subset \mathbb{T}$ by

$$c_\mu(F) = \inf\{c_\mu(U) : U \ \text{ open},\ \ F \subset U\}.$$

Since the L^2 norm dominates the Dirichlet type norm, it is completely obvious that sets having c_μ–capacity 0 have Lebesgue measure 0. We say that a property holds c_μ-quasi-everywhere (c_μ–q.e.) if it holds everywhere outside a set of c_μ–capacity 0. So, c_μ–q.e. implies a.e.. A closed set of capacity zero will be called, throughout this paper, *μ–polar* set. If $d\mu(e^{it}) = dt/2\pi$, the normalized arc measure on $\mathbb{T}$, then $\mathcal{D}(\mu)$ is the classical Dirichlet space $\mathcal{D}$ and c_μ is comparable to the logarithmic capacity, see [10, Theorem 14] and [1, Theorem 2.5.5].

Our first result on μ-capacity gives an estimate of capacity of arcs in terms of the kernel. More precisely we have:

Theorem 2. *Let $I \subset \mathbb{T}$ be the arc of length $|I| = 1-\rho$ with the midpoint at $\zeta \in \mathbb{T}$. Then*

$$c_\mu(I) \asymp \frac{1}{k^\mu(\rho\zeta, \rho\zeta)}.$$

where the implied constants are absolute.

As a consequence,

$$c_\mu(\{e^{i\theta}\}) = 0 \iff \int_0^1 \frac{dr}{(1-r)P[\mu](re^{i\theta}) + (1-r)^2} = \infty. \tag{2}$$

In the sequel we will suppose that E is a closed set which has Lebesgue measure zero and μ is a finite positive measure on $\mathbb{T}$. Now our goal is to give sufficient condition on E to be μ–polar. Let us introduce the local modulus of continuity of μ on E which will play a crucial role in this paper. It is defined by

$$\rho_{\mu,E}(t) := \sup\{\mu(\zeta e^{-it}, \zeta e^{it}) : \zeta \in E\}. \tag{3}$$

Note that $\rho_{\mu,\mathbb{T}} = \rho_\mu$ is the classical modulus of continuity of μ. Let us write

$$E_t := \{\zeta \in \mathbb{T}\colon\ d(\zeta, E) \leq t\},$$

where d denotes the distance with respect to arc-length, and denote by $|E_t|$ the Lebesgue measure of E_t. We can express the function $|E_t|$ in terms of

$$N_E(t) := 2\sum_j \mathbf{1}_{\{|I_j| > 2t\}},$$

where (I_j) are the components of $\mathbb{T}\backslash E$, as follows

$$\int_0^t N_E(s)ds = |E_t|.$$

In Theorem 5.4, we give a sufficient conditions on a closed subset E, in terms of $\rho_{\mu,E}$ and N_E, to be μ-polar. To illustrate this theorem we give here some of its corollaries.

(i) If

$$\int_0^\pi \frac{dt}{\int_0^t (\rho_\mu(s)N_E(s)/s)ds} = +\infty,$$

then $c_\mu(E) = 0$.

This result can be considered as an extension of Carleson's Theorem [4, section IV, Theorem 2]. In fact if $\mu = m$ is the Lebesgue measure then $\rho_{\mu,E}(t) = t$ and $c_\mu \asymp c$ (c is the logarithmic capacity). We obtain Carleson's theorem which says that if $\int_0^\pi dt/|E_t| = \infty$, then $c(E) = 0$.

(ii) Suppose that $\rho_{\mu,E}(t) = O(t^\alpha)$ with $1 \leq \alpha < 2$. If

$$\int_0^\pi \frac{dt}{t^{\alpha-1}|E_t|} = +\infty,$$

then $c_\mu(E) = 0$.

(iii) If $\rho_{\mu,E}(t) = O(t^\alpha)$ with $\alpha > 2$, we have $c_\mu(E) = 0$.

Note also that if $t^\alpha = O(\rho_{\mu,\{1\}}(t))$ with $\alpha < 1$, then by (2), $c_\mu(\{1\}) > 0$.

The proof of Theorem 5.4 uses an idea analogous to the proof of Theorem 1. However, our test functions must peak on the whole set E and the desired weighted Sobolev spaces will depend on μ and E. In fact, we prove that there is no bounded point evaluation at 0 for $W^2(\varphi)$ (where φ depends on $\rho_{\mu,E}$ and N_E), then $c_\mu(E) = 0$. Note finally, that there is no bounded point evaluation at 0 for $W^2(\varphi)$ if and only if $\lim\limits_{t\to 0^+} K_\varphi(t,t) = \infty$.

The plan of the paper is the following. In the next section we recall two formulas of the Dirichlet type norm; we also give punctual estimates of some outer functions. In Section 3 we give norm estimates of our test functions. In Section 4 we give diagonal asymptotic estimates of reproducing kernel. In section 5, we prove the announced results on capacity.

Throughout the paper, we use the following notations:

- $A \lesssim B$ means that there is an absolute constant C such that $A \leq CB$.
- $A \asymp B$ if both $A \lesssim B$ and $B \lesssim A$.

- $C(x_1, \ldots, x_n)$ denote a constant which depends only on variables $x_1, \ldots, x_n$.

2. Preliminaries

2.1. **Norm formulas.** In this subsection we recall some results about norm formulas in Dirichlet type spaces which will be used in what follows.

For a finite positive measure μ on $\mathbb{T}$, the harmonic Dirichlet space $\mathcal{D}^h(\mu)$ consists of functions $f \in L^2(\mathbb{T})$ such that

$$\mathcal{D}_\mu(f) := \int_{\mathbb{T}} \mathcal{D}_\xi(f) d\mu(\xi) < \infty,$$

where $\mathcal{D}_\xi(f)$ is the *local Dirichlet integral* of f at $\xi \in \mathbb{T}$ given by

$$\mathcal{D}_\xi(f) := \int_{\mathbb{T}} \frac{|f(e^{it}) - f(\xi)|^2}{|e^{it} - \xi|^2} \frac{dt}{2\pi}.$$

The Douglas' formula, see [6, Theorem 7.1.3], expresses the Dirichlet integral of a function f in terms of the Poisson transform of μ, namely

$$D_\mu(f) = \int_{\mathbb{D}} |\nabla P[f]|^2 P[\mu] dA, \qquad f \in D^h(\mu),$$

where $dA(z) = dxdy/\pi$ stands for the normalized area measure in $\mathbb{D}$. In particular, if $f \in \mathcal{D}(\mu) (= \mathcal{D}^h(\mu) \cap H^2)$, Douglas' Formula becomes

$$\mathcal{D}_\mu(f) := \int_{\mathbb{D}} |f'(z)|^2 P[\mu](z) dA(z) < \infty.$$

Another useful formula, due to Richter and Sundberg [13, Theorem 3.1], gives the local Dirichlet integral of function f in terms of their zeros sequence, their singular measure and the modulus of their radial limit. We will need, throughout this paper, the Richter–Sundberg formula mainly for outer functions. Recall that outer functions are given by

$$f(z) = \exp \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log \varphi(\zeta) \frac{|d\zeta|}{2\pi}, \qquad (z \in \mathbb{D}),$$

where φ is a positive function such that $\log \varphi \in L^1(\mathbb{T})$. Note that the radial limit of f, noted also by f, exists a.e. and $|f| = \varphi$ a.e. on $\mathbb{T}$.

Let $f \in H^2$ be an outer function such that $f(\zeta)$ exists at $\zeta \in \mathbb{T}$. We have the following Richter–Sundberg formula :

$$\mathcal{D}_\zeta(f) = \int_{\mathbb{T}} \frac{|f(\lambda)|^2 - |f(\zeta)|^2 - 2|f(\zeta)|^2 \log |f(\lambda)/f(\zeta)|}{|\lambda - \zeta|^2} \frac{|d\lambda|}{2\pi}. \tag{4}$$

2.2. **Punctual estimates of test functions.** The result obtained in this subsection will be used in the proof of the lower estimate of the kernel.

Lemma 2.1. *Let $1/2 < r = 1 - a < 1$ and let $I_k = [e^{ia_k}, e^{ia_{k+1}}]$ with $a_0 = 0$, $a_k = 2^k a$ $(k \geq 1)$. Let N be the integer such that $2^N a \leq \pi < 2^{N+1} a$, then*

$$\sum_{k=0}^{N-1} (k+1)\varpi(r, I_k, \mathbb{D}) \asymp 1,$$

where $\varpi(r, I_k, \mathbb{D})$ denotes the harmonic measure of I_k at r in $\mathbb{D}$.

Proof. Without loss of generality, we may suppose that $2^N a = \pi$. Note that

$$\sum_{k=0}^{N-1} (k+1)\varpi(r, I_k, \mathbb{D}) \geq \sum_{k=0}^{N-1} \varpi(r, I_k, \mathbb{D}) \asymp 1.$$

For the reverse inequality, let $g(z) = \log 1/|1 - rz|$. Since g is harmonic in the neighbourhood of $\overline{\mathbb{D}}$,

$$g(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - |z|^2}{|1 - ze^{-i\theta}|^2} \log \frac{1}{|1 - re^{i\theta}|} d\theta.$$

So,

$$g(r) = \log \frac{1}{1 - r^2} = \sum_{k=0}^{N-1} \frac{1}{\pi} \int_{a_k}^{a_{k+1}} \frac{1 - r^2}{|1 - re^{-i\theta}|^2} \log \frac{1}{|1 - re^{i\theta}|} d\theta. \tag{5}$$

For $k = 0, \ldots, N-1$, and $\theta \in (a_k, a_{k+1})$, we have

$$\frac{1}{|1 - re^{i\theta}|} \asymp \frac{1}{2^k} \frac{1}{1 - r}.$$

By (5), we get

$$\log \frac{1}{1 - r^2} = \log \frac{1}{1 - r} - \log 2 \sum_{k=0}^{N-1} k\varpi(r, I_k, \mathbb{D}) + O(1)$$

and our result follows. □

Let $w : (0, \pi) \to (0, +\infty)$ be a continuous positive function such that $\log w \in L^1(\mathbb{T})$. As before f_w denote the outer function satisfying

$$|f_w(e^{it})| = w(|t|) \quad \text{a.e. on } (-\pi, \pi). \tag{6}$$

Proposition 2.2. *Let $w : (-\pi, \pi) \to (0, +\infty)$ be an even continuous positive decreasing function such that $\log w \in L^1(\mathbb{T})$. Suppose that $w(x) \leq 2w(2x)$. Let f_w be an outer function given by* (6)*. Then*

$$w(1 - r) \lesssim |f_w(r)|, \qquad 0 \leq r < 1.$$

Proof. Let a , I_k and N as in lemma 2.1, and suppose that $a_N = \pi$. We have

$$\begin{aligned}
|f_w(r)| &= \exp\Big\{ \sum_{k=0}^{N-1} \frac{1}{\pi} \int_{I_k} \frac{1-r^2}{|1-re^{i\theta}|^2} \log w(\theta) d\theta \Big\} \\
&\geq \exp\Big\{ \sum_{k=0}^{N-1} \log \omega(2^{k+1}a) \frac{1}{\pi} \int_{I_k} \frac{1-r^2}{|1-re^{i\theta}|^2} d\theta \Big\} \\
&\geq \exp\Big\{ \sum_{k=0}^{N-1} (\log \omega(a) - (k+1)\log 2) \frac{1}{\pi} \int_{I_k} \frac{1-r^2}{|1-re^{i\theta}|^2} d\theta \Big\} \\
&\geq \exp\Big\{ \log w(a) \sum_{k=0}^{N-1} \varpi(r, I_k, \mathbb{D}) - \log 2 \sum_{k=0}^{N-1} (k+1) \varpi(r, I_k, \mathbb{D}) \Big\}.
\end{aligned}$$

We obtain from Lemma 2.1, that $w(1-r) \lesssim |f_w(r)|$. The case $a_N < \pi$ can be treated in the same way by taking into account the interval $[e^{ia_N}, e^{i\pi}]$. □

2.3. **Regularization lemma.** Let μ be a positive finite measure on $\mathbb{T}$, we set $d\mu(s) = d\mu(e^{is})$. Denote by

$$\widehat{\mu}(s) = \mu([e^{-is}, e^{is}]) \ \ (0 \leq s \leq \pi) \ \text{ and } \ \widehat{\mu}(s) = \widehat{\mu}(\pi) \ \ (s > \pi).$$

Let

$$F_\mu(x) = \int_{-\pi}^{\pi} \frac{x^2}{x^2+s^2} d\mu(s) \qquad x > 0. \tag{7}$$

Note that F_μ is increasing and $F_\mu(x)/x^2$ is decreasing. We extend F_μ at the origin by $F_\mu(0) = F_\mu(0^+)$. In the following lemma we collect some elementary properties of F_μ which will be used in the sequel.

Lemma 2.3. *Let ν be a positive finite measure on $\mathbb{T}$. We have the following*

(1) $F_\nu(x) \asymp xP[\nu](1-x)$, for $x > 0$,

(2) $\widehat{\nu}(x) \lesssim F_\nu(x)$ for $x \geq 0$,

(3) $\displaystyle\int_{x \leq |s| \leq \pi} \frac{d\nu(s)}{s^2} \lesssim \frac{F_\nu(x)}{x^2}$, for $x > 0$,

(4) If h is a positive monotone function on $(0, \pi)$. Then

$$\int_{-a}^{a} h(|x|) d\nu(x) \lesssim \int_0^a \frac{h(x)}{x} \widehat{\nu}(2x) dx,$$

(5) If $\widehat{\nu}(2x) \leq c\widehat{\nu}(x)$ for some constant $c < 4$, then $F_\nu(x) \lesssim \dfrac{\widehat{\nu}(x)}{4-c}$.

Proof. (1), (2) and (3) are obvious. To prove (4) suppose that h is a decreasing function. Clearly if $\nu(\{0\}) > 0$, then (4) is obvious. So, suppose that $\nu(\{0\}) = 0$. We have

$$\begin{aligned}\int_{-a}^{a} h(|x|)d\nu(x) &= \sum_{\substack{n\geq 0\\ m}} \int_{2^{-(n+1)}a}^{2^{-n}a} h(x)(d\nu(x)+d\nu(-x))\\ &\lesssim \sum_{n=0} h(2^{-n}a)\widehat{\nu}(2^{-n}a)\\ &\lesssim \int_0^a \frac{h(x)}{x}\widehat{\nu}(2x)dx.\end{aligned}$$

Analogue argument works if h is increasing.
Finally to prove (5), suppose that $\widehat{\nu}(2x) \leq c\widehat{\nu}(x)$ with $c < 4$. We have $\widehat{\nu}(2^n x) \leq c^n\widehat{\nu}(x)$ and

$$\int_{|t|\geq x} \frac{d\nu(t)}{t^2}dt = \sum_{n\geq 0} \int_{2^n x\leq |t|\leq 2^{n+1}x} \frac{d\nu(t)}{t^2} \leq \sum_{n\geq 0} \frac{\widehat{\nu}(2^{n+1}x)}{2^{2n}x^2} \leq \frac{4c}{4-c}\frac{\widehat{\nu}(x)}{x^2}.$$

So

$$F_\nu(x) \leq \widehat{\nu}(x) + \int_{|\theta|>x} \frac{x^2}{x^2+\theta^2}d\nu(\theta) \leq \widehat{\nu}(x) + x^2\int_{\theta\geq x} \frac{d\nu(x)}{\theta^2} \lesssim \frac{\widehat{\nu}(x)}{4-c}.$$

□

3. Norm estimate of test functions

3.1. **Norm estimate of analytic test functions.** The purpose of this subsection is to give estimate of norms of some outer functions which play an important role in what follows.

The following lemma is the first step to prove Theorem 3.2.

Lemma 3.1. *Let $w : [0,\pi] \to (0,+\infty)$ be a C^1 decreasing convex function such that $w(x) \leq 2w(2x)$. Suppose that $x^2|w'(x)|$ is increasing and let f_w be the outer function given by* (6)*. Then*

$$\mathcal{D}_\mu(f_w) \lesssim \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3,$$

where

$$\begin{aligned}\mathcal{J}_1 &:= \int_{x=0}^{\pi}\int_{y=0}^{x} |w'(y)|w(y)\frac{|w'(x)|}{w(x)}\frac{\widehat{\mu}(y)}{x}dxdy,\\ \mathcal{J}_2 &:= \int_{s=-\pi}^{\pi} w'(s)^2 s d\mu(s),\\ \mathcal{J}_3 &:= \int_{x=0}^{\pi}\int_{y=x}^{\pi} x|w'(y)|w(y)\frac{|w'(x)|}{w(x)}\Big(\int_{y\leq|s|\leq\pi}\frac{d\mu(s)}{s^2}\Big)dxdy.\end{aligned}$$

Proof. Without loss of generality, we may assume that $d\mu(s) = d\mu(-s)$. By Richter-Sundberg formula (4) we have

$$\mathcal{D}_{e^{is}}(f_w) = \frac{8}{2\pi}\int_{t=0}^{\pi}\int_{x=s}^{t}\int_{y=s}^{x} w'(y)w(y)\frac{w'(x)}{w(x)}dydx\frac{dt}{|e^{is}-e^{it}|^2}.$$

So,

$$\begin{aligned}\mathcal{D}_{\mu}(f_w) &= \frac{16}{2\pi}\int_{s=0}^{\pi}\int_{t=0}^{\pi}\int_{x=s}^{t}\int_{y=s}^{x} w'(y)w(y)\frac{w'(x)}{w(x)}dydxdt\frac{d\mu(s)}{|e^{is}-e^{it}|^2}\\ &= \underbrace{\int_{s=0}^{\pi}\int_{t=2s}^{\pi}\dots}_{\mathcal{I}_1}+\underbrace{\int_{s=0}^{\pi}\int_{t=s/2}^{2s}\dots}_{\mathcal{I}_2}+\underbrace{\int_{s=0}^{\pi}\int_{t=0}^{s/2}\dots}_{\mathcal{I}_3}.\end{aligned}$$

To complete the proof we will estimate each term separately.

If $2s \le t \le \pi$, we have $|t-s| \ge t/2$,

$$\begin{aligned}\mathcal{I}_1 &\lesssim \int_{s=0}^{\pi}\int_{t=s}^{\pi}\int_{x=s}^{t}\int_{y=s}^{x}|w'(y)|w(y)\frac{|w'(x)|}{w(x)}dydx\frac{dt}{t^2}d\mu(s)\\ &\lesssim \int_{t=0}^{\pi}\int_{x=0}^{t}\int_{y=0}^{x}|w'(y)|w(y)\frac{|w'(x)|}{w(x)}\Big(\int_{|s|\le y}d\mu(s)\Big)\frac{dt}{t^2}dydx\\ &\lesssim \int_{x=0}^{\pi}\int_{y=0}^{x}|w'(y)|w(y)\frac{|w'(x)|}{w(x)}\frac{\widehat{\mu}(y)}{x}dydx\end{aligned}$$

Since $w(2x) \asymp w(x)$ and $w'(2x) \asymp w'(x)$, we have

$$\mathcal{I}_2 \lesssim \int_{s=-\pi}^{\pi} w'(s)^2 s d\mu(s).$$

If $0 \le t \le s/2$, then $|t-s| \ge s/2$,

$$\begin{aligned}\mathcal{I}_3 &\lesssim \int_{s=0}^{\pi}\int_{t=0}^{s}\int_{x=t}^{s}\int_{y=x}^{s}|w'(y)|w(y)\frac{|w'(x)|}{w(x)}dydx\frac{d\mu(s)}{s^2}dt\\ &\lesssim \int_{t=0}^{\pi}\int_{x=t}^{\pi}\int_{y=x}^{\pi}|w'(y)|w(y)\frac{|w'(x)|}{w(x)}\Big(\int_{y\le|s|\le\pi}\frac{d\mu(s)}{s^2}\Big)dt\\ &\lesssim \int_{x=0}^{\pi}\int_{y=x}^{\pi}|w'(y)|w(y)\frac{|w'(x)|}{w(x)}\Big(\int_{y\le|s|\le\pi}^{\pi}\frac{d\mu(s)}{s^2}\Big)xdxdy.\end{aligned}$$

□

Theorem 3.2. *Let $w : [0,\pi] \to (0,+\infty)$ be a C^1 decreasing convex function such that $w(x) \le 2w(2x)$. Suppose that $x^2|w'(x)|$ is increasing and let f_w be the outer function given by (6). Then*

$$\mathcal{D}_{\mu}(f_w) \lesssim \|F_{\mu}w'\|_{\infty}\|w\|_{\infty},$$

where F_{μ} is given by (7).

Proof. By (3) of Lemma 2.3, we have $\widehat{\mu} \leq F_\mu$. Now Lemma 3.1 gives

$$\begin{aligned}
\mathcal{J}_3 &\lesssim \int_{x=0}^{\pi}\int_{y=x}^{\pi} |w'(y)|w(y)\frac{|w'(x)|}{w(x)}\frac{F_\mu(y)}{y^2}xdydx \\
&\lesssim \|F_\mu w'\|_\infty \int_{x=0}^{\pi}\int_{y=x}^{\pi} |w'(x)|\frac{w(y)x}{w(x)y^2}dydx \\
&\lesssim \|F_\mu w'\|_\infty \int_{x=0}^{\pi}\int_{y=x}^{\pi} |w'(x)|\frac{x}{y^2}dydx \\
&\lesssim \|F_\mu w'\|_\infty \|w\|_\infty.
\end{aligned}$$

Note that $x|w'(x)| \lesssim w(x)$ for all $x \in]0,\pi]$. Indeed, since $w(2x) \asymp w(x)$ and $|w'(2x)| \asymp |w'(x)|$, it suffices to prove the inequality for $x \in [0,\pi/2]$. We have

$$w(x) \geq \int_x^{\pi} t^2|w'(t)|\frac{dt}{t^2} \geq x^2|w'(x)|\Big(\frac{1}{x}-\frac{1}{\pi}\Big) \geq \frac{x}{2}|w'(x)|.$$

So, again by Lemma 2.3 we get

$$\begin{aligned}
\mathcal{J}_1 &\lesssim \|F_\mu w'\|_\infty \int_{x=0}^{\pi}\int_{y=0}^{x} |w'(x)|\frac{w(y)}{w(x)x}dydx \\
&\lesssim \|F_\mu w'\|_\infty \int_{x=0}^{\pi}\int_{y=0}^{x}\int_{u=y}^{\pi} \frac{|w'(x)||w'(u)|}{w(x)x}dudydx \\
&+ \|F_\mu w'\|_\infty \int_{x=0}^{\pi}\int_{y=0}^{x} |w'(x)|\frac{w(\pi)}{w(x)x}dydx \\
&= \mathcal{J}_{12}+\mathcal{J}_{22}.
\end{aligned}$$

We have

$$\begin{aligned}
\mathcal{J}_{12} &\lesssim \|F_\mu w'\|_\infty \int_{u=0}^{\pi}\int_{x=0}^{\pi}\int_{y=0}^{\min(u,x)} \frac{|w'(x)||w'(u)|}{w(x)x}dudydx \\
&\lesssim \|F_\mu w'\|_\infty \Big(\int_{u=0}^{\pi}\int_{x=0}^{u} \frac{|w'(x)w'(u)|}{w(x)}dudx + \int_{u=0}^{\pi}\int_{x=u}^{\pi} \frac{u|w'(x)w'(u)|}{xw(x)}dudx\Big) \\
&\lesssim \|F_\mu w'\|_\infty \Big(\int_{x=0}^{\pi} |w'(x)|dx + \int_{u=0}^{\pi}\int_{x=u}^{\pi} \frac{u|w'(x)w'(u)|}{xw(x)}dudx)\Big) \\
&\lesssim \|F_\mu w'\|_\infty \Big(\|w\|_\infty + \int_{u=0}^{\pi}\int_{x=u}^{\pi} \frac{u|w'(u)|}{x^2}dudx\Big) \\
&\lesssim \|F_\mu w'\|_\infty \|w\|_\infty.
\end{aligned}$$

Since w is decreasing, we get

$$\begin{aligned}\mathcal{J}_{22} &\leq \|F_\mu w'\|_\infty \int_{x=0}^{\pi}\int_{y=0}^{x} |w'(x)|\frac{dx}{x}dy \\ &\leq \|F_\mu w'\|_\infty \|w\|_\infty.\end{aligned}$$

Finally, applying (4) of Lemma 2.3 with $d\nu(s) = sd\mu(s)$, we have $\widehat{\nu}(t) \leq t\widehat{\mu}(t)$ and

$$\mathcal{J}_2 \lesssim \int_0^\pi \frac{w'(t)^2}{t}\widehat{\nu}(2t)dt \lesssim \|F_\mu w'\|_\infty \|w\|_\infty.$$

□

3.2. **Norm estimate of test functions in $\mathcal{D}^h(\mu)$.** Our goal here is to give an estimate of the norm of some distance functions in $\mathcal{D}^h(\mu)$ (For analytic distance functions see [7]).The result of this subsection will be used in the proof of Theorem 5.4.
Let E be a closed subset of $\mathbb{T}$, μ be a positive finite measure and denote by $\rho_{\mu,E}$ the local modulus of continuity of μ on E given by (3). Note that $\rho_{\mu,\{1\}}(t) = \widehat{\mu}(t)$. Recall that $N_E(t) := 2\sum_j \mathbf{1}_{\{|I_j|>2t\}}$. where (I_j) are the components of $\mathbb{T}\backslash E$.

Lemma 3.3. *Let $\Omega : (0,\pi] \to \mathbb{R}^+$ be a positive decreasing function, then*

$$\int_{\mathbb{T}} \Omega(\mathrm{dist}(\zeta, E))d\mu(\zeta) \lesssim \int_0^1 \Omega(t)\frac{\rho_{\mu,E}(t)}{t}N_E(t)dt.$$

Proof. Write $\mathbb{T}\setminus E = \cup_n I_n$, where $(I_n)_n = (e^{i\alpha_n}, e^{i\beta_n})_n$ are the components of $\mathbb{T}\setminus E$. Let $d\mu_n(t) = d\mu(t+\alpha_n) + d\mu(\beta_n - t)$, By Lemma 2.3

$$\begin{aligned}\int_{I_n} \Omega(\mathrm{dist}(\zeta, E))d\mu(\zeta) &\asymp \int_0^{|I_n|/2} \Omega(t)d\mu_n(t) \\ &\lesssim \int_0^{|I_n|/2} \Omega(t)\frac{\widehat{\mu_n}(t)}{t}dt \\ &\lesssim \int_0^{|I_n|/2} \Omega(t)\frac{\rho_{\mu,E}(t)}{t}dt.\end{aligned}$$

Summing over all I_n, we get

$$\int_{\mathbb{T}} \Omega(\mathrm{dist}(\zeta, E))d\mu(\zeta) \lesssim \int_0^1 \Omega(t)\frac{\rho_{\mu,E}(t)}{t}N_E(t)dt,$$

and the proof is complete. □

Lemma 3.4. *Let E be a closed subset of $\mathbb{T}$. Let w be a convex decreasing function and let $\Omega(\zeta) = w(d(\zeta, E))$. Then*

$$\mathcal{D}_\mu(\Omega) \lesssim \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3,$$

where

$$\mathcal{I}_1 := \int_0^\pi \int_0^\pi \frac{(w(t)-w(t+s))^2}{s^2}\frac{\rho_{\mu,E}(t)}{t}N_E(t)dsdt,$$
$$\mathcal{I}_2 = \int_0^\pi w'(t)^2\rho_{\mu,E}(2t)N_E(t)dt,$$
$$\mathcal{I}_3 := \int_{t=0}^\pi \int_{s=t}^\pi \frac{(w(t)-w(t+s))^2}{s^2}\frac{\rho_{\mu,E}(s)}{s}N_E(t)dsdt,$$

Proof. Set $\delta = d(\zeta, E)$ and $\delta' = d(\zeta', E)$. By Lemma 3.3 we have

$$\begin{aligned}
\mathcal{J}_1 &= \frac{1}{2\pi}\int_{\mathbb{T}}\int_{\delta'\le\delta}\frac{(w(\delta)-w(\delta'))^2}{|\zeta-\zeta'|^2}|d\zeta|d\mu(\zeta') \\
&\lesssim \int_{\mathbb{T}}\int_{\delta'\le\delta}\frac{(w(\delta')-w(\delta'+|\zeta-\zeta'|))^2}{|\zeta-\zeta'|^2}|d\zeta|d\mu(\zeta') \\
&\lesssim \int_{\mathbb{T}}\int_{s=0}^\pi \frac{(w(\delta')-w(\delta'+s))^2}{s^2}dsd\mu(\zeta') \\
&\lesssim \int_0^\pi\int_0^\pi \frac{(w(t)-w(t+s))^2}{s^2}\frac{\rho_{\mu,E}(t)}{t}N_E(t)dsdt,
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{J}_2 &= \frac{1}{2\pi}\int_{\mathbb{T}}\int_{\delta\le\delta'}\frac{(w(\delta)-w(\delta'))^2}{|\zeta-\zeta'|^2}d\mu(\zeta')|d\zeta| \\
&\lesssim \int_{\mathbb{T}}\int_{\mathbb{T}}\frac{(w(\delta)-w(\delta+|\zeta-\zeta'|))^2}{|\zeta-\zeta'|^2}d\mu(\zeta')|d\zeta| \\
&\lesssim \int_{\zeta\in\mathbb{T}}\int_{|\zeta'-\zeta|\le\delta} + \int_{\zeta\in\mathbb{T}}\int_{|\zeta'-\zeta|\ge\delta} = \mathcal{J}_{21}+\mathcal{J}_{22}.
\end{aligned}$$

Clearly we have

$$\mathcal{J}_{21} \lesssim \int_{\mathbb{T}} w'(\delta)\rho_{\mu,E}(2\delta)|d\zeta| \lesssim \int_0^\pi w'(t)^2\rho_{\mu,E}(2t)N_E(t)dt.$$

With the same calculation, as in Lemma 3.3, we have

$$\mathcal{J}_{22} \lesssim \int_{t=0}^\pi\int_{s=t}^\pi \frac{(w(t)-w(t+s))^2}{s^2}\frac{\rho_{\mu,E}(s)}{s}N_E(t)dsdt.$$

Since $\mathcal{D}_\mu(w) = \mathcal{J}_1 + \mathcal{J}_2$, we get our result. □

For a positive increasing function ψ such that $\psi(0) = 0$, we set

$$M_{\psi,E}(s) = \max\Big(\int_0^s \frac{\psi(t)}{t}N_E(t)dt,\ \frac{\psi(s)}{s}|E_s|\Big), \tag{8}$$

where $E_t = \{\zeta \in E : \ \text{dist}(\zeta, E) \leq t\}$. If ψ is concave, then $\psi(x)/x$ is decreasing and

$$\frac{\psi(s)}{s}|E_s| \leq \frac{\psi(s)}{s}\int_0^s N_E(t)dt \leq \int_0^s \frac{\psi(t)}{t}N_E(t)dt = M_{\psi,E}(s).$$

And if ψ is convex then $\psi(x)/x$ is increasing, so

$$\int_0^s \frac{\psi(t)}{t}N_E(t)dt \leq \frac{\psi(s)}{s}\int_0^s N_E(t)dt = \frac{\psi(s)}{s}|E_s| = M_{\psi,E}(s).$$

The function ψ is called α–admissible if ψ is concave or convex and $\psi(s)/s^\alpha$ is decreasing for some $\alpha > 0$. Now we can state the main result of this subsection

Theorem 3.5. *Let E be a closed subset of $\mathbb{T}$. Let w be a convex decreasing function and let $\Omega(\zeta) = w(d(\zeta, E))$. Suppose that there exists an α–admissible function ψ, with $\alpha < 2$, such that $\rho_{\mu,E}(s) \leq \psi(s)$. Then*

$$\mathcal{D}_\mu(\Omega) \leq C(\alpha)\|w' M_{\psi,E}\|_\infty \|w\|_\infty.$$

Proof. We apply lemma 3.4. An analogue calculation, as in the proof of Theorem 3.2, gives

$$\mathcal{I}_1 + \mathcal{I}_2 \lesssim \|w' M_{\rho_{\mu,E},E}\|_\infty \|w\|_\infty.$$

Now we consider the integral $\mathcal{I}_3$. We have

$$\begin{aligned}
\mathcal{I}_3 &= 2\int_{t=0}^{\pi}\int_{s=t}^{\pi}\int_{u=t}^{t+s}\int_{v=t}^{t+s} w'(u)w'(v)\frac{\rho_{\mu,E}(s)}{s^3}N_E(t)dvdudsdt \\
&\lesssim \int_{t=0}^{\pi}\int_{s=t}^{\pi}\int_{u=t}^{2s}\int_{v=t}^{2s} |w'(u)||w'(v)|\frac{\rho_{\mu,E}(s)}{s^3}N_E(t)dvdudsdt \\
&\lesssim \int_{v=0}^{\pi}|w'(v)|\frac{\psi(v)}{v^2}\int_{u=0}^{v}|w'(u)||E_u|dudv \\
&\lesssim \int_{v=0}^{\pi}|w'(v)|\frac{1}{v^{2-\alpha}}\int_{u=0}^{v}|w'(u)|\frac{\psi(u)}{u^\alpha}|E_u|dudv \\
&\leq C(\alpha)\sup_u\left(|w'(u)|\frac{\psi(u)}{u}|E_u|\right)\int_{v=0}^{\pi}|w'(v)|\frac{1}{v^{2-\alpha}}v^{2-\alpha}dv \\
&\leq C(\alpha)\sup_u\left(|w'(u)|\frac{\psi(u)}{u}|E_u|\right)\|w\|_\infty.
\end{aligned}$$

□

Corollary 3.6. *Let E be a closed subset of $\mathbb{T}$. Let w be a convex decreasing function and let $\Omega(\zeta) = w(d(\zeta, E))$. Suppose that $\rho_{\mu,E}(t) = O(t^\alpha)$ for some $\alpha > 0$. Then*

(1) $\mathcal{D}_\mu(\Omega) \leq C(\alpha)\sup_{t\geq 0}\left|w'(t)\int_t^\pi t^{\alpha-1}N_E(t)dt\right|\|w\|_\infty$, *if* $0 < \alpha \leq 1$,

(2) $\mathcal{D}_\mu(\Omega) \leq C(\alpha)\sup_{t\geq 0}\left|w'(t)t^{\alpha-1}|E_t|\right|\|w\|_\infty$, *if* $1 \leq \alpha < 2$,

(3) $\mathcal{D}_\mu(\Omega) \leq C(\alpha)\sup_{t\geq 0}\left|w'(t)|\log t||E_t|\right|\|w\|_\infty$, *if* $\alpha = 2$,

(4) $\mathcal{D}_\mu(\Omega) \leq C(\alpha,h)\sup_{t\geq 0}\big|w'(t)h(t)|E_t|\big|\|w\|_\infty$, *if* $\alpha>2$, *where* h *is a positive increasing function such that* $h(0)=0$ *and* $\int_0 ds/h(s)<\infty$.

Proof. (1) and (2) are direct consequences of Theorem 3.5.
Now we prove (3). The proof of Theorem 3.5 gives

$$\mathcal{I}_3 \lesssim \int_{v=0}^{\pi} |w'(v)| \log\frac{1}{v}\int_{u=0}^{v}|w'(u)||E_u|dudv,$$

and we get our estimate.
Finally we prove (4). Since $|E_t|\to 0$, then there exists a positive increasing function h, $h(0)=0$, such that $\int_0 ds/h(s)<\infty$. Again, by the proof of Theorem 3.5 we have

$$\mathcal{I}_3 \lesssim C(\alpha)\int_{v=0}^{\pi} |w'(v)| \int_{u=0}^{v}\frac{h(u)}{h(u)}|w'(u)||E_u|dudv,$$

which gives the desired estimate. □

4. KERNEL ESTIMATE

In this section we will prove Theorem 1. The reproducing kernel k^μ of $\mathcal{D}(\mu)$ is defined by,

$$f(z)=\langle f, k^\mu(\cdot,z)\rangle, \qquad f\in\mathcal{D}(\mu), \quad z\in\mathbb{D}.$$

So,

$$k^\mu(z,z)=\sup\{|f(z)|^2 :\ f\in\mathcal{D}(\mu),\ \|f\|_\mu^2\leq 1\}. \tag{9}$$

It follows obviously that for $|z|\leq 1/2$ we have

$$k^\mu(z,z)\asymp 1+\int_0^{|z|}\frac{dr}{(1-r)P[\mu](rz/|z|)+(1-r)^2}.$$

By Littlewood–Paley identity, we have

$$\begin{aligned}\|f\|_\mu^2 &= \|f\|_{H^2}^2+\mathcal{D}_\mu(f)\\ &= |f(0)|^2+\int_{\mathbb{D}}|f'(w)|^2|[|\log|w||+P[\mu](w)]dA(w)\\ &\asymp |f(0)|^2+\int_{\mathbb{D}}|f'(w)|^2|[(1-|w|)+P[\mu](w)]dA(w).\end{aligned}$$

Let $f\in\mathcal{D}(\mu)\backslash\{0\}$ and let $f=I_fO_f$ be the inner–outer factorization of f. Then $O_f\in\mathcal{D}(\mu)$ and $\mathcal{D}_\mu(O_f)\leq\mathcal{D}_\mu(f)$ (see [13]). Thus by (9), we get

$$k^\mu(z,z)=\sup\{|f(z)|^2 :\ f\in\mathcal{D}(\mu)\text{ outer function and }\|f\|_\mu\leq 1\}. \tag{10}$$

This observation will be useful in the proof of the lower estimate.

4.1. **Proof of the upper estimate.** Let $z=\rho\in[1/2,1)$. By Lemma 2.3, it suffices to prove that

$$k^\mu(\rho,\rho)\lesssim 1+\int_{1-\rho}^1\frac{dx}{F_\mu(x)+x^2}.$$

Let $f\in\mathcal{D}(\mu)$, since $f\in H^2$,

$$|f(iy)|\le\frac{\|f\|_{H^2}}{\sqrt{1-y}}\le\sqrt{2}\|f\|_{H^2},\qquad 0<y<1/2.$$

So, for $0<y<1/2$, we have

$$|f(\rho+i(1-\rho)y)|=\Big|f(iy)+\int_0^\rho f'(t+i(1-t)y)dt\Big|\lesssim\int_0^\rho|f'(t+i(1-t)y)|dt+\|f\|_{H^2}.$$

Let Δ be the triangle with vertices $-i/2$, 1, $i/2$ and let $\Delta_\rho=\{x+iy\in\Delta:\,0\le x\le\rho\}$. By change of variables $w=u+iv=t+i(1-t)y$, we get

$$\begin{aligned}
\frac{1}{1-\rho}\int_{-(1-\rho)/2}^{(1-\rho)/2}|f(\rho+i\eta)|d\eta &= \int_{-1/2}^{1/2}|f(\rho+i(1-\rho)y)|dy\\
&\lesssim \int_0^\rho\int_{-1/2}^{1/2}|f'(t+i(1-t)y)|dydt+\|f\|_{H^2}\\
&\le \int_{\Delta_\rho}|f'(w)|\frac{dudv}{1-u}+\|f\|_{H^2}\\
&\lesssim \mathcal{D}_\mu(f)^{1/2}\Big[\int_{\Delta_\rho}\frac{dA(w)}{(1-u)^2((1-|w|)+P[\mu](w))}\Big]^{1/2}+\|f\|_{H^2}\\
&\lesssim \mathcal{D}_\mu(f)^{1/2}\Big[\int_0^\rho\frac{du}{(1-u)^2+(1-u)P[\mu](u)}\Big]^{1/2}+\|f\|_{H^2}\\
&\lesssim \mathcal{D}_\mu(f)^{1/2}\Big[\int_{1-\rho}^1\frac{dx}{F_\mu(x)+x^2}\Big]^{1/2}+\|f\|_{H^2}. \qquad (11)
\end{aligned}$$

Denote by $D(\lambda,r)$ the disc of radius r centered at λ. Since

$$D(\rho,(1-\rho)/4)\subset\{z=x+iy:\,|x-\rho|\le(1-\rho)/4\text{ and }|y|\le(1-x)/4\},$$

by (11) and the subharmonicity of $|f|$ we obtain

$$\begin{aligned}
|f(\rho)| &\lesssim \frac{1}{(1-\rho)^2}\int_{x=\rho-\frac{1-\rho}{4}}^{\rho+\frac{1-\rho}{4}}\Big(\int_{y=-\frac{1-x}{2}}^{\frac{1-x}{2}}|f(x+iy)dy\Big)dx\\
&\lesssim \mathcal{D}_\mu(f)^{1/2}\Big[\int_{\frac{5}{4}(1-\rho)}^1\frac{dx}{F_\mu(x)+x^2}\Big]^{1/2}+\|f\|_{H^2} \qquad (12)
\end{aligned}$$

Now from (9), we get

$$k^\mu(\rho,\rho)\lesssim 1+\int_{1-\rho}^1\frac{dx}{F_\mu(x)+x^2}\asymp 1+\int_0^{|z|}\frac{dr}{(1-r)P[\mu](rz/|z|)+(1-r)^2}.$$

4.2. **Lower estimate.**

4.2.1. *Weighted Sobolev spaces.* In this subsection we introduce weighted Sobolev spaces which will be used in the proof of lower estimate of the norm of the kernel of $\mathcal{D}(\mu)$ and in the proof of Theorem 5.4. Let φ be a nondecreasing continuous function. The Sobolev space associated with φ, $W^\infty(\varphi)$, consists of real continuous functions f on $]0, 2\pi]$ given by

$$f(x) = f(2\pi) + \int_x^{2\pi} g(s)ds, \qquad 0 < x \leq 2\pi, \tag{13}$$

where g is a measurable function satisfying

$$||g\varphi||_\infty = \sup_{0<x} |g(x)|\varphi(x) < \infty.$$

As usual g will be denoted by f'. Equipped with the following norm

$$\|f\|_{W^\infty(\varphi)} = \|f'\varphi\|_\infty,$$

$W^\infty(\varphi)$ is a Banach space. It becomes a topological Banach algebra, if and only if,

$$\int_0 \frac{dx}{\varphi(x)} < \infty.$$

We say that f is regular, and write $f \in \mathcal{R}$, if f is a C^1 convex decreasing function on $[0, 2\pi]$, satisfying $f(2t) \leq 2f(t)$ and $t^2|f'(t)|$ is increasing.

Our goal is to estimate

$$\gamma_\varphi(a) = \sup\big\{f^2(a) : f \in W^\infty(\varphi) \cap \mathcal{R},\ \|f\|_{W^\infty(\varphi)}\|f\|_\infty + \|f\|_2^2 \leq 1\big\}.$$

First we will examine the hilbertian case

$$W^2(\varphi) = \Big\{f \text{ of the form (13)} \ : \|f\|^2_{W^2(\varphi)} = \|f\|_2^2 + \int_0^{2\pi} |f'(t)|^2\varphi(t)dt < \infty\Big\}.$$

We also need the following subspace of $W^2(\varphi)$

$$W_0^2(\varphi) = \Big\{f \text{ of the form (13)} \ : f(2\pi) = 0, \text{ and } \|f\|^2_{W_0^2(\varphi)} = \int_0^{2\pi} |f'(t)|^2\varphi(t)dt < \infty\Big\}.$$

Clearly, evaluations at points of $]0, 2\pi]$ define continuous linear functionals on $W^2(\varphi)$ and on $W_0^2(\varphi)$. Let K_φ, L_φ be the reproducing kernels of $W^2(\varphi)$ and of $W_0^2(\varphi)$ respectively. A simple computation gives the following expression of the reproducing kernel L_φ of $W_0^2(\varphi)$. We have

$$L_\varphi(t,s) = \begin{cases} \displaystyle\int_t^{2\pi} \frac{dx}{\varphi(x)} & t \geq s, \\ \\ L_\varphi(s,s) & t \leq s. \end{cases}$$

The estimates of K_φ, on the diagonal is given by

$$K_\varphi(a,a) \asymp 1 + L_\varphi(a,a) = 1 + \int_a^{2\pi} \frac{dx}{\varphi(x)}.$$

It means that

$$\sup\left\{f(a)^2 : f \in W^2(\varphi)\ ,\ \|f\|^2_{W^2(\varphi)} \le 1\right\} \asymp 1 + \int_a^{2\pi} \frac{dx}{\varphi(x)}.$$

We have the following proposition.

Proposition 4.1. *Suppose that $t^2/\varphi(t)$ is increasing and $\varphi(t) \ge t^2$. Let $a < 1/2$, we have*

$$\gamma_\varphi(a) \asymp K_\varphi(a,a).$$

Proof. Since $\|f\|^2_{W^2(\varphi)} \le \|f\|_{W^\infty(\varphi)}\|f\|_\infty$, then $\gamma_\varphi(a) \le K_\varphi(a,a)$.
Conversely, let $f = f_0 + 1$, where

$$f_0(x) = \int_{2\pi\frac{x+a}{2\pi+a}}^{2\pi} \frac{ds}{\varphi(s)}, \qquad 0 < x \le 2\pi.$$

Clearly $f \in \mathcal{R}$. Since $\varphi(2t) \asymp \varphi(t)$, we have

$$\begin{aligned}
\|f_0\|_2^2 &= \int_0^{2\pi} f_0(t)^2 dt \\
&\lesssim \int_0^{\pi}\int_{u=t+a}^{2\pi}\int_{v=t+a}^{2\pi} \frac{1}{\varphi(u)\varphi(v)} dudvdt \\
&\lesssim \int_{t=0}^{a}\int_{u=a}^{2\pi}\int_{v=a}^{2\pi} \frac{1}{\varphi(u)\varphi(v)} dudvdt + \int_{t=a}^{\pi}\int_{u=t}^{2\pi}\int_{v=t}^{2\pi} \frac{1}{\varphi(u)\varphi(v)} dudvdt \\
&\lesssim \int_{u=a}^{2\pi}\int_{v=a}^{2\pi} \frac{a}{\varphi(u)v^2} dudv + \int_{u=a}^{2\pi}\int_{v=u}^{2\pi} \frac{u}{\varphi(u)v^2} dudv \\
&\lesssim \|f_0\|_\infty.
\end{aligned}$$

Then we obtain

$$K_\varphi(a,a) \lesssim \frac{f(a)^2}{\|f\|_{W^\infty(\varphi)}\|f\|_\infty + \|f\|_2^2} \le \gamma_\varphi(a).$$

And the proof is complete. □

4.2.2. *Proof of the lower estimates.* Let $z = r \in]1/2, 1[$ and let $w \in \mathcal{R}$. We consider the outer function f_w given by

$$|f_w(e^{it})| = w(|t|), \qquad \text{a.e on } [-\pi, \pi].$$

Let $\varphi(x) = F_\mu(x) + x^2$. By (10), Proposition 2.2 and Theorem 3.2 we have

$$\begin{aligned} k^\mu(r,r) &\gtrsim \frac{|f_w(r)|^2}{\|f_w\|^2_{\mathcal{D}(\mu)}} \\ &\gtrsim \frac{w^2(1-r)}{\|w\|_{W^\infty(F_\mu)}\|w\|_\infty + \|w\|_2^2} \\ &\gtrsim \frac{w^2(1-r)}{\|w\|_{W^\infty(\varphi)}\|w\|_\infty + \|w\|_2^2}. \end{aligned}$$

Thus

$$k^\mu(r,r) \gtrsim \gamma_\varphi(1-r).$$

By Proposition 4.1, we obtain the result.

5. Capacity

Let μ be a positive finite measure on $\mathbb{T}$ and let c_μ be the associated capacity given by (1). The capacity c_μ is a Choquet capacity [4, 9] and so for every borelian set of $\mathbb{T}$ we have

$$c_\mu(E) = \sup\{c_\mu(K) : K \text{ compact }, K \subset E\}.$$

Note that c_μ satisfies a weak-type inequality. Namely:

$$c_\mu(\{\zeta \in \mathbb{T} : |f(\zeta)| \geq t \quad c_\mu\text{-q.e.}\}) \leq \frac{\|f\|^2_\mu}{t^2}, \qquad f \in \mathcal{D}^h(\mu).$$

As consequence of this inequality we have the following properties.

Proposition 5.1. *The following properties are satisfied*

- *Let $E \subset \mathbb{T}$ be a Borel set and let $\mathcal{M}_\mu(E) := \{f \in \mathcal{D}(\mu) : g|E = 0 \quad c_\mu\text{-q.e}\}$. Then the set $\mathcal{M}_\mu(E)$ is closed in $\mathcal{D}(\mu)$.*
- *If $f \in \mathcal{D}(\mu)$ is cyclic for $\mathcal{D}(\mu)$ then f is outer function and $c_\mu(Z_\mathbb{T}(f)) = 0$.*
- *Every function $f \in \mathcal{D}(\mu)$ has non-tangential limits c_μ-q.e on $\mathbb{T}$, more precisely The radial limit $\lim_{r\to 1-} f(r\zeta)$ exists and is finite for every $f \in \mathcal{D}^h(\mu)$ if and only if $c_\mu(\zeta) > 0$.*

Proof. See [5, 9]. □

Now we will give the proof of the estimate of the capacity of arcs.

Proof Theorem 2. Suppose that $\zeta = 1$. Let $w \in \mathcal{R}$ and let f_w be the outer function satisfying

$$|f_w(e^{it})| = w(|t|), \qquad \text{a.e on } [-\pi, \pi].$$

It's clear that $w(|I|) \lesssim w(x)$ for $|x| \leq 2|I|$. We have

$$\gamma_\varphi(|I|) \lesssim 1/c_\mu(I).$$

By proposition 4.1, we obtain

$$c_\mu(I) \lesssim 1/k^\mu(\rho, \rho), \quad (\rho = 1 - |I|).$$

For the reverse inequality note that

$$c_\mu(I) = \inf\{\|f\|_\mu^2 : f \in C^1, 0 \leq f \leq 1 \text{ and } f = 1 \text{ on } I\}.$$

Consider the function $u \in \mathcal{D}^h(\mu)$ such that $0 \leq u \leq 1$ and $u_{|I} = 1$. Hence $u \in \mathcal{D}^h(\mu) \cap C^1$. We have $P[u](1 - |I|) \asymp 1$. Let $\rho = 1 - |I|$, a similar argument, as in the proof of (12), gives

$$P[u](\rho) \lesssim \mathcal{D}_\mu(u)^{1/2} \sqrt{k^\mu(\rho, \rho)} + \|u\|_{L^2(\mathbb{T})}.$$

So $k^\mu(\rho, \rho)^{-1/2} \lesssim \mathcal{D}_\mu(u)^{1/2}$, and

$$c_\mu(I) \geq \frac{1}{k^\mu(\rho, \rho)}.$$

□

As an immediate consequence, we obtain

Corollary 5.2. *Let $\lambda \in \mathbb{T}$.*

$$c_\mu(\{\lambda\}) = 0 \iff \int_0^1 \frac{dx}{(1-x)P[\mu](x\lambda) + (1-x)^2} = \infty.$$

Now we will give some sufficient conditions on a closed subset of $\mathbb{T}$ to be μ–polar. Let E be a closed subset of $\mathbb{T}$. We define $n_E(\varepsilon)$, the *ε–covering number of E*, to be the smallest number of the closed arcs of length 2ε that cover E. Note that

$$\varepsilon n_E(\varepsilon) \leq |E_\varepsilon| \leq 4\varepsilon n_E(\varepsilon), \qquad 0 < \varepsilon \leq \pi.$$

Let

$$\kappa_\mu(r) = \inf\{k^\mu(r\zeta, r\zeta) : \zeta \in \operatorname{supp} \mu\}.$$

It 's easy to see that κ_μ is unbounded if and only if, for each $\zeta \in \mathbb{T}$, we have $c_\mu(\zeta) = 0$. In this case one can prove easily, by the sub-additivity property of capacity, that

Corollary 5.3. *Let E be a closed subset of $\mathbb{T}$ such that $\lim_{r \to 1^-} \kappa_\mu(r) = \infty$. If*

$$n_E(\varepsilon) = o(\kappa_\mu(1 - \varepsilon)), \qquad \varepsilon \to 0,$$

then $c_\mu(E) = 0$.

Recall that for a positive increasing function ψ on $(0, 2\pi)$ such that $\psi(0) = 0$, we set

$$M_{\psi,E}(s) = \max\Big(\int_0^s \frac{\psi(t)}{t} N_E(t)dt,\ \frac{\psi(s)}{s}|E_s|\Big), \qquad s \in (0, 2\pi).$$

Now we can state the main result of this section.

Theorem 5.4. *Let E be closed subset of $\mathbb{T}$ such that $\rho_{\mu,E} \leq \psi$ where ψ is α–admissible for some $\alpha < 2$. If*

$$\int_0^\pi \frac{dt}{M_{\psi,E}(t)} = +\infty,$$

then $c_\mu(E) = 0$.

Proof. Note that $M_{\psi,E}$ is given by (5), so $M_{\psi,E}(s)$ is increasing. Let $a > 0$. By the definition of capacity and using Theorem 3.5 and Proposition 4.1 we have

$$K_{M_{\psi,E}}(a, a) \asymp \gamma^2_{M_{\psi,E}}(a) \lesssim \frac{1}{c_\mu(E)}.$$

When a goes to zero, we get

$$\int_0^\pi \frac{dt}{M_{\psi,E}(t)} \lesssim \frac{1}{c_\mu(E)}.$$

And the proof is complete. □

Remarks. Now we give some examples:

(i) Let E be a closed subset of $\mathbb{T}$ and let $\widetilde{M}_{\mu,E}(t) = \int_0^t (\rho_\mu(s)N_E(s)/s)ds$. If

$$\int_0^\pi \frac{dt}{\widetilde{M}_{\mu,E}(t)} = \infty,$$

then $c_\mu(E) = 0$.

(ii) If $\mu = m$ is the Lebesgue measure, then c_μ is comparable to the logarithmic capacity and $\widetilde{M}_{\mu,E}(s) = |E_s|$. Theorem 5.4 says that if

$$\int_0 \frac{dt}{|E_t|} = +\infty,$$

then $c(E) = 0$. This result is du to Carleson [4, Theorem 2, p.30].

(iii) Let K be a closed subset of $\mathbb{T}$ such that $\rho_{\mu,K}(s) = O(s^{1+\beta})$ for some $0 < \beta$. If $\beta < 1$, then every subset of K with Hausdorff dimension less than β is μ-polar. If $\beta > 1$, then every subset E of K with $|E| = 0$ is μ-polar. The following measures $d\mu(\zeta) = d(\zeta, K)^\beta dm(\zeta)$ provide such examples.

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