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Luck Darnière

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# Cell decomposition and dimension theory on $p$ -optimal fields

Luck Darnière

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## Abstract

We prove that for  $p$ -optimal fields (a very large subclass of  $p$ -minimal fields containing all the known examples) a cell decomposition theorem follows from methods going back to Denef’s paper [Den84]. We derive from it the existence of definable Skolem functions and strong  $p$ -minimality, thus providing a new and simpler proof of the main result of [vdDHM99]. The topological dimension introduced in [HM97] for strongly  $p$ -minimal fields is shown to be a “dimension function” in the sense of [vdD89]. In particular it coincides in  $p$ -optimal fields with the topological rank. Finally we obtain for the restricted class of  $p^2$ -optimal fields a preparation theorem for definable functions, from which it follows that infinite sets definable over such fields are isomorphic iff they have the same dimension.

## 1 Introduction

This paper is an attempt to continue the road opened by Haskell and Macpherson in [HM97] toward a  $p$ -adic version of  $o$ -minimality, by isolating large subclasses of  $p$ -minimal fields to which Denef’s methods of [Den84] apply with striking efficiency.

Recall that a  **$p$ -adically closed** field is a field  $K$  elementarily equivalent to a finite extension of the field  $\mathbf{Q}_p$  of  $p$ -adic numbers. An expansion of  $K$  (that is an  $\mathcal{L}$ -structure  $\mathcal{K}$  extending the ring structure of  $K$  for some language  $\mathcal{L}$  containing the language of rings) is  **$p$ -minimal** if every definable subset of the affine line  $K$  is semi-algebraic. By “definable” we always mean definable in the language  $\mathcal{L}$  with parameters from  $K$ . For sets and functions definable (with parameters as always) in the language of rings, we use the locution “semi-algebraic” instead. Abusing the notation we will identify  $K$  and  $\mathcal{K}$ , and call  $K$  itself a  $p$ -minimal field.  $K$  is **strongly  $p$ -minimal** ( $P$ -minimal in [HM97]) if every elementarily equivalent  $\mathcal{L}$ -structure is  $p$ -minimal.

Strong  $p$ -minimality was introduced by Haskell and MacPherson in [HM97]. Among other things they built a reasonably good dimension theory for definable sets over strongly  $p$ -minimal fields. They also left open several questions, such as the existence of a cell decomposition, or if their dimension was a “dimension function” in the sense of [vdD89]. Since their proofs make extensive use of the model-theoretic compactness theorem, very little is known on  $p$ -minimal fields without the “strong” assumption. In particular it is not known if every  $p$ -minimal field is strongly  $p$ -minimal.

Mourgues proved in [Mou09] that a cell decomposition similar to the one of [Den84] holds for a strongly  $p$ -minimal field  $K$  if and only if it has **definable Skolem functions** (“definable selection” in [Mou09]), that is if for every positive integers  $m, n$  and every definable subset  $S$  of  $K^{m+n}$  the coordinate projection of  $S$  onto  $K^m$  has a definable section. It is not known at the moment if strongly  $p$ -minimal fields always have definable Skolem functions.

As Cluckers noted in [Clu04], it was lacking in [Mou09] a preparation theorem for definable functions. He filled this lacuna for the classical analytic structure on  $K$  (see below), and derived from his preparation theorem several important applications, for parametric integrals and classification of subanalytic sets up to isomorphism.

In this paper we consider natural restrictions of  $p$ -minimality defined by mean of  $(m, d)$ -**basic functions**, that is  $m$ -ary functions which are polynomial in their last  $d$  variables with as coefficients global definable functions in the other variables. Thus  $(m, m)$ -basic functions are just ordinary polynomial functions in  $m$  variable with coefficients in  $K$ . An  $(m, d)$ -**basic set** of order  $N$  is a set of the form

$$S = \{x \in K^m : f(x) \in P_N\}$$

with  $f$  a  $(m, d)$ -basic function,  $N \geq 1$  an integer and

$$P_N = \{x \in K : \exists y \in K, x = y^N\}.$$

When  $m$  is clear from the context, we will simply talk of  $d$ -basic functions and sets.

**Remark 1.1** By the argument of lemma 2.1 in [Den84], the following subsets of  $K^m$  are  $d$ -basic, for every  $d$ -basic  $m$ -ary functions  $f, g$ .

$$\{x \in K^m : f(x) = 0\} \quad \text{and} \quad \{x \in K^m : |g(x)| \leq |f(x)|\}$$

Moreover, since  $P_N^\times = P_N \setminus \{0\}$  is a subgroup of finite index in  $K^\times$ , the complement in  $K^m$  of a  $d$ -basic set is then a finite union of  $d$ -basic sets. Hence every (finite) boolean combination of basic sets is the union of intersections of finitely many basic sets. All of them can be taken the same larger order, because  $P_{N'}^\times$  is a subgroup of  $P_N^\times$  of finite index for every  $N'$  which is divisible by  $N$ .

We say that  $K$  is  **$p$ -optimal** if every definable subset of  $K^m$  (for every  $m$ ) is a boolean combination of  $(m, 1)$ -basic sets. This is a special case of  $p^d$ -optimality, which will be defined in section 2 by the condition that certain definable subsets of  $K^m \times K^{d-1}$  (for every  $m$  and fixed  $d \geq 1$ ) are boolean combinations of  $(m, d)$ -basic sets. In this paper we will concentrate on  $p^2$  and  $p$ -optimal fields.

**(Strong)  $p$ -minimality versus  $p$ -optimality.** Note that  $p$ -optimal fields are *not* supposed to be strongly  $p$ -minimal. They are  $p$ -minimal because 1-basic subsets of the affine line  $K$  are semi-algebraic. Moreover it is difficult to imagine any proof of  $p$ -minimality which does not involve in a way or another a quantifier elimination result similar to Macintyre’s theorem 2.1. The condition defining  $p$ -optimality is actually very close to such kind of elimination. So close that we can expect it to be proved simultaneously in most cases, if not all, without additional effort.

This is indeed what happens in the subanalytic case (see below) as well as every known example such as the non-standard analytic structure on  $\mathbf{Q}_p((t^{\mathbf{Q}}))$  studied in [Ble10], or the expansions of  $\mathbf{Q}_p$  with Weierstrass systems of [Mar08]: all of them are indeed examples of  $p$ -optimal fields. Although not surprising, it is then quite remarkable that every  $p$ -optimal field is strongly  $p$ -minimal. More precisely (theorem 4.2):

**Theorem 1.2** *For every expansion of a  $p$ -adically closed field  $F$ , the following are equivalent:*

1.  $F$  is  $p$ -optimal.
2. Denef's cell decomposition theorem 3.6 holds in  $F$ .
3.  $F$  is strongly  $p$ -minimal and has definable Skolem function.

Of course (3) $\Rightarrow$ (2) follows from Mourgues [Mou09] (not the other implications, since Mourgues considers only strongly  $p$ -minimal fields). The most interesting part of theorem 1.2 is perhaps (1) $\Rightarrow$ (3). Let us see how it applies to a fundamental example.

**Application to the subanalytic case.** In the classical analytic structure, initially introduced on  $\mathbf{Z}_p$  by Denef and van den Dries in [DvdD88] (see definition 1.3 and further in [Clu04] for its adaptation to the field case)  $p$ -minimality was derived from the quantifier elimination theorem 1.1 in [DvdD88], the proof of which is based on the Weierstrass preparation and division theorem for analytic functions. Strong  $p$ -minimality was proved later in [vdDHM99] by mean of an intricate parametric version of this same Weierstrass division. But a detailed study of the original proof of theorem 1.1 in [DvdD88] shows that it directly proves (a very strong form of)  $p$ -optimality for the classical analytic structure. Thus our result, that  $p$ -optimal fields are strongly  $p$ -minimal, applies to this structure and gives an alternate proof of the main result of [vdDHM99], both simpler and much more general.

**Question 1.3** Does there exist a (strongly)  $p$ -minimal expansion of a  $p$ -adically closed field which is not  $p$ -optimal?

**Main other results.** In the remaining of this paper  $K$  will always denote a  $p$ -optimal field, and  $F$  an expansion of a  $p$ -adically closed field. However in the last section 8 we will restrict to  $p^2$ -optimal fields, in order to generalize to them all the results of [Clu04]. Indeed we will show the cell preparation theorem for subanalytic functions of [Clu04] applies to every definable function in a  $p^2$ -optimal field (theorem 8.4) from which the next result follows (theorem 8.6):

**Theorem 1.4** *Two infinite sets definable over a  $p^2$ -optimal field are isomorphic if and only if they have the same dimension.*

In the mean time, sections 5 to 7 will be devoted to various improvement of the dimension theory of [HM97] over  $p$ -optimal fields. In particular we prove that in this context their dimension is indeed a “dimension function” (theorem 6.3) and derive from it (theorem 7.1):

**Theorem 1.5** *For every definable subset  $A$  of  $K^m$ ,  $\dim \bar{A} \setminus A < \dim A$ .*

Topological notions such as continuity, interior, closure and so on refer to the topology of the  $p$ -valuation. For  $A \subseteq K^m$  we let  $\bar{A}$  denote the topological closure of  $A$ . Expression such as “open in  $A$ ” or “neighbourhood in  $A$ ” always refer to the topology induced on  $A$  by  $K^m$ , whose open sets are the sets  $U \cap A$  with  $U$  an open subset of  $K^m$ . In general the sets open in  $A$  are not open in  $K^m$ , unless  $A$  itself is open in  $K^m$ . The following results on relative topology (theorem 5.2, proposition 5.1 and corollary 7.5) extend theorems 3.2 and 5.4 of [HM97].

**Theorem 1.6** *Let  $A_1, \dots, A_r \subseteq A$  be a family of definable subsets of  $K^m$ . If the union of the  $A_k$ 's has non empty interior in  $A$  then at least one of them has non empty interior in  $A$ .*

**Theorem 1.7** *For every definable function  $f : X \subseteq K^m \rightarrow K^n$  let  $\mathcal{C}(f)$  denote the set of points  $x$  in  $X$  such that  $f$  is continuous on a neighbourhood of  $x$  in  $X$ . Then  $\mathcal{C}(f)$  is dense in  $X$  and  $\dim X \setminus \mathcal{C}(f) < \dim X$ .*

**Remark 1.8** If  $X = K^m$  proposition 5.1 follows from Theorem 5.4 in [HM97]. Similarly if  $A = K^m$  theorem 5.2 follows from Theorem 3.2 in [HM97]. However the present “relative” cases do seem to us follow easily from [HM97]. In particular it is worth mentioning that Remark 5.5 in [HM97] which outlines an argument proving theorem 1.7 in strongly  $p$ -minimal fields is probably misleading. By mean of Lemma 7.1 in [Den84] it seems to prove it only piecewise: there is a finite partition of  $X$  in definable pieces  $A$  on each of which the restriction  $f|_A$  of  $f$  satisfies  $\dim A \setminus \mathcal{C}(f|_A) < \dim A$ . But in the lack of theorem 1.5 below one can not ensure that the pieces  $A$  of dimension  $\dim X$  are open in  $X$ , hence that  $\mathcal{C}(f)$  contains  $\mathcal{C}(f|_A)$ . Thus theorem 1.7 does not follow from this argument, and Theorem 5.4 of [HM97] (which asserts that  $\dim X \setminus \mathcal{C}(f) < m$ ) seems to be the best we can say at the moment for strongly  $p$ -minimal fields

**Acknowledgement.** This paper is based on [HM97] and [Den84], with which the reader is expected to be familiar. However I borrowed ideas from papers of many other authors, specially Lou van den Dries in [vdD84] and [vdD98] and Raf Cluckers in [Clu04]. The concept of  $p$ -optimal field seems to be new but appears implicitly in many papers on  $p$ -adic fields, specially [Den86] which has been a source of inspiration for me. I would like to thank Raf Cluckers for pointing me a nasty error in the previous version of this paper.

**Terminology and notation.** For every  $a$  in  $K$ ,  $v(a)$  and  $|a|$  denote the  $p$ -valuation of  $a$  and its norm. The norm is nothing but the valuation, but with a multiplicative notation so that  $|0| = 0$ ,  $|ab| = |a| \cdot |b|$ ,  $|a+b| \leq \max(|a|, |b|)$  and of course  $|a| \leq |b|$  if and only if  $v(a) \geq v(b)$ . The valuation ring of  $v$  is  $R$ , and we fix some  $\pi$  in  $R$  such that  $\pi R$  is the maximal ideal of  $R$ .

For convenience we will sometimes add to  $K$  one more element  $\infty$ , with the property that  $|x| < |\infty|$  for every  $x$  in  $K$ . We also denote  $\infty$  any partial function with constant value  $\infty$ .

For every subset  $X$  of  $K$  we let  $X^\times = X \setminus \{0\}$ . Note the difference between  $R^\times = R \setminus \{0\}$  and  $R^* =$  the set of units in  $R$ .

Recall that  $K^0$  is a one-point set. When a tuple  $a = (x, t)$  is given in  $K^{m+1}$  it is understood that  $x = (x_1, \dots, x_m)$  and  $t$  is the last coordinate. We let  $\widehat{a} = x$  denote the projection of  $a$  onto  $K^m$ . Similarly, the projection of a subset  $S$  of  $K^{m+1}$  onto  $K^m$  is denoted  $\widehat{S}$ .

We extend  $|\cdot|$  (or  $v$ ) to  $K^m$  coordinatewise. That is, for every  $x \in K^m$  we let:

$$|(x_1, \dots, x_m)| = (|x_1|, \dots, |x_m|).$$

For every  $A \subseteq K^m$  we let  $|A|$  denote the image of  $A$  by this extension of the valuation.

Analogously to Landau's notation  $\mathcal{O}(x^n)$  of calculus, we let  $\mathcal{U}_n(x)$  denote any definable function in the multi-variable  $x$  with values in  $1 + \pi^n R$ . So, given a family of functions  $f_i, g_i$  on the same domain  $X$ , we write that  $f_i = \mathcal{U}_n g_i$  for every  $i$ , when there are definable functions  $\omega_i : X \rightarrow R$  and such that for every  $x$  in  $X$ ,  $f_i(x) = (1 + \pi^n \omega_i(x))g_i(x)$ .

If  $K^\circ$  is a finite extension of  $\mathbf{Q}_p$  to which  $K$  is elementarily equivalent as a ring, and  $R^\circ$  is the  $p$ -valuation ring of  $K^\circ$ , then the following set is semi-algebraic (see lemma 2.1, point 4, in [Den86])

$$Q_{N,M}^\circ = \{0\} \cup \bigcup_{k \in \mathbf{Z}} \pi^{kN} (1 + \pi^M R^\circ).$$

We let  $Q_{N,M}$  denote the semi-algebraic subset of  $K^\circ$  corresponding<sup>1</sup> by elementary equivalence to  $Q_{N,M}^\circ$  in  $K$ . If  $M > v(N)$  Hensel's lemma implies that  $1 + \pi^M R$  is contained in  $P_N^\times$ . Note that  $Q_{N,M}^\times$  is then a clopen subgroup of  $P_N^\times$  with finite index. The next property also follows from Hensel's lemma (see for example lemma 1 and corollary 1 in [Clu01]).

**Lemma 1.9** *The function  $x \mapsto x^e$  is a group endomorphism of  $Q_{N_0, M_0}^\times$ . If  $M_0 > v(e)$  this endomorphism is injective and its image is  $Q_{eN_0, v(e)+M_0}^\times$ .*

In particular  $x \mapsto x^N$  defines a continuous bijection from  $Q_{1, v(N)+1}$  to  $Q_{N, 2v(N)+1}$ . We let  $x \mapsto x^{\frac{1}{N}}$  denote the reverse bijection.

## 2 $p(m, d)$ -minimality

In this section  $F$  is an expansion of  $p$ -adically closed field (without any special assumption of minimality nor optimality). Recall the following celebrated result, stated for  $\mathbf{Q}_p$  by Macintyre in [Mac76]. The generalization to  $p$ -adically closed fields can be found in [PR84].

**Theorem 2.1 (Macintyre)** *The semi-algebraic subsets of  $F^m$  are exactly the boolean combination of  $(m, m)$ -basic sets.*

Let us say that  $A \subseteq F^m \times |F|^d$  is definable (resp. semi-algebraic) if  $[A]$  defined below is so

$$[A] = \{(x, y) \in F^{m+d} : (x, |y|) \in A\}.$$

<sup>1</sup>For a more intrinsic definition of  $Q_{N,M}$  inside  $K$ , see [CL12].

We say that  $F$  is  $p(m, d)$ -**minimal** if for every definable subset  $A$  of  $F^m \times |F|^d$ ,  $[A]$  is a boolean combination of  $(m + d, d + 1)$ -basic sets. Thus by Macintyre's theorem 2.1,  $F$  is  $p(1, d)$ -minimal iff every definable subset of  $K \times |K|^d$  is semi-algebraic. In particular it is  $p(1, 0)$ -minimal iff it is  $p$ -minimal. And  $F$  is  $p$ -optimal iff it is  $p(m, 0)$ -minimal for every  $m$ . If  $F$  is  $p(m, d)$ -minimal for every  $m$  and a fixed  $d$ , we say that it is  $p^{d+1}$ -**optimal**. As usual we will say that  $F$  is strongly  $p(m, d)$ -minimal if every elementarily extension of  $F$  is  $p(m, d)$ -minimal, and similarly for strongly  $p^{d+1}$ -optimal fields.

**Remark 2.2** It is an easy exercise to check that  $p(m + 1, d)$ -minimality or  $p(m, d + 1)$ -minimality both imply  $p(m, d)$ -minimal. In particular  $p^{d+1}$ -optimality implies  $p^d$ -optimality.

**Proposition 2.3** *Assume that  $F$  is strongly  $p(1, d)$ -minimal and has  $m$ -ary definable Skolem functions. Then  $F$  is  $p(m + 1, d)$ -minimal.*

Thus if  $F$  is strongly  $p(1, d)$ -minimal and has definable Skolem functions (for every  $m$ ) then it is  $p^{d+1}$ -optimal by remark 2.2. In particular, if  $F$  is strongly  $p$ -minimal with definable Skolem functions then it is  $p$ -optimal.

**Question 2.4** Is every  $p^{d+1}$ -optimal field strongly  $p(1, d)$ -minimal? Is it strongly  $p^{d+1}$ -optimal?

We will prove in section 4 that the first question has a positive answer for  $d = 1$ .

*Proof:* Let  $S$  be a definable subset of  $F^{m+1} \times |F|^d$ , and  $S'$  the corresponding definable set over an elementary extension  $F'$  of  $F$ . For every  $x'$  in  $F'^m$  let  $[S']_{x'}$  denote the fiber of  $[S']$  over  $x'$ :

$$[S']_{x'} = \{(t', y') \in F' \times F'^d : (x', t', y') \in [S']\}$$

For every  $x'$  in  $F'^m$ ,  $[S']_{x'} = [S'_{x'}]$  where  $S'_{x'}$  is the set of  $(t', y') \in F' \times |F'|^d$  such that  $(x', t', y') \in S'$ . The  $p(1, d)$ -minimality of  $F'$  then gives a tuple  $a'_{x'}$  of coefficients of a description of  $[S']_{x'}$  as a boolean combination of  $(d + 1)$ -basic subsets of  $F'^{d+1}$ . The model-theoretic compactness theorem then gives definable subsets  $S_1, \dots, S_q$  of  $F^m$  and for every  $i \leq q$  an  $\mathcal{L}$ -formula  $\varphi_i(\alpha, \tau, \psi)$  with  $n_i + 1 + d$  free variables which is a boolean combination of formulas  $f(\alpha, \tau, \psi) \in P_N$  with  $f \in \mathbf{Z}[\alpha, \tau, \psi]$ , such that for every  $x$  in  $S_i$  there is a list of coefficients  $a_x$  such that

$$[S]_x = \{(t, y) \in F \times F^d : F \models \varphi_i(a_x, t, y)\}.$$

With other words, for every  $x$  in  $S_i$

$$F \models \exists a \forall t, y ((x, t, y) \in [S] \leftrightarrow \varphi_i(a, t, y)).$$

By assumption  $F$  has  $m$ -ary definable Skolem functions, hence for each  $i \leq q$  there is a definable function  $\sigma_i : S_i \rightarrow F^{n_i}$  such that for every  $x \in S_i$

$$F \models \forall t, y [(x, t, y) \in [S] \leftrightarrow \varphi_i(\sigma(x), t, y)].$$

Let  $B_i = \{(x, t, y) \in F^{m+1+d} : F \models \varphi_i(\sigma(x), t, y)\}$ . By construction this is a boolean combination of  $(d + 1)$ -basic subsets of  $F^{m+1+d}$ . On the other hand

$S_i \times F^{d+1}$  is obviously a  $(d+1)$ -basic subset of  $F^{m+1+d}$ . Indeed, if  $c_i(x)$  denotes the indicator function of  $S_i$ , then  $h_i(x, t, y) = c_i(x) - 1$  is  $(d+1)$ -basic and we have

$$S_i \times F^{d+1} = \{(x, t, y) \in F^{m+1+d} : h_i(x, t, y) = 0\}$$

which is a  $(d+1)$ -basic set by remark 1.1. The conclusion follows, since  $S$  is the union of the sets  $B_i \cap (S_i \times F^{d+1})$ .

■

### 3 Cell decomposition

In this section  $K$  denotes a  $p$ -optimal field.

The cells which usually appear in the literature on  $p$ -adic fields are non empty subsets of  $K^{m+1}$  of the form:

$$\{(x, t) \in X \times K : |\nu(x)| \square_1 |t - c(x)| \square_2 |\mu(x)| \text{ and } t - c(x) \in \lambda G\} \quad (1)$$

where  $X \subseteq K^m$  is a definable set,  $c, \mu, \nu$  are definable functions from  $X$  to  $K$ ,  $\square_1, \square_2$  are  $\leq, <$  or no condition,  $\lambda \in K$  and  $G$  is a semi-algebraic subgroup of  $K^\times$  with finite index. In this paper we will only consider the cases when  $G$  is  $K^\times$  (theorem 3.4),  $P_N^\times$  (theorem 3.6) or  $Q_{N,M}^\times$  (theorem 8.4).

In its simplest form, Denef's cell decomposition theorem asserts that every semi-algebraic subset of  $K^m$  is the disjoint union of finitely many cells. It will be convenient to fix a few more conditions on our cells, but most of all we want to pay attention on *how the functions defining the output cells depend on the input data*.

So we define **presented cells** in  $K^{m+1}$  as tuples  $A = (c_A, \nu_A, \mu_A, \lambda_A, G_A)$  with  $c_A$  a definable function on a non-empty domain  $X \subseteq K^m$  with values in  $K$ ,  $\nu_A$  and  $\mu_A$  either definable functions on  $X$  with values in  $K^\times$  or constant functions on  $X$  with values 0 or  $\infty$ ,  $\lambda_A$  an element of  $K$  and  $G_A$  semi-algebraic subgroup of  $K^\times$  with finite index, such that for every  $x \in X$  there is  $t \in K$  such that:

$$|\nu_A(x)| \leq |t - c_A(x)| \leq |\mu_A(x)| \text{ and } t - c_A(x) \in \lambda_A G_A. \quad (2)$$

We call it a **presented cell mod**  $G$  when  $G_A = G$ . Of course the set of tuples  $(x, t) \in X \times K$  satisfying (2) is then a cell of  $K^{m+1}$  in the usual sense of (1). We call it the **underlying cellular set** of  $A$ . Abusing the notation we will also denote it  $A$  most often. The existence, for every  $x \in X$ , of  $t$  satisfying (2) simply means that  $X$  is exactly  $\widehat{A}$ . We call it the **basement** of  $A$ . The function  $c_A$  is called its **center**,  $\mu_A$  and  $\nu_A$  its **bounds**,  $G_A$  its **modulo**.

$A$  is said to be of **type 0** if  $\lambda_A = 0$ , and of type 1 otherwise. Contrary to its center, bounds, and modulo, the type of  $A$  only depends on its underlying set.

*The word "cell" will usually refer to presented cells.* However, for sake of simplicity, we will freely talk of disjoint cells, bounded cells, families of cells partitioning some set and so on, meaning that the underlying cellular sets of these (presented) cells have the corresponding properties. For instance, it is clear that every cellular set as in (1) is in that sense the disjoint union of finitely many (presented) cells mod  $G$ .



**Lemma 3.1 (Denef)** *Let  $S$  be a definable subset of  $K^{m+n}$ . Assume that there is an integer  $\alpha \geq 1$  such that for every  $x$  in  $K^m$  the fiber*

$$S_x = \{y \in K^n : (x, y) \in S\}$$

*has cardinality  $\leq \alpha$ . Then the coordinate projection of  $S$  on  $K^m$  has a definable section.*

*Proof:* Identical to the proof of lemma 7.1 in [Den84].

■

**Lemma 3.2 (Denef)** *Let  $f$  be an  $(m+1, 1)$ -basic function in  $(x, t) = (x_1, \dots, x_m, t)$ . Let  $n \geq 1$  be a fixed integer. Then there exists a partition of  $K^{m+1}$  into sets  $A$  of the form*

$$A = \bigcap_{j \in S} \bigcap_{l \in S_j} \{(x, t) \in K^{m+1} : x \in C \text{ and } |t - c_j(x)| \square_{j,l} |a_{j,l}(x)|\}$$

*where  $S$  and  $S_j$  are finite index sets,  $C$  is a definable subset of  $K^m$ , and  $c_j, a_{j,l}$  are definable functions from  $K^m$  to  $K$ , such that for all  $(x, t)$  in  $A$  we have*

$$f(x, t) = \mathcal{U}_n(x, t) h(x) \prod_{j \in S} (t - c_j(x))^{e_j}$$

*with  $h : K^m \rightarrow K$  a definable function and  $e_j \in \mathbf{N}$ .*

It is sufficient to check it for every  $n$  large enough so we can assume that:

$$1 + \pi^n R \subseteq P_N \cap R^* \tag{3}$$

Thus  $\mathcal{U}_n(x, t)$  in the conclusion could be replaced by  $u(x, t)^N$  with  $u$  a definable function from  $A$  to  $R^*$ . This is indeed how this result is stated in lemma 7.2 of [Den84]. However it is the above equivalent (but slightly more precise) form which appears in Denef's proof, and which we retain in this paper.

*Proof:* Follow the proof of lemma 7.2 of [Den84], using the  $p$ -minimality assumption and basic functions in place of Macintyre's quantifier elimination and polynomial functions. Of course, lemma 7.1 used in Denef's proof has to be replaced with the analogous lemma 3.1.

■

**Remark 3.3 (co-algebraic functions)** A remarkable by-product of Denef's proof is that the functions  $c_j$  and  $a_{j,l}$  in the conclusion of lemma 3.2 belong to  $\text{coalg}(f)$ , which we define now.

Given an  $(m+1, 1)$ -basic function  $f$ , we say that a function  $h : X \subseteq K^m \rightarrow K$  belongs to  $\text{coalg}(f)$  if there exists a finite partition of  $X$  in definable pieces  $H$ , on each of which the degree in  $t$  of  $f(x, t)$  is constant, say  $e_H$ , and such that the following holds. If  $e_H \leq 0$  then  $h(x)$  is identically equal to 0 on  $H$ . Otherwise there is a family  $(\xi_1, \dots, \xi_{r_H})$  of  $K$ -linearly independent elements in an algebraic closure of  $K$  and a family of definable functions  $b_{i,j} : H \rightarrow K$  for  $1 \leq i \leq e_H$  and  $1 \leq j \leq r_H$ , such that for every  $x$  in  $H$

$$f(x, T) = a_{e_H}(x) \prod_{1 \leq i \leq e_H} \left( T - \sum_{1 \leq j \leq r_H} b_{i,j}(x) \xi_j \right)$$

and

$$h(x) = \sum_{1 \leq i \leq e_H} \sum_{1 \leq j \leq r_H} \alpha_{i,j} b_{i,j}(x)$$

With the  $\alpha_{i,j}$ 's in  $K$ . If  $\mathcal{F}$  is any family of  $(m+1, 1)$ -basic functions we let  $\text{coalg}(\mathcal{F})$  denote the set of linear combinations of functions in  $\text{coalg}(f)$  for  $f$  in  $\mathcal{F}$ .

**Theorem 3.4 (Denef)** *Let  $\mathcal{F}$  be a finite family of  $(m+1, 1)$ -basic functions. Let  $n \geq 1$  be a fixed integer. Then there exists a finite partition of  $K^{m+1}$  into presented cells  $H \bmod K^\times$  such that the center and bounds of  $H$  belong to  $\text{coalg}(\mathcal{F}) \cup \{\infty\}$  and for every  $(x, t)$  in  $H$  and every  $f$  in  $\mathcal{F}$*

$$f(x, t) = \mathcal{U}_n(x, t) h_{f,H}(x) (t - c_H(x))^{\alpha_{f,H}} \quad (4)$$

with  $h_{f,H} : \widehat{H} \rightarrow K$  a definable function and  $\alpha_{f,H} \in \mathbf{N}$ .

*Proof:* Follow the proof of theorem 7.3 in [Den84], using once again the  $p$ -minimality assumption and basic functions in place of Macintyre's quantifier elimination and polynomial functions.

■

Given two families  $\mathcal{A}, \mathcal{B}$  of subsets of  $K^m$ , recall that  $\mathcal{B}$  **refines**  $\mathcal{A}$  if  $\mathcal{B}$  is a partition of  $\bigcup \mathcal{A}$  such that every  $A$  in  $\mathcal{A}$  which meets some  $B$  in  $\mathcal{B}$  contains it.

**Corollary 3.5 (Denef)** *Let  $\mathcal{F}$  be a finite family of  $(m, 1)$ -basic functions,  $N \geq 1$  an integer and  $\mathcal{A}$  a family of boolean combinations of subsets of  $K^m$  defined by  $f(x) \in P_N$  with  $f$  in  $\mathcal{F}$ . Then there exists a finite family  $\mathcal{H}$  of cells mod  $P_N^\times$  with center and bounds in  $\text{coalg}(\mathcal{F})$  which refines  $\mathcal{A}$ .*

*Proof:* Theorem 3.4 applies to  $\mathcal{F}$  with  $n > v(N)$ , so that  $1 + \pi^n R \subseteq P_N$ . It gives a partition of  $K^m$  into presented cells  $B \bmod K^\times$ . Every such cell  $B$  is the disjoint union of finitely many presented cells  $H \bmod P_N^\times$ , whose centers and bounds are the restrictions to  $\widehat{H}$  of the center and bounds of  $A$  (hence belong to  $\text{coalg}(\mathcal{F})$ ), on which  $h_{f,B}(x)P_N^\times$  and  $(t - c_B(x))P_N^\times$  are constant, simultaneously for every  $f$  in  $\mathcal{F}$ . Thus every  $A$  in  $\mathcal{A}$  either contains  $H$  or is disjoint from  $H$  by (4) and our choice of  $n$ , which proves the result.

■

The following simpler statement, which follows directly from corollary 3.5 by  $p$ -minimality, is sufficient in most cases.

**Theorem 3.6 (Denef's cell decomposition)** *For every finite family  $\mathcal{A}$  of definable subsets of  $K^m$  there is for some  $N$  a finite family of presented cells mod  $P_N^\times$  refining  $\mathcal{A}$ .*

## 4 Skolem functions

**Lemma 4.1** *Assume that Denef's cell decomposition theorem 3.6 holds true for an expansion of a  $p$ -adically closed field  $F$ . Then it has definable Skolem functions.*

The proof is taken from the appendix of [DvdD88].

*Proof:* By a straightforward induction it suffices to prove that for every definable subset  $A$  of  $F^{m+1}$  the coordinate projection of  $A$  onto  $\widehat{A}$  has a definable section. If  $A$  is a union of finitely many definable sets  $B$  and if a definable section  $\sigma_B : \widehat{B} \rightarrow B$  has been found for each projection of  $B$  onto  $\widehat{B}$  we are done. Thus, by cell decomposition, we can assume that  $A$  is a presented cell mod  $P_N^\times$  for some  $N$ . We deal with the case when  $A = (c_A, \nu_A, \mu_A, \lambda_A)$  is of type 1 and  $\nu_A \neq 0$  or  $\mu_A \neq \infty$ , the other cases being trivial.

If  $\nu_A \neq 0$ , w.l.o.g. we can assume that  $v \circ \nu_A$  is equal to  $v(\lambda_A)$  modulo  $N$ , so  $\nu_A/\lambda_A$  takes values in  $v^{-1}(Nv(F^\times))$ . As  $Q_{N,2v(N)+1}$  is a definable subgroup of  $v^{-1}(Nv(F^\times))$  with finite index, there is a partition of  $X$  in finitely many definable pieces  $X$  on which  $\nu_A/\lambda_A$  has constant residue class modulo  $Q_{N,2v(N)+1}$ . Again it suffices to prove the result for each piece  $A \cap (X \times F)$  of  $A$ , so we can assume that  $X = \widehat{A}$ . Pick any  $u$  in  $R^*$  such that  $\nu_A/\lambda_A$  belongs constantly to  $u \cdot Q_{N,2v(N)+1}$  on  $\widehat{A}$ . Then  $\xi = (\nu_A/(u\lambda_A))^{1/N}$  is a definable function on  $\widehat{A}$  and the map

$$\sigma : x \mapsto (x, c_A(x) + \lambda_A \xi(x)^N)$$

is a definable section of the projection of  $A$  onto  $\widehat{A}$ . If  $\nu_A = 0$  and  $\mu_A \neq \infty$  a similar argument on  $\mu_A$  gives the conclusion.  $\blacksquare$

**Theorem 4.2** *For every expansion of a  $p$ -adically closed field  $F$ , the following are equivalent:*

1.  $F$  is  $p$ -optimal.
2. Denef's cell decomposition theorem 3.6 holds in  $F$ .
3.  $F$  is strongly  $p$ -minimal and has definable Skolem function.

*Proof:* (1) $\Rightarrow$ (2) is theorem 3.6, and (3) $\Rightarrow$ (1) follows from proposition 2.3. So let us prove that (2) $\Rightarrow$ (3). By lemma 4.1 it only remains to derive strong  $p$ -minimality from the cell decomposition theorem 3.6.

Let  $\Phi(\xi, \sigma)$  be a parameter-free formula with  $m+1$  variables. It defines a subset  $S$  of  $F^{m+1}$  which splits into finitely many cells  $C$  mod  $P_N^\times$  for some  $N$ . Let  $\mathcal{C}$  be the family of these cells, and  $X_1, \dots, X_r$  a finite partition of  $\widehat{S}$  refining the  $\widehat{C}$ 's for  $C \in \mathcal{C}$ . For each  $i \leq r$  let  $\theta_i(\alpha_i, \xi)$  be a parameter-free formula in  $n_i + m$  variables and  $\alpha_i \in F^{n_i}$  such that

$$X_i = \{x \in F^m : F \models \theta(\alpha_i, x)\}.$$

Let  $\Theta(\alpha_1, \dots, \alpha_r)$  be the parameter-free formula in  $n_1 + \dots + n_r$  saying that, given any values  $a'_i$  of the parameters  $\alpha_i$ , the formulas  $\theta_i(a'_i, \xi)$  define a partition of  $\widehat{S}$ . In particular we have  $F \models \Theta(a_1, \dots, a_r)$ .

Let  $\mathcal{C}_i$  be the family of all the cells  $C \cap (X_i \times K)$  for  $C \in \mathcal{C}$ . This is a finite partition of  $S \cap (X_i \times K)$  into cells mod  $P_N^\times$ , which consists in  $k_0^i$  cells of type 0,  $k_1^i$  cells  $D$  of type 1 with  $\mu_D \neq \infty$ , and  $k_\infty^i$  cells  $D$  of type 1 with  $\mu_D = \infty$ . We let  $k^i = (k_0^i, k_1^i, k_\infty^i)$ . For every  $x \in X_i$ , the fiber  $S_x = \{t \in F : (x, t) \in S\}$  is the disjoint union of the fibers  $C_x$  for  $C \in \mathcal{C}_i$ , each of which is of the same type as  $C$ . Given a tuple  $k = (k_0, k_1, k_\infty)$  it is an easy exercise to write a parameter-free formula  $\Psi_{k,N}(\xi)$  in  $m$  free variables saying that, given any value

$x'$  of the parameter  $\xi$ , the set of points  $t'$  in  $F$  such that  $F \models \Phi(x', t')$  is the disjoint union of  $k_0$  cells mod  $P_N^\times$  of type 0,  $k_1$  cells  $D'$  mod  $P_N^\times$  of type 1 with  $\mu_{D'} \neq \infty$ , and  $k_\infty$  cells  $D'$  mod  $P_N^\times$  of type 1 with  $\mu_{D'} = \infty$ . By construction we have

$$F \models \exists \alpha_1, \dots, \alpha_r \Theta(\alpha_1, \dots, \alpha_r) \wedge \bigwedge_{i \leq r} \forall \xi [\theta(\alpha_i, \xi) \rightarrow \Psi_{k^i, N}(\xi)]$$

This formula is satisfied in every  $\tilde{F} \equiv F$ . So there are  $\tilde{a}_i$  in  $\tilde{F}^{n_i}$  for  $i \leq r$  such that the sets

$$\tilde{X}_i = \{\tilde{x} \in \tilde{F}^m : \tilde{F} \models \theta(\tilde{a}_i, \tilde{x})\}$$

form a partition of  $\{\tilde{x} \in \tilde{F}^m : \exists \tilde{t} \in \tilde{F}, \tilde{F} \models \theta(\tilde{x}, \tilde{t})\}$ , and for every  $\tilde{x} \in \tilde{X}_i$  the set of  $\tilde{t} \in \tilde{F}$  such that  $\tilde{F} \models \theta(\tilde{x}, \tilde{t})$  is the disjoint union of  $k_0^i + k_1^i + k_\infty^i$  cells of  $\tilde{F}$ . In particular the formula  $\Phi(\tilde{x}, \sigma)$  defines a semi-algebraic subset of  $F$ , whatever is the value of the parameter  $\tilde{x}$  in  $\tilde{F}^m$ . This being true for every formula  $\Phi$ , it follows that  $\tilde{F}$  is  $p$ -minimal hence that  $F$  is strongly  $p$ -minimal.

■

**Corollary 4.3 (Mourgues)** *Let  $K$  be a  $p$ -optimal field. Then for every definable function  $f : X \subseteq K^m \rightarrow K$  there is a finite partition of  $X$  in definable pieces on each of which the restriction of  $f$  is continuous.*

*Proof:* Mourgues has proved in [Mou09] that the cell decomposition theorem holds in strongly  $p$ -minimal fields with definable Skolem functions (hence in  $p$ -optimal fields) and that moreover the cells can be chosen with *continuous* center and bounds. The conclusion follows immediately, by applying this “continuous cell decomposition” to the graph of  $f$ .

■

## 5 Dense intersections and continuity

Recall that, until the end of this paper,  $K$  denotes a  $p$ -optimal field.

For every definable subsets  $B \subseteq A$  of  $K^m$  let  $\text{Int}_A B$  denote the relative interior of  $B$  inside  $A$ , that is the largest set  $U \cap A$  contained in  $B$  with  $U$  an open subset of  $K^m$ . We simply note  $\text{Int} B$  for  $\text{Int}_{K^m} B$ .

We say that a function  $h$  with domain  $X \subseteq K^m$  is **locally constant** if every point of  $X$  has a neighbourhood in  $X$  on which  $h$  is constant. It is **piecewise** locally constant (resp. continuous) if there is a finite partition of  $X$  in definable pieces on which the restriction of  $h$  is locally constant (resp. continuous). By corollary 4.3 we know that every definable function  $f : X \subseteq K^m \rightarrow K$  is piecewise continuous. We are going to show that  $f$  is actually continuous “almost everywhere”.

**Proposition 5.1** *Every definable function  $f : X \subseteq K^m \rightarrow K^n$ ,  $f$  is continuous on a definable set  $U$  dense and open in  $X$ .*

Let  $\text{LCD}_m$  denote this statement. It is closely related to the following one.

**Theorem 5.2** *Let  $A_1, \dots, A_r \subseteq A$  be a family of definable subsets of  $K^m$ .*

1. *If  $A_1, \dots, A_r$  are dense in  $A$  then so is their intersection.*

2. If the union of the  $A_k$ 's has non empty interior in  $A$  then at least one of them has non empty interior in  $A$ .

Let  $DI_m$  (for ‘‘Dense Intersection’’) denote this result. Note that the two points of  $DI_m$  are obviously equivalent: the second one follows by applying the first one to the complements of the  $A_k$ 's in  $A$ , and conversely.

$DI_1$  follows immediately from  $p$ -minimality. We are going to show that  $DI_m \Rightarrow LCD_m$  (lemma 5.3) and that  $LCD_m + DI_m \Rightarrow DI_{m+1}$  (lemma 5.4), from which both results follow by induction on  $m$ .

**Lemma 5.3** *For every  $m$ ,  $DI_m \Rightarrow LCD_m$ .*

*Proof:* Let  $f : X \subseteq K^m \rightarrow K$  be a definable function. Corollary 4.3 gives a finite partition  $\mathcal{A}$  in definable pieces on each of which the restriction of  $f$  is continuous. Then  $U = \bigcup \{\text{Int}_X A : A \in \mathcal{A}\}$  is a definable set open in  $X$  on which  $f$  is continuous. Clearly  $X \setminus U$  is the union of the sets  $A \setminus \text{Int}_X A$  as  $A$  ranges over  $\mathcal{A}$ . Each of them has empty interior in  $X$ , hence so does their union by  $DI_m$ . So  $X \setminus U$  has empty interior in  $X$ , that is  $U$  is dense in  $X$ . ■

**Lemma 5.4** *For every  $m$ ,  $LCD_m + DI_m \Rightarrow DI_{m+1}$ .*

The argument is adapted from [HM97], Theorem 3.2.

*Proof:* It suffices to prove the second point of  $DI_{m+1}$  under the additional hypothesis that the  $A_k$ 's are disjoint, and for  $r = 2$ . So let  $A_1, A_2$  be two disjoint definable sets whose union has non empty interior inside a definable subset  $A_0$  of  $K^{m+1}$ . There is a box  $U = X \times Y$  with  $X$  (resp.  $Y$ ) a non empty open subset of  $K^m$  (resp.  $K$ ) such that  $A_0 \cap U$  is non empty and contained in  $A_1 \cup A_2$ . Replacing each  $A_k$  by  $A_k \cap U$  if necessary, we can assume that  $A_0 \subseteq U$  hence  $A_1 \cup A_2 = A_0$ .

For  $k = 0, 1, 2$  and every  $x$  in  $X$ , let  $A_{k,x} = \{y \in K : (x, y) \in A_k\}$ . This is a definable subset of  $K$  hence  $A_{k,x} \setminus \text{Int}(A_{k,x})$  is finite by  $p$ -minimality. So the coordinate projection of the definable set

$$B = \bigcup_{\substack{x \in X \\ k=1,2}} \{x\} \times (A_{k,x} \setminus \text{Int}(A_{k,x}))$$

onto  $K^m$  has finite fibres  $B_x$ . By cell decomposition, the cardinality of these fibres is bounded by some integer  $N$ . Let  $(y_i)_{0 \leq i \leq N}$  be a sequence of distinct elements in the open set  $Y$ . For each  $i \leq N$  let

$$V_i = \{x \in \widehat{A}_0 : y_i \notin B_x\}.$$

For each  $x$  in  $\widehat{A}_0$  at least one of the  $N+1$  elements  $y_i$  does not belong to  $B_x$  hence  $\widehat{A}_0$  is covered by the  $V_i$ 's. By  $DI_m$  at least one of them, say  $V_0$ , has non empty interior  $Z_0$  inside  $\widehat{A}_0$ . Then  $y_0 \notin B_x$  for every  $x$  in  $Z_0$ , hence  $Z_0 = W_1 \cup W_2$  where  $W_k$  is defined as

$$W_k = \{x \in Z_0 : y_0 \in \text{Int } A_{k,x}\}.$$

By  $DI_m$  at least one of the  $W_k$ 's, for example  $W_1$ , has non empty interior  $Z_1$  inside  $Z_0$  hence inside  $\widehat{A}_0$ . For every  $x$  in  $Z_1$  there is  $\rho_x$  in  $K^\times$  such that the

ball  $B(y_0, \rho_x)$  is contained in  $\text{Int}(A_{1,x})$ . Let  $\rho : Z_1 \rightarrow K^\times$  be a definable Skolem function corresponding to this property. By  $\text{LCD}_m$  there is a definable set  $Z_2$  dense and open in  $Z_1$ , hence open in  $\widehat{A}_0$ , on which  $\rho$  is continuous, hence  $|\rho|$  is locally constant. Fix an element  $a$  in  $Z_2$  and some  $\delta$  in  $K^\times$  such that  $B(a, \delta) \cap \widehat{A}_0$  is contained in  $Z_1$  and  $|\rho|$  is constant on  $B(a, \delta) \cap \widehat{A}_0$ . We have

$$A_0 \cap [B(a, \delta) \times B(y_0, \rho(a))] \subseteq A_1.$$

Thus  $(a, y_0)$  belongs to  $\text{Int}_{A_0} A_1$ , which is then non-empty.

■

**Corollary 5.5** *Let  $B \subseteq A \subseteq K^m$  be definable. If  $B$  is dense in  $A$  then so is  $\text{Int}_A B$ .*

*Proof:*  $B \setminus \text{Int} B$  has empty interior in  $A$ , and so does  $A \setminus B$  by assumption. Hence their union  $A \setminus \text{Int} B$  has empty interior in  $A$  by theorem 5.2, that is  $\text{Int}_A B$  is dense in  $A$ .

■

## 6 Dimension

Recall that  $K$  is a  $p$ -optimal field. The **dimension** (the “topological dimension” in [HM97]) of a non-empty definable subset  $S$  of  $K^m$ , denoted  $\dim S$ , is the greatest integer  $d$  such that there exists a subset  $I$  of  $\{1, \dots, m\}$  such that  $\pi_I^m(S)$  has non-empty interior, where  $\pi_I^m : F^m \rightarrow F^d$  is defined by

$$\pi_I^m : (x_i)_{1 \leq i \leq m} \mapsto (x_{i_k})_{1 \leq k \leq d}$$

with  $i_1 < \dots < i_d$  an enumeration of  $I$ . This projection will be denoted  $\pi_I$  when  $m$  is clear from the context. By convention  $\dim \emptyset = -\infty$ .

Given a permutation  $\sigma$  of  $\{1, \dots, m\}$ , for every  $x = (x_1, \dots, x_m)$  in  $K^m$  we let  $x^\sigma = (x_{\sigma(1)}, \dots, x_{\sigma(m)})$ . Similarly we let  $S^\sigma = \{x^\sigma : x \in S\}$  for every definable subset  $S$  of  $K^m$ .

Following [vdD89], a dimension function on the definable sets over  $K$  is a function  $d$  with values in  $\mathbf{N} \cup \{-\infty\}$  such that for every positive integer  $m$  and every definable sets  $S, S_1, S_2$  of  $K^m$  we have:

**(Dim 1)**  $d(S) = -\infty \Leftrightarrow S = \emptyset$ ,  $d(\{a\}) = 0$  for each  $a \in K$ ,  $d(K) = 1$ .

**(Dim 2)**  $d(S_1 \cup S_2) = \max(d(S_1), d(S_2))$ .

**(Dim 3)**  $d(S^\sigma) = d(S)$  for each permutation  $\sigma$  of  $\{1, \dots, m\}$ .

**(Dim 4)** For  $k = 0, 1$  and every definable set  $T \subseteq K^{m+1}$  the set  $T(k)$  of  $x$  in  $\widehat{T}$  such that  $d(T_x) = k$ , where  $T_x$  denotes the fiber of  $T$  over  $x$ , is definable and

$$d(\{(x, t) \in T : x \in T(k)\}) = \dim(T(k)) + k.$$

The aim of this section is to prove that in  $p$ -optimal fields the topological dimension introduced in [HM97] is a dimension function. It satisfies (Dim 1) by definition, (Dim 2) by theorem 3.2 in [HM97], and obviously (Dim 3). Moreover, if  $T$  is a definable subset of  $K^{m+1}$  then by  $p$ -minimality  $x \in T(0)$  if and only if the points in  $T_x$  are isolated, hence  $T(0)$  is indeed definable and so is  $T(1) = \widehat{T} \setminus T(0)$ . Thus we only have to prove the dimension formula of (Dim 4).

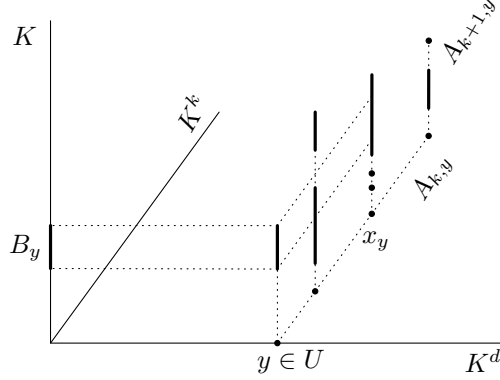
**Lemma 6.1** *Let  $A$  be a definable subset of  $K^m$ ,  $d = \dim A$  and  $I$  a subset of  $\{1, \dots, m\}$  with  $d$  elements such that  $\pi_I(A)$  has non empty interior. Let*

$$Z = \{y \in \pi_I(A) : \pi_I^{-1}(\{y\}) \cap A \text{ is infinite}\}.$$

*Then  $Z$  has empty interior.*

*Proof:* Let  $i_1 < \dots < i_d$  be an enumeration of  $I$ , and  $\sigma$  a permutation of  $\{1, \dots, m\}$  such that  $\sigma(k) = i_k$  for every  $k \leq d$ . Replacing  $A$  by  $A^\sigma$  if necessary we can assume that  $I = \{1, \dots, d\}$ .

For every  $d \leq k \leq m$  let  $A_k = \pi_{\{1, \dots, k\}}^m(A)$ . For every  $y$  in  $A_d = \pi_I^m(A)$  let  $A_{k,y}$  be the set of  $x$  in  $A_k$  such that  $\pi_I^k(x) = y$ . We let  $Z_k$  be the set of  $y$  in  $A_d$  such that  $A_{k,y}$  is infinite. Clearly  $Z_d$  is empty,  $Z_m = Z$  and  $Z_k \subseteq Z_{k+1}$  for every  $k$  in between. Assume for a contradiction that  $Z$  has non empty interior. Let  $k < m$  be then the greatest index such that  $Z_k$  lacks interior, and let  $J = I \cup \{k+1\}$ .



By theorem 5.2  $Z_{k+1} \setminus Z_k$  has non empty interior, so let  $U$  be a non empty open subset of  $K^d$  contained in it. For every  $y$  in  $U$ ,  $A_{k,y}$  is finite and  $A_{k+1,y}$  is infinite, so there is an element  $x_y$  in  $A_{k,y}$  and by  $p$ -minimality a ball  $B_y$  in  $K$  such that  $\{x_y\} \times B_y$  is contained in  $A_{k+1,y}$ . *A fortiori* we have

$$\{y\} \times B_y \subseteq \pi_J^{d+1}(A_{k+1,y}) \subseteq \pi_J^{d+1}(A_{k+1}) = \pi_J^m(A).$$

By lemma 4.1 there is a pair of definable functions  $c : U \rightarrow K$  and  $\rho : U \rightarrow K^\times$  such that every  $B_y$  contains the ball of center  $c(y)$  and radius  $\rho(y)$ . As  $U$  is open in  $K^d$ , by proposition 5.1 there is a non empty open set  $V$  in  $K^d$  contained in  $U$  such that  $c$  and  $\rho$  are continuous on  $V$ . We have

$$\{(y, t) \in V \times K : |t - c(y)| < |\rho(y)|\} \subseteq \pi_J^m(A).$$

This is a non empty open subset of  $K^{d+1}$  because  $c, \rho$  are continuous and  $V$  is non empty open in  $K^d$ . It follows that  $\dim A \geq d + 1$ , a contradiction.

■

**Lemma 6.2** *For every cell  $A \subseteq K^{m+1}$ ,  $\dim A = \dim \hat{A} + \text{tp } A$ .*

*Proof:* Let  $A$  be a presented cell mod  $G$  in  $K^{m+1}$ , for some large definable subgroup of  $K^\times$ . Let  $d$  be its dimension.

We first prove that  $\dim A \geq \dim \widehat{A} + \text{tp } A$ . Let  $e = \dim \widehat{A}$  and  $J \subseteq \{1, \dots, m\}$  an index set with  $e$  elements such that  $\pi_J^m(\widehat{A})$  has non empty interior in  $K^e$ . Then  $\pi_J^{m+1}(A) = \pi_J^m(\widehat{A})$  so  $\dim A \geq \dim \widehat{A}$ . If  $A$  is of type 0 we are done. Otherwise let  $I = J \cup \{m+1\}$ , let  $\sigma : \pi_I^m(\widehat{A}) \rightarrow \widehat{A}$  be a definable section of  $\pi_I^m$  and let  $B$  be the presented cell in  $K^{d+1}$  defined by

$$B = (c_A \circ \sigma, \nu_A \circ \sigma, \mu_A \circ \sigma, \lambda_A, G).$$

This is a cell of type 1 contained in  $\pi_I^{m+1}(A)$ . Moreover  $\widehat{B} = \pi_J^m(\widehat{A})$  has non empty interior in  $K^e$ . Thus by proposition 5.1 the center and bounds of  $B$  are continuous on a non-empty definable set  $V$  open in  $K^e$ . This continuity implies that  $B \cap (V \times K)$  is open in  $K^{e+1}$ . As it is contained in  $\pi_I^{m+1}(A)$  it follows that  $\dim A \geq e + 1$  so we are done.

For the reverse inequality, note first that  $\dim \widehat{A} + 1 \geq \dim A$ . Indeed, let  $I$  be a subset of  $\{1, \dots, m+1\}$  with  $d$  elements such that  $\pi_I^{m+1}(A)$  has non empty interior. If  $m+1 \notin I$  then  $\pi_I^m(\widehat{A}) = \pi_I^{m+1}(A)$  hence  $\dim \widehat{A} \geq d$  and *a fortiori*  $\dim \widehat{A} + 1 \geq d$ . Otherwise  $m+1 \in I$  so  $J = I \setminus \{m+1\}$  has  $d-1$  elements. Then  $\pi_J^m(\widehat{A})$  has non empty interior in  $K^{d-1}$  since

$$\pi_J^m(\widehat{A}) = \pi_I^{m+1}(A).$$

So  $\dim \widehat{A} \geq d-1$  hence  $\dim \widehat{A} + 1 \geq d$ .

We have proved that  $\dim \widehat{A} + \text{tp } A \leq \dim A \leq \dim \widehat{A} + 1$ . If  $A$  is of type 1 the conclusion follows. Now assume that  $A$  is of type 0. It remains to prove that in this case  $\dim A \leq \dim \widehat{A}$ . So let  $I$  be a subset of  $\{1, \dots, m+1\}$  with  $d$  elements such that  $\pi_I^{m+1}(A)$  has non empty interior. Assume for a contradiction that  $\dim \widehat{A} < d$ . This implies that  $m+1$  belongs to  $I$  (otherwise  $\pi_I^m(\widehat{A}) = \pi_I^{m+1}(A)$  has non empty interior hence  $\dim \widehat{A} \geq d$ ). Let  $J = I \setminus \{m+1\}$ ,  $B = \pi_I^{m+1}(A)$  and  $Y = \widehat{B} = \pi_J^m(\widehat{A})$ .

By assumption  $B$  contains a non empty open subset  $U$  of  $K^d$ . Since  $J$  has  $d-1$  elements and  $\dim \widehat{A} = d-1$ , lemma 6.1 implies that the set

$$Z = \{y \in Y : \pi_J^{-1}(\{y\}) \cap \widehat{A} \text{ is infinite}\}$$

has empty interior. Now  $\widehat{U}$  is a non empty open subset of  $K^{d-1}$  contained in  $Y$ , so it cannot be contained in  $Z$ . Pick any  $y$  in  $\widehat{U} \setminus Z$ . As  $y \notin Z$  there are finitely many points  $x_1, \dots, x_N$  in  $\widehat{A}$  such that  $\pi_J^m(x_i) = y$ . For every  $s$  in  $K$  such that  $(y, s)$  belongs to  $U$  there is  $(x, t)$  in  $A$  such that  $\pi_I^{m+1}(x, t) = (y, s)$ . But this implies that  $\pi_J^m(x) = y$  and  $s = t = c_A(x)$ , so  $s = c_A(x_i)$  for some  $i$ . Thus  $U \cap (\{y\} \times K)$  is finite, contradicting that  $U$  is open in  $K^d$ .  
■

**Theorem 6.3** *The function  $\dim$ , defined for definable sets over a  $p$ -optimal field, is a dimension function.*

*Proof:* Let  $T$  be a definable subset of  $K^{m+1}$ . By cell decomposition there is for some  $N$  a partition  $\mathcal{A}$  of  $T$  into cells mod  $P_N^\times$ . Refining the basements of



the cells in  $\mathcal{A}$  if necessary, we can assume that  $\{\widehat{A} : A \in \mathcal{A}\}$  is a partition of  $\widehat{T}$  refining  $\{T(0), T(1)\}$ . For  $k = 0, 1$  let  $\mathcal{A}_k$  denote the cells in  $\mathcal{A}$  such that  $\widehat{A}_k \subseteq T(k)$ . Then  $\{(x, t) \in T : x \in T(k)\}$  is the union of the cells in  $\mathcal{A}_k$  and  $T(k)$  is the union of their basements, so we have

$$\dim\{(x, t) \in T : x \in T(k)\} = \max_{A \in \mathcal{A}_k} \dim A \quad (5)$$

$$\dim T(k) = \max_{A \in \mathcal{A}_k} \dim \widehat{A} \quad (6)$$

For  $k = 0$ , every cell  $A$  in  $\mathcal{A}_0$  is of type 0 hence  $\dim A = \dim \widehat{A}$  by lemma 6.2. The conclusion follows from (5) and (6) in that case.

For  $k = 1$ , there is at least one cell  $B$  in  $\mathcal{A}_1$  such that  $\dim \widehat{B} = \dim T(1)$  by (6). Since  $\widehat{B} \subseteq T(1)$ ,  $A \cap (\widehat{B} \times K)$  projects with infinite fiber onto  $\widehat{B}$ . Hence there is at least one cell  $C$  in  $\mathcal{A}$  of type 1 such that  $\widehat{C} = \widehat{B}$  (hence  $C \in \mathcal{A}_1$ ). By lemma 6.2 we have

$$\dim C = \dim \widehat{C} + 1 = \dim T(1) + 1.$$

Now for every  $A$  in  $\mathcal{A}_1$  we have by lemma 6.2 and (6)

$$\dim A = \dim \widehat{A} + \text{tp } A \leq \dim T(1) + 1 = \dim C$$

So  $\max_{A \in \mathcal{A}_k} \dim A = \dim C = \dim T(1) + 1$  as expected.

■

**Corollary 6.4** *Let  $f : A \subseteq K^m \rightarrow K^n$  be a definable map.*

1.  $\dim f(A) \leq \dim A$  and equality holds when  $f$  is injective.
2. For every positive integer  $d$  the set  $S(d)$  defined by

$$S(d) = \{y \in f(A) : \dim f^{-1}(\{y\}) = d\}$$

*is definable and  $\dim f^{-1}(S(d)) = d + \dim S(d)$ .*

*Proof:* This follows directly from theorem 6.3, as general properties of dimension functions (see corollary 1.5 in [vdD89]).

■

It is worth mentioning the following consequence of corollary 6.4, which will be needed in the next section.

**Corollary 6.5** *Let  $f : A \subseteq K^m \rightarrow K^n$  be a definable map, and  $B$  a definable subset of  $A$ . If  $\dim B \cap f^{-1}(\{y\}) < \dim f^{-1}(\{y\})$  for every  $y$  in  $f(A)$  then  $\dim B < \dim A$ .*

*Proof:* Let  $S = f(A)$ ,  $g$  the restriction of  $f$  to  $B$  and  $T = g(B)$ . Let  $S(d)$  (resp.  $T(e)$ ) be as in corollary 6.4 for  $f$  (resp.  $g$ ) and for every positive integers  $d, e$ . As  $B = \bigcup_{e \leq n} g^{-1}(T(e))$  there is some  $e \leq n$  such that  $\dim B = \dim g^{-1}(T(e))$ . By assumption  $\dim g^{-1}(\{y\}) < \dim f^{-1}(\{y\})$  for every  $y$  in  $B$  hence  $T(e)$  is contained in the union of  $S(d)$  for  $d > e$ . It follows that  $\dim T(e) \leq \dim S(d)$  for some  $d > e$ , hence  $e + \dim T(e) < d + \dim S(d)$ . Thus by corollary 6.4,  $\dim B = \dim g^{-1}(T(e)) < \dim f^{-1}(S(d)) \leq \dim A$ .

■

## 7 Boundary and topological rank

**Theorem 7.1** *For every non empty definable subset  $A$  of  $K^m$*

$$\dim \overline{A} \setminus A < \dim A.$$

*As a consequence  $\dim A = \dim \overline{A}$ .*

**Remark 7.2** This result does not follow from proposition 2.23 of [vdD89] since  $p$ -optimal fields are *not* algebraically bounded in the restricted sense of [vdD89]. The classical analytic structure on  $\mathbf{Q}_p$  is a counter-example (see [vdD89], page 191).

The following proof is borrowed from the  $\mathcal{o}$ -minimal analogous theorem 1.8 in [vdD98].

**Notation** Given an index  $i \in \{1, \dots, m\}$  we let  $\pi_i : K^m \rightarrow K$  denote the coordinate projection onto the  $i$ -th axis. For every  $S \subseteq K^m$  and  $z \in K$  we let

$$S_z^{(i)} = \{x \in K^{m-1} : (x_1, \dots, x_{i-1}, z, x_i, \dots, x_{m-1}) \in S\}.$$

**Lemma 7.3** *For every definable subset  $A$  of  $K^m$  there are only finitely many  $z$  in  $K$  such that*

$$(\overline{A})_z^{(i)} \neq \overline{A_z^{(i)}}.$$

*Proof:* Permuting the axes if necessary we can assume that  $i = 1$ . For  $z \in K$  and  $S \subseteq K^n$  with  $n \geq 1$  we simply write  $S_z$  for  $S_z^{(1)}$ .

Let  $Z$  be the set of elements  $z$  in  $K$  such that  $(\overline{A})_z \neq \overline{A_z}$ . Replacing  $A$  by  $A \cap \pi_1^{-1}(Z)$  if necessary we can assume that  $Z = \pi_1(A)$ . Let  $C$  be the set of  $(z, x, \rho)$  in  $Z \times K^{m-1} \times K$  such that

$$B(x, \rho) \cap (\overline{A})_z \neq \emptyset \quad \text{and} \quad B(x, \rho) \cap A_z = \emptyset$$

As  $\overline{A_z}$  is always contained in  $(\overline{A})_z$ , by construction  $C_z \neq \emptyset$  for every  $z$  in  $Z$ . Moreover, if  $(x, \rho)$  belongs to  $C_z$  and we fix any element  $y$  in  $B(x, \rho) \cap (\overline{A})_z$  then for every  $(x', \rho')$  in  $K^{m-1} \times K$  such that

$$\|x' - y\| < |\rho'| < |\rho| \tag{7}$$

we have  $y \in B(x', \rho') \subseteq B(y, \rho) = B(x, \rho)$  hence  $(x', \rho') \in C_z$ . The above condition (7) defines an open subset of  $K^m$ , hence  $\dim C_z = m$ . By corollary 6.4 it follows that

$$\dim C = m + \dim Z. \tag{8}$$

On the other hand, let us show that the projection  $\Pi : C \rightarrow K^m$  which maps  $(z, x, \rho)$  to  $(x, \rho)$  has finite fibers. Assume the contrary. Then for some  $(x, \rho)$  in  $K^{m-1} \times K^\times$  there is a ball  $B_0$  contained in  $Z$  such that  $(z, x, \rho)$  belongs to  $C$  for every  $z$  in  $B_0$ . Then  $B(x, \rho)$  is disjoint from  $A_z$  for every  $z$  in  $B_0$ , hence  $B_0 \times B(x, \rho)$  is disjoint from  $A \cap \pi_1^{-1}(B)$ . It is disjoint as well from its closure since  $B_0 \times B(x, \rho)$  is open. On the other hand  $B(x, \rho)$  meets  $(\overline{A})_z$  for every  $z$  in  $Z$ , and in particular it meets  $(\overline{A})_z$  for some  $z$  in  $B_0$ . Then  $B_0 \times B(x, \rho)$  meets  $(\overline{A})_z$ , which is contained in the closure of  $A \cap \pi_1^{-1}(B_0)$ , a contradiction.

So  $\Pi : C \rightarrow K^m$  has finite fibers, which implies that  $\dim C \leq m$ . By (8) we conclude that  $\dim Z = 0$  hence  $Z$  is finite.

■

We can turn now to theorem 7.1.

*Proof:* (of theorem 7.1). The result is obvious for  $m \leq 1$  so we can assume, by induction, that  $m \geq 2$  and it has been proved for  $m - 1$ . For every definable set  $S$  we let  $\text{Fr } S = \overline{S} \setminus S$ .

Let  $A$  be a definable subset of  $K^m$ . For each  $i$  in  $\{1, \dots, m\}$  let

$$F_i = \left\{ z \in K : (\overline{A})_z^{(i)} \neq \overline{A_z^{(i)}} \right\}.$$

Note that for every  $z$  in  $K$

$$(\text{Fr } A)_z^{(i)} \setminus \text{Fr}(A_z^{(i)}) = (\overline{A})_z^{(i)} \setminus \overline{A_z^{(i)}} \quad (9)$$

so  $(\text{Fr } A)_z^{(i)} = \text{Fr}(A_z^{(i)})$  for every  $z \in K \setminus F_i$ .

Let  $H_i = \pi_i^{-1}(F_i)$  and  $H = \bigcap_{i=1}^m H_i = F_1 \times \dots \times F_m$ . Each  $F_i$  is finite by lemma 7.3 hence so is  $H$ . So  $\dim \text{Fr } A = \dim(\text{Fr } A) \setminus H$  and moreover

$$(\text{Fr } A) \setminus H = \bigcup_{i=1}^m (\text{Fr } A) \setminus H_i.$$

Thus it suffices to prove that  $\dim \text{Fr } A \setminus H_i < \dim A$  for each  $i$ . By symmetry we can assume that  $i = 1$  and remove the exponents  $^{(i)}$  in order to ease the notation. By (9) we have then

$$(\text{Fr } A) \setminus H_1 = \bigcup_{z \in K \setminus F_1} \{z\} \times (\text{Fr } A)_z = \bigcup_{z \in K \setminus F_1} \{z\} \times \text{Fr}(A_z). \quad (10)$$

By induction hypothesis  $\dim \text{Fr}(A_z) < \dim A_z$  for every  $z$ . Thus (10) implies that  $\dim((\text{Fr } A) \setminus H_1)_z < \dim A_z$  for every  $z$ . The conclusion follows by corollary 6.5.

■

**Corollary 7.4** *Let  $Y \subseteq X$  be a pair of definable subsets of  $K^m$ . If  $\dim Y = \dim X$  then  $\dim \text{Int}_X Y = \dim X$ .*

*Proof:* As  $\text{Int}_X Y = Y \setminus \overline{X \setminus Y}$  we have  $Y \setminus \text{Int}_X Y = Y \cap \overline{X \setminus Y}$  hence  $Y \setminus \text{Int}_X Y$  is contained in  $\overline{X \setminus Y} \setminus (X \setminus Y)$ . It follows from theorem 7.1 that

$$\dim(Y \setminus \text{Int}_X Y) < \dim(X \setminus Y) \leq \dim X$$

hence  $\dim X = \max(\dim \text{Int}_X Y, \dim Y \setminus \text{Int}_X Y) = \dim \text{Int}_X Y$ .

■

**Corollary 7.5** *For every definable function  $f : X \subseteq K^m \rightarrow K^n$  let  $\mathcal{C}(f)$  denote the set of points  $x$  in  $X$  such that  $f$  is continuous on a neighbourhood of  $x$  in  $X$ . Then we have  $\dim X \setminus \mathcal{C}(f) < \dim X$ .*

*Proof:* If  $f = (f_1, \dots, f_n)$  then  $X \setminus \mathcal{C}(f)$  is the union of the  $X \setminus \mathcal{C}(f_i)$ 's. So it suffices to prove the result for each  $f_i$  separately, that is we can assume  $n = 1$ . By proposition 5.1,  $X \setminus \mathcal{C}(f)$  has empty interior in  $X$ . So it has dimension  $< \dim X$  by corollary 7.4.

■

**Topological rank** For every subsets  $A, B$  of  $K^m$  we write  $B \ll A$  when  $B$  is a subset of  $A$  with empty interior in  $A$ , that is:

$$B \ll A \iff B \subseteq A \subseteq \overline{A \setminus B}.$$

It is a strict partial order on the non empty sets. We call **topological rank** and denote  $\text{rk } A$  the corresponding rank on the non empty definable subsets of  $K^m$ . So  $\text{rk } A = 0$  if  $A$  is minimal for  $\ll$ , that is if every point of  $A$  is isolated;  $\text{rk } A \geq k + 1$  if  $A \gg B$  for some non empty definable set  $B$  of rank  $\geq k$ . Of course  $\text{rk } A = k$  if  $\text{rk } A \geq k$  but  $\text{rk } A \not\geq k + 1$ . By convention we let  $\text{rk } \emptyset = -\infty$ .

**Proposition 7.6** *For every definable set  $A \subseteq K^m$ ,  $\dim A = \text{rk } A$ .*

*Proof:* We have to prove that for every positive integers  $m, d$  and every definable subset  $A$  of  $K^m$

$$\dim A \geq d \iff \text{rk } A \geq d.$$

Assume that it has been proved for every  $(m, d)$  with  $m \leq n$  (it is obvious in  $K^0$ ) or  $m = n + 1$  and  $d \leq e$  (it is obvious for finite sets). Let  $A$  be a definable subset of  $K^{n+1}$ .

If  $\text{rk } A \geq e + 1$  then  $A$  contains a definable set  $B$  with rank  $e$  such that  $\text{Int}_A B = \emptyset$ . Then  $\dim B \geq e$  by induction hypothesis and  $\dim A > \dim B$  by corollary 7.4, so  $\dim A \geq e + 1$ .

Conversely, if  $\dim A \geq e + 1$  then  $A$  contains a cell  $C$  of dimension  $\geq e + 1$ . It can be chosen with continuous center and bounds by corollary 4.3. Let  $X = \widehat{C}$ , by lemma 6.2  $\dim X = \dim C - \text{tp } C$ . As  $X$  is contained in  $K^m$  the induction hypothesis gives  $Y \ll X$  with rank and dimension  $\dim X - 1$ . Then  $D = C \cap (Y \times K)$  is a cell contained in  $C$  with the same type as  $C$ , hence by lemma 6.2 and the induction hypothesis

$$\dim D = \dim Y + \text{tp } C = (\dim X - 1) + \text{tp } C = \dim C - 1 \geq e$$

By induction hypothesis  $\text{rk } D \geq e$ . But  $Y \subseteq \overline{X \setminus Y}$  and the continuity of the center and bounds of  $C$  imply that  $D \subseteq \overline{C \setminus D}$ , hence  $D \ll C$ . *A fortiori*  $D \ll A$  so  $\text{rk } A \geq e + 1$ .

■

## 8 Cell preparation in $p^2$ -optimal fields

We are going to show that  $p^2$ -optimality is sufficient to obtain a cell preparation theorem for definable functions analogous to theorem 2.8 of [Clu04] for subanalytic functions, with pretty much the same proof and the same consequences.

Actually we will prove it under the assumption (valid in every  $p^2$ -optimal field) that the conclusion of the following lemma is sufficient. Note that this conclusion obviously implies  $p$ -optimality (by applying it the indicator function of any definable subset of  $K^m$ , and using remark 1.1).

**Question 8.1** Does the conclusion of lemma 8.2 imply that  $K$  is  $p^2$ -optimal?

**Lemma 8.2 (Denef)** *Assume that  $K$  is  $p^2$ -optimal. Then for every definable function  $f : X \subseteq K^m \rightarrow K$  there is an integer  $e \geq 1$  such that for every  $x$  in  $X$*

$$|f(x)|^e = \left| \frac{p(x)}{q(x)} \right|$$

with  $p, q$  a pair of 1-basic functions such that  $q(x) \neq 0$  for every  $x$  in  $X$ .

*Proof:* By  $p^2$ -optimality,  $\{(x, t) \in K^m \times K : |t| = |f(x)|\}$  is a boolean combination of 2-basic subsets of  $K^{m+1}$ . The proof of Denef's theorem 6.3 in [Den84] then applies word-for-word, with basic functions instead of polynomial functions. It gives a partition of  $X$  in finitely many definable pieces  $A$ , on each of  $|f|^e = |p_A/q_A|$  for some 1-basic functions such that  $q_A(x) \neq 0$  for every  $x$  in  $A$ . The global functions  $p$  and  $q$  which take value 1 outside  $X$  and are equal respectively to  $p_A, q_A$  on each  $A$  are then 1-basic and give the conclusion. ■

**Remark 8.3** Given an integer  $n_0 \geq 1$ , the set  $1 + \pi^{n_0}R$  is a definable subgroup of  $R^*$  with finite index. Thus in lemma 8.2 we can always assume, refining if necessary the partition of  $X$  involved in the proof (but keeping the same integer  $e$  independently of  $n_0$ ), that for every  $x$  in  $A$

$$f(x)^e = \mathcal{U}_{n_0}(x) \frac{p_A(x)}{q_A(x)}$$

Moreover we can assume as well that  $A$  is a boolean combination of 1-basic sets of the same power  $N$  (see remark 1.1).

**Theorem 8.4 (Cell preparation)** *Assume that the conclusion of lemma 8.2 holds true. Let  $(\theta_i : A_i \subseteq K^{m+1} \rightarrow K)_{i \in I}$  be a finite family of definable functions and  $N_0 \geq 1$  an integer. Then there exists an integer  $e \geq 1$  and, for every  $n \in \mathbf{N}^\times$ , a pair of integers  $M, N$  and a finite family  $\mathcal{H}$  of presented cells mod  $Q_{N,M}^\times$  with continuous centers and bounds such that  $M > 2v(e)$ ,  $eN_0$  divides  $N$ ,  $\mathcal{H}$  refines  $(A_i)_{i \in I}$ , and for every  $(x, t) \in H$ ,*

$$\theta_i(x, t) = \mathcal{U}_{e,n}(x, t) h(x) [\lambda_H^{-1}(t - c_H(x))]^{\frac{\alpha}{e}} \quad (11)$$

for every  $i \in I$  and every  $H \in \mathcal{H}$  contained in  $A_i$ , with  $h : \widehat{H} \rightarrow K$  a continuous definable function and  $\alpha \in \mathbf{Z}$  (both depending on  $i$  and  $H$ )<sup>2</sup>.

All the basic sets and functions involved in the proof are  $(m, 1)$ -basic. So we will omit the prefix  $(m, 1)$  for seek of simplicity.

*Proof:* For each  $i$  let  $e_i$  be an integer,  $\mathcal{A}_i$  a partition of  $A_i$  and  $\mathcal{F}_i$  a family of basic functions, all given by the proof of lemma 8.2 applied to  $\theta_i$ . By replacing each  $e_i$  with a common multiple we can assume that all of them are equal to some integer  $e \geq 1$ . Given an integer  $n \geq 1$  we can refine the partition  $\mathcal{A}_i$  as in remark 8.3 with  $n_0 = n + 2v(e)$ .

Let  $\mathcal{A}$  be a finite family of definable sets refining  $\bigcup_{i \in I} \mathcal{A}_i$ . We can assume that each of them is a boolean combination of basic sets of the same power  $N$ ,

<sup>2</sup>If  $H$  is of type 0 then it is understood that  $\alpha = 0$  and we use the conventions that in this case  $\lambda_H^{-1} = 0$  and  $0^0 = 1$ .

with  $N$  a multiple of  $eN_0$ . For every  $A$  in  $\mathcal{A}$ , every  $i \in I$  such that  $A_i$  contains  $A$  and every  $(x, t)$  in  $A$  we have

$$\theta_i(x, t)^e = \mathcal{U}_{n_0}(x, t) \frac{p_{i,A}(x, t)}{q_{i,A}(x, t)} \quad (12)$$

with  $p_{i,A}$  and  $q_{i,A}$  a pair of basic functions such that  $q_{i,A}(x, t) \neq 0$  on  $A$ .

For each  $A$  in  $\mathcal{A}$  let  $\mathcal{F}_A$  be the set of basic functions involved in a description of  $A$  as a boolean combination of basic sets of power  $N$ . Theorem 3.4 applies to the family  $\mathcal{F}$  of all the basic functions  $p_{i,A}$ ,  $q_{i,A}$  and the functions in  $\mathcal{F}_A$ , for every possible  $A$ 's and  $i$ 's. It gives a partition of  $K^{m+1}$  in finitely many presented cells  $B \bmod K^\times$  such that for every  $f$  in  $\mathcal{F}$  and every  $(x, t)$  in  $B$

$$f(x, t) = \mathcal{U}_M(x, t) h_{f,B}(x) (t - c_B(x))^{\beta_{f,B}} \quad (13)$$

with  $M = n_0 + v(N)$ ,  $h_{f,B} : \widehat{B} \rightarrow K$  a definable function and  $\beta_{f,B}$  a positive integer.

Partitioning  $\widehat{B}$  if necessary, we can assume that the cosets  $h_{f,B}(x)Q_{N,M}^\times$  are constant on  $\widehat{B}$ . Since  $M > v(N)$ ,  $1 + \pi^M R$  is contained in  $Q_{N,M}^\times$ , so  $B$  itself can be partitioned into cells  $H \bmod Q_{N,M}^\times$  such that  $\widehat{H} = \widehat{B}$ ,  $c_H = c_B$  and  $f(x, t)Q_{N,M}^\times$  is constant on  $H$  by (13), for every  $f$  in  $\mathcal{F}$ . *A fortiori*  $f(x, t)P_N^\times$  is constant on  $H$  for every  $f$  in  $\mathcal{F}$  hence each  $A$  in  $\mathcal{A}$  either contains  $H$  or is disjoint from  $H$ , for every  $A$  in  $\mathcal{A}$ . So the family  $\mathcal{H}$  of all among these cells  $H$  which are contained in  $\bigcup \mathcal{A}$  refines  $\mathcal{A}$ , hence refines  $\{A_i : i \in I\}$  as well.

For every cell  $H$  in  $\mathcal{H}$  there is a unique cell  $B$  as above containing  $H$ . For every  $i \in I$  such that  $H$  is contained in  $A_i$ , the unique  $A$  in  $\mathcal{A}$  containing  $B$  is also contained in  $A_i$ . By (13) applied to  $f = p_{i,A}$  and to  $f = q_{i,A}$ , and by (12) we have for every  $(x, t) \in H$

$$\theta_i(x, t)^e = \mathcal{U}_{n_0}(x, t) \frac{\mathcal{U}_M(x, t) h_{p_{i,A},B}(x) (t - c_B(x))^{\beta_{p_{i,A},B}}}{\mathcal{U}_M(x, t) h_{q_{i,A},B}(x) (t - c_B(x))^{\beta_{q_{i,A},B}}} \quad (14)$$

The  $\mathcal{U}_{n_0}$  and  $\mathcal{U}_M$  factors simplify in a single  $\mathcal{U}_{n_0}$  since  $M \geq n_0$ . By construction  $c_H = c_B$  and  $\widehat{H} = \widehat{B}$ . So, for every  $(x, t)$  in  $H$  we get

$$\theta_i(x, t)^e = \mathcal{U}_{n_0}(x, t) g(x) [\lambda_H^{-1}(t - c_H(x))]^\alpha \quad (15)$$

with  $g : \widehat{H} \rightarrow K$  a definable function and  $\alpha \in \mathbf{Z}$  (both depending on  $i$  and  $H$ ). In turn  $\mathcal{U}_{n_0} = \mathcal{U}_{e, n_0 - v(e)}^e$  because  $n_0 > 2v(e)$  (see lemma 1.9). The later can be replaced by  $\mathcal{U}_n^e$  because  $n_0 - v(e) = n + v(e) \geq n$ . So (15) becomes

$$\theta_i(x, t)^e = \mathcal{U}_{e,n}(x, t)^e g(x) \left( [\lambda_H^{-1}(t - c_B(x))]^{\frac{\alpha}{e}} \right)^e \quad (16)$$

This implies that  $g$  takes values in  $P_e$ , hence  $g = h^e$  for some definable function  $h : \widehat{H} \rightarrow K$ , from which (11) follows. By corollary 4.3 these functions  $h$  as well as the center and bounds of every cell in  $\mathcal{H}$ , are piecewise continuous. Thus they can be made continuous by partitioning the cells in  $\mathcal{H}$  *via* an appropriate refinement of their basements.

■

**Corollary 8.5** *Assume that the conclusion of lemma 8.2 holds true. Let  $(\theta_i : A \subseteq K^m \rightarrow K)_{i \in I}$  be a finite family of definable functions with the same domain. Then for every integer  $n \geq 1$ , there exists an integer  $e$ , a semi-algebraic set  $\tilde{A} \subseteq K^m$  and a definable bijection  $\varphi : \tilde{A} \rightarrow A$  such that for every  $i \in I$  and every  $x$  in  $\tilde{A}$*

$$\theta_i \circ \varphi(x) = \mathcal{U}_{e,n}(x) \tilde{\theta}_i(x)$$

with  $\tilde{\theta}_i : \tilde{A} \subseteq K^m \rightarrow K$  a semi-algebraic functions.

*Proof:* The proof goes by induction on  $m$ . Let us assume that it has been proved for some  $m \geq 0$  (it is trivial for  $m = 0$ ) and that a finite family  $(\theta_i)_{i \in I}$  of definable functions is given with domain  $A \subseteq K^{m+1}$ . If  $A$  is a disjoint union of sets  $B$  it suffices to prove the result for the restrictions of the  $\theta_i$ 's to  $B$ . So, for any given integer  $n \geq 1$ , theorem 8.4 with  $N_0 = 1$  reduces to the case when  $A$  is a presented cell mod  $Q_{N,M}^\times$  for some  $N, M$  such that for some  $e_0 \geq 1$  dividing  $N, M > 2v(e_0)$  and for every  $i \in I$  and every  $(x, t)$  in  $A$

$$\theta_i(x, t) = \mathcal{U}_{e_0,n}(x, t) h_i(x) [\lambda_A^{-1}(t - c_A(x))]^{\frac{\alpha_i}{e_0}} \quad (17)$$

with  $h_i : \hat{A} \rightarrow K$  a definable function and  $\alpha_i \in \mathbf{Z}$ .

Let  $e_1 \geq 1$  be an integer,  $Y \subseteq K^m$  a semi-algebraic set,  $\psi : Y \rightarrow \hat{A}$  a definable bijection,  $\tilde{f} : Y \rightarrow K$  a semi-algebraic function for each  $f$  in  $\mathcal{F}$ , all of this given by the induction hypothesis applied to  $\mathcal{F} = \{\mu_A, \nu_A\} \cup \{h_i\}_{i \in I}$ . Let  $\tilde{A}$  be the set of  $(y, s) \in Y \times K$  such that

$$|\tilde{\nu}_A(y)| \leq |s| \leq |\tilde{\mu}_A(x)| \quad \text{and} \quad s \in \lambda_A Q_{N,M}^\times.$$

Then  $\varphi : (y, s) \mapsto (\psi(y), c_A(\psi(y)) + s)$  defines a bijection from  $\tilde{A}$  to  $A$ . For every  $i \in I$  and every  $(y, s) \in \tilde{A}$  we have

$$\theta_i \circ \varphi(y, s) = \mathcal{U}_{e_0,n}(y, s) \mathcal{U}_{e_1,n}(y, s) \tilde{h}_i(y) (\lambda_A^{-1} s)^{\frac{\alpha_i}{e_0}}$$

The first two factors can be replaced by  $\mathcal{U}_{e,n}$  with  $e$  any common multiple of  $e_0$  and  $e_1$ . Since  $\tilde{\theta} : (y, s) \mapsto \tilde{h}_i(y) (\lambda_A^{-1} s)^{\frac{\alpha_i}{e_0}}$  is a semi-algebraic function on  $\tilde{A}$  the conclusion follows.

■

Theorem 8.4 and corollary 8.5 are exactly analogous to theorems 2.8 and 3.1 in [Clu04], except that we obtain a slightly more precise equality of functions mod  $(1 + \pi^n R) \cdot \mathbf{U}_e$  instead of equality of their norm (which is the same as equality of functions mod  $R^*$ ). Thus all the important consequences that are derived from these theorems in [Clu04] for the classical analytic structure remain valid in every  $p^2$ -optimal field.

For applications to parametric integrals, which require numerous specific definitions, we refer the reader to the proofs of theorems 4.2 and 4.4 in [Clu04]. For the classification of definable sets up to isomorphisms, which only uses the ‘‘topological dimension’’ defined in [HM97] for definable sets over any strongly  $p$ -minimal field we have the following.

**Theorem 8.6** *Assume that the conclusion of lemma 8.2 holds true. Then there exists a definable bijection between two infinite definable sets  $A \subseteq K^m$  and  $B \subseteq K^n$  if and only if they have the same dimension.*

*Proof:* If there is a definable bijection (an “isomorphism”) between  $A$  and  $B$  they have the same dimension by corollary 6.4 in [HM97] (as already mentioned, the conclusion of lemma 8.2 implies that  $K$  is  $p$ -optimal hence strongly  $p$ -minimal). Conversely, if  $A$  and  $B$  have the same dimension  $d$  then by corollary 8.5 they are isomorphic to infinite semi-algebraic sets  $\tilde{A}$  and  $\tilde{B}$  respectively, both of which have dimension  $d$  by corollary 6.4 in [HM97] again. Then  $\tilde{A}$  and  $\tilde{B}$  are semi-algebraically isomorphic by the main result of [Clu01], hence  $A$  and  $B$  are isomorphic.

■

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