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Perimeters, uniform enlargement and high dimensions

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Abstract

We study the isoperimetric problem in product spaces equipped with the uniform distance. Our main result is a characterization of isoperimetric inequalities which, when satisfied on a space, are still valid for the product spaces, up to a constant which does not depend on the number of factors. Such dimension free bounds have applications to the study of influences of variables.

1 Introduction

Let (X, d, μ) denote a metric probability space, where X is separable and μ is a Borel probability measure on (X, d) . For a Borel subset A of X , we define, for $r > 0$, the open r -neighbourhood of A by $A_r = \{x \in X \mid d(x, A) < r\}$, and its outer and inner boundary measures (also called Minkowski contents) by

$$\mu^+(A) = \liminf_{r \rightarrow 0^+} \frac{\mu(A_r) - \mu(A)}{r}, \quad \mu^-(A) = \mu^+(X \setminus A).$$

The isoperimetric problem consists in obtaining sharp lower bounds on the above quantities in terms of the measure $\mu(A)$. The isoperimetric function of (X, d, μ) , denoted by $I_{(X, d, \mu)}$ (or simply I_μ when there is no ambiguity on the underlying metric space), is defined for $p \in [0, 1]$ as follows :

$$I_\mu(p) = \inf_{A \subseteq X; \mu(A)=p} \min(\mu^+(A), \mu^-(A)) \tag{1}$$

$$= \inf_{A \subseteq X; \mu(A) \in \{p, 1-p\}} \mu^+(A) \tag{2}$$

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where the infimum is taken over all Borel subsets A of X . As we can see from the definition, I_μ is the largest function such that, for every $A \subseteq X$, $\mu^+(A) \geq I_\mu(\mu(A))$ and for every $t \in [0, 1]$, $I_\mu(t) = I_\mu(1 - t)$. Notice also that $I_\mu(0) = I_\mu(1) = 0$.

Given metric probability spaces (X_i, d_i, μ_i) , $i = 1, \dots, n$, several metric structures can be considered on the product probability space $(X_1 \times \dots \times X_n, \mu_1 \otimes \dots \otimes \mu_n)$. Throughout this paper, we equip this product with the supremum distance $d = d_\infty^{(n)}$ defined by

$$d_\infty^{(n)}((x_1, \dots, x_n), (y_1, \dots, y_n)) := \max_i d_i(x_i, y_i).$$

We shall also say that $d_\infty^{(n)}$ is the ℓ_∞ -combination of the distances d_i , $1 \leq i \leq n$. The isoperimetric problem has been intensively studied in the Riemannian setting, where the geodesic distance on a product manifold is the ℓ_2 -combination of the geodesic distance on the factors. Hence, from a geometric viewpoint, the choice of the ℓ_∞ -combination is less natural than the one of the ℓ_2 -combination $d_2^{(n)}((x_i)_{i=1}^n, (y_i)_{i=1}^n) = (\sum_i d(x_i, y_i)^2)^{1/2}$. Nevertheless, the study of the uniform enlargement has various motivations. We briefly explain some of them.

Firstly the isoperimetric problem for the uniform enlargement is technically easier to deal with in the setting of product spaces, due to the product structure of metric balls. This often allows to work by comparisons. For instance Bollobás and Leader study this problem for the uniform measure on the cube in order to solve the discrete isoperimetric problem on the grid [8]. Since $d_\infty^{(n)} \leq d_2^{(n)} \leq \sqrt{n} d_\infty^{(n)}$, it easily follows that

$$\frac{1}{\sqrt{n}} I_{(X^n, d_\infty^{(n)}, \mu^n)} \leq I_{(X^n, d_2^{(n)}, \mu^n)} \leq I_{(X^n, d_\infty^{(n)}, \mu^n)}.$$

This approach was used e.g. by Morgan [16] for products of two Riemannian manifolds.

Another motivation for studying the isoperimetric problem for the uniform enlargement is that it amounts to the study of the usual isoperimetric problem for a special class of sets. Let us explain this briefly in the setting of \mathbb{R}^n equipped with a probability measure $d\mu(x) = \rho(x)dx$ and the ℓ_∞ distance. If ρ is continuous and $A \subset \mathbb{R}^n$ is a domain with Lipschitz boundary, its outer Minkowski content is

$$\mu^+(A) = \int_{\partial A} \|n_A(x)\|_1 \rho(x) d\mathcal{H}_{n-1}(x),$$

where $n_A(x)$ is a unit outer normal to A at x (unit for the Euclidean length). Consequently, the boundary measure for the uniform enlargement coincides with the usual one $\int_{\partial A} \rho(x) d\mathcal{H}_{n-1}(x)$, for sets A such that almost surely on ∂A the outer normal is equal to a vector of the canonical basis of \mathbb{R}^n (or its opposite). These so-called rectilinear sets comprise cartesian products of intervals $I_1 \times \dots \times I_n$, their finite unions and their complements. Hence the isoperimetric problem for the uniform enlargement is closely connected to the usual isoperimetric problem restricted to the class of rectilinear sets (actually, a smooth domain A can be approximated by rectilinear sets in such

a way that their boundary measures approach the one of A for the uniform enlargement). Note that rectilinear sets naturally appear when studying the supremum of random variables, as $\{x \in \mathbb{R}^n \mid \max_i x_i \in [a, b]\}$ is rectilinear. This was one of the original motivations of Bobkov and Bobkov-Houdré [5, 7] for studying isoperimetry for the uniform enlargement.

Eventually, let us mention that isoperimetric inequalities for the uniform enlargement naturally appear in the recent extension by Keller, Mossel and Sen [12] of the theory of influences of variables to the continuous setting.

Computing exactly the isoperimetric profile is a hard task, even in simple product spaces (see e.g. the survey article [18]). However, various probabilistic questions involve sequences of independent random variables and require lower estimates on the isoperimetric profile of n -fold product spaces, which actually do not depend on the value of n . First observe that for all integers $n \geq 1$,

$$I_{(X^{n+1}, d_\infty^{(n+1)}, \mu^{n+1})} \leq I_{(X^n, d_\infty^{(n)}, \mu^n)},$$

which holds because for every set $A \subset X^n$, $\mu^{n+1}(A \times X) = \mu^n(A)$ and $(\mu^{n+1})^+(A \times X) = (\mu^n)^+(A)$. Therefore one may define the so-called infinite dimensional isoperimetric profile of (X, d, μ) as follows : for $t \in [0, 1]$,

$$I_{\mu^\infty}(t) := \inf_{n \geq 1} I_{(X^n, d_\infty^{(n)}, \mu^n)} \leq I_{(X, d, \mu)}.$$

This quantity has been investigated by Bobkov [5], Bobkov and Houdré [7] and Barthe [3]. In particular, Bobkov has put forward a sufficient condition for the equality $I_{\mu^\infty} = I_\mu$ to hold. This condition depends only on the function I_μ but it is rather restrictive. However it allowed to get a natural family of isoperimetric inequalities for which there exists $K > 1$ such that $I_\mu \geq I_{\mu^\infty} \geq \frac{1}{K} I_\mu$. We shall say in this case that the isoperimetric inequality with profile I_μ tensorizes, up to a factor K .

The goal of this article is to provide a workable necessary and sufficient condition for the latter property to hold. We were inspired by a sufficient condition for tensorization, given by E. Milman [15] in the setting of ℓ_2 -distances on products. We now describe the plan of the paper. In the next section, we recall the known sufficient condition for $I_{\mu^\infty} = I_\mu$ and propose a new one. Building on this, we provide a sufficient condition for tensorization up to a factor in the third section. By a careful study of product sets, we actually show that this condition is also necessary. The final section draws consequences of our isoperimetric inequalities to the theory of influences of variables : following the argument of [12], we obtain an extension of the Kahn-Kalai-Linial theorem about the existence of a coordinate with a large influence.

Let us conclude this introduction with some useful notation. If (Y, ρ) is a metric space we define the modulus of gradient of a locally Lipschitz function $f : Y \rightarrow \mathbb{R}$ by :

$$|\nabla f|(x) = \limsup_{\rho(x, y) \rightarrow 0^+} \frac{|f(x) - f(y)|}{\rho(x, y)},$$

this quantity being zero at isolated points. Note that when the distance is given by a norm on a vector space, that is $\rho(x, y) = \|x - y\|$, and when f is differentiable, then the modulus of gradient coincides with $\|Df(x)\|_*$. We shall work under the following Hypothesis (\mathcal{H}): for every $m, n \in \mathbb{N}^*$ and for every locally Lipschitz function $f : X^{m+n} \rightarrow \mathbb{R}$, for μ^{m+n} -almost every point $(x, y) \in X^m \times X^n$:

$$|\nabla f|(x, y) = |\nabla_x f|(x, y) + |\nabla_y f|(x, y).$$

This assumption holds in various cases : when (X, d) is an open metric subset of a Minkowski space $(\mathbb{R}^n, \|\cdot\|)$ and when μ is absolutely continuous with respect to Lebesgue's measure, or for Riemannian manifolds when the measure is absolutely continuous with respect to the volume form (as a consequence of Rademacher's theorem of almost everywhere differentiability of Lipschitz functions). On the contrary, this hypothesis often fails in discrete settings.

2 Sharp isoperimetric inequalities

We start by recalling a couple of important results about extremal half-spaces for the isoperimetric problem. The first one below is due to Bobkov and Houdré [6] and deals with the real line. Before stating it, we need to introduce some notations. Let \mathcal{M} be the set of Borel probability measures on \mathbb{R} which are concentrated on a possibly unbounded interval (a, b) and have a density f which is positive and continuous on (a, b) . For $\mu \in \mathcal{M}$, the distribution function $F_\mu(x) := \mu((-\infty, x])$ is one-to-one from (a, b) to $(0, 1)$ and one may define

$$J_\mu(t) = f(F_\mu^{-1}(t)), \quad t \in (0, 1).$$

We may as well consider J_μ as a function on $[0, 1]$ by setting $J_\mu(0) = J_\mu(1) = 0$. The value of $J_\mu(t)$ represents the boundary measure of the half-line of measure t starting at $-\infty$. Let $\mathcal{L} \subset \mathcal{M}$ denote the set of (non-Dirac) log-concave probability measures on \mathbb{R} (the density f is of the form e^{-c} for some convex function c).

Proposition 1 ([6]). *The map $\mu \mapsto J_\mu$ is one-to-one between the set \mathcal{M} and the set of positive continuous functions on $(0, 1)$. It is also one-to-one between the subset \mathcal{L} of log-concave probability measures and the set of positive concave functions on $(0, 1)$. Moreover for $\mu \in \mathcal{M}$, the following properties are equivalent :*

(i) $I_\mu = J_\mu$ (meaning for any $p \in (0, 1)$, the infimum in (1) is attained for the set $(-\infty, F_\mu^{-1}(p)]$),

(ii) the measure μ is symmetric around its median, i.e. J_μ is symmetric around $\frac{1}{2}$, and for all $p, q > 0$ such that $p + q < 1$,

$$J_\mu(p + q) \leq J_\mu(p) + J_\mu(q).$$

The next basic lemma allows to compare the various conditions on isoperimetric profiles that appear in the rest of the article. In particular, it shows that the above result encompasses a classical theorem of Borell, asserting that for even log-concave probability measures on \mathbb{R} , half-lines are solutions to the isoperimetric problem.

Lemma 1. *Let $T \in (0, +\infty]$ and $K : [0, T) \rightarrow \mathbb{R}^+$ be a non-negative function. Consider the following properties that K may verify :*

(i) K is concave,

(ii) $t \mapsto K(t)/t$ is non-increasing,

(iii) for all $x, y \in [0, T)$ with $a + b < T$, it holds $K(a + b) \leq K(a) + K(b)$.

Then (i) \implies (ii) and (ii) \implies (iii).

Proof. If K is concave then $t \mapsto (K(t) - K(0))/t$ is non-increasing. Since $t \mapsto K(0)/t$ is non-increasing as well, the first implication follows. Assuming (ii) and without loss of generality $a \leq b$,

$$K(a + b) \leq (a + b) \frac{K(b)}{b} = a \frac{K(b)}{b} + K(b) \leq K(a) + K(b).$$

□

The next result provides sharp isoperimetric inequalities in high dimensions. It goes back to the dissertation thesis of S. Bobkov. See also [5].

Theorem 1. *Let $J : [0, 1] \rightarrow \mathbb{R}^+$ be a concave function, with $J(t) = J(1 - t)$ for all $t \in [0, 1]$. Assume that for all $a, b \in [0, 1]$,*

$$J(ab) \leq aJ(b) + bJ(a). \tag{3}$$

Then for every space (X, d, μ) verifying Hypothesis (\mathcal{H}) ,

$$I_\mu \geq J \implies I_{\mu^\infty} \geq J.$$

Moreover there exists an even log-concave probability measure ν on \mathbb{R} such that $I_\nu = I_{\nu^\infty} = J$ and for every n , coordinate half-spaces are solutions of the isoperimetric problem for ν^n .

Condition (3) may be verified in a few instances as $J(t) = t(1 - t)$. However, it is not so easy to deal with, in particular in conjunction with the symmetry assumption. For these reasons, stronger conditions of more local nature are useful. In [3], it is shown that (3) is verified when J is concave, twice differentiable and $-1/J''$ is concave. Observe that condition (3) amounts to the subadditivity of the function $u \mapsto e^u J(e^{-u})$ on \mathbb{R}^+ . Hence, using the second part of Lemma 1, we obtain that the condition " $t \mapsto J(t)/(t \log(1/t))$ is non-decreasing" implies (3) as well. By a tedious but straightforward calculation, this yields a neat variant of one of the main results of [3]:

Corollary 1. For $\beta \in [0, 1]$, the function K_β defined for $t \in [0, 1]$ by

$$K_\beta(t) := t(1-t) \log^\beta \left(\frac{3}{t(1-t)} \right),$$

satisfies that for every space (X, d, μ) verifying Hypothesis (\mathcal{H}) and all $c \geq 0$,

$$I_\mu \geq cK_\beta \implies I_{\mu^\infty} \geq cK_\beta.$$

Let us point out that (3) is not the best sufficient condition for the conclusion of the above theorem to hold. The optimal condition given by Bobkov's approach is the following : for every Borel probability measure N on $[0, 1]$,

$$J \left(\int t dN(t) \right) \leq \int J(t) dN(t) + \int_0^1 J(N([0, t])) dt.$$

Actually when $\mu \in \mathcal{F}$ is a probability measure on \mathbb{R} and $J = J_\mu = I_\mu$, it is not hard to check, considering subgraphs, that the above condition is necessary and sufficient for having $I_\mu = I_{\mu^\infty}$. However this condition is hard to verify in practice, and most of the work in Bobov's proof consists in showing that when J is concave, it boils down to (3).

Next, we develop a different approach to dimension free isoperimetric inequalities. We use classical methods to make a link between isoperimetric inequalities, and some Beckner-type functional inequalities, which nicely tensorize.

Lemma 2. Let $a \in (0, 1]$ and (X, d, μ) be a metric probability space. Let $c > 0$, then the following assertions are equivalent :

(i) For all $p \in [0, 1]$, $cI_\mu(p) \geq p - p^{\frac{1}{a}}$,

(ii) For every locally Lipschitz function $f : X \rightarrow [0, 1]$, $c \int |\nabla f| d\mu \geq \int f d\mu - (\int f^a d\mu)^{\frac{1}{a}}$.

Proof. Assuming (i), we apply the co-area inequality to an arbitrary locally Lipschitz function f (see e.g. [6]); next we take advantage of the isoperimetric inequality for μ :

$$\begin{aligned} c \int |\nabla f| d\mu &\geq c \int_0^1 \mu^+(\{f \geq t\}) dt \\ &\geq \int_0^1 \left(\mu(\{f \geq t\}) - \mu(\{f \geq t\})^{\frac{1}{a}} \right) dt \\ &= \int f d\mu - \int_0^1 \mu(\{f \geq t\})^{\frac{1}{a}} dt. \end{aligned}$$

In order to conclude that the second assertion is valid, we apply the Minkowski inequality with exponent $1/a \geq 1$:

$$\begin{aligned} \left(\int_0^1 \mu(\{f \geq t\})^{\frac{1}{a}} dt \right)^a &= \left(\int_0^1 \left(\int \mathbf{1}_{f(s) \geq t} d\mu(s) \right)^{\frac{1}{a}} dt \right)^a \\ &\leq \int \left(\int_0^1 (\mathbf{1}_{f(s) \geq t})^{\frac{1}{a}} dt \right)^a d\mu(s) = \int f^a d\mu. \end{aligned}$$

The fact that the second assertion implies the first one is rather standard : one applies the functional inequalities to Lipschitz approximations of the characteristic function of an arbitrary Borel set $A \subset X$ (see Lemma 3.7 in [6]). This yields $c\mu^+(A) \geq \mu(A) - \mu(A)^{\frac{1}{a}}$. Applying the inequality to $1 - f$ instead of f and using $|\nabla f| = |\nabla(1 - f)|$ and then taking approximations of $\mathbf{1}_A$ gives $c\mu^+(A) \geq 1 - \mu(A) - (1 - \mu(A))^{\frac{1}{a}}$ for all A , which is equivalent to $c\mu^-(A) \geq \mu(A) - \mu(A)^{\frac{1}{a}}$ for all Borel sets A . \square

The following extension of the classical subadditivity property of the variance is due to Latała and Oleszkiewicz [14]. It allowed them to devise functional inequalities with the tensorization property. Actually, they focused on Sobolev inequalities involving L_2 -norms of gradients, with applications to concentration inequalities. Here we aim at functional inequalities involving L_1 -norms of gradients and provide information about isoperimetric inequalities.

Lemma 3. *Let (Ω_1, μ_1) and (Ω_2, μ_2) be probability spaces and let $(\Omega, \mu) = (\Omega_1 \times \Omega_2, \mu_1 \otimes \mu_2)$ be their product probability space. For any non-negative random variable Z defined on (Ω, μ) and having finite first moment and for any strictly convex function ϕ on $[0, +\infty)$ such that $\frac{1}{\phi'}$ is a concave function, the following inequality holds true :*

$$\mathbb{E}_\mu \phi(Z) - \phi(\mathbb{E}_\mu Z) \leq \mathbb{E}_\mu \left([\mathbb{E}_{\mu_1} \phi(Z) - \phi(\mathbb{E}_{\mu_1} Z)] + [\mathbb{E}_{\mu_2} \phi(Z) - \phi(\mathbb{E}_{\mu_2} Z)] \right).$$

Theorem 2. *Let (X, d, μ) be a metric probability space verifying hypothesis (\mathcal{H}) . Let $a \in [\frac{1}{2}, 1]$ and $c > 0$. If for all $p \in (0, 1)$, $I_\mu \geq c(p - p^{\frac{1}{a}})$, then for all $p \in (0, 1)$,*

$$I_{\mu^\infty}(p) \geq c(p - p^{\frac{1}{a}}).$$

Proof. By Lemma 2, we know that for every locally Lipschitz function $f : X \rightarrow [0, 1]$

$$\frac{1}{c} \int |\nabla f| d\mu \geq \int f d\mu - \left(\int f^a d\mu \right)^{\frac{1}{a}}. \quad (4)$$

We shall prove that this functional inequality tensorizes, meaning that for all n the same property is verified by μ^n . Applying Lemma 2 again will give the claimed dimension-free isoperimetric inequality.

Checking the tensorization property is done along the same lines as in [14]. Assume that (X_1, ν_1, d_1) and (X_2, ν_2, d_2) satisfy (4). Since $a \in [\frac{1}{2}, 1]$, Lemma 3 applies to $\Phi(t) = t^{\frac{1}{a}}$ and gives

$$\begin{aligned} & \int f d\nu_1 d\nu_2 - \left(\int f^a d\nu_1 d\nu_2 \right)^{\frac{1}{a}} = \int \Phi(f^a) d\nu_1 d\nu_2 - \Phi \left(\int f^a d\nu_1 d\nu_2 \right) \\ & \leq \int \left(\int \Phi(f^a) d\nu_1 - \Phi \left(\int f^a d\nu_1 \right) \right) d\nu_2 + \int \left(\int \Phi(f^a) d\nu_2 - \Phi \left(\int f^a d\nu_2 \right) \right) d\nu_1 \\ & \leq \frac{1}{c} \int (|\nabla_1 f| + |\nabla_2 f|) d\nu_1 d\nu_2, \end{aligned}$$

where $|\nabla_i f|$ is the norm of the gradient of f taken with respect to the i -th variable. When $\nu_1 = \mu^m$ and $\nu_2 = \mu^n$, we may apply Hypothesis (\mathcal{H}) to replace the function in the latter integral by the norm of the full gradient $|\nabla f|$. This allows to show by induction that for all n , μ^n verifies the claimed functional inequality. \square

This result readily generalizes :

Theorem 3. *Let $c : [\frac{1}{2}, 1] \rightarrow \mathbb{R}^+$, and consider for $p \in [0, 1]$,*

$$L(p) := \sup_{a \in [\frac{1}{2}, 1]} c(a) \max \left\{ p - p^{\frac{1}{a}}, 1 - p - (1 - p)^{\frac{1}{a}} \right\}.$$

If (X, d, μ) satisfies (\mathcal{H}) and $I_\mu \geq L$ then

$$I_{\mu^\infty} \geq L.$$

Moreover there exists an even probability measure ν on \mathbb{R} such that $I_\nu = I_{\nu^\infty} = L$ and such that for all n , coordinate half-spaces are solutions to the isoperimetric problem for ν^n .

Proof. Observe that since, by definition, isoperimetric functions of probability measures are symmetric with respect to $\frac{1}{2}$, the property for all $p \in [0, 1]$, $I_\mu \geq c(p - p^{\frac{1}{a}})$ is equivalent to $I_\mu(p) \geq cM_a(p)$, for all p , where

$$M_a(p) := \max \left\{ p - p^{\frac{1}{a}}, 1 - p - (1 - p)^{\frac{1}{a}} \right\}.$$

Hence the fact that $I_\mu \geq L$ implies $I_{\mu^\infty} \geq L$ is a direct consequence of the previous theorem, applied for all values of a .

Next, it is not hard to check that for $a \in [\frac{1}{2}, 1]$, M_a is subadditive, being a supremum of two concave functions defined on $[0, 1]$. And, since the property " $J(x + y) \leq J(x) + J(y)$ for all x, y " is stable under supremum, it follows that L is also subadditive.

Hence, by Proposition 1, there exists an even probability measure ν on \mathbb{R} such that $I_\nu = L$ and half-lines solve the isoperimetric problem for ν . As we just proved, $I_\nu \geq L$ ensures that $I_{\nu^\infty} \geq L$. Combining this with $L = I_\nu \geq I_{\mu^\infty}$ yields $I_{\nu^\infty} = L$. The coordinate halfspace $\{x \in \mathbb{R}^n \mid x_1 \leq t\}$

has same measure and boundary measure (for ν^n), as the set $(-\infty, t]$ (for ν). It is then clear that it solves the isoperimetric problem. □

Remark that for $a \in (\frac{1}{2}, 1)$, the function $M_a(p) = \max \{p - p^{\frac{1}{a}}, 1 - p - (1 - p)^{\frac{1}{a}}\}$ is not concave, hence the measure ν_a is not log-concave. Actually, M_a does not even have its maximum at $\frac{1}{2}$. Hence it cannot be obtained as a supremum of concave functions which are in addition symmetric around $1/2$. Therefore it gives a genuinely new example of a measure for which coordinate half-spaces solve the isoperimetric problem in any dimension (that could not be deduced from Theorem 1).

3 Approximate inequalities

Let us start with some notations. Given two non-negative functions f, g defined on a set $S \subset \mathbb{R}$ and $D \geq 1$, we write $f \approx_D g$ and say that f and g are equivalent up to a factor D if there exists $a > 0$ such that for all $x \in S$, $a g(x) \leq f(x) \leq D a g(x)$. We write $f \approx g$ when there exists D such that $f \approx_D g$.

We say that a non-negative function f defined on a set $S \subset \mathbb{R}$ is essentially non-decreasing (with constant $D \geq 1$) when there exists a non-decreasing function g on S such that $f \approx_D g$. In the same way, we may define the notion of essentially non-increasing functions.

Also, a non-negative function f defined on an interval is said to be essentially concave (or pseudo-concave) if it is equivalent to a concave function.

The next proposition provides workable formulations of the above definitions. The part about essentially concave functions is due to Peetre [17].

Lemma 4. *Let f be a non-negative function defined on $S \subset \mathbb{R}$. Then f is essentially non-decreasing (resp. essentially non-increasing) with constant $D \geq 1$ if and only if for every $s \leq t$ in S ,*

$$f(s) \leq D f(t) \quad (\text{resp. } f(t) \geq D f(s)).$$

When f is defined on $(0, +\infty)$, the following assertions are equivalent :

- (i) *f is essentially concave with some constant C_1 ,*
- (ii) *There exists $C_2 \geq 1$ such that for all $s, t \in \mathbb{R}_+^*$, $f(s) \leq C_2 \max\left(1, \frac{s}{t}\right) f(t)$.*
- (iii) *There exists $C_3 \geq 1$ such that on \mathbb{R}_+^* , f is essentially non-decreasing and $t \mapsto \frac{f(t)}{t}$ is essentially non-increasing, both with constant C_3 .*

Moreover, the smallest possible constants verify $C_1/2 \leq C_2 = C_3 \leq C_1$.

Proof. The argument for essentially non-decreasing functions is very simple and we skip it. Let us just point out that it involves the least non-decreasing function above f , which is given by $\hat{f}(t) := \sup\{f(x) \mid x \in S \cap (-\infty, t]\}$.

Next let us focus on concavity issues. The equivalence of the last two statements is obvious.

Assume f is essentially concave on \mathbb{R}_+^* . Then there exists a concave function h on \mathbb{R}_+^* which is equivalent to f . And as f is positive, h is positive, therefore, being concave, h is necessarily non-decreasing on \mathbb{R}_+^* . Moreover $t \mapsto \frac{h(t)}{t}$ is non-increasing on \mathbb{R}_+^* . So f satisfies the third condition.

Eventually, let us assume the second condition and show that f is equivalent to a concave function. The natural guess is the least concave majorant of f , which is explicitly given for $t > 0$ by

$$\hat{f}(t) := \sup \left\{ \sum_{i=1}^n \lambda_i f(t_i) \mid n \in \mathbb{N}^*, \lambda_i \geq 0, t_i > 0, \sum_{i=1}^n \lambda_i = 1 \text{ and } \sum_{i=1}^n \lambda_i t_i = t \right\}.$$

By definition $f \leq \hat{f}$. Let $n \in \mathbb{N}^*$, $t \in \mathbb{R}_+^*$, $(\lambda_i)_{1 \leq i \leq n}$ and $(t_i)_{1 \leq i \leq n}$ such that, for all i , $\lambda_i \geq 0$, $\sum_{i=1}^n \lambda_i = 1$ and $\sum_{i=1}^n \lambda_i t_i = t$. Using the hypothesis, we obtain

$$\sum_{i=1}^n \lambda_i f(t_i) \leq C_2 \sum_{i=1}^n \lambda_i \max\left(1, \frac{t_i}{t}\right) f(t) \leq C_2 \left(\sum_{i=1}^n \lambda_i + \sum_{i=1}^n \frac{\lambda_i t_i}{t} \right) f(t) = 2C_2 f(t)$$

Therefore $f \leq \hat{f} \leq 2C_2 f$ and we have shown that f is essentially concave. \square

We are now ready to state our main results :

Theorem 4. *Let J be a non-negative function defined on $[0, 1]$ with $J(0) = 0$. Assume that it is symmetric around $\frac{1}{2}$ (i.e. for every $t \in [0, 1]$, $J(t) = J(1 - t)$) and that the function*

$$t \in (0, 1) \mapsto \frac{J(t)}{t \log(1/t)}$$

is essentially non-decreasing with constant D . Then for every metric probability space (X, d, μ) satisfying Hypothesis (\mathcal{H}) :

$$I_\mu \geq J \implies I_{\mu^\infty} \geq \frac{1}{c_D} J,$$

with $c_D = 2(D/\log 2)^2 \leq 5D^2$. Moreover, there exists a symmetric log-concave measure ν on the real line such that, on $[0, 1]$, $J \approx I_\nu \approx I_{\nu^\infty}$.

If in addition J is concave one can take $c_D = 2D$ for $D > 1$ and $c_1 = 1$.

Remark 1. This result should be compared to a theorem of E. Milman in [15], where a similar condition is given for dimension-free isoperimetric inequalities for the ℓ_2 combination of distances on products (in other words for the Euclidean enlargement). His condition involves an essential monotonicity property of J/I_γ where γ is the one-dimensional standard Gaussian measure. On $(0, 1/2]$ it is known that $I_\gamma(t) \approx t\sqrt{\log(1/t)}$.

In order to formulate a converse statement, we introduce the following hypothesis : we say that (X, d, μ) enjoys the regularity property (\mathcal{R}) if for all $t \in (0, 1)$, $I_\mu(t) < +\infty$ and for all $n \in \mathbb{N}^*$, $t \in (0, 1)$ and $\varepsilon > 0$ there exists a Borel set $A \subset X^n$ with $\mu^n(A) = t$, $(\mu^n)^+(A) \leq I_{\mu^n}(t) + \varepsilon$ and

$$(\mu^n)^+(A) = \lim_{h \rightarrow 0^+} \frac{\mu^n(A_h \setminus A)}{h},$$

where the products X^n are equipped with the uniform distance. This hypothesis means that there are almost solutions of the isoperimetric problems for which the liminf in the definition of the Minkowski content is actually a real limit. Thanks to Theorem 15 in [2] it is not hard to check this property for log-concave measures on the real line. We will give more comments on this hypothesis in Remark 3 below.

Theorem 5. *Let (X, d, μ) satisfy hypothesis (\mathcal{R}) . Then the map*

$$t \in (0, 1) \mapsto \frac{I_{\mu^\infty}(t)}{t \log(1/t)}$$

is continuous and essentially non-decreasing.

Combining these two theorems, we can formulate our results as an equivalence :

Corollary 2. *Let (X, d, μ) denote a metric space equipped with a Borel probability measure μ and satisfying hypothesis (\mathcal{R}) and (\mathcal{H}) . Then the following assertions are equivalent :*

- (i) *There exists a constant C such that $\frac{I_\mu}{J_1}$ is essentially non-decreasing on $(0, 1)$ with constant C ,*
- (ii) *There exists a constant $K \geq 1$ such that, on $[0, 1]$, $\frac{1}{K}I_\mu \leq I_{\mu^\infty} \leq I_\mu$.*

We introduce two functions, both defined on $[0, 1]$ by : $J_0(t) = t$ and $J_1(t) = t \log \frac{1}{t}$. The next lemma gives a different formulation of the main condition appearing in the previous theorems.

Lemma 5. *Let $K : [0, 1] \rightarrow \mathbb{R}_+$ be a non-negative function such that K is symmetric with respect to $\frac{1}{2}$ (i.e. for $t \in [0, 1]$, $K(t) = K(1 - t)$). Then the following assertions are equivalent :*

- (i) *There is a constant C such that $\frac{K}{J_1}$ is essentially non-decreasing on $(0, 1)$ with constant C .*
- (ii) *There exists constants C_0 and C_1 such that $\frac{K}{J_0}$ is essentially non-increasing on $(0, \frac{1}{2}]$ with constant C_0 and $\frac{K}{J_1}$ is essentially non-decreasing on $(0, \frac{1}{2}]$ with constant C_1 .*

Moreover, the smallest possible constants verify $C \leq \frac{C_0 C_1}{\log 2}$ and $C_0 \leq \frac{C}{\log 2}$, $C_1 \leq C$.

Proof. We use the concavity of the map $t \mapsto (1-t) \log \frac{1}{1-t}$, which yields, for every $t \in [0, \frac{1}{2}]$, $t \log 2 \leq (1-t) \log \frac{1}{1-t} \leq t$. Assuming (i), $\frac{K}{J_1}$ is essentially non-decreasing on $(0, \frac{1}{2}]$ with constant C . For the second part of the assertion, let $0 < s \leq t \leq \frac{1}{2}$. Then,

$$\frac{K(t)}{t} \leq \frac{K(1-t)}{(1-t) \log \frac{1}{1-t}} \leq C \frac{K(1-s)}{(1-s) \log \frac{1}{1-s}} \leq \frac{C}{\log 2} \frac{K(s)}{s}. \quad (5)$$

For the converse implication: assuming (ii), we first check that $\frac{K}{J_1}$ is essentially non-decreasing on $[\frac{1}{2}, 1)$. Let $\frac{1}{2} \leq s \leq t < 1$, then

$$\frac{K(s)}{s \log \frac{1}{s}} \leq \frac{1}{\log 2} \frac{K(1-s)}{1-s} \leq \frac{C_0}{\log 2} \frac{K(1-t)}{1-t} \leq \frac{C_0}{\log 2} \frac{K(t)}{t \log \frac{1}{t}}.$$

To get the property on the whole interval $(0, 1)$, it suffices to use $\frac{1}{2}$ as an intermediate point. \square

The next corollary describes the possible size of an infinite dimensional isoperimetric profile:

Corollary 3. *Let (X, d, μ) denote a metric space equipped with a Borel probability measure μ and satisfying hypotheses (\mathcal{R}) and (\mathcal{H}) .*

If $\inf_{t \in (0, \frac{1}{2}]} \frac{I_\mu(t)}{t} = 0$ then I_{μ^∞} is identically 0, else there exist $\alpha, \beta > 0$ such that for all $t \in [0, 1]$,

$$\alpha \min(t, 1-t) \leq I_{\mu^\infty}(t) \leq \beta \min\left(t \log \frac{1}{t}, (1-t) \log \frac{1}{1-t}\right).$$

Remark 2. The function defined on $[0, 1]$ by $t \mapsto \min(t, 1-t)$ is the isoperimetric function of the double-sided exponential measure on \mathbb{R} , $e^{-|x|} dx / 2$. Using the notation and results of Corollary 1, we observe that it is equivalent to the function $K_0(t) = t(1-t)$. Moreover there is a log-concave probability measure ℓ_0 on the real line for which $K_0 = I_{\ell_0} = I_{\ell_0^\infty}$ (actually, ℓ_0 is the standard logistic measure ℓ with density $\frac{e^{-x}}{(1+e^{-x})^2}$ with respect to Lebesgue's measure). Hence the lower bound is optimal up to the multiplicative factor.

The upper bound of I_{μ^∞} given in the above corollary is due to Bobkov and Houdré [6]. A similar remark applies to it: the quantity in the upper estimate is equivalent to the function K_1 of Corollary 1, which is also an infinite dimensional isoperimetric profile (of a measure which is reminiscent of Gumble laws, as its distribution function is of the order of $e^{-\beta e^{-y}}$ when $y \rightarrow -\infty$, for some $\beta > 0$).

The fact that the infinite dimensional isoperimetric profile is either trivial, or at least as big as the one of the exponential measure was already discovered, in slightly different forms, by Talagrand [19] and by Bobkov and Houdré [7].

Proof of Corollary 3. By Theorem 5, there exists $C \geq 1$ such that for all $t \in (0, 1/2]$,

$$I_{\mu^\infty}(t) \leq Ct \log\left(\frac{1}{t}\right) \times \frac{2}{\log 2} I_{\mu^\infty}\left(\frac{1}{2}\right). \quad (6)$$

Applying Theorem 5 again, together with Lemma 5, we get that there exists $D \geq 1$ such that for all $t \in (0, 1/2]$,

$$D \frac{I_{\mu^\infty}(t)}{t} \geq 2I_{\mu^\infty}\left(\frac{1}{2}\right).$$

Therefore, assuming $\inf_{t \in (0, \frac{1}{2}]} \frac{I_\mu(t)}{t} = 0$, and using that $I_\mu \geq I_{\mu^\infty}$, we can deduce that $I_{\mu^\infty}(\frac{1}{2}) = 0$. Then (6) and the symmetry of isoperimetric functions yield $I_{\mu^\infty} = 0$ pointwise.

Next assume that there exists $\kappa > 0$ such that $I_\mu(t) \geq \kappa t$ for all $t \in (0, 1/2]$. Then Theorem 4 applies to $J(t) := \kappa \min(t, 1 - t)$ (Lemma 5 gives a quick way to check the hypothesis) and gives $I_{\mu^\infty} \geq cJ$ for some $c > 0$. \square

Remark 3. Our results are stated for general metric spaces, but are devised for continuous settings (e.g. for which the values taken by the measure cover all $[0, 1]$). This is why additional hypotheses appear in our statements. One may find Hypothesis (\mathcal{H}) quite natural (it is related to a.e. differentiability of Lipschitz functions). On the other hand, Hypothesis (\mathcal{R}) is more demanding, as it seems to require approximation theorems by smooth sets.

Let us point out a possible variant of Theorem 5 where all the hypotheses are incorporated in the structure of the ambient space: assume that X is a finite dimensional vector space of dimension p , that the distance d is induced by a norm N on X and that μ has a positive C^1 density h with respect to Lebesgue's measure, $\mu = h \cdot \mathcal{L}^p$. We equip the product spaces X^n with d_∞ , the ℓ_∞ -combination of N , i.e. for $x, y \in X^n$, $d_\infty(x, y) = \max_{1 \leq i \leq n} N(x_i, y_i)$. Then, instead of using the Minkowski content as a definition of the boundary measure, let us chose the notion of generalized perimeter instead : if $A \subseteq X^n$ is measurable, then

$$P_{\mu^{\otimes n}, \infty} = \sup \left\{ \int_A \sum_{i=1}^n |\nabla_i(\varphi h)| d\mathcal{L}^{np} \mid \varphi \in C_c^1(X^n) \text{ and } \sup_{x \in X^n} d_\infty(\varphi(x), 0) \leq 1 \right\},$$

where $|\nabla f|$ is the modulus of gradient of f .

Since the perimeter is defined as a supremum (recall that the Minkowski content is an inferior limit), the proof of Lemma 8 below does not require any regularity assumption. Hence the proof of Theorem 5 applies without any changes and does not require (\mathcal{R}). The proof of Theorem 4 also applies to this new setting, without assuming (\mathcal{H}), with the following main modification: instead of using functional inequalities for locally Lipschitz functions, we work in the class of functions of bounded variations. We refer the reader to the book of Ambrosio, Fusco and Pallara [1] for an exhaustive study of this approach in the Euclidean case. This requires to use various results about these functions: co-area inequality (Theorem 3.40), approximation by smooth functions (Theorem 3.9), approximate differentiability (Theorem 3.83 and Proposition 3.92 among others).

3.1 Proof of Theorem 4

We start with a few preliminary statements.

Lemma 6. Consider a function $K : [0, 1] \rightarrow \mathbb{R}^+$ with $K(0) = 0$. Assume that K is symmetric with respect to $\frac{1}{2}$ and that $\frac{K}{J_1}$ is essentially non-decreasing on $(0, 1)$ with constant D . Then

(i) K is essentially non-decreasing on $[0, \frac{1}{2}]$ with constant $\frac{2D}{e \log 2}$,

(ii) K is essentially concave. More precisely there exists a concave function $I : [0, 1] \rightarrow \mathbb{R}^+$, which is symmetric with respect to $\frac{1}{2}$, and is equivalent to K up to a factor $2D/\log 2$.

Proof. Observe that the function $J_1(t) = t \log(1/t)$ is increasing on $(0, 1/e]$ and decreasing on $[1/e, 1)$. Its maximum is therefore $J(1/e) = 1/e$.

Assume that $0 \leq s \leq t \leq \frac{1}{2}$. Then, by hypothesis $K(s) \leq D \frac{J_1(s)}{J_1(t)} K(t)$. If $t \leq \frac{1}{e}$, we can conclude that $K(s) \leq DK(t)$. If $t \in (1/e, 1/2]$, we argue differently

$$K(s) \leq D \frac{J_1(s)}{J_1(t)} K(t) \leq D \frac{J_1(1/e)}{J_1(1/2)} K(t) = \frac{2D}{e \log 2} K(t).$$

This concludes the proof of (i).

Next, let us prove (ii). Consider the map \tilde{K} defined on \mathbb{R}_+ by :

$$\tilde{K}(t) = \begin{cases} K(t) & \text{if } t \in [0, \frac{1}{2}) \\ K(\frac{1}{2}) & \text{if } t \geq \frac{1}{2} \end{cases}$$

Combining (i) and the second part of (ii) in Lemma 5, one readily checks that \tilde{K} satisfies the hypothesis of Assertion (iii) in Lemma 4 with constant $\frac{D}{\log 2} (\geq \frac{2D}{e \log 2})$. Hence there exists a concave function H which is equivalent to \tilde{K} on $(0, +\infty)$, up to a factor $2D/\log 2$. Define I to be the restriction of H to $(0, \frac{1}{2}]$, extended at 0 by $I(0) = 0$ and to $[0, 1]$ by symmetry with respect to $\frac{1}{2}$. Since H is concave and non-negative on $(0, +\infty)$, it is also non-decreasing. Therefore, the function I is concave as well. As $\tilde{K} \approx H$ on $(0, +\infty)$, we obtain by restriction that $K \approx I$ on $(0, 1/2]$, up to the same constant. Since $K(0) = I(0) = 0$, and both I and K are symmetric with respect to $1/2$, we can conclude that $I \approx K$ on $[0, 1]$, up to a factor $2D/\log 2$. \square

The following result shows how we exploit the essentially monotonicity properties of J/J_0 and J/J_1 where $J_0(t) = t$ and $J_1(t) = t \log(1/t)$.

Proposition 2. Let $J : (0, 1) \rightarrow \mathbb{R}^+$ such that for all t , $J(t) = J(1 - t)$. Assume that on $(0, 1/2]$, J/J_0 is essentially non-increasing with constant D_0 and J/J_1 is essentially non-decreasing with constant D_1 . Then there exists a function $c : [1/2, 1) \rightarrow \mathbb{R}^+$ such that for all $t \in (0, 1)$,

$$J(t) \geq \sup_{a \in [1/2, 1)} c(a) (t - t^{\frac{1}{a}}),$$

and for all $t \in (0, \frac{1}{2}]$,

$$J(t) \leq 2D_0 \max(D_0, D_1) \sup_{a \in [1/2, 1)} c(a) (t - t^{\frac{1}{a}}).$$

The proof of this proposition relies on the following statement, which is related to [4, Lemma 19].

Lemma 7. *Let $\Phi : (0, \frac{1}{\log 2}] \rightarrow \mathbb{R}^+$. If Φ is essentially non-increasing with constant C_0 , then for all $y \in (0, \frac{1}{2}]$,*

$$\sup_{\alpha \in (0,1]} \Phi(\alpha)(1 - y^\alpha) \geq \frac{1}{2C_0} \Phi\left(\frac{1}{\log(1/y)}\right).$$

If, in addition, $s \mapsto s\Phi(s)$ is essentially non-decreasing with constant C_1 , then for all $y \in (0, 1/2]$,

$$\sup_{\alpha \in (0,1]} \Phi(\alpha)(1 - y^\alpha) \leq \max(C_0, C_1) \Phi\left(\frac{1}{\log(1/y)}\right).$$

Proof. In order to bound the supremum from below, we just select an appropriate value for α : if $y \in (0, 1/e]$, choosing $\alpha = 1/\log(1/y) \in (0, 1]$, we get

$$\sup_{\alpha \in (0,1]} \Phi(\alpha)(1 - y^\alpha) \geq (1 - e^{-1})\Phi\left(\frac{1}{\log(1/y)}\right).$$

If $y \in [1/e, 1/2]$, we choose $\alpha = 1$ and get

$$\sup_{\alpha \in (0,1]} \Phi(\alpha)(1 - y^\alpha) \geq (1 - y)\Phi(1) \geq \frac{1}{2}\Phi(1).$$

However, $y \geq 1/e$ ensures $1/\log(1/y) \geq 1$. Since Φ is essentially non-increasing, $C_0\Phi(1) \geq \Phi(1/\log(1/y))$. Therefore we have proved the first claim, with a constant $\min(1 - e^{-1}, 1/(2C_0)) = 1/(2C_0)$.

To prove the converse inequality, we change variables as follows: setting $c = \alpha \log(1/y)$,

$$\sup_{\alpha \in (0,1]} \Phi(\alpha)(1 - y^\alpha) = \sup_{c \in (0, \log(1/y)]} (1 - e^{-c})\Phi\left(\frac{c}{\log(1/y)}\right).$$

For $c \geq 1$, we know that $\Phi\left(\frac{c}{\log(1/y)}\right) \leq C_0\Phi\left(\frac{1}{\log(1/y)}\right)$ and we bound $1 - e^{-c}$ from above by 1. For $c \in (0, 1]$, we take advantage of the hypothesis on $x \mapsto x\Phi(x)$, in the form $c\Phi(cx) \leq C_1\Phi(x)$ for $x > 0$:

$$(1 - e^{-c})\Phi\left(\frac{c}{\log(1/y)}\right) \leq C_1 \frac{1 - e^{-c}}{c} \Phi\left(\frac{1}{\log(1/y)}\right) \leq C_1 \Phi\left(\frac{1}{\log(1/y)}\right).$$

These two estimates readily give the claim. □

Proof of Proposition 2. For $\alpha \in (0, 1/\log 2]$, we define

$$\Phi(\alpha) := \frac{J(e^{-1/\alpha})}{e^{-1/\alpha}} = \frac{J}{J_0}(e^{-1/\alpha}).$$

Since $e^{-1/\alpha} \in (0, 1/2]$, our hypothesis ensures that Φ is essentially non-increasing with constant D_0 . Notice that

$$\alpha\Phi(\alpha) := \frac{J(e^{-1/\alpha})}{e^{-1/\alpha} \log\left(\frac{1}{e^{-1/\alpha}}\right)} = \frac{J}{J_1}(e^{-1/\alpha})$$

Hence by hypothesis, it is essentially non-decreasing with constant D_1 . Therefore we may apply the previous lemma to Φ . Since by definition, $\Phi(1/\log(1/y)) = J(y)/y$, it gives that for all $y \in (0, 1/2]$,

$$\frac{J(y)}{y} \geq \frac{1}{\max(D_0, D_1)} \sup_{\alpha \in (0, 1]} \Phi(\alpha)(1 - y^\alpha),$$

and for all $y \in (0, 1/2]$,

$$\frac{J(y)}{y} \leq 2D_0 \sup_{\alpha \in (0, 1]} \Phi(\alpha)(1 - y^\alpha).$$

Multiplying these inequalities by y , and setting $a := 1/(1 + \alpha)$, the former estimate gives for $y \in (0, 1/2]$

$$J(y) \geq \frac{1}{\max(D_0, D_1)} \sup_{\alpha \in (0, 1]} \Phi(\alpha)(y - y^{1+\alpha}) = \frac{1}{\max(D_0, D_1)} \sup_{a \in [1/2, 1]} \Phi\left(\frac{1}{a} - 1\right)(y - y^{\frac{1}{a}}). \quad (7)$$

Hence we have prove the claimed lower bound on J with $c(a) = \Phi(a^{-1} - 1)/\max(D_0, D_1)$. We proceed in the same way with the upper bound on J . The ratio of the upper bound to the lower bound is $2D_0 \max(D_0, D_1)$.

It remains to extend the lower bound (7) to values $y \in (1/2, 1)$. To do this we use the symmetry of J and the fact that for all $a \in [1/2, 1]$ and all $s \in [1/2, 1)$, $1 - s - (1 - s)^{\frac{1}{a}} \geq s - s^{\frac{1}{a}}$ (this follows from the comparison of second derivatives, observing that equality holds at $1/2$ and 1): for $y \in (1/2, 1)$,

$$J(y) = J(1 - y) \geq c(a)(1 - y - (1 - y)^{\frac{1}{a}}) \geq c(a)(y - y^{\frac{1}{a}}).$$

□

Proof of Theorem 4. Let us denote by $D_0 \geq 1$ the smallest constant such that J/J_0 is essentially non-increasing on $(0, 1/2]$ with constant D_0 . Similarly let $D_1 \geq 1$ be the smallest constant such that J/J_1 is essentially non-decreasing on $(0, 1/2]$ with constant D_1 .

First, we apply Proposition 2. With the notation of the proposition, it follows that for all $a \in [1/2, 1]$ and all $t \in [0, 1]$,

$$I_\mu(t) \geq J(t) \geq c(a)(t - t^{\frac{1}{a}}).$$

Note that for $t = 0$ or $t = 1$ all quantities vanish. Next, Theorem 2 tell us that $I_{\mu^\infty}(t) \geq c(a)(t - t^{\frac{1}{a}})$. This is true for all a and all t , hence applying the second part of Proposition 2, we deduce that for all $t \in [0, 1/2]$,

$$I_{\mu^\infty}(t) \geq \sup_{a \in [1/2, 1]} c(a)(t - t^{\frac{1}{a}}) \geq \frac{1}{2D_0 \max(D_0, D_1)} J(t).$$

Since both I_{μ^∞} and J are symmetric with respect to $1/2$, we can conclude that for all $t \in [0, 1]$, it holds $I_{\mu^\infty}(t) \geq J(t)/(2D_0 \max(D_0, D_1))$.

In the general case, we know by Lemma 5 that $D_0 \leq D/\log 2$ and $D_1 \leq D$. Therefore we get that $I_{\mu^\infty} \geq J/c_D$ with $c_D = 2D^2/(\log 2)^2$.

In the particular case where J is concave, we know that $J(t)/t$ is non-increasing, so that $D_0 = 1$ and $I_{\mu^\infty} \geq J/(2D_1) \geq J/(2D)$. The paragraph after Theorem 1 explains that when J is concave, symmetric and such that J/J_1 is non-decreasing, the inequality $I_\mu \geq J$ implies $I_{\mu^\infty} \geq J$. Hence in this case the conclusion of Theorem 4 is valid with $c_1 = 1$. \square

3.2 Proof of Theorem 5

First, we recall a classical property of infinite dimensional profiles, which comes from testing isoperimetric inequalities on product sets. It was put forward by Bobkov in [5].

Lemma 8. *Let (X, d, μ) be a metric space equipped with a Borel probability measure μ and satisfying the regularity property (\mathcal{R}) . Then the infinite-dimensional isoperimetric profile of (X, d, μ) satisfies, for every $a, b \in [0, 1]$:*

$$I_{\mu^\infty}(ab) \leq aI_{\mu^\infty}(b) + bI_{\mu^\infty}(a). \quad (8)$$

Proof. The inequality is obvious if a or b is equal to 1, or to 0 since $I_{\mu^\infty}(0) = 0$. Let $a, b \in (0, 1)$ and $\varepsilon > 0$. Let $m, n \in \mathbb{N}^*$. Let $A \subset X^m$ be any set with $\mu^m(A) = a$. Let $B \subset X^n$ be any set with $\mu^n(B) = b$, $(\mu^n)^+(B) \leq I_{\mu^n}(b) + \varepsilon$ and

$$(\mu^n)^+(B) = \lim_{h \rightarrow 0} \frac{\mu^n(B_h \setminus B)}{h},$$

which is possible thanks to Hypothesis (\mathcal{R}) .

Then consider $A \times B \subseteq \mathbb{R}^{m+n}$. Obviously, $\mu^{m+n}(A \times B) = ab$. The uniform enlargement of a product set is still a product: $(A \times B)_h = A_h \times B_h$ for any $h > 0$. Therefore

$$\begin{aligned} \frac{1}{h} \mu^{m+n}((A \times B)_h \setminus (A \times B)) &= \frac{\mu^m(A_h) \mu^n(B_h) - \mu(A) \mu(B)}{h} \\ &= \frac{\mu^m(A_h \setminus A)}{h} \mu^n(B_h) + \mu^m(A) \frac{\mu^n(B_h \setminus B)}{h} \end{aligned}$$

Since by hypothesis $\lim_{h \rightarrow 0} (\mu^n(B_h) - \mu^n(B))/h < +\infty$, we know that $\lim_{h \rightarrow 0} \mu^n(B_h) = \mu^n(B)$ (note the convergence holds by monotonicity). Taking upper limits in $h \rightarrow 0$, and observing that two of the three terms have limits, we deduce from the latter inequality that

$$(\mu^{m+n})^+(A \times B) \leq (\mu^m)^+(A) \mu^n(B) + \mu^m(A) (\mu^n)^+(B). \quad (9)$$

Since $I_{\mu^{m+n}}(ab) \leq (\mu^{m+n})^+(A \times B)$, we obtain after optimizing on sets A of measure a and using the hypothesis on the boundary measure of B :

$$I_{\mu^{m+n}}(ab) \leq I_{\mu^m}(a) b + a(I_{\mu^n}(b) + \varepsilon).$$

Letting ε tend to 0, and m, n tend to $+\infty$ gives the claim (8). \square

The symmetry property ($I_{\mu^\infty}(t) = I_{\mu^\infty}(1-t)$ for all $t \in (0, 1]$) and the two-points inequality (8) are enough to deduce Theorem 5, as the next statement shows:

Proposition 3. *Let $I : [0, 1] \rightarrow [0, +\infty]$ be an application satisfying that for all $a, b \in [0, 1]$*

$$I(a) = I(1-a) \quad \text{and} \quad I(ab) \leq aI(b) + bI(a), \quad (10)$$

with the convention that $+\infty \times 0 = 0$. If there exists $x_0 \in [0, 1]$ such that $\limsup_{x \rightarrow x_0} I(x) < +\infty$ then I is continuous and $t \mapsto I(t)/(t \log(1/t))$ is essentially non-decreasing on $(0, 1)$.

The condition of local boundedness around some point cannot be removed as shown by the following example: $I(t) = 0$ if $t \in \mathbb{Q}$ and $I(t) = +\infty$ otherwise.

The proof of the proposition uses the next two easy lemmas.

Lemma 9. *Let $S \subset (0, 1)$ be a set with the following stability property:*

$$(x \in S \quad \text{and} \quad y \in S) \implies (1-x \in S \quad \text{and} \quad xy \in S).$$

If S is not empty then it is dense in $(0, 1)$. Moreover, if S has non-empty interior then $S = (0, 1)$.

In other words, if S is neither \emptyset nor $(0, 1)$ then S and $(0, 1) \setminus S$ are dense in $(0, 1)$. This is the case for instance of $S = \mathbb{Q} \cap (0, 1)$.

Proof. Let $t \in (0, 1)$ be an element of S , then for all $n \in \mathbb{N}^*$, $x_n := 1 - t^n$ belong to S and the sequence (x_n) tends to 1. Given $0 < a < b < 1$, let us show that there is a point of S between a and b . Choose k large enough such that $x_k > \max(b, a/b)$. Then for all $n \geq 1$, $(x_k)^n \in S$. Obviously $x_k \geq b$ and $\lim_n (x_k)^n = 0$. Let n_0 be maximal with $(x_k)^{n_0} \geq b$. Then

$$b > x_k^{n_0+1} = x_k^{n_0} x_k > b \times \frac{a}{b} = a.$$

Hence $x_k^{n_0+1} \in S \cap (a, b)$. This completes the proof of the density of S .

Assume now that $(a, b) \subset S$ for some $0 < a < b < 1$. Consider an arbitrary $x \in (0, 1)$ and let us show that $x \in S$. If $x \in (a, b)$, we have nothing to prove. If $x \in (0, a]$, we use the fact that S being non-empty is dense: there exists $y \in S \cap (x/b, x/a)$. Since S contains y and (a, b) , the stability by product ensures that S also contains (ya, yb) . Hence $x \in (ya, yb) \subset S$. Eventually, if $x \in [b, 1)$, we consider $1 - x \in (0, 1 - b]$. By the symmetry assumption $(1 - b, 1 - a) \subset S$, so the latter argument yields $1 - x \in S$, and using symmetry again $x \in S$. \square

The next lemma is a classical result about subadditive functions on \mathbb{R}_+ (see e.g. [13]) :

Lemma 10. *Let $K : \mathbb{R}_+ \rightarrow \mathbb{R}$ be a subadditive function with $\lim_0 K = 0$. Then*

$$\lim_{h \rightarrow 0^+} \frac{K(h)}{h} = \sup_{t > 0} \frac{K(t)}{t}.$$

Proof. Denote $S = \sup_{t > 0} \frac{K(t)}{t}$. Given any $u < S$, there exists $x_0 \in \mathbb{R}_+^*$ such that $K(x_0) > ux_0$. For any $h \in (0, x_0)$, write $x_0 = nh + \delta$ with $n = \lfloor \frac{x_0}{h} \rfloor$ and $\delta \in [0, h)$. By subadditivity of K ,

$$ux_0 < K(x_0) \leq nK(h) + K(\delta) = (x_0 - \delta) \frac{K(h)}{h} + K(\delta).$$

Next, we let h tend to 0^+ . In this case $\delta \rightarrow 0^+$ and $K(\delta) \rightarrow 0$, hence (for any $u < S$)

$$u \leq \liminf_{h \rightarrow 0^+} \frac{K(h)}{h}.$$

Therefore $S \leq \liminf_{h \rightarrow 0^+} \frac{K(h)}{h}$. On the other hand, $\limsup_{h \rightarrow 0^+} \frac{K(h)}{h} \leq S$ holds by definition. \square

Proof of Proposition 3. There is nothing to prove if I is identically 0, so we assume that I does not vanish everywhere. Observe that the two-points inequality in (10), applied for $a = b = 0$, yields $I(0) = 0$.

Consider the subset S_1 of $(0, 1)$ of points x such that the function I is bounded on a neighbourhood of x . Our hypothesis $\limsup_{x \rightarrow x_0} I(x) < +\infty$ ensures that S_1 has non-empty interior. Thanks to (10), one readily checks that S_1 is stable by product and by symmetry with respect to $1/2$. Hence Lemma 9 applies to S_1 and shows that $S_1 = (0, 1)$. This means that I is locally bounded at every point of $(0, 1)$. By compactness, we deduce that I is bounded on any segment $[a, b] \subset (0, 1)$.

The next step of the proof is an argument of Bobkov and Houdré, that we include for completeness. The two-points inequality implies by induction that for all $a \in [0, 1]$ and any integer $k \geq 1$, $I(a^k) \leq ka^{k-1}I(a)$. Let $t \in (0, 1/e]$. Choosing $k = \lfloor \log(1/t) \rfloor \geq 1$ and $a = t^{1/k}$ in the latter inequality leads to

$$I(t) \leq kt^{1-\frac{1}{k}}I(t^{\frac{1}{k}}) = t \lfloor \log(1/t) \rfloor \frac{I(t^{1/k})}{t^{1/k}}.$$

Using that for $x \geq 1$, $x/\lfloor x \rfloor \in [1, 2]$, we obtain that $t^{1/k} = \exp(-\log(1/t)/\lfloor \log(1/t) \rfloor) \in [e^{-2}, e^{-1}]$. Hence for $t \in (0, e^{-1}]$, $I(t) \leq Ct \log(1/t)$ where $C = e^2 \sup\{I(s); s \in [e^{-2}, e^{-1}]\}$ is finite (by the previous point). In particular, this estimates implies that $I(t)$ tends to 0 when $t \neq 0$ tends to 0. Since $I(0) = 0$, the function I is continuous at 0. By symmetry it is also continuous at 1, with $I(1) = 0$.

Consider the map $K : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ defined by $K(x) = e^x I(e^{-x})$. Then for all $x, y \in \mathbb{R}_+$, by (10)

$$K(x+y) = \frac{I(e^{-x}e^{-y})}{e^{-x}e^{-y}} \leq \frac{e^{-x}I(e^{-y}) + e^{-x}I(e^{-y})}{e^{-x}e^{-y}} = K(y) + K(x),$$

which means that K is subadditive. Moreover, since I is continuous at 1, K is continuous at 0, with $K(0) = I(1) = 0$.

For all $a > \epsilon > 0$, we have by subadditivity $K(a+\epsilon) \leq K(a) + K(\epsilon)$ and $K(a) \leq K(a-\epsilon) + K(\epsilon)$. Letting ϵ tend to zero, we obtain for all $a > 0$

$$\limsup_{x \rightarrow a^+} K(x) \leq K(a) \leq \liminf_{x \rightarrow a^-} K(x).$$

In words, on $(0, +\infty)$, the function K is right-upper-semicontinuous and left-lower-semicontinuous. Since for all $t \in (0, 1)$, $I(t) = tK(\log(1/t))$, it follows that on $(0, 1)$ the function I is left-upper-semicontinuous and right-lower-semicontinuous. Note that "left" and "right" were exchanged, since $t \mapsto \log(1/t)$ is continuous decreasing. The symmetry assumption $I(t) = I(1-t)$ allows to exchange once more: so I is also right-upper-semicontinuous and left-lower-semicontinuous on $(0, 1)$. Thus I is continuous on $(0, 1)$, and actually on $[0, 1]$. Indeed, the continuity at the endpoints has already been established.

Next, let us draw another consequence of the above properties of K . Lemma 10 directly applies and gives that

$$\lim_{h \rightarrow 0^+} \frac{K(h)}{h} = \sup_{x > 0} \frac{K(x)}{x} = \sup_{x > 0} \frac{e^x I(e^{-x})}{x}.$$

Since we assume that I is not identically 0, the above limit, denoted by L , belongs to $(0, +\infty]$. We now translate this convergence in terms of I : using symmetry, for $t \in (0, 1)$,

$$\frac{I(t)}{t} = \frac{I(1-t)}{t} = \frac{1-t}{t} K\left(\log\left(\frac{1}{1-t}\right)\right) = \frac{(1-t) \log\left(\frac{1}{1-t}\right)}{t} \times \frac{K\left(\log\left(\frac{1}{1-t}\right)\right)}{\log\left(\frac{1}{1-t}\right)}.$$

When $t > 0$ tends to 0, the first ratio tends to 1, and the second to $L = \lim_{h \rightarrow 0^+} K(h)/h$. Therefore we can deduce that $\lim_{t \rightarrow 0^+} I(t)/t = L \in (0, +\infty]$.

In order to turn this limit into a lower bound on $I(t)/t$ for $t \in (0, 1/2]$, we need to check that I does not vanish in $(0, 1)$. To do this, let us consider the set $S_0 = \{x \in (0, 1); I(x) = 0\}$. By (10), it is stable by product and symmetry around 1/2. If it were non-empty, the first part of Lemma 9

would imply that S_0 is dense in $(0, 1)$. By continuity of I , we would conclude that I is identically 0. Since, we assumed that I does not vanish everywhere, it follows that $S_0 = \emptyset$. As a conclusion, the function I vanishes only at 0 and 1.

On $(0, 1/2]$ the map $t \mapsto I(t)/t$ is continuous, with positive values. Moreover it has a positive (maybe infinite) limit at 0^+ . As a consequence, there exists $c > 0$ such that $I(t) \geq ct$ for all $t \leq 1/2$.

Let us deduce that I is essentially non-decreasing on $[0, 1/2]$. Let $0 \leq s < t \leq \frac{1}{2}$. Using the two-points inequality (10)

$$I(s) = I\left(t \times \frac{s}{t}\right) \leq \frac{s}{t}I(t) + tI\left(\frac{s}{t}\right) \leq I(t) + t \max I \leq \left(1 + \frac{\max I}{c}\right) I(t),$$

where we have used that I is continuous on $[0, 1]$ and $I(t) \geq ct$.

Eventually, let us prove that $\frac{I}{J_1}$ is essentially non-decreasing on $(0, 1)$. Let $0 < s < t < 1$. Then one can write $s = t^{k+\alpha}$ with $k = \left\lfloor \frac{\log \frac{1}{s}}{\log \frac{1}{t}} \right\rfloor \in \mathbb{N}^*$ and $\alpha \in [0, 1)$. By the two-points inequality for I :

$$\frac{I(s)}{s} = \frac{I(t^{k+\alpha})}{t^{k+\alpha}} \leq k \frac{I(t)}{t} + \frac{I(t^\alpha)}{t^\alpha}. \quad (11)$$

Assume first that $t \geq \frac{1}{2}$. Then $t^\alpha \geq t \geq \frac{1}{2}$. We have shown that I is essentially non-decreasing on $[0, \frac{1}{2}]$ (with a constant denoted by D). By symmetry, it follows that I is essentially non-increasing on $[\frac{1}{2}, 1]$ with constant D . Hence

$$\frac{I(t^\alpha)}{t^\alpha} \leq D \frac{I(t)}{t^\alpha} \leq D \frac{I(t)}{t}.$$

Combining this estimate with (11) gives

$$\frac{I(s)}{s} \leq (k + D) \frac{I(t)}{t} \leq (1 + D)k \frac{I(t)}{t} \leq (1 + D) \frac{\log \frac{1}{s}}{\log \frac{1}{t}} \frac{I(t)}{t},$$

that is $\frac{I(s)}{J_1(s)} \leq (1 + D) \frac{I(t)}{J_1(t)}$. In particular, we have shown that I/J_1 is essentially non-decreasing on $[1/2, 1]$. Using the symmetry of I , this implies that on $[0, 1/2]$ the function $I(t)/J_0(t) = I(t)/t$ is essentially non-increasing. This is actually explained in the first part of the proof of Lemma 5, see Equation (5). We have already shown that I is essentially non-decreasing on $(0, 1/2]$. Thus by symmetry, I is essentially non-increasing on $[1/2, 1]$, and so is the map $t \mapsto I(t)/t = I(t)/J_0(t)$. Therefore, I/J_0 is essentially non-increasing on the whole interval $(0, 1]$. Let us denote by D_0 the corresponding constant.

The latter fact allows to conclude: Let $0 < s < t < 1$. Since $t^\alpha \geq t$, we know that $\frac{I(t^\alpha)}{t^\alpha} \leq D_0 \frac{I(t)}{t}$. Combining this estimate with (11) gives,

$$\frac{I(s)}{s} \leq k \frac{I(t)}{t} + \frac{I(t^\alpha)}{t^\alpha} \leq (k + D_0) \frac{I(t)}{t} \leq (1 + D_0)k \frac{I(t)}{t} \leq (1 + D_0) \frac{\log \frac{1}{s}}{\log \frac{1}{t}} \frac{I(t)}{t}.$$

The proof is now complete. □

4 An application to geometric influences

This section is devoted to an application of Theorem 4 to geometric influences. The notion of influence of a variable on a boolean function plays an important role in discrete harmonic analysis, with applications to various fields (see e.g. the survey article [10] on threshold phenomena). Let us recall the definition: for a function $f : \{0, 1\}^n \rightarrow \{0, 1\}$, which can be viewed as a subset $A = \{x; f(x) = 1\}$ of $\{0, 1\}^n$, the influence of the i -th variable with respect to a probability measure ν on the discrete cube $\{0, 1\}^n$ is

$$I_i(f) = I_i(A) := \mathbb{P}_{x \sim \nu}(f(x) \neq f(\tau_i(x))) = \mathbb{P}_{x \sim \nu}(x \in A \text{ xor } \tau_i(x) \in A),$$

where $\tau_i(x)$ is the neighbour of x having different i -th coordinate, $(\tau_i(x))_i = 1 - x_i$. Geometrically speaking, $I_i(A)$ measures the size of the edge boundary of A in the i -th direction. A seminal result in the theory of influences is the KKL theorem (by Kahn, Kalai and Linial [9]). Based on the hypercontractivity inequality, it ensures the existence of a coordinate with a large influence for non-constant boolean functions.

Recent papers by Keller [11] and Keller, Mossel and Sen [12] develop the theory of influences in the case of a continuous space. They propose two different definitions: h -influences [11] involve the measures of the intersections of a given set with all lines in the i -th canonical direction, while geometric influences [12] involve the boundary measures of the intersections with lines in the i -th direction.

Definition. Let $n \in \mathbb{N}^*$, $i \in \{1, \dots, n\}$, $x \in \mathbb{R}^n$ and A a Borel subset of \mathbb{R}^n . For $z \in \mathbb{R}^{n-1}$, we set

$$A_i^z = \left\{ y \in \mathbb{R} \mid (z_1, \dots, z_{i-1}, y, z_i, \dots, z_{n-1}) \in A \right\}.$$

Let $\nu = \nu_1 \otimes \dots \otimes \nu_n$ be a product probability measure on \mathbb{R}^n .

If $h : [0, 1] \rightarrow \mathbb{R}_+$ is a measurable function, the h -influence of the i -th coordinate on A with respect to ν is defined by

$$\mathcal{I}_{\nu, i}^h(A) = \int_{\mathbb{R}^{n-1}} h(\nu_i(A_i^z)) d\widehat{\nu}^i(z),$$

where $\widehat{\nu}^i = \nu_1 \otimes \dots \otimes \nu_{i-1} \otimes \nu_{i+1} \otimes \dots \otimes \nu_n$.

The geometric influence of the i -th coordinate on A with respect to the measure ν is given by

$$\mathcal{I}_{\nu, i}^{\mathcal{G}}(A) = \int_{\mathbb{R}^{n-1}} (\nu_i)^+(A_i^z) d\widehat{\nu}^i(z).$$

When the choice of the underlying measure is obvious, we simply write $\mathcal{I}_i^h(A)$ and $\mathcal{I}_i^{\mathcal{G}}(A)$.

Keller was able to prove an analogue of the KKL theorem for h -influences provided h is larger than the entropy function Ent defined by $\text{Ent}(x) = x \log \frac{1}{x} + (1 - x) \log \frac{1}{1-x}$ for $x \in (0, 1)$ and $\text{Ent}(0) = \text{Ent}(1) = 0$. His result [11] is stated for functions on the unit cube, equipped with Lebesgue's measure. Using a standard transportation argument yields the following formulation:

Theorem 6. *Let μ be a probability measure on \mathbb{R} . Then, for every Borel set $A \subseteq \mathbb{R}^n$*

$$\max_{1 \leq i \leq n} \mathcal{I}_i^{\text{Ent}}(A) \geq \gamma \mu^n(A) \mu^n(A^c) \frac{\log n}{n},$$

where $\gamma > 0$ is a universal constant.

Keller, Mossel and Sen [12] establish an analogue of the KKL theorem for geometric influences for Boltzmann measures $d\mu_\rho(t) = \exp(-|t|^\rho) dt / Z_\rho dt$ with $\rho \geq 1$ (and under mild assumptions for log-concave measures enjoying the same isoperimetric inequality as μ_ρ). Thanks to Theorem 4 we can propose a more general result:

Theorem 7. *Let μ be an even log-concave probability measure on \mathbb{R} , with positive and C^1 -bounded density φ_μ . Assume that $I_\mu \geq J$ where J is a non-negative function on $[0, 1]$, which is symmetric with respect to $1/2$, verifies $J(0) = 0$ and $t \mapsto J(t)/(t \log(1/t))$ is essentially non-decreasing on $(0, 1)$ with constant D . Then for every Borel set $A \subset \mathbb{R}^n$,*

$$\max_{1 \leq i \leq n} \mathcal{I}_i^{\mathcal{G}}(A) \geq \alpha_D \mu^n(A) \mu^n(A^c) J\left(\frac{1}{n}\right),$$

where $\alpha_D \geq \frac{\kappa}{D^3}$ and $\kappa > 0$ is a universal constant.

Remark 4. Actually the conclusion of Theorem 7 holds under less restrictive conditions on the measure. In particular the log-concavity assumption can be removed, either by using a different symmetrization argument than in [12] or by introducing another definition of the geometric influence based on notions of geometric measure theory. These modifications require a substantial and technical work that will appear in the PhD dissertation of the second-named author. In the present paper, we simply explain how the argument of Keller, Mossel and Sen can be adapted, putting forward the parts of the reasoning where the conditions on J are used.

The next lemma follows from Proposition 1.3 in [12]. It explains the connection between geometric influences and boundary measure for the uniform enlargement:

Lemma 11. *Let μ be as in Theorem 7. Let $A \subseteq \mathbb{R}^n$ be a monotone increasing set (in the following sense: if $x \in A$ and for all i , $x_i \leq y_i$, then $y \in A$). Then*

$$\sum_{i=1}^n \mathcal{I}_i^{\mathcal{G}}(A) = (\mu^n)^+(A).$$

Proof of Theorem 7. We follow the argument of Keller, Mossel and Sen. Assume $n \geq 2$. Let $A \subseteq \mathbb{R}^n$ be as in the statement of the theorem. Lemma 3.7 of [12] ensures that, without loss of generality, one can assume that A is increasing. Set $t = \mu^n(A)$. Since A and A^c have the same influences, we may assume that $t \leq 1/2$ (A^c is monotone decreasing, but passing to its image by the symmetry

with respect to the origin we may ensure that we work is an increasing set of measure at most $1/2$. We distinguish two cases :

First case : $t \leq 1/n$. Thanks to Lemma 11 and to the isoperimetric inequality of Theorem 4

$$\sum_{i=1}^n \mathcal{I}_i^{\mathcal{G}}(A) = (\mu^n)^+(A) \geq \frac{1}{c_D} J(\mu^n(A)) = \frac{1}{c_D} J(t),$$

with $c_D = \frac{2D^2}{(\log 2)^2}$. Lemma 5 asserts that $s \mapsto J(s)/s$ is essentially non-increasing on $(0, 1/2]$ with constant $D/\log 2$, therefore $J(t)/t \geq nJ(1/n) \log 2/D$. Consequently

$$\max_i \mathcal{I}_i^{\mathcal{G}}(A) \geq \frac{1}{n} \sum_{i=1}^n \mathcal{I}_i^{\mathcal{G}}(A) \geq \frac{\log 2}{D c_D} t J\left(\frac{1}{n}\right) \geq \frac{(\log 2)^3}{2D^3} t(1-t) J\left(\frac{1}{n}\right).$$

Second case : $t \in (1/n, 1/2]$. The argument uses three main ingredients. The first one is the isoperimetric inequality $I_\mu \geq J$, which implies that for all $i \leq n$,

$$\mathcal{I}_i^{\mathcal{G}}(A) = \int_{\mathbb{R}^{n-1}} \mu^+(A_i^z) d\mu^{n-1}(z) \geq \int_{\mathbb{R}^{n-1}} J(\mu(A_i^z)) d\mu^{n-1}(z) = \mathcal{I}_i^J(A). \quad (12)$$

The second ingredient is a comparison between J -influences and Ent-influences: observe that Ent is symmetric with respect to $1/2$ and is increasing and one-to-one on $[0, 1/2]$. Let $\text{Ent}^{-1} : [0, \log 2] \rightarrow [0, 1/2]$ be its reciprocal function. Then for all $i \leq n$,

$$\mathcal{I}_i^J(A) \geq \frac{J(s)}{2D} \quad \text{with} \quad s := \text{Ent}^{-1}\left(\frac{\mathcal{I}_i^{\text{Ent}}(A)}{2}\right) \in \left[0, \frac{1}{2}\right]. \quad (13)$$

We postpone the proof of this inequality, and explain how to conclude. The last ingredient is Keller's version of the KKL inequality (Theorem 6). It provides an index i such that $\mathcal{I}_i^{\text{Ent}}(A) \geq \gamma t(1-t) \log(n)/n$, where $\gamma > 0$ is a universal constant. Observe for further use that necessarily $\gamma < 8$ (indeed, $\text{Ent} \leq \log 2$ and one can choose $n = 2$ and $t = 1/2$ in Keller's theorem). Let us show that the i -th coordinate has a large geometric influence.

Observe that for every $y \in [0, \frac{1}{2}] \subset [0, \log 2]$, $\theta(y) := \frac{y}{2 \log \frac{1}{y}} \leq \text{Ent}^{-1}(y)$. Since $\mathcal{I}_i^{\text{Ent}}(A)/2 \leq \log(2)/2 \leq 1/2$, and θ is increasing on $(0, 1)$,

$$\begin{aligned} s &= \text{Ent}^{-1}\left(\frac{\mathcal{I}_i^{\text{Ent}}(A)}{2}\right) \geq \theta\left(\frac{\mathcal{I}_i^{\text{Ent}}(A)}{2}\right) \geq \theta\left(\frac{\gamma}{2} t(1-t) \frac{\log n}{n}\right) \\ &= \frac{\gamma t(1-t)}{4n} \frac{\log n}{\log \frac{2n}{\gamma t(1-t) \log n}} \geq \frac{\gamma t(1-t)}{4n} \times \frac{\log n}{\log \frac{4n^2}{\gamma \log n}}, \end{aligned}$$

where the latter inequality relies on $t(1-t) \geq (1-t)/n \geq 1/(2n)$. Since $\gamma \leq 8$, the last fraction in the lower bound of s is a positive function of $n \geq 2$ with a positive limit when n tends to infinity.

Hence there exists $c = c(\gamma) > 0$ such that $s \geq ct(1-t)/n$. It remains to combine this estimate with $\mathcal{I}_i^{\mathcal{G}}(A) \geq J(s)/(2D)$, a consequence of (12) and (13):

If $s \leq \frac{1}{n}$, then we also use Lemma 5, which asserts that $u \mapsto J(u)/u$ is essentially non-increasing on $(0, \frac{1}{2}]$ with constant $\frac{D}{\log 2}$:

$$\mathcal{I}_i^{\mathcal{G}}(A) \geq \frac{J(s)}{2D} \geq \frac{\log 2}{2D^2} s \frac{J(\frac{1}{n})}{\frac{1}{n}} \geq \frac{\log 2}{2D^2} ct(1-t)J\left(\frac{1}{n}\right).$$

If $s > \frac{1}{n}$ then, using the fact that J is essentially non-decreasing on $(0, \frac{1}{2})$ with constant $\frac{2D}{e \log 2}$ (see Lemma 6), we get

$$\mathcal{I}_i^{\mathcal{G}}(A) \geq \frac{J(s)}{2D} \geq \frac{e \log 2}{4D^2} J\left(\frac{1}{n}\right) \geq \frac{e \log 2}{D^2} t(1-t)J\left(\frac{1}{n}\right).$$

Eventually, we give a proof for (13). By hypothesis, J/J_1 is essentially non-decreasing with constant D , where $J_1(x) = x \log(1/x)$. Observe that for $x \in [0, \frac{1}{2}]$,

$$J_1(x) \leq \text{Ent}(x) = J_1(x) + J_1(1-x) \leq 2J_1(x).$$

It follows that $\frac{J}{\text{Ent}}$ is essentially non-decreasing on $(0, \frac{1}{2}]$ with constant $2D$. Recall that $s \in (0, \frac{1}{2})$ verifies $\text{Ent}(s) = \mathcal{I}_i^{\text{Ent}}(A)/2$. Note that, if $x \notin [s, 1-s]$, then $\text{Ent}(x) < \mathcal{I}_i^{\text{Ent}}(A)/2$. This yields

$$\int_{\mu(A_i^z) \in [s, 1-s]} \text{Ent}(\mu(A_i^z)) d\mu^{n-1}(z) = \mathcal{I}_i^{\text{Ent}}(A) - \int_{\mu(A_i^z) \notin [s, 1-s]} \text{Ent}(\mu(A_i^z)) d\mu^{n-1}(z) \geq \frac{\mathcal{I}_i^{\text{Ent}}(A)}{2}.$$

Therefore, using in addition the symmetry with respect to $1/2$ of J and Ent and the fact that $\frac{J}{\text{Ent}}$ is essentially non-decreasing on $(0, 1/2]$ with constant $2D$, we get

$$\begin{aligned} \mathcal{I}_i^J(A) &\geq \int_{\mu(A_i^z) \in [s, 1-s]} J(\mu(A_i^z)) d\mu^{n-1}(z) \\ &= \int_{\mu(A_i^z) \in [s, 1-s]} J(\min(\mu(A_i^z), 1 - \mu(A_i^z))) d\mu^{n-1}(z) \\ &\geq \frac{1}{2D} \frac{J(s)}{\text{Ent}(s)} \int_{\mu(A_i^z) \in [s, 1-s]} \text{Ent}(\min(\mu(A_i^z), 1 - \mu(A_i^z))) d\mu^{n-1}(z) \\ &= \frac{1}{2D} \frac{J(s)}{\text{Ent}(s)} \int_{\mu(A_i^z) \in [s, 1-s]} \text{Ent}(\mu(A_i^z)) d\mu^{n-1}(z) \\ &\geq \frac{1}{2D} \frac{J(s)}{\text{Ent}(s)} \frac{\mathcal{I}_i^{\text{Ent}}(A)}{2} = \frac{J(s)}{2D}. \end{aligned}$$

The proof is complete. □

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