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Mohammed Bachir. A classification of isomorphisms and isometries on functions spaces.. 2014. hal-01082632

HAL Id: hal-01082632

<https://hal.science/hal-01082632>

Preprint submitted on 14 Nov 2014

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A classification of isomorphisms and isometries on functions spaces.

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November 14, 2014

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Abstract. We establish in this article a formula which will allow to classify isometries as well as partial isometries between spaces of functions. Our result applies in a non compact framework and for abstract class of functions spaces included in the space of continuous and bounded functions on complete metric space. The result of this article is in particular a generalization of the Banach-Stone theorem. We use the techniques of differentiability as in the original proof of Banach for convex functions generalizing the norm of uniform convergence. We use also a duality result in a non convex framework.

Keyword, phrase: The Banach-Stone theorem, isometries and isomorphisms, differentiability, duality.

1 Introduction.

The classical Banach-Stone theorem say that if K and L are compact spaces and T is an isomorphism from $(C(K), \|\cdot\|_\infty)$ onto $(C(L), \|\cdot\|_\infty)$ (where $C(K)$ and $C(L)$ denotes the spaces of all real valued continuous functions on K and L respectively) then, the map T is an isometry if and only if there exists an homeomorphism $\pi : L \rightarrow K$ and a continuous map $\epsilon : L \rightarrow \{\pm 1\}$ such that

$$T\varphi(y) = \epsilon(y)\varphi \circ \pi(y); \forall y \in L, \forall \varphi \in C(K). \quad (1)$$

The Banach-Stone theorem was invested by several authors in diverse directions making way to several advances and publications for example by D. Amir [1], J. Araujo [2], [3], J. Araujo and J. Font [4], E. Behrends [10], [11], M. Cambern [12], B. Cengiz, [13], M. Garrido and J. Jaramillo [18], [19], K. Jarosz [23], K. Jarosz and V. Pathak, [24], D. Vieira [26], N. Weaver [27]. The list remaining still long, we send back to the article of M. Garrido and J. Jaramillo [17] for a history of contributions and a more complete list.

We establish in this article a general formula which will allow to give a classification of isomorphisms between functions spaces resulting “totally or partially” from homeomorphisms. We establish our result in the non compact framework for abstract class of functions spaces included classical spaces. To be more clear, we give the following example. Suppose that X and Y are two Banach spaces and suppose that there exists an homeomorphism $\pi : Y \rightarrow X$ preserving norms i.e $\|\pi(y)\|_X = \|y\|_Y$ for all $y \in Y$. Let us denote by $\mathcal{H}_{\|\cdot\|_X, \|\cdot\|_Y}(Y, X)$ the class of all homeomorphism from Y onto X preserving norms and by $C_b(X)$ the space of all continuous bounded functions on X . Let $T : (C_b(X), \|\cdot\|_\infty) \rightarrow (C_b(Y), \|\cdot\|_\infty)$ be an isomorphism.

Question 1. Under what condition the isomorphism T comes canonically as in the formula (1) from the class of the homeomorphisms $\pi \in \mathcal{H}_{\|\cdot\|_X, \|\cdot\|_Y}(X, Y)$?

The answer to this question is that T comes canonically from $\pi \in \mathcal{H}_{\|\cdot\|_X, \|\cdot\|_Y}(X, Y)$ if and only if, it satisfies the following condition

$$\sup_{y \in Y} \{|T\varphi(y)| - \|y\|_Y\} = \sup_{x \in X} \{|\varphi(x)| - \|x\|_X\}, \forall \varphi \in C_b(X). \quad (2)$$

Thus the class of all isomorphisms T satisfying (2) is in bijection with the set

$$\{\epsilon : Y \rightarrow \{\pm 1\} \text{ continuous}\} \times \mathcal{H}_{\|\cdot\|_X, \|\cdot\|_Y}(Y, X).$$

For example, when $X = Y = \mathbb{R}$ (which are in particular connected spaces), we have that the isomorphisms $T : C_b(\mathbb{R}) \rightarrow C_b(\mathbb{R})$ that satisfy $\sup_{y \in \mathbb{R}} \{|T\varphi(y)| - |y|\} = \sup_{x \in \mathbb{R}} \{|\varphi(x)| - |x|\}$, $\forall \varphi \in C_b(\mathbb{R})$ are exactly four isometries corresponding to the homeomorphisms satisfying $|\pi(x)| = |x|$ for all $x \in \mathbb{R}$ which are $\pi(x) = x$ or $\pi(x) = -x$ for all $x \in \mathbb{R}$:

- (1) $T_1\varphi(y) = \varphi(y)$ for all $\varphi \in C_b(\mathbb{R})$ and all $y \in \mathbb{R}$. The identity map.
- (2) $T_2\varphi(y) = -\varphi(y)$ for all $\varphi \in C_b(\mathbb{R})$ and all $y \in \mathbb{R}$.
- (3) $T_3\varphi(y) = \varphi(-y)$ for all $\varphi \in C_b(\mathbb{R})$ and all $y \in \mathbb{R}$.
- (4) $T_4\varphi(y) = -\varphi(-y)$ for all $\varphi \in C_b(\mathbb{R})$ and all $y \in \mathbb{R}$.

Our main result Theorem 1 gives a general classification in the spirit mentioned in the example above. More generally, a class of homeomorphism π given by a functional constraint $f \circ \pi = g$ for fixed lower semicontinuous and bounded from below functions f and g , characterize the class of isomorphism satisfying $\sup_{y \in Y} \{|T\varphi(y)| - g(y)\} = \sup_{x \in X} \{|\varphi(x)| - f(x)\}$ for all $\varphi \in C_b(X)$. In the particular case where $f = 0$ on X and $g = 0$ on Y we recover the class of all isometries for $\|\cdot\|_\infty$. Before giving our main result Theorem 1 below, we are going to introduce explanatory commutative diagrams corresponding to our classification.

1.1 The commutative diagrams for the classical Banach-Stone theorem.

Let $T : C(K) \rightarrow C(L)$ be an isomorphism, and let $\mathcal{H}_{0,0}(L, K)$ be the set of all homeomorphism from L onto K if such homeomorphism exists. The left diagram commutes

if and only that of right also.

$$\begin{array}{ccc}
C(K) & \xrightarrow{T} & C(L) \\
& \searrow \|\cdot\|_\infty & \downarrow \|\cdot\|_\infty \\
& & \mathbb{R}
\end{array}
\quad \Leftrightarrow \quad
\begin{array}{ccc}
\exists \pi : L & \xrightarrow{\pi} & K \\
& \searrow 0 & \downarrow 0 \\
& & \mathbb{R}
\end{array}$$

$$\|\cdot\|_\infty \circ T = \|\cdot\|_\infty \Leftrightarrow \exists \pi \in \mathcal{H}_{0,0}(L, K) \text{ such that } T \text{ comes canonically from } \pi$$

1.2 The commutative diagrams for the classification result.

Let (X, d) and (Y, d') be a complete metric spaces. Let $A \subset C_b(X)$ and $B \subset C_b(Y)$ be a Banach spaces satisfying certain general axioms (This axioms are verified for classical and known function spaces. See Axioms 1 and the examples in Proposition 2). Let $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ and $g : Y \rightarrow \mathbb{R} \cup \{+\infty\}$ be two lower semicontinuous and bounded from below functions with non empty domains i.e $\text{dom}(f) := \{x \in X : f(x) < +\infty\} \neq \emptyset$ and $\text{dom}(g) \neq \emptyset$. We denote by $\overline{\text{dom}(f)}$ and $\overline{\text{dom}(g)}$ the closure of $\text{dom}(f)$ and $\text{dom}(g)$ in X and Y respectively. We denote by $\mathcal{H}_{f,g}(Y, X)$ the class of all homeomorphism from $\overline{\text{dom}(g)}$ onto $\overline{\text{dom}(f)}$ if such homeomorphism exists. We give below the diagrams of our classification result. The left diagram commutes if and only that of right also.

Let $T : A \rightarrow B$ be an isomorphism,

$$\begin{array}{ccc}
A & \xrightarrow{T} & B \\
& \searrow \mathcal{F} & \downarrow \mathcal{G} \\
& & \mathbb{R}
\end{array}
\quad \Leftrightarrow \quad
\begin{array}{ccc}
\exists \pi : \overline{\text{dom}(g)} & \xrightarrow{\pi} & \overline{\text{dom}(f)} \\
& \searrow g & \downarrow f \\
& & \mathbb{R} \cup \{+\infty\}
\end{array}$$

$\mathcal{G} \circ T = \mathcal{F} \Leftrightarrow \exists \pi \in \mathcal{H}_{f,g}(Y, X)$ such that $f \circ \pi = g$ on $\overline{\text{dom}(g)}$ and T comes canonically from π on $\overline{\text{dom}(g)}$. Where

$$\mathcal{F}(\varphi) := \sup_{x \in X} \{|\varphi(x)| - f(x)\}, \forall \varphi \in A$$

$$\mathcal{G}(\psi) := \sup_{y \in Y} \{|\psi(y)| - g(y)\}, \forall \psi \in B.$$

We recover the classical real valued Banach-Stone theorem in the case of complete metric spaces X and Y with $A = C_b(X)$, $B = C_b(Y)$, $f = 0$ on X and $g = 0$ on Y . The situation in Question 1. correspond to $f = \|\cdot\|_X$ and $g = \|\cdot\|_Y$. Our main result is then the following theorem.

1.3 The main result.

Theorem 1 *Let X and Y be two complete metric spaces and $A \subset C_b(X)$ and $B \subset C_b(Y)$ be two Banach spaces satisfying the axioms A1-A4 (with the same property P^β . See Axioms 1 and the examples in Proposition 2). Let $T : A \rightarrow B$ be an isomorphism and let $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ and $g : Y \rightarrow \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous and bounded from below with non empty domains. Then (1) \Leftrightarrow (2).*

- (1) For all $\varphi \in A$, we have $\sup_{y \in Y} \{|T\varphi(y)| - g(y)\} = \sup_{x \in X} \{|\varphi(x)| - f(x)\}$.
- (2) There exist an homeomorphism $\pi : \overline{\text{dom}(g)} \rightarrow \overline{\text{dom}(f)}$ and a continuous function $\varepsilon : \text{dom}(g) \rightarrow \{+1, -1\}$ such that, for all $y \in \text{dom}(g)$ and all $\varphi \in A$ we have

$$T\varphi(y) = \varepsilon(y)\varphi \circ \pi(y)$$

and

$$g(y) = f \circ \pi(y).$$

Remark 1 Note that from (2) we obtain $\sup_{y \in \overline{\text{dom}(g)}} |T\varphi(y)| = \sup_{x \in \overline{\text{dom}(f)}} |\varphi(x)|$, $\forall \varphi \in A$. It follows that, if $\overline{\text{dom}(f)} = X$ and $\overline{\text{dom}(g)} = Y$ then the condition (1) implies in particular that T is isometric for the norm $\|\cdot\|_\infty$. So using lower semicontinuous functions such that $\overline{\text{dom}(f)} = X$ and $\overline{\text{dom}(g)} = Y$ will permit to classify isometries for the norm $\|\cdot\|_\infty$. In the general case Theorem 1 permit to classify isomorphisms not necessary isometries satisfying $\sup_{y \in E} |T\varphi(y)| = \sup_{x \in F} |\varphi(x)|$, $\forall \varphi \in A$ for some $E \subset X$ and $F \subset Y$.

As an immediate consequence of the above theorem is the following general Banach-Stone result.

Corollary 1 Let X and Y be two complete metric spaces. Let $A \subset C_b(X)$ and $B \subset C_b(Y)$ be two Banach spaces satisfying the axioms A_1 - A_4 (with the same property P^β). Let E be a closed subset of X and F be a closed subset of Y . Let $T : A \rightarrow B$ be an isomorphism. Then

$$\sup_{y \in F} |T\varphi(y)| = \sup_{x \in E} |\varphi(x)|$$

for all $\varphi \in A$ if and only if there exists an homeomorphism $\pi : F \rightarrow E$ and a continuous map $\varepsilon : F \rightarrow \{-1, +1\}$ such that for all $y \in F$ and all $\varphi \in A$ we have $T\varphi(y) = \varepsilon(y)\varphi \circ \pi(y)$.

Proof. It suffices to apply Theorem 1 with the indicator functions $f = i_E$ and $g = i_F$ where i_E (respectively, i_F) is equal to 0 on E (respectively, on F) and $+\infty$ otherwise.

Remark 2 We recover the classical version of the Banach-Stone theorem from the above corollary by taking $E = X$ and $F = Y$.

Note that there are situation where an isomorphism is not an isometry but ‘‘partially’’ isometric. Indeed, Let K and L be two metric and compact non homeomorphic spaces such that $C(K)$ and $C(L)$ are isomorphic and let $T_1 : C(K) \rightarrow C(L)$ be an isomorphism. This situation exists for example for $C[0, 1]$ and $C([0, 1] \times [0, 1])$. A. A. Milutin [25] proved that if K and L are both uncountable compact metric spaces, then $C(K)$ and $C(L)$ are always linearly isomorphic. Let R and S two homeomorphic complete metric spaces such that $R \cap K = \emptyset$ and $S \cap L = \emptyset$ and let $\pi : S \rightarrow R$ an homeomorphism. Let us consider the map $T : C_b(K \cup R) \rightarrow C_b(L \cup S)$ defined by $T(\varphi)(y) = T_1(\varphi|_K)(y)$ if $y \in L$ and $T(\varphi)(y) = \varphi \circ \pi(y)$ if $y \in S$ for all $\varphi \in C_b(K \cup R)$. Where $\varphi|_K$ denotes the restriction of φ to K . The map T is an isomorphism not isometric satisfying $\sup_{y \in S} |T\varphi(y)| = \sup_{x \in R} |\varphi(x)|$ for all $\varphi \in C_b(K \cup R)$.

The proof of the above theorem will be given in Section 4. It is based on differentiability

argument as in the original proof given by Banach [9], with duality result, Theorem 3 and Theorem 4 (See Section 2.4) which were established in [7] together with Lemma 1 (See Section 2.3) which is a consequence of the well known Deville-Godefroy-Ziler variational principle [14]. We use the general version of this principle which also works for complete metric spaces and is due to R. Deville and J. Rivaliski [16]. Also let us note that a similar result of our main result Theorem 1 can be established in a vectorial frame for abstract subspaces of the space of all continuous and bounded Z -valued functions $C_b(X, Z)$. Some conditions on the Banach space Z are necessary like the smoothness of the norm of Z . The proof being more technical and requiring others intermediate results, we omit it in this article but we send back to the thesis [5] (see also [6]) for a vectorial Banach-Stone theorem. Let us mention here that the first result about the vectorial Banach-Stone theorem is due to E. Behrends [10], [11]. Others results about the vector valued Banach-Stone theorem was established by J. Araujo in [2], [3] and by K. Jaros in [23].

Remark 3 *In Theorem 1 above, we can have more information about the homeomorphism π . More the spaces A and B are regular more the homeomorphism π is it also. For example if $A = C_b^u(X)$ and $B = C_b^u(Y)$ (the spaces of uniformly continuous and bounded) we obtain that π is uniformly continuous as well as π^{-1} . If $A = Lip_b(X)$ and $B = Lip_b(Y)$ (the spaces of all Lipschitz bounded map) and X and Y are quasi-convex then π is bi-Lipschitz. This follows from the fact that if $\varphi \circ \pi$ is Lipschitz for all $\varphi \in Lip_b(X)$ then π is Lipschitz. See Theorem 44 in the paper of M. Garrido and J. Jaramillo [17]. It is also known from the paper of J. Gutiérrez and J. Llavona [21] (See also [22]) that if $p \circ \pi$ is C^k smooth function (in this case X is assumed to be a Banach space) for all $p \in X^*$ (the topological dual of X) then π is of class C^{k-1} . So in the case where $A = C_b^k(X)$ and $B = C_b^k(Y)$ (the space of k -times continuously Fréchet differentiable function f such that $f, f', \dots, f^{(k)}$ are uniformly bounded) we can deduce, using the composition $\alpha \circ p$ where $p \in X^*$ and $\alpha : \mathbb{R} \rightarrow \mathbb{R}$ be a desirable “very smooth” function, that π is C^{k-1} diffeomorphism. Note that if moreover X has the Schur property then from the paper of M. Bachir and G. Lancien [8] we can deduce that π is of class C^k . We draw the attention here on the fact that the Schur property and the axiom A_3 (See Axioms 1) are not still compatible in infinite dimension, for example the space $X = l^1(\mathbb{N})$ which has the Schur property does not admit a Lipschitz and C^1 smooth bump function (See [15]).*

This paper is organised as follows. In section 2. we introduce the axioms which we shall use in this article and we give examples satisfying them. We also give tools and preliminaries result which will permit us to give the proof of our main theorem. In section 3. We give the proof of our main result Theorem 1.

2 Tools and preliminaries results.

To prove our main result we need to establish certain lemmas and to recall other results already established.

2.1 The Dirac masses.

Let (X, d) be a complete metric spaces and $(A, \|\cdot\|)$ a Banach space included in $C_b(X)$. By δ we denote the Dirac map and by δ_x the Dirac mass associated to the point $x \in X$. By A^* we denote the topological dual of A . We have

$$\begin{aligned}\delta : X &\rightarrow A^* \\ x &\rightarrow \delta_x\end{aligned}$$

and for all $x \in X$,

$$\begin{aligned}\delta_x : A &\rightarrow \mathbb{R} \\ \varphi &\rightarrow \varphi(x).\end{aligned}$$

Definition 1 (The property P^β) Let (X, d) be a complete metric space and $(A, \|\cdot\|)$ a Banach space included in $C_b(X)$. We say that A has the property $\mathbf{P}^\mathbf{F}$ (respectively, $\mathbf{P}^\mathbf{G}$) if, for each sequence $(x_n)_n \subset X$ we have:

$(x_n)_n$ converges in (X, d) if and only if the associated sequence of the Dirac masses $(\delta_{x_n})_n$ converges in $(A^*, \|\cdot\|^*)$ (respectively, in (A^*, w^*)). Where $\|\cdot\|^*$ denotes the dual norm and w^* the weak-star topology.

We denote by τ_β the weak-star topology if $\beta = G$ and the norm topology if $\beta = F$.

The crucial property P^β is related to the geometry of the Banach space A and is connected to the differentiability to the supremum norm $\|\cdot\|_\infty$ for more detail see the article [7]. The following proposition follow easily from the above definition.

Proposition 1 Let (X, d) be a complet metric space and $(A, \|\cdot\|)$ be a Banach space included in $C_b(X)$ which separate the points of X .

(1) Suppose that A has the property P^F then $\delta(X) := \{\delta_x : x \in X\}$ is closed for the norm topology in A^* and the map

$$\begin{aligned}\delta : (X, d) &\rightarrow (\delta(X), \|\cdot\|^*) \\ x &\mapsto \delta_x\end{aligned}$$

is an homeomorphism.

(2) Suppose that A has the property P^G then $\delta(X) := \{\delta_x : x \in X\}$ is sequentially closed for the weak-star topology in A^* and the map

$$\begin{aligned}\delta : (X, d) &\rightarrow (\delta(X), w^*) \\ x &\mapsto \delta_x\end{aligned}$$

is an sequential homeomorphism.

The map

$$\begin{aligned}\delta : X &\rightarrow A^* \\ x &\rightarrow \delta_x\end{aligned}$$

is a non linear analogous to the canonical linear isometry $i : Z \rightarrow Z^{**}$ where Z is Banach space and Z^{**} its bidual. The map δ permit to linearize the metric space X in A^* (See the paper of Godefroy-Kalton [20] when A is the set of all Lipschitz map on X that vanich at some point).

2.2 Axioms and examples.

We give now the general axioms that the space A has to satisfy in our results.

Axiome 1 *In all the article (X, d) is a complete metric space and A is a class of functions spaces included in $C_b(X)$ such that :*

(A₁) *the space $(A, \|\cdot\|)$ is a Banach space and $\|\cdot\| \geq \|\cdot\|_\infty$.*

(A₂) *the space A separate the points of X and contain the constants.*

(A₃) *for each $n \in \mathbb{N}^*$, there exists a positive constant M_n such that for each $x \in X$ there exists a function $h_n : X \rightarrow [0, 1]$ such that $h_n \in A$, $\|h_n\| \leq M_n$, $h_n(x) = 1$ and $\text{diam}(\text{supp}(h_n)) < \frac{1}{n}$.*

(A₄) *the space A has the property P^β ($\beta = F$ or $\beta = G$).*

Remark 4 : *These axioms are satisfied by various and classical spaces of functions. We give below some examples. Let us mention here, that the axiom (A₃) is related to the variational principle of Deville-Godefroy-Zizler [14] and Deville-Rivalski [16] and the axiom (A₄) was introduced and studied in [7].*

By the space $C_b^u(X)$ we denote the Banach space of all bounded uniformly continuous function on X , by $\text{Lip}_b^\alpha(X)$ the Banach space of all α -Holder and bounded functions on X ($0 < \alpha \leq 1$), by $C_b^k(X)$ the Banach space of all k -times continuously Fréchet differentiable functions f such that $f, f', \dots, f^{(k)}$ are uniformly bounded, by $C_b^{1,\alpha}(X)$ ($0 < \alpha \leq 1$) the Banach space of all Fréchet differentiable functions f on X such that f and f' are uniformly bounded on X and f' is α -Holder. Finally by $C_b^{1,u}(X)$ we denote the Banach space of all Fréchet differentiable functions f on X such that f and f' are uniformly bounded on X and f' is uniformly continuous. All these spaces are provided with their natural norm $\|\cdot\|$ of Banach space that satisfy $\|\cdot\| \geq \|\cdot\|_\infty$ (See [7]).

Proposition 2 *We have,*

(1) *For every complete metric space X , the spaces $C_b(X)$, $C_b^u(X)$ and $\text{Lip}_b^\alpha(X)$ ($0 < \alpha \leq 1$) satisfy the axioms (A₁)-(A₄)*

(2) *If X is a Banach space having a bump function (that is a function with a non empty and bounded support) in $A = C_b^k(X)$ ($k \in \mathbb{N}^*$) (respectively, in $A = C_b^{1,\alpha}(X)$ ($0 < \alpha \leq 1$) or $A = C_b^{1,u}(X)$), then A satisfy the axioms (A₁)-(A₄).*

Proof. The proof is an consequence of Proposition 3, Proposition 4 and Proposition 5 below.

Proposition 3 *All of the spaces A mentioned in the examples above, satisfy the axioms (A₁) and (A₂).*

Proof. The axiom (A₁) follow from the definitions of the spaces and there norms (See [7]). These spaces contain the constants. To see that these spaces separate the points of X , let $x \neq y$ in X . In the case of the spaces $C_b(X)$, $C_b^u(X)$ and $\text{Lip}_b^\alpha(X)$ where (X, d) is a metric space, Let us set $\phi(\cdot) := \min(d(x, \cdot), 1) \in \text{Lip}_b^\alpha(X) \subset C_b^u(X) \subset C_b(X)$. We have $\phi(x) = 0$ and $\phi(y) = \min(d(x, y), 1) > 0$. So $\phi(x) \neq \phi(y)$ and thus these spaces separate the points of X . In the case where X is a Banach space and $A = C_b^k(X)$

($k \in \mathbb{N}^*$) or $C_b^{1,\alpha}(X)$ ($0 < \alpha \leq 1$) or $C_b^{1,u}(X)$, since $x \neq y$, then by the Hahn-Banach theorem, there exists $p \in X^*$ such that $p(x) \neq p(y)$. We construct then a map $\gamma : \mathbb{R} \rightarrow \mathbb{R}$ such that $\gamma \circ p \in A$ and $\gamma(p(x)) = p(x)$ and $\gamma(p(y)) = p(y)$. Thus A separate the point of X .

The following proposition is in [7].

Proposition 4 [Proposition 2.5 and 2.6, [7]] *Let (X, d) be a complete metric space. We have :*

- (1) *If $A = C_b(X)$ or $C_b^u(X)$ then A satisfy \mathbf{P}^G (but not \mathbf{P}^F).*
- (2) *If $A = \text{Lip}_b^\alpha(X)$ with $0 < \alpha \leq 1$ then A satisfy \mathbf{P}^F .*
- b) *Suppose that X is Banach space and $A = C_b^k(X)$ ($k \in \mathbb{N}^*$) or $C_b^{1,\alpha}(X)$ ($0 < \alpha \leq 1$) or $C_b^{1,u}(X)$ then A satisfy \mathbf{P}^F .*

We have proved that the above examples of spaces satisfy the axioms (A_1) , (A_2) and (A_4) . These examples satisfy also the axiom (A_3) whenever the space X has a bump function in A (that is a function with a non empty and bounded support). The existence of a bump function in $C_b(X)$, $C_b^u(X)$ or in $\text{Lip}_b^\alpha(X)$ with $0 < \alpha \leq 1$ is always true by using the metric d on X . This is note always the case when X is a Banach space for the spaces of smooth functions $A = C_b^k(X)$ ($k \in \mathbb{N}^*$) or $C_b^{1,\alpha}(X)$ ($0 < \alpha \leq 1$) or $C_b^{1,u}(X)$. In the last examples the existence of a bump function is connected to the geometry of the Banach space X . For more information on the existence of a bump function in $C_b^k(X)$ ($k \in \mathbb{N}^*$) or $C_b^{1,\alpha}(X)$ ($0 < \alpha \leq 1$) or $C_b^{1,u}(X)$ we refer to the book of R. Deville, G. Godefroy and V. Zizler [15].

Proposition 5 *we have:*

- (1) *For every complete metric space X , the spaces $C_b(X)$, $C_b^u(X)$ and $\text{Lip}_b^\alpha(X)$ satisfy the axiom (A_3)*
- (2) *If X is a Banach space having a bump function in $A = C_b^k(X)$ ($k \in \mathbb{N}^*$) (respectively in $C_b^{1,\alpha}(X)$ ($0 < \alpha \leq 1$) or $C_b^{1,u}(X)$), then A satisfy the axiom (A_3) .*

Proof. The proof is easy and can be found in Remark 2.5 (See also Proposition 1.4) in [16].

2.3 A useful Lemmas.

We need the following Lemma 1 that is a consequence of the Variational principle of Deville-Godefroy-Zizler [14] generalized by Deville-Rivalski [16].

Definition 2 *We say that a function f has a strong minimum at x if $\inf_X f = f(x)$ and $d(x_n, x) \rightarrow 0$ whenever $f(x_n) \rightarrow f(x)$.*

Theorem 2 ([Deville-Godefroy-Zizler [14]] ; [Deville-Rivalski [16]]):

Let (X, d) be a complete metric space and $A \subset C_b(X)$ be a space satisfying the axioms (A_1) and (A_3) . Let f be a lower semicontinuous and bounded from below function with non empty domain. Then,

$$\{\varphi \in A / f - \varphi \text{ does not attain a strong minimum on } X\}$$

is σ -porous in particular it is of the first Baire category.

Lemma 1 *Under the hypothesis of Theorem 2, we have that for every lower semicontinuous and bounded from below with non empty domain $f : X \rightarrow \mathbb{R} \cup +\infty$ ($f \not\equiv +\infty$) the set*

$$D(f) := \{x \in X / \exists \varphi_x \in A : f - \varphi_x \text{ has a strong minimum at } x\}$$

is dense in $\overline{\text{dom}(f)}$.

Proof. Let $x \in \text{dom}(f)$ and $n \in \mathbb{N}^*$. Let $h_n \in A$ as in Theorem 2 with $h_n(x) = 1$. let us set $\lambda_x^n := f(x) - \inf_X(f) + \frac{3}{n}$ and applying Theorem 2 to the function $f + \lambda_x^n h_n$, there exists $x_n \in X$ and $\varphi \in A$ such that $\|\varphi\| < \frac{1}{n}$ and $f + \lambda_x^n h_n - \varphi$ has a strong minimum at x_n . Suppose that $d(x, x_n) \geq \frac{1}{n}$. since $\text{diam}(\text{supp}(h_n)) < \frac{1}{n}$ and $x \in \text{supp}(h_n)$, then $h(x_n) = 0$. Thus, we have

$$\begin{aligned} \inf_X(f) - \varphi(x_n) &\leq f(x_n) - \varphi(x_n) \\ &= f(x_n) - \lambda_x^n h_n(x_n) - \varphi(x_n) \\ &< f(x) - \lambda_x^n h_n(x) - \varphi(x) \\ &= f(x) - \lambda_x^n - \varphi(x) \end{aligned}$$

We deduct that $\lambda_x^n < f(x) - \inf_X(f) + \frac{2}{n}$ which is a contradiction with the choice of λ_x^n . So $d(x, x_n) < \frac{1}{n}$ and $x_n \in D(f)$. It follows that $D(f)$ is dense in $\text{dom}(f)$.

We need also the following two Lemmas.

Lemma 2 *Let Z be a Banach space and $h, k : Z \rightarrow \mathbb{R}$ two continuous and convex functions. Suppose that the function $z \rightarrow l(z) := \max(h(z), k(z))$ is Fréchet (respectively, Gâteaux) differentiable at z_0 . Then h is Fréchet (respectively, Gâteaux) differentiable at z_0 or k is Fréchet (respectively, Gâteaux) differentiable at z_0 and $l'(z_0) = h'(z_0)$ or $l'(z_0) = k'(z_0)$.*

Proof. We give the proof for the fréchet differentiability, the Gâteaux differentiability is similar. Suppose without loose of generality that $l(z_0) = h(z_0)$ and let us prove that h is Fréchet differentiable at z_0 and that $l'(z_0) = h'(z_0)$. For each $z \neq 0$ we have:

$$0 \leq \frac{h(z_0 + z) + h(z_0 - z) - 2h(z_0)}{\|z\|} \leq \frac{l(z_0 + z) + l(z_0 - z) - 2l(z_0)}{\|z\|}$$

The right member tends to 0 when z tends to 0 since l is convex and Fréchet differentiable at z_0 . This implies that h is Fréchet differentiable at z_0 by convexity. Now, since $l(z) \geq h(z)$ for all $z \in Z$ and $l(z_0) = h(z_0)$, then for all $t > 0$ we have

$$\begin{aligned} h'(z_0)(z) - l'(z_0)(z) &\leq \frac{h'(z_0)(tz) - l'(z_0)(tz) + l(z_0 + tz) - h(z_0 + tz) - l(z_0) + h(z_0)}{t} \\ &= \frac{l(z_0 + tz) - l(z_0) - l'(z_0)(tz)}{t} + \frac{h(z_0 + tz) - h(z_0) - h'(z_0)(tz)}{t} \end{aligned}$$

So, sending t to 0^+ , we have $h'(z_0)(z) - l'(z_0)(z) \leq 0$, for all $z \in Z$. Thus $h'(z_0) = l'(z_0)$.

Lemma 3 *Let (X, d) be a complete metric space and $A \subset C_b(X)$ satisfying the axioms A_1, A_2 and A_4 . Let $(\lambda)_n \subset \mathbb{R}$ such that $|\lambda_n| = 1$ for all $n \in \mathbb{N}$ and let $(x_n)_n \subset X$. Suppose that $\lambda_n \delta_{x_n}$ converges for the topology τ_β (the norm topology or the weak-star topology) to some point $Q \in A^*$. Then $(\lambda_n)_n$ converges in \mathbb{R} to some real number λ such that $|\lambda| = 1$ and $(x_n)_n$ converges to some point x in (X, d) and we have $Q = \lambda \delta_x$.*

Proof. Since $\lambda_n \delta_{x_n}$ converges for the topology τ_β to some point $Q \in A^*$, then $\lambda_n \delta_{x_n}(\varphi) \rightarrow Q(\varphi)$ for all $\varphi \in A$. Since A contain the constants, we have $\lambda_n \rightarrow Q(1) := \lambda$ with $|\lambda| = 1$. Now, since $(\lambda_n)_n$ converges to λ and $\lambda_n \delta_{x_n}$ converges for the topology τ_β to Q . By dividing by λ_n we have that δ_{x_n} converges for the topology τ_β to $\frac{Q}{\lambda} \in A^*$. The property P^β implies that $(x_n)_n$ converges to some point $x \in X$ and in consequence that δ_{x_n} converges for the topology τ_β to δ_x . By the uniqueness of the limit we have that $Q = \lambda \delta_x$.

2.4 Duality results.

We recall from [7] the following results which are the key of the results of this article. Let (X, d) be a complete metric space and A be a class of functions space satisfying the axioms A_1 - A_4 . For the proof of Theorem 1, we need the following results about the conjugacy f^\times of lower semicontinuous bounded from below function f defined by

$$\begin{aligned} f^\times : A &\rightarrow \mathbb{R} \\ \varphi &\rightarrow \sup_{x \in X} \{\varphi(x) - f(x)\} \end{aligned}$$

We define the second conjugacy of f by $f^{\times\times} : X \rightarrow \mathbb{R} \cup \{+\infty\}$;

$$f^{\times\times}(x) := \sup_{\varphi \in A} \{\varphi(x) - f^\times(\varphi)\}; \forall x \in X$$

Proposition 6 ([Proposition 2.1, [7]]): *The map f^\times is convex and 1-Lipschitz on A .*

The second conjugate $f^{\times\times}$ is not convex in general, but we have:

Theorem 3 ([Theorem 2.2, [7]]): *Let f be a bounded from below function on X and suppose that A satisfy the axiom A_4 . Then, f is lower semicontinuous if and only if $f^{\times\times} = f$.*

Finally, we need the following duality theorem between differentiability and the well posed problems.

Theorem 4 ([Theorem 2.8, [7]]): *Let $A \subset C_b(X)$ be a class of Banach space satisfying the axioms A_1, A_2 and the property \mathbf{P}^F (respectively \mathbf{P}^G). Let $\varphi \in A$ and f be a lower semicontinuous function. Then (1) \Leftrightarrow (2).*

- (1) *the function $f - \varphi$ has a strong minimum at some point in X .*
- (2) *the function f^\times is Fréchet differentiable (respectively, Gâteaux differentiable) at φ on A .*

Moreover, in this case the Fréchet derivative (respectively, the Gâteaux derivative) of f^\times at φ is $(f^\times)'(\varphi) = \delta_x$ where $x \in X$ is the point where $f - \varphi$ has its strong minimum.

We also need the following elementary lemma.

Lemma 4 *We have*

$$\sup_{x \in X} \{|\varphi(x)| - f(x)\} = \max(f^\times(\varphi), f^\times(-\varphi))$$

for all $\varphi \in A$.

Proof. Since $|t| = \max(t, -t)$ on \mathbb{R} , by inverting the supremum and the maximum we have

$$\begin{aligned} \sup_{x \in X} \{|\varphi(x)| - f(x)\} &= \max(\sup_{x \in X} \{\varphi(x) - f(x)\}, \sup_{x \in X} \{-\varphi(x) - f(x)\}) \\ &= \max(f^\times(\varphi), f^\times(-\varphi)). \end{aligned}$$

3 The proof of Theorem 1.

The proof is given after some steps. The part (2) \Rightarrow (1) is easy. We prove the part (1) \Rightarrow (2).

Let X and Y be two complete metric spaces and $A \subset C_b(X)$ and $B \subset C_b(Y)$ be two Banach spaces satisfying the axioms A1-A4 with the same property P^β . (See Axioms 1 and the examples in Proposition 2). Let $T : A \rightarrow B$ be an isomorphism and let $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ and $g : Y \rightarrow \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous and bounded from below functions with non empty domains. Suppose that

$$\sup_{y \in Y} \{|T\varphi(y)| - g(y)\} = \sup_{x \in X} \{|\varphi(x)| - f(x)\}$$

For all $\varphi \in A$ and denote by $T^* : B^* \rightarrow A^*$ the adjoint of T .

3.1 The map T has the canonical form.

Lemma 5 : *There exists a map $\pi : \overline{\text{dom}(g)} \rightarrow \overline{\text{dom}(f)}$ and a map $\varepsilon : \overline{\text{dom}(g)} \rightarrow \{-1, 1\}$ such that for all $y \in \overline{\text{dom}(g)}$ we have : $T^*\delta_y = \varepsilon(y)\delta_{\pi(y)}$.*

Remark 5 *The formula above is equivalent to $T\varphi(y) = \varepsilon(y)\varphi \circ \pi(y)$ for all $y \in \overline{\text{dom}(g)}$ and all $\varphi \in A$.*

Proof. By lemma 1, the set $D(g)$ is dense in $\overline{\text{dom}(g)}$. Let $y \in D(g)$ and $\tilde{\psi}_y \in B$ such that $g - \tilde{\psi}_y$ has a strong minimum at y . Let $c \in \mathbb{R}$ such that $c > \frac{1}{2} \left(g^\times(-\tilde{\psi}_y) - g^\times(\tilde{\psi}_y) \right)$ and put $\psi_y = c + \tilde{\psi}_y$. The function $g - \psi_y$ has also a strong minimum at y and satisfies by the choice of c the inequality $g^\times(\psi_y) > g^\times(-\psi_y)$. Since g^\times and so also $g^\times \circ (-I_Y)$ (where I_Y denotes the identity map on B) are continuous (Lipschitz) by Proposition 6, there exists an open neighborhood $O(\psi_y) \subset B$ of ψ_y such that $g^\times(\psi) > g^\times(-\psi)$ for all $\psi \in O(\psi_y)$. Thus, we have $\max(g^\times(\psi), g^\times(-\psi)) = g^\times(\psi)$ on the open set $O(\psi_y)$. Since $g - \psi_y$ has a strong minimum at y , Theorem 4 guarantees the β -differentiability of g^\times at ψ_y with $(g^\times)'(\psi_y) = \delta_y$. Since the functions

$\psi \rightarrow \max(g^\times(\psi), g^\times(-\psi))$ and $\psi \rightarrow g^\times(\psi)$ coincide on the open set $O(\psi_y)$ we conclude that $\psi \rightarrow \max(g^\times(\psi), g^\times(-\psi))$ is β -differentiable at ψ_y with derivative equal to $(g^\times)'(\psi_y) = \delta_y$. On the other hand, there exists $\varphi_y \in A$ such that $\psi_y = T\varphi_y$ by the surjectivity of T . The composition of β -differentiable function with linear an continuous map is again β -differentiable. Thus, we have the β -differentiability of the composite map $\varphi \rightarrow \max(g^\times(T\varphi), g^\times(-T\varphi))$ at φ_y on A and the chain rule formula give $\delta_y \circ T$ as derivative of the function $\varphi \rightarrow \max(g^\times(T\varphi), g^\times(-T\varphi))$ at φ_y . But by hypothesis and by Lemma 4, $\max(g^\times(T\varphi), g^\times(-T\varphi)) = \max(f^\times(\varphi), f^\times(-\varphi))$ for all $\varphi \in A$. We deduce that the function $\varphi \rightarrow \max(f^\times(\varphi), f^\times(-\varphi))$ is also β -differentiable at φ_y on A with the same derivative $\delta_y \circ T$. Lemma 2 implies that either the function $\varphi \rightarrow f^\times(\varphi)$ or the function $\varphi \rightarrow f^\times(-\varphi)$ is β -differentiable at φ_y with the derivative given by the derivative of $\varphi \rightarrow \max(f^\times(\varphi), f^\times(-\varphi))$ at φ_y that is here $\delta_y \circ T$. If it is the function $\varphi \rightarrow f^\times(\varphi)$, Theorem 4 assert that there exists $\pi(y) \in D(f)$ such that $(f^\times)'(\varphi_y) = \delta_{\pi(y)}$. If it is the function $\varphi \rightarrow f^\times(-\varphi)$, Theorem 4 assert the existence of point $\pi(y) \in X$ such that $(f^\times \circ (-I_X))'(\varphi_y) = (f^\times)'(-\varphi_y) \circ (-I_X) = \delta_{\pi(y)} \circ (-I_X) = -\delta_{\pi(y)}$ where I_X denotes the identity map on A . By identifying the derivatives of the two equal functions $\varphi \rightarrow \max(g^\times(T\varphi), g^\times(-T\varphi)) = \max(f^\times(\varphi), f^\times(-\varphi))$ we have : $\delta_y \circ T = \delta_{\pi(y)}$ or $\delta_y \circ T = -\delta_{\pi(y)}$. Let us put $\varepsilon(y) = \pm 1$. Then we have

$$\forall y \in D(g) \exists \pi(y) \in D(f) / \quad T^*\delta_y := \delta_y \circ T = \varepsilon(y)\delta_{\pi(y)} \quad . \quad (3)$$

Now, let y be any point of $\overline{\text{dom}(g)}$, there exists by Lemma 1 a sequences $(y_n)_n \subset D(g)$ such that $y_n \rightarrow y$. The property P^β (axiom A_4) implies that $\delta_{y_n} \xrightarrow{\tau_\beta} \delta_y$. Since T^* is τ_β to τ_β continue (here τ_β is the norm or the weak-star topology) then $T^*\delta_{y_n} \xrightarrow{\tau_\beta} T^*\delta_y$. Since $(y_n)_n \subset D(g)$, then from (3) there exists $\pi(y_n) \in D(f)$ such that $T^*\delta_{y_n} = \varepsilon(y_n)\delta_{\pi(y_n)}$. So we have $\varepsilon(y_n)\delta_{\pi(y_n)} \xrightarrow{\tau_\beta} T^*\delta_y$. Lemma 3 implies the existence of a real number $\varepsilon(y) = \pm 1$ and some point $\pi(y) \in X$ such that $\varepsilon(y_n) \rightarrow \varepsilon(y)$ in \mathbb{R} and $\pi(y_n) \rightarrow \pi(y)$ in X . Thus $\pi(y) \in \overline{D(f)} = \overline{\text{dom}(f)}$. The Lemma 3 implies also that $T^*\delta_y = \varepsilon(y)\delta_{\pi(y)}$. Thus we have proved that there exists a map $\pi : \overline{\text{dom}(g)} \rightarrow \overline{\text{dom}(f)}$ and a map $\varepsilon : \overline{\text{dom}(g)} \rightarrow \{-1, 1\}$ such that for all $y \in \overline{\text{dom}(g)}$, we have $T^*\delta_y = \varepsilon(y)\delta_{\pi(y)}$.

3.2 The map π is bijective.

Lemma 5 applied to T^{-1} implies the existence of a map $\pi' : \overline{\text{dom}(f)} \rightarrow \overline{\text{dom}(g)}$ and a map $\varepsilon' : \overline{\text{dom}(f)} \rightarrow \{-1, 1\}$ such that for all $x \in \overline{\text{dom}(f)}$ we have $(T^{-1})^*\delta_x = \varepsilon'(x)\delta_{\pi'(x)}$. We obtain then ,

$$\begin{aligned} \delta_x &= T^*(\varepsilon'(x)\delta_{\pi'(x)}) \\ &= \varepsilon'(x)T^*(\delta_{\pi'(x)}) \\ &= \varepsilon'(x)\varepsilon(\pi(x))\delta_{\pi(\pi'(x))} \end{aligned}$$

By applying the above identity to the constant function 1 we obtain $\varepsilon'(x)\varepsilon(\pi(x)) = 1$. On the other hand, since the space A separate the points of X (axiom (A_2)), we obtain $\pi(\pi'(x)) = x$. This reasoning applies for all $x \in \overline{\text{dom}(f)}$. By inverting the roles of T et T^{-1} , we have also $\pi'(\pi(y)) = y$ for all $y \in \overline{\text{dom}(g)}$. Thus π is bijective.

3.3 The maps ε , π and π^{-1} are continuous.

Let $y_n \in \overline{\text{dom}(g)}$ such that $y_n \rightarrow y \in \overline{\text{dom}(g)}$. Let us prove that $\pi(y_n) \rightarrow \pi(y)$ in $\overline{\text{dom}(f)}$. Indeed, by the property P^β (axiom A_4) we have that $\delta_{y_n} \xrightarrow{\tau_\beta} \delta_y$. Since T is τ_β to τ_β continuous then $T^*\delta_{y_n} \xrightarrow{\tau_\beta} T^*\delta_y$ for the τ_β topology in A^* . In other words $\varepsilon(y_n)\delta_{\pi(y_n)} \xrightarrow{\tau_\beta} \varepsilon(y)\delta_{\pi(y)}$ which implies by Lemma 3 that $\varepsilon(y_n) \rightarrow \varepsilon(y)$ in \mathbb{R} and $\delta_{\pi(y_n)} \xrightarrow{\tau_\beta} \delta_{\pi(y)}$ in A^* . Again by the property P^β , we have $\pi(y_n) \rightarrow \pi(y)$ in $\overline{\text{dom}(g)}$. Thus, ε and π are continuous. The same argument applied to T^{-1} shows that π^{-1} is also continuous.

3.4 The formula $g = f \circ \pi$ on $\overline{\text{dom}(g)}$.

This formula follow from the previous affirmations together with Theoreme 4. Indeed, by hypothesis we have

$$\sup_{y \in Y} \{|T\varphi(y)| - g(y)\} = \sup_{x \in X} \{|\varphi(x)| - f(x)\}$$

for all $\varphi \in A$. Since g (resp f) is equal to $+\infty$ on $Y \setminus \overline{\text{dom}(g)}$ (respectively, on $X \setminus \overline{\text{dom}(f)}$) we have

$$\sup_{y \in \overline{\text{dom}(g)}} \{|T\varphi(y)| - g(y)\} = \sup_{x \in \overline{\text{dom}(f)}} \{|\varphi(x)| - f(x)\}$$

for all $\varphi \in A$. Since $\overline{\text{dom}(g)}$ and $\overline{\text{dom}(f)}$ are homeomorphic, then by replacing $T\varphi(y)$ in the above formula with its expression $\varepsilon(y)\varphi(\pi(y))$ for $y \in \overline{\text{dom}(g)}$, we obtain

$$\sup_{y \in \overline{\text{dom}(g)}} \{|\varphi(\pi(y))| - g(y)\} = \sup_{x \in \overline{\text{dom}(f)}} \{|\varphi(x)| - f(x)\},$$

for all $\varphi \in A$. Since π is bijective, we obtain by the change of variable $x = \pi^{-1}(y)$

$$\sup_{x \in \overline{\text{dom}(f)}} \{|\varphi(x)| - g(\pi^{-1}(x))\} = \sup_{x \in \overline{\text{dom}(f)}} \{|\varphi(x)| - f(x)\},$$

for all $\varphi \in A$. The above formula is also true for the functions $\varphi - \inf_X(\varphi) \geq 0$ for all $\varphi \in A$ since A contain the constants. Replacing φ by $\varphi - \inf_X(\varphi) \geq 0$ we obtain

$$\sup_{x \in \overline{\text{dom}(f)}} \{\varphi(x) - g(\pi^{-1}(x))\} - \inf_X(\varphi) = \sup_{x \in \overline{\text{dom}(f)}} \{\varphi(x) - f(x)\} - \inf_X(\varphi),$$

for all $\varphi \in A(X)$. So

$$\sup_{x \in \overline{\text{dom}(f)}} \{\varphi(x) - g(\pi^{-1}(x))\} = \sup_{x \in \overline{\text{dom}(f)}} \{\varphi(x) - f(x)\},$$

for all $\varphi \in A$. Let us denote by $i_{\overline{\text{dom}(f)}}$ the lower semicontinuous indicator function which is equal to 0 on $\overline{\text{dom}(f)}$ and equal to $+\infty$ otherwise. The above formula can be written as follow

$$\sup_{x \in X} \left\{ \varphi(x) - \left(g(\pi^{-1}(x)) + i_{\overline{\text{dom}(f)}} \right) \right\} = \sup_{x \in X} \left\{ \varphi(x) - \left(f(x) + i_{\overline{\text{dom}(f)}} \right) \right\},$$

for all $\varphi \in A$. In other words, by using the notation of the conjugacy (see section 1.4)

$$\left(g \circ \pi^{-1} + i_{\overline{\text{dom}(f)}}\right)^\times(\varphi) = (f + i_{\overline{\text{dom}(f)}})^\times(\varphi)$$

for all $\varphi \in A$. By passing to the second conjugacy we obtain

$$(g \circ \pi^{-1} + i_{\overline{\text{dom}(f)}})^{\times\times}(x) = \left(f + i_{\overline{\text{dom}(f)}}\right)^{\times\times}(x)$$

for all $x \in X$. Since the functions $f + i_{\overline{\text{dom}(f)}}$ and $g \circ \pi^{-1} + i_{\overline{\text{dom}(f)}}$ are bounded from below and lower semicontinuous on X , then by Theoreme 4, each of these functions coincide with his second conjugacy. Thus we have

$$g \circ \pi^{-1} + i_{\overline{\text{dom}(f)}} = f + i_{\overline{\text{dom}(f)}}$$

which is equivalent to $g \circ \pi^{-1} = f$ on $\overline{\text{dom}(f)}$ as well as $g = f \circ \pi$ on $\overline{\text{dom}(g)}$.

Remark 6 *By imitating the above prof, we obtain the following version.*

Theorem 5 *Let X and Y be two complete metric spaces and $A \subset C_b(X)$ and $B \subset C_b(Y)$ be two Banach spaces satisfying the axioms A1-A4 (with the same property P^β . See Axioms 1 and the examples in Proposition 2). Let $T : A \rightarrow B$ be an isomorphism and let $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ and $g : Y \rightarrow \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous and bounded from below with non empty domains. Then (1) \Leftrightarrow (2).*

(1) $g^\times \circ T = f^\times$ (i.e for all $\varphi \in A$, $\sup_{y \in Y} \{T\varphi(y) - g(y)\} = \sup_{x \in X} \{\varphi(x) - f(x)\}$).

(2) *There exist an homeomorphism $\pi : \overline{\text{dom}(g)} \rightarrow \overline{\text{dom}(f)}$ such that, for all $y \in \overline{\text{dom}(g)}$ and all $\varphi \in A$ we have*

$$T\varphi(y) = \varphi \circ \pi(y)$$

and

$$g(y) = f \circ \pi(y).$$

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