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# THE SYMMETRIC INVARIANTS OF CENTRALIZERS AND SLODOWY GRADING

JEAN-YVES CHARBONNEL AND ANNE MOREAU

Abstract. Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra of rank $\ell$ over an algebraically closed field $\Bbbk$ of characteristic zero, and let $e$ be a nilpotent element of $\mathfrak{g}$. Denote by $\mathfrak{g}^e$ the centralizer of $e$ in $\mathfrak{g}$ and by $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ the algebra of symmetric invariants of $\mathfrak{g}^e$. We say that $e$ is *good* if the nullvariety of some $\ell$ homogeneous elements of $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ in $(\mathfrak{g}^e)^*$ has codimension $\ell$. If $e$ is good then $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is a polynomial algebra. The main result of this paper stipulates that if for some homogeneous generators of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$, the initial homogeneous components of their restrictions to $e+\mathfrak{g}^f$ are algebraically independent, with $(e,h,f)$ an $\mathfrak{sl}_2$-triple of $\mathfrak{g}$, then $e$ is good. As applications, we pursue the investigations of [PPY07] and we produce (new) examples of nilpotent elements that verify the above polynomiality condition, in simple Lie algebras of both classical and exceptional types. We also give a counter-example in type $\mathbf{D}_7$.

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## 1. Introduction

1.1. Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra of rank $\ell$ over an algebraically closed field $\Bbbk$ of characteristic zero, let $\langle .,. \rangle$ be the Killing form of $\mathfrak{g}$ and let $G$ be the adjoint group of $\mathfrak{g}$. If $\mathfrak{a}$ is a subalgebra of $\mathfrak{g}$, we denote by $\mathrm{S}(\mathfrak{a})$ the symmetric algebra of $\mathfrak{a}$. For $x \in \mathfrak{g}$, we denote by $\mathfrak{g}^x$ the centralizer of $x$ in $\mathfrak{g}$ and by $G^x$ the stabilizer of $x$ in $G$. Then $\mathrm{Lie}(G^x) = \mathrm{Lie}(G^x_0) = \mathfrak{g}^x$ where $G^x_0$ is the identity component of $G^x$. Moreover, $\mathrm{S}(\mathfrak{g}^x)$ is a $\mathfrak{g}^x$-module and $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x} = \mathrm{S}(\mathfrak{g}^x)^{G^x_0}$. An interesting question, first raised by A. Premet, is the following:

**Question 1.** *Is* $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$ *a polynomial algebra in* $\ell$ *variables?*

In order to answer this question, thanks to the Jordan decomposition, we can assume that $x$ is nilpotent. Besides, if $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$ is polynomial for some $x \in \mathfrak{g}$, then it is so for any element in the adjoint orbit $G.x$ of $x$. If $x = 0$, it is well-known since Chevalley that $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x} = \mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ is polynomial in $\ell$ variables. At the

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opposite extreme, if $x$ is a regular nilpotent element of $\mathfrak{g}$, then $\mathfrak{g}^x$ is abelian of dimension $\ell$, [DV69], and $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x} = \mathrm{S}(\mathfrak{g}^x)$ is polynomial in $\ell$ variables too.

For the introduction, let us say most simply that $x \in \mathfrak{g}$ *verifies the polynomiality condition* if $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$ is a polynomial algebra in $\ell$ variables.

A positive answer to Question 1 was suggested in [PPY07, Conjecture 0.1] for any simple $\mathfrak{g}$ and any $x \in \mathfrak{g}$. O. Yakimova has since discovered a counter-example in type $\mathbf{E}_8$, [Y07], disconfirming the conjecture. More precisely, the elements of the minimal nilpotent orbit in $\mathbf{E}_8$ do not verify the polynomiality condition. The present paper contains another counter-example in type $\mathbf{D}_7$ (cf. Example 7.8). In particular, we cannot expect a positive answer to [PPY07, Conjecture 0.1] for the simple Lie algebras of classical type. Question 1 still remains interesting and has a positive answer for a large number of nilpotent elements $e \in \mathfrak{g}$ as it is explained below.

1.2. We briefly review in this paragraph what has been achieved so far about Question 1. Recall that the *index* of a finite-dimensional Lie algebra $\mathfrak{q}$, denoted by $\operatorname{ind}\mathfrak{q}$, is the minimal dimension of the stabilizers of linear forms on $\mathfrak{q}$ for the coadjoint representation, (cf. [Di74]):

$$\operatorname{ind}\mathfrak{q} := \min\{\dim \mathfrak{q}^\xi \ ; \ \xi \in \mathfrak{q}^*\} \ \text{ where } \ \mathfrak{q}^\xi := \{x \in \mathfrak{q} \ ; \ \xi([x, \mathfrak{q}]) = 0\}.$$

By [R63], if $\mathfrak{q}$ is algebraic, i.e., $\mathfrak{q}$ is the Lie algebra of some algebraic linear group $Q$, then the index of $\mathfrak{q}$ is the transcendental degree of the field of $Q$-invariant rational functions on $\mathfrak{q}^*$. The following result will be important for our purpose.

**Theorem 1.1** ([CMo10, Theorem 1.2])**.** *The index of $\mathfrak{g}^x$ is equal to $\ell$ for any $x \in \mathfrak{g}$.*

Theorem 1.1 was first conjectured by Elashvili in the 90's motivated by a result of Bolsinov, [Bol91, Theorem 2.1]. It was proven by O. Yakimova when $\mathfrak{g}$ is a simple Lie algebra of classical type, [Y06], and checked by a computer programme by W. de Graaf when $\mathfrak{g}$ is a simple Lie algebra of exceptional type, [DeG08]. Before that, the result was established for some particular classes of nilpotent elements by D. Panyushev, [Pa03].

Theorem 1.1 is deeply related to Question 1. Indeed, thanks to Theorem 1.1, [PPY07, Theorem 0.3] applies and by [PPY07, Theorems 4.2 and 4.4], if $\mathfrak{g}$ is simple of type $\mathbf{A}$ or $\mathbf{C}$, then all nilpotent elements of $\mathfrak{g}$ verify the polynomiality condition. The result for the type $\mathbf{A}$ was independently obtained by Brown and Brundan, [BB09]. In [PPY07], the authors also provide some examples of nilpotent elements satisfying the polynomiality condition in the simple Lie algebras of types $\mathbf{B}$ and $\mathbf{D}$, and a few ones in the simple exceptional Lie algebras.

More recently, the analogue question to Question 1 for the positive characteristic was dealt with by L. Topley for the simple Lie algebras of types $\mathbf{A}$ and $\mathbf{C}$, [T12].

1.3. The main goal of this paper is to continue the investigations of [PPY07]. Let us describe the main results. The following definition is central in our work (cf. Definition 3.2):

**Definition 1.2.** An element $x \in \mathfrak{g}$ is called a *good element of* $\mathfrak{g}$ if for some graded sequence $(p_1, \dots, p_\ell)$ in $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$, the nullvariety of $p_1, \dots, p_\ell$ in $(\mathfrak{g}^x)^*$ has codimension $\ell$ in $(\mathfrak{g}^x)^*$.

For example, by [PPY07, Theorem 5.4], all nilpotent elements of a simple Lie algebra of type $\mathbf{A}$ are good, and by [Y09, Corollary 8.2], the *even*[1] nilpotent elements of $\mathfrak{g}$ are good if $\mathfrak{g}$ is of type $\mathbf{B}$ or $\mathbf{C}$ or if $\mathfrak{g}$ is of type $\mathbf{D}$ with odd rank. We rediscover these results in a more general setting (cf. Theorem 5.1 and Corollary 5.8). The good elements verify the polynomiality condition (cf. Theorem 3.3):

[1] i.e., this means that the Dynkin grading of $\mathfrak{g}$ associated with the nilpotent element has no odd term.

**Theorem 1.3.** *Let $x$ be a good element of $\mathfrak{g}$. Then $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$ is a polynomial algebra and $\mathrm{S}(\mathfrak{g}^x)$ is a free extension of $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$.*

Furthermore, $x$ is good if and only if so is its nilpotent component in the Jordan decomposition (cf. Proposition 3.5). As a consequence, we can restrict the study to the case of nilpotent elements.

Let $e$ be a nilpotent element of $\mathfrak{g}$. By the Jacobson-Morosov Theorem, $e$ is embedded into a $\mathfrak{sl}_2$-triple $(e, h, f)$ of $\mathfrak{g}$. Denote by $\mathcal{S}_e := e + \mathfrak{g}^f$ the *Slodowy slice associated with e*. Identify $\mathfrak{g}^*$ with $\mathfrak{g}$, and $(\mathfrak{g}^e)^*$ with $\mathfrak{g}^f$, through the Killing form $\langle .,. \rangle$. For $p$ in $\mathrm{S}(\mathfrak{g}) \simeq \Bbbk[\mathfrak{g}^*] \simeq \Bbbk[\mathfrak{g}]$, denote by ${}^e p$ the initial homogeneous component of its restriction to $\mathcal{S}_e$. According to [PPY07, Proposition 0.1], if $p$ is in $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$, then ${}^e p$ is in $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$. The main result of the paper is the following (cf. Theorem 3.6) whose proof is outlined in Subsection 1.4:

**Theorem 1.4.** *Suppose that for some homogeneous generators $q_1, \dots, q_\ell$ of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$, the polynomial functions ${}^e q_1, \dots, {}^e q_\ell$ are algebraically independent. Then e is a good element of $\mathfrak{g}$. In particular, $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is a polynomial algebra and $\mathrm{S}(\mathfrak{g}^e)$ is a free extension of $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$. Moreover, $({}^e q_1, \dots, {}^e q_\ell)$ is a regular sequence in $\mathrm{S}(\mathfrak{g}^e)$.*

Theorem 1.4 can be applied to a great number of nilpotent orbits in the simple classical Lie algebras (cf. Section 5), and for some nilpotent orbits in the exceptional Lie algebras (cf. Section 6).

To state our results for the simple Lie algebras of types **B** and **D**, let us introduce some more notations. Assume that $\mathfrak{g} = \mathfrak{so}(\mathbb{V}) \subset \mathfrak{gl}(\mathbb{V})$ for some vector space $\mathbb{V}$ of dimension $2\ell + 1$ or $2\ell$. For an endomorphism $x$ of $\mathbb{V}$ and for $i \in \{1, \dots, \dim \mathbb{V}\}$, denote by $Q_i(x)$ the coefficient of degree $\dim \mathbb{V} - i$ of the characteristic polynomial of $x$. Then for any $x$ in $\mathfrak{g}$, $Q_i(x) = 0$ whenever $i$ is odd. Define a generating family $q_1, \dots, q_\ell$ of the algebra $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ as follows. For $i = 1, \dots, \ell - 1$, set $q_i := Q_{2i}$. If $\dim \mathbb{V} = 2\ell + 1$, set $q_\ell := Q_{2\ell}$, and if $\dim \mathbb{V} = 2\ell$, let $q_\ell$ be a homogeneous element of degree $\ell$ of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ such that $Q_{2\ell} = q_\ell^2$. Denote by $\delta_1, \dots, \delta_\ell$ the degrees of ${}^e q_1, \dots, {}^e q_\ell$ respectively. By [PPY07, Theorem 2.1], if

$$\dim \mathfrak{g}^e + \ell - 2(\delta_1 + \cdots + \delta_\ell) = 0,$$

then the polynomials ${}^e q_1, \dots, {}^e q_\ell$ are algebraically independent. In that event, by Theorem 1.4, $e$ is good and we will say that $e$ is *very good* (cf. Corollary 5.8 and Definition 5.10). The very good nilpotent elements of $\mathfrak{g}$ can be characterized in term of their associated partitions of $\dim \mathbb{V}$ (cf. Lemma 5.11). Theorem 1.4 also allows to obtain examples of good, but not very good, nilpotent elements of $\mathfrak{g}$; for them, there are a few more work to do (cf. Subsection 5.3).

In this way, we obtain a large number of good nilpotent elements, including all even nilpotent elements in type **B**, or in type **D** with odd rank (cf. Corollary 5.8). For the type **D** with even rank, we obtain the statement for some particular cases (cf. Theorem 5.23). On the other hand, there are examples of elements that verify the polynomiality condition but that are not good; see Examples 7.5 and 7.6. To deal with them, we use different techniques, more similar to those used in [PPY07]. These alternative methods are presented in Section 7.

As a result of all this, we observe for example that all nilpotent elements of $\mathfrak{so}(\Bbbk^7)$ are good, and that all nilpotent elements of $\mathfrak{so}(\Bbbk^n)$, with $n \leqslant 13$, verify the polynomiality condition (cf. Table 5). In particular, by [PPY07, §3.9], this provides examples of good nilpotent elements for which the codimension of $(\mathfrak{g}^e)^*_{\mathrm{sing}}$ in $(\mathfrak{g}^e)^*$ is 1 (cf. Remark 7.7). Here, $(\mathfrak{g}^e)^*_{\mathrm{sing}}$ stands for the set of nonregular linear forms $x \in (\mathfrak{g}^e)^*$, i.e.,

$$(\mathfrak{g}^e)^*_{\mathrm{sing}} := \{x \in (\mathfrak{g}^e)^* \mid \dim (\mathfrak{g}^e)^x > \operatorname{ind} \mathfrak{g}^e = \ell\}.$$

For such nilpotent elements, note that [PPY07, Theorem 0.3] cannot be applied.

Our results do not cover all nilpotent orbits in type **B** and **D**. As a matter of fact, we obtain a counter-example in type $\mathbf{D}_7$ to Premet's conjecture (cf. Example 7.8).

**Proposition 1.5.** *The nilpotent elements of* $\mathfrak{so}(\Bbbk^{14})$ *associated with the partition* $(3,3,2,2,2,2)$ *of* 14 *do not satisfy the polynomiality condition.*

1.4. We outline in this paragraph the proof of Theorem 1.4.

Let $q_1,\dots,q_\ell$ be homogeneous generators of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ of degrees $d_1,\dots,d_\ell$ respectively. The sequence $(q_1,\dots,q_\ell)$ is ordered so that $d_1\leqslant\cdots\leqslant d_\ell$. Assume that the polynomial functions ${}^eq_1,\dots,{}^eq_\ell$ are algebraically independent.

According to Theorem 1.3, it suffices to show that $e$ is good, and more accurately that the nullvariety of ${}^eq_1,\dots,{}^eq_\ell$ in $\mathfrak{g}^f$ has codimension $\ell$, since ${}^eq_1,\dots,{}^eq_\ell$ are invariant homogeneous polynomials. To this end, it suffices to prove that $\mathrm{S}(\mathfrak{g}^e)$ is a free extension of the $\Bbbk$-algebra generated by ${}^eq_1,\dots,{}^eq_\ell$ (see Proposition 2.5,(ii)). We are lead to find a subspace $V_0$ of $S$ such that the linear map

$$V_0\otimes_{\Bbbk}\Bbbk[{}^eq_1,\dots,{}^eq_\ell]\longrightarrow S,\qquad v\otimes a\longmapsto va$$

is a linear isomorphism. We explain below the construction of the subspace $V_0$.

Let $x_1,\dots,x_r$ be a basis of $\mathfrak{g}^e$ such that for $i=1,\dots,r$, $[h,x_i]=n_ix_i$ for some nonnegative integer $n_i$. For $\mathbf{j}=(j_1,\dots,j_r)$ in $\mathbb{N}^r$, set:

$$|\mathbf{j}|:=j_1+\cdots+j_r,\qquad |\mathbf{j}|_e:=j_1n_1+\cdots+j_rn_r+2|\mathbf{j}|,\qquad x^{\mathbf{j}}=x_1^{j_1}\cdots x_r^{j_r}.$$

The algebra $\mathrm{S}(\mathfrak{g}^e)$ has two gradations: the standard one and the *Slodowy gradation*. For all $\mathbf{j}$ in $\mathbb{N}^r$, $x^{\mathbf{j}}$ is homogeneous with respect to these two gradations. It has standard degree $|\mathbf{j}|$ and, by definition, it has Slodowy degree $|\mathbf{j}|_e$. For $m$ nonnegative integer, denote by $\mathrm{S}(\mathfrak{g}^e)^{[m]}$ the subspace of $\mathrm{S}(\mathfrak{g}^e)$ of Slodowy degree $m$.

Let us simply denote by $S$ the algebra $\mathrm{S}(\mathfrak{g}^e)$ and let $t$ be an indeterminate. For any subspace $V$ of $S$, set:

$$V[t]:=\Bbbk[t]\otimes_{\Bbbk}V,\qquad V[t,t^{-1}]:=\Bbbk[t,t^{-1}]\otimes_{\Bbbk}V,\qquad V[[t]]:=\Bbbk[[t]]\otimes_{\Bbbk}V,\qquad V((t)):=\Bbbk((t))\otimes_{\Bbbk}V,$$

with $\Bbbk((t))$ the fraction field of $\Bbbk[[t]]$. For $V$ a subspace of $S[[t]]$, denote by $V(0)$ the image of $V$ by the quotient morphism

$$S[t]\longrightarrow S,\qquad a(t)\longmapsto a(0).$$

The Slodowy gradation of $S$ induces a gradation of the algebra $S((t))$ with $t$ having degree 0. Let $\tau$ be the morphism of algebras

$$S\longrightarrow S[t],\qquad x_i\mapsto tx_i,\quad i=1,\dots,r.$$

The morphism $\tau$ is a morphism of graded algebras. Denote by $\delta_1,\dots,\delta_\ell$ the standard degrees of ${}^eq_1,\dots,{}^eq_\ell$ respectively, and set for $i=1,\dots,\ell$:

$$Q_i:=t^{-\delta_i}\tau(\kappa(q_i))\quad\text{ with }\quad\kappa(q_i)(x):=q_i(e+x),\quad\forall\,x\in\mathfrak{g}^f.$$

Let $A$ be the subalgebra of $S[t]$ generated by $Q_1,\dots,Q_\ell$. Then $A(0)$ is the subalgebra of $S$ generated by ${}^eq_1,\dots,{}^eq_\ell$. For $(j_1,\dots,j_\ell)$ in $\mathbb{N}^\ell$, $\kappa(q_1^{j_1})\cdots\kappa(q_\ell^{j_\ell})$ and ${}^eq_1^{j_1}\cdots{}^eq_\ell^{j_\ell}$ are Slodowy homogeneous of Slodowy degree $2d_1j_1+\cdots+2d_\ell j_\ell$ (cf. Proposition 4.1,(i)). Hence, $A$ and $A(0)$ are graded subalgebras of $S[t]$ and $S$ respectively. Denote by $A(0)_+$ the augmentation ideal of $A(0)$, and let $V_0$ be a graded complement to $SA(0)_+$ in $S$.

We first obtain that $S[[t]]$ is a free extension of $A$ (cf. Corollary 4.17), and that $S[[t]]$ is a free extension of the subalgebra $\tilde{A}$ of $S[[t]]$ generated by $\Bbbk[[t]]$ and $A$ (cf. Theorem 4.21,(i)). From these results, we deduce that the linear map

$$V_0\otimes_{\Bbbk}A(0)\longrightarrow S,\qquad v\otimes a\longmapsto va$$

is a linear isomorphism, as expected; see Theorem 4.21,(iii).

1.5. The remainder of the paper will be organized as follows.

Section 2 is about general facts on commutative algebra, useful for the Sections 3 and 4. In Section 3, the notions of good elements and good orbits are introduced, and some properties of good elements are described. Theorem 3.3 asserts that the good elements verify the polynomiality condition. The main result (Theorem 3.6) is also stated in this section. Section 4 is devoted to the proof of Theorem 3.6. In Section 5, we give applications of Theorem 3.6 to the simple classical Lie algebras. In Section 6, we give applications to the exceptional Lie algebras of types $\mathbf{E}_6$, $\mathbf{F}_4$ and $\mathbf{G}_2$. This allows us to exhibit a great number of good nilpotent orbits. Other examples, counter-examples, remarks and a conjecture are discussed in Section 7. In this last section, other techniques are developed.

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## 2. General facts on commutative algebra

We state in this section preliminary results on commutative algebra. Theorem 2.20 will be particularly important in Sections 3 for the proof of Theorem 3.3. As for Proposition 2.5, it will be used in the proof of Theorem 3.6.

2.1. As a rule, for $A$ a graded algebra over $\mathbb{N}$, we denote by $A_+$ the ideal of $A$ generated by its homogeneous elements of positive degree. For $M$ a graded $A$-module, we set $M_+ := A_+M$.

Let $S$ be a finitely generated regular graded $\Bbbk$-algebra over $\mathbb{N}$. If $E$ is a finite dimensional vector space over $\Bbbk$, we denote by $\mathrm{S}(E)$ the polynomial algebra generated by $E$. It is a finitely generated regular $\Bbbk$-algebra, graded over $\mathbb{N}$ by the standard gradation. Let $A$ be a graded subalgebra of $S$, different from $S$ and such that $A = \Bbbk + A_+$. Let $X_A$ and $X_S$ be the affine varieties $\mathrm{Specm}(A)$ and $\mathrm{Specm}(S)$ respectively, and let $\pi_{A,S}$ be the morphism from $X_S$ to $X_A$ whose comorphism is the canonical injection from $A$ into $S$. Let $\mathcal{N}_0$ be the nullvariety of $A_+$ in $X_S$ and set

$$N := \dim S - \dim A.$$

**Lemma 2.1.** (i) *The irreducible components of the fibers of $\pi_{A,S}$ have dimension at least $N$.*

(ii) *If $\mathcal{N}_0$ has dimension $N$, then the fibers of $\pi_{A,S}$ are equidimensional of dimension $N$.*

(iii) *Suppose that $S = \mathrm{S}(E)$ for some finite dimensional $\Bbbk$-vector space $E$. If $\mathcal{N}_0$ has dimension $N$, then for some $x_1, \dots, x_N$ in $E$, the nullvariety of $x_1, \dots, x_N$ in $\mathcal{N}_0$ is equal to $\{0\}$.*

*Proof.* (i) Let $F$ be a fiber of $\pi_{A,S}$ and let $U$ be an open subset of $X_S$ whose intersection with $F$ is not empty and irreducible. The restriction of $\pi_{A,S}$ to $U$ is a dominant morphism from $U$ to $X_A$. So, $N$ is the minimal dimension of the fibers of the restriction of $\pi_{A,S}$ to $U$, whence the assertion.

(ii) Denote by $x_0$ the element $A_+$ of $X_A$. Since $A$ is a graded algebra, there exists a regular action of the one dimensional multiplicative group $\mathrm{G_m}$ on $X_A$. Furthermore, for all $x$ in $X_A$, $x_0$ is in the closure of $\mathrm{G_m}.x$. Hence the dimension of the fiber of $\pi_{A,S}$ at $x$ is at most $\dim \mathcal{N}_0$. As a result, when $\dim \mathcal{N}_0$ is the minimal dimension of the fibers of $\pi_{A,S}$, all the fibers of $\pi_{A,S}$ are equidimensional of dimension $N$ by (i).

(iii) For $x = (x_i)_{i\in I}$ a family of elements of $E$, denote by $A[x]$ the subalgebra of $\mathrm{S}(E)$ generated by $A$ and $x$, and denote by $\mathcal{N}_0(x)$ its nullvariety in $\mathcal{N}_0$. Since $\mathcal{N}_0$ is a cone, $\mathcal{N}_0(x)$ is equal to $\{0\}$ if it has dimension 0. So it suffices to find $N$ elements $x_1, \dots, x_N$ of $E$ such that $\mathcal{N}_0(x_1, \dots, x_N)$ has dimension 0. Let us prove by induction on $i$ that for $i = 1, \dots, N$, there exist $i$ elements $x_1, \dots, x_i$ of $E$ such that $\mathcal{N}_0(x_1, \dots, x_i)$ has dimension $N - i$. By induction on $i$, with $A[x_1, \dots, x_i]$ instead of $A$, it suffices to find $x$ in $E$ such that $\mathcal{N}_0(x)$ has dimension $N - 1$.

Let $Z_1, \dots, Z_m$ be the irreducible components of $\mathcal{N}_0$ and let $I_i$ be the ideal of definition of $Z_i$ in $\mathrm{S}(E)$. Since $\mathcal{N}_0$ is equidimensional of dimension $N$ by hypothesis, for $i = 1, \dots, m$, $Z_i$ has dimension $N$. In particular, $I_i$ does not contain $E$ since $A \neq \mathrm{S}(E)$. So, there exists $x$ in $E$, not in the union of $I_1, \dots, I_m$. Then, for $i = 1, \dots, m$, the nullvariety of $x$ in $Z_i$ is equidimensional of dimension $N-1$. As a result, the nullvariety of the ideal of $\mathrm{S}(E)$ generated by $A_+$ and $x$ is equidimensional of dimension $N-1$, whence the assertion. □

Let $\overline{A}$ be the algebraic closure of $A$ in $S$.

**Lemma 2.2.** *Let $M$ be a graded $A$-module and let $V$ be a graded subspace of $M$ such that $M = V \oplus M_+$. Denote by $\tau$ the canonical map $A \otimes_{\Bbbk} V \longrightarrow M$. Then $\tau$ is surjective. Moreover, $\tau$ is bijective if and only if $M$ is a flat $A$-module.*

*Proof.* Let $M'$ be the image of $\tau$. Since $M = V \oplus M_+ = V + A_+M \subset M' + A_+M$, we get by induction on $k$,

$$M \subset M' + A_+^k M.$$

Since $M$ is graded and since $A_+$ is generated by elements of positive degree, $M = M'$.

If $\tau$ is bijective, then all basis of $V$ is a basis of the $A$-module $M$. In particular, it is a flat $A$-module. Conversely, let us suppose that $M$ is a flat $A$-module. For $v$ in $M$, denote by $\overline{v}$ the element of $V$ such that $v - \overline{v}$ is in $A_+M$.

*Claim* 2.3. Let $(v_1, \dots, v_n)$ be a graded sequence in $M$ such that $\overline{v_1}, \dots, \overline{v_n}$ is linearly free over $\Bbbk$. Then $v_1, \dots, v_n$ is linearly free over $A$.

*Proof of Claim 2.3.* Since the sequence $(v_1, \dots, v_n)$ is graded, it suffices to prove that for a graded sequence $(a_1, \dots, a_n)$ in $A$,

$$a_1v_1 + \cdots + a_nv_n = 0 \Longrightarrow a_1 = \cdots = a_n = 0.$$

Prove the statement by induction on $n$. First of all, by flatness, for some graded sequence $(y_1, \dots, y_k)$ in $M$ and for some graded sequence $(b_{i,j}, i = 1, \dots, n, j = 1, \dots, k)$,

$$v_i = \sum_{j=1}^{k} b_{i,j} y_j \quad \text{and} \quad \sum_{l=1}^{n} a_l b_{l,m} = 0$$

for $i = 1, \dots, n$ and $m = 1, \dots, k$. For $n = 1$, since $\overline{v_1} \neq 0$, for some $j$, $b_{1,j}$ is in $\Bbbk^*$ since $A = \Bbbk + A_+$. So $a_1 = 0$. Suppose the statement true for $n-1$. Since $\overline{v_n} \neq 0$, for some $j$, $b_{n,j}$ is in $\Bbbk^*$, whence

$$a_n = -\sum_{i=1}^{n-1} \frac{b_{i,j}}{b_{n,j}} a_i \quad \text{and} \quad \sum_{i=1}^{n-1} a_i (v_i - \frac{b_{i,j}}{b_{n,j}} v_n) = 0.$$

Since $\overline{v_1}, \dots, \overline{v_n}$ are linearly free over $\Bbbk$, so is the sequence

$$\left( \overline{v_i - (b_{i,j}/b_{n,j}) v_n}, \quad i = 1, \dots, n-1 \right).$$

By induction hypothesis, $a_1 = \cdots = a_{n-1} = 0$, whence $a_n = 0$. □

According to Claim 2.3, any graded basis of $V$ is linearly free over $A$. Hence any graded basis of $V$ is a basis of the $A$-module $M$ since $M = AV$. □

**Corollary 2.4.** *Suppose that $S = \mathrm{S}(E)$ for some finite dimensional $\Bbbk$-vector space $E$, and suppose that $\dim \mathcal{N}_0 = N$. Then $\overline{A}$ is the integral closure of $A$ in $\mathrm{S}(E)$. In particular, $\overline{A}$ is finitely generated.*

*Proof.* Since $A$ is finitely generated, so is its integral closure in $\mathrm{S}(E)$ by [Ma86, §33, Lemma 1]. According to the hypothesis on $\mathcal{N}_0$ and Lemma 2.1,(iii), for some $x_1, \dots, x_N$ in $E$, the nullvariety of $x_1, \dots, x_N$ in $\mathcal{N}_0$ is equal to $\{0\}$. In particular, $x_1, \dots, x_N$ are algebraically independent over $A$ since $E$ has dimension $N + \dim A$. Let $J$ be the ideal of $\mathrm{S}(E)$ generated by $A_+$ and $x_1, \dots, x_N$. Then the radical of $J$ is the augmentation ideal of $\mathrm{S}(E)$ so that $J$ has finite codimension in $\mathrm{S}(E)$. For $V$ a graded complement to $J$ in $\mathrm{S}(E)$, $\mathrm{S}(E)$ is the $A[x_1, \dots, x_N]$-submodule generated by $V$ by Lemma 2.2. Hence $\mathrm{S}(E)$ is a finite extension of $A[x_1, \dots, x_N]$.

Let $p$ be in $\overline{A}$. Since $A[x_1, \dots, x_N]$ is finitely generated, $A[x_1, \dots, x_N][p]$ is a finite extension of $A[x_1, \dots, x_N]$. Let

$$p^m + a_{m-1}p^{m-1} + \cdots + a_0 = 0$$

an integral dependence equation of $p$ over $A[x_1, \dots, x_N]$. For $i = 0, \dots, m$, $a_i$ is a polynomial in $x_1, \dots, x_N$ with coefficients in $A$ since $x_1, \dots, x_N$ are algebraically independent over $A$. Denote by $a_i(0)$ its constant coefficient. Since $p$ is in $\overline{A}$, $x_1, \dots, x_N$ are algebraically independent over $A[p]$, whence

$$p^m + a_{m-1}(0)p^{m-1} + \cdots + a_0(0) = 0.$$

As a result, $\overline{A}$ is the integral closure of $A$ in $\mathrm{S}(E)$. □

**Proposition 2.5.** *Let us consider the following conditions on $A$:*

1) *$A$ is a polynomial algebra,*
2) *$A$ is a regular algebra,*
3) *$A$ is a polynomial algebra generated by* $\dim A$ *homogeneous elements,*
4) *the $A$-module $S$ is faithfully flat,*
5) *the $A$-module $S$ is flat,*
6) *the $A$-module $S$ is free.*

(i) *The conditions* (1), (2), (3) *are equivalent.*

(ii) *The conditions* (4), (5), (6) *are equivalent. Moreover, Condition* (4) *implies Condition* (2) *and, in that event, $\mathcal{N}_0$ is equidimensional of dimension $N$.*

(iii) *If $\mathcal{N}_0$ is equidimensional of dimension $N$, then the conditions* (1), (2), (3), (4), (5), (6) *are all equivalent.*

*Proof.* Let $n$ be the dimension of $A$.

(i) The implications (3) ⇒ (1), (1) ⇒ (2) are straightforward. Let us suppose that $A$ is a regular algebra. Since $A$ is graded and finitely generated, there exists a graded sequence $(x_1, \dots, x_n)$ in $A_+$ representing a basis of $A_+/A_+^2$. Let $A'$ be the subalgebra of $A$ generated by $x_1, \dots, x_n$. Then

$$A_+ \subset A' + A_+^2.$$

So by induction on $m$,

$$A_+ \subset A' + A_+^m$$

for all positive integer $m$. Then $A = A'$ since $A$ is graded and $A_+$ is generated by elements of positive degree. Moreover, $x_1, \dots, x_n$ are algebraically independent over $\Bbbk$ since $A$ has dimension $n$. Hence $A$ is a polynomial algebra generated by $n$ homogeneous elements.

(ii) The implications (4) ⇒ (5), (6) ⇒ (5) are straightforward and (5) ⇒ (6) is a consequence of Lemma 2.2.

(5) ⇒ (4): Recall that $x_0 = A_+$. Let us suppose that $S$ is a flat $A$-module. Then $\pi_{A,S}$ is an open morphism whose image contains $x_0$. Moreover, $\pi(X_S)$ is stable under the action of $\mathrm{G_m}$. So $\pi_{A,S}$ is surjective. Hence, by [Ma86, Ch. 3, Theorem 7.2], $S$ is a faithfully flat extension of $A$.

(4)$\Rightarrow$(2): Since $S$ is regular and since $S$ is a faithfully flat extension of $A$, all finitely generated $A$-module has finite projective dimension. So by [Ma86, Ch. 7, §19, Lemma 2], the global dimension of $A$ is finite. Hence by [Ma86, Ch. 7, Theorem 19.2], $A$ is regular.

If Condition (4) holds, by [Ma86, Ch. 5, Theorem 15.1], the fibers of $\pi_{A,S}$ are equidimensional of dimension $N$. So $\mathcal{N}_0$ is equidimensional of dimension $N$.

(iii) Suppose that $\mathcal{N}_0$ is equidimensional of dimension $N$. By (i) and (ii), it suffices to prove that (2)$\Rightarrow$(5). By Lemma 2.1,(ii), the fibers of $\pi_{A,S}$ are equidimensional of dimension $N$. Hence by [Ma86, Ch. 8, Theorem 23.1], $S$ is a flat extension of $A$ since $S$ and $A$ are regular. □

2.2. We present in this paragraph some results about algebraic extensions, that are independent of Subsection 2.1. These results are used only in the proof of Proposition 2.15. Our main reference is [Ma86]. For $A$ an algebra and $\mathfrak{p}$ a prime ideal of $A$, $A_{\mathfrak{p}}$ denotes the localization of $A$ at $\mathfrak{p}$.

Let $t$ be an indeterminate, and let $L$ be a field containing $\Bbbk$. Let $B$, $L_1$, $B_1$ verifying the following conditions:

(I) $L_1$ is an algebraic extension of $L(t)$ of finite degree,
(II) $L$ is algebraically closed in $L_1$,
(III) $B$ is a finitely generated subalgebra of $L$, $L$ is the fraction field of $B$ and $B$ is integrally closed in $L$,
(IV) $B_1$ is the integral closure of $B[t]$ in $L_1$,
(V) $tB_1$ is a prime ideal of $B_1$.

For $C$ a subalgebra of $L$, containing $B$, we set:

$$R(C) := C \otimes_B B_1,$$

and we denote by $\mu_C$ the canonical morphism $R(C) \to CB_1$. Since $C$ and $B_1$ are integral algebras, the morphisms $c \mapsto c\otimes 1$ and $b \mapsto 1\otimes b$ from $C$ and $B_1$ to $R(C)$ respectively are embeddings. So, $C$ and $B_1$ are identified to subalgebras of $R(C)$ by these embeddings. We now investigate some properties of the algebras $R(C)$.

**Lemma 2.6.** *Let $\mu_L$ be the canonical morphism $R(L) \to LB_1$.*
(i) *The algebra $R(L)$ is reduced and $\mu_L$ is an isomorphism.*
(ii) *The ideal $tLB_1$ of $LB_1$ is maximal. Furthermore $B_1$ is a finite extension of $B[t]$.*
(iii) *The algebra $LB_1$ is the direct sum of $L$ and $tLB_1$.*
(iv) *The ring $LB_1$ is integrally closed in $L_1$.*

*Proof.* (i) Let $a$ be in the kernel of $\mu_L$. Since $L$ is the fraction field of $B$, for some $b$ in $B$, $ba = 1\otimes\mu_L(ba)$ so that $ba = 0$ and $a = 0$. As a result, $\mu_L$ is an isomorphism and $R(L)$ is reduced since $LB_1$ is integral.

(ii) Since $t$ is not algebraic over $L$ and since $LB_1$ is integral over $L[t]$ by Condition (IV), $tLB_1$ is strictly contained in $LB_1$. Let $a$ and $b$ be in $LB_1$ such that $ab$ is in $tLB_1$. By Condition (III), for some $c$ in $B \setminus \{0\}$, $ca$ and $cb$ are in $B_1$. So, by Condition (V), $ca$ or $cb$ is in $tB_1$. Hence $a$ or $b$ is in $tLB_1$. As a result, $tLB_1$ is a prime ideal and the quotient $Q$ of $LB_1$ by $tLB_1$ is an integral domain. Denote by $\iota$ the quotient morphism. Since $L$ is a field, the restriction of $\iota$ to $L$ is an embedding of $L$ into $Q$. According to Conditions (I) and (IV) and [Ma86, §33, Lemma 1], $B_1$ is a finite extension of $B[t]$. Then $Q$ is a finite extension of $L$ and $tLB_1$ is a maximal ideal of $LB_1$.

(iii) Since $L$ is algebraically closed in $L_1$, $Q$ and $L_1$ are linearly disjoint over $L$. So, $Q\otimes_L L_1$ is isomorphic to the extension of $L_1$ generated by $Q$. Denoting this extension by $QL_1$, $QB_1$ is a subalgbera of $QL_1$ and we

have the exact sequences

$$0 \longrightarrow tLB_1 \longrightarrow LB_1 \longrightarrow Q \longrightarrow 0$$

$$0 \longrightarrow tQB_1 \longrightarrow QB_1 \longrightarrow Q \otimes_L Q \longrightarrow 0$$

$$0 \longrightarrow tQB_1 \longrightarrow tQB_1 + LB_1 \longrightarrow Q \otimes 1 \longrightarrow 0$$

As a result,

$$Q \subset LB_1 + tQB_1.$$

By (ii), $QB_1$ is a finite $L[t]$-module. So, by Nakayama's Lemma, for some $a$ in $L[t]$, $(1+ta)QB_1$ is contained in $LB_1$. As a result, $Q$ is contained in $L_1$, whence $Q = L$ since $L$ is algebraically closed in $L_1$. The assertion follows since $Q$ is the quotient of $LB_1$ by $tLB_1$.

(iv) Let $a$ be in the integral closure of $LB_1$ in $L_1$ and let

$$a^m + a_{m-1}a^{m-1} + \cdots + a_0 = 0$$

an integral dependence equation of $a$ over $LB_1$. For some $b$ in $L \setminus \{0\}$, $ba_i$ is in $B_1$ for $i = 0, \ldots, m-1$. Then, by Condition (IV), $ba$ is in $B_1$ since it verifies an integral dependence equation over $B_1$. As a result, $LB_1$ is integrally closed in $L_1$. □

Let $L_2$ be the Galois extension of $L(t)$ generated by $L_1$, and let $\Gamma$ be the Galois group of the extension $L_2$ of $L(t)$. Denote by $B_2$ the integral closure of $B[t]$ in $L_2$. For $C$ subalgebra of $L$, containing $B$, set

$$R_2(C) := C \otimes_B B_2,$$

and denote by $\mu_{C,2}$ the canonical morphism $R_2(C) \to CB_2$. The action of $\Gamma$ in $B_2$ induces an action of $\Gamma$ in $R_2(C)$ given by $g.(c \otimes b) = c \otimes g(b)$.

**Lemma 2.7.** *Let $x$ be a primitive element of $L_1$, and let $\Gamma_x$ be the stabilizer of $x$ in $\Gamma$.*

(i) *The subfield $L_1$ of $L_2$ is the set of fixed points under the action of $\Gamma_x$ in $L_2$, and $B_1$ is the set of fixed points under the action of $\Gamma_x$ in $B_2$.*

(ii) *For $C$ subalgebra of $L$ containing $B$, the canonical morphism $R(C) \to R_2(C)$ is an embedding and its image is the set of fixed points under the action of $\Gamma_x$ in $R_2(C)$.*

(iii) *For $C$ subalgebra of $L$, containing $B$, $C[t]$ is embedded in $R(C)$ and $R_2(C)$. Moreover, $C[t]$ is the set of fixed points under the action of $\Gamma$ in $R_2(C)$.*

*Proof.* (i) Let $L_1'$ be the set of fixed points under the action of $\Gamma_x$ in $L_2$,

$$L_1' = \{y \in L_2 \mid \Gamma_x.y = y\}.$$

Then $L_1$ is contained in $L_1'$, and $L_2$ is an extension of degree $|\Gamma_x|$ of $L_1'$. Since $x$ is a primitive element of $L_1$, the degree of this extension is equal to $|\Gamma.x|$ so that $L_2$ is an extension of degree $|\Gamma_x|$. Hence $L_1' = L_1$.

Since $B_2$ is the integral closure of $B[t]$ in $L_2$, $B_2$ is invariant under $\Gamma$. Moreover, the intersection of $B_2$ and $L_1$ is equal to $B_1$ by Condition (IV). Hence $B_1$ is the set of fixed points under the action of $\Gamma_x$ in $B_2$.

(ii) For $a$ in $B_2$ and $b$ in $R_2(C)$, set:

$$a^{\#} := \frac{1}{|\Gamma_x|} \sum_{g \in \Gamma_x} g(a), \quad \overline{b} := \frac{1}{|\Gamma_x|} \sum_{g \in \Gamma_x} g.b.$$

Then $a \mapsto a^{\#}$ is a projection of $B_2$ onto $B_1$. Moreover, it is a morphism of $B_1$-module. Denote by $\iota$ the canonical morphism $R(C) \to R_2(C)$, and by $\varphi$ the morphism

$$R_2(C) \longrightarrow R(C), \quad c{\otimes}a \longmapsto c{\otimes}a^{\#}.$$

For $b$ in $R_2(C)$,

$$\varphi(b) = \varphi(\overline{b}) \quad \text{and} \quad \iota{\circ}\varphi(b) = \overline{b}$$

Then $\varphi$ is a surjective morphism and the image of $\iota{\circ}\varphi$ is the set of fixed points under the action of $\Gamma_x$ in $R_2(C)$. Moreover $\iota$ is injective, whence the assertion.

(iii) From the equalities

$$R(C) = (C \otimes_B B[t]) \otimes_{B[t]} B_1 \quad \text{and} \quad C[t] = C \otimes_B B[t]$$

we deduce that $R(C) = C[t] \otimes_{B[t]} B_1$. In the same way, $R_2(C) = C[t] \otimes_{B[t]} B_2$. Then, since $C[t]$ is an integral algebra, the morphism $c \mapsto c{\otimes}1$ is an embedding of $C[t]$ in $R(C)$ and $R_2(C)$. Moreover, $C[t]$ is invariant under the action of $\Gamma$ in $R_2(C)$.

Let $a$ be in $R_2(C)$ invariant under $\Gamma$. Then $a$ has an expansion

$$a = \sum_{i=1}^{k} c_i{\otimes}b_i$$

with $c_1, \dots, c_k$ in $C[t]$ and $b_1, \dots, b_k$ in $B_2$. Since $a$ is invariant under $\Gamma$,

$$a = \frac{1}{|\Gamma|} \sum_{g\in\Gamma} g.a = \frac{1}{|\Gamma|} \sum_{g\in\Gamma} \sum_{i=1}^{k} c_i{\otimes}g.b_i.$$

For $i = 1, \dots, k$, set:

$$b'_i := \frac{1}{|\Gamma|} \sum_{g\in\Gamma} g.b_i$$

The elements $b'_1, \dots, b'_k$ are in $B[t]$, and

$$a = (\sum_{i=1}^{k} c_i b'_i){\otimes}1 \in C[t],$$

whence the assertion. □

From now on, we fix a finitely generated subalgebra $C$ of $L$ containing $B$. Denote by $\mathfrak{n}$ the nilradical of $R(C)$.

**Lemma 2.8.** *Let $\mathfrak{k}$ be the kernel of $\mu_{C,2}$ and let $\mathfrak{n}_2$ be the nilradical of $R_2(C)$.*

(i) *The algebras $R(C)$ and $R_2(C)$ are finitely generated. Furthermore, they are finite extensions of $C[t]$.*

(ii) *For $a$ in $\mathfrak{k}$, $ba = 0$ for some $b$ in $B \setminus \{0\}$.*

(iii) *The ideal $\mathfrak{k}$ is the minimal prime ideal of $R_2(C)$ such that $\mathfrak{k} \cap B = \{0\}$. Moreover, $\mathfrak{k} \cap B[t] = \{0\}$.*

(iv) *The ideal $\mathfrak{n}$ is the kernel of $\mu_C$. Moreover, $\mathfrak{n}_2 = \mathfrak{k}$ and $\mathfrak{n}$ is a prime ideal.*

(v) *The local algebra $R(C)_{\mathfrak{n}}$ is isomorphic to $L_1$.*

*Proof.* (i) According to Lemma 2.7,(iii), $R(C)$ is an extension of $C[t]$ and $R(C) = C[t] \otimes_{B[t]} B_1$. Then, by Lemma 2.6,(ii), $R(C)$ is a finite extension of $C[t]$. In particular, $R(C)$ is a finitely generated algebra since so is $C$. In the same way, $R_2(C)$ is a finite extension of $C[t]$ and it is finitely generated.

(ii) Let $a$ be in $\mathfrak{k}$. Then $a$ has an expansion

$$a = \sum_{i=1}^{k} c_i \otimes b_i$$

with $c_1, \dots, c_k$ in $C$ and $b_1, \dots, b_k$ in $B_2$. Since $C$ and $B$ have the same fraction field, for some $b$ in $B \setminus \{0\}$, $bc_i$ is in $B$, whence

$$ba = 1 \otimes (\sum_{i=1}^{k} bc_i b_i).$$

So $ba = 0$ since $\mathfrak{k}$ is the kernel of $\mu_{C,2}$.

(iii) By (i) there are finitely many minimal prime ideals of $R_2(C)$. Denote them by $\mathfrak{p}_1, \dots, \mathfrak{p}_k$. Since $C[t]$ is an integral algebra, $\mathfrak{n}_2 \cap C[t] = \{0\}$ so that $\mathfrak{p}_i \cap C = \{0\}$ for some $i$. Let $i$ be such that $\mathfrak{p}_i \cap B = \{0\}$ and let $a$ be in $\mathfrak{k}$. By (ii), for some $b$ in $B \setminus \{0\}$, $ba$ is in $\mathfrak{p}_i$. Hence $\mathfrak{k}$ is contained in $\mathfrak{p}_i$. Since $CB_2$ is an integral algebra, $\mathfrak{k}$ is a prime ideal. Then $\mathfrak{p}_i = \mathfrak{k}$ since $\mathfrak{p}_i$ is a minimal prime ideal, whence the assertion since for some $j$, $\mathfrak{p}_j \cap C[t] = \{0\}$.

(iv) By (iii), there is only one minimal prime ideal of $R_2(C)$ whose intersection with $B$ is equal to $\{0\}$. So, it is invariant under $\Gamma$. Hence $\mathfrak{k}$ is invariant under $\Gamma$. As a result, for $a$ in $\mathfrak{k}$,

$$0 = \prod_{g \in \Gamma} (a - g.a) = a^m + a_{m-1} a^{m-1} + \cdots + a_0$$

with $m = |\Gamma|$ and $a_0, \dots, a_{m-1}$ in $\mathfrak{k}$. Moreover, by Lemma 2.7,(iii), $a_0, \dots, a_{m-1}$ are in $C[t]$. So, by (iii), they are all equal to zero so that $a$ is a nilpotent element. Hence $\mathfrak{k}$ is contained in $\mathfrak{n}_2$. Then $\mathfrak{n}_2 = \mathfrak{k}$ by (iii).

By Lemma 2.7,(ii), $R(C)$ identifies with a subalgebra of $R_2(C)$ so that $\mathfrak{n} = \mathfrak{n}_2 \cap R(C)$, and $\mu_C$ is the restriction of $\mu_{C,2}$ to $R(C)$. Hence $\mathfrak{n}$ is the kernel of $\mu_C$ and $\mathfrak{n}$ is a prime ideal of $R(C)$.

(v) By (iii), $\mathfrak{n} \cap C = \{0\}$. So, by (ii), $\mathfrak{n}R(C)_{\mathfrak{n}} = \{0\}$. As a result, $R(C)_{\mathfrak{n}}$ is a field since $\mathfrak{n}R(C)_{\mathfrak{n}}$ is a maximal ideal of $R(C)_{\mathfrak{n}}$. Moreover, by (iii), it is isomorphic to a subfield of $L_1$, containing $B_1$. So, $R(C)_{\mathfrak{n}}$ is isomorphic to $L_1$. □

For $c$ in $L[t]$, denote by $c(0)$ the constant term of $c$ as a polynomial in $t$ with coefficients in $L$.

**Lemma 2.9.** *Assume that $C$ is integrally closed in $L$. Denote by $\overline{CB_1}$ the integral closure of $CB_1$ in $L_1$.*

(i) *Let $i \in \{1, 2\}$. For all positive integer $j$, the intersection of $C[t]$ and $t^j LB_i$ equals $t^j C[t]$.*

(ii) *The intersection of $tLB_1$ and $\overline{CB_1}$ equals $t\overline{CB_1}$.*

(iii) *The algebra $\overline{CB_1}$ is contained in $C + t\overline{CB_1}$.*

(iv) *The algebra $B_1$ is the direct sum of $B$ and $tB_1$.*

*Proof.* First of all, $CB_1$ and $\overline{CB_1}$ are finite extensions of $C[t]$ by Lemma 2.7,(i), and [Ma86, §33, Lemma 1]. So $\overline{CB_1}$ is the integral closure of $C[t]$ in $L_1$ by Condition (IV). Denote by $\overline{CB_2}$ the integral closure of $C[t]$ in $L_2$. Since $C$ is integrally closed in $L$, $C[t]$ is integally closed in $L[t]$. Hence $C[t]$ is the set of fixed points under the action of $\Gamma$ in $\overline{CB_2}$. Let $a$ be in $\overline{CB_2}$. Then

$$0 = \prod_{g \in \Gamma} (a - g(a)) = a^m + a_{m-1} a^{m-1} + \cdots + a_0$$

with $a_0, \dots, a_{m-1}$ in $C[t]$.

(i) Since $t^j LB_1$ is contained in $t^j LB_2$ and contains $t^j C[t]$, it suffices to prove the assertion for $i = 2$. Let us prove it by induction on $j$. Let $c$ be in $C[t]$. Then $c - c(0)$ is in $tLB_2$. By Lemma 2.6,(ii), $L \cap tLB_2 = \{0\}$ since $L$ is a field, whence $C \cap tLB_2 = \{0\}$ since $C$ is contained in $L$. As a result, if $c$ is in $tLB_2$, $c(0) = 0$ and $c$ is in $tC[t]$, whence the assertion for $j = 1$. Suppose the assertion true for $j - 1$. Let $c$ be in $C[t] \cap t^j LB_2$.

By induction hypothesis, $c = t^{j-1}c'$ with $c'$ in $C[t]$. Then $c'$ is in $C[t] \cap tLB_2$, whence $c$ is in $t^jC[t]$ by the assertion for $j = 1$.

(ii) Suppose that $a$ is in $tLB_1$. Since $tLB_2$ is invariant under $\Gamma$, for $i = 0, \dots, m-1$, $a_i$ is in $t^{m-i}LB_2$. Set for $i = 0, \dots, m-1$,

$$a'_i := \frac{a_i}{t^{m-i}}.$$

Then by (i), $a'_0, \dots, a'_{m-1}$ are in $C[t]$. Moreover,

$$(\frac{a}{t})^m + a'_{m-1}(\frac{a}{t^{m-1}})^{m-1} + \cdots + a'_0 = 0,$$

so that $a/t$ is in $\overline{CB_1}$, whence the assertion.

(iii) Suppose that $a$ is in $\overline{CB_1}$. By Lemma 2.6,(iii), $L$ is the quotient of $LB_1$ by $tLB_1$. So, denoting by $\overline{a}$ the image of $a$ by the quotient morphism,

$$\overline{a}^m + a_{m-1}(0)\overline{a}^{m-1} + \cdots + a_0(0) = 0.$$

Then $\overline{a}$ is in $C$ since $C$ is integrally closed. Hence $a$ is in $C + tLB_1$. As a result, by (ii), $\overline{CB_1}$ is contained in $C + t\overline{CB_1}$.

(iv) By Condition (III), $B$ is integrally closed in $L$. So the assertion results from (iii) and Condition (IV) for $C = B$. □

**Corollary 2.10.** *The ideal $R(C)t$ of $R(C)$ is prime and $t$ is not a zero divisor in $R(C)$.*

*Proof.* According to Lemma 2.9,(iv), $R(C) = C + R(C)t$. Furthermore, this sum is direct since $C \cap tCB_1 = \{0\}$ by Lemma 2.6,(ii) and since the restriction of $\mu_C$ to $C$ is injective. Then $R(C)t$ is a prime ideal of $R(C)$ since $C$ is an integral algebra.

Since $R(C)t$ is a prime ideal, $\mathfrak{n}$ is contained in $R(C)t$. According to Lemma 2.8,(iv), $\mathfrak{n}$ is the kernel of $\mu_C$. Let $a$ be in $\mathfrak{n}$. Then $a = a't$ for some $a'$ in $R(C)$. Since $0 = \mu_C(a't) = \mu_C(a')t$, $a'$ is in $\mathfrak{n}$. As a result, by induction on $m$, for all positive integer $m$, $a = a_m t^m$ for some $a_m$ in $\mathfrak{n}$.

For $k$ positive integer, denote by $J_k$ the subset of elements $a$ of $R(C)$ such that $at^k = 0$. Then $(J_1, J_2, \dots)$ is an increasing sequence of ideals of $R(C)$. For $a$ in $J_k$, $0 = \mu_C(at^k) = \mu_C(a)t^k$. Hence $J_k$ is contained in $\mathfrak{n}$. According to Lemma 2.8,(i), the $\Bbbk$-algebra $R(C)$ is finitely generated. So for some positive integer $k_0$, $J_k = J_{k_0}$ for all $k$ bigger than $k_0$. Let $a$ be in $J_1$. Then $a = a_{k_0}t^{k_0}$ for some $a_{k_0}$ in $\mathfrak{k}$. Since $a_{k_0}t^{k_0+1} = 0$, $a_{k_0}$ is in $J_{k_0}$ so that $a = 0$. Hence $t$ is not a zero divisor in $R(C)$. □

**Proposition 2.11.** *Suppose that $C$ is integrally closed and Cohen-Macaulay. Let $\mathfrak{p}$ be a prime ideal of $CB_1$, containing $t$ and let $\tilde{\mathfrak{p}}$ be its inverse image by $\mu_C$.*

(i) *The local algebra $(CB_1)_{\mathfrak{p}}$ is normal.*

(ii) *The local algebra $R(C)_{\tilde{\mathfrak{p}}}$ is Cohen-Macaulay and reduced. In particular, the canonical morphism $R(C)_{\tilde{\mathfrak{p}}} \to (CB_1)_{\mathfrak{p}}$ is an isomorphism.*

(iii) *The local algebra $(CB_1)_{\mathfrak{p}}$ is Cohen-Macaulay.*

*Proof.* (i) Let $\overline{CB_1}$ be the integral closure of $CB_1$ in $L_1$. Setting $S := CB_1 \setminus \mathfrak{p}$, $(CB_1)_{\mathfrak{p}}$ is the localization of $CB_1$ with respect to $S$. Denote by $(\overline{CB_1})_{\mathfrak{p}}$ the localization of $\overline{CB_1}$ with respect to $S$. Then $(\overline{CB_1})_{\mathfrak{p}}$ is a finite $(CB_1)_{\mathfrak{p}}$-module since $\overline{CB_1}$ is a finite extension of $CB_1$. According to Lemma 2.8,(iii),

$$\overline{CB_1} \subset CB_1 + t\overline{CB_1}.$$

Then since $t$ is in $\mathfrak{p}$,

$$(\overline{CB_1})_{\mathfrak{p}}/(CB_1)_{\mathfrak{p}} = \mathfrak{p}(\overline{CB_1})_{\mathfrak{p}}/(CB_1)_{\mathfrak{p}}.$$

So, by Nakayama's Lemma, $(\overline{CB_1})_{\mathfrak{p}} = (CB_1)_{\mathfrak{p}}$, whence the assertion.

(ii) According to Corollary 2.10, $R(C)t$ is a prime ideal containing $\mathfrak{n}$. Denote by $\overline{\mathfrak{p}}$ the intersection of $\mathfrak{p}$ and $C$. Since $\tilde{\mathfrak{p}}$ is the inverse image of $\mathfrak{p}$ by $\mu_C$, $C_{\overline{\mathfrak{p}}}$ is the quotient of $R(C)_{\tilde{\mathfrak{p}}}$ by $R(C)_{\tilde{\mathfrak{p}}}t$. Since $C$ is Cohen-Macaulay, so is $C_{\overline{\mathfrak{p}}}$. As a result, $R(C)_{\tilde{\mathfrak{p}}}$ is Cohen-Macaulay since $t$ is not a zero divisor in $R(C)$ by Corollary 2.10 and since $R(C)_{\tilde{\mathfrak{p}}}t$ is a prime ideal of height 1.

Denote by $\mu_{C,\tilde{\mathfrak{p}}}$ the canonical extension of $\mu_C$ to $R(C)_{\tilde{\mathfrak{p}}}$. Then $(CB_1)_{\mathfrak{p}}$ is the image of $\mu_{C,\tilde{\mathfrak{p}}}$. According to Lemma 2.8,(iv), the nilradical $\mathfrak{n}R(C)_{\tilde{\mathfrak{p}}}$ of $R(C)_{\tilde{\mathfrak{p}}}$ is the minimal prime ideal of $R(C)_{\tilde{\mathfrak{p}}}$ and it is the kernel of $\mu_{C,\tilde{\mathfrak{p}}}$. By Lemma 2.8,(v), the localization of $R(C)_{\tilde{\mathfrak{p}}}$ at $\mathfrak{n}R(C)_{\tilde{\mathfrak{p}}}$ is a field. In particular, it is regular. Then, by [Bou98, §1, Proposition 15], $R(C)_{\tilde{\mathfrak{p}}}$ is a reduced algebra since it is Cohen-Macaulay. As a result, $\mu_{C,\tilde{\mathfrak{p}}}$ is an isomorphism onto $(CB_1)_{\mathfrak{p}}$.

(iii) results from (ii). □

2.3. Return to the situation of Subsection 2.1, and keep its notations. From now on, and until the end of the section, we assume that $S = \mathrm{S}(E)$ for some finite dimensional $\Bbbk$-vector space $E$. As a rule, if $B$ is a subalgebra of $\mathrm{S}(E)$, we denote by $K(B)$ its fraction field, and we set for simplicity

$$K := K(\mathrm{S}(E)).$$

Furthermore we assume until the end of the section that the following conditions hold:

(a) $\dim \mathcal{N}_0 = N$,
(b) $A$ is a polynomial algebra,
(c) $K(\overline{A})$ is algebraically closed in $K$.

We aim to prove Theorem 2.20 (see Subsection 2.4). Let $(v_1, \dots, v_N)$ be a sequence of elements of $E$ such that its nullvariety in $\mathcal{N}_0$ equals $\{0\}$. Such a sequence does exist by Lemma 2.1,(iii). Set

$$C := \overline{A}[v_1, \dots, v_N].$$

By Proposition 2.5,(ii), $C$ is a polynomial algebra if and only if so is $\overline{A}$ since $C$ is a faithfully flat extension of $\overline{A}$. Therefore, in order to prove Theorem 2.20, it suffices to prove that $\mathrm{S}(E)$ is a free extension of $C$, again by Proposition 2.5,(ii). This is now our goal.

Condition (c) is actually not useful for the following lemma:

**Lemma 2.12.** *The algebra $C$ is integrally closed and $\mathrm{S}(E)$ is the integral closure of $C$ in $K$.*

*Proof.* Since $\overline{A}$ has dimension $\dim E - N$ and since the nullvariety of $v_1, \dots, v_N$ in $\mathcal{N}_0$ is $\{0\}$, $v_1, \dots, v_N$ are algebraically independent over $A$ and $\overline{A}$. By Serre's normality criterion [Bou98, §1, n°10, Théorème 4], any polynomial algebra over a normal ring is normal. So $C$ is integrally closed since so is $\overline{A}$ by definition. Moreover, $C$ is a graded finitely generated subalgebra of $\mathrm{S}(E)$ since so is $\overline{A}$ by Corollary 2.4. Since $C$ has dimension $\dim E$, $\mathrm{S}(E)$ is algebraic over $C$. Then, by Corollary 2.4, $\mathrm{S}(E)$ is the integral closure of $C$ in $K$. Indeed, $\mathrm{S}(E)$ is integrally closed as a polynomial algebra and $\{0\}$ is the nullvariety of $C_+$ in $E^*$. □

Set $Z_0 := \mathrm{Specm}(\overline{A})$ and $Z := Z_0 \times \Bbbk^N$. Then $Z$ is equal to $\mathrm{Specm}(C)$. Let $X_0$ be a desingularization of $Z_0$ and let $\pi_0$ be the morphism of desingularization. Such a desingularization does exist by [Hir64]. Set $X := X_0 \times \Bbbk^N$ and denote by $\pi$ the morphism

$$X \longrightarrow Z, \qquad (x, v) \longmapsto (\pi_0(x), v).$$

Then $(X, \pi)$ is a desingularization of $Z$.

Fix $x_0$ in $\pi_0^{-1}(C_+)$. For $i = 0, \dots, N$, set $X_i := X_0 \times \Bbbk^i$ and let $x_i := (x_0, 0_{\Bbbk^i})$. Define $K_i$, $C'_i$, $C_i$ by the induction relations:

(1) $C'_0 := C_0 := \overline{A}$ and $K_0$ is the fraction field of $\overline{A}$,
(2) $C'_i := C'_{i-1}[v_i]$,
(3) $K_i$ is the algebraic closure of $K_{i-1}(v_i)$ in $K$ and $C_i$ is the integral closure of $C_{i-1}[v_i]$ in $K_i$.

**Lemma 2.13.** *Let $i = 1, \dots, N$.*
(i) *The field $K_i$ is a finite extension of $K_{i-1}(v_i)$ and $K_{i-1}$ is algebraically closed in $K_i$.*
(ii) *The algebra $C_i$ is finitely generated and integrally closed in $K$. Moreover, $K_i$ is the fraction field of $C_i$.*
(iii) *The algebra $C_i$ is contained in $\mathrm{S}(E)$ and $C_N = \mathrm{S}(E)$. Moreover, $K_N = K$.*
(iv) *The algebra $C_i$ is a finite extension of $C'_i$.*
(v) *The algebra $C_i$ is the intersection of $\mathrm{S}(E)$ and $K_i$. Moreover, $v_iC_i$ is a prime ideal of $C_i$.*

*Proof.* (i) By Condition (c), $K_0$ is algebraically closed in $K$. So $K_0$ is algebraically closed in $K_1$. By definition, for $i > 1$, $K_{i-1}$ is algebraically closed in $K$. So it is in $K_i$. Since the nullvariety of $v_1, \dots, v_N$ in $\mathcal{N}_0$ equals $\{0\}$, $v_1, \dots, v_N$ are algebraically independent over $K_0$. Hence $K_{i-1}(v_i, \dots, v_N)$ is a field of rational fractions over $K_{i-1}$. Moreover, $K$ is an algebraic extension of $K_{i-1}(v_i, \dots, v_N)$ by Lemma 2.12. Since $\mathrm{S}(E)$ is a finitely generated $\Bbbk$-algebra, $K$ is a finite extension of $K_{i-1}(v_i, \dots, v_N)$. By definition, $K_i$ is the algebraic closure of $K_{i-1}(v_i)$ in $K$. Hence $K_i$ is a finite extension of $K_{i-1}(v_i)$.

(ii) Prove the assertion by induction on $i$. By definition, it is true for $i = 0$ and $C_i$ is the integral closure of $C_{i-1}[v_i]$ in $K_i$ for $i = 1, \dots, N$, whence the assertion by (i) and [Ma86, §33, Lemma 1].

(iii) Since $\mathrm{S}(E)$ is integrally closed in $K$, $C_i$ is contained in $\mathrm{S}(E)$ by induction on $i$. By definition, the field $K_N$ is algebraically closed in $K$ and it contains $C$. So $K_N = K$ by Lemma 2.12. Since $C_N$ is integrally closed in $K_N$ and it contains $C$, $C_N = \mathrm{S}(E)$ by Lemma 2.12.

(iv) Prove the assertion by induction on $i$. By definition, it is true for $i = 0$. Suppose that it is true for $i - 1$. Then $C_i$ is a finite extension of $C'_{i-1}[v_i] = C'_i$.

(v) Prove by induction on $i$ that $C_{N-i}$ is the intersection of $\mathrm{S}(E)$ and $K_{N-i}$ for $i = 0, \dots, N$. By (iii), it is true for $i = 0$. Suppose that it is true for $i - 1$. By induction hypothesis, it suffices to prove that $C_{N-i}$ is the intersection of $C_{N-i+1}$ and $K_{N-i}$. Let $a$ be in this intersection. Then $a$ verifies an integral dependence equation over $C_{N-i}[v_{N-i+1}]$:

$$a^m + a_{m-1}a^{m-1} + \cdots + a_0 = 0.$$

Denoting by $a_j(0)$ the constant term of $a_j$ as a polynomial in $v_{N-i+1}$ with coefficients in $C_{N-i}$,

$$a^m + a_{m-1}(0)a^{m-1} + \cdots + a_0(0) = 0$$

since $a$ is in $K_{N-i}$ and $v_{N-i+1}$ is algebraically independent over $K_{N-i}$. Hence $a$ is in $C_{N-i}$ since $C_{N-i}$ is integrally closed in $K_{N-i}$ by (ii).

Let $a$ and $b$ be in $C_i$ such that $ab$ is in $v_iC_i$. Since $v_i$ is in $E$, $v_i\mathrm{S}(E)$ is a prime ideal of $\mathrm{S}(E)$. So $a$ or $b$ is in $v_i\mathrm{S}(E)$ since $C_i$ is contained in $\mathrm{S}(E)$. Hence $a/v_i$ or $b/v_i$ are in the intersection of $\mathrm{S}(E)$ and $K_i$. So $a$ or $b$ is in $v_iC_i$. □

*Remark* 2.14. According to Lemma 2.13,(i),(ii),(iv), for $i = 1, \dots, N$, $K_{i-1}$, $v_i$, $C_{i-1}$, $K_i$, $C_i$ verify Conditions (I), (II), (III), (V) verifed by $L$, $t$, $B$, $L_1$, $B_1$ in Subsection 2.2. Moreover, Condition (IV) is verified by construction (cf. Lemma 2.13,(v)).

**Proposition 2.15.** *Let $i = 1, \dots, N$.*
(i) *The semi-local algebra $\mathcal{O}_{X_i,x_i}C_i$ is normal and Cohen-Macaulay.*
(ii) *The canonical morphism $\mathcal{O}_{X_i,x_i} \otimes_{C'_i} C_i \to \mathcal{O}_{X_i,x_i}C_i$ is an isomorphism.*

*Proof.* (i) The local ring $\mathcal{O}_{X_i,x_i}$ is an extension of $C'_i$ and $C_i$ is a finite extension of $C'_i$ by Lemma 2.13,(iv). So $\mathcal{O}_{X_i,x_i}C_i$ is a semi-local ring as a finite extension of the local ring $\mathcal{O}_{X_i,x_i}$. Prove the assertion by induction on $i$. For $i = 0$, $\mathcal{O}_{X_0,x_0}C_0 = \mathcal{O}_{X_0,x_0}$ and $\mathcal{O}_{X_0,x_0}$ is a regular local algebra. Suppose that it is true for $i-1$ and set $\mathfrak{A}_{i-1} := \mathcal{O}_{X_{i-1},x_{i-1}}C_{i-1}$. Then $\mathfrak{A}_{i-1}$ is a subalgebra of $K_{i-1}$ since $\mathcal{O}_{X_{i-1},x_{i-1}}$ is contained in the fraction field of $C'_{i-1}$. Let $\mathfrak{m}$ be a maximal ideal of $\mathcal{O}_{X_i,x_i}C_i$. The local ring $\mathcal{O}_{X_i,x_i}$ is the localization of $\mathcal{O}_{X_{i-1},x_{i-1}}[v_i]$ at $\mathfrak{m} \cap \mathcal{O}_{X_{i-1},x_{i-1}}[v_i]$. Hence $v_i$ is in $\mathfrak{m}$, and $\mathfrak{m} \cap \mathfrak{A}_{i-1}C_i$ is a prime ideal of $\mathfrak{A}_{i-1}C_i$ such that the localization of $\mathfrak{A}_{i-1}C_i$ at this prime ideal is the localization of $\mathcal{O}_{X_i,x_i}C_i$ at $\mathfrak{m}$. By the induction hypothesis, $\mathfrak{A}_{i-1}$ is normal and Cohen-Macaulay. According to Remark 2.14 and Proposition 2.11,(i) and (iii), the localization of $\mathfrak{A}_{i-1}C_i$ at $\mathfrak{m} \cap \mathfrak{A}_{i-1}C_i$ is normal and Cohen-Macaulay, whence the assertion.

(ii) Prove the assertion by induction on $i$. For $i = 0$, $C_0$ is contained in $\mathcal{O}_{X_0,x_0}$. Suppose that it is true for $i-1$. For $j \in \{i-1, i\}$, denote by $\nu_j$ the canonical morphism

$$\mathcal{O}_{X_j,x_j} \otimes_{C'_j} C_j \longrightarrow \mathcal{O}_{X_j,x_j}C_j.$$

Recall that $\mathfrak{A}_{i-1} := \mathcal{O}_{X_{i-1},x_{i-1}}C_{i-1}$. By induction hypothesis, the morphism $\nu_{i-1}\otimes \mathrm{id}_{C_i}$,

$$(\mathcal{O}_{X_{i-1},x_{i-1}} \otimes_{C'_{i-1}} C_{i-1}) \otimes_{C_{i-1}} C_i \longrightarrow \mathfrak{A}_{i-1} \otimes_{C_{i-1}} C_i$$

is an isomorphism. Since $C'_{i-1}$ is contained in $\mathcal{O}_{X_{i-1},x_{i-1}}$,

$$\mathcal{O}_{X_{i-1},x_{i-1}} \otimes_{C'_{i-1}} C'_{i-1}[v_i] = \mathcal{O}_{X_{i-1},x_{i-1}}[v_i].$$

Furthermore,

$$(\mathcal{O}_{X_{i-1},x_{i-1}} \otimes_{C'_{i-1}} C_{i-1}) \otimes_{C_{i-1}} C_i = \mathcal{O}_{X_{i-1},x_{i-1}} \otimes_{C'_{i-1}} C_i = (\mathcal{O}_{X_{i-1},x_{i-1}} \otimes_{C'_{i-1}} C'_{i-1}[v_i]) \otimes_{C'_{i-1}[v_i]} C_i,$$

whence an isomorphism

$$\mathcal{O}_{X_{i-1},x_{i-1}}[v_i] \otimes_{C'_{i-1}[v_i]} C_i \longrightarrow \mathfrak{A}_{i-1} \otimes_{C_{i-1}} C_i.$$

Let $\mathfrak{m}$ be as in (i). Set

$$\mathfrak{p} := \mathfrak{m} \cap \mathfrak{A}_{i-1}C_i, \qquad \tilde{\mathfrak{m}} := \nu_i^{-1}(\mathfrak{m}),$$

and denote by $\tilde{\mathfrak{p}}$ the inverse image of $\mathfrak{p}$ by the canonical morphism

$$\mathfrak{A}_{i-1} \otimes_{C_{i-1}} C_i \longrightarrow \mathfrak{A}_{i-1}C_i.$$

According to Proposition 2.11,(ii), the canonical morphism

$$(\mathcal{O}_{X_{i-1},x_{i-1}}C_{i-1} \otimes_{C_{i-1}} C_i)_{\tilde{\mathfrak{p}}} \longrightarrow (\mathcal{O}_{X_{i-1},x_{i-1}}C_i)_{\mathfrak{p}}$$

is an isomorphism since $\mathcal{O}_{X_{i-1},x_{i-1}}C_{i-1}$ is a finitely generated subalgebra of $K_{i-1}$, containing $C_{i-1}$, which is Cohen-Macaulay and integrally closed. Let $\mathfrak{p}^{\#}$ be the inverse image of $\tilde{\mathfrak{p}}$ by the isomorphism

$$\mathcal{O}_{X_{i-1},x_{i-1}}[v_i] \otimes_{C'_{i-1}[v_i]} C_i \longrightarrow \mathcal{O}_{X_{i-1},x_{i-1}}C_{i-1} \otimes_{C_{i-1}} C_i.$$

Then the canonical morphism

$$(\mathcal{O}_{X_{i-1},x_{i-1}}[v_i] \otimes_{C'_i} C_i)_{\mathfrak{p}^{\#}} \longrightarrow (\mathcal{O}_{X_{i-1},x_{i-1}}C_i)_{\mathfrak{p}}$$

is an isomorphism. From the equalities

$$(\mathcal{O}_{X_{i-1},x_{i-1}}[v_i] \otimes_{C'_i} C_i)_{\mathfrak{p}^{\#}} = (\mathcal{O}_{X_i,x_i} \otimes_{C'_i} C_i)_{\tilde{\mathfrak{m}}}, \qquad (\mathcal{O}_{X_{i-1},x_{i-1}}C_i)_{\mathfrak{p}} = (\mathcal{O}_{X_i,x_i}C_i)_{\mathfrak{m}}$$

we deduce that the support of the kernel of $\nu_i$ in $\mathrm{Spec}(\mathcal{O}_{X_i,x_i} \otimes_{C'_i} C_i)$ does not contain $\tilde{\mathfrak{m}}$. As a result, denoting by $\mathcal{S}_i$ this support, $\mathcal{S}_i$ does not contain the inverse images by $\nu_i$ of the maximal ideals of $\mathcal{O}_{X_i,x_i}C_i$.

According to Lemma 2.8,(iv), the kernel of the canonical morphism

$$\mathfrak{A}_{i-1} \otimes_{C_{i-1}} C_i \longrightarrow \mathcal{O}_{X_{i-1},x_{i-1}}C_i$$

is the nilradical of $\mathfrak{A}_{i-1} \otimes_{C_{i-1}} C_i$. Hence, the kernel of the canonical morphism

$$\mathcal{O}_{X_{i-1},x_{i-1}}[v_i] \otimes_{C'_i} C_i \to \mathcal{O}_{X_{i-1},x_{i-1}}[v_i]C_i$$

is the nilradical of $\mathcal{O}_{X_{i-1},x_{i-1}}[v_i] \otimes_{C'_i} C_i$ since the canonical map

$$\mathcal{O}_{X_{i-1},x_{i-1}}[v_i] \otimes_{C'_{i-1}[v_i]} C_i \longrightarrow \mathfrak{A}_{i-1} \otimes_{C_{i-1}} C_i$$

is an isomorphism by induction hypothesis. As a result, all element of $\mathcal{S}_i$ is the inverse image of a prime ideal in $\mathcal{O}_{X_i,x_i}C_i$. Hence $\mathcal{S}_i$ is empty, and $\nu_i$ is an isomorphism. □

The following Corollary results from Proposition 2.15 and Lemma 2.13,(iii) since $\pi^{-1}(C_+) = \pi_0^{-1}(C_+) \times \{0\}$.

**Corollary 2.16.** *Let $x$ be in $\pi^{-1}(C_+)$.*

(i) *The semi-local algebra $\mathcal{O}_{X,x}\mathrm{S}(E)$ is normal and Cohen-Macaulay.*

(ii) *The canonical morphism $\mathcal{O}_{X,x} \otimes_C \mathrm{S}(E) \to \mathcal{O}_{X,x}\mathrm{S}(E)$ is an isomorphism.*

Let $d$ be the degree of the extension $K$ of $K(C)$. Let $x$ be in $\pi^{-1}(C_+)$, and denote by $Q_x$ the quotient of $\mathcal{O}_{X,x}\mathrm{S}(E)$ by $\mathfrak{m}_x\mathrm{S}(E)$, with $\mathfrak{m}_x$ the maximal ideal of $\mathcal{O}_{X,x}$.

**Lemma 2.17.** *Let $V$ be a graded complement to $\mathrm{S}(E)C_+$ in $\mathrm{S}(E)$.*

(i) *The $\Bbbk$-space $V$ has finite dimension, $\mathrm{S}(E) = CV$ and $K = K(C)V$.*

(ii) *The $\Bbbk$-space $Q_x$ has dimension $d$. Furthermore, for all subspace $V'$ of dimension $d$ of $V$ such that $Q_x$ is the image of $V'$ by the quotient map, the canonical map*

$$\mathcal{O}_{X,x} \otimes_{\Bbbk} V' \longrightarrow \mathcal{O}_{X,x}\mathrm{S}(E)$$

*is bijective.*

*Proof.* (i) According to Lemma 2.12, $\mathrm{S}(E)$ is a finite extension of $C$. Hence, the $\Bbbk$-space $V$ is finite dimensional. On the other hand, we have $\mathrm{S}(E) = V + \mathrm{S}(E)C_+$. Hence, by induction on $m$, $\mathrm{S}(E) = CV + \mathrm{S}(E)C_+^m$ for any $m$, whence $\mathrm{S}(E) = CV$ since $C_+$ is generated by elements of positive degree. As a result, $K = K(C)V$ since the $\Bbbk$-space $V$ is finite dimensional.

(ii) Let $d'$ be the dimension of $Q_x$. By (i), since $C_+$ is contained in $\mathfrak{m}_x$,

$$\mathcal{O}_{X,x}\mathrm{S}(E) = V + \mathfrak{m}_x\mathrm{S}(E).$$

As a result, for some subspace $V'$ of dimension $d'$ of $V$, $Q_x$ is the image of $V'$ by the quotient map. Then,

$$\mathcal{O}_{X,x}\mathrm{S}(E) = \mathcal{O}_{X,x}V' + \mathfrak{m}_x\mathrm{S}(E),$$

and by Nakayama's Lemma, $\mathcal{O}_{X,x}\mathrm{S}(E) = \mathcal{O}_{X,x}V'$. Let $(v_1, \dots, v_{d'})$ be a basis of $V'$. Suppose that the sequence $(v_1, \dots, v_{d'})$ is not linearly free over $\mathcal{O}_{X,x}$. A contradiction is expected. Let $l$ be the smallest integer such that

$$a_1v_1 + \cdots + a_{d'}v_{d'} = 0$$

for some sequence $(a_1, \dots, a_{d'})$ in $\mathfrak{m}_x^l$, not contained in $\mathfrak{m}_x^{l+1}$. According to Corollary 2.16,(i) and [Ma86, Ch. 8, Theorem 23.1], $\mathcal{O}_{X,x}\mathrm{S}(E)$ is a flat extension of $\mathcal{O}_{X,x}$ since $\mathcal{O}_{X,x}\mathrm{S}(E)$ is a finite extension of $\mathcal{O}_{X,x}$. So, for some $w_1, \dots, w_m$ in $\mathrm{S}(E)$ and for some sequences $(b_{i,1}, \dots, b_{i,m}, i = 1, \dots, d')$ in $\mathcal{O}_{X,x}$,

$$v_i = \sum_{j=1}^{m} b_{i,j}w_j \quad \text{and} \quad \sum_{j=1}^{d'} a_jb_{j,k} = 0$$

for all $i = 1, \dots, d'$ and for $k = 1, \dots, m$. Since $\mathcal{O}_{X,x}\mathrm{S}(E) = \mathcal{O}_{X,x}V'$,

$$w_j = \sum_{k=1}^{d'} c_{j,k} v_k$$

for some sequence $(c_{j,k}, j = 1, \dots, m, i = 1, \dots, d')$ in $\mathcal{O}_{X,x}$. Setting

$$u_{i,k} = \sum_{j=1}^{m} b_{i,j} c_{j,k}$$

for $i, k = 1, \dots, d'$, we have

$$v_i = \sum_{k \in I} u_{i,k} v_k \quad \text{and} \quad \sum_{j \in I} a_j u_{j,i} = 0$$

for all $i = 1, \dots, d'$. Since $v_1, \dots, v_{d'}$ is linearly free modulo $\mathfrak{m}_x \mathrm{S}(E)$,

$$u_{i,k} - \delta_{i,k} \in \mathfrak{m}_x$$

for all $(i, k)$, with $\delta_{i,k}$ the Kronecker symbol. As a result, $a_i$ is in $\mathfrak{m}_x^{l+1}$ for all $i$, whence a contradiction. Then the canonical map

$$\mathcal{O}_{X,x} \otimes_{\Bbbk} V' \longrightarrow \mathcal{O}_{X,x}\mathrm{S}(E)$$

is bijective. Since $K = K(C)\mathrm{S}(E)$ and since $K(C)$ is the fraction field of $\mathcal{O}_{X,x}$, $v_1, \dots, v_{d'}$ is a basis of $K$ over $K(C)$. Hence, $d' = d$ and the assertion follows. □

Recall that $K_0$ is the fraction field of $\overline{A}$. Let $v_{N+1}, \dots, v_{N+r}$ be elements of $E$ such that $v_1, \dots, v_{N+r}$ is a basis of $E$. Denoting by $t_1, \dots, t_r$ some indeterminates, let $\vartheta$ be the morphism of $C$-algebras

$$C[t_1, \dots, t_r] \longrightarrow \mathrm{S}(E), \qquad t_i \longmapsto v_{N+i},$$

and let $\tilde{\vartheta}$ be the morphism of $K_0[v_1, \dots, v_N]$-algebras

$$K_0[v_1, \dots, v_N, t_1, \dots, t_r] \longrightarrow K_0 \otimes_{\overline{A}} \mathrm{S}(E), \qquad t_i \longmapsto v_{N+i}.$$

For $\mathbf{i} = (i_1, \dots, i_N)$ in $\mathbb{N}^N$ and for $\mathbf{j} = (j_1, \dots, j_r)$ in $\mathbb{N}^r$, set:

$$v^{\mathbf{i}} := v_1^{i_1} \cdots v_N^{i_N}, \qquad t^{\mathbf{j}} := t_1^{j_1} \cdots t_r^{j_r}.$$

For $a$ in $\overline{A}$, denote by $\overline{a}$ the polynomial in $\Bbbk[v_1, \dots, v_N, t_1, \dots, t_r]$ such that $\vartheta(\overline{a}) = a$.

**Lemma 2.18.** *Let $I$ be the ideal of $C[t_1, \dots, t_r]$ generated by the elements $a - \overline{a}$ with $a$ in $\overline{A}$.*

(i) *For all graded generating family $(a_1, \dots, a_m)$ of $\overline{A}_+$, $I$ is the ideal generated by the sequence $(a_i - \overline{a_i}\,,\ i = 1, \dots, m)$.*

(ii) *The ideal $I$ is the kernel of $\vartheta$.*

*Proof.* (i) Let $I'$ be the ideal of $C[t_1, \dots, t_r]$ generated by the sequence $(a_i - \overline{a_i}, i = 1, \dots, m)$. Since the map $a \mapsto \overline{a}$ is linear, it suffices to prove that $a - \overline{a}$ is in $I'$ for all homogeneous element $a$ of $\overline{A}_+$. Prove it by induction on the degree of $a$. For some graded sequence $(b_1, \dots, b_m)$ in $\overline{A}$,

$$a = b_1 a_1 + \cdots + b_m a_m$$

so that

$$a - \overline{a} = \sum_{i=1}^{m} b_i (a_i - \overline{a_i}) + \sum_{i=1}^{m} \overline{a_i}(b_i - \overline{b_i}).$$

If $a$ has minimal degree, $b_1, \dots, b_m$ are in $\Bbbk$ and $b_i = \overline{b_i}$ for $i = 1, \dots, m$. Otherwise, for $i = 1, \dots, m$, if $b_i$ is not in $\Bbbk$, $b_i$ has degree smaller than $a$, whence the assertion by induction hypothesis.

(ii) By definition, $I$ is contained in the kernel of $\vartheta$. Let $a$ be in $C[t_1,\dots,t_r]$. Then $a$ has an expansion

$$a = \sum_{(\mathbf{i},\mathbf{j})\in\mathbb{N}^N\times\mathbb{N}^r} a_{\mathbf{i},\mathbf{j}} v^{\mathbf{i}} t^{\mathbf{j}}$$

with the $a_{\mathbf{i},\mathbf{j}}$'s in $\overline{A}$, whence

$$a = \sum_{(\mathbf{i},\mathbf{j})\in\mathbb{N}^N\times\mathbb{N}^r} (a_{\mathbf{i},\mathbf{j}} - \overline{a_{\mathbf{i},\mathbf{j}}}) v^{\mathbf{i}} t^{\mathbf{j}} + \sum_{(\mathbf{i},\mathbf{j})\in\mathbb{N}^N\times\mathbb{N}^r} \overline{a_{\mathbf{i},\mathbf{j}}} v^{\mathbf{i}} t^{\mathbf{j}}.$$

If $\vartheta(a) = 0$, then

$$\sum_{(\mathbf{i},\mathbf{j})\in\mathbb{N}^N\times\mathbb{N}^r} \overline{a_{\mathbf{i},\mathbf{j}}} v^{\mathbf{i}} t^{\mathbf{j}} = 0$$

since the restriction of $\vartheta$ to $\Bbbk[v_1,\dots,v_N,t_1,\dots,t_r]$ is injective, whence the assertion. □

For $x$ in $\pi^{-1}(C_+)$, denote by $\vartheta_x$ the morphism

$$\mathcal{O}_{X,x}[t_1,\dots,t_r] \longrightarrow K, \qquad at^{\mathbf{j}} \longmapsto a v_{N+1}^{j_1}\cdots v_{N+r}^{j_r}.$$

**Proposition 2.19.** *Let $x$ be in $\pi^{-1}(C_+)$.*

(i) *The kernel of $\vartheta_x$ is the ideal of $\mathcal{O}_{X,x}[t_1,\dots,t_r]$ generated by $I$. Furthermore, the image of $\vartheta_x$ is the subalgebra $\mathcal{O}_{X,x}\mathrm{S}(E)$ of $K$.*

(ii) *The intersection of $\mathfrak{m}_x\mathrm{S}(E)$ and $\mathrm{S}(E)$ is equal to $C_+\mathrm{S}(E)$.*

*Proof.* (i) From the short exact sequence

$$0 \longrightarrow I \longrightarrow C[t_1,\dots,t_r] \longrightarrow \mathrm{S}(E) \longrightarrow 0$$

we deduce the exact sequence

$$\mathcal{O}_{X,x}\otimes_C I \longrightarrow \mathcal{O}_{X,x}\otimes_C C[t_1,\dots,t_r] \longrightarrow \mathcal{O}_{X,x}\otimes_C \mathrm{S}(E) \longrightarrow 0.$$

Moreover, we have a commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & \\
\mathcal{O}_{X,x}\otimes_C I \arrow[r, "\mathrm{d}"] \arrow[d, "\delta"] & \mathcal{O}_{X,x}\otimes_C C[t_1,\dots,t_r] \arrow[r, "\mathrm{d}"] \arrow[d, "\delta"] & \mathcal{O}_{X,x}\otimes_C \mathrm{S}(E) \arrow[r] \arrow[d, "\delta"] & 0 \\
\mathcal{O}_{X,x}I \arrow[r, "\mathrm{d}"] \arrow[d] & \mathcal{O}_{X,x}[t_1,\dots,t_r] \arrow[r, "\mathrm{d}"] \arrow[d] & \mathcal{O}_{X,x}\mathrm{S}(E) \arrow[r] \arrow[d] & 0 \\
0 & 0 & 0 &
\end{tikzcd}$$

with exact columns by Corollary 2.16,(ii). For $a$ in $\mathcal{O}_{X,x}[t_1,\dots,t]$ such that $\mathrm{d}a = 0$,

$$a = \delta b, \quad b = \mathrm{d}c \quad \text{with} \quad b \in \mathcal{O}_{X,x}\otimes_C C[t_1,\dots,t_r], \quad c \in \mathcal{O}_{X,x}\otimes_C I,$$

so that $a = \mathrm{d}\circ\delta c$. Hence $\mathcal{O}_{X,x}I$ is the kernel of $\vartheta_x$.

(ii) Let $a_1,\dots,a_m$ be a graded generating family of $\overline{A}_+$. For $i = 1,\dots,m$,

$$\overline{a_i} = \sum_{(\mathbf{j},\mathbf{k})\in\mathbb{N}^N\times\mathbb{N}^r} a_{i,\mathbf{j},\mathbf{k}} v^{\mathbf{j}} t^{\mathbf{k}},$$

with the $a_{i,\mathbf{j},\mathbf{k}}$'s in $\Bbbk$. Set:

$$a'_i := \sum_{\mathbf{k}\in\mathbb{N}^N} a_{i,0,\mathbf{k}} t^{\mathbf{k}}.$$

For $i = 1, \dots, m$,

$$a'_i \in \overline{a_i} - a_i + C_+[t_1, \dots, t_r]$$

since $a_i$ is in $\overline{A}_+$ so that $\vartheta(a'_i)$ is in $C_+\mathrm{S}(E)$.

Since $C_+$ is contained in $\mathfrak{m}_x$, $C_+\mathrm{S}(E)$ is contained in $\mathfrak{m}_x\mathrm{S}(E) \cap \mathrm{S}(E)$. Let $a$ be in $\mathfrak{m}_x[t_1, \dots, t_r]$ such that $\vartheta_x(a)$ is in $\mathrm{S}(E)$. According to (i),

$$a \in C[t_1, \dots, t_r] + \mathcal{O}_{X,x} I.$$

So, by Lemma 2.17,(i),

$$a = b + b_1(a_1 - \overline{a_1}) + \cdots + b_m(a_m - \overline{a_m}),$$

with $b$ in $C[t_1, \dots, t_r]$ and $b_1, \dots, b_m$ in $\mathcal{O}_{X,x}$. Then,

$$b = b_0 + b_+, \quad \text{with} \quad b_0 \in \Bbbk[t_1, \dots, t_r] \quad \text{and} \quad b_+ \in C_+[t_1, \dots, t_r]$$

$$b_i = b_{i,0} + b_{i,+}, \quad \text{with} \quad b_{i,0} \in \Bbbk \quad \text{and} \quad b_{i,+} \in \mathfrak{m}_x$$

for $i = 1, \dots, m$. Since $a$ is in $\mathfrak{m}_x[t_1, \dots, t_r]$ and $a_1, \dots, a_m$ are in $C_+$,

$$b_0 - b_{1,0}\overline{a_1} - \cdots - b_{m,0}\overline{a_m} \in \mathfrak{m}_x[t_1, \dots, t_r]$$

Moreover, for $i = 1, \dots, m$,

$$\overline{a_i} - a'_i \in C_+[t_1, \dots, t_r].$$

Hence

$$b_0 - b_{1,0}a'_1 - \cdots - b_{m,0}a'_m = 0 \quad \text{since} \quad \mathfrak{m}_x[t_1, \dots, t_r] \cap \Bbbk[t_1, \dots, t_r] = 0.$$

As a result, $\vartheta_x(a)$ is in $C_+\mathrm{S}(E)$ since $\vartheta_x(a) = \vartheta_x(b_0) + \vartheta_x(b_+)$. □

2.4. We are now in a position to prove the main result of the section. Recall the main notations: $E$ is a finite dimensional vector space over $\Bbbk$, $A$ is a graded subalgebra of $\mathrm{S}(E)$, different from $\mathrm{S}(E)$ and such that $A = \Bbbk + A_+$, $\mathcal{N}_0$ is the nullvariety of $A_+$ in $E^*$, $K$ is the fraction field of $\mathrm{S}(E)$ and $K(\overline{A})$ that one of $\overline{A}$, the algebraic closure of $A$ in $\mathrm{S}(E)$.

**Theorem 2.20.** *Suppose that the following conditions are verified:*

(a) $\mathcal{N}_0$ *has dimension* $N$,
(b) $A$ *is a polynomial algebra,*
(c) $K(\overline{A})$ *is algebraically closed in* $K$.

*Then* $\overline{A}$ *is a polynomial algebra. Moreover,* $\mathrm{S}(E)$ *is a free extension of* $\overline{A}$.

*Proof.* Use the notations of Subsection 2.3. In particular, set

$$C = \overline{A}[v_1, \dots, v_N],$$

with $(v_1, \dots, v_N)$ a sequence of elements of $E$ such that its nullvariety in $\mathcal{N}_0$ is equal to $\{0\}$ (cf. Lemma 2.1,(iii)), and let $K(C)$ be the fraction field of $C$. As already explained, according to Proposition 2.5,(ii), it suffices to prove that $\mathrm{S}(E)$ is a free extension of $C$. Let $V$ be as in Lemma 2.18, a graded complement to $\mathrm{S}(E)C_+$ in $\mathrm{S}(E)$. Recall that $X$ is a desingularization of $Z = \mathrm{Specm}(C)$ and that $\pi$ is the morphism of desingularization. Let $x$ be in $\pi^{-1}(C_+)$. According to Proposition 2.19,(ii), for some subspace $V'$ of $V$, $V'$ is a complement to $\mathfrak{m}_x\mathrm{S}(E)$ in $\mathcal{O}_{X,x}\mathrm{S}(E)$. Then, by Lemma 2.17,(ii), $V'$ has dimension the degree of the extension $K$ of $K(C)$ and the canonical map

$$\mathcal{O}_{X,x} \otimes_{\Bbbk} V' \longrightarrow \mathcal{O}_{X,x}\mathrm{S}(E)$$

is bijective. Moreover,

$$V' \oplus \mathrm{S}(E)C_+ = \mathrm{S}(E) \quad \text{and} \quad V' = V.$$

Indeed, for $a \in \mathrm{S}(E)$, write $a = b + c$ with $b \in V'$ and $c \in \mathfrak{m}_x\mathrm{S}(E)$. Since $V'$ is contained in $\mathrm{S}(E)$, $c$ is in $\mathrm{S}(E)$, whence $c$ in $\mathrm{S}(E)C_+$ by Proposition 2.19,(ii). In addition, $\mathrm{S}(E) = CV$ as it has been observed in the proof of Lemma 2.17,(i). As a result, the canonical map

$$C \otimes_{\Bbbk} V \longrightarrow \mathrm{S}(E)$$

is bijective. This concludes the proof of the theorem. □

## 3. Good elements and good orbits

Recall that $\Bbbk$ is an algebraically closed field of characteristic zero. As in the introduction, $\mathfrak{g}$ is a simple Lie algebra over $\Bbbk$ of rank $\ell$, $\langle .,. \rangle$ denotes the Killing form of $\mathfrak{g}$, and $G$ denotes the adjoint group of $\mathfrak{g}$.

3.1. The notions of good element and good orbit in $\mathfrak{g}$ are introduced in this paragraph.

For $x$ in $\mathfrak{g}$, denote by $\mathfrak{g}^x$ its centralizer in $\mathfrak{g}$, by $G^x$ its stabilizer in $G$, by $G_0^x$ the identity component of $G^x$ and by $K_x$ the fraction field of the symmetric algebra $\mathrm{S}(\mathfrak{g}^x)$. Then $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$ and $K_x^{\mathfrak{g}^x}$ denote the sets of $G_0^x$-invariant elements of $\mathrm{S}(\mathfrak{g}^x)$ and $K_x$ respectively.

**Lemma 3.1.** *Let $x$ be in $\mathfrak{g}$. Then $K_x^{\mathfrak{g}^x}$ is the fraction field of $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$ and $K_x^{\mathfrak{g}^x}$ is algebraically closed in $K_x$.*

*Proof.* Let $a$ be in $K_x$, algebraic over $K_x^{\mathfrak{g}^x}$. For all $g$ in $G_0^x$, $g.a$ verifies the same equation of algebraic dependence over $K_x^{\mathfrak{g}^x}$ as $a$. Since a polynomial in one indeterminate has a finite number of roots, the $G_0^x$-orbit of $a$ is finite. But this orbit is then reduced to $\{a\}$, $G_0^x$ being connected. Hence $a$ is in $K_x^{\mathfrak{g}^x}$. This shows that $K_x^{\mathfrak{g}^x}$ is algebraically closed in $K_x$.

Let $K_0$ be the fraction field of $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$. According to [CMo10, Theorem 1.2] (see also Theorem 1.1), the index of $\mathfrak{g}^x$ is equal to $\ell$. So, by [R63], the transcendence degree of $K_x^{\mathfrak{g}^x}$ over $\Bbbk$ is equal to $\ell$. For $x$ nilpotent, $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$ contains $\ell$ algebraically independent elements by [PPY07, Proposition 0.1 and Corollary 2.3]. So for all $x$ in $\mathfrak{g}$, $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$ contains $\ell$ algebraically independent elements. Thus $K_x^{\mathfrak{g}^x}$ is an algebraic extension of $K_0$. Let $A$ be the integral closure of $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$ in $K_x^{\mathfrak{g}^x}$. We have $K_0A = K_x^{\mathfrak{g}^x}$. Indeed, for $a$ in $K_x^{\mathfrak{g}^x}$, by multiplying the coefficients of an algebraic dependance equation of $a$ over $K_0$ by a power of some, judiciously chosen, $b \in \mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$, we observe that $ba$ is integral over $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$. Hence $ba \in A$. As, in addition, $K_0A$ is clearly contained in $K_x^{\mathfrak{g}^x}$, we get the announced equality. As a result, $K_x^{\mathfrak{g}^x}$ is the fraction field of $A$. Furthermore, since $\mathrm{S}(\mathfrak{g}^x)$ is integrally closed, $A$ is contained in $\mathrm{S}(\mathfrak{g}^x)$. Hence $A = \mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$ because $\mathrm{S}(\mathfrak{g}^x) \cap K_x^{\mathfrak{g}^x} = \mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$ and $K_0 = K_x^{\mathfrak{g}^x}$. □

**Definition 3.2.** An element $x \in \mathfrak{g}$ is called a *good element of* $\mathfrak{g}$ if for some homogeneous elements $p_1, \ldots, p_\ell$ of $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$, the nullvariety of $p_1, \ldots, p_\ell$ in $(\mathfrak{g}^x)^*$ has codimension $\ell$ in $(\mathfrak{g}^x)^*$. A $G$-orbit in $\mathfrak{g}$ is called *good* if it is the orbit of a good element.

Since the nullvariety of $\mathrm{S}(\mathfrak{g})_+^{\mathfrak{g}}$ in $\mathfrak{g}$ is the nilpotent cone of $\mathfrak{g}$, 0 is a good element of $\mathfrak{g}$. For $(g, x)$ in $G \times \mathfrak{g}$ and for $a$ in $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$, $g(a)$ is in $\mathrm{S}(\mathfrak{g}^{g(x)})^{\mathfrak{g}^{g(x)}}$. So, an orbit is good if and only if all its elements are good.

**Theorem 3.3.** *Let $x$ be a good element of $\mathfrak{g}$. Then $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$ is a polynomial algebra and $\mathrm{S}(\mathfrak{g}^x)$ is a free extension of $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$.*

*Proof.* Let $p_1, \ldots, p_\ell$ be homogeneous elements of $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$ such that the nullvariety of $p_1, \ldots, p_\ell$ in $(\mathfrak{g}^x)^*$ has codimension $\ell$. Denote by $A$ the subalgebra of $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$ generated by $p_1, \ldots, p_\ell$. Then $A$ is a graded subalgebra of $\mathrm{S}(\mathfrak{g}^x)$ and the nullvariety of $A_+$ in $(\mathfrak{g}^x)^*$ has codimension $\ell$. So, by Lemma 2.1,(ii), $A$ has

dimension $\ell$. Hence, $p_1, \dots, p_\ell$ are algebraically independent and $A$ is a polynomial algebra. Denote by $\overline{A}$ the algebraic closure of $A$ in $\mathrm{S}(\mathfrak{g}^x)$. By Lemma 3.1, $\overline{A}$ is contained in $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$ and the fraction field of $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$ is algebraically closed in $K_x$. As a matter of fact, $\overline{A} = \mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$ since the fraction fields of $A$ and $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$ have the same transcendence degree. Hence, by Theorem 2.20, $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$ is a polynomial algebra and $\mathrm{S}(\mathfrak{g}^x)$ is free extension of $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$. □

*Remark* 3.4. The algebra $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$ may be polynomial even though $x$ is not good. Indeed, let us consider a nilpotent element $e$ of $\mathfrak{g} = \mathfrak{so}(\Bbbk^{10})$ in the nilpotent orbit associated with the partition $(3, 3, 2, 2)$. Then the algebra $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is polynomial, generated by elements of degrees $1, 1, 2, 2, 5$. But the nullcone has an irreducible component of codimension at most 4. So, $e$ is not good. We refer the reader to Example 7.5 for more details.

For $x \in \mathfrak{g}$, denote by $x_{\mathrm{s}}$ and $x_{\mathrm{n}}$ the semisimple and the nilpotent components of $x$ respectively.

**Proposition 3.5.** *Let $x$ be in $\mathfrak{g}$. Then $x$ is good if and only if $x_{\mathrm{n}}$ is a good element of the derived algebra of $\mathfrak{g}^{x_{\mathrm{s}}}$.*

*Proof.* Let $\mathfrak{z}$ be the center of $\mathfrak{g}^{x_{\mathrm{s}}}$ and let $\mathfrak{a}$ be the derived algebra of $\mathfrak{g}^{x_{\mathrm{s}}}$. Then

$$\mathfrak{g}^x = \mathfrak{z} \oplus \mathfrak{a}^{x_{\mathrm{n}}}, \qquad \mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x} = \mathrm{S}(\mathfrak{z}) \otimes_{\Bbbk} \mathrm{S}(\mathfrak{a}^{x_{\mathrm{n}}})^{\mathfrak{a}^{x_{\mathrm{n}}}}.$$

By the first equality, $(\mathfrak{a}^{x_{\mathrm{n}}})^*$ identifies with the orthogonal complement to $\mathfrak{z}$ in $(\mathfrak{g}^x)^*$. Set $d := \dim \mathfrak{z}$. Suppose that $x_{\mathrm{n}}$ is a good element of $\mathfrak{a}$ and let $p_1, \dots, p_{\ell-d}$ be homogeneous elements of $\mathrm{S}(\mathfrak{a}^{x_{\mathrm{n}}})^{\mathfrak{a}^{x_{\mathrm{n}}}}$ whose nullvariety in $(\mathfrak{a}^{x_{\mathrm{n}}})^*$ has codimension $\ell - d$. Denoting by $v_1, \dots, v_d$ a basis of $\mathfrak{z}$, the nullvariety of $v_1, \dots, v_d, p_1, \dots, p_{\ell-d}$ in $(\mathfrak{g}^x)^*$ is the nullvariety of $p_1, \dots, p_{\ell-d}$ in $(\mathfrak{a}^{x_{\mathrm{n}}})^*$. Hence, $x$ is a good element of $\mathfrak{g}$.

Conversely, let us suppose that $x$ is a good element of $\mathfrak{g}$. By Theorem 3.3, $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$ is a polynomial algebra generated by homogeneous polynomials $p_1, \dots, p_\ell$. Since $\mathfrak{z}$ is contained in $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$, $p_1, \dots, p_\ell$ can be chosen so that $p_1, \dots, p_d$ are in $\mathfrak{z}$ and $p_{d+1}, \dots, p_\ell$ are in $\mathrm{S}(\mathfrak{a}^{x_{\mathrm{n}}})^{\mathfrak{a}^{x_{\mathrm{n}}}}$. Then the nullvariety of $p_{d+1}, \dots, p_\ell$ in $(\mathfrak{a}^{x_{\mathrm{n}}})^*$ has codimension $\ell - d$. Hence, $x_{\mathrm{n}}$ is a good element of $\mathfrak{a}$. □

3.2. In view of Theorem 3.3, we wish to find a sufficient condition for that an element $x \in \mathfrak{g}$ is good. According to Proposition 3.5, it is enough to consider the case where $x$ is nilpotent.

Let $e$ be a nilpotent element of $\mathfrak{g}$, embedded into an $\mathfrak{sl}_2$-triple $(e, h, f)$ of $\mathfrak{g}$. Identify the dual of $\mathfrak{g}$ with $\mathfrak{g}$, and the dual of $\mathfrak{g}^e$ with $\mathfrak{g}^f$ through the Killing form $\langle .,. \rangle$ of $\mathfrak{g}$. For $p$ in $\mathrm{S}(\mathfrak{g}) \simeq \Bbbk[\mathfrak{g}]$, denote by $\kappa(p)$ the restriction to $\mathfrak{g}^f$ of the polynomial function $x \mapsto p(e + x)$ and denote by ${}^e p$ its initial homogeneous component. According to [PPY07, Proposition 0.1], for $p$ in $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$, ${}^e p$ is in $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$.

The proof of the following theorem will be achieved in Subsection 4.4.

**Theorem 3.6.** *Suppose that for some homogeneous generators $q_1, \dots, q_\ell$ of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$, the polynomial functions ${}^e q_1, \dots, {}^e q_\ell$ are algebraically independent. Then $e$ is a good element of $\mathfrak{g}$. In particular, $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is a polynomial algebra and $\mathrm{S}(\mathfrak{g}^e)$ is a free extension of $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$. Moreover, $({}^e q_1, \dots, {}^e q_\ell)$ is a regular sequence in $\mathrm{S}(\mathfrak{g}^e)$.*

The overall idea of the proof is the following.

According to Theorem 3.3, it suffices to prove that $e$ is good, and more accurately that the nullvariety of ${}^e q_1, \dots, {}^e q_\ell$ in $\mathfrak{g}^f$ has codimension $\ell$ since ${}^e q_1, \dots, {}^e q_\ell$ are invariant homogeneous polynomials. As explained in the introduction, we will use the Slodowy grading on $\mathrm{S}(\mathfrak{g}^e)[[t]]$ and $\mathrm{S}(\mathfrak{g}^e)((t))$, induced from that on $\mathrm{S}(\mathfrak{g}^e)$, to deal with this problem. This is the main purpose of Section 4.

## 4. Slodowy grading and proof of Theorem 1.4

This section is devoted to the proof of Theorem 3.6 (or Theorem 1.4). The proof is achieved in Subsection 4.5. As in the previous section, $\mathfrak{g}$ is a simple Lie algebra over $\Bbbk$ and $(e, h, f)$ is an $\mathfrak{sl}_2$-triple of $\mathfrak{g}$. Let us simply denote by $S$ the algebra $\mathrm{S}(\mathfrak{g}^e)$.

Let $q_1, \dots, q_\ell$ be homogeneous generators of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ of degrees $d_1, \dots, d_\ell$ respectively. The sequence $(q_1, \dots, q_\ell)$ is ordered so that $d_1 \leqslant \cdots \leqslant d_\ell$. We assume in the whole section that the polynomial functions ${}^e q_1, \dots, {}^e q_\ell$ are algebraically independent. The aim is to show that $e$ is good (cf. Definition 3.2).

4.1. Let $x_1, \dots, x_r$ be a basis of $\mathfrak{g}^e$ such that for $i = 1, \dots, r$, $[h, x_i] = n_i x_i$ for some nonnegative integer $n_i$. For $\mathbf{j} = (j_1, \dots, j_r)$ in $\mathbb{N}^r$, set:

$$|\mathbf{j}| := j_1 + \cdots + j_r, \qquad |\mathbf{j}|_e := j_1 n_1 + \cdots + j_r n_r + 2|\mathbf{j}|, \qquad x^{\mathbf{j}} = x_1^{j_1} \cdots x_r^{j_r}.$$

The algebra $S$ has two gradations: the standard one and the *Slodowy gradation*. For all $\mathbf{j}$ in $\mathbb{N}^r$, $x^{\mathbf{j}}$ is homogeneous with respect to these two gradations. It has standard degree $|\mathbf{j}|$ and Slodowy degree $|\mathbf{j}|_e$. In this section, we only consider the Slodowy gradation. So, by gradation we will always mean Slodowy gradation. For $m$ nonnegative integer, denote by $S^{[m]}$ the subspace of $S$ of degree $m$.

Let $t$ be an indeterminate. For all subspace $V$ of $S$, set:

$$V[t] := \Bbbk[t] \otimes_{\Bbbk} V, \qquad V[t, t^{-1}] := \Bbbk[t, t^{-1}] \otimes_{\Bbbk} V, \qquad V[[t]] := \Bbbk[[t]] \otimes_{\Bbbk} V, \qquad V((t)) := \Bbbk((t)) \otimes_{\Bbbk} V,$$

with $\Bbbk((t))$ the fraction field of $\Bbbk[[t]]$. For $V$ a subspace of $S[[t]]$, denote by $V(0)$ the image of $V$ by the quotient morphism

$$S[t] \longrightarrow S, \qquad a(t) \longmapsto a(0).$$

The gradation of $S$ induces a gradation of the algebra $S((t))$ with $t$ having degree 0. For $V$ a graded subspace of $S((t))$ and for $m$ a nonnegative integer, let $V^{[m]}$ be its component of degree $m$. In particular, for $V$ a graded subspace of $S$, $V((t))$ is a graded subspace of $S((t))$ and

$$V((t))^{[m]} = V^{[m]}((t)).$$

Let $\tau$ be the morphism of algebras,

$$\tau \colon S \longrightarrow S[t], \quad x_i \mapsto t x_i \quad \text{for} \quad i = 1, \dots, r.$$

The morphism $\tau$ is a morphism of graded algebras. Denote by $\delta_1, \dots, \delta_\ell$ the standard degrees of ${}^e q_1, \dots, {}^e q_\ell$ respectively, and set for $i = 1, \dots, \ell$

$$Q_i := t^{-\delta_i} \tau(\kappa(q_i)).$$

Let $A$ be the subalgebra of $S[t]$ generated by $Q_1, \dots, Q_\ell$. Then observe that $A(0)$ is the subalgebra of $S$ generated by ${}^e q_1, \dots, {}^e q_\ell$. For $\mathbf{j} = (j_1, \dots, j_\ell)$ in $\mathbb{N}^\ell$, set

$$q^{\mathbf{j}} := q_1^{j_1} \cdots q_\ell^{j_\ell}, \qquad \kappa(q)^{\mathbf{j}} := \kappa(q_1^{j_1}) \cdots \kappa(q_\ell^{j_\ell}), \qquad {}^e q^{\mathbf{j}} := {}^e q_1^{j_1} \cdots {}^e q_\ell^{j_\ell}, \qquad Q^{\mathbf{j}} := Q_1^{j_1} \cdots Q_\ell^{j_\ell}.$$

**Proposition 4.1.** (i) *For* $\mathbf{j}$ *in* $\mathbb{N}^\ell$, $\kappa(q)^{\mathbf{j}}$ *and* ${}^e q^{\mathbf{j}}$ *are homogeneous of degree* $2d_1 j_1 + \cdots + 2d_\ell j_\ell$.

(ii) *The map* $Q \mapsto Q(0)$ *is an isomorphism of graded algebras from* $A$ *onto* $A(0)$.

*Proof.* (i) Denote by $t \mapsto h(t)$ the one-parameter subgroup of $G$ generated by $\mathrm{ad}\, h$. Then for $i = 1, \dots, r$, for $y$ in $\mathfrak{g}^f$ and $t$ in $\Bbbk^*$,

$$\langle x_i, t^2 h(t^{-1})(e + y) \rangle = t^2 \langle h(t)(x_i), e + y \rangle = t^{2+n_i} \langle x_i, e + y \rangle,$$

whence

$$\langle x^{\mathbf{j}}, t^2 h(t^{-1})(e + y) \rangle = t^{|\mathbf{j}|_e} \langle x^{\mathbf{j}}, e + y \rangle$$

for all $\mathbf{j}$ in $\mathbb{N}^r$. For $\mathbf{j}$ in $\mathbb{N}^\ell$, $q^{\mathbf{j}}$ is homogeneous of standard degree $d_1 j_1 + \cdots + d_\ell j_\ell$ and $q^{\mathbf{j}}$ is invariant under the one parameter subgroup $t \mapsto h(t)$. So, for all $(t, y)$ in $\Bbbk^* \times \mathfrak{g}^f$,

$$\kappa(q)^{\mathbf{j}}(t^2 h(t^{-1})(y)) = t^{2d_1 j_1 + \cdots + 2d_\ell j_\ell} \kappa(q)(y).$$

Hence, $\kappa(q)^{\mathbf{j}}$ is homogeneous of degree $2d_1 j_1 + \cdots + 2d_\ell j_\ell$ so that $\kappa(q)^{\mathbf{j}}$ is a linear combination of monomials $x^{\mathbf{k}}$ of degree $2d_1 j_1 + \cdots + 2d_\ell j_\ell$. As a result, ${}^e q^{\mathbf{j}}$ is homogeneous of degree $2d_1 j_1 + \cdots + 2d_\ell j_\ell$ either.

(ii) The set $(Q^{\mathbf{j}}, \mathbf{j} \in \mathbb{N}^\ell)$ is a basis of the $\Bbbk$-space $A$ and the image of $Q^{\mathbf{j}}$ by the map $Q \mapsto Q(0)$ is equal to ${}^e q^{\mathbf{j}}$. Moreover, by (i), $Q^{\mathbf{j}}$ and ${}^e q^{\mathbf{j}}$ are homogeneous of degree $2d_1 j_1 + \cdots + 2d_\ell j_\ell$ so that $Q \mapsto Q(0)$ is a morphism of graded algebras. By definition, its image is $A(0)$. Since ${}^e q_1, \ldots, {}^e q_\ell$ are algebraically independent, it is injective. □

By Proposition 4.1,(ii), $A$ and $A(0)$ are isomorphic graded subalgberas of $S[t]$ and $S$ respectively. In particular, $A$ is a polynomial algebra since $A(0)$ is polynomial by our hypothesis.

Denote by $A_+$ and $A(0)_+$ the ideals of $A$ and $A(0)$ generated by the homogeneous elements of positive degree respectively, and denote by $\tilde{A}$ the subalgebra of $S[[t]]$ generated by $\Bbbk[[t]]$ and $A$, i.e.,

$$\tilde{A} := \Bbbk[[t]]A.$$

**Lemma 4.2.** (i) *The algebra $\tilde{A}$ is isomorphic to $\Bbbk[[t]] \otimes_\Bbbk A$. In particular, it is regular.*

(ii) *The element $t$ of $\tilde{A}$ is prime.*

(iii) *All prime element of $A$ is a prime element of $\tilde{A}$.*

*Proof.* (i) Let $a_m, m \in \mathbb{N}$, be in $A$ such that

$$\sum_{m \in \mathbb{N}} t^m a_m = 0.$$

If $a_m \neq 0$ for some $m$, then $a_p(0) = 0$ if $p$ is the smallest one such that $a_p \neq 0$. By Proposition 4.1,(ii), it is not possible. Hence, the canonical map

$$\Bbbk[[t]] \otimes_\Bbbk A \longrightarrow \tilde{A}$$

is an isomorphism. As observed just above, $A$ is a polynomial algebra. Then $\tilde{A}$ is a regular algebra by [Ma86, Ch. 7, Theorem 19.5].

(ii) By (i), $A$ is the quotient of $\tilde{A}$ by $t\tilde{A}$ so that $t$ is a prime element of $\tilde{A}$.

(iii) By (i), for $a$ in $A$, the quotient $\tilde{A}/\tilde{A}a$ is isomorphic to $\Bbbk[[t]] \otimes_\Bbbk A/Aa$. Hence $a$ is a prime element of $\tilde{A}$ if it is a prime element of $A$. □

As it has been explained in Subsection 3.2, in order to prove Theorem 3.6, we aim to prove that $S$ is a free extension that $A(0)$. As a first step, we describe in Subsections 4.2, 4.3 and 4.4 some properties of the algebra $A$. We show in Subsection 4.3 that $S((t))$ is a free extension of $A$ (cf. Proposition 4.9,(iii)), and we show in Subsection 4.4 that $S[[t]]$ is a free extension of $A$ (cf. Corollary 4.17). In Subsection 4.5, we consider the algebra $\tilde{A}$ and prove that $S[[t]]$ is a free extension of $\tilde{A}$ (cf. Theorem 4.21,(i)). The expected result will follow from this (cf. Theorem 4.21,(iii)).

4.2. Let $\theta_e$ be the map

$$G \times (e + \mathfrak{g}^f) \longrightarrow \mathfrak{g}, \qquad (g, x) \mapsto g(x),$$

and let $\mathcal{J}_e$ be the ideal of $\mathrm{S}(\mathfrak{g}^e)$ generated by the elements $\kappa(q_1), \ldots, \kappa(q_\ell)$.

**Lemma 4.3.** (i) *The map $\theta_e$ is a smooth morphism onto a dense open subset of $\mathfrak{g}$, containing $G.e$.*

(ii) *The nullvariety of $\mathcal{J}_e$ in $\mathfrak{g}^f$ is equidimensional of dimension $r - \ell$.*

(iii) *The ideal $\mathcal{J}_e$ of $\mathrm{S}(\mathfrak{g}^e)$ is radical.*

*Proof.* (i) Since $\mathfrak{g} = \mathfrak{g}^f + [e, \mathfrak{g}]$, $\theta_e$ is a submersion at $(1_\mathfrak{g}, e)$. Then, $\theta_e$ is a submersion at all points of $G \times (e + \mathfrak{g}^f)$ since it is $G$-equivariant for the left multiplication in $G$ and since

$$\lim_{t\to\infty} t^{-2}h(t)(x) = e$$

for all $x$ in $e + \mathfrak{g}^f$. So, by [H77, Ch. III, Proposition 10.4], $\theta_e$ is a smooth morphism onto a dense open subset of $\mathfrak{g}$, containing $G.e$.

(ii) Let $\mathcal{S}_e$ be the Slodowy slice $e + \mathfrak{g}^f$ associated with $e$. By (i), $\theta_e$ is equidimensional of dimension $\dim \mathcal{S}_e$. Denote by $\mathcal{V}_e$ the nullvariety of $\mathcal{J}_e$ in $\mathfrak{g}^f$. Then $G \times (e + \mathcal{V}_e)$ is the inverse image by $\theta_e$ of the nilpotent cone of $\mathfrak{g}$. Hence, $G \times (e + \mathcal{V}_e)$ is equidimensional of dimension

$$\dim \mathfrak{g} - \ell + \dim \mathcal{S}_e = \dim \mathfrak{g}^e - \ell + \dim \mathfrak{g}$$

since the nilpotent cone is irreducible of codimension $\ell$ and contains $G.e$. From this, we deduce that $\mathcal{V}_e$ is equidimensional of dimension $r - \ell$.

(iii) Let $\mathcal{X}_e$ be the subscheme of $e + \mathfrak{g}^f$ defined by the ideal $\mathcal{J}_e$ of $S$. In particular, $e + \mathcal{V}_e$ is the underlying variety of $\mathcal{X}_e$. By (ii), $\mathcal{X}_e$ is a complete intersection. In particular, it is Cohen-Macaulay. Let $X$ be an irreducible component of $\mathcal{V}_e$. Denoting by $U$ an open subset of $\mathfrak{g}^f$ such that $U \cap \mathcal{V}_e$ is not empty and contained in $X$, $G.(e+U)$ is an open subset of $\mathfrak{g}$ by (i). Moreover, its intersection with the nilpotent cone of $\mathfrak{g}$ is not empty and contained in $G.(e+X)$. As a result, for some $x$ in $X$, $e+x$ is a regular nilpotent element of $\mathfrak{g}$. For $q$ in $\mathrm{S}(\mathfrak{g})$, denote by $\mathrm{d}q$ and $\mathrm{d}\kappa(q)$ the differentials of $q$ and $\kappa(q)$ respectively. So, by [Ko63, Theorem 9], $\mathrm{d}q_1(e+x), \dots, \mathrm{d}q_\ell(e+x)$ are linearly independent. For $i = 1, \dots, \ell$, $\mathrm{d}\kappa(q_i)(e+x)$ is the restriction of $\mathrm{d}q_i(e+x)$ to $\mathfrak{g}^f$ since $q_i$ is $G$-invariant. So, the orthogonal complement of $\mathrm{d}\kappa(q_1)(e+x), \dots, \mathrm{d}\kappa(q_\ell)(e+x)$ in $\mathfrak{g}^f$ is the intersection of $\mathfrak{g}^f$ and $[e+x, \mathfrak{g}]$. In particular, it has dimension $\dim \mathfrak{g}^f - \ell$ so that $\mathrm{d}\kappa(q_1)(e+x), \dots, \mathrm{d}\kappa(q_\ell)(e+x)$ are linearly independent. As a result, $e + x$ is a smooth point of $\mathcal{X}_e$. Then, by [Bou98, §1, Proposition 15], $\mathcal{X}_e$ is reduced and $\mathcal{J}_e$ is radical. □

Denote by $\mathcal{V}$ the nullvariety of $A_+$ in $\mathfrak{g}^f \times \Bbbk$, and by $\mathcal{V}_0$ the nullvariety of $A(0)_+$ in $\mathfrak{g}^f$. Then denote by $\mathcal{V}_*$ the union of the irreducible components of $\mathcal{V}$ which are not contained in $\mathfrak{g}^f \times \{0\}$. Note that $\mathcal{V}_0 \times \{0\}$ is the nullvariety of $t$ in $\mathcal{V}$, and that

$$\mathcal{V} = \mathcal{V}_* \cup \mathcal{V}_0 \times \{0\}.$$

**Corollary 4.4.** (i) *The variety $\mathcal{V}_*$ is equidimensional of dimension $r + 1 - \ell$. Moreover, for $X$ an irreducible component of $\mathcal{V}_*$ and for $z$ in $\Bbbk$, the nullvariety of $t - z$ in $X$ has dimension $r - \ell$.*

(ii) *The algebra $S[t, t^{-1}]$ is a free extension of $A$.*

(iii) *The ideal $S[t, t^{-1}]A_+$ of $S[t, t^{-1}]$ is radical.*

*Proof.* (i) Let $\mathcal{V}'_*$ be the intersection of $\mathcal{V}_*$ and $\mathfrak{g}^f \times \Bbbk^*$ and let $X$ be an irreducible component of $\mathcal{V}'_*$. Then $\mathcal{V}'_*$ is the nullvariety of $Q_1, \dots, Q_\ell$ in $\mathfrak{g}^f \times \Bbbk^*$ since $A_+$ is the ideal of $A$ generated by $Q_1, \dots, Q_\ell$. In particular, $X$ has dimension at least $r+1-\ell$. For $z$ in $\Bbbk^*$, denote by $X_z$ the subvariety of $\mathfrak{g}^f$ such that $X_z \times \{z\} = X \cap \mathfrak{g}^f \times \{z\}$. By definition, for $i = 1, \dots, \ell$, $Q_i = t^{-\delta_i}\tau\circ\kappa(q_i)$. Hence $\mathcal{V}'_*$ is the nullvariety of $\tau\circ\kappa(q_1), \dots, \tau\circ\kappa(q_\ell)$ in $\mathfrak{g}^f \times \Bbbk^*$ and $X_z$ is the image of $X_1$ by the homothety $v \mapsto z^{-1}v$. By Lemma 4.3,(ii), $X_1$ has dimension $r - \ell$. Hence $X_z$ has dimension $r - \ell$ and $X$ has dimension at most $r + 1 - \ell$. As a result, $X$ has dimension $r + 1 - \ell$ and $X_z$ is strictly contained in $X$, whence the assertion since $X$ is not contained in $\mathfrak{g}^f \times \{0\}$ by definition.

(ii) The algebra $S[t, t^{-1}]$ is graded and $A$ is a graded polynomial subalgebra of $S[t, t^{-1}]$. According to (i), the fiber at $A_+$ of the extension $S[t, t^{-1}]$ of $A$ is equidimensional of dimension $r + 1 - \ell$. Hence, by Proposition 2.5, $S[t, t^{-1}]$ is a free extension of $A$.

(iii) Let $\mathcal{I}_e$ be the ideal of $S[t,t^{-1}]$ generated by $\tau\circ\kappa(q_1),\dots,\tau\circ\kappa(q_\ell)$. Since $t^{\delta_i}Q_i=\tau\circ\kappa(q_i)$ for $i=1,\dots,\ell$, we get $\mathcal{I}_e=S[t,t^{-1}]A_+$. Denote by $\overline{\tau}$ the endomorphism of the algebra $S[t,t^{-1}]$ defined by

$$\overline{\tau}(t)=t,\quad \overline{\tau}(x_1)=tx_1,\dots,\overline{\tau}(x_r)=tx_r.$$

Then $\overline{\tau}$ is an automorphism and $\mathcal{I}_e=\overline{\tau}(S[t,t^{-1}]\mathcal{J}_e)$. So, it suffices to prove that the ideal $S[t,t^{-1}]\mathcal{J}_e$ is radical.

Let $\mathcal{J}'_e$ be the radical of $S[t,t^{-1}]\mathcal{J}_e$. For $a$ in $S[t,t^{-1}]$, $a$ has a unique expansion

$$a=\sum_{m\in\mathbb{Z}}t^m a_m$$

with $(a_m, m\in\mathbb{Z})$ a sequence of finite support in $S$. Denote by $\nu(a)$ the cardinality of this finite support. Moreover, $a$ is in $S[t,t^{-1}]\mathcal{J}_e$ if and only if $a_m$ is in $\mathcal{J}_e$ for all $m$. Suppose that $S[t,t^{-1}]\mathcal{J}_e$ is strictly contained in $\mathcal{J}'_e$. A contradiction is expected. Let $a$ be in $\mathcal{J}'_e\setminus S[t,t^{-1}]\mathcal{J}_e$ such that $\nu(a)$ is minimal. Denote by $m_0$ the smallest integer such that $a_{m_0}\neq 0$. For some positive integer, $a^k$ and $(t^{-m_0}a)^k$ are in $S[t,t^{-1}]\mathcal{J}_e$ and we have

$$(t^{-m_0}a)^k=a_{m_0}^k+\sum_{m>0}t^m b_m$$

with the $b_m$'s in $\mathcal{J}_e$. Then $a_{m_0}^k$ is in $\mathcal{J}_e$ and by Lemma 4.3,(iii), $a_{m_0}$ is in $\mathcal{J}_e$. As a result $a':=a-t^{m_0}a_{m_0}$ is an element of $\mathcal{J}'_e$ such that $\nu(a')<\nu(a)$. By the minimality of $\nu(a)$, $a'$ is in $S[t,t^{-1}]\mathcal{J}_e$ and so is $a$, whence the contradiction. □

Let $\mathcal{I}_*$ be the ideal of definition of $\mathcal{V}_*$ in $S[t]$. Then $\mathcal{I}_*$ is an ideal of $S[t]$ containing the radical of $S[t]A_+$. It will be shown that $\mathcal{V}_*=\mathcal{V}$ and that $S[t]A_+$ is radical (cf. Theorem 4.21). Thus, $\mathcal{I}_*$ will be at the end equal to $S[t]A_+$.

Let $\mathfrak{p}_1,\dots,\mathfrak{p}_m$ be the minimal prime ideals containing $S[t]A_+$ and let $\mathfrak{q}_1,\dots,\mathfrak{q}_m$ be the primary decomposition of $S[t]A_+$ such that $\mathfrak{p}_i$ is the radical of $\mathfrak{q}_i$ for $i=1,\dots,m$.

**Lemma 4.5.** (i) *For $a$ in $S[t]$, $a$ is in $\mathcal{I}_*$ if and only if $t^m a$ is in $S[t]A_+$ for some positive integer $m$. Moreover, for some nonnegative integer $l$, $t^l\mathcal{I}_*$ is contained in $S[t]A_+$.*

(ii) *The ideal $\mathcal{I}_*$ is the intersection of the prime ideals $\mathfrak{p}_i$ which do not contain $t$. Furthermore, for such $i$, $\mathfrak{p}_i=\mathfrak{q}_i$, i.e. $\mathfrak{q}_i$ is radical.*

*Proof.* (i) Let $a$ be in $S[t]$. If $t^l a$ is in $S[t]A_+$ for some positive integer $l$, then $a$ is equal to 0 on $\mathcal{V}_*$ so that $a$ is in $\mathcal{I}_*$. Conversely, if $a$ is in $\mathcal{I}_*$, then $ta$ is in the radical of $S[t]A_+$ since $\mathcal{V}$ is contained in the union of $\mathcal{V}_*$ and $\mathfrak{g}^f\times\{0\}$. According to Corollary 4.4,(iii), for some positive integer $m$, $t^m(ta)$ is in $S[t]A_+$. Since $\mathcal{I}_*$ is finitely generated as an ideal of $S[t]$, we deduce that for some nonnegative integer $l$, $t^l\mathcal{I}_*$ is contained in $S[t]A_+$, whence the assertion.

(ii) Let $i\in\{1,\dots,m\}$. Then $\mathfrak{p}_i$ does not contain $t$ if and only if the nullvariety of $\mathfrak{p}_i$ in $\mathfrak{g}^f\times\mathbb{k}$ is an irreducible component of $\mathcal{V}_*$, whence the first part of the statement.

By (i), for some nonnegative integer $l$, $t^l\mathcal{I}_*$ is contained in $S[t]A_*$. Let $l$ be the minimal nonnegative integer verifying this condition. If $l=0$, $\mathcal{I}_*=S[t]A_+$, whence the assertion. Suppose $l$ positive. Denote by $\mathcal{I}'_*$ the ideal of $S[t]$ generated by $t^l$ and $S[t]A_+$. It suffices to prove that $S[t]A_+$ is the intersection of $\mathcal{I}_*$ and $\mathcal{I}'_*$. As a matter of fact, if so, the primary decomposition of $S[t]A_+$ is the union of the primary decompositions of $\mathcal{I}_*$ and $\mathcal{I}'_*$ since the minimal prime ideals containing $\mathcal{I}_*$ do not contain $t$.

Let $a$ be in the intersection of $\mathcal{I}_*$ and $\mathcal{I}'_*$. Then

$$a=t^l b+\sum_{i=1}^{\ell}a_iQ_i$$

with $b, a_1, \ldots, a_l$ in $S[t]$. Since $S[t]A_+$ is contained in $\mathcal{I}_*$, $t^l b$ is in $\mathcal{I}_*$ and $b$ is in $\mathcal{I}_*$ by (i). Hence $t^l b$ and $a$ are in $S[t]A_+$. As a result, $S[t]A_+$ is the intersection of $\mathcal{I}_*$ and $\mathcal{I}'_*$ since $S[t]A_+$ is contained in this intersection. □

4.3. Let $V_0$ be a graded complement to $SA(0)_+$ in $S$. We will show that the linear map

$$V_0 \otimes_{\Bbbk} A(0) \longrightarrow S, \qquad v \otimes a \longmapsto va$$

is a linear isomorphism (cf. Theorem 4.21).

**Lemma 4.6.** *We have $S[[t]] = V_0[[t]] + S[[t]]A_+$ and $S((t)) = V_0((t)) + S((t))A_+$.*

*Proof.* The equality $S((t)) = V_0((t)) + S((t))A_+$ will follow from the equality $S[[t]] = V_0[[t]] + S[[t]]A_+$. Since $S[[t]]$, $V_0[[t]]$ and $S[[t]]A_+$ are graded, it suffices to show that for $d$ a positive integer,

$$S[[t]]^{[d]} \subset V_0[[t]]^{[d]} + (S[[t]]A_+)^{[d]},$$

the inclusion $V_0[[t]] + S[[t]]A_+ \subset S[[t]]$ being obvious.

Let $d$ be a positive integer and let $a$ be in $S[[t]]^{[d]}$. Let $(\varphi_1, \ldots, \varphi_m)$ be a basis the $\Bbbk[[t]]$-module $(S[[t]]A_+)^{[d]}$. Such a basis does exist since $\Bbbk[[t]]$ is a principal ring and $S[[t]]^{[d]}$ is a finite free $\Bbbk[[t]]$-module. Then $\varphi_1(0), \ldots, \varphi_m(0)$ generate $(SA(0)_+)^{[d]}$. Since $S^{[d]} = V_0^{[d]} \oplus (SA(0)_+)^{[d]}$,

$$a - a_0 - \sum_{j=1}^{m} a_{0,j}\varphi_j = t\psi_0,$$

with $a_0$ in $V_0^{[d]}$, $a_{0,1}, \ldots, a_{0,m}$ in $\Bbbk$ and $\psi_0 \in S[[t]]^{[d]}$. Suppose that there are sequences $(a_0, \ldots, a_n)$ and $(a_{i,1}, \ldots, a_{i,m})$, for $i = 0, \ldots, n$, in $V_0^{[d]}$ and $\Bbbk$ respectively such that

$$a - \sum_{i=0}^{n} a_i t^i - \sum_{i=0}^{n}\sum_{j=1}^{m} t^i a_{i,j}\varphi_j = t^{n+1}\psi_n$$

for some $\psi_n$ in $S[[t]]^{[d]}$. Then for some $a_{n+1}$ in $V_0^{[d]}$ and $a_{n+1,1}, \ldots, a_{n+1,m}$ in $\Bbbk$,

$$\psi_n - a_{n+1} - \sum_{j=1}^{m} a_{n+1,j}\varphi_j \in tS[[t]]$$

so that

$$a - \sum_{i=0}^{n+1} a_i t^i - \sum_{i=0}^{n+1}\sum_{j=1}^{m} a_{i,j}\varphi_j t^i \in t^{n+2}S[[t]].$$

As a result,

$$a \in V_0[[t]]^{[d]} + (S[[t]]A_+)^{[d]}$$

since $S[[t]]^{[d]}$ is a finite $\Bbbk[[t]]$-module. □

Recall that $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ are the minimal prime ideals of $S[t]$ containing $S[t]A_+$. Since $A_+$ is a graded subspace of $S[t]$, $S[t]A_+$ is a graded ideal of $S[t]$, and so are $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$. According to Lemma 4.5,(ii), $\mathcal{I}_*$ is the intersection of the $\mathfrak{p}_i$'s which do not contain $t$. Hence, $\mathcal{I}_*$ is graded. Thereby, $\mathcal{I}_* \cap V_0[t]$ has a graded complement in $V_0[t]$. Set

$$W := \mathcal{I}_* \cap V_0[t].$$

Then $W(0)$ is a graded subspace of $V_0$. Denote by $V'_0$ a graded complement to $W(0)$ in $V_0$. Then set

$$V''_0 := W(0)$$

so that $V_0 = V'_0 \oplus V''_0$.

**Lemma 4.7.** *Let $(v_i, i \in J)$ be a graded basis of $V_0'$.*

(i) *The sequence $(v_i, i \in J)$ is linearly free over $\Bbbk[t]$.*

(ii) *The sum of $W$ and of $V_0'[t]$ is direct.*

*Proof.* We prove (i) and (ii) all together.

Let $(c_i, i \in J)$ be a sequence in $\Bbbk[t]$, with finite support $J_c$, such that

$$\sum_{i \in J} c_i v_i = w$$

for some $w$ in $W$. Suppose that $J_c$ is not empty. A contradiction is expected. Since $V_0'$ is a complement to $V_0''$, $c_i(0) = 0$ for all $i$ in $J$. Then, for $i$ in $J_c$, $c_i = t^{m_i} c_i'$ with $m_i > 0$ and $c_i'(0) \neq 0$. Denote by $m$ the smallest of the integers $m_i$, for $i \in J_c$. Then $w = t^m w'$ for some $w'$ in $V_0[t]$, and

$$\sum_{i \in J_c} t^{m_i - m} c_i' v_i = w'.$$

According to Lemma 4.5,(i), $w'$ is in $\mathcal{I}_*$. So, $c_i'(0) = 0$ for $i$ such that $m_i = m$, whence the contradiction. □

As a rule, for $M$ a $\Bbbk[t]$-submodule of $S[t]$, we denote by $\widehat{M}$ the $\Bbbk[[t]]$-module generated by $M$, i.e.,

$$\hat{M} = \Bbbk[[t]]M.$$

**Lemma 4.8.** *Let $M$ be a $\Bbbk[t]$-submodule of $S[t]$.*

(i) *Let $a$ be in the intersection of $S[t]$ and $\widehat{M}$. For some $q$ in $\Bbbk[t]$ such that $q(0) \neq 0$, $qa$ is in $M$.*

(ii) *For $N$ a $\Bbbk[t]$-submodule of $S[t]$, the intersection of $\widehat{M}$ and $\widehat{N}$ is the $\Bbbk[[t]]$-module generated by $M \cap N$.*

*Proof.* (i) Denote by $\overline{a}$ the image of $a$ in $S[t]/M$ by the quotient map and by $J$ its annihilator in $\Bbbk[t]$. Then we have a commutative diagram with exact lines and columns:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "\mathrm{d}"] & S[t] \arrow[r, "\mathrm{d}"] & S[t]/M \arrow[r] & 0 \\
0 \arrow[r] & J \arrow[r, "\mathrm{d}"] & \Bbbk[t] \arrow[r, "\mathrm{d}"] \arrow[u, "\delta"] & \Bbbk[t]\overline{a} \arrow[r] \arrow[u, "\delta"] & 0 \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Since $\Bbbk[[t]]$ is a flat extension of $\Bbbk[t]$, tensoring this diagram by $\Bbbk[[t]]$ gives the following diagram with exact lines and columns:

$$\begin{tikzcd}
0 \arrow[r] & \widehat{M} \arrow[r, "\mathrm{d}"] & S[[t]] \arrow[r, "\mathrm{d}"] & \Bbbk[[t]] \otimes_{\Bbbk[t]} S[t]/M \arrow[r] & 0 \\
0 \arrow[r] & \Bbbk[[t]]J \arrow[r, "\mathrm{d}"] & \Bbbk[[t]] \arrow[r, "\mathrm{d}"] \arrow[u, "\delta"] & \Bbbk[[t]]\overline{a} \arrow[r] \arrow[u, "\delta"] & 0 \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

For $b$ in $\Bbbk[[t]]$, $(\delta \circ \mathrm{d})b = (\mathrm{d} \circ \delta)b = 0$ since $a$ is in $\widehat{M}$, whence $\mathrm{d}b = 0$. As a result, $\Bbbk[[t]]J = \Bbbk[[t]]$. So $qa$ is in $M$ for some $q$ in $\Bbbk[t]$ such that $q(0) \neq 0$.

(ii) Since $\Bbbk[[t]]$ is a flat extension of $\Bbbk[t]$, the canonical morphism

$$\Bbbk[[t]] \otimes_{\Bbbk[t]} M \longrightarrow \widehat{M}.$$

is an isomorphism and from the short exact sequence

$$0 \longrightarrow M \cap N \longrightarrow M \oplus N \longrightarrow M + N \longrightarrow 0$$

we deduce the short exact sequence

$$0 \longrightarrow \Bbbk[[t]] \otimes_{\Bbbk[t]} M \cap N \longrightarrow \Bbbk[[t]] \otimes_{\Bbbk[t]} (M \oplus N) \longrightarrow \Bbbk[[t]] \otimes_{\Bbbk[t]} (M + N) \longrightarrow 0,$$

whence the short exact sequence

$$0 \longrightarrow \widehat{M \cap N} \longrightarrow \widehat{M} \oplus \widehat{N} \longrightarrow \widehat{M + N} \longrightarrow 0,$$

and whence the assertion. □

**Proposition 4.9.** (i) *The space $V_0[[t]]$ is the direct sum of $V_0'[[t]]$ and $\widehat{W}$.*

(ii) *The space $S[[t]]$ is the direct sum of $V_0'[[t]]$ and of $W + S[[t]]A_+$.*

(iii) *The linear map*

$$V_0'((t)) \otimes_{\Bbbk} A \longrightarrow S((t)), \qquad v \otimes a \longmapsto v \otimes a$$

*is a graded isomorphism onto $S((t))$.*

(iv) *For all nonnegative integer $d$,*

$$\dim S^{[d]} = \sum_{i=0}^{d} \dim V_0'^{[d-i]} \times \dim A^{[i]}.$$

*Proof.* (i) According to Lemma 4.8,(ii), the intersection of $V_0'[[t]]$ and $\widehat{W}$ is the $\Bbbk[[t]]$-submodule generated by the intersection of $V_0'[t]$ and $W$. So, by Lemma 4.7,(iii), the sum of $V_0'[[t]]$ and $\widehat{W}$ is direct.

Let $(v_i, i \in J)$ be a graded basis of $V_0'$. Let $d$ be a positive integer and let $v$ be in $V_0^{[d]}$. Denote by $J_d$ the set of indices $i$ such that $v_i$ has degree $d$. Since $V_0$ is the direct sum of $V_0'$ and $V_0''$, for some $w$ in $W^{[d]}$ and for some $c_i, i \in J_d$, in $\Bbbk$,

$$v - \sum_{i \in J} c_i v_i = w(0).$$

Since $w - w(0)$ is in $tV_0[t]^{[d]}$,

$$v - \sum_{i \in J_d} c_i v_i - w \in tV_0[t]^{[d]}.$$

As a result,

$$V_0^{[d]}[[t]] \subset V_0'^{[d]}[[t]] + \widehat{W}^{[d]} + tV_0^{[d]}[[t]].$$

Then by induction on $m$,

$$V_0^{[d]}[[t]] \subset V_0'^{[d]}[[t]] + \widehat{W}^{[d]} + t^m V_0^{[d]}[[t]].$$

So, since $V_0^{[d]}[[t]]$ is a finitely generated $\Bbbk[[t]]$-module,

$$V_0^{[d]}[[t]] = V_0'^{[d]}[[t]] + \widehat{W}^{[d]},$$

whence the assertion.

(ii) According to Lemma 4.5,(i), for some nonnegative integer $l$, $t^l \mathcal{I}_*$ is contained in $S[t]A_+$. Hence $\widehat{W} + S[[t]]A_+$ is equal to $W + S[[t]]A_+$. So, by (i) and Lemma 4.6,

$$S[[t]] = V_0'[[t]] + W + S[[t]]A_+.$$

According to Lemma 4.7,(ii), the intersection of $V_0'[t]$ and $S[t]A_+$ is equal to $\{0\}$ since $S[t]A_+$ is contained in $\mathcal{I}_*$. As a result, by Lemma 4.8,(ii), the intersection of $V_0'[[t]]$ and $S[[t]]A_+$ is equal to $\{0\}$. If $a$ is in the intersection of $V_0'[[t]]$ and $W + S[[t]]A_+$, $t^l a$ is in the intersection of $V_0'[[t]]$ and $S[[t]]A_+$. So the sum of $V_0'[[t]]$ and $W + S[[t]]A_+$ is direct.

(iii) According to Lemma 4.5,(i), $W$ is contained in $S((t))A_+$. So, by (ii),

$$S((t)) = V_0'((t)) \oplus S((t))A_+.$$

Since $\Bbbk[[t]]$ is a flat extension of $\Bbbk[t]$, and since

$$S((t)) = \Bbbk[[t]] \otimes_{\Bbbk[t]} S[t, t^{-1}],$$

we deduce that $S((t))$ is a flat extension of $A$ by Corollary 4.4,(ii). So, by Lemma 2.2, all basis of $V_0'[[t]]$ over $\Bbbk$ is linearly free over $A$. The assertion follows.

(iv) First of all, the canonical map

$$\Bbbk((t)) \otimes_{\Bbbk} A \longrightarrow \Bbbk((t))A$$

is an isomorphism by Lemma 4.2,(i). As a result, we have the canonical isomorphism

$$V_0'((t)) \otimes_{\Bbbk((t))} \Bbbk((t))A \longrightarrow V_0'((t)) \otimes_{\Bbbk((t))} (\Bbbk((t)) \otimes_{\Bbbk} A),$$

and for all nonnegative integer $i$,

$$\dim A^{[i]} = \dim_{\Bbbk((t))}(\Bbbk((t))A)^{[i]}.$$

From the above isomorphism, it results that the canonical morphism

$$V_0'((t)) \otimes_{\Bbbk((t))} \Bbbk((t))A \longrightarrow V_0'((t)) \otimes_{\Bbbk} A$$

is an isomorphism of graded spaces since $V_0'((t)) \otimes_{\Bbbk((t))} \Bbbk((t)) = V_0'((t))$. As a result, by (iii), the canonical morphism

$$V_0'((t)) \otimes_{\Bbbk((t))} \Bbbk((t))A \longrightarrow S((t))$$

is a graded isomorphism. So, for all nonnegative integer $d$,

$$\dim_{\Bbbk((t))} S((t))^{[d]} = \sum_{i=0}^{d} \dim_{\Bbbk((t))} V_0'((t))^{[d-i]} \times \dim_{\Bbbk((t))}(\Bbbk((t))A)^{[i]},$$

whence the assertion since $\dim S^{[i]} = \dim_{\Bbbk((t))} S((t))^{[i]}$ and $\dim V_0'^{[i]} = \dim_{\Bbbk((t))} V_0'((t))^{[i]}$ for all $i$. □

4.4. Let $(w_k, k \in K)$ be a graded sequence in $W$ such that $(w_k(0), k \in K)$ is a basis of $V_0'' = W(0)$. For $k$ in $K$, denote by $m_k$ the smallest integer such that $t^{m_k} w_k$ is in $S[t]A_+$. According to Lemma 4.5,(i), $m_k$ is finite for all $k$. Moreover, $m_k$ is positive since $W(0) \cap SA(0)_+ = \{0\}$. Set

$$\Theta := \{(k, i) \mid k \in K,\ i \in \{0, \dots, m_k - 1\}\},$$

and set for all $(k, i)$ in $\Theta$,

$$w_{k,i} := t^i w_k.$$

Let $E_\Theta$ be the $\Bbbk$-subspace of $V_0[t]$ generated by the elements $w_{k,i}, (k, i) \in \Theta$.

Set

$$\widehat{\mathcal{J}_*} := \Bbbk[[t]]\mathcal{J}_*.$$

It is an ideal of $S[[t]]$.

**Lemma 4.10.** (i) *For some $q$ in $\Bbbk[t]$ such that $q(0) \neq 0$, $q\mathcal{J}_*$ is contained in $W + S[t]A_+$.*

(ii) *The space $W$ is contained in $E_\Theta + S[t]A_+$. Moreover, $\widehat{\mathcal{J}_*}$ is the sum of $E_\Theta$ and $S[[t]]A_+$.*

(iii) *The sequence $(w_{k,i}, (k, i) \in \Theta)$ is a graded basis of $E_\Theta$.*

(iv) *For all nonnegative integer $i$, $E_\Theta^{[i]}$ has finite dimension.*

(v) *For $i$ a nonnegative integer, there exists a nonnegative integer $l_i$ such that $t^{l_i} E_\Theta^{[i]}$ is contained in $V_0'[[t]]A_+$.*

*Proof.* (i) Let $a$ be in $\mathcal{J}_*$. According to Lemma 4.6 and Lemma 4.8,(i), for some $q$ in $\Bbbk[t]$ such that $q(0) \neq 0$, $qa \in \mathcal{J}_*$ and $qa = a_1 + a_2$ with $a_1$ in $V_0[t]$ and $a_2$ in $S[t]A_+$. Then $a_1$ is in $\mathcal{J}_*$ since so are $a_2$ and $qa$. So $a_1 \in \mathcal{J}_* \cap V_0[t] = W$. The assertion follows because $\mathcal{J}_*$ is finitely generated.

(ii) Let us prove the first assertion. It suffices to prove

$$W \subset E_\Theta + S[t]A_+ + t^m S[t]$$

for all $m$. Indeed, $W$, $E_\Theta$, $S[t]A_+$ are contained in $\mathcal{J}_*$. So, if $w = e + a + t^m b$, with $w \in W$, $e \in E_\Theta$ and $b \in S[t]$, then $b$ is in $\mathcal{J}_*$ and so, for $m$ big enough, it is in $S[t]A_+$ by Lemma 4.5,(i).

Prove now the inclusion by induction on $m$. The inclusion is tautological for $m = 0$, and it is true $m = 1$ because $E_\Theta(0) = V_0''$. Suppose that it is true for $m > 0$. Let $w$ be in $W$. By induction hypothesis,

$$w = a + b + t^m c, \quad \text{with} \quad a \in E_\Theta,\ b \in S[t]A_+,\ c \in S[t].$$

Since $E_\Theta$ and $S[t]A_+$ are contained in $\mathcal{J}_*$, $c$ is in $\mathcal{J}_*$ by Lemma 4.5,(i). According to (i), for some $q$ in $\Bbbk[t]$ such that $q(0) \neq 0$, $qc = a' + b'$ with $a'$ in $W$ and $b'$ in $S[t]A_+$. Since the inclusion is true for $m = 1$,

$$t^m(a' + b') \in t^m E_\Theta + S[t]A_+ + t^{m+1} S[[t]],$$

and by definition, $t^m E_\Theta$ is contained in $E_\Theta + S[t]A_+$. Moreover, $q(0)c$ is in $qc + tS[t]$. Then

$$t^m c \in E_\Theta + S[t]A_+ + t^{m+1} S[t] \quad \text{and} \quad w \in E_\Theta + S[t]A_+ + t^{m+1} S[t],$$

whence the statement.

Turn to the second assertion. By (i), $\widehat{\mathcal{J}_*}$ is the sum of $\widehat{W}$ and $S[[t]]A_+$. An element of $\hat{W}$ is the sum of terms $t^m w_m$, with $m \in \mathbb{N}$ and $w_n \in W$. For $m$ big enough, $t^m w_m \in S[t]A_+$ by Lemma 4.5,(i). So $\widehat{\mathcal{J}_*}$ is the sum of $W$ and $S[[t]]A_+$, whence the assertion by the previous inclusion.

(iii) By definition, the sequence $(w_{k,i}, (k,i) \in \Theta)$ is graded. So it suffices to prove that it is free over $\Bbbk$. Let $(c_{k,i}, (k,i) \in \Theta)$ be a sequence in $\Bbbk$, with finite support, such that

$$\sum_{k \in K} \sum_{i=0}^{m_k - 1} c_{k,i} w_{k,i} = 0.$$

Let us prove that $c_{k,i} = 0$ for all $(k,i)$. Suppose $c_{k,i} \neq 0$ for some $(k,i)$. A contradiction is expected. Let $K'$ be the set of $k$ such that $c_{k,i} \neq 0$ for some $i$. Denote by $i_0$ the smallest integer such that $c_{k,i_0} \neq 0$ for some $k$ in $K'$ and set:

$$K_0' := \{k \in K' \mid c_{k,i_0} \neq 0\}.$$

Then

$$\sum_{k \in K_0'} c_{k,i_0} w_k(0) = 0,$$

whence the contradiction since the elements $(w_k(0), k \in K)$ are linearly independent.

(iv) Let $K_i$ be the set of $k$ such that $w_k$ is in $S[t]^{[i]}$. For such $k$, $w_k(0)$ is in $S^{[i]}$. Hence $K_i$ is finite since $S^{[i]}$ has finite dimension and since the elements $(w_k(0), k \in K)$ are linearly independent. For $k$ in $K$, $\Bbbk[t]w_k \cap E_\Theta$ has dimension $m_k$ by (iii). Hence $E_\Theta^{[i]}$ has finite dimension.

(v) Let $k$ be in $K_i$. Set

$$\Theta^{[i]} := \Theta \cap (K_i \times \mathbb{N}).$$

By Proposition 4.9,(iii), $t^{l+m_k} w_k$ is in $V_0'[[t]]A_+$ since $t^{m_k} w_k$ is in $S[t]A_+$ by definition, whence the assertion since $E_\Theta^{[i]}$ is generated by the finite sequence $(w_{k,j}, (k,j) \in \Theta^{[i]})$. □

**Definition 4.11.** We say that a subset $T$ of $\Theta$ is *complete* if

$$(k,i) \in T \Longrightarrow (k,j) \in T, \ \forall j \in \{0,\dots,i\}.$$

For $T$ subset of $\Theta$, denote by $K_T$ the image of $T$ by the projection $(k,i) \mapsto k$, and by $E_T$ the subspace of $E_\Theta$ generated by the elements $w_{k,i}$, $(k,i) \in T$. In particular, $K_\Theta = K$.

**Lemma 4.12.** *For some complete subset $T$ of $\Theta$ such that $K_T = K$, the subspace $E_T$ is a complement to $S[t]A_+$ in $E_\Theta + S[t]A_+$. In particular, the sum of $E_T$ and $S[t]A_+$ is direct.*

*Proof.* Since $V_0'' \cap SA(0)_+ = \{0\}$, the sum of $E_{K\times\{0\}}$ and $S[t]A_+$ is direct. Let $\mathcal{T}$ be the set of complete subsets $T$ of $\Theta$ verifying the following conditions:

(1) for all $k$ in $K$, $(k,0)$ is in $T$,
(2) the sum of $E_T$ and $S[t]A_+$ is direct.

Since the sum of $E_{K\times\{0\}}$ and $S[t]A_+$ is direct, $\mathcal{T}$ is not empty. If $(T_j, j \in \mathfrak{J})$ is an increasing sequence of elements of $\mathcal{T}$, with respect to the inclusion, its union is in $\mathcal{T}$. Then, by Zorn's Lemma, $\mathcal{T}$ has a maximal element. Denote it by $T_*$. It remains to prove that $w_{k,i}$ is in $E_{T_*} + S[t]A_+$ for all $(k,i)$ in $\Theta$.

Let $k$ be in $K$. Denote by $i$ the biggest integer such that $(k,i)$ is in $T_*$. Prove by induction on $i'$ that for $m_k > i' > i$, $w_{k,i'}$ is in $E_{T_*} + S[t]A_+$. By maximality of $T_*$ and $i$, $w_{k,i+1}$ is in $E_{T_*} + S[t]A_+$. Suppose that $w_{k,i'}$ is in $E_{T_*} + S[t]A_+$. Then, for some $a$ in $S[t]A_+$ and $c_{m,j}$, $(m,j) \in T_*$ in $\Bbbk$,

$$w_{k,i'} = \sum_{(m,j)\in T_*} c_{m,j} w_{m,j} + a,$$

whence

$$w_{k,i'+1} = \sum_{(m,j)\in T_*} c_{m,j} t^{j+1} w_m + ta.$$

By maximality of $T_*$, $t^{j+1}w_m$ is in $E_{T_*} + S[t]A_+$ for all $(m,j)$ such that $t^j w_m$ is in $T_*$. Hence $w_{k,i'+1}$ is in $E_{T_*} + S[t]A_+$. The lemma follows. □

Fix a complete subset $T_*$ of $\Theta$ such that

$$K_{T_*} = K \qquad \text{and} \qquad E_\Theta + S[t]A_+ = E_{T_*} \oplus S[t]A_+,$$

and set

$$E := E_{T_*}.$$

Such a set $T_*$ does exist by Lemma 4.12.

**Corollary 4.13.** (i) *The space $S[[t]]$ is the direct sum of $V_0'[[t]]$, $E$ and $S[[t]]A_+$.*
(ii) *The space $S[[t]]$ is the sum of $EA$ and $V_0'[[t]]A$.*

*Proof.* (i) According to Proposition 4.9,(ii), $S[[t]]$ is the direct sum of $V_0'[[t]]$ and $W + S[[t]]A_+$. By Lemma 4.10,(ii) (and its proof), $W + S[[t]]A_+$ is equal to $E_\Theta + S[[t]]A_+$. Since $E_\Theta + S[t]A_+$ is the direct sum of $E$ and $S[t]A_+$, we deduce that $W + S[[t]]A_+$ is the direct sum of $E$ and $S[[t]]A_+$. Hence, $S[[t]]$ is the direct sum of $V_0'[[t]]$, $E$ and $S[[t]]A_+$.

(ii) By (i) and by induction on $m$,

$$S[[t]] \subset V_0'[[t]]A + EA + S[[t]]A_+^m.$$

Hence $S[[t]]$ is the sum of $V_0'[[t]]A$ and $EA$ since $S[[t]]$ is graded and $A_+$ is generated by elements of positive degree. □

**Definition 4.14.** For $k$ in $K$, denote by $\nu_k$ the degree of $w_k$. For $T$ and $T'$ subsets of $\Theta$, we say that $T$ *is smaller than* $T'$, and we denote $T \prec T'$, if the following conditions are satisfied:

(1) $T$ is contained in $T'$
(2) if for $k$ in $K_T$ and $k'$ in $K_{T'}$, we have $\nu_{k'} < \nu_k$, then $k'$ is in $K_T$.

Let $\mu$ be the linear map

$$E \otimes_{\Bbbk} A \oplus V_0'[[t]] \otimes_{\Bbbk} A \longrightarrow S[[t]], \qquad w\otimes a + v\otimes b \longmapsto wa + vb.$$

For $T$ a subset of $T_*$, denote by $\mu_T$ the restriction of $\mu$ to the subspace

$$E_T \otimes_{\Bbbk} A \oplus V_0'[[t]] \otimes_{\Bbbk} A.$$

**Lemma 4.15.** *Let $\mathcal{T}_*$ be the set of subsets $T$ of $T_*$ such that $\mu_T$ is injective.*
(i) *The set $\mathcal{T}_*$ is not empty.*
(ii) *The set $\mathcal{T}_*$ has a maximal element with respect to the order $\prec$.*
(iii) *The set $T_*$ is in $\mathcal{T}_*$.*

*Proof.* (i) For $k$ in $K$, set $T_k := \{(k, 0)\}$. Suppose that $T_k$ is not in $\mathcal{T}_*$. A contradiction is expected. Then for some $a$ in $A \setminus \{0\}$, $w_k a$ is in $V_0'[[t]]A_+$, whence

$$w_k a = \sum_{i\in J} v_i b_i$$

with $(b_i, i \in J)$ in $\Bbbk[[t]]A_+$ with finite support. By Lemma 4.10,(v), for some positive integer, $t^l w_k$ is in $V_0'[[t]]A_+$. Then

$$t^l w_k = \sum_{i\in J} v_i c_i$$

with $(c_i, i \in J)$ in $\Bbbk[[t]]A_+$ with finite support. Hence

$$\sum_{i\in J} v_i t^l b_i = \sum_{i\in J} v_i c_i a.$$

According to Proposition 4.9,(iii), $t^l b_i = c_i a$ for all $i$. Since $a \neq 0$, $a(0) \neq 0$ by Proposition 4.1,(ii). Then, by Lemma 4.2,(ii), $c_i = t^l c_i'$ for some $c_i'$ in $\tilde{A} = \Bbbk[[t]]A$. As a result,

$$w_k = \sum_{i\in J} v_i c_i',$$

whence the contradiction by Corollary 4.13,(i).

(ii) Let $(T_l, l \in L)$ be a net in $\mathcal{T}_*$ with respect to $\prec$. Let $T$ be the union of the sets $T_l$, $l \in L$. Since $E_T$ is the space generated by the subspaces $E_{T_l}, l \in L$, the map $\mu_T$ is injective. Let $l_0$ be in $L$ and $k$ in $K_T$ such that $\nu_k < \nu_{k'}$ for some $k'$ in $K_{T_{l_0}}$. Since $K_T$ is the union of the sets $K_{T_l}$, $l \in L$, we deduce that $k$ is in $K_{T_l}$ for some $l$ in $L$. By properties of the nets, for some $l'$ in $L$, $T_l \prec T_{l'}$ and $T_{l_0} \prec T_{l'}$ so that $k$ is in $K_{T_{l'}}$. Hence, $k$ is in $K_{T_{l_0}}$, whence $T_{l_0} \prec T$. As a result, $\prec$ is an inductive order in $\mathcal{T}_*$, and by Zorn's Theorem, it has a maximal element.

(iii) Let $T$ be a maximal element of $\mathcal{T}_*$ with respect to $\prec$. Suppose $T$ strictly contained in $T_*$. A contradiction is expected. Let $k$ be in $K$ such that $(k, i)$ is not in $T$ and $(k, i)$ is in $T_*$ for some $i$. We can suppose that $\nu_k$ is minimal under this condition. Let $i_*$ be the smallest integer such that $(k, i_*)$ is not in $T$ and $(k, i_*)$ is in $T_*$. Then $T \prec T \cup \{(k, i_*)\}$. So, by the maximality of $T$, for some $a$ in $A \setminus \{0\}$,

$$w_{k,i_*} a \in E_T A + V_0'[[t]]A.$$

Since $E_T$, $V_0'[[t]]$, $A$, $w_{k,i_*}$ are graded, we can suppose that $a$ is homogeneous. Then $a$ has positive degree. Otherwise, $w_{k,i_*} \in E_T A + V_0'[[t]]A \subset E_T + V_0'[[t]] + S[[t]]A_+$, and we deduce from Corollary 4.13,(i), that $w_{k,i_*} \in E_T$ since $w_{k,i_*} \in E_{T_*}$. This is impossible by the choice of $(k, i_*)$. Thus, by Corollary 4.13,(ii),

$$w_{k,i_*} a \in E_T A_+ + V_0'[[t]]A_+.$$

Hence

$$w_{k,i_*} a = \sum_{(n,j)\in T} w_{n,j} a_{n,j} + \sum_{i\in J} v_i b_i$$

with $(a_{n,j}, (n, j) \in T)$ in $A_+$ and $(b_i, i \in J)$ in $\tilde{A}_+$ with finite support.

By Corollary 4.13,(ii),

$$t^{m_k} w_k = \sum_{(l,s)\in T_*} w_{l,s} a_{l,s,k} + \sum_{i\in J} v_i b_{i,k}$$

with $(a_{l,s,k}, (l, s) \in T_*)$ in $A_+$ and $(b_{i,k}, i \in J)$ in $\tilde{A}_+$ with finite support. Moreover these two sequences are graded, so that $a_{l,s,k} = 0$ if $\nu_l \geqslant \nu_k$. By minimality of $\nu_k$, $(l, s)$ is in $T$ if $a_{l,s,k} \neq 0$. For $(n, j)$ in $T$ such that $m_k - i_* + j \geqslant m_n$,

$$t^{m_k - i_*} w_{n,j} = \sum_{(l,s)\in T_*} w_{l,s} a_{l,s,n,j} + \sum_{i\in J} v_i b_{i,n,j}$$

with $(a_{l,s,n,j}, (l, s) \in T_*)$ in $A_+$ and $(b_{i,n,j}, i \in J)$ in $\tilde{A}_+$ with finite support. Moreover these two sequences are graded, so that $a_{l,s,n,j} = 0$ if $\nu_l \geqslant \nu_n$. So, by minimality of $\nu_k$, $(l, s)$ is in $T$ if $a_{l,s,n,j} \neq 0$ and $\nu_n \leqslant \nu_k$. As a result,

$$\begin{aligned}
\sum_{(l,s)\in T} w_{l,s} a_{l,s,k} a + \sum_{i\in J} v_i b_{i,k} a &= \sum_{(n,j)\in T} w_{n,j} t^{m_k - i_*} a_{n,j} + \sum_{i\in J} v_i t^{m_k - i_*} b_i \\
&= \sum_{\substack{(n,j)\in T \\ m_k - i_* + j < m_n}} w_{n, m_k - i_* + j} a_{n,j} + \sum_{\substack{(n,j)\in T \\ m_k - i_* + j \geqslant m_n}} w_{l,s} a_{l,s,n,j} a_{n,j} \\
&\quad + \sum_{i\in J} v_i t^{m_k - i^*} b_i + \sum_{\substack{(n,j)\in T \\ m_k - i_* + j \geqslant m_n}} \sum_{i\in J} v_i b_{i,n,j} a_{n,j}
\end{aligned}$$

whence

$$\begin{aligned}
\sum_{(l,s)\in T} w_{l,s} a_{l,s,k} a + \sum_{i\in J} v_i b_{i,k} a &= \sum_{\substack{(n,j)\in T \\ m_k - i_* + j < m_n}} w_{n, m_k - i_* + j} a_{n,j} + \sum_{\substack{(n,j)\in T \\ m_k - i_* + j \geqslant m_n}} \sum_{(l,s)\in T} w_{l,s} a_{l,s,n,j} a_{n,j} \\
&\quad + \sum_{i\in J} v_i (t^{m_k - i_*} b_i + \sum_{\substack{(n,j)\in T \\ m_k - i_* + j \geqslant m_n}} b_{i,n,j} a_{n,j}).
\end{aligned}$$

Since $\mu_T$ is injective, for all $i$ in $J$,

$$t^{m_k - i_*} b_i + \sum_{\substack{(n,j)\in T \\ m_k - i_* + j \geqslant m_n}} b_{i,n,j} a_{n,j} - b_{i,k} a = 0, \tag{1}$$

and for all $(l, s)$ in $T$,

$$a_{l,s+i_*-m_k} + \sum_{\substack{(n,j)\in T \\ m_k - i_* + j \geqslant m_n}} a_{n,j} a_{l,s,n,j} - a_{l,s,k} a = 0. \tag{2}$$

with $a_{l,s} = 0$ if $s < 0$.

*Claim* 4.16. For all $(l, s)$ in $T$, $a$ divides $a_{l,s}$ in $A$.

*Proof of Claim 4.16.* Prove the claim by induction on $\nu_l$. Let $l$ be in $K_T$ such that

$$\nu_{l'} > \nu_l \quad \text{and} \quad (l', s') \in T \Longrightarrow a_{l',s'} = 0.$$

Then by Equality (2), $a_{l,s+i_*-m_k} = a_{l,s,k}a$, whence the satement for $l$. Suppose that $a$ divides $a_{l',s'}$ in $A$ for all $(l', s')$ in $T$ such that $\nu_{l'} > \nu_l$. By Equality (2) and the induction hypothesis, $a$ divides $a_{l,s+i_*-m_k}$ in $A$ since $a_{l,s,n,j} = 0$ for $\nu_n \leqslant \nu_l$, whence the claim. □

By Claim 4.16 and Equality (1), for all $i$ in $J$, $a$ divides $t^{m_k-i_*}b_i$ in $\Bbbk[[t]]A$. Since $a$ has positive degree, all prime divisor of $a$ in $A$ has positive degree and does not divide $t$ since $t$ has degree 0. Then, by Lemma 4.2,(iii), $a$ divides $b_i$ in $\Bbbk[[t]]A$. As a result,

$$w_{k,i_*} \in E_T A + V_0'[[t]]A$$

whence

$$w_{k,i_*} \in V_0'[[t]] + E_T + S[[t]]A_+.$$

Since $w_{k,i_*}$ is in $E$, $w_{k,i_*}$ is in $E_T$ by Corollary 4.13,(i). We get a contradiction because $(k, i_*)$ is not in $T$. □

**Corollary 4.17.** *The canonical map*

$$E \otimes_{\Bbbk} A \oplus V_0'[[t]] \otimes_{\Bbbk} A \longrightarrow S[[t]]$$

*is an isomorphism. In particular, $S[[t]]$ is a free extension of $A$.*

*Proof.* By Lemma 4.15, $T_*$ is the biggest element of $\mathcal{T}_*$. Hence $\mu$ is injective. Then, by Corollary 4.13,(ii), $\mu$ is bijective. As a matter of fact, $\mu$ is an isomorphism of $A$-modules, whence the corollary. □

4.5. Recall that $\tilde{A}$ is the subalgebra of $S[[t]]$ generated by $\Bbbk[[t]]$ and $A$. Our next aim is to show that $S[[t]]$ is a free extension of $\tilde{A}$ (cf. Theorem 4.21). Theorem 3.6 will then follows.

For $I$ an ideal of $\tilde{A}$, denote by $\sigma_I$ and $\nu_I$ the canonical morphisms

$$S[[t]] \otimes_A I \xrightarrow{\sigma_I} S[[t]] \otimes_A \tilde{A} \qquad\qquad S[[t]] \otimes_{\tilde{A}} I \xrightarrow{\nu_I} S[[t]]I\ .$$

Consider on $S[[t]]\otimes_A I$ and $S[[t]]\otimes_{\tilde{A}} I$ the linear topologies such that $\{t^n(S[[t]]\otimes_A I)\}_{n\in\mathbb{N}}$ and $\{t^n(S[[t]]\otimes_{\tilde{A}} I)\}_{n\in\mathbb{N}}$ are systems of neighborhood of 0 in these $S[[t]]$-modules. Denote by $\varphi_I$ the canonical morphism

$$S[[t]] \otimes_A I \xrightarrow{\varphi_I} S[[t]] \otimes_{\tilde{A}} I$$

and by $\mathcal{K}_I$ its kernel. Then $\varphi_I$ is continuous with respect to the above topologies.

**Lemma 4.18.** *Let $I$ be an ideal of $\tilde{A}$.*

(i) *The morphism $\sigma_I$ is injective.*

(iii) *The module $\mathcal{K}_I$ is the $S[[t]]$-submodule of $S[[t]] \otimes_A I$ generated by the elements $r\otimes a - 1\otimes ra$ with $r$ in $\Bbbk[[t]]$ and $a$ in $I$.*

*Proof.* (i) According to Corollary 4.17, $S[[t]]$ is a flat extension of $A$. The assertion follows since $I$ is contained in $\tilde{A}$.

(ii) Let $\mathcal{K}_I'$ the $S[[t]]$-submodule of $S[[t]] \otimes_A I$ generated by the elements $r\otimes a - 1\otimes ra$ with $r$ in $\Bbbk[[t]]$ and $a$ in $I$. Then $\mathcal{K}_I'$ is contained in $\mathcal{K}_I$. Let us prove the other inclusion.

Let $(x, y)$ be in $S[[t]] \times I$ and let $a$ be in $\tilde{A}$. According to (i), $a$ has an expansion

$$a = \sum_{i=1} r_i a_i$$

with $r_1, \dots, r_m$ in $\Bbbk[[t]]$ and $a_1, \dots, a_m$ in $A$. Then, in $S[[t]] \otimes_A I$,

$$x \otimes ay - ax \otimes y = \sum_{i=1}^{m} x \otimes r_i a_i y - r_i x \otimes a_i y = \sum_{i=1}^{m} x(1 \otimes r_i a_i y - r_i \otimes a_i y) \in \mathcal{K}'_I.$$

As a result, $\mathcal{K}_I = \mathcal{K}'_I$ since $\mathcal{K}_I$ is the S[[t]]-submodule of $S[[t]] \otimes_A I$ generated by the $xa \otimes y - x \otimes ay$'s. □

**Corollary 4.19.** *Let $I$ be an ideal of $\tilde{A}$. The module $\mathcal{K}_I$ is the closure of the $S[[t]]$-submodule of $S[[t]] \otimes_A I$ generated by the set $\{t \otimes a - 1 \otimes ta\}_{a \in I}$.*

*Proof.* Let $\mathcal{L}_I$ the $S[[t]]$-submodule generated by the set $\{t \otimes a - 1 \otimes ta\}_{a \in I}$. Prove by induction on $n$ that $t^n \otimes a - 1 \otimes t^n a$ is in $\mathcal{L}_I$ for all $a$ in $I$. The statement is straightforward for $n = 0, 1$. Suppose $n \geqslant 2$ and the statement true for $n - 1$. For $a$ in $I$,

$$t^n a - 1 \otimes t^n a = t^{n-1}(t \otimes a - 1 \otimes ta) + t^{n-1} \otimes ta - 1 \otimes t^{n-1} ta.$$

By induction hypothesis, $t^{n-1} \otimes ta - 1 \otimes t^{n-1} ta$ is in $\mathcal{L}_I$, whence $t^n \otimes a - 1 \otimes t^n a$ is in $\mathcal{L}_I$. As a result, for $r$ in $\Bbbk[t]$, $r \otimes a - 1 \otimes ra$ is in $\mathcal{L}_I$. So, for $r$ in $\Bbbk[[t]]$, $r \otimes a - 1 \otimes ra$ is in the closure of $\mathcal{L}_I$ in $S[[t]] \otimes_A I$. Since $\varphi_I$ is continuous, $\mathcal{K}_I$ is a closed submodule of $S[[t]] \otimes_A I$, whence the corollary by Lemma 4.18,(iii). □

**Proposition 4.20.** *Let $I$ be an ideal of $\tilde{A}$.*

(i) *The canonical morphism*

$$V'_0 \tilde{A} \otimes_{\tilde{A}} I \longrightarrow S[[t]] \otimes_{\tilde{A}} I$$

*is an embedding.*

(ii) *For the structure of $S[[t]]$-module on $S[[t]] \otimes_{\tilde{A}} I$, $t$ is not a divisor of $0$ in $S[[t]] \otimes_{\tilde{A}} I$.*

*Proof.* (i) We have the commutative diagram

$$\begin{array}{ccc} V'_0 \tilde{A} \otimes_{\tilde{A}} I & \xrightarrow{\mathrm{d}} & S[[t]] \otimes_{\tilde{A}} I \\ \downarrow \delta & & \downarrow \delta \\ V'_0 I & \xrightarrow{\mathrm{d}} & S[[t]] I \end{array}$$

with canonical arrows d and $\delta$. According to Proposition 4.9,(iii), the left down arrow $\delta$ is an isomorphism. Let $a$ be in $V'_0 \tilde{A} \otimes_{\tilde{A}} I$ such that $\mathrm{d}a = 0$. Then $\mathrm{d} \circ \delta a = 0$, whence $\delta a = 0$ since the bottom horizontal arrow d is an embedding so that $a = 0$.

(ii) Let $a$ be in $S[[t]] \otimes_A I$ such that $t\varphi_I(a) = 0$. According to Corollary 4.19, for $l$ in $\mathbb{N}$ such that $l \geqslant 2$,

$$ta - \sum_{i=1}^{m} b_i(t \otimes a_i - 1 \otimes ta_i) \in t^l S[[t]] \otimes_A I$$

for some $b_1, \dots, b_m$ in $S[[t]]$ and for some $a_1, \dots, a_m$ in $I$. For $i = 1, \dots, m$,

$$b_i = b_{i,0} + tb'_i$$

with $b_{i,0}$ in $S$ and $b'_i$ in $S[[t]]$, whence

$$t(a - \sum_{i=1}^{m} b'_i(t \otimes a_i - 1 \otimes ta_i)) - \sum_{i=1}^{m} b_{i,0}(t \otimes a_i - 1 - \otimes ta_i) \in t^l S[[t]] \otimes_A I.$$

Set:

$$a' := a - \sum_{i=1}^{m} b'_i(t \otimes a_i - 1 \otimes ta_i) \quad \text{and} \quad a'' = \sum_{i=1}^{m} b_{i,0}(t \otimes a_i - 1 \otimes ta_i).$$

Then $\varphi_I(a) = \varphi_I(a')$ and by Lemma 4.18,(i), $\sigma_I(a'')$ is in $tS[[t]] \otimes_{\Bbbk} \Bbbk[[t]]$. Moreover, for $i = 1, \dots, m$, $a_i$ has a unique expansion

$$a_i = \sum_{n\in\mathbb{N}} t^n a_{i,n}$$

with $a_{i,n}$, $n \in \mathbb{N}$, in $A$. Then

$$\begin{aligned}\sigma_I(a'') &= \sum_{i=1}^{m} b_{i,0}(\sum_{n\in\mathbb{N}} ta_{i,n}\otimes t^n - a_{i,n}\otimes t^{n+1}) \\ &= t\sum_{i=1}^{m} a_{i,0}b_{i,0}\otimes 1 + \sum_{n\in\mathbb{N}^*}\sum_{i=1}^{m} b_{i,0}(ta_{i,n} - a_{i,n-1})\otimes t^n.\end{aligned}$$

Since the right hand side is divisible by $t$ in $S[[t]] \otimes_{\Bbbk} \Bbbk[[t]]$, for all positive integer $n$,

$$\sum_{i=1}^{m} b_{i,0}a_{i,n-1} = 0$$

since $b_{i,0}$ and $a_{i,n-1}$ are in $S$ for all $i$. Hence $\sigma_I(a'') = 0$ and $a'' = 0$ by Lemma 4.18,(ii). Thus,

$$a' \in t^{l-1}S[[t]] \otimes_A I.$$

As a result, $\varphi_I(a)$ is in $t^l S[[t]] \otimes_{\tilde{A}} I$ for all positive integer $l$. Since the $S[[t]]$-module $S[[t]] \otimes_{\tilde{A}} I$ is finitely generated, by a Krull's theorem [Ma86, Ch. 3, Theoreom 8.9], for some $b$ in $S[[t]]$, $(1 + tb)\varphi_I(a) = 0$, whence $\varphi_I(a) = 0$ since $t\varphi_I(a) = 0$. □

Remind that $\mathcal{V}_0$ is the nullvariety of $A(0)_+$ in $\mathfrak{g}^f$, and that $\tilde{A} = \Bbbk[[t]]A$.

**Theorem 4.21.** (i) *The algebra $S[[t]]$ is a free extension of $\tilde{A}$.*

(ii) *The varieties $\mathcal{V}$ and $\mathcal{V}_*$ are equal. Moreover, $\mathcal{V}_0$ is equidimensional of dimension $r - \ell$.*

(iii) *The $A(0)$-module $S$ is free and $V_0 = V_0'$. In particular, the canonical morphism*

$$V_0 \otimes_{\Bbbk} A(0) \longrightarrow S, \quad v\otimes a \longmapsto va$$

*is an isomorphism.*

*Proof.* (i) First of all, prove that $S[[t]]$ is a flat extension of $\tilde{A}$. Then the freeness of the extension will result from the equality $V_0 = V_0'$, Lemma 4.6 and Proposition 4.9,(iii).

By the criterion of flatness [Ma86, Ch. 3, Theorem 7.7], it is equivalent to say that for all ideal $I$ of $\tilde{A}$, the canonical morphism $\nu_I$,

$$S[[t]] \otimes_{\tilde{A}} I \longrightarrow S[[t]]I$$

is injective. Let $a$ be in the kernel of $\nu_I$. Consider the commutative diagram

$$\begin{array}{ccc} V_0'\tilde{A} \otimes_{\tilde{A}} I & \xrightarrow{\mathrm{d}} & S[[t]] \otimes_{\tilde{A}} I \\ \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} \\ V_0'I & \xrightarrow{\mathrm{d}} & S[[t]]I \end{array}$$

of the proof of Proposition 4.20,(i). According to Lemma 4.10,(v), for $l$ sufficiently big, $t^l a = \mathrm{d}b$ for some $b$ in $V_0'\tilde{A} \otimes_{\tilde{A}} I$. Then $\delta b = 0$ since $\nu_I(t^l a) = 0$. By Proposition 4.9,(iii), $\delta$ is an isomorphism. Hence $b = 0$ and $t^l a = 0$. Then, by Proposition 4.20,(ii), $a = 0$, whence the the flatness.

(ii) Denote by $\Bbbk[t]_0$ the localization of $\Bbbk[t]$ at $t\Bbbk[t]$. Then $\Bbbk[[t]]$ is a faithfully flat extension of $\Bbbk[t]_0$. Hence, $S[[t]]$ is a faithfully flat extension of

$$S[t]_0 := \Bbbk[t]_0 \otimes_{\Bbbk[t]} S.$$

Set

$$\tilde{A}_0 := \Bbbk[t]_0 \otimes_{\Bbbk} A.$$

Then

$$\tilde{A} = \Bbbk[[t]] \otimes_{\Bbbk[t]_0} \tilde{A}_0$$

so that $\tilde{A}$ is faithfully flat extension of $\tilde{A}_0$. For $M$ a $\tilde{A}_0$-module, we have

$$\Bbbk[[t]] \otimes_{\Bbbk[t]_0} (S[t]_0 \otimes_{\tilde{A}_0} M) = (\Bbbk[[t]] \otimes_{\Bbbk[t]_0} S[t]_0) \otimes_{\tilde{A}_0} M = S[[t]] \otimes_{\tilde{A}} (\tilde{A} \otimes_{\tilde{A}_0} M).$$

Hence, $S[t]_0$ is a flat extension of $\tilde{A}_0$ since so is the extension $S[[t]]$ of $\tilde{A}$.

The variety $\mathcal{V}$ is the union of $\mathcal{V}_*$ and $\mathcal{V}_0 \times \{0\}$. Moreover $\mathcal{V}_0 \times \{0\}$ is the nullvariety in $\mathfrak{g}^f \times \Bbbk$ of the ideal of $\Bbbk[t]A$ generated by $t$ and $A_+$. Then, by [Ma86, Ch. 5, Theorem 15.1], $\mathcal{V}_0$ is equidimensional of dimension $r-\ell$ since $S[t]_0$ is a flat extension of $\tilde{A}_0$ by (i) and since $\tilde{A}_0$ has dimension $\ell+1$. Since $\mathcal{V}$ is the nullvariety of $\ell$ functions, all irreducible component of $\mathcal{V}$ has dimension at least $r+1-\ell$ by [Ma86, Ch. 5, Theorem 13.5]. Hence any irreducible component of $\mathcal{V}_0 \times \{0\}$ is not an irreducible component of $\mathcal{V}$. As a result, $\mathcal{V}_0 \times \{0\}$ is contained in $\mathcal{V}_*$ and so $\mathcal{V} = \mathcal{V}_*$.

(iii) Since $A(0)$ is a polynomial algebra, $S$ is a free extension of $A(0)$ by (ii) and Proposition 2.5. Moreover, by Lemma 2.2, the linear map

$$V_0 \otimes_{\Bbbk} A(0) \longrightarrow S, \qquad v \otimes a \longmapsto va$$

is a graded isomorphism with respect to the gradation of $V_0 \otimes_{\Bbbk} A(0)$ induced by those of $V_0$ and $A(0)$. As a result, for all nonnegative integer $i$,

$$\dim S^{[i]} = \sum_{j=0}^{i} \dim V_0^{[i-j]} \times \dim A(0)^{[j]},$$

whence $\dim V_0^{[i]} = \dim V_0'^{[i]}$ for all $i$ by Proposition 4.9,(iv) since $\dim A^{[i]} = \dim A(0)^{[i]}$ for all $i$ by Proposition 4.1,(ii). Then $V_0 = V_0'$. □

As explained in Subsection 3.2, by Theorem 3.3 and Proposition 2.5,(ii), Theorem 3.6 results from Theorem 4.21,(ii).

*Remark* 4.22. According to the part (ii) of Theorem 4.21, $\mathcal{J}_*$ is the radical of $S[t]A_+$. Hence $S[t]A_+$ is radical by Lemma 4.5,(ii), and then $\mathcal{J}_* = S[t]A_+$.

## 5. Consequences of Theorem 1.4 for the simple classical Lie algebras

This section concerns some applications of Theorem 1.4 to the simple classical Lie algebras.

5.1. The first consequence of Theorem 3.6 is the following.

**Theorem 5.1.** *Assume that $\mathfrak{g}$ is simple of type* **A** *or* **C**. *Then all the elements of $\mathfrak{g}$ are good.*

*Proof.* This follows from [PPY07, Theorems 4.2 and 4.4], Theorem 3.6 and Proposition 3.5. □

5.2. In this subsection and the next one, $\mathfrak{g}$ is assumed to be simple of type **B** or **D**. More precisely, we assume that $\mathfrak{g}$ is the simple Lie algebra $\mathfrak{so}(\mathbb{V})$ for some vector space $\mathbb{V}$ of dimension $2\ell+1$ or $2\ell$. Then $\mathfrak{g}$ is embedded into $\tilde{\mathfrak{g}} := \mathfrak{gl}(\mathbb{V}) = \mathrm{End}(\mathbb{V})$. For $x$ an endomorphism of $\mathbb{V}$ and for $i \in \{1, \dots, \dim \mathbb{V}\}$, denote by $Q_i(x)$ the coefficient of degree $\dim \mathbb{V} - i$ of the characteristic polynomial of $x$. Then, for any $x$ in $\mathfrak{g}$, $Q_i(x) = 0$ whenever $i$ is odd. Define a generating family $(q_1, \dots, q_\ell)$ of the algebra $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ as follows. For $i = 1, \dots, \ell-1$, set $q_i := Q_{2i}$. If $\dim \mathbb{V} = 2\ell+1$, set $q_\ell = Q_{2\ell}$ and if $\dim \mathbb{V} = 2\ell$, let $q_\ell$ be a homogeneous element of degree $\ell$ of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ such that $Q_{2\ell} = q_\ell^2$.

Let $(e, h, f)$ be an $\mathfrak{sl}_2$-triple of $\mathfrak{g}$. Following the notations of Subsection 3.2, for $i \in \{1, \dots, \ell\}$, denote by ${}^e q_i$ the initial homogeneous component of the restriction to $\mathfrak{g}^f$ of the polynomial function $x \mapsto q_i(e+x)$, and by $\delta_i$ the degree of ${}^e q_i$. According to [PPY07, Theorem 2.1], ${}^e q_1, \dots, {}^e q_\ell$ are algebraically independent if and only if

$$\dim \mathfrak{g}^e + \ell - 2(\delta_1 + \cdots + \delta_\ell) = 0.$$

Our first aim in this subsection is to describe the sum $\dim \mathfrak{g}^e + \ell - 2(\delta_1 + \cdots + \delta_\ell)$ in term of the partition of $\dim \mathbb{V}$ associated with $e$.

*Remark* 5.2. The sequence of the degrees $(\delta_1, \dots, \delta_\ell)$ is described by [PPY07, Remark 4.2].

For $\lambda = (\lambda_1, \dots, \lambda_k)$ a sequence of positive integers, with $\lambda_1 \geqslant \cdots \geqslant \lambda_k$, set:

$$|\lambda| := k, \qquad r(\lambda) := \lambda_1 + \cdots + \lambda_k.$$

Assume that the partition $\lambda$ of $r(\lambda)$ is associated with a nilpotent orbit of $\mathfrak{so}(\Bbbk^{r(\lambda)})$. Then the even integers of $\lambda$ have an even multiplicity, [CMc93, §5.1]. Thus $k$ and $r(\lambda)$ have the same parity. Moreover, there is an involution $i \mapsto i'$ of $\{1, \dots, k\}$ such that $i = i'$ if $\lambda_i$ is odd, and $i' \in \{i-1, i+1\}$ if $\lambda_i$ is even. Set:

$$S(\lambda) := \sum_{i=i',\, i \text{ odd}} i - \sum_{i=i',\, i \text{ even}} i$$

and denote by $n_\lambda$ the number of even integers in the sequence $\lambda$.

From now on, assume that $\lambda$ is the partition of $\dim \mathbb{V}$ associated with the nilpotent orbit $G.e$.

**Lemma 5.3.** (i) *If* $\dim \mathbb{V}$ *is odd, i.e., $k$ is odd, then*

$$\dim \mathfrak{g}^e + \ell - 2(\delta_1 + \cdots + \delta_\ell) = \frac{n_\lambda - k - 1}{2} + S(\lambda).$$

(ii) *If* $\dim \mathbb{V}$ *is even, i.e., $k$ is even, then*

$$\dim \mathfrak{g}^e + \ell - 2(\delta_1 + \cdots + \delta_\ell) = \frac{n_\lambda + k}{2} + S(\lambda).$$

*Proof.* (i) If $\dim \mathbb{V}$ is odd, then by [PPY07, §4.4, (14)],

$$2(\delta_1 + \cdots + \delta_\ell) = \dim \mathfrak{g}^e + \frac{\dim \mathbb{V}}{2} + \frac{k - n_\lambda}{2} - S(\lambda),$$

whence

$$\dim \mathfrak{g}^e + \ell - 2(\delta_1 + \cdots + \delta_\ell) = \frac{n_\lambda - k - 1}{2} + S(\lambda)$$

since $\dim \mathbb{V} = 2\ell + 1$.

(ii) If $\dim \mathbb{V}$ is even, then $\delta_\ell = k/2$ by [PPY07, Remark 4.2] and by [PPY07, §4.4, (14)],

$$2(\delta_1 + \cdots + \delta_\ell) + k = \dim \mathfrak{g}^e + \frac{\dim \mathbb{V}}{2} + \frac{k - n_\lambda}{2} - S(\lambda)$$

whence

$$\dim \mathfrak{g}^e + \ell - 2(\delta_1 + \cdots + \delta_\ell) = \frac{n_\lambda + k}{2} + S(\lambda)$$

since $\dim \mathbb{V} = 2\ell$. $\square$

The sequence $\lambda = (\lambda_1, \dots, \lambda_k)$ verifies one of the following five conditions:

1) $\lambda_k$ and $\lambda_{k-1}$ are odd,
2) $\lambda_k$ and $\lambda_{k-1}$ are even,
3) $k > 3$, $\lambda_k$ and $\lambda_1$ are odd and $\lambda_i$ is even for any $i \in \{2, \dots, k-1\}$,
4) $k > 4$, $\lambda_k$ is odd and there is $k' \in \{2, \dots, k-2\}$ such that $\lambda_{k'}$ is odd and $\lambda_i$ is even for any $i \in \{k'+1, \dots, k-1\}$,
5) $k = 1$ or $\lambda_k$ is odd and $\lambda_i$ is even for any $i < k$.

For example, $(4, 4, 3, 1)$ verifies Condition (1); $(6, 6, 5, 4, 4)$ verifies Condition (2); $(7, 6, 6, 4, 4, 4, 4, 3)$ verifies Condition (3); $(8, 8, 7, 5, 4, 4, 2, 2, 3)$ verifies Condition (4) with $k' = 4$; (9) and $(6, 6, 4, 4, 3)$ verify Condition (5). If $k = 2$, then one of the conditions (1) or (2) is satisfied.

**Definition 5.4.** Define a sequence $\lambda^*$ of positive integers, with $|\lambda^*| \leqslant |\lambda|$, as follows:

- if $k = 2$ or if Condition (3) or (5) is verified, then set $\lambda^* := \lambda$,
- if Condition (1) or (2) is verified, then set:
$$\lambda^* := (\lambda_1, \dots, \lambda_{k-2}),$$
- if $k > 3$ and if the Condition (4) is verified, then set
$$\lambda^* := (\lambda_1, \dots, \lambda_{k'-1}).$$

In any case, the partition of $r(\lambda^*)$ corresponding to $\lambda^*$ is associated with a nilpotent orbit of $\mathfrak{so}(\mathbb{k}^{r(\lambda^*)})$. Recall that $n_\lambda$ is the number of even integers in the sequence $\lambda$.

**Definition 5.5.** Denote by $d_\lambda$ the integer defined by:

- if $k = 2$, then $d_\lambda := n_\lambda$,
- if $k > 2$ and if Condition (1) or (4) is verified, then $d_\lambda := d_{\lambda^*}$,
- if $k > 2$ and if Condition (2) is verified, then $d_\lambda := d_{\lambda^*} + 2$,
- if $k > 2$ and if Condition (3) is verified, then $d_\lambda := 0$,
- if Condition (5) is verified, then $d_\lambda := 0$.

**Lemma 5.6.** (i) *Assume that k is odd. If Condition* (1), (2) *or* (5) *is verified, then*
$$\frac{n_\lambda - k - 1}{2} + S(\lambda) = \frac{n_{\lambda^*} - |\lambda^*| - 1}{2} + S(\lambda^*).$$
*Otherwise,*
$$\frac{n_\lambda - k - 1}{2} + S(\lambda) = \frac{n_{\lambda^*} - |\lambda^*| - 1}{2} + S(\lambda^*) + k - |\lambda^*| - 2.$$
(ii) *If k is even, then*
$$\frac{n_\lambda + k}{2} + S(\lambda) = \frac{n_{\lambda^*} + |\lambda^*|}{2} + S(\lambda^*) + d_\lambda - d_{\lambda^*}.$$

*Proof.* (i) If Condition (3) or (5) is verified, there is nothing to prove. If Condition (1) is verified,
$$n_\lambda = n_{\lambda^*}, \qquad S(\lambda) = S(\lambda^*) + 1.$$
Then
$$\frac{n_\lambda - k - 1}{2} + S(\lambda) = \frac{n_{\lambda^*} - |\lambda^*| - 1}{2} - 1 + S(\lambda^*) + 1$$
whence the assertion. If Condition (2) is verified,
$$n_\lambda = n_{\lambda^*} + 2, \qquad S(\lambda) = S(\lambda^*).$$

Then,

$$\frac{n_\lambda - k - 1}{2} + S(\lambda) = \frac{n_{\lambda^*} - |\lambda^*| - 1}{2} + S(\lambda^*)$$

whence the assertion. If Condition (4) is verified,

$$n_\lambda = n_{\lambda^*} + k - |\lambda^*| - 2, \qquad S(\lambda) = S(\lambda^*) + k - |\lambda^*| - 1.$$

Then,

$$\frac{n_\lambda - k - 1}{2} + S(\lambda) = \frac{n_{\lambda^*} - |\lambda^*| - 1}{2} - 1 + S(\lambda^*) + k - |\lambda^*| - 1$$

whence the assertion.

(ii) If $k = 2$ or if $k > 2$ and Condition (3) or (5) is verified, there is nothing to prove. Let us suppose that $k > 3$. If Condition (1) is verified,

$$n_\lambda = n_{\lambda^*}, \qquad S(\lambda) = S(\lambda^*) - 1.$$

Then

$$\frac{n_\lambda + k}{2} + S(\lambda) = \frac{n_{\lambda^*} + |\lambda^*|}{2} + 1 + S(\lambda^*) - 1$$

whence the assertion since $d_\lambda = d_{\lambda^*}$. If Condition (2) is verified,

$$n_\lambda = n_{\lambda^*} + 2, \qquad S(\lambda) = S(\lambda^*).$$

Then,

$$\frac{n_\lambda + k}{2} + S(\lambda) = \frac{n_{\lambda^*} + |\lambda^*|}{2} + 2 + S(\lambda^*)$$

whence the assertion since $d_\lambda - d_{\lambda^*} = 2$. If Condition (4) is verified,

$$n_\lambda = n_{\lambda^*} + k - |\lambda^*| - 2, \qquad S(\lambda) = S(\lambda^*) + |\lambda^*| + 1 - k.$$

Then,

$$\frac{n_\lambda + k}{2} + S(\lambda) = \frac{n_{\lambda^*} + |\lambda^*|}{2} + k - |\lambda^*| - 1 + S(\lambda^*) + |\lambda^*| - k + 1$$

whence the assertion since $d_\lambda = d_{\lambda^*}$. □

**Lemma 5.7.** (i) *If $\lambda_1$ is odd and if $\lambda_i$ is even for $i \geqslant 2$, then* $\dim \mathfrak{g}^e + \ell - 2(\delta_1 + \cdots + \delta_\ell) = 0$.

(ii) *If $k$ is odd, then* $\dim \mathfrak{g}^e + \ell - 2(\delta_1 + \cdots + \delta_\ell) = n_\lambda - d_\lambda$.

(iii) *If $k$ is even, then* $\dim \mathfrak{g}^e + \ell - 2(\delta_1 + \cdots + \delta_\ell) = d_\lambda$.

*Proof.* (i) By the hypothesis, $n_\lambda = k - 1$ and $S(\lambda) = 1$, whence the assertion by Lemma 5.3,(i).

(ii) Let us prove the assertion by induction on $k$. For $k = 3$, if $\lambda_1$ and $\lambda_2$ are even, $n_\lambda = 2$, $d_\lambda = 0$ and $S(\lambda) = 3$, whence the equality by Lemma 5.3,(i). Assume that $k > 3$ and suppose that the equality holds for the integers smaller than $k$. If Condition (1) or (2) is verified, then by Lemma 5.3,(i), Lemma 5.6,(i) and by induction hypothesis,

$$\dim \mathfrak{g}^e + \ell - 2(\delta_1 + \cdots + \delta_\ell) = n_{\lambda^*} - d_{\lambda^*}.$$

But if Condition (1) or (2) is verified, then $n_\lambda - d_\lambda = n_{\lambda^*} - d_{\lambda^*}$. If Condition (5) is verified, then

$$n_\lambda = k - 1, \qquad S(\lambda) = k, \qquad d_\lambda = 0,$$

whence the equality by Lemma 5.3,(i). Let us suppose that Condition (4) is verified. By Lemma 5.3,(i), Lemma 5.6,(i) and by induction hypothesis,

$$\dim \mathfrak{g}^e + \ell - 2(\delta_1 + \cdots + \delta_\ell) = n_{\lambda^*} - d_{\lambda^*} + k - |\lambda^*| - 2 = n_\lambda - d_\lambda$$

whence the assertion since Condition (3) is never verified when $k$ is odd.

(iii) The statement is clear for $k = 2$ by Lemma 5.3,(ii). Indeed, if Condition (1) is verified, then $d_\lambda = n_\lambda = 0$ and $S(\lambda) = -1$ and if Condition (2) is verified, then $d_\lambda = n_\lambda = 2$ and $S(\lambda) = 0$. If Condition (3) is verified, $n_\lambda = k - 2$ and $S(\lambda) = 1 - k$, whence the statement by Lemma 5.3,(ii). When Condition (4) is verified, by induction on $|\lambda|$, the statement results from Lemma 5.3,(ii) and Lemma 5.6,(ii), whence the assertion since Condition (5) is never verified when $k$ is even. □

**Corollary 5.8.** (i) *If $\lambda_1$ is odd and if $\lambda_i$ is even for all $i \geqslant 2$, then $e$ is good.*

(ii) *If $k$ is odd and if $n_\lambda = d_\lambda$, then $e$ is good. In particular, if $\mathfrak{g}$ is of type* **B**, *then the even nilpotent elements of $\mathfrak{g}$ are good.*

(iii) *If $k$ is even and if $d_\lambda = 0$, then $e$ is good. In particular, if $\mathfrak{g}$ is of type* **D** *and of odd rank, then the even nilpotent elements of $\mathfrak{g}$ are good.*

*Proof.* As it has been already noticed, by [PPY07, Theorem 2.1], the polynomials ${}^e q_1, \dots, {}^e q_\ell$ are algebraically independent if and only if

$$\dim \mathfrak{g}^e + \ell - 2(\delta_1 + \cdots + \delta_\ell) = 0.$$

So, by Theorem 3.6 and Lemma 5.7, if either $\lambda_1$ is odd and $\lambda_i$ is even for all $i \geqslant 2$, or if $k$ is odd and $n_\lambda = d_\lambda$, or if $k$ is even and $d_\lambda = 0$, then $e$ is good.

Suppose that $e$ is even. Then the integers $\lambda_1, \dots, \lambda_k$ have the same parity, cf. e.g. [Ca85, §1.3.1]. Moreover, $n_\lambda = d_\lambda = 0$ whenever $\lambda_1, \dots, \lambda_k$ are all odd (cf. Definition 5.5). This in particular occurs if either $\mathfrak{g}$ is of type **B**, or if $\mathfrak{g}$ is of type **D** with odd rank. □

*Remark* 5.9. The fact that even nilpotent elements of $\mathfrak{g}$ are good if either $\mathfrak{g}$ is of type **B**, or is $\mathfrak{g}$ is of type **D** with odd rank, was already observed by O. Yakimova in [Y09, Corollary 8.2] with a different formulation.

**Definition 5.10.** A sequence $\lambda = (\lambda_1, \dots, \lambda_k)$ is said to be *very good* if $n_\lambda = d_\lambda$ whenever $k$ is odd and if $d_\lambda = 0$ whenever $k$ is even. A nilpotent element of $\mathfrak{g}$ is said to be *very good* if it is associated with a very good partition of $\dim \mathbb{V}$.

According to Corollary 5.8, if $e$ is very good then $e$ is good. The following lemma characterizes the very good sequences.

**Lemma 5.11.** (i) *If $k$ is odd then $\lambda$ is very good if and only if $\lambda_1$ is odd and if $(\lambda_2, \dots, \lambda_k)$ is a concatenation of sequences verifying Conditions* (1) *or* (2) *with $k = 2$.*

(ii) *If $k$ is even then $\lambda$ is very good if and only if $\lambda$ is a concatenation of sequences verifying Condition* (3) *or Condition* (1) *with $k = 2$.*

For example, the partitions $(5, 3, 3, 2, 2)$ and $(7, 5, 5, 4, 4, 3, 1, 1)$ of 15 and 30 respectively are very good.

*Proof.* (i) Assume that $\lambda_1$ is odd and that $(\lambda_2, \dots, \lambda_k)$ is a concatenation of sequences verifying Conditions (1) or (2) with $k = 2$. So, if $k > 1$, then $n_\lambda - d_\lambda = n_{\lambda^*} - d_{\lambda^*}$. Then, a quick induction proves that $n_\lambda - d_\lambda = n_{(\lambda_1)} - d_{(\lambda_1)} = 0$ since $\lambda_1$ is odd. The statement is clear for $k = 1$.

Conversely, assume that $n_\lambda - d_\lambda = 0$. If $\lambda$ verifies Conditions (1) or (2), then $n_\lambda - d_\lambda = n_{\lambda^*} - d_{\lambda^*}$ and $|\lambda^*| < |\lambda|$. So, we can assume that $\lambda$ does not verify Conditions (1) or (2). Since $k$ is odd, $\lambda$ cannot verify Condition (3). If $\lambda$ verifies Condition (4), then $n_\lambda - d_\lambda = n_\lambda - d_{\lambda^*} > n_{\lambda^*} - d_{\lambda^*} \geqslant 0$. This is impossible since $n_\lambda - d_\lambda = 0$. If $\lambda$ verifies Condition (5), then $n_\lambda - d_\lambda = n_\lambda$. So, $n_\lambda - d_\lambda = 0$ if and only if $k = 1$. Thereby, the direct implication is proven.

(ii) Assume that $\lambda$ is a concatenation of sequences verifying Condition (3) or Condition (1) with $k = 2$. In particular, $\lambda$ does not verify Condition (2). Moreover, Condition (5) is not verified since $k$ is even. Then $d_\lambda = 0$ by induction on $|\lambda|$, whence $e$ is very good.

Conversely, suppose that $d_\lambda = 0$. If $k = 2$, Condition (1) is verified and if $k = 4$, then either Condition (3) is verified, or $\lambda_1, \dots, \lambda_4$ are all odd. Suppose $k > 4$. Condition (2) is not verified since $d_\lambda = d_{\lambda_*} + 2$ in this case. If Condition (1) is verified then $d_{\lambda_*} = 0$ and $\lambda$ is a concatenation of $\lambda^*$ and $(\lambda_{k-1}, \lambda_k)$. If Condition (4) is verified, then $d_{\lambda_*} = 0$ and $\lambda$ is a concatenation of $\lambda_*$ and a sequence verifying Condition (3), whence the assertion by induction on $|\lambda|$ since Condition (5) is not verified when $k$ is even. □

5.3. Assume in this subsection that $\lambda = (\lambda_1, \dots, \lambda_k)$ verifies the following condition:

(∗) *For some $k' \in \{2, \dots, k\}$, $\lambda_i$ is even for all $i \leqslant k'$, and $(\lambda_{k'+1}, \dots, \lambda_k)$ is very good.*

In particular, $k'$ is even and by Lemma 5.11, $\lambda_{k'+1}$ is odd and $\lambda$ is not very good. For example, the sequences $\lambda = (6, 6, 4, 4, 3, 2, 2)$ and $(6, 6, 4, 4, 3, 3, 3, 2, 2, 1)$ satisfy the condition (∗) with $k' = 4$. Define a sequence $\nu = (\nu_1, \dots, \nu_k)$ of integers of $\{1, \dots, \ell\}$ by

$$\forall\, i \in \{1, \dots, k'\}, \qquad \nu_i := \frac{\lambda_1 + \cdots + \lambda_i}{2}.$$

If $k' = k$, then $\nu_k = (\lambda_1 + \cdots + \lambda_k)/2 = r(\lambda)/2 = \ell$. Define elements $p_1, \dots, p_{k'}$ of $\mathrm{S}(\mathfrak{g}^e)$ as follows:

- if $k' < k$, set for $i \in \{1, \dots, k'\}$, $p_i := {}^e q_{\nu_i}$,
- if $k' = k$, set for $i \in \{1, \dots, k' - 1\}$, $p_i := {}^e q_{\nu_i}$ and set $p_k := ({}^e q_{\nu_k})^2$. In this case, set also $\tilde{p}_k := {}^e q_{\nu_k}$.

Remind that $\delta_i$ is the degree of ${}^e q_i$ for $i = 1, \dots, \ell$. The following lemma is a straightforward consequence of [PPY07, Remark 4.2]:

**Lemma 5.12.** (i) *For all $i \in \{1, \dots, k'\}$,* $\deg\, p_i = i$.

(ii) *Set $\nu_0 := 0$. Then for $i \in \{1, \dots, k'\}$ and $r \in \{1, \dots, \nu_{k'} - 1\}$,*

$$\delta_r = i \iff \nu_{i-1} < r \leqslant \nu_i.$$

*In particular, for $r \in \{1, \dots, \nu_{k'} - 2\}$, $\delta_r < \delta_{r+1}$ if and only if $r$ is a value of the sequence $\nu$.*

*Example* 5.13. Consider the partition $\lambda = (8, 8, 4, 4, 4, 4, 2, 2, 1, 1)$ of 38. Then $k = 10$, $k' = 8$ and $\nu = (4, 8, 10, 12, 14, 16, 17, 18)$. We represent in Table 1 the degrees of the polynomials $p_1, \dots, p_8$ and ${}^e q_1, \dots, {}^e q_{18}$. Note that $\deg\, {}^e q_{19} = 5$. In the table, the common degree of the polynomials appearing on the $i$th column is $i$.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | ${}^e q_4 = p_1$ | ${}^e q_8 = p_2$ | | | | | | |
| | ${}^e q_3$ | ${}^e q_7$ | | | | | | |
| | ${}^e q_2$ | ${}^e q_6$ | ${}^e q_{10} = p_3$ | ${}^e q_{12} = p_4$ | ${}^e q_{14} = p_5$ | ${}^e q_{16} = p_6$ | | |
| | ${}^e q_1$ | ${}^e q_5$ | ${}^e q_9$ | ${}^e q_{11}$ | ${}^e q_{13}$ | ${}^e q_{15}$ | ${}^e q_{17} = p_7$ | ${}^e q_{18} = p_8$ |
| degrees | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |

TABLE 1.

Let $\mathfrak{s}$ be the subalgebra of $\mathfrak{g}$ generated by $e, h, f$ and decompose $\mathbb{V}$ into simple $\mathfrak{s}$-modules $\mathbb{V}_1, \dots, \mathbb{V}_k$ of dimension $\lambda_1, \dots, \lambda_k$ respectively. One can order them so that for $i \in \{1, \dots, k'/2\}$, $\mathbb{V}_{(2(i-1)+1)'} = \mathbb{V}_{2i}$. For $i \in \{1, \dots, k\}$, denote by $e_i$ the restriction to $\mathbb{V}_i$ of $e$ and set $\varepsilon_i := e_i^{\lambda_i - 1}$. Then $e_i$ is a regular nilpotent element of $\mathfrak{gl}(\mathbb{V}_i)$ and $(\mathrm{ad}\, h)\varepsilon_i = 2(\lambda_i - 1)\varepsilon_i$.

For $i \in \{1, \dots, k'/2\}$, set

$$\mathbb{V}[i] := \mathbb{V}_{2(i-1)+1} + \mathbb{V}_{2i}$$

and set

$$\mathbb{V}[0] := \mathbb{V}_{k'+1} \oplus \cdots \oplus \mathbb{V}_k.$$

Then for $i \in \{0, 1, \dots, k'/2\}$, denote by $\mathfrak{g}_i$ the simple Lie algebra $\mathfrak{so}(\mathbb{V}[i])$. For $i \in \{1, \dots, k'/2\}$, $e_{2(i-1)+1} + e_{2i}$ is an even nilpotent element of $\mathfrak{g}_i$ with Jordan blocks of size $(\lambda_{2(i-1)+1}, \lambda_{2i})$. Let $i \in \{1, \dots, k'/2\}$ and set:

$$z_i := \varepsilon_{2(i-1)+1} + \varepsilon_{2i}.$$

Then $z_i$ lies in the center of $\mathfrak{g}^e$ and

$$(\mathrm{ad}\, h) z_i = 2(\lambda_{2(i-1)+1} - 1) z_i = 2(\lambda_{2i} - 1) z_i.$$

Moreover, $2(\lambda_{2i} - 1)$ is the highest weight of $\mathrm{ad}\, h$ acting on $\mathfrak{g}_i^e := \mathfrak{g}_i \cap \mathfrak{g}^e$, and the intersection of the $2(\lambda_{2i} - 1)$-eigenspace of $\mathrm{ad}\, h$ with $\mathfrak{g}_i^e$ is spanned by $z_i$, see for instance [Y09, §1]. Set

$$\overline{\mathfrak{g}} \;:=\; \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{k'/2} \;=\; \mathfrak{so}(\mathbb{V}[0]) \oplus \mathfrak{so}(\mathbb{V}[1]) \oplus \cdots \oplus \mathfrak{so}(\mathbb{V}[k'/2])$$

and denote by $\overline{\mathfrak{g}}^e$ (resp. $\overline{\mathfrak{g}}^f$) the centralizer of $e$ (resp. $f$) in $\overline{\mathfrak{g}}$. For $p \in \mathrm{S}(\mathfrak{g}^e)$, denote by $\overline{p}$ its restriction to $\overline{\mathfrak{g}}^f \simeq (\overline{\mathfrak{g}}^e)^*$; it is an element of $\mathrm{S}(\overline{\mathfrak{g}}^e)$. Our goal is to describe the elements $\overline{p}_1, \dots, \overline{p}_{k'}$ (see Proposition 5.18). The motivation comes from Lemma 5.14.

Let $\mathfrak{g}^f_{\mathrm{reg}}$ (resp. $\overline{\mathfrak{g}}^f_{\mathrm{reg}}$) be the set of elements $x \in \mathfrak{g}^f$ (resp. $\overline{\mathfrak{g}}^f$) such that $x$ is a regular linear form on $\mathfrak{g}^e$ (resp. $\overline{\mathfrak{g}}^e$).

**Lemma 5.14.** (i) *The intersection $\mathfrak{g}^f_{\mathrm{reg}} \cap \overline{\mathfrak{g}}^f$ is a dense open subset of $\overline{\mathfrak{g}}^f_{\mathrm{reg}}$.*

(ii) *The morphism*

$$\theta : \quad G_0^e \times \overline{\mathfrak{g}}^f \longrightarrow \mathfrak{g}^f, \quad (g, x) \longmapsto g.x$$

*is a dominant morphism from $G_0^e \times \overline{\mathfrak{g}}^f$ to $\mathfrak{g}^f$.*

*Proof.* (i) Since $\lambda$ verifies the condition $(*)$, it verifies the condition (1) of the proof of [Y06, §4, Lemma 3] and so, $\mathfrak{g}^f_{\mathrm{reg}} \cap \overline{\mathfrak{g}}^f$ is a dense open subset of $\overline{\mathfrak{g}}^f$. Moreover, since $\mathfrak{g}^e$ and $\overline{\mathfrak{g}}^e$ have the same index by [Y06, Theorem 3], $\mathfrak{g}^f_{\mathrm{reg}} \cap \overline{\mathfrak{g}}^f$ is contained in $\overline{\mathfrak{g}}^f_{\mathrm{reg}}$.

(ii) Let $\mathfrak{m}$ be the orthogonal complement to $\overline{\mathfrak{g}}$ in $\mathfrak{g}$ with respect to the Killing form $\langle .,. \rangle$. Since the restriction to $\overline{\mathfrak{g}} \times \overline{\mathfrak{g}}$ of $\langle .,. \rangle$ is nondegenerate, $\mathfrak{g} = \overline{\mathfrak{g}} \oplus \mathfrak{m}$ and $[\overline{\mathfrak{g}}, \mathfrak{m}] \subset \mathfrak{m}$. Set $\mathfrak{m}^e := \mathfrak{m} \cap \mathfrak{g}^e$. Since the restriction to $\overline{\mathfrak{g}}^f \times \overline{\mathfrak{g}}^e$ of $\langle .,. \rangle$ is nondegenerate, we get the decomposition

$$\mathfrak{g}^e = \overline{\mathfrak{g}}^e \oplus \mathfrak{m}^e$$

and $\mathfrak{m}^e$ is the orthogonal complement to $\overline{\mathfrak{g}}^f$ in $\mathfrak{g}^e$. Moreover, $[\overline{\mathfrak{g}}^e, \mathfrak{m}^e] \subset \mathfrak{m}^e$.

By (i), $\mathfrak{g}^f_{\mathrm{reg}} \cap \overline{\mathfrak{g}}^f \neq \varnothing$. Let $x \in \mathfrak{g}^f_{\mathrm{reg}} \cap \overline{\mathfrak{g}}^f$. The tangent map at $(1_{\mathfrak{g}}, x)$ of $\theta$ is the linear map

$$\mathfrak{g}^e \times \overline{\mathfrak{g}}^f \longrightarrow \mathfrak{g}^f, \qquad (u, y) \longmapsto u.x + y,$$

where $u.$ denotes the coadjoint action of $u$ on $\mathfrak{g}^f \simeq (\mathfrak{g}^e)^*$. The index of $\overline{\mathfrak{g}}^e$ is equal to the index of $\mathfrak{g}^e$ and $[\overline{\mathfrak{g}}^e, \mathfrak{m}^e] \subset \mathfrak{m}^e$. So, the stabilizer of $x$ in $\overline{\mathfrak{g}}^e$ coincides with the stabilizer of $x$ in $\mathfrak{g}^e$. In particular, $\dim \mathfrak{m}^e.x = \dim \mathfrak{m}^e$. As a result, $\theta$ is a submersion at $(1_{\mathfrak{g}}, x)$ since $\dim \mathfrak{g}^f = \dim \mathfrak{m}^e + \dim \overline{\mathfrak{g}}^f$. In conclusion, $\theta$ is a dominant morphism from $G_0^e \times \overline{\mathfrak{g}}^f$ to $\mathfrak{g}^f$. □

Let $(\mu_1,\dots,\mu_m)$ be the strictly decreasing sequence of the values of the sequence $(\lambda_1,\dots,\lambda_{k'})$ and let $k_1,\dots,k_m$ be the multiplicity of $\mu_1,\dots,\mu_m$ respectively in this sequence. By our assumption, the integers $\mu_1,\dots,\mu_m,k_1,\dots,k_m$ are all even. Notice that $k_1+\cdots+k_m=k'$. The set $\{1,\dots,k'\}$ decomposes into parts $K_1,\dots,K_m$ of cardinality $k_1,\dots,k_m$ respectively given by:

$$\forall\, s\in\{1,\dots,m\},\qquad K_s:=\{k_0+\cdots+k_{s-1}+1,\dots,k_0+\cdots+k_s\}.$$

Here, the convention is that $k_0:=0$.

*Remark* 5.15. For $s\in\{1,\dots,m\}$ and $i\in K_s$,

$$\nu_i:=k_0(\frac{\mu_0}{2})+\cdots+k_{s-1}(\frac{\mu_{s-1}}{2})+j(\frac{\mu_s}{2}),$$

where $j=i-(k_0+\cdots+k_{s-1})$ and $\mu_0=0$.

Decompose also the set $\{1,\dots,k'/2\}$ into parts $I_1,\dots,I_m$ of cardinality $k_1/2,\dots,k_m/2$ respectively, with

$$\forall\, s\in\{1,\dots,m\},\qquad I_s:=\{\frac{k_0+\cdots+k_{s-1}}{2}+1,\dots,\frac{k_0+\cdots+k_s}{2}\}.$$

For $p\in\mathrm{S}(\mathfrak{g}^e)$ an eigenvector of $\mathrm{ad}\,h$, denote by $\mathrm{wt}(p)$ its $\mathrm{ad}\,h$-weight.

**Lemma 5.16.** *Let $s\in\{1,\dots,m\}$ and $i\in K_s$.*

(i) *Set $j=i-(k_0+\cdots+k_{s-1})$. Then,*

$$\mathrm{wt}(\overline{p}_i)=2(2\nu_i-i)=\sum_{l=1}^{s-1}2k_l(\mu_l-1)+2j(\mu_s-1).$$

*Moreover, if $p\in\{{}^eq_1,\dots,{}^eq_{\ell-1},({}^eq_\ell)^2\}$ is of degree $i$, then $\mathrm{wt}(p)=\mathrm{wt}(\overline{p})\leqslant 2(2\nu_i-i)$ and the equality holds if and only if $p=p_i$.*

(ii) *The polynomial $\overline{p}_i$ is in $\Bbbk[z_l,\ l\in I_1\cup\ldots\cup I_s]$.*

*Proof.* (i) This is a consequence of [PPY07, Lemma 4.3] (or [Y09, Theorem 6.1]), Lemma 5.12 and Remark 5.15.

(ii) Let $\tilde{\mathfrak{g}}^f$ be the centralizer of $f$ in $\tilde{\mathfrak{g}}=\mathfrak{gl}(\mathbb{V})$, and let ${}^e\overline{Q}_{2\nu_i}$ be the initial homogeneous component of the restriction to

$$(\mathfrak{gl}(\mathbb{V}[0])\oplus\mathfrak{gl}(\mathbb{V}[1])\oplus\cdots\oplus\mathfrak{gl}(\mathbb{V}[k'/2]))\ \cap\ \tilde{\mathfrak{g}}^f$$

of the polynomial function $x\mapsto Q_{2\nu_i}(e+x)$. Since $\overline{p}_i\neq 0$, $\overline{p}_i$ is the restriction to $\overline{\mathfrak{g}}^f$ of ${}^e\overline{Q}_{2\nu_i}$ and we have

$$\mathrm{wt}({}^e\overline{Q}_{2\nu_i})=\mathrm{wt}(\overline{p}_i)=2(2\nu_i-i),\qquad \deg{}^e\overline{Q}_{2\nu_i}=\deg\overline{p}_i=i.$$

Then, by (i) and [PPY07, Lemma 4.3], ${}^e\overline{Q}_{2\nu_i}$ is a sum of monomials whose restriction to $\overline{\mathfrak{g}}^f$ is zero and of monomials of the form

$$(\varepsilon_{\varsigma^{(1)}1}\dots\varepsilon_{\varsigma^{(1)}k_1})\cdots(\varepsilon_{\varsigma^{(s-1)}1}\dots\varepsilon_{\varsigma^{(s-1)}k_{s-1}})(\varepsilon_{\varsigma^{(s)}j_1}\dots\varepsilon_{\varsigma^{(s)}j_i})$$

where $j_1<\cdots<j_i$ are integers of $K_s$, and $\varsigma^{(1)},\dots,\varsigma^{(s-1)}$, $\varsigma^{(s)}$ are permutations of $K_1,\dots,K_{s-1}$, $\{j_1,\dots,j_i\}$ respectively. Hence, $\overline{p}_i$ is in $\Bbbk[z_l,\ l\in I_1\cup\ldots\cup I_s]$. More precisely, for $l\in I_1\cup\ldots\cup I_s$, the element $z_l$ appears in $\overline{p}_i$ with a multiplicity at most 2 since $z_l=\varepsilon_{2(l-1)+1}+\varepsilon_{2l}$. □

Let $s\in\{1,\dots,m\}$ and $i\in K_s$. In view of Lemma 5.16,(ii), we aim to give an explicit formula for $\overline{p}_i$ in term of the elements $z_1,\dots,z_{k'/2}$. Besides, according to Lemma 5.16,(ii), we can assume that $s=m$. As a first step, we state inductive formulae. If $k'>2$, set

$$\overline{\mathfrak{g}}':=\mathfrak{so}(\mathbb{V}[1])\oplus\cdots\oplus\mathfrak{so}(\mathbb{V}[k'/2-1]),$$

and let $\overline{p}'_1, \dots, \overline{p}'_{k'}$ be the restrictions to $(\overline{\mathfrak{g}}')^f := \overline{\mathfrak{g}}' \cap \mathfrak{g}^f$ of $\overline{p}_1, \dots, \overline{p}_{k'}$ respectively. Note that $\overline{p}'_{k'-1} = \overline{p}'_{k'} = 0$. Set by convention $k_0 := 0$, $p_0 := 1$, $p'_0 := 1$ and $p_{-1} := 0$. It will be also convenient to set

$$k^* := k_0 + \cdots + k_{m-1}.$$

**Lemma 5.17.** (i) *If $k_m = 2$, then*

$$\overline{p}_{k^*+1} = -2\,\overline{p}'_{k^*}\, z_{k'/2} \quad \textit{and} \quad \overline{p}_{k^*+2} = \overline{p}'_{k^*}\,(z_{k'/2})^2.$$

(ii) *If $k_m > 2$, then*

$$\overline{p}_{k^*+1} = \overline{p}'_{k^*+1} - 2\,\overline{p}'_{k^*}\, z_{k'/2}$$

*and for $j = 2, \dots, k_m$,*

$$\overline{p}_{k^*+j} = \overline{p}'_{k^*+j} - 2\,\overline{p}'_{k^*+j-1}\, z_{k'/2} + \overline{p}'_{k^*+j-2}\,(z_{k'/2})^2.$$

*Proof.* For $i = 1, \dots, k'/2$, let $w_i$ be the element of $\mathfrak{g}_i^f := \mathfrak{g}_i \cap \mathfrak{g}^f$ such that

$$(\operatorname{ad} h)w_i = -2(\lambda_{2i} - 1)w_i \quad \text{and} \quad \det(e_i + w_i) = 1.$$

Remind that $p_i(y)$, for $y \in \mathfrak{g}^f$, is the initial homogeneous component of the coefficient of the term $T^{\dim \mathbb{V} - 2\nu_i}$ in the expression $\det(T - e - y)$. By Lemma 5.16,(ii), in order to describe $\overline{p}_i$, it suffices to compute $\det(T - e - s_1 w_1 - \cdots - s_{k'/2} w_{k'/2})$, with $s_1, \dots, s_{k'/2}$ in $\Bbbk$.

1) To start with, consider the case $k' = k_m = 2$. By Lemma 5.16, $p_1 = a z_1$ and $p_2 = b z_1^2$ for some $a, b \in \Bbbk$. One has,

$$\det(T - e - s_1 w_1) = T^{2\mu_1} - 2 s_1 T^{\mu_1} + s_1^2.$$

As a result, $a = -2$ and $b = 1$. This proves (i) in this case.

2) Assume from now that $k' > 2$. Setting $e' := e_1 + \cdots + e_{k'/2-1}$, observe that

$$\begin{aligned} \det(T - e - s_1 w_1 - \cdots - s_{k'/2} w_{k'/2}) \\ &= \det(T - e' - s_1 w_1 - \cdots - s_{k'/2-1} w_{k'/2-1}) \det(T - e_{k'/2} - s_{k'/2} w_{k'/2}) \\ &= \det(T - e' - s_1 w_1 - \cdots - s_{k'/2-1} w_{k'/2-1})\,(T^{2\mu_m} - 2 s_{k'/2} T^{\mu_m} + s_{k'/2}^2) \end{aligned} \tag{3}$$

where the latter equality results from Step (1).

(i) If $k_m = 2$, then $k^* = k' - 2$ and the constant term in $\det(T - e' - s_1 w_1 - \cdots - s_{k'/2-1} w_{k'/2-1})$ is $\overline{p}'_{k^*}$. By Lemma 5.16,(i),

$$\operatorname{wt}(\overline{p}_{k^*+1}) = \operatorname{wt}(\overline{p}'_{k^*}) + \operatorname{wt}(z_{k'/2})$$

and $\overline{p}'_{k^*}$ is the only element appearing in the coefficients of $\det(T - e' - s_1 w_1 - \cdots - s_{k'/2-1} w_{k'/2-1})$ of this weight. Similarly,

$$\operatorname{wt}(\overline{p}_{k^*+2}) = \operatorname{wt}(\overline{p}'_{k^*}) + \operatorname{wt}((z_{k'/2})^2)$$

and $\overline{p}'_{k^*}$ is the only element appearing in the coefficients of $\det(T - e' - s_1 w_1 - \cdots - s_{k'/2-1} w_{k'/2-1})$ of this weight. As a consequence, the equalities follow.

(ii) Suppose $k_m > 2$. Then by Lemma 5.16,(i),

$$\operatorname{wt}(\overline{p}_{k^*+1}) = \operatorname{wt}(\overline{p}'_{k^*+1}) = \operatorname{wt}(\overline{p}'_{k^*}) + \operatorname{wt}(z_{k'/2}).$$

Moreover, $\overline{p}'_{k^*+1}$ and $\overline{p}'_{k^*}$ are the only elements appearing in the coefficients of $\det(T - e' - s_1 w_1 - \cdots - s_{k'/2-1} w_{k'/2-1})$ of this weight with degree $k^* + 1$ and $k^*$ respectively. Similarly, by Lemma 5.16,(i), for $j \in \{2, \dots, k_m\}$,

$$\operatorname{wt}(\overline{p}_{k^*+j}) = \operatorname{wt}(\overline{p}'_{k^*+j}) = \operatorname{wt}(\overline{p}'_{k^*+j-1}) + \operatorname{wt}(z_{k'/2}) = \operatorname{wt}(\overline{p}'_{k^*+j-2}) + \operatorname{wt}((z_{k'/2})^2).$$

Moreover, $\overline{p}'_{k^*+j}$, $\overline{p}'_{k^*+j-1}$ and $\overline{p}'_{k^*+j-2}$ are the only elements appearing in the coefficients of $\det(T - e' - s_1w_1 - \cdots - s_{k'/2-1}w_{k'/2-1})$ of this weight with degree $k^* + j$, $k^* + j - 1$ and $k^* + j - 2$ respectively.

In both cases, this forces the inductive formula (ii) through the factorization (3). □

For a subset $I = \{i_1, \ldots, i_l\} \subseteq \{1, \ldots, k'/2\}$ of cardinality $l$, denote by $\sigma_{I,1}, \ldots, \sigma_{I,l}$ the elementary symmetric functions of $z_{i_1}, \ldots, z_{i_l}$:

$$\forall\, j \in \{1, \ldots, l\}, \qquad \sigma_{I,j} = \sum_{1 \leqslant a_1 < a_2 < \cdots < a_j \leqslant l} z_{i_{a_1}} z_{i_{a_2}} \ldots z_{i_{a_j}}.$$

Set also $\sigma_{I,0} := 1$ and $\sigma_{I,j} := 0$ if $j > l$ so that $\sigma_{I,j}$ is well defined for any nonnegative integer $j$. Set at last $\sigma_{I,j} := 1$ for any $j$ if $I = \varnothing$. If $I = I_s$, with $s \in \{1, \ldots, m\}$, denote by $\sigma_j^{(s)}$, for $j \geqslant 0$, the elementary symmetric function $\sigma_{I_s,j}$.

**Proposition 5.18.** *Let $s \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, k_s\}$. Then*

$$\overline{p}_{k_0+\cdots+k_{s-1}+j} = (-1)^j\, \overline{p}_{k_0+\cdots+k_{s-1}} \sum_{r=0}^{j} \sigma_{j-r}^{(s)} \sigma_r^{(s)} = (-1)^j\, (\sigma_{k_0/2}^{(1)} \ldots \sigma_{k_{s-1}/2}^{(s-1)})^2 \sum_{r=0}^{j} \sigma_{j-r}^{(s)} \sigma_r^{(s)}.$$

*Example* 5.19. If $m = 1$, then $k' = k_1$ and

$$p_1 = -\sigma_1^{(1)}\sigma_0^{(1)} - \sigma_0^{(1)}\sigma_1^{(1)} = -2\sigma_1^{(1)} = -2(z_1 + \cdots + z_{k'/2}),$$

$$p_2 = \sigma_2^{(1)}\sigma_0^{(1)} + (\sigma_1^{(1)})^2 + \sigma_0^{(1)}\sigma_2^{(1)} = 2\sigma_2^{(1)} + (\sigma_1^{(1)})^2,$$

$$\cdots,$$

$$\overline{p}_{k'} = (\sigma_{k'/2}^{(1)})^2 = (z_1 z_2 \ldots z_{k'/2})^2.$$

*Proof.* By Lemma 5.16,(ii), we can assume that $s = m$. Retain the notations of Lemma 5.17. In particular, set again

$$k^* := k_0 + \cdots + k_{m-1}.$$

We prove the statement by induction on $k'/2$. If $k' = 2$, then $m = 1$, $k_m = k' = 2$ and the statement follows from by Lemma 5.17,(i). Assume now that $k' > 2$ and the statement true for the polynomials $\overline{p}'_1, \ldots, \overline{p}'_{k'-1}$.

If $k_m = 2$, the statement follows from Lemma 5.17,(i).

Assume $k_m > 2$. For any $r \geqslant 0$, we set $\sigma'_r := \sigma_{I',r}$ where $I' = \{\frac{k^*}{2} + 1, \ldots, \frac{k'}{2} - 1\} \subset I_m$. In particular, $\sigma'_0 = 1$ by convention. Observe that for any $r \geqslant 1$,

$$\sigma_r^{(m)} = \sigma'_r + \sigma'_{r-1} z_{k'/2}.$$

Setting $\sigma'_{-1} := 0$, the above equality remains true for $r = 0$. By the induction hypothesis and by Lemma 5.17,(ii), for $j \in \{2, \ldots, k_m\}$,

$$\begin{aligned}
\overline{p}_{k^*+j} &= \overline{p}'_{k^*+j} - 2\,\overline{p}'_{k^*+j-1}\, z_{k'/2} + \overline{p}'_{k^*+j-2}\, (z_{k'/2})^2 \\
&= \overline{p}_{k^*}((-1)^j \sum_{r=0}^{j} \sigma'_{j-r}\sigma'_r - 2(-1)^{j-1} \sum_{r=0}^{j-1} \sigma'_{j-r-1}\sigma'_r\, z_{k'/2} + (-1)^{j-2} \sum_{r=0}^{j-2} \sigma'_{j-r-2}\sigma'_r\, z_{k'/2}^2). \\
&= (-1)^j\, \overline{p}_{k^*}(\sum_{r=0}^{j} \sigma'_{j-r}\sigma'_r + 2\,(\sum_{r=0}^{j-1} \sigma'_{j-r-1}\sigma'_r)\, z_{k'/2} + (\sum_{r=0}^{j-2} \sigma'_{j-r-2}\sigma'_r)\, z_{k'/2}^2)
\end{aligned}$$

since $\overline{p}'_{k^*} = \overline{p}_{k^*}$. On the other hand, we have

$$\begin{aligned}
\sum_{r=0}^{j} \sigma^{(m)}_{j-r}\sigma^{(m)}_{r} &= \sum_{r=0}^{j}(\sigma'_{j-r} + \sigma'_{j-r-1}z_{k'/2})(\sigma'_{r} + \sigma'_{r-1}z_{k'/2}) \\
&= \sum_{r=0}^{j}\sigma'_{j-r}\sigma'_{r} + (\sum_{r=0}^{j}\sigma'_{j-r-1}\sigma'_{r} + \sum_{r=0}^{j}\sigma'_{j-r}\sigma'_{r-1})\, z_{k'/2} + (\sum_{r=0}^{j}\sigma'_{j-r-1}\sigma'_{r-1})\, z^2_{k'/2} \\
&= \sum_{r=0}^{j}\sigma'_{j-r}\sigma'_{r} + 2\,(\sum_{r=0}^{j-1}\sigma'_{j-r-1}\sigma'_{r})\, z_{k'/2} + (\sum_{r=0}^{j-2}\sigma'_{j-r-2}\sigma'_{r})\, z^2_{k'/2}.
\end{aligned}$$

Thereby, for any $j \in \{2, \dots, k_m\}$, we get

$$\overline{p}_{k^*+j} = (-1)^j\, \overline{p}_{k^*} \sum_{r=0}^{j} \sigma^{(m)}_{j-r}\sigma^{(m)}_{r}.$$

For $j = 1$, since $\overline{p}'_{k^*} = \overline{p}_{k^*}$, by Lemma 5.17,(ii), and our induction hypothesis,

$$\overline{p}_{k^*+1} = \overline{p}'_{k^*+1} - 2\,\overline{p}'_{k^*}\, z_{k'/2} = \overline{p}_{k^*}(-2\sigma'_1) - 2\,\overline{p}_{k^*}\, z_{k'/2} = \overline{p}_{k^*}(-2\sigma^{(m)}_1).$$

This proves the first equality of the proposition.

For the second one, it suffices to prove by induction on $s \in \{1, \dots, m\}$ that

$$\overline{p}_{k_0+\cdots+k_{s-1}} = (\sigma^{(1)}_{k_0/2} \dots \sigma^{(s-1)}_{k_{s-1}/2})^2.$$

For $s = 1$, then $\overline{p}_{k_0+\cdots+k_{s-1}} = \overline{p}_0 = 1$ and $\sigma_{\varnothing,0} = 1$ by convention. Assume $s > 2$ and the statement true for $1, \dots, s-1$. By the first equality with $j = k_s$, $\overline{p}_{k_0+\cdots+k_s} = (-1)^{k_s}\, \overline{p}_{k_0+\cdots+k_{s-1}}(\sigma^{(s)}_{k_s/2})^2$, whence the statement by induction hypothesis since $k_s$ is even. □

*Remark* 5.20. Remind that the polynomial $\tilde{p}_k$ was defined before Lemma 5.12. As a by product of the previous formula, whenever $k' = k$, we obtain

$$\overline{\tilde{p}}_k = \sigma^{(1)}_{k_0/2} \dots \sigma^{(m)}_{k_m/2}.$$

For $s \in \{1, \dots, m\}$ and $j \in \{1, \dots, k_s\}$, set

$$\rho_{k_0+\cdots+k_{s-1}+j} := \frac{\overline{p}_{k_0+\cdots+k_{s-1}+j}}{\overline{p}_{k_0+\cdots+k_{s-1}}}.$$

Proposition 5.18 says that $\rho_{k_0+\cdots+k_{s-1}+j}$ is an element of $\mathrm{Frac}(\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}) \cap \mathrm{S}(\mathfrak{g}^e) = \mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$.

**Lemma 5.21.** *Let $s \in \{1, \dots, m\}$ and $j \in \{k_s/2 + 1, \dots, k_s\}$. There is a polynomial $R^{(s)}_j$ of degree $j$ such that*

$$\rho_{k_0+\cdots+k_{s-1}+j} = R^{(s)}_j(\rho_{k_0+\cdots+k_{s-1}+1}, \dots, \rho_{k_0+\cdots+k_{s-1}+k_s/2}).$$

*In particular, for any $j \in \{k_1/2 + 1, \dots, k_1\}$, we have*

$$\overline{p}_j = R^{(1)}_j(\overline{p}_1, \dots, \overline{p}_{k_1/2}).$$

*Proof.* 1) Prove by induction on $j \in \{1, \dots, k_s/2\}$ that for some polynomial $T^{(s)}_j$ of degree $j$,

$$\sigma^{(s)}_j = T^{(s)}_j(\rho_{k_0+\cdots+k_{s-1}+1}, \dots, \rho_{k_0+\cdots+k_{s-1}+j}).$$

By Proposition 5.18, $\rho_{k_0+\cdots+k_{s-1}+1} = -(\sigma_1^{(s)}\sigma_0^{(s)} + \sigma_0^{(s)}\sigma_1^{(s)}) = -2\sigma_1^{(s)}$. Hence, the statement is true for $j = 1$. Suppose $j \in \{2, \dots, k_s/2\}$ and the statement true for $\sigma_1^{(s)}, \dots, \sigma_{j-1}^{(s)}$. Since $j \leqslant k_s/2$, $\sigma_j^{(s)} \neq 0$, and by Proposition 5.18,

$$\rho_{k_0+\cdots+k_{s-1}+j} = (-1)^j(\sigma_j^{(s)}\sigma_0^{(s)} + \sigma_0\sigma_j^{(s)}) + (-1)^j \sum_{r=1}^{j-1} \sigma_{j-r}^{(s)}\sigma_r^{(s)} = 2(-1)^j\sigma_j^{(s)} + (-1)^j \sum_{r=1}^{j-1} \sigma_{j-r}^{(s)}\sigma_r^{(s)}.$$

So, the statement for $j$ follows from our induction hypothesis.

2) Let $j \in \{k_s/2 + 1, \dots, k_s\}$. Proposition 5.18 shows that $\rho_{k_0+\cdots+k_{s-1}+j}$ is a polynomial in $\sigma_1^{(s)}, \dots, \sigma_{k_s/2}^{(s)}$. Hence, by Step 1), $\rho_{k_0+\cdots+k_{s-1}+j}$ is a polynomial in

$$\rho_{k_0+\cdots+k_{s-1}+1}, \dots, \rho_{k_0+\cdots+k_{s-1}+k_s/2}.$$

Furthermore, by Proposition 5.18 and Step (1), this polynomial has degree $j$. □

*Remark* 5.22. By Remark 5.20 and the above proof, if $k' = k$ then for some polynomial $\tilde{R}$ of degree $k_m/2$,

$$\frac{\overline{\tilde{p}}_k}{\sigma_{k_0/2}^{(1)} \dots \sigma_{k_{m-1}/2}^{(m-1)}} = \sigma_{k_m/2}^{(m)} = \tilde{R}(\rho_{k_0+\cdots+k_{m-1}+1}, \dots, \rho_{k_0+\cdots+k_{m-1}+k_m/2}).$$

**Theorem 5.23.** (i) *Assume that $\lambda$ verifies the condition $(*)$ and that $\lambda_1 = \cdots = \lambda_{k'}$. Then $e$ is good.*

(ii) *Assume that $k = 4$ and that $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are even. Then $e$ is good.*

For example, $(6, 6, 6, 6, 5, 3)$ satisfies the hypothesis of (i) and $(6, 6, 4, 4)$ satisfies the hypothesis of (ii).

*Remark* 5.24. If $\lambda$ verifies the condition $(*)$ then by Lemma 5.7,

$$\dim \mathfrak{g}^e + \ell - 2(\delta_1 + \cdots + \delta_\ell) = k'.$$

Indeed, if $k$ is odd, then $n_\lambda - d_\lambda = n_{\lambda'} - d_{\lambda'}$ where $\lambda' = (\lambda_1, \dots, \lambda_{k'}, \lambda_{k'+1})$ so that $n_\lambda - d_\lambda = n_{\lambda'} - d_{\lambda'} = n_{\lambda'} = k'$ since $\lambda_{k'+1}$ is odd. If $k$ is even, then $d_\lambda = n_{\lambda'} = k'$ where $\lambda' = (\lambda_1, \dots, \lambda_{k'})$.

*Proof.* (i) In the previous notations, the hypothesis means that $m = 1$ and $k' = k_m$. According to Lemma 5.21 and Lemma 5.14, for $j \in \{k'/2 + 1, \dots, k' - 1\}$,

$$p_j = R_j^{(1)}(p_1, \dots, p_{k'/2}),$$

where $R_j^{(1)}$ is a polynomial of degree $j$. Moreover, if $k' = k$, then by Remark 5.22 and Lemma 5.14,

$$\tilde{p}_k = \tilde{R}(p_1, \dots, p_{k/2}),$$

where $\tilde{R}$ is a polynomial of degree $k/2$.

- If $k' < k$, set for any $j \in \{k'/2 + 1, \dots, k'\}$,

$$r_j := q_{\nu_j} - R_j^{(1)}(q_{\nu_1}, \dots, q_{\nu_{k'/2}}).$$

Then by Lemma 5.12,

$$\forall\, j \in \{k'/2 + 1, \dots, k'\}, \quad \deg {}^e r_j \geqslant j + 1.$$

- If $k' = k$, set for $j \in \{k/2 + 1, \dots, k' - 1\}$,

$$r_j := q_{\nu_j} - R_j^{(1)}(q_{\nu_1}, \dots, q_{\nu_{k'/2}}) \text{ and } r_k := q_{\nu_k} - \tilde{R}(q_{\nu_1}, \dots, q_{\nu_{k/2}}).$$

Then by Lemma 5.12,

$$\forall\, j \in \{k/2 + 1, \dots, k - 1\}, \quad \deg {}^e r_j \geqslant j + 1 \quad \text{and} \quad \deg {}^e r_k \geqslant k/2 + 1.$$

In both cases,

$$\{q_j \mid j \in \{1,\dots,\ell\} \smallsetminus \{\nu_{k'/2+1},\dots,\nu_{k'}\}\} \ \cup \ \{r_{k'/2+1},\dots,r_{k'}\}$$

is a homogeneous generating system of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$. Denote by $\hat{\delta}$ the sum of the degrees of the polynomials

$${}^e q_j,\ \ j \in \{1,\dots,\ell\} \smallsetminus \{\nu_{k'/2+1},\dots,\nu_{k'}\}, \qquad {}^e r_{k'/2+1},\dots,{}^e r_{k'}.$$

The above discussion shows that $\hat{\delta} \geqslant \delta_1 + \cdots + \delta_\ell + k'/2$. By Remarks 5.24, we obtain

$$\dim \mathfrak{g}^e + \ell - 2\hat{\delta} \leqslant 0.$$

In conclusion, by [PPY07, Theorem 2.1] and Theorem 3.6, $e$ is good.

(ii) In the previous notations, the hypothesis means that $k' = k = 4$. If $m = 1$ the statement is a consequence of (i). Assume that $m = 2$. Then by Proposition 5.18, $\overline{p}_1 = -2z_1$, $\overline{p}_2 = z_1^2$, $\overline{p}_3 = -2z_1^2 z_2$ and $\overline{p}_4 = (z_1 z_2)^2$. Moreover, $\overline{\tilde{p}}_4 = z_1 z_2$. Hence, by Lemma 5.14, $p_2 = \frac{1}{4}p_1^2$ and $p_3 = p_1 \tilde{p}_4$. Set $r_2 := q_{\nu_2} - \frac{1}{4}q_{\nu_1}^2$ and $r_3 := q_{\nu_3} - q_{\nu_1}q_{\nu_4}$. Then $\deg {}^e r_2 \geqslant 3$ and $\deg {}^e r_3 \geqslant 4$. Moreover,

$$\{q_1,\dots,q_\ell\} \smallsetminus \{q_{\nu_2}, q_{\nu_3}\} \cup \{r_2, r_3\}$$

is a homogeneous generating system of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$. Denoting by $\hat{\delta}$ the sum of the degrees of the polynomials

$$\{{}^e q_1,\dots,{}^e q_\ell\} \smallsetminus \{{}^e q_{\nu_2}, {}^e q_{\nu_3}\} \cup \{{}^e r_2, {}^e r_3\},$$

we obtain that $\hat{\delta} \geqslant \delta_1 + \cdots + \delta_\ell + 2$. But $\dim \mathfrak{g}^e + \ell - 2(\delta_1 + \cdots + \delta_\ell) = k' = 4$ by Remark 5.24. So, $\dim \mathfrak{g}^e + \ell - 2\hat{\delta} \leqslant 0$. In conclusion, by [PPY07, Theorem 2.1] and Theorem 3.6, $e$ is good. □

## 6. Examples in simple exceptional Lie algebras

We give in this section examples of good nilpotent elements in simple exceptional Lie algebras (of type $\mathbf{E}_6$, $\mathbf{F}_4$ or $\mathbf{G}_2$) which are not covered by [PPY07]. These examples are all obtained through Theorem 3.6.

*Example* 6.1. Suppose that $\mathfrak{g}$ has type $\mathbf{E}_6$. Let $\mathbb{V}$ be the module of highest weight the fundamental weight $\varpi_1$ with the notation of Bourbaki. Then $\mathbb{V}$ has dimension 27 and $\mathfrak{g}$ identifies with a subalgebra of $\mathfrak{sl}_{27}(\Bbbk)$. For $x$ in $\mathfrak{sl}_{27}(\Bbbk)$ and for $i = 2,\dots,27$, let $p_i(x)$ be the coefficient of $T^{27-i}$ in $\det(T - x)$ and denote by $q_i$ the restriction of $p_i$ to $\mathfrak{g}$. Then $(q_2, q_5, q_6, q_8, q_9, q_{12})$ is a generating family of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ since these polynomials are algebraically independent, [Me88]. Let $(e, h, f)$ be an $\mathfrak{sl}_2$-triple of $\mathfrak{g}$. Then $(e, h, f)$ is an $\mathfrak{sl}_2$-triple of $\mathfrak{sl}_{27}(\Bbbk)$. We denote by ${}^e p_i$ the initial homogeneous component of the restriction to $e + \tilde{\mathfrak{g}}^f$ of $p_i$ where $\tilde{\mathfrak{g}}^f$ is the centralizer of $f$ in $\mathfrak{sl}_{27}(\Bbbk)$. As usual, ${}^e q_i$ denotes the initial homogeneous component of the restriction to $e + \mathfrak{g}^f$ of $q_i$. For $i = 2, 5, 6, 8, 9, 12$,

$$\deg {}^e p_i \leqslant \deg {}^e q_i.$$

In some cases, from the knowledge of the maximal eigenvalue of the restriction of $\mathrm{ad}\, h$ to $\mathfrak{g}$ and the $\mathrm{ad}\, h$-weight of ${}^e p_i$, it is possible to deduce that $\deg {}^e p_i < \deg {}^e q_i$. On the other hand,

$$\deg {}^e q_2 + \deg {}^e q_5 + \deg {}^e q_6 + \deg {}^e q_8 + \deg {}^e q_9 + \deg {}^e q_{12} \leqslant \frac{1}{2}(\dim \mathfrak{g}^e + 6),$$

with equality if and only if ${}^e q_2$, ${}^e q_5$, ${}^e q_6$, ${}^e q_8$, ${}^e q_9$, ${}^e q_{12}$ are algebraically independent. From this, it is possible to deduce in some cases that $e$ is good. These cases are listed in Table 2 where the nine columns are indexed in the following way:

1: the label of the orbit $G.e$ in the Bala-Carter classification,
2: the weighted Dynkin diagram of $G.e$,
3: the dimension of $\mathfrak{g}^e$,
4: the partition of 27 corresponding to the nilpotent element $e$ of $\mathfrak{sl}_{27}(\Bbbk)$,

5: the degrees of ${}^ep_2$, ${}^ep_5$, ${}^ep_6$, ${}^ep_8$, ${}^ep_9$, ${}^ep_{12}$,
6: their $\operatorname{ad} h$-weights,
7: the maximal eigenvalue $\nu$ of the restriction of $\operatorname{ad} h$ to $\mathfrak{g}$,
8: the sum $\Sigma$ of the degrees of ${}^ep_2$, ${}^ep_5$, ${}^ep_6$, ${}^ep_8$, ${}^ep_9$, ${}^ep_{12}$,
9: the sum $\Sigma' = \frac{1}{2}(\dim \mathfrak{g}^e + \ell)$.

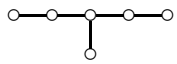
| | Label |  | dim $\mathfrak{g}^e$ | partition | deg ${}^ep_i$ | weights | $\nu$ | $\Sigma$ | $\Sigma'$ |
|---|---|---|---|---|---|---|---|---|---|
| **1.** | $E_6$ | 2 2 2 2 2 | 6 | (17,9,1) | 1,1,1,1,1,1 | 2,8,10,14,16,22 | 16 | 6 | 6 |
| **2.** | $E_6(a_1)$ | 2 2 0 2 2 | 8 | (13,9,5) | 1,1,1,1,1,1 | 2,8,10,14,16,22 | 16 | 6 | 7 |
| **3.** | $D_5$ | 2 0 2 0 2 | 10 | (11,9,5,1,1) | 1,1,1,1,1,1 | 2,8,10,14,16,22 | 14 | 6 | 8 |
| **4.** | $A_5 + A_1$ | 2 0 2 0 2 | 12 | $(9, 7, 5^2, 1)$ | 1,1,1,1,1,2 | 2,8,10,14,16,20 | 10 | 7 | 9 |
| **5.** | $D_5(a_1)$ | 1 1 0 1 1 | 14 | (8,7,6,3,2,1) | 1,1,1,1,2,2 | 2,8,10,14,14,20 | 10 | 8 | 10 |
| **6.** | $A_5$ | 2 1 0 1 2 | 14 | $(9, 6^2, 5, 1)$ | 1,1,1,1,1,2 | 2,8,10,14,16,20 | 10 | 7 | 10 |
| **7.** | $A_4 + A_1$ | 1 1 0 1 1 | 16 | $(7, 6, 5, 4, 3, 2)$ | 1,1,1,2,2,2 | 2,8,10,12,14,20 | 8 | 9 | 11 |
| **8.** | $D_4$ | 0 0 2 0 0 | 18 | $(7^3, 1^6)$ | 1,1,1,2,2,2 | 2,8,10,12,14,20 | 10 | 9 | 12 |
| **9.** | $A_3 + 2A_1$ | 0 0 2 0 0 | 20 | $(5^3, 3^3, 1^3)$ | 1,1,2,2,2,3 | 2,8,8,12,14,18 | 6 | 11 | 13 |
| **10.** | $A_1 + 2A_2$ | 1 0 1 0 1 | 24 | $(5, 4^2, 3^3, 2^2, 1)$ | 1,1,2,2,2,3 | 2,8,8,12,14,18 | 5 | 11 | 15 |

TABLE 2. Data for $\mathbf{E}_6$

For the orbit **1**, $\Sigma = \Sigma'$. Hence, ${}^eq_2$, ${}^eq_5$, ${}^eq_6$, ${}^eq_8$, ${}^eq_9$, ${}^eq_{12}$ are algebraically independent and by Theorem 3.6, $e$ is good. For the orbits **2**,**3**,...,**10**, we observe that $\Sigma < \Sigma'$, i.e.,

$$\deg {}^ep_2 + \deg {}^ep_5 + \deg {}^ep_6 + \deg {}^ep_8 + \deg {}^ep_9 + \deg {}^ep_{12} < \frac{1}{2}(\dim \mathfrak{g}^e + 6).$$

So, we need some more arguments that we give below.

**2.** Since $16 < 22$, $\deg {}^ep_{12} < \deg {}^eq_{12}$.
**3.** Since $14 < 16$, $\deg {}^ep_i < \deg {}^eq_i$ for $i = 9, 12$.
**4.** Since $10 < 14$, $\deg {}^ep_i < \deg {}^eq_i$ for $i = 8, 9$.
**5.** Since $10 < 14$, $\deg {}^ep_8 < \deg {}^eq_8$. Moreover, the multiplicity of the weight 10 equals 1. So, either $\deg {}^eq_6 > 1$, or $\deg {}^eq_{12} > 2$, or ${}^eq_{12} \in \Bbbk\, {}^eq_6^2$.
**6.** Since $10 < 14$, $\deg {}^ep_i < \deg {}^eq_i$ for $i = 8, 9$. Moreover, the multiplicity of the weight 10 equals 1. So, either $\deg {}^eq_6 > 1$, or $\deg {}^eq_{12} > 2$, or ${}^eq_{12} \in \Bbbk\, {}^eq_6^2$.
**7.** Since $8 < 10$ and $2\times 8 < 20$, $\deg {}^ep_i < \deg {}^eq_i$ for $i = 6, 12$.

**8.** Since the center of $\mathfrak{g}^e$ has dimension 2 and the weights of $h$ in the center are 2 and 10, $\deg {}^e p_5 < \deg {}^e q_5$. Moreover, since the weights of $h$ in $\mathfrak{g}^e$ are $0, 2, 6, 10$, $\deg {}^e p_9 < \deg {}^e q_9$ and since the multiplicity of the weight 10 equals 1, either $\deg {}^e q_6 > 1$, or $\deg {}^e q_{12} > 2$, or ${}^e q_{12} \in \Bbbk\, {}^e q_6^2$.

**9.** Since $6 < 8$ and $2\times 6 < 14$, $\deg {}^e p_i < \deg {}^e q_i$ for $i = 5, 9$.

**10.** Since $5 < 8$, $2\times 5 < 12$ and $3\times 5 < 18$, $\deg {}^e p_i < \deg {}^e q_i$ for $i = 5, 8, 9, 12$.

In cases **2**, **3**, **4**, **7**, **9**, **10**, the discussion shows that

$$\deg {}^e q_2 + \deg {}^e q_5 + \deg {}^e q_6 + \deg {}^e q_8 + \deg {}^e q_9 + \deg {}^e q_{12} = \frac{1}{2}(\dim \mathfrak{g}^e + 6).$$

Hence, ${}^e q_2$, ${}^e q_5$, ${}^e q_6$, ${}^e q_8$, ${}^e q_9$, ${}^e q_{12}$ are algebraically independent and by Theorem 3.6, $e$ is good. In cases **5**, **6**, **8**, if the above equality does not hold, then for some $a$ in $\Bbbk^*$,

$$\deg {}^e q_2 + \deg {}^e q_5 + \deg {}^e q_6 + \deg {}^e q_8 + \deg {}^e q_9 + \deg {}^e(q_{12} - a q_6^2) = \frac{1}{2}(\dim \mathfrak{g}^e + 6).$$

Hence ${}^e q_2$, ${}^e q_5$, ${}^e q_6$, ${}^e q_8$, ${}^e q_9$, ${}^e(q_{12} - a q_6^2)$ are algebraically independent and by Theorem 3.6, $e$ is good.

In addition, according to [PPY07, Theorem 0.4] and Theorem 3.6, the elements of the minimal orbit of $\mathbf{E}_6$, labelled $A_1$, are good. In conclusion, it remains nine unsolved nilpotent orbits in type $\mathbf{E}_6$.

*Example* 6.2. Suppose that $\mathfrak{g}$ is simple of type $\mathbf{F}_4$. Let $\mathbb{V}$ be the module of highest weight the fundamental weight $\varpi_4$ with the notation of Bourbaki. Then $\mathbb{V}$ has dimension 26 and $\mathfrak{g}$ identifies with a subalgebra of $\mathfrak{sl}_{26}(\Bbbk)$. For $x$ in $\mathfrak{sl}_{26}(\Bbbk)$ and for $i = 2, \dots, 26$, let $p_i(x)$ be the coefficient of $T^{26-i}$ in $\det(T - x)$ and denote by $q_i$ the restriction of $p_i$ to $\mathfrak{g}$. Then $(q_2, q_6, q_8, q_{12})$ is a generating family of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ since these polynomials are algebraically independent, [Me88]. Let $(e, h, f)$ be an $\mathfrak{sl}_2$-triple of $\mathfrak{g}$. Then $(e, h, f)$ is an $\mathfrak{sl}_2$-triple of $\mathfrak{sl}_{26}(\Bbbk)$. As in Example 6.1, in some cases, it is possible to deduce that $e$ is good. These cases are listed in Table 3, indexed as in Example 6.1.

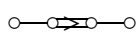
| | Label |  | | | | $\dim \mathfrak{g}^e$ | partition | $\deg {}^e p_i$ | weights | $\nu$ | $\Sigma$ | $\Sigma'$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **1.** | $F_4$ | 2 | 2 | 2 | 2 | 4 | (17,9) | 1,1,1,1 | 2,10,14,22 | 22 | 4 | 4 |
| **2.** | $B_4$ | 2 | 2 | 0 | 2 | 6 | (11,9,5,1) | 1,1,1,1 | 2,10,14,22 | 14 | 4 | 5 |
| **3.** | $C_3 + A_1$ | 0 | 2 | 0 | 2 | 8 | $(9, 7, 5^2)$ | 1,1,1,2 | 2,10,14,20 | 10 | 5 | 6 |
| **4.** | $C_3$ | 1 | 0 | 1 | 2 | 10 | $(9, 6^2, 5)$ | 1,1,1,2 | 2,10,14,20 | 10 | 5 | 7 |
| **5.** | $B_3$ | 2 | 2 | 0 | 0 | 10 | $(7^3, 1^5)$ | 1,1,2,2 | 2,10,12,20 | 10 | 6 | 7 |
| **6.** | $\tilde{A}_2 + A_2$ | 0 | 2 | 0 | 0 | 12 | $(5^3, 3^3, 1^2)$ | 1,2,2,3 | 2,8,12,18 | 6 | 8 | 8 |
| **7.** | $B_2 + A_1$ | 1 | 0 | 1 | 0 | 14 | $(5^2, 4^2, 3, 2^2, 1)$ | 1,2,2,3 | 2,8,12,18 | 6 | 8 | 9 |
| **8.** | $\tilde{A}_2 + A_1$ | 0 | 1 | 0 | 1 | 16 | $(5, 4^2, 3^3, 2^2)$ | 1,2,2,3 | 2,8,12,18 | 5 | 8 | 10 |

TABLE 3. Data for $\mathbf{F}_4$

For the orbits **2**, **3**, **4**, **5**, **7**,**8**, we observe that $\Sigma < \Sigma'$. So, we need some more arguments to conclude as in Example 6.1.

- **2.** Since $14 < 22$, $\deg {}^e p_{12} < \deg {}^e q_{12}$.
- **3.** Since $10 < 14$, $\deg {}^e p_8 < \deg {}^e q_8$.
- **4.** Since $10 < 14$, $\deg {}^e p_8 < \deg {}^e q_8$. Moreover, the multiplicity of the weight 10 equals 1 so that $\deg {}^e q_6 > 1$ or $\deg {}^e q_{12} > 2$ or ${}^e q_{12} \in \Bbbk\, {}^e q_6^2$.
- **5.** The multiplicity of the weight 10 equals 1. So, either $\deg {}^e q_6 > 1$, or $\deg {}^e q_{12} > 2$, or ${}^e q_{12} \in \Bbbk\, {}^e q_6^2$.
- **7.** Suppose that ${}^e q_2$, ${}^e q_6$, ${}^e q_8$, ${}^e q_{12}$ have degree $1, 2, 2, 3$. We expect a contradiction. Since the center has dimension 2 and since the multiplicity of the weight 6 equals 1, for $z$ of weight 6 in the center, ${}^e q_6 \in \Bbbk e z$, ${}^e q_8 \in \Bbbk z^2$, ${}^e q_{12} \in \Bbbk z^3$. So, for some $a$ and $b$ in $\Bbbk^*$,

$$ {}^e q_2^2\, {}^e q_8 - a\, {}^e q_6^2 = 0, \qquad {}^e q_{12}^2 - b\, {}^e q_8^3 = 0 $$

  Hence, $q_2, q_6, q_2^2 q_8 - a q_6^2, q_{12}^2 - b q_8^3$ are algebraically independent element of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ such that

$$ \deg {}^e q_2 + \deg {}^e q_6 + \deg {}^e(q_2^2 q_8 - a q_6^2) + \deg {}^e(q_{12}^2 - b q_8^3) \geqslant 1 + 2 + 5 + 7 > 2 + 3 + 9 $$

  whence a contradiction by [PPY07, Theorem 2.1] (see Lemma 7.1).
- **8.** Since $2\times 5 < 12$ and $3\times 5 < 18$, $\deg {}^e q_8 > \deg {}^e p_8$ and $\deg {}^e q_{12} > \deg {}^e p_{12}$.

In addition, according to [PPY07, Theorem 0.4] and Theorem 3.6, the elements of the minimal orbit of $\mathbf{F}_4$, labelled $A_1$, are good. In conclusion, it remains six unsolved nilpotent orbits in type $\mathbf{F}_4$.

*Example* 6.3. Suppose that $\mathfrak{g}$ is simple of type $\mathbf{G}_2$. Let $\mathbb{V}$ be the module of highest weight the fundamental weight $\varpi_1$ with the notation of Bourbaki. Then $\mathbb{V}$ has dimension 7 and $\mathfrak{g}$ identifies with a subalgebra of $\mathfrak{sl}_7(\Bbbk)$. For $x$ in $\mathfrak{sl}_7(\Bbbk)$ and for $i = 2, \ldots, 7$, let $p_i(x)$ be the coefficient of $T^{7-i}$ in $\det(T - x)$ and denote by $q_i$ the restriction of $p_i$ to $\mathfrak{g}$. Then $q_2, q_6$ is a generating family of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ since these polynomials are algebraically independent, [Me88]. Let $(e, h, f)$ be an $\mathfrak{sl}_2$-triple of $\mathfrak{g}$. Then $(e, h, f)$ is an $\mathfrak{sl}_2$-triple of $\mathfrak{sl}_7(\Bbbk)$. In all cases, we deduce that $e$ is good from Table 4, indexed as in Example 6.1.

| | Label | | | $\dim \mathfrak{g}^e$ | partition | $\deg {}^e p_i$ | weights | $\nu$ | $\Sigma$ | $\Sigma$ |
|---|---|---|---|---|---|---|---|---|---|---|
| **1.** | $G_2$ | 2 | 2 | 2 | (7) | 1,1 | 2,10 | 10 | 2 | 2 |
| **2.** | $A_1 + \tilde{A}_1$ | 0 | 2 | 4 | $(3^2, 1)$ | 1,2 | 2,8 | 4 | 3 | 3 |
| **3.** | $\tilde{A}_1$ | 1 | 0 | 6 | $(3, 2^2)$ | 1,3 | 2,6 | 3 | 4 | 4 |
| **4.** | $A_1$ | 0 | 1 | 8 | $(2^2, 1^3)$ | 1,4 | 2,4 | 2 | 5 | 5 |

TABLE 4. Data for $\mathbf{G}_2$

## 7. Other examples, remarks and a conjecture

This section provides examples of nilpotent elements which verify the polynomiality condition but that are not good. We also obtain an example of nilpotent element in type $\mathbf{D}_7$ which does not verify the polynomiality condition (cf. Example 7.8). Then we conclude with some remarks and a conjecture.

7.1. **Some general results.** In this subsection, $\mathfrak{g}$ is a simple Lie algebra over $\Bbbk$ and $(e, h, f)$ is an $\mathfrak{sl}_2$-triple of $\mathfrak{g}$. For $p$ in $\mathrm{S}(\mathfrak{g})$, ${}^e p$ is the initial homogeneous component of the restriction of $p$ to the Slodowy slice $e + \mathfrak{g}^f$. Recall that $\Bbbk[e + \mathfrak{g}^f]$ identifies with $\mathrm{S}(\mathfrak{g}^e)$ by the Killing form $\langle .,. \rangle$ of $\mathfrak{g}$.

Let $\eta_0 \in \mathfrak{g}^e \otimes_\Bbbk \bigwedge^2 \mathfrak{g}^f$ be the bivector defining the Poisson bracket on $\mathrm{S}(\mathfrak{g}^e)$ induced from the Lie bracket. According to the main theorem of [Pr02], $\mathrm{S}(\mathfrak{g}^e)$ is the graded algebra relative to the Kazhdan filtration of the finite $W$-algebra associated with $e$ so that $\mathrm{S}(\mathfrak{g}^e)$ inherits another Poisson structure. The so-obtained graded algebra structure is the Slodowy graded algebra structure (see Subsection 4.1). Let $\eta \in \mathrm{S}(\mathfrak{g}^e) \otimes_\Bbbk \bigwedge^2 \mathfrak{g}^f$ be the bivector defining this other Poisson structure. According to [Pr02, Proposition 6.3] (see also [PPY07, §2.4]), $\eta_0$ is the initial homogeneous component of $\eta$. Denote by $r$ the dimension of $\mathfrak{g}^e$ and set:

$$\omega := \eta^{(r-\ell)/2} \in \mathrm{S}(\mathfrak{g}^e) \otimes_\Bbbk \bigwedge^{r-\ell} \mathfrak{g}^f, \qquad \omega_0 := \eta_0^{(r-\ell)/2} \in \mathrm{S}(\mathfrak{g}^e) \otimes_\Bbbk \bigwedge^{r-\ell} \mathfrak{g}^f.$$

Then $\omega_0$ is the initial homogeneous component of $\omega$.

Let $v_1, \dots, v_r$ be a basis of $\mathfrak{g}^f$. For $\mu$ in $\mathrm{S}(\mathfrak{g}^e) \otimes_\Bbbk \bigwedge^i \mathfrak{g}^e$, denote by $j(\mu)$ the image of $v_1 \wedge \cdots \wedge v_r$ by the right interior product of $\mu$ so that

$$j(\mu) \in \mathrm{S}(\mathfrak{g}^e) \otimes_\Bbbk \bigwedge^{r-i} \mathfrak{g}^f.$$

**Lemma 7.1.** *Let $q_1, \dots, q_\ell$ be some homogeneous generators of $\mathrm{S}(\mathfrak{g})^\mathfrak{g}$ and let $r_1, \dots, r_\ell$ be algebraically independent homogeneous elements of $\mathrm{S}(\mathfrak{g})^\mathfrak{g}$.*

(i) *For some homogeneous element $p$ of $\mathrm{S}(\mathfrak{g})^\mathfrak{g}$,*

$$\mathrm{d}r_1 \wedge \cdots \wedge \mathrm{d}r_\ell = p\, \mathrm{d}q_1 \wedge \cdots \wedge \mathrm{d}q_\ell.$$

(ii) *The following inequality holds:*

$$\sum_{i=1}^{\ell} \deg {}^e r_i \leqslant \deg {}^e p + \frac{1}{2}(\dim \mathfrak{g}^e + \ell).$$

(iii) *The polynomials ${}^e r_1, \dots, {}^e r_\ell$ are algebraically independent if and only if*

$$\sum_{i=1}^{\ell} \deg {}^e r_i = \deg {}^e p + \frac{1}{2}(\dim \mathfrak{g}^e + \ell).$$

*Proof.* (i) Since $q_1, \dots, q_\ell$ are generators of $\mathrm{S}(\mathfrak{g})^\mathfrak{g}$, for $i \in \{1, \dots, \ell\}$, $r_i = R_i(q_1, \dots, q_\ell)$ where $R_i$ is a polynomial in $\ell$ indeterminates, whence the assertion with

$$p = \det\left(\frac{\partial R_i}{\partial q_j},\ 1 \leqslant i, j \leqslant \ell\right).$$

(ii) Remind that for $p$ in $\mathrm{S}(\mathfrak{g})$, $\kappa(p)$ denotes the restriction to $\mathfrak{g}^f$ of the polynomial function $x \mapsto p(e + x)$. According to [PPY07, Theorem 1.2],

$$j(\mathrm{d}\kappa(q_1) \wedge \cdots \wedge \mathrm{d}\kappa(q_\ell)) = a\omega$$

for some $a$ in $\Bbbk^*$. Hence by (i),

$$j(\mathrm{d}\kappa(r_1) \wedge \cdots \wedge \mathrm{d}\kappa(r_\ell)) = a\kappa(p)\omega.$$

The initial homogeneous component of the right-hand side is $a\, {}^e p \omega_0$ and the degree of the initial homogeneous component of the left-hand side is at least

$$\deg {}^e r_1 + \cdots + \deg {}^e r_\ell - \ell.$$

The assertion follows since $\omega_0$ has degree

$$\frac{1}{2}(\dim \mathfrak{g}^e - \ell).$$

(iii) If ${}^e r_1, \dots, {}^e r_\ell$ are algebraically independent, then the degree of the initial homogeneous component of $j(\mathrm{d}r_1 \wedge \cdots \wedge \mathrm{d}r_\ell)$ equals

$$\deg {}^e r_1 + \cdots + \deg {}^e r_\ell - \ell$$

whence

$$\deg {}^e r_1 + \cdots + \deg {}^e r_\ell = \deg {}^e p + \frac{1}{2}(\dim \mathfrak{g}^e + \ell)$$

by the proof of (ii). Conversely, if the equality holds, then

$$j(\mathrm{d}\,{}^e r_1 \wedge \cdots \wedge \mathrm{d}\,{}^e r_\ell) = a\,{}^e p \omega_0 \tag{4}$$

by the proof of (ii). In particular, ${}^e r_1, \dots, {}^e r_\ell$ are algebraically independent. □

**Corollary 7.2.** *For $i = 1, \dots, \ell$, let $r_i := R_i(q_1, \dots, q_i)$ be a homogeneous element of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ such that $\dfrac{\partial R_i}{\partial q_i} \neq 0$. Then ${}^e r_1, \dots, {}^e r_\ell$ are algebraically independent if and only if*

$$\deg {}^e r_1 + \cdots + \deg {}^e r_\ell = \sum_{i=1}^{\ell} \deg {}^e p_i + \frac{1}{2}(\dim \mathfrak{g}^e + \ell)$$

*with $p_i = \dfrac{\partial R_i}{\partial q_i}$ for $i = 1, \dots, \ell$.*

*Proof.* Since $\dfrac{\partial R_i}{\partial q_i} \neq 0$ for all $i$, $r_1, \dots, r_\ell$ are algebraically independent and

$$\mathrm{d}r_1 \wedge \cdots \wedge \mathrm{d}r_\ell = \prod_{i=1}^{\ell} \frac{\partial R_i}{\partial q_i} \mathrm{d}q_1 \wedge \cdots \wedge \mathrm{d}q_\ell$$

whence the corollary by Lemma 7.1,(iii). □

Let $\mathfrak{g}^f_{\mathrm{sing}}$ be the set of nonregular elements of the dual $\mathfrak{g}^f$ of $\mathfrak{g}^e$. If $\mathfrak{g}^f_{\mathrm{sing}}$ has codimension at least 2 in $\mathfrak{g}^f$, we will say that $\mathfrak{g}^e$ is *nonsingular*.

**Corollary 7.3.** *Let $r_1, \dots, r_\ell$ and $p$ be as in Lemma 7.1 and such that ${}^e r_1, \dots, {}^e r_\ell$ are algebraically independent.*

(i) *If ${}^e p$ is a greatest divisor of $\mathrm{d}\,{}^e r_1 \wedge \cdots \wedge \mathrm{d}\,{}^e r_\ell$ in $\mathrm{S}(\mathfrak{g}^e) \otimes_{\Bbbk} \bigwedge^\ell \mathfrak{g}^e$, then $\mathfrak{g}^e$ is nonsingular.*

(ii) *Assume that there are homogeneous polynomials $p_1, \dots, p_\ell$ in $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ verifying the following conditions:*

1) *${}^e r_1, \dots, {}^e r_\ell$ are in $\Bbbk[p_1, \dots, p_\ell]$,*
2) *if $d$ is the degree of a greatest divisor of $\mathrm{d}p_1 \wedge \cdots \wedge \mathrm{d}p_\ell$ in $\mathrm{S}(\mathfrak{g}^e)$, then*

$$\deg p_1 + \cdots + \deg p_\ell = d + \frac{1}{2}(\dim \mathfrak{g}^e + \ell).$$

*Then $\mathfrak{g}^e$ is nonsingular.*

*Proof.* (i) Suppose that ${}^e p$ is a greatest divisor of $\mathrm{d}\,{}^e r_1 \wedge \cdots \wedge \mathrm{d}\,{}^e r_\ell$ in $\mathrm{S}(\mathfrak{g}^e) \otimes_{\Bbbk} \bigwedge^\ell \mathfrak{g}^e$. Then for some $\omega_1$ in $\mathrm{S}(\mathfrak{g}^e) \otimes_{\Bbbk} \bigwedge^\ell \mathfrak{g}^e$ whose nullvariety in $\mathfrak{g}^f$ has codimension at least 2,

$$\mathrm{d}\,{}^e r_1 \wedge \cdots \wedge \mathrm{d}\,{}^e r_\ell = {}^e p\, \omega_1.$$

Therefore $j(\omega_1) = a\omega_0$ by Equality (4). Since $x$ is in $\mathfrak{g}^f_{\mathrm{sing}}$ if and only if $\omega_0(x) = 0$, we get (i).

(ii) By Condition (1),

$$\mathrm{d}\,{}^e r_1 \wedge \cdots \wedge \mathrm{d}\,{}^e r_\ell = q\,\mathrm{d}p_1 \wedge \cdots \wedge \mathrm{d}p_\ell$$

for some $q$ in $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$, and for some greatest divisor $q'$ of $\mathrm{d}p_1 \wedge \cdots \wedge \mathrm{d}p_\ell$ in $\mathrm{S}(\mathfrak{g}^e) \otimes_{\Bbbk} \bigwedge^\ell \mathfrak{g}^e$,

$$\mathrm{d}p_1 \wedge \cdots \wedge \mathrm{d}p_\ell = q'\omega_1.$$

So, by Equality (4),

$$qq'\,j(\omega_1) = a\,{}^e p\,\omega_0, \tag{5}$$

so that ${}^e p$ divides $qq'$ in $\mathrm{S}(\mathfrak{g}^e)$. By Condition (2) and Equality (5), $\omega_0$ and $\omega_1$ have the same degree. Then $qq'$ is in $\Bbbk^*\,{}^e p$, and for some $a'$ in $\Bbbk^*$,

$$j(\omega_1) = a'\omega_0,$$

whence (ii), again since $x$ is in $\mathfrak{g}^f_{\mathrm{sing}}$ if and only if $\omega_0(x) = 0$. □

The following proposition is a particular case of [JS10, §5.7]. More precisely, part (i) follows from [JS10, Remark 5.7] and part (ii) follows from [JS10, Theorem 5.7].

**Proposition 7.4.** *Suppose that $\mathfrak{g}^e$ is nonsingular.*

(i) *If there exist algebraically independent homogeneous polynomials $p_1, \ldots, p_\ell$ in $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ such that*

$$\deg p_1 + \cdots + \deg p_\ell = \frac{1}{2}(\dim \mathfrak{g}^e + \ell)$$

*then $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is a polynomial algebra generated by $p_1, \ldots, p_\ell$.*

(ii) *Suppose that the semiinvariant elements of $\mathrm{S}(\mathfrak{g}^e)$ are invariant. If $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is a polynomial algebra then it is generated by homogeneous polynomials $p_1, \ldots, p_\ell$ such that*

$$\deg p_1 + \cdots + \deg p_\ell = \frac{1}{2}(\dim \mathfrak{g}^e + \ell).$$

7.2. **New examples.** To produce new examples, our general strategy is to apply Proposition 7.4,(i). To that end, we first apply Corollary 7.3 in order to prove that $\mathfrak{g}^e$ is nonsingular. Then, we search for independent homogeneous polynomials $p_1, \ldots, p_\ell$ in $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ satisfying the conditions of Corollary 7.3,(ii) with $d = 0$.

*Example* 7.5. Let $e$ be a nilpotent element of $\mathfrak{so}(\Bbbk^{10})$ associated with the partition $(3, 3, 2, 2)$. Then $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is a polynomial algebra but $e$ is not good as explained below.

In this case, $\ell = 5$ and let $q_1, \ldots, q_5$ be as in Subsection 5.2. The degrees of ${}^e q_1, \ldots, {}^e q_5$ are $1, 2, 2, 3, 2$ respectively. By a computation performed by Maple, ${}^e q_1, \ldots, {}^e q_5$ verify the algebraic relation:

$${}^e q_4^2 - 4\,{}^e q_3\,{}^e q_5^2 = 0.$$

Set:

$$r_i := \begin{cases} q_i & \text{if} \quad i = 1, 2, 3, 5 \\ q_4^2 - 4q_3q_5^2 & \text{if} \quad i = 4. \end{cases}$$

The polynomials $r_1, \ldots, r_5$ are algebraically independent over $\Bbbk$ and

$$\mathrm{d}r_1 \wedge \cdots \wedge \mathrm{d}r_5 = 2\,q_4\,\mathrm{d}q_1 \wedge \cdots \wedge \mathrm{d}q_5$$

Moreover, ${}^e r_4$ has degree at least 7. Then, by Corollary 7.2, ${}^e r_1, \ldots, {}^e r_5$ are algebraically independent since

$$\frac{1}{2}(\dim \mathfrak{g}^e + 5) + 3 = 14 = 1 + 2 + 2 + 2 + 7,$$

and by Lemma 7.1,(ii) and (iii), ${}^e r_4$ has degree 7.

A precise computation performed by Maple shows that ${}^e r_3 = p_3^2$ for some $p_3$ in the center of $\mathfrak{g}^e$, and that ${}^e r_4 = p_4\, {}^e r_5$ for some polynomial $p_4$ of degree 5 in $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$. Setting $p_i := {}^e r_i$ for $i = 1, 2, 5$, the polynomials $p_1, \dots, p_5$ are algebraically independent homogeneous polynomials of degree $1, 2, 1, 5, 2$ respectively. Furthermore, a computation performed by Maple proves that the greatest divisors of $\mathrm{d}p_1 \wedge \cdots \wedge \mathrm{d}p_5$ in $\mathrm{S}(\mathfrak{g}^e)$ have degree 0, and that $p_4$ is in the ideal of $\mathrm{S}(\mathfrak{g}^e)$ generated by $p_3$ and $p_5$. So, by Corollary 7.3,(ii), $\mathfrak{g}^e$ is nonsingular, and by Proposition 7.4,(i), $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is a polynomial algebra generated by $p_1, \dots, p_5$. Moreover, $e$ is not good since the nullvariety of $p_1, \dots, p_5$ in $(\mathfrak{g}^e)^*$ has codimension at most 4.

*Example* 7.6. In the same way, for the nilpotent element $e$ of $\mathfrak{so}(\Bbbk^{11})$ associated with the partition $(3, 3, 2, 2, 1)$, we can prove that $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is a polynomial algebra generated by polynomials of degree $1, 1, 2, 2, 7$, $\mathfrak{g}^e$ is nonsingular but $e$ is not good.

We also obtain that for the nilpotent element $e$ of $\mathfrak{so}(\Bbbk^{12})$ (resp. $\mathfrak{so}(\Bbbk^{13})$) associated with the partition (5,3,2,2) or (3,3,2,2,1,1) (resp. (5,3,2,2,1), (4,4,2,2,1), or (3,3,2,2,1,1,1)), $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is a polynomial algebra, $\mathfrak{g}^e$ is nonsingular but $e$ is not good.

We can summarize our conclusions for the small ranks. Assume that $\mathfrak{g} = \mathfrak{so}(\mathbb{V})$ for some vector space $\mathbb{V}$ of dimension $2\ell + 1$ or $2\ell$ and let $e \in \mathfrak{g}$ be a nilpotent element of $\mathfrak{g}$ associated with the partition $\lambda = (\lambda_1, \dots, \lambda_k)$ of $\dim \mathbb{V}$. If $\ell \leqslant 6$, our previous results (Corollary 5.8, Lemma 5.11, Theorem 5.23, Examples 7.5 and 7.6) show that either $e$ is good, or $e$ is not good but $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is nevertheless a polynomial algebra and $\mathfrak{g}^e$ is nonsingular. We describe in Table 5 the partitions $\lambda$ corresponding to good $e$, and those corresponding to the case where $e$ is not good. The third column of the table gives the degrees of the generators in the latter case.

| Type | $e$ is good | $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is polynomial, $\mathfrak{g}^e$ is nonsingular but $e$ is not good | degrees of the generators |
|---|---|---|---|
| $\mathbf{B}_n$, $\mathbf{D}_n$, $n \leqslant 4$ | any $\lambda$ | $\varnothing$ | |
| $\mathbf{B}_5$ | $\lambda \neq (3, 3, 2, 2, 1)$ | $\lambda = (3, 3, 2, 2, 1)$ | $1, 1, 2, 2, 7$ |
| $\mathbf{D}_5$ | $\lambda \neq (3, 3, 2, 2)$ | $\lambda = (3, 3, 2, 2)$ | 1,1,2,2,5 |
| $\mathbf{B}_6$ | $\lambda \notin \{(5, 3, 2, 2, 1), (4, 4, 2, 2, 1), (3, 3, 2, 2, 1, 1, 1)\}$ | $\lambda \in \{(5, 3, 2, 2, 1), (4, 4, 2, 2, 1), (3, 3, 2, 2, 1, 1, 1)\}$ | $\{1, 1, 1, 2, 2, 7; 1, 1, 2, 2, 3, 6; 1, 1, 2, 2, 6, 7\}$ |
| $\mathbf{D}_6$ | $\lambda \notin \{(5, 3, 2, 2), (3, 3, 2, 2, 1, 1)\}$ | $\lambda \in \{(5, 3, 2, 2), (3, 3, 2, 2, 1, 1)\}$ | $\{1,1,1,2,2,5; 1,1,2,2,3,7\}$ |

TABLE 5. Conclusions for $\mathfrak{g}$ of type $\mathbf{B}_\ell$ or $\mathbf{D}_\ell$ with $\ell \leqslant 6$

*Remark* 7.7. The above discussion shows that there are good nilpotent elements for which the codimension of $(\mathfrak{g}^e)^*_{\mathrm{sing}}$ in $(\mathfrak{g}^e)^*$ is 1. Indeed, by [PPY07, §3.9], for some nilpotent element $e'$ in $\mathbf{B}_3$, the codimension of $(\mathfrak{g}^{e'})^*_{\mathrm{sing}}$ in $(\mathfrak{g}^{e'})^*$ is 1 but, in $\mathbf{B}_3$, all nilpotent elements are good (cf. Table 5).

7.3. **A counter-example.** From the rank 7, there are elements that do no satisfy the polynomiality condition. The following example disconfirms a conjecture of Premet that any nilpotent element of a simple Lie algebra of classical type satisfies the polynomiality condition.

*Example* 7.8. Let $e$ be a nilpotent element of $\mathfrak{so}(\Bbbk^{14})$ associated with the partition $(3, 3, 2, 2, 2, 2)$. Then $e$ does not satisfy the polynomiality condition.

In this case, $\ell = 7$ and let $q_1, \dots, q_7$ be as in Subsection 5.2. The degrees of ${}^e q_1, \dots, {}^e q_7$ are $1, 2, 2, 3, 4, 5, 3$ respectively. By a computation performed by Maple, we can prove that ${}^e q_1, \dots, {}^e q_7$ verify the two following algebraic relations:

$$16\, {}^e q_3^2\, {}^e q_5{}^2 + {}^e q_4^4 - 8\, {}^e q_3\, {}^e q_5\, {}^e q_4^2 - 64\, {}^e q_3^3\, {}^e q_7{}^2 = 0, \qquad {}^e q_3\, {}^e q_6^2 - {}^e q_7^2\, {}^e q_4{}^2 = 0$$

Set:

$$r_i := \begin{cases} q_i & \text{if} \quad i = 1,2,3,4,7 \\ 16\, q_3^2 q_5{}^2 + q_4^4 - 8\, q_3 q_5 q_4^2 - 64\, q_3^3 q_7{}^2 & \text{if} \quad i = 5 \\ q_3 q_6^2 - q_7^2 q_4^2 & \text{if} \quad i = 6 \end{cases}$$

The polynomials $r_1, \dots, r_7$ are algebraically independent over $\Bbbk$ and

$$\mathrm{d}r_1 \wedge \cdots \wedge \mathrm{d}r_7 = 2q_3 q_6\, (32 q_3^2 q_5 - 8 q_3 q_4^2)\, \mathrm{d}q_1 \wedge \cdots \wedge \mathrm{d}q_7$$

Moreover, ${}^e r_5$ and ${}^e r_6$ have degree at least 13 and ${}^e(2q_3 q_6 (32 q_3^2 q_5 - 8 q_3 q_4^2))$ has degree 15. Then, by Corollary 7.2, ${}^e r_1, \dots, {}^e r_7$ are algebraically independent since

$$\frac{1}{2}(\dim \mathfrak{g}^e + 7) + 15 = 37 = 1 + 2 + 2 + 3 + 3 + 26$$

and by Lemma 7.1,(ii) and (iii), ${}^e r_5$ and ${}^e r_6$ have degree 13.

A precise computation performed by Maple shows that ${}^e r_3 = p_3^2$ for some $p_3$ in the center of $\mathfrak{g}^e$, ${}^e r_4 = p_3 p_4$ for some polynomial $p_4$ of degree 2 in $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$, ${}^e r_5 = p_3^3\, {}^e q_7 p_5$ for some polynomial $p_5$ of degree 7 in $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$, and ${}^e r_6 = p_4\, {}^e r_7 p_6$ for some polynomial $p_6$ of degree 8 in $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$. Setting $p_i := {}^e r_i$ for $i = 1, 2, 7$, the polynomials $p_1, \dots, p_7$ are algebraically independent homogeneous polynomials of degree $1, 2, 1, 2, 7, 8, 3$ respectively. Let $\mathfrak{l}$ be a reductive factor of $\mathfrak{g}^e$. According to [Ca85, Ch. 13],

$$\mathfrak{l} \simeq \mathfrak{so}_2(\Bbbk) \times \mathfrak{sp}_4(\Bbbk) \simeq \Bbbk \times \mathfrak{sp}_4(\Bbbk).$$

In particular, the center of $\mathfrak{l}$ has dimension 1. Let $\{x_1, \dots, x_{37}\}$ be a basis of $\mathfrak{g}^e$ such that $x_{37}$ lies in the center of $\mathfrak{l}$ and such that $x_1, \dots, x_{36}$ are in $[\mathfrak{l}, \mathfrak{l}] + \mathfrak{g}^e_{\mathrm{u}}$ with $\mathfrak{g}^e_{\mathrm{u}}$ the nilpotent radical of $\mathfrak{g}^e$. Then $p_2$ is a polynomial in $\Bbbk[x_1, \dots, x_{37}]$ depending on $x_{37}$. As a result, by [DDV74, Theorems 3.3 and 4.5], the semiinvariant polynomials of $\mathrm{S}(\mathfrak{g}^e)$ are invariant.

*Claim* 7.9. The algebra $\mathfrak{g}^e$ is nonsingular.

*Proof.* [Proof of Claim 7.9] The space $\Bbbk^{14}$ is the orthogonal direct sum of two subspaces $\mathbb{V}_1$ and $\mathbb{V}_2$ of dimension 6 and 8 respectively and such that $e, h, f$ are in $\overline{\mathfrak{g}} := \mathfrak{so}(\mathbb{V}_1) \oplus \mathfrak{so}(\mathbb{V}_2)$. Then $\overline{\mathfrak{g}}^e = \overline{\mathfrak{g}} \cap \mathfrak{g}^e$ is a subalgebra of dimension 21 containing the center of $\mathfrak{g}^e$. For $p$ in $\mathrm{S}(\mathfrak{g}^e)$, denote by $\overline{p}$ its restriction to $\overline{\mathfrak{g}}^f$. The partition $(3, 3, 2, 2, 2, 2)$ verifies the condition (1) of the proof of [Y06, §4, Lemma 3]. So, the proof of Lemma 5.14 remains valid, and the morphism

$$G_0^e \times \overline{\mathfrak{g}}^f \longrightarrow \mathfrak{g}^f, \qquad (g, x) \longmapsto g(x)$$

is dominant. As a result, for $p$ in $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$, the differential of $\overline{p}$ is the restriction to $\overline{\mathfrak{g}}^f$ of the differential of $p$. A computation performed by Maple proves that $\overline{p_3}^{10}$ is a greatest divisor of $\mathrm{d}\overline{p_1} \wedge \cdots \wedge \mathrm{d}\overline{p_7}$ in $\mathrm{S}(\overline{\mathfrak{g}^e})$. If $q$ is a greatest divisor of $\mathrm{d}p_1 \wedge \cdots \wedge \mathrm{d}p_7$ in $\mathrm{S}(\mathfrak{g}^e)$, then $q$ is in $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ since the semiinvariant polynomials are invariant. So $q = p_3^d$ for some nonnegative integer $d$. One can suppose that $\{x_1, \dots, x_{16}\}$ is a basis of the orthogonal complement to $\overline{\mathfrak{g}}^f$ in $\mathfrak{g}^e$. Then the Pfaffian of the matrix

$$\Big([x_i, x_j],\ 1 \leqslant i, j \leqslant 16\Big)$$

is in $\Bbbk^* p_3^8$ so that $p_3^2$ is a greatest divisor of $\mathrm{d}p_1 \wedge \cdots \wedge \mathrm{d}p_7$ in $\mathrm{S}(\mathfrak{g}^e)$. Since

$$\deg p_1 + \cdots + \deg p_7 = 2 + 22 = 2 + \frac{1}{2}(\dim \mathfrak{g}^e + \ell),$$

we conclude that $\mathfrak{g}^e$ is nonsingular by Corollary 7.3,(ii). □

*Claim* 7.10. Suppose that $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is a polynomial algebra. Then for some homogeneous polynomials $p'_5$ and $p'_6$ of degrees at least 5 and at most 8 respectively, $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is generated by $p_1$, $p_2$, $p_3$, $p_4$, $p'_5$, $p'_6$, $p_7$. Furthermore, the possible values for $(\deg p'_5, \deg p'_6)$ are $(5, 8)$ or $(6, 7)$.

*Proof.* [Proof of Claim 7.10] Since the semiinvariants are invariants, by Claim 7.9 and Proposition 7.4,(ii), there are homogeneous generators $\varphi_1, \dots, \varphi_\ell$ of $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ such that

$$\deg \varphi_1 \leqslant \cdots \leqslant \deg \varphi_\ell,$$

and

$$\deg \varphi_1 + \cdots + \deg \varphi_\ell = \frac{1}{2}(\dim \mathfrak{g}^e + \ell) = 22.$$

According to [Mo06c, Theorem 1.1.8] or [Y06b], the center of $\mathfrak{g}^e$ has dimension 2. Hence, $\varphi_1$ and $\varphi_2$ have degree 1. Thereby, we can suppose that $\varphi_1 = p_1$ and $\varphi_2 = p_3$ since $p_1$ and $p_3$ are linearly independent elements of the center of $\mathfrak{g}^e$. Since $p_2$ and $p_4$ are homogneous elements of degree 2 such that $p_1, \dots, p_4$ are algebraically indepent, $\varphi_3$ and $\varphi_4$ have degree 2 and we can suppose that $\varphi_3 = p_2$ and $\varphi_4 = p_4$. Since $p_7$ has degree 3, $\varphi_5$ has degree at most 3 and at least 2 since the center of $\mathfrak{g}^e$ has dimension 2. Suppose that $\varphi_5$ has degree 2. A contradiction is expected. Then

$$\deg \varphi_6 + \deg \varphi_7 = 22 - (1 + 1 + 2 + 2 + 2) = 14.$$

Moreover, since $p_1, \dots, p_7$ are algebraically independent, $\varphi_7$ has degree at most 8 and $\varphi_6$ has degree at least 6. Hence $p_7$ is in the ideal of $\Bbbk[p_1, p_3, \varphi_3, \varphi_4, \varphi_5]$ generated by $p_1$ and $p_3$. But a computation shows that the restriction of $p_7$ to the nullvariety of $p_1$ and $p_3$ in $\mathfrak{g}^f$ is different from 0, whence the expected contradiction. As a result, $\varphi_5$ has degree 3 and

$$\deg \varphi_6 + \deg \varphi_7 = 13.$$

One can suppose $\varphi_5 = p_7$ and the possible values for $(\deg \varphi_6, \deg \varphi_7)$ are $(5, 8)$ and $(6, 7)$ since $\varphi_7$ has degree at most 8. □

Suppose that $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ is a polynomial algebra. A contradiction is expected. Let $p'_5$ and $p'_6$ be as in Claim 7.10 and such that $\deg p'_5 < \deg p'_6$. Then $(\deg p'_5, \deg p'_6)$ equals $(5, 8)$ or $(6, 7)$. A computation shows that we can choose a basis $\{x_1, \dots, x_{37}\}$ of $\mathfrak{g}^e$ with $x_{37} = p_3$, with $p_1$, $p_2$, $p_3$, $p_4$, $p_7$ in $\Bbbk[x_3, \dots, x_{37}]$ and with $p_5$, $p_6$ of degree 1 in $x_1$. Moreover, the coefficient of $x_1$ in $p_5$ is a prime element of $\Bbbk[x_3, \dots, x_{37}]$, the coefficient of $x_1$ in $p_6$ is a prime element of $\Bbbk[x_2, \dots, x_{37}]$ having degree 1 in $x_2$, and the coefficient of $x_1x_2$ in $p_6$ equals $a^2p_3^2$ with $a$ a prime homogeneous polynomial of degree 2 such that $a, p_1, p_2, p_3, p_4$ are algebraically independent. In particular, $a$ is not invariant. If $p'_5$ has degree 5, then

$$p_5 = p'_5 r_0 + r_1$$

with $r_0$ in $\Bbbk[p_1, p_2, p_3, p_4]$ and $r_1$ in $\Bbbk[p_1, p_2, p_3, p_4, p_7]$ so that $p'_5$ has degree 1 in $x_1$, and the coefficient of $x_1$ in $p_5$ is the product of $r_0$ and the coefficient of $x_1$ in $p'_5$. But this is impossible since this coefficient is prime. So, $p'_5$ has degree 6 and $p'_6$ has degree 7. We can suppose that $p'_6 = p_5$. Then

$$p_6 = p_5 r_0 + p'_6 r_1 + r_2$$

with $r_0$ homogeneous of degree 1 in $\Bbbk[p_1, p_3]$, $r_1$ homogeneous of degree 2 in $\Bbbk[p_1, p_2, p_3, p_4]$, and $r_2$ homogeneous of degree 8 in $\Bbbk[p_1, p_2, p_3, p_4, p_7]$. According to the above remarks on $p_5$ and the coefficient of $x_1x_2$ in $p_6$, $r_1$ is in $\Bbbk^* p_3^2$ since $r_1$ has degree 2.

For $p$ in $\mathrm{S}(\mathfrak{g}^e)$, denote by $\overline{p}$ its image in $\mathrm{S}(\mathfrak{g}^e)/p_3\mathrm{S}(\mathfrak{g}^e)$. A computation shows that for some $u$ in $\mathrm{S}(\mathfrak{g}^e)/p_3\mathrm{S}(\mathfrak{g}^e)$,

$$\overline{p_5} = \overline{p_4}^2 u, \qquad \overline{p_6} = -\overline{p_4 p_7} u.$$

Furthermore, $\overline{p_4}$ and $\overline{p_7}$ are different prime elements of $\mathrm{S}(\mathfrak{g}^e)/p_3\mathrm{S}(\mathfrak{g}^e)$ and the coefficient $u_1$ of $x_1$ in $u$ is the product of two different polynomials of degree 1. The coefficient of $x_1$ in $\overline{p_6}$ is $u_1\overline{p_4}^2\overline{r_0}$ since

$$\overline{p_6} = \overline{p_5 r_0} + \overline{r_2}.$$

On the other hand, the coefficient of $x_1$ in $\overline{p_6}$ is $-u_1\overline{p_4 p_7}$, whence the contradiction since $r_0$ has degree 1.

7.4. **A conjecture.** All examples of good elements we achieved satisfy the hypothesis of Theorem 3.6.

**Conjecture 7.11.** *Let $\mathfrak{g}$ be a simple Lie algebra and let $e$ be a nilpotent of $\mathfrak{g}$. If $e$ is good then for some graded generating sequence $(q_1, \dots, q_\ell)$ in $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$, ${}^e q_1, \dots, {}^e q_\ell$ are algebraically independent over $\Bbbk$. In other words, the converse implication of Theorem 3.6 holds.*

Notice that it may happen that for some $r_1, \dots, r_\ell$ in $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$, the elements ${}^e r_1, \dots, {}^e r_\ell$ are algebraically independent over $\Bbbk$, and that however $e$ is not good. This is the case for instance for the nilpotent elements in $\mathfrak{so}(\Bbbk^{12})$ associated with the partition $(5, 3, 2, 2)$, cf. Example 7.6.

In fact, according to [PPY07, Corollary 2.3], for any nilpotent $e$ of $\mathfrak{g}$, there exist $r_1, \dots, r_\ell$ in $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ such that ${}^e r_1, \dots, {}^e r_\ell$ are algebraically independent over $\Bbbk$.

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