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# ON THE KOSTANT SECTION AND THE UNIVERSAL CENTRALIZER IN POSITIVE CHARACTERISTIC AND OVER THE INTEGERS

SIMON RICHE

ABSTRACT. In this paper we prove analogues of fundamental results of Kostant on the universal centralizer of a connected reductive algebraic group for algebraically closed fields of positive characteristic (with mild assumptions). We also extend some of these results to integral coefficients.

## 1. INTRODUCTION

1.1. This note is concerned with the generalization of some fundamental results concerning the universal centralizer group scheme  $\mathbf{I}$  associated with a connected reductive group  $\mathbf{G}$  (and its Lie algebra  $\mathfrak{J}$ ) to the case where the coefficients are an algebraically closed field  $\mathbb{F}$  of positive characteristic or a finite localization  $\mathfrak{A}$  of the integers (with mild assumptions).

The case of complex coefficients is due to Kostant [Ko]. The case of fields of positive characteristic is probably well known (at least in part) to experts, but is not treated in detail in the literature as far we know (except for some results in [Ve]), and is usually used only under rather strong assumptions (see e.g. [Ng]). The case of integral coefficients might be new.

In the “classical” case (field of characteristic either zero or large), these results play an important role in various aspects of the geometric Langlands program, see e.g. [G1, BF, Do, Ng]. We expect our generalizations to play a similar role in modular (or integral) versions of these results. In fact, our main motivation comes from our desire to generalize the definition of the “Kostant–Whittaker reduction functor” used e.g. in [BF, Do] to the case of integral coefficients. This functor plays an important technical role in [MR].

1.2. The first part of the paper (Section 3) is concerned with the case of algebraically closed fields. In this case we have tried to prove the various results in the maximal reasonable generality, and to introduce our assumptions only when they become necessary.

First we recall the construction of Kostant’s section to the adjoint quotient  $\mathfrak{g}/\mathbf{G}$ , where  $\mathfrak{g} = \mathcal{L}ie(\mathbf{G})$  (see Theorem 3.2.2). Then we use these results to prove that the universal centralizer  $\mathbf{I}$  is smooth over the regular locus  $\mathfrak{g}_{\text{reg}}$  (see Corollary 3.3.6). We also describe the Lie algebra of the restriction  $\mathbf{I}_{\text{reg}}$  of  $\mathbf{I}$  to  $\mathfrak{g}_{\text{reg}}$  in terms of the cotangent bundle to  $\mathfrak{g}/\mathbf{G}$  (see Theorem 3.4.2), generalizing results in characteristic zero due (to the best of our knowledge) to Bezrukavnikov–Finkelberg [BF]. Finally we consider variants of these results when the Lie algebra  $\mathfrak{g}$  is replaced by the Grothendieck resolution  $\tilde{\mathfrak{g}}$  (see §3.5).

1.3. The second part of the paper (Section 4) is concerned with the case when  $\mathbf{G}$  is a split, connected, simply-connected, semi-simple algebraic group over the finite localization  $\mathfrak{R}$  of  $\mathbb{Z}$  obtained by inverting all prime which are not very good.

In this setting we define and study a Kostant section  $\mathcal{S}$  (see in particular Theorem 4.2.3), prove that the restriction of the universal centralizer to  $\mathcal{S}$  is smooth (see Proposition 4.3.1), and describe the Lie algebra of this restriction in terms of the cotangent bundle to  $\mathcal{S}$  (see Theorem 4.3.4). Finally we study analogues of these objects for the Grothendieck resolution (see §4.4).

The proofs in this section mainly proceed by reduction to the case of algebraically closed fields.

1.4. **Some notation and conventions.** All rings in this paper are tacitly assumed to be unital and commutative. If  $A$  is a  $B$ -algebra, we denote by  $\Omega_{A/B}$  the  $A$ -module of relative differential forms, see [Ha, §II.8]. If  $M$  is an  $A$ -module, we denote by  $S_A(M)$  (or sometimes simply  $S(M)$ ) the symmetric algebra of  $M$ .

If  $X$  is a scheme, we denote by  $\mathcal{O}_X$  its structure sheaf, and by  $\mathcal{O}(X)$  the global sections of  $\mathcal{O}_X$ . If  $Y$  is a scheme and  $X$  is a  $Y$ -scheme, we denote by  $\Omega_{X/Y}$  the  $\mathcal{O}_X$ -module of relative differential forms. If  $X$  is smooth over  $Y$ , we denote by  $\mathbb{T}^*(X/Y)$  (or simply  $\mathbb{T}^*(X)$ ) the cotangent bundle to  $X$  over  $Y$ , in other words the relative spectrum of the symmetric algebra of the locally free  $\mathcal{O}_X$ -module  $\mathcal{T}_{X/Y} := \mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/Y}, \mathcal{O}_X)$ .

If  $\mathfrak{R}$  is a finite localization of  $\mathbb{Z}$ , we define a *geometric point* of  $\mathfrak{R}$  to be an algebraically closed field whose characteristic is not invertible in  $\mathfrak{R}$ . If  $\mathbb{F}$  is a such a geometric point, then there exists a unique algebra morphism  $\mathfrak{R} \rightarrow \mathbb{F}$ , so that tensor products of the form  $\mathbb{F} \otimes_{\mathfrak{R}} (-)$  make sense.

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## 2. PRELIMINARY RESULTS

2.1. **Reduction to algebraically closed fields.** Let  $\mathfrak{R}$  be a finite localization of  $\mathbb{Z}$ . In this subsection we prove various results that allow to deduce results over  $\mathfrak{R}$  from their analogues over algebraically closed fields of positive characteristic. These results will be used crucially in Section 4.

**Lemma 2.1.1.** *Let  $A$  be a finitely generated  $\mathfrak{R}$ -algebra, and let  $A'$  be a finitely generated  $A$ -algebra. Assume that  $A'$  is flat over  $\mathfrak{R}$  and that for any geometric point  $\mathbb{F}$  of  $\mathfrak{R}$  of positive characteristic,  $\mathbb{F} \otimes_{\mathfrak{R}} A'$  is flat over  $\mathbb{F} \otimes_{\mathfrak{R}} A$ . Then  $A'$  is flat over  $A$ .*

*Proof.* By [SP, Tag 00HT, Item (7)], it suffices to prove that for all maximal ideals  $\mathfrak{m} \subset A'$ , with  $\mathfrak{p}$  the inverse image of  $\mathfrak{m}$  in  $A$ , the  $A_{\mathfrak{p}}$ -module  $A'_{\mathfrak{m}}$  is flat. Let also  $\mathfrak{q}$  be the inverse image of  $\mathfrak{m}$  in  $\mathfrak{R}$ . Then, by [SP, Tag 00GB],  $\mathfrak{q}$  is a maximal ideal of  $\mathfrak{R}$ , i.e. is of the form  $\ell \cdot \mathfrak{R}$  for some prime number  $\ell$  not invertible in  $\mathfrak{R}$ . From our assumption on geometric points one can easily deduce that  $A'/(\ell \cdot A')$  is flat over  $A/(\ell \cdot A)$ . Hence, using [SP, Tag 00HT, Item (7)] again, we deduce that  $(A'/(\ell \cdot A'))_{\mathfrak{m}/\ell \cdot A'} = A'_{\mathfrak{m}}/\mathfrak{q} \cdot A'_{\mathfrak{m}}$  is flat over  $(A/(\ell \cdot A))_{\mathfrak{p}/\ell \cdot A} = A_{\mathfrak{p}}/\mathfrak{q} \cdot A_{\mathfrak{p}}$ . By the

same result and our other assumption we know that  $A'_m$  is flat over  $\mathfrak{R}_q$ . Hence we can apply [SP, Tag 00MP] to the morphisms  $\mathfrak{R}_q \rightarrow A_p \rightarrow A'_m$  to deduce that  $A'_m$  is flat over  $A_p$ , which finishes the proof.  $\square$

In the next statements, if  $X$  is an  $\mathfrak{R}$ -scheme and  $\mathfrak{S}$  is an  $\mathfrak{R}$ -algebra, we set  $X_{\mathfrak{S}} := X \times_{\mathrm{Spec}(\mathfrak{R})} \mathrm{Spec}(\mathfrak{S})$ .

**Corollary 2.1.2.** *Let  $X$  and  $Y$  be schemes which are locally of finite type over  $\mathfrak{R}$ , and let  $f: X \rightarrow Y$  be a morphism. Assume that  $X$  is flat over  $\mathfrak{R}$  and that for any geometric point  $\mathbb{F}$  of  $\mathfrak{R}$  of positive characteristic, the morphism  $X_{\mathbb{F}} \rightarrow Y_{\mathbb{F}}$  induced by  $f$  is flat. Then  $f$  is flat.*

*Proof.* Since flatness is a property which is local both on  $X$  and  $Y$ , we can assume that both of them are affine. Then the claim follows from Lemma 2.1.1.  $\square$

**Proposition 2.1.3.** *Let  $X$  and  $Y$  be schemes which are of finite type over  $\mathfrak{R}$ , and let  $f: X \rightarrow Y$  be a morphism. Assume that  $X$  is flat over  $\mathfrak{R}$  and that for any geometric point  $\mathbb{F}$  of  $\mathfrak{R}$  of positive characteristic, the morphism  $X_{\mathbb{F}} \rightarrow Y_{\mathbb{F}}$  induced by  $f$  is smooth. Then  $f$  is smooth.*

*Proof.* First, by Corollary 2.1.2,  $f$  is flat.

The smooth locus of  $f$  is open (see e.g. the definition of smoothness in [SP, Tag 01V5]) and the closed points are dense in every closed subset of  $X$  (see [SP, Tag 00G3]), hence it is enough to prove that  $f$  is smooth at closed points of  $X$ . Let  $x$  be such a closed point. By [SP, Tag 00GB], the residue field  $\kappa(x)$  has characteristic  $\ell$  for some prime number  $\ell$  invertible in  $\mathfrak{R}$ . Let  $\mathbb{F}$  be an algebraic closure of  $\kappa(x)$ , let  $x'$  be the point in  $X_{\kappa(x)}$  whose image in  $X$  is  $x$ , and let  $x''$  be the unique closed point of  $X_{\mathbb{F}}$  lying over  $x'$ . Let also  $f': X_{\kappa(x)} \rightarrow Y_{\kappa(x)}$  and  $f'': X_{\mathbb{F}} \rightarrow Y_{\mathbb{F}}$  be the morphisms induced by  $f$ . Then  $f''$  is smooth by assumption. Moreover, one can see (using e.g. the characterization of smoothness in [SP, Tag 01V9, Item (3)] and the invariance of dimension under field extension, see [GW, Proposition 5.38]) that the smoothness of  $f''$  at  $x''$  is equivalent to the smoothness of  $f'$  at  $x'$ , which is itself equivalent to the smoothness of  $f$  at  $x$ . This finishes the proof.  $\square$

**Lemma 2.1.4.** *Let  $A$  be a finitely generated  $\mathfrak{R}$ -algebra. Let  $X$  be a Noetherian  $A$ -scheme which is projective over  $A$  and flat over  $\mathfrak{R}$ . Assume that for any geometric point  $\mathbb{F}$  of  $\mathfrak{R}$  of positive characteristic, the scheme  $X_{\mathbb{F}}$  is affine. Then  $X$  is affine.*

*Proof.* By Serre's criterion (see [Ha, Theorem III.3.7]) it suffices to prove that for any coherent sheaf of ideals  $\mathcal{I} \subset \mathcal{O}_X$ , the complex of  $A$ -modules  $R\Gamma(X, \mathcal{I})$  is concentrated in degree 0. However by Grothendieck's vanishing theorem [Ha, Theorem III.2.7] this complex is bounded, and since  $X$  is projective over  $A$  its cohomology sheaves are finitely generated over  $A$  (see [Ha, Theorem III.5.2]). Hence by [BR2, Lemma 1.4.1(2)] it suffices to prove that for any geometric point  $\mathbb{F}$  of  $\mathfrak{R}$ , the complex  $\mathbb{F} \otimes_{\mathfrak{R}}^L R\Gamma(X, \mathcal{I})$  is concentrated in degree 0.

Now one can easily check that we have

$$\mathbb{F} \otimes_{\mathfrak{R}}^L R\Gamma(X, \mathcal{I}) \cong R\Gamma(X, \mathbb{F} \otimes_{\mathfrak{R}}^L \mathcal{I}).$$

Moreover, since  $\mathcal{O}_X$  is flat over  $\mathfrak{R}$ , the complex  $\mathbb{F} \otimes_{\mathfrak{R}}^L \mathcal{I}$  is concentrated in degree 0, and isomorphic to the direct image of a coherent sheaf  $\mathcal{I}_{\mathbb{F}}$  on  $X_{\mathbb{F}}$ . Hence we have

$$\mathbb{F} \otimes_{\mathfrak{R}}^L R\Gamma(X, \mathcal{I}) \cong R\Gamma(X_{\mathbb{F}}, \mathcal{I}_{\mathbb{F}}),$$

and the desired claim follows from the assumption that  $X_{\mathbb{F}}$  is affine and the other direction in Serre's criterion.  $\square$

**2.2. Lie algebra of a group scheme.** Let  $X$  be a Noetherian scheme, and let  $G$  be a Noetherian group scheme over  $X$ . We denote by  $\varepsilon: X \rightarrow G$  the identity, and consider the  $\mathcal{O}_X$ -coherent sheaf

$$\omega_{G/X} := \varepsilon^*(\Omega_{G/X}).$$

Then, by definition (see e.g. [SGA, Exposé II, §4.11]; see also [Wa, Chap. 12] for the case  $G$  and  $X$  are affine), the Lie algebra of  $G$  is the quasi-coherent  $\mathcal{O}_X$ -module

$$\mathcal{L}ie(G/X) := \mathcal{H}om_{\mathcal{O}_X}(\omega_{G/X}, \mathcal{O}_X),$$

with its natural bracket. We also denote by  $\mathbb{L}ie(G/X)$  the scheme over  $X$  which is the relative spectrum of the symmetric algebra of the  $\mathcal{O}_X$ -module  $\omega_{G/X}$ . This scheme is naturally a Lie algebra over  $X$ . When  $X$  is clear, we will sometimes abbreviate  $\mathcal{L}ie(G/X)$ , resp.  $\mathbb{L}ie(G/X)$ , to  $\mathcal{L}ie(G)$ , resp.  $\mathbb{L}ie(G)$ .

We will use the following easy consequences of the definition.

- Lemma 2.2.1.** (1) *If  $G$  is smooth over  $X$ , then the  $\mathcal{O}_X$ -module  $\mathcal{L}ie(G/X)$  is locally free of finite rank, and  $\mathbb{L}ie(G/X)$  is a vector bundle over  $X$ .*  
(2) *Let  $f: Y \rightarrow X$  be a morphism. Assume that either  $G$  is smooth over  $X$ , or that  $f$  is flat. Then there exists a canonical isomorphism*

$$\mathcal{L}ie(G \times_X Y/Y) \cong f^* \mathcal{L}ie(G/X)$$

*of  $\mathcal{O}_Y$ -Lie algebras.*

*Proof.* (1) Since  $G \rightarrow X$  is smooth, the  $\mathcal{O}_G$ -module  $\Omega_{G/X}$  is locally free of finite rank (see [SP, Tag 02G1]), which implies that  $\omega_{G/X}$  and  $\mathcal{L}ie(G/X)$  are locally free of finite rank over  $\mathcal{O}_X$ . The fact that  $\mathbb{L}ie(G/X)$  is a vector bundle follows.

(2) We have

$$f^* \mathcal{L}ie(G/X) = f^* \mathcal{H}om_{\mathcal{O}_X}(\varepsilon^* \Omega_{G/X}, \mathcal{O}_X) \cong \mathcal{H}om_{\mathcal{O}_Y}(f^* \varepsilon^* \Omega_{G/X}, \mathcal{O}_Y)$$

under our assumptions (see [Ma, Theorem 7.11] for the second case). Now using [Ha, Proposition II.8.10] one can easily check that  $f^* \varepsilon^* \Omega_{G/X}$  is the restriction to  $Y$  of  $\Omega_{G \times_X Y/Y}$ , which finishes the proof.  $\square$

**2.3. Notation and assumptions.** In Section 3 we will work in the following setting.

Let  $\mathbb{F}$  be an algebraically closed field of characteristic  $\ell \geq 0$ . Let  $\mathbf{G}$  be a connected reductive group over  $\mathbb{F}$ , of rank  $r$ . Let  $\mathbf{B} \subset \mathbf{G}$  be a Borel subgroup and  $\mathbf{T} \subset \mathbf{B}$  be a maximal torus. We denote by  $\mathfrak{t} \subset \mathfrak{b} \subset \mathfrak{g}$  the Lie algebras of  $\mathbf{T} \subset \mathbf{B} \subset \mathbf{G}$ . We also denote by  $\mathbf{U}$  the unipotent radical of  $\mathbf{B}$ , by  $\mathfrak{n}$  its Lie algebra, and by  $\mathcal{D}\mathbf{G} \subset \mathbf{G}$  the derived subgroup. We denote by  $\mathbf{B}^+$  the Borel subgroup of  $\mathbf{G}$  which is opposite to  $\mathbf{B}$  with respect to  $\mathbf{T}$ , by  $\mathbf{U}^+$  its unipotent radical, and by  $\mathfrak{b}^+$  and  $\mathfrak{n}^+$  their respective Lie algebras.

We let  $\Phi$  be the root system of  $(\mathbf{G}, \mathbf{T})$ , and  $W$  be its Weyl group. For  $\alpha \in \Phi$ , we denote by  $\mathfrak{g}_\alpha \subset \mathfrak{g}$  the corresponding root subspace.

We denote by  $\Phi^+ \subset \Phi$  the system of positive roots consisting of the *opposites* of the  $\mathbf{T}$ -weights in  $\mathfrak{n}$ , and by  $\Delta$  the corresponding basis of  $\Phi$ . We also denote by  $\check{\Phi} \subset X_*(\mathbf{T})$  the coroots of  $\Phi$ , and by  $\check{\Delta} \subset \Phi^+ \subset \check{\Phi}$  the coroots corresponding to  $\Delta$  and  $\Phi^+$  respectively. We denote by  $\mathbb{Z}\Phi \subset X^*(\mathbf{T})$ , resp.  $\mathbb{Z}\check{\Phi} \subset X_*(\mathbf{T})$ , the lattice

generated by the roots, resp. coroots. For  $\alpha \in \Phi$ , resp.  $\check{\alpha} \in \check{\Phi}$ , we denote by  $d\alpha$ , resp.  $d\check{\alpha}$ , the differential of  $\alpha$ , resp.  $\check{\alpha}$ , considered as an element in  $\mathfrak{t}^*$ , resp.  $\mathfrak{t}$ .

Let us consider the following conditions:

- (C1) for all  $\alpha \in \Phi$ ,  $d\alpha \neq 0$ ;
- (C2)  $\ell$  is good for  $\mathbf{G}$ , and  $X_*(\mathbf{T})/\mathbb{Z}\check{\Phi}$  has no  $\ell$ -torsion;
- (C3)  $\ell$  is good for  $\mathbf{G}$ , and neither  $X_*(\mathbf{T})/\mathbb{Z}\check{\Phi}$  nor  $X^*(\mathbf{T})/\mathbb{Z}\Phi$  has  $\ell$ -torsion;
- (C4)  $\ell$  is good for  $\mathbf{G}$ ,  $X_*(\mathbf{T})/\mathbb{Z}\check{\Phi}$  has no  $\ell$ -torsion, and there exists a  $\mathbf{G}$ -equivariant isomorphism  $\mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$ .

We claim that

$$(C4) \Rightarrow (C3) \Rightarrow (C2) \quad \text{and} \quad (C3) \Rightarrow (C1).$$

Indeed, the condition that  $X^*(\mathbf{T})/\mathbb{Z}\Phi$  has no  $\ell$ -torsion means that the vectors  $d\alpha$  ( $\alpha \in \Delta$ ) are linearly independent in  $\mathfrak{t}^* \cong \mathbb{F} \otimes_{\mathbb{Z}} X^*(\mathbf{T})$ . This implies in particular that they are non zero. Since any root is  $W$ -conjugate to a simple root, this justifies the implication (C3)  $\Rightarrow$  (C1).

The implication (C3)  $\Rightarrow$  (C2) is obvious. Now, if  $\kappa: \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$  is a  $\mathbf{G}$ -equivariant isomorphism, restricting to  $\mathbf{T}$ -fixed points we obtain an isomorphism  $\mathfrak{t} \xrightarrow{\sim} \mathfrak{t}^*$ . It is not difficult to deduce from the  $\mathbf{G}$ -equivariance of  $\kappa$  that this isomorphism must send each differential of a simple root to a non-zero multiple of the differential of the corresponding coroot. Hence the former are linearly independent iff the latter are linearly independent, which justifies the implication (C4)  $\Rightarrow$  (C3).

Let us note also that the condition that  $X_*(\mathbf{T})/\mathbb{Z}\check{\Phi}$  has no  $\ell$ -torsion is automatic if  $\mathcal{D}\mathbf{G}$  is simply connected. Indeed, Let  $\mathbf{T}'$  be the maximal torus of  $\mathcal{D}\mathbf{G}$  contained in  $\mathbf{T}$ . Then, as explained in [J2, §II.1.18], we have

$$X_*(\mathbf{T}') = X_*(\mathbf{T}) \cap \left( \sum_{\check{\alpha} \in \check{\Phi}} \mathbb{Q} \cdot \check{\alpha} \right).$$

In particular,  $X_*(\mathbf{T}')/\mathbb{Z}\check{\Phi}$  is the torsion part of  $X_*(\mathbf{T})/\mathbb{Z}\check{\Phi}$ . If  $\mathcal{D}\mathbf{G}$  is simply connected then  $X_*(\mathbf{T}')/\mathbb{Z}\check{\Phi} = 0$ , hence  $X_*(\mathbf{T})/\mathbb{Z}\check{\Phi}$  is a free  $\mathbb{Z}$ -module. This remark shows that condition (C4) holds (hence also the other conditions) provided  $\mathbf{G}$  satisfies Jantzen's "standard hypotheses", see e.g. [J1, §6.4].

- Remark 2.3.1.* (1) Condition (C1) is discussed in [J3, §13.3]. The only cases when this condition might not be satisfied occurs when  $\ell = 2$  and  $\mathbf{G}$  has a component of type  $C_n$  ( $n \geq 1$ ). In particular, it is satisfied if  $\ell$  is good for  $\mathbf{G}$  except possibly when  $\ell = 2$  and  $\mathbf{G}$  has a component of type  $A_1$ .
- (2) Primes satisfying (C3) are called "pretty good" in [He]. See also [He, Theorem 5.2] for the relation between this condition and other variants of the "standard hypotheses." In the case  $\mathbf{G}$  is semisimple, this condition is equivalent to  $\ell$  being very good, and (C4) is equivalent to (C3), see the proof of Lemma 4.1.3 below.

**2.4. Regular elements and the adjoint quotient.** We continue with the notation of §2.3. For any closed subgroup  $\mathbf{H} \subset \mathbf{G}$  and any  $x \in \mathfrak{g}$ , we denote by  $\mathbf{H}_x$  the scheme-theoretic centralizer of  $x$  in  $\mathbf{H}$ , i.e. the closed subgroup of  $\mathbf{H}$  defined by the fiber product  $x \times_{\mathfrak{g}} \mathbf{H}$ , where the morphism  $\mathbf{H} \rightarrow \mathfrak{g}$  sends  $h$  to  $h \cdot x$ . If  $\mathbf{H}$  is reduced, we also denote by  $C_{\mathbf{H}}(x)$  the reduced part of  $\mathbf{H}_x$ , i.e. the reduced subgroup in  $\mathbf{H}$  whose closed points are  $\{g \in \mathbf{H} \mid g \cdot x = x\}$ . If  $\mathfrak{h}$  is the Lie algebra of  $\mathbf{H}$ , we set  $\mathfrak{h}_x := \{y \in \mathfrak{h} \mid [x, y] = 0\}$ .

Recall that for  $x \in \mathbf{G}$  we have  $\dim(\mathbf{G}_x) \geq r$ . (This follows, by standard arguments, from the fact that any element in  $\mathfrak{g}$  is contained in the Lie algebra of a Borel subgroup, see [Bo, Proposition 14.25], and that  $\mathbf{B}$  acts trivially on  $\mathfrak{b}/\mathfrak{n}$ .) An element  $x \in \mathfrak{g}$  is called *regular* if  $\dim(\mathbf{G}_x) = r$ . We denote by  $\mathfrak{g}_{\text{reg}} \subset \mathfrak{g}$  the subset consisting of regular elements; it is open (see e.g. [Hu, Proposition 1.4]) and non empty (see Lemma 3.1.1 below). We will also consider the subset  $\mathfrak{g}_{\text{rs}} \subset \mathfrak{g}_{\text{reg}}$  consisting of regular *semi-simple* elements. This subset is open, and non-empty if (and only if) (C1) holds, see [J3, §13.3]. For any subset  $\mathfrak{k} \subset \mathfrak{g}$ , we set  $\mathfrak{k}_{\text{reg}} := \mathfrak{k} \cap \mathfrak{g}_{\text{reg}}$ ,  $\mathfrak{k}_{\text{rs}} := \mathfrak{k} \cap \mathfrak{g}_{\text{rs}}$ . Then by [J3, §13.3], if (C1) holds we have

$$(2.4.1) \quad \mathfrak{t}_{\text{rs}} = \{x \in \mathfrak{t} \mid \forall \alpha \in \Phi, d\alpha(x) \neq 0\}.$$

We will denote by

$$\chi: \mathfrak{g} \rightarrow \mathfrak{g}/\mathbf{G}$$

the quotient morphism, and by  $\chi_{\text{reg}}: \mathfrak{g}_{\text{reg}} \rightarrow \mathfrak{g}/\mathbf{G}$  its restriction to regular elements. (Here,  $\mathfrak{g}/\mathbf{G} := \text{Spec}(\mathcal{O}(\mathfrak{g})^{\mathbf{G}})$ .) In Section 3 we will need a few well-known results concerning this morphism.

**Lemma 2.4.1.** *If  $x, y \in \mathfrak{g}_{\text{reg}}$  and  $\chi(x) = \chi(y)$ , then  $x$  and  $y$  are  $\mathbf{G}$ -conjugate.*

*Proof.* This follows from [J3, Proposition 7.13]. Alternatively, one can prove this result directly using the facts that  $\chi$  separates semi-simple  $\mathbf{G}$ -orbits (since they are closed, see [Bo, Proposition 11.8]), that  $C_{\mathbf{G}}(x)^0$  is reductive if  $x \in \mathfrak{g}$  is semi-simple (see [Bo, Proposition 13.19]) and that the nilpotent cone of a connected reductive group is irreducible (see [J3, Lemma 6.2]).  $\square$

**Proposition 2.4.2.** *Assume that (C1) holds. Then the inclusion  $\mathfrak{t} \subset \mathfrak{g}$  induces an algebra isomorphism  $\mathcal{O}(\mathfrak{g})^{\mathbf{G}} \xrightarrow{\sim} \mathcal{O}(\mathfrak{t})^W$ ; in other words an isomorphism of schemes  $\mathfrak{t}/W \xrightarrow{\sim} \mathfrak{g}/\mathbf{G}$ .*

*Proof.* In [J3, §7.12] it is explained that the proof given in [SS, §II.3.17'] applies if (C1) is satisfied.  $\square$

When (C1) is satisfied, we will usually use Proposition 2.4.2 to consider  $\chi$  as a morphism from  $\mathfrak{g}$  to  $\mathfrak{t}/W$ .

**Lemma 2.4.3.** *Assume that (C1) holds, and that  $X_*(\mathbf{T})/\mathbb{Z}\check{\Phi}$  has no  $\ell$ -torsion. Then*

- (1) *the action of  $W$  on  $\mathfrak{t}_{\text{rs}}$  is free (in the sense that the stabilizer of any  $x \in \mathfrak{t}_{\text{rs}}$  is trivial);*
- (2) *for any  $x \in \mathfrak{t}_{\text{rs}}$  we have  $\mathbf{G}_x = \mathbf{T}$ .*

*Proof.* (1) Our assumptions imply that the stabilizer in  $\mathfrak{t}$  of the reflection  $s_\alpha$  associated with  $\alpha \in \Phi$  is the hyperplane  $\text{Ker}(d\alpha)$ . Using (2.4.1) we deduce that if  $x \in \mathfrak{t}_{\text{rs}}$ , then  $x$  is not fixed by any  $s_\alpha$ . Hence to conclude it is enough to prove that if  $w \in W$  and  $x \in \mathfrak{t}$  satisfy  $w \cdot x = x$ , then  $w$  is a product of reflections stabilizing  $x$ .

This property follows from a well-known argument (see e.g. [Hu, Lemma on p. 32]). In fact, let us first assume that  $\ell > 0$ . Then we have natural isomorphisms

$$\mathfrak{t} \cong X_*(\mathbf{T}) \otimes_{\mathbb{Z}} \mathbb{F} \cong (X_*(\mathbf{T})/\ell \cdot X_*(\mathbf{T})) \otimes_{\mathbb{F}_\ell} \mathbb{F}.$$

Hence it is enough to prove a similar property for the  $W$ -action on  $X_*(\mathbf{T})/\ell \cdot X_*(\mathbf{T})$ . Now let  $w \in W$  and  $\check{\mu} \in X_*(\mathbf{T})$  be such that  $w \cdot \check{\mu} = \check{\mu} \pmod{\ell \cdot X_*(\mathbf{T})}$ . Then

$w \cdot \check{\mu} - \check{\mu} \in \mathbb{Z}\check{\Phi} \cap \ell \cdot X_*(\mathbf{T})$ . Since  $X_*(\mathbf{T})/\mathbb{Z}\check{\Phi}$  has no  $\ell$ -torsion, this implies that  $w \cdot \check{\mu} - \check{\mu} \in \ell \cdot \mathbb{Z}\check{\Phi}$ . Let us write  $w \cdot \check{\mu} - \check{\mu} = \ell\check{\lambda}$ . Then the image of  $\check{\mu}$  in  $X_*(\mathbf{T}) \otimes_{\mathbb{Z}} \mathbb{R}$  is fixed by the element  $(\ell\check{\lambda}) \cdot w$  of the group  $W \ltimes \ell\mathbb{Z}\check{\Phi}$ , which is the affine Weyl group of [J2, §II.6.1] (for the reductive group which is Langlands dual to  $\mathbf{G}$ ). By a well-known result on groups generated by reflections in real affine spaces (see [Bo, V, §3, Proposition 1]),  $(\ell\check{\lambda}) \cdot w$  is a product of reflections in  $W \ltimes \ell\mathbb{Z}\check{\Phi}$  stabilizing  $\check{\mu}$ . Projecting on  $W$ , we deduce that indeed  $w$  is a product of reflections in  $W$  which stabilize  $\check{\mu} \bmod \ell \cdot X_*(\mathbf{T})$ , which finishes the proof in the case  $\ell > 0$ .

The case  $\ell = 0$  is similar and simpler; details are left to the reader.

(2) First we show that  $\mathbf{G}_x$  is smooth. In fact, since  $x$  is regular we have  $\dim(\mathbf{G}_x) = \dim(C_{\mathbf{G}}(x)) = r$ . On the other hand, it is easily checked using the triangular decomposition of  $\mathfrak{g}$  that  $\mathfrak{g}_x = \mathfrak{t}$ . We deduce that the inclusions  $\mathcal{L}ie(\mathbf{T}) \subset \mathcal{L}ie(\mathbf{G}_x) \subset \mathfrak{g}_x$  must be equalities. We have proved that  $\dim(\mathbf{G}_x) = \dim(\mathcal{L}ie(\mathbf{G}_x))$ , which implies that  $\mathbf{G}_x$  is smooth (see [Wa, Corollary on p. 94]).

It follows from [J3, Lemma 13.3] that  $\mathbf{G}_x = C_{\mathbf{G}}(x) \subset N_{\mathbf{G}}(\mathbf{T})$ . Since no element of  $W = N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$  stabilizes  $x$  by (1), we deduce that  $\mathbf{G}_x = \mathbf{T}$ .  $\square$

We will finally use the following easy observation.

**Lemma 2.4.4.** *If  $\kappa: \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$  is a  $\mathbf{G}$ -equivariant isomorphism, then for any  $x \in \mathfrak{g}$  the image  $\kappa(\mathfrak{g}_x)$  of  $\mathfrak{g}_x$  is the subspace  $(\mathfrak{g}/[x, \mathfrak{g}])^* \subset \mathfrak{g}^*$ .*

*Proof.* Using the  $\mathbf{G}$ -equivariance of  $\kappa$  we observe that  $\kappa(\mathfrak{g}_x) \subset (\mathfrak{g}/[x, \mathfrak{g}])^*$ . Then we conclude by a dimension argument.  $\square$

### 3. THE CASE OF FIELDS

In this section we use the notation of §§2.3–2.4.

**3.1. The principal nilpotent element and the Kostant section.** For any  $\alpha \in \Delta$  we choose a non-zero root vector  $e_\alpha \in \mathfrak{g}_\alpha$ , and set

$$e := \sum_{\alpha \in \Delta} e_\alpha.$$

**Lemma 3.1.1.** *The element  $e \in \mathfrak{g}$  is regular.*

*Proof.* First we observe that regular nilpotent elements exist: this follows from the fact that the nilpotent cone in  $\mathfrak{g}$  has dimension  $\dim(\mathfrak{g}) - r$  (see [J3, Theorem 6.4]) and consists of finitely many  $\mathbf{G}$ -orbits (see [J3, §2.8] and references therein). Then the claim follows from [Sp, Lemma 5.8]. (In *loc. cit.* it is assumed that  $\mathbf{G}$  is semisimple, but this assumption does not play any role in the proof of this lemma.)  $\square$

**Lemma 3.1.2.** *Assume that (C2) holds. Then the morphism*

$$\mathfrak{n} \rightarrow \mathfrak{b}, \quad x \mapsto [e, x]$$

*is injective.*

*Proof.* This result is a consequence of the considerations in [Sp, Section 2]. In fact, let  $\mathbf{G}_{\mathbb{Z}} \supset \mathbf{B}_{\mathbb{Z}} \supset \mathbf{T}_{\mathbb{Z}}$  be a split connected reductive group over  $\mathbb{Z}$ , a Borel subgroup and a (split) maximal torus such that the base change of  $\mathbf{G}_{\mathbb{Z}}$ , resp.  $\mathbf{B}_{\mathbb{Z}}$ , resp.  $\mathbf{T}_{\mathbb{Z}}$ , to  $\mathbb{F}$  is  $\mathbf{G}$ , resp.  $\mathbf{B}$ , resp.  $\mathbf{T}$ . We denote by  $\mathfrak{t}_{\mathbb{Z}} \subset \mathfrak{b}_{\mathbb{Z}} \subset \mathfrak{g}_{\mathbb{Z}}$  the Lie algebras of  $\mathbf{T}_{\mathbb{Z}} \subset \mathbf{B}_{\mathbb{Z}} \subset \mathbf{G}_{\mathbb{Z}}$ . Then we have  $\mathfrak{g} = \mathbb{F} \otimes_{\mathbb{Z}} \mathfrak{g}_{\mathbb{Z}}$ . Moreover, since the morphism

$$\prod_{\alpha \in \Delta} \alpha: \mathbf{T} \rightarrow (\mathbb{F}^\times)^\Delta$$

is surjective, we can assume that each  $e_\alpha$  is the image in  $\mathfrak{g}$  of a vector  $x_\alpha \in \mathfrak{g}_\mathbb{Z}$  which forms a  $\mathbb{Z}$ -basis of the  $\alpha$ -weight space of  $\mathfrak{g}_\mathbb{Z}$  (with respect to the action of  $\mathbf{T}_\mathbb{Z}$ ).

As in [Sp] we consider the  $\mathbb{Z}$ -grading  $\mathfrak{g}_\mathbb{Z} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_\mathbb{Z}^i$  induced by the height<sup>1</sup> of roots, and denote by  $t_i: \mathfrak{g}_\mathbb{Z}^i \rightarrow \mathfrak{g}_\mathbb{Z}^{i+1}$  the morphism  $y \mapsto [\sum_{\alpha \in \Delta} x_\alpha, y]$ . Then the morphism of the lemma is the morphism obtained from

$$\bigoplus_{i < 0} t_i: \bigoplus_{i < 0} \mathfrak{g}_\mathbb{Z}^i \rightarrow \bigoplus_{i \leq 0} \mathfrak{g}_\mathbb{Z}^i$$

by applying the functor  $\mathbb{F} \otimes_{\mathbb{Z}} (-)$ .

It is clear that  $t_{-1}$  identifies with the inclusion  $\mathbb{Z}\check{\Phi} \hookrightarrow X_*(\mathbf{T})$ . Hence the induced morphism  $\mathbb{F} \otimes_{\mathbb{Z}} \mathfrak{g}_\mathbb{Z}^{-1} \rightarrow \mathbb{F} \otimes_{\mathbb{Z}} \mathfrak{g}_\mathbb{Z}^0$  is injective if (C2) holds (more precisely, iff the second condition in (C2) holds).

Now if  $i < -1$ , then by [Sp, Proposition 2.2] the morphism  $t_i$  is injective, and by [Sp, Theorem 2.6] its cokernel has no  $p$ -torsion if  $p$  is good. (In [Sp, Section 2] it is assumed that the root system of  $\mathbf{G}_\mathbb{Z}$  is simple. However the result we need follows from Springer's results applied to each simple factor in a simply-connected cover of the derived subgroup of  $\mathbf{G}_\mathbb{Z}$ .) Hence  $\mathbb{F} \otimes_{\mathbb{Z}} t_i$  is injective, which finishes the proof.  $\square$

From now on we will assume that (C2) holds, so that we can apply Lemma 3.1.2. Let us consider the cocharacter  $\check{\lambda}_o := \sum_{\check{\alpha} \in \check{\Phi}^+} \check{\alpha}: \mathbb{G}_m \rightarrow \mathbf{T}$ . This cocharacter defines (via the adjoint action) a  $\mathbb{G}_m$ -action on  $\mathfrak{g}$ . The vector  $e$  is a weight vector for this action, of weight 2. Let us choose a  $\mathbb{G}_m$ -stable complement  $\mathfrak{s}$  to  $[e, \mathfrak{n}]$  in  $\mathfrak{b}$ , and set

$$\mathcal{S} := e + \mathfrak{s}, \quad \Upsilon := e + \mathfrak{b}.$$

Then  $\Upsilon$  is endowed with a  $\mathbb{G}_m$ -action defined by  $t \cdot x := t^{-2} \check{\lambda}_o(t) \cdot x$ . This action contracts  $\Upsilon$  to  $e$  (as  $t \rightarrow \infty$ ) and stabilizes  $\mathcal{S}$ .

Let us note that

$$(3.1.1) \quad \Upsilon \subset \mathfrak{g}_{\text{reg}}.$$

Indeed, by Lemma 3.1.1,  $e \in \mathfrak{g}_{\text{reg}}$ . Hence an open neighborhood of  $e$  in  $\Upsilon$  is included in  $\mathfrak{g}_{\text{reg}}$ . Using the contracting  $\mathbb{G}_m$ -action on  $\Upsilon$  defined above, we deduce that the whole of  $\Upsilon$  is included in  $\mathfrak{g}_{\text{reg}}$ .

**Lemma 3.1.3.** *Assume that (C3) holds. Then we have*

$$\mathfrak{g} = \mathfrak{s} \oplus [e, \mathfrak{g}].$$

*Proof.* By the same argument as for centralizers in  $\mathbf{G}$  (see §2.4), one can check that  $\dim(\mathfrak{g}_x) \geq r$  for all  $x \in \mathfrak{g}$ . In particular, it follows that  $\dim([e, \mathfrak{g}]) \leq \dim(\mathbf{G}) - r$ . Hence to prove the lemma it suffices to prove that  $\mathfrak{g} = \mathfrak{s} + [e, \mathfrak{g}]$ . And for this it suffices to prove that  $\mathfrak{n}^+ \subset [e, \mathfrak{g}]$ .

This fact will again be deduced from [Sp]. We use the same notation as in the proof of Lemma 3.1.2. As in this proof, one can assume that each  $e_\alpha$  is obtained from a  $\mathbb{Z}$ -basis of the corresponding root space in  $\mathfrak{g}_\mathbb{Z}$ , and consider the morphisms  $t_i$ . Then  $\mathbb{F} \otimes t_0: \mathbb{F} \otimes_{\mathbb{Z}} \mathfrak{g}_\mathbb{Z}^0 \rightarrow \mathbb{F} \otimes_{\mathbb{Z}} \mathfrak{g}_\mathbb{Z}^1$  identifies with the morphism

$$\mathfrak{t} \rightarrow \mathbb{F}^\Delta, \quad t \mapsto (d\alpha(t))_{\alpha \in \Delta}.$$

<sup>1</sup>Recall that the height of a positive root  $\alpha$  is the number of simple roots occurring in the decomposition of  $\alpha$  as a sum of simple roots (counted with multiplicities). The height of a negative root  $\alpha$  is the opposite of the height of  $-\alpha$ .

Hence it is surjective if and only if the linear forms  $d\alpha$  ( $\alpha \in \Delta$ ) are linearly independent, i.e. if and only if the third condition in (C3) holds.

And for  $i \geq 1$ , combining [Sp, Proposition 2.2] and [Sp, Theorem 2.6] we obtain that  $\text{Coker}(t_i)$  is finite and has no  $p$ -torsion if  $p$  is good. Hence  $\mathbb{F} \otimes_{\mathbb{Z}} t_i$  is surjective, which finishes the proof.  $\square$

*Remark 3.1.4.* More generally, if (C3) holds, for any  $x \in \mathcal{S}$  we have  $\mathfrak{g} = \mathfrak{s} \oplus [x, \mathfrak{g}]$ . (In particular, we have  $\dim(\mathfrak{g}_x) = r$  in this case.) Indeed, as in the proof of Lemma 3.1.3, it suffices to prove that  $\mathfrak{g} = \mathfrak{s} + [x, \mathfrak{g}]$ , i.e. that the morphism  $\mathfrak{s} \oplus \mathfrak{g} \rightarrow \mathfrak{g}$  defined by  $(s, y) \mapsto s + [x, y]$  is surjective. This property is an open condition on  $x$ . Since it is satisfied by  $e$ , it is satisfied in a neighborhood of  $e$  in  $\mathcal{S}$ . We conclude using the contracting  $\mathbb{G}_m$ -action.

### 3.2. Kostant's theorem.

**Proposition 3.2.1.** *Assume that (C2) holds. Then the morphism*

$$\mathbf{U} \times \mathcal{S} \rightarrow \Upsilon$$

*induced by the adjoint action is an isomorphism of varieties.*

*Proof.* The proof is copied from that of [GG, Lemma 2.1].

Let us denote by  $\psi$  the morphism of the lemma. First, we remark that the differential  $d_{(1,e)}\psi$  of  $\psi$  at  $(1, e)$  can be identified with the morphism

$$\begin{cases} \mathfrak{n} \oplus \mathfrak{s} & \rightarrow & \mathfrak{b} \\ (n, s) & \mapsto & [n, e] + s \end{cases} .$$

By Lemma 3.1.2 and the definition of  $\mathfrak{s}$ , this morphism is an isomorphism. It follows that  $\psi$  is étale in a neighborhood of  $(1, e)$ , and in particular dominant. We deduce that the morphism  $\psi^*: \mathcal{O}(\Upsilon) \rightarrow \mathcal{O}(\mathbf{U} \times \mathcal{S})$  is injective.

To prove that  $\psi^*$  is an isomorphism, we consider the  $\mathbb{G}_m$ -actions on  $\mathcal{S}$  and  $\Upsilon$  defined in §3.1. In fact we also define a  $\mathbb{G}_m$ -action on  $\mathbf{U} \times \mathcal{S}$  by setting  $t \cdot (u, s) = (\check{\lambda}_\circ(t)u\check{\lambda}_\circ(t)^{-1}, t \cdot s)$ . Then the actions on  $\mathbf{U} \times \mathcal{S}$  and  $\Upsilon$  are contracting as  $t \rightarrow \infty$  (to  $(1, e)$  and to  $e$  respectively), and  $\psi$  is  $\mathbb{G}_m$ -equivariant. Hence to conclude we only have to prove that the characters of the action of  $\mathbb{G}_m$  on  $\mathcal{O}(\Upsilon)$  and  $\mathcal{O}(\mathbf{U} \times \mathcal{S})$  coincide. However, if  $X$  is any of these spaces,  $x$  is its  $\mathbb{G}_m$ -fixed point,  $\mathfrak{m}_x \subset \mathcal{O}(X)$  the ideal of  $x$ , and  $T_x$  the tangent space of  $X$  at  $x$ , then, as a  $\mathbb{G}_m$ -module,  $\mathcal{O}(X)$  is isomorphic to the associated graded of the  $\mathfrak{m}_x$ -adic filtration, which is itself isomorphic (again as a  $\mathbb{G}_m$ -module) to  $\mathcal{O}(T_x)$ . Since  $d_{(1,e)}\psi$  is an isomorphism, we deduce the equality of characters, which finishes the proof.  $\square$

Now we can prove the main result of this subsection. Our proof is essentially identical to that in [Ve, Proposition 6.3] (which is due to Springer), and is partially based on the same idea as for Proposition 3.2.1. The fact that the proof in [Ve] can be generalized to very good primes using the results of Demazure is mentioned in [Sl, §3.14], see also [J3, §7.14]. In fact, slightly weaker assumptions are sufficient.

**Theorem 3.2.2** (Kostant's theorem). *Assume that (C1) and (C2) hold. Then the natural morphisms*

$$\mathcal{S} \rightarrow \Upsilon/\mathbf{U} \rightarrow \mathfrak{g}/\mathbf{G}$$

*are isomorphisms.*

*Proof.* Proposition 3.2.1 implies that the first morphism is an isomorphism. So to complete the proof it is enough to prove that the composition  $\mathcal{S} \rightarrow \mathfrak{g}/\mathbf{G}$  is an isomorphism. First, let us prove that this morphism is dominant.

For this it suffices to prove that  $\mathbf{G} \cdot \mathcal{S}$  contains the open set of regular semisimple elements. So, let  $x \in \mathfrak{g}$  be a regular semisimple element. Then  $x$  is  $\mathbf{G}$ -conjugate to a regular element  $y \in \mathfrak{t}$  by [Bo, Proposition 11.8]. Now  $\mathbf{U}^+ \cdot y = y + \mathfrak{n}^+ \ni y + e$  (see e.g. [J3, §13.3]), hence  $x$  is  $\mathbf{G}$ -conjugate to  $y + e$ . Finally, since  $y + e \in e + \mathfrak{b} = \Upsilon$ , it follows from Proposition 3.2.1 that  $x \in \mathbf{G} \cdot \mathcal{S}$ .

Now we consider again the  $\mathbb{G}_m$ -action on  $\mathcal{S}$  defined in §3.1. Then the morphism  $\mathcal{S} \rightarrow \mathfrak{g}/\mathbf{G}$  is  $\mathbb{G}_m$ -equivariant, where the  $\mathbb{G}_m$ -action on  $\mathfrak{g}/\mathbf{G}$  is induced by the action  $t \cdot x = t^{-2}x$  on  $\mathfrak{g}$ . As in the proof of Proposition 3.2.1, to conclude it is enough to prove that the  $\mathbb{G}_m$ -modules  $\mathcal{O}(\mathcal{S})$  and  $\mathcal{O}(\mathfrak{g}/\mathbf{G})$  have the same character.

To prove this we remark that  $\mathcal{S}$  is isomorphic, as a  $\mathbb{G}_m$ -variety, to  $\mathfrak{s}$ , hence to  $\mathfrak{b}/[e, \mathfrak{n}]$ . (Here the  $\mathbb{G}_m$ -action on  $\mathfrak{b}$  is given by  $t \cdot x = t^{-2}\check{\lambda}_\circ(t) \cdot x$ .) Now recall the notation introduced in the proof of Lemma 3.1.2. As in this lemma we can assume that each  $e_\alpha$  is obtained from an element  $x_\alpha \in \mathfrak{g}_\mathbb{Z}$ . Then one can consider the base change  $\mathbf{G}_\mathbb{C}$  of  $\mathbf{G}_\mathbb{Z}$  to  $\mathbb{C}$ , and the corresponding objects  $\mathbf{B}_\mathbb{C}$ ,  $\mathbf{T}_\mathbb{C}$ ,  $\mathfrak{g}_\mathbb{C}$ ,  $\mathfrak{b}_\mathbb{C}$ ,  $\mathfrak{t}_\mathbb{C}$ . If  $e_\mathbb{C}$  denotes the image of  $\sum_{\alpha \in \Delta} x_\alpha$  in  $\mathfrak{g}_\mathbb{C}$ , then it follows from the proof of Lemma 3.1.2 that the  $(\mathbb{G}_m)_\mathbb{F}$ -character of  $\mathfrak{b}/[e, \mathfrak{n}]$  coincides with the  $(\mathbb{G}_m)_\mathbb{C}$ -character of  $\mathfrak{b}_\mathbb{C}/[e_\mathbb{C}, \mathfrak{n}_\mathbb{C}]$  (where the  $(\mathbb{G}_m)_\mathbb{C}$ -action on  $\mathfrak{b}_\mathbb{C}$  is defined by the same formula as the  $(\mathbb{G}_m)_\mathbb{F}$ -action on  $\mathfrak{b}_\mathbb{F}$  considered above). By Lemma 3.1.3 applied to the field  $\mathbb{C}$ , we have a  $\mathbb{G}_m$ -equivariant isomorphism  $\mathfrak{b}_\mathbb{C}/[e_\mathbb{C}, \mathfrak{n}_\mathbb{C}] \xrightarrow{\sim} \mathfrak{g}_\mathbb{C}/[e_\mathbb{C}, \mathfrak{g}_\mathbb{C}]$ . Using Kostant's theorem over  $\mathbb{C}$  (see [Ko, Theorem 7]), we deduce that the  $\mathbb{G}_m$ -character of  $\mathcal{O}(\mathcal{S})$  coincides with the  $(\mathbb{G}_m)_\mathbb{C}$ -character of  $\mathcal{O}(\mathfrak{t}_\mathbb{C}/W)$ , where the  $(\mathbb{G}_m)_\mathbb{C}$ -action is induced by the action on  $\mathfrak{t}_\mathbb{C}$  given by  $t \cdot x = t^{-2}x$ .

From these considerations we obtain that to conclude the proof it suffices to prove that the  $\mathbb{G}_m$ -character of  $\mathcal{O}(\mathfrak{g}/\mathbf{G})$  coincides with the  $(\mathbb{G}_m)_\mathbb{C}$ -character of  $\mathcal{O}(\mathfrak{t}_\mathbb{C}/W)$ . By Proposition 2.4.2, it suffices to show that the  $\mathbb{G}_m$ -character of  $\mathcal{O}(\mathfrak{t}/W)$  coincides with the  $(\mathbb{G}_m)_\mathbb{C}$ -character of  $\mathcal{O}(\mathfrak{t}_\mathbb{C}/W)$ . However, if  $N$  is the product of the prime numbers  $p$  which are bad primes for  $\mathbf{G}$  or such that  $X_*(\mathbf{T})/\mathbb{Z}\check{\Phi}$  has  $p$ -torsion, then we have

$$\begin{aligned} \mathcal{O}(\mathfrak{t}_\mathbb{C}/W) &= \mathbb{C} \otimes_{\mathbb{Z}[1/N]} \mathbb{S}_{\mathbb{Z}[1/N]}(X^*(\mathbf{T}) \otimes_{\mathbb{Z}} \mathbb{Z}[1/N])^W \\ \mathcal{O}(\mathfrak{t}_\mathbb{F}/W) &= \mathbb{F} \otimes_{\mathbb{Z}[1/N]} \mathbb{S}_{\mathbb{Z}[1/N]}(X^*(\mathbf{T}) \otimes_{\mathbb{Z}} \mathbb{Z}[1/N])^W \end{aligned}$$

by [De, Corollaire on p. 296]. (Note that the  $\mathbb{Z}$ -module  $\text{Coker}(i)$ , where  $i$  is the map considered in [De, §5] for the root system  $\Phi \subset X^*(\mathbf{T})$ , is isomorphic to  $\text{Ext}_{\mathbb{Z}}^1(X_*(\mathbf{T})/\mathbb{Z}\check{\Phi}, \mathbb{Z})$  with our notation; in particular, this  $\mathbb{Z}$ -module has the same torsion as  $X_*(\mathbf{T})/\mathbb{Z}\check{\Phi}$ .) By [De, Théorème 2 on p. 295], the  $\mathbb{Z}[1/N]$ -module  $\mathbb{S}_{\mathbb{Z}[1/N]}(X^*(\mathbf{T}) \otimes_{\mathbb{Z}} \mathbb{Z}[1/N])^W$  is graded free. We deduce the equality of characters, which finishes the proof.  $\square$

*Remark 3.2.3.* If (C1) and (C2) hold, it follows from Theorem 3.2.2 that  $\mathfrak{g}/\mathbf{G} = \mathfrak{t}/W$  is smooth. In fact this follows more directly from the ingredients of the proof: see [De, Corollaire and Théorème 3 on p. 296].

**3.3. The universal centralizer.** The main player of this subsection and the next one is the “universal centralizer” over  $\mathfrak{g}$ , i.e. the closed subgroup of the group scheme  $\mathbf{G} \times \mathfrak{g}$  over  $\mathfrak{g}$  defined as the fiber product

$$\mathbf{I} := \mathfrak{g} \times_{\mathfrak{g} \times \mathfrak{g}} (\mathbf{G} \times \mathfrak{g}).$$

Here the morphism  $\mathfrak{g} \rightarrow \mathfrak{g} \times \mathfrak{g}$  is the diagonal embedding, and the morphism  $\mathbf{G} \times \mathfrak{g} \rightarrow \mathfrak{g} \times \mathfrak{g}$  is defined by  $(g, x) \mapsto (g \cdot x, x)$ . By definition, for  $x \in \mathfrak{g}$ , the (scheme-theoretic) fiber of  $\mathbf{I}$  over  $x$  identifies with the group scheme  $\mathbf{G}_x$ .

We will denote by  $\mathbf{I}_{\text{reg}}$  the restriction of  $\mathbf{I}$  to  $\mathfrak{g}_{\text{reg}}$ , by  $\mathbf{I}_{\text{rs}}$  its restriction to  $\mathfrak{g}_{\text{rs}}$ , and by  $\mathbf{I}_{\mathcal{S}}$  its restriction to  $\mathcal{S}$ .

**Lemma 3.3.1.** *Assume that (C3) holds. Then the morphism*

$$a: \mathbf{G} \times \mathcal{S} \rightarrow \mathfrak{g}_{\text{reg}}$$

*induced by the adjoint action is smooth and surjective.*

*Proof.* Let us first prove smoothness. The differential of  $a$  at  $(1, e)$  can be identified with the morphism  $\mathfrak{g} \times \mathfrak{s} \rightarrow \mathfrak{g}$  sending  $(x, s)$  to  $[x, e] + s$ . By Lemma 3.1.3 this differential is surjective, hence  $a$  is smooth in a neighborhood of  $(1, e)$ . In fact, since this morphism is  $\mathbf{G}$ -equivariant, there exists a neighborhood  $V$  of  $e$  in  $\mathcal{S}$  such that  $a$  is smooth on  $\mathbf{G} \times V$ . Then using the  $\mathbf{G}_m$ -action on  $\mathcal{S}$  as in the proof of (3.1.1), we conclude that  $a$  is smooth.

To prove surjectivity, using smoothness it is enough to prove that closed points of  $\mathfrak{g}_{\text{reg}}$  belong to the image of  $a$ . In other words, we have to prove that any element of  $\mathfrak{g}_{\text{reg}}$  is  $\mathbf{G}$ -conjugate to an element of  $\mathcal{S}$ . Let  $x \in \mathfrak{g}_{\text{reg}}$ . Then it follows from Theorem 3.2.2 that there exists  $y \in \mathcal{S}$  such that  $\chi(x) = \chi(y)$ . We conclude using Lemma 2.4.1 and (3.1.1).  $\square$

*Remark 3.3.2.* Assume that (C3) holds. Then for all  $x \in \mathfrak{g}_{\text{reg}}$  we have  $\dim(\mathfrak{g}_x) = r$ . In fact, this property was observed for  $x \in \mathcal{S}$  in Remark 3.1.4. But since any element of  $\mathfrak{g}_{\text{reg}}$  is  $\mathbf{G}$ -conjugate to an element in  $\mathcal{S}$  by Lemma 3.3.1, it holds on the whole of  $\mathfrak{g}_{\text{reg}}$ . In the terminology of [Sp, §5.7] this means that, in this case, all regular elements in  $\mathfrak{g}$  are *smoothly regular*.

Let us note the following consequence. (See also [BG, Corollary on p. 746] for a different proof, assuming that Jantzen's "standard hypotheses" hold and that  $\ell$  is odd.)

**Proposition 3.3.3.** *Assume that (C3) holds. Then the morphism  $\chi_{\text{reg}}: \mathfrak{g}_{\text{reg}} \rightarrow \mathfrak{g}/\mathbf{G}$  is smooth and surjective.*

*Proof.* Consider the composition

$$\mathbf{G} \times \mathcal{S} \xrightarrow{a} \mathfrak{g}_{\text{reg}} \xrightarrow{\chi_{\text{reg}}} \mathfrak{g}/\mathbf{G}.$$

Using Theorem 3.2.2 this morphism identifies with the projection  $\mathbf{G} \times \mathcal{S} \rightarrow \mathcal{S}$  on the second factor. In particular it is smooth and surjective. By [SP, Tag 02K5], we deduce that  $\chi_{\text{reg}}$  is smooth and surjective.  $\square$

*Remark 3.3.4.* Assume that (C3) holds. The fact that  $\chi_{\text{reg}}$  is smooth implies that, for all  $x \in \mathfrak{g}_{\text{reg}}$ , the morphism  $d_x(\chi_{\text{reg}}): \mathbb{T}_x(\mathfrak{g}_{\text{reg}}) \rightarrow \mathbb{T}_{\chi(x)}(\mathfrak{g}/\mathbf{G})$  is surjective; in other words we have a surjection  $\mathfrak{g} \rightarrow \mathbb{T}_{\chi(x)}(\mathfrak{g}/\mathbf{G})$ . (Here  $\mathbb{T}_x(\mathfrak{g}_{\text{reg}})$ , resp.  $\mathbb{T}_{\chi(x)}(\mathfrak{g}/\mathbf{G})$ , denotes the tangent space to  $\mathfrak{g}_{\text{reg}}$  at  $x$ , resp. to  $\mathfrak{g}/\mathbf{G}$  at  $\chi(x)$ .) Since the composition of  $\chi_{\text{reg}}$  with the morphism  $\mathbf{G} \rightarrow \mathfrak{g}_{\text{reg}}$  defined by  $g \mapsto g \cdot x$  is constant, the differential  $d_x(\chi_{\text{reg}})$  must vanish on  $[x, \mathfrak{g}]$ . Since  $\dim(\mathfrak{g}/[x, \mathfrak{g}]) = r$  (see Remark 3.3.2), we finally obtain that  $d_x(\chi_{\text{reg}})$  induces an isomorphism  $\mathfrak{g}/[x, \mathfrak{g}] \xrightarrow{\sim} \mathbb{T}_{\chi(x)}(\mathfrak{g}/\mathbf{G})$ .

**Proposition 3.3.5.** *Assume that (C3) holds. Then the group scheme  $\mathbf{I}_{\mathcal{S}}$  is smooth over  $\mathcal{S}$ .*

*Proof.* By definition we have  $\mathbf{I}_{\mathcal{S}} = \mathcal{S} \times_{\mathfrak{g} \times \mathcal{S}} (\mathbf{G} \times \mathcal{S})$ . Since each of the morphisms which define this fiber product factors through  $\mathfrak{g} \times_{\mathfrak{g}/\mathbf{G}} \mathcal{S}$ , we also have  $\mathbf{I}_{\mathcal{S}} \cong \mathcal{S} \times_{\mathfrak{g} \times_{\mathfrak{g}/\mathbf{G}} \mathcal{S}} (\mathbf{G} \times \mathcal{S})$ . Now using Theorem 3.2.2 we obtain that the first projection induces an isomorphism  $\mathfrak{g} \times_{\mathfrak{g}/\mathbf{G}} \mathcal{S} \xrightarrow{\sim} \mathfrak{g}$ , which provides an isomorphism

$$\mathbf{I}_{\mathcal{S}} \xrightarrow{\sim} \mathcal{S} \times_{\mathfrak{g}} (\mathbf{G} \times \mathcal{S}).$$

In this fiber product the morphism  $\mathbf{G} \times \mathcal{S} \rightarrow \mathfrak{g}$  is the composition of the morphism  $a$  of Lemma 3.3.1 with the open embedding  $\mathfrak{g}_{\text{reg}} \hookrightarrow \mathfrak{g}$ , hence it is smooth; this implies our claim.  $\square$

**Corollary 3.3.6.** *Assume that (C3) holds. Then the group scheme  $\mathbf{I}_{\text{reg}}$  is smooth over  $\mathfrak{g}_{\text{reg}}$ .*

*Proof.* The following diagram (where each map is the natural one) is Cartesian by the  $\mathbf{G}$ -equivariance of  $\mathbf{I}$ :

$$\begin{array}{ccc} G \times \mathbf{I}_{\mathcal{S}} & \longrightarrow & \mathbf{I}_{\text{reg}} \\ \downarrow & & \downarrow \\ G \times \mathcal{S} & \xrightarrow{a} & \mathfrak{g}_{\text{reg}}. \end{array}$$

By Lemma 3.3.1 and Proposition 3.3.5, the composition  $G \times \mathbf{I}_{\mathcal{S}} \rightarrow \mathfrak{g}_{\text{reg}}$  is smooth. Since  $a$  is smooth and surjective, so is the morphism  $G \times \mathbf{I}_{\mathcal{S}} \rightarrow \mathbf{I}_{\text{reg}}$ . Using [SP, Tag 02K5], we deduce that the morphism  $\mathbf{I}_{\text{reg}} \rightarrow \mathfrak{g}_{\text{reg}}$  is smooth.  $\square$

*Remark 3.3.7.* Assume that (C3) holds. Then it follows from Corollary 3.3.6 that for any  $x \in \mathfrak{g}_{\text{reg}}$ , the stabilizer  $\mathbf{G}_x$  is a smooth group scheme over  $\mathbb{F}$ , i.e. an algebraic group in the “traditional” sense. In particular we have  $\dim_{\mathbb{F}}(\mathcal{L}ie(\mathbf{G}_x)) = \dim(\mathbf{G}_x) = r$ , see [Wa, Second corollary on p. 94], and therefore the embedding  $\mathcal{L}ie(\mathbf{G}_x) \hookrightarrow \mathfrak{g}_x$  is an isomorphism (see Remark 3.3.2).

**Corollary 3.3.8.** *Assume that (C3) holds. Then the group scheme  $\mathbf{I}_{\text{reg}}$  is commutative.*

*Proof.* By Corollary 3.3.6,  $\mathbf{I}_{\text{reg}}$  is flat over  $\mathfrak{g}_{\text{reg}}$ . Hence it is enough to prove that  $\mathbf{I}_{\text{rs}}$  is commutative. Since any element in  $\mathfrak{g}_{\text{rs}}$  is  $\mathbf{G}$ -conjugate to an element in  $\mathfrak{t}_{\text{rs}}$  (see [Bo, Proposition 11.8]), it is enough to prove that for any  $x \in \mathfrak{t}_{\text{rs}}$ , the group  $\mathbf{G}_x$  is commutative. However in this case we have  $\mathbf{G}_x = \mathbf{T}$  (see Lemma 2.4.3(2)), which is clearly commutative.  $\square$

Let us also note the following property.

**Proposition 3.3.9.** *Assume that (C3) holds. Then there exists a unique smooth affine commutative group scheme  $\mathbf{J}$  over  $\mathfrak{t}/W$  such that the pullback of  $\mathbf{J}$  under  $\chi_{\text{reg}}: \mathfrak{g}_{\text{reg}} \rightarrow \mathfrak{t}/W$  is  $\mathbf{I}_{\text{reg}}$ .*

*Sketch of proof.* The group scheme  $\mathbf{J}$  is constructed by descent (for schemes affine over a base) along the smooth and surjective morphism  $\chi_{\text{reg}}$  (see Proposition 3.3.3), following the arguments in [Ng, Lemma 2.1.1].

These arguments use the fact that the morphism  $\mu: G \times \mathfrak{g}_{\text{reg}} \rightarrow \mathfrak{g}_{\text{reg}} \times_{\mathfrak{t}/W} \mathfrak{g}_{\text{reg}}$  defined by  $(g, x) \mapsto (x, g \cdot x)$  is smooth and surjective. To check this property, we consider the composition

$$(3.3.1) \quad \mathbf{G} \times \mathbf{G} \times \mathcal{S} \xrightarrow{\text{id}_{\mathbf{G}} \times a} \mathbf{G} \times \mathfrak{g}_{\text{reg}} \xrightarrow{\mu} \mathfrak{g}_{\text{reg}} \times_{\mathfrak{t}/W} \mathfrak{g}_{\text{reg}}.$$

To prove that  $\mu$  is smooth and surjective, it suffices to prove that this composition has the same properties: then surjectivity is clear, and smoothness follows from Lemma 3.3.1 and [SP, Tag 02K5]. Now, to prove that (3.3.1) is smooth and surjective, it is enough to prove that its composition with the automorphism of  $\mathbf{G} \times \mathbf{G} \times \mathcal{S}$  defined by  $(g, h, x) \mapsto (hg^{-1}, g, x)$  has the same properties. The latter morphism sends  $(g, h, x)$  to  $(g \cdot x, h \cdot x)$ . Hence it identifies with the smooth and surjective morphism

$$(\mathbf{G} \times \mathcal{S}) \times_{\mathfrak{t}/W} (\mathbf{G} \times \mathcal{S}) \xrightarrow{a \times a} \mathfrak{g}_{\text{reg}} \times_{\mathfrak{t}/W} \mathfrak{g}_{\text{reg}},$$

which finishes the proof.  $\square$

*Remark 3.3.10.* The group scheme  $\mathbf{I}_{\mathcal{S}}$  is naturally isomorphic to the pullback of  $\mathbf{J}$  along the isomorphism  $\mathcal{S} \xrightarrow{\sim} \mathfrak{t}/W$  given by the composition  $\mathcal{S} \hookrightarrow \mathfrak{g}_{\text{reg}} \rightarrow \mathfrak{t}/W$ .

Using the group scheme  $\mathbf{J}$  one can obtain a more “concrete” description of the category of  $\mathbf{G}$ -equivariant coherent sheaves on  $\mathfrak{g}_{\text{reg}}$ , as follows.

**Proposition 3.3.11.** *Assume that (C3) holds. Then there exists a canonical equivalence of categories*

$$\text{Coh}^{\mathbf{G}}(\mathfrak{g}_{\text{reg}}) \cong \text{Rep}(\mathbf{J}),$$

where the right-hand side is the category of representations of  $\mathbf{J}$  which are of finite type over  $\mathcal{O}(\mathfrak{t}/W)$ .

*Sketch of proof.* We only indicate how to construct the functors in both directions.

First, consider a  $\mathbf{G}$ -equivariant coherent sheaf  $\mathcal{F}$  on  $\mathfrak{g}_{\text{reg}}$ . Then it is naturally endowed with the structure of a representation of  $\mathbf{I}$ , see e.g. [MR, §2.3]. Hence its restriction to  $\mathcal{S}$  is a representation of  $\mathbf{I}_{\mathcal{S}}$ . Identifying  $\mathcal{S}$  with  $\mathfrak{t}/W$  (see Theorem 3.2.2) and  $\mathbf{I}_{\mathcal{S}}$  with  $\mathbf{J}$  (see Remark 3.3.10), we obtain a representation of  $\mathbf{J}$ .

In the other direction, consider some (finite type) representation  $\mathcal{G}$  of  $\mathbf{J}$ . Then, to define the corresponding  $\mathbf{G}$ -equivariant coherent sheaf  $\mathcal{F}$  on  $\mathfrak{g}_{\text{reg}}$ , since  $a$  is smooth and surjective (see Lemma 3.3.1) it suffices to define a  $\mathbf{G}$ -equivariant coherent sheaf  $\mathcal{F}'$  on  $\mathbf{G} \times \mathcal{S}$  together with a descent datum. However  $\mathcal{G}$  defines (as above) a representation  $\mathcal{G}'$  of  $\mathbf{I}_{\mathcal{S}}$ , and we can consider the coherent sheaf  $\mathcal{F}' := \mathcal{O}_{\mathbf{G}} \boxtimes \mathcal{G}'$  on  $\mathbf{G} \times \mathcal{S}$ . One can identify the fiber product  $(\mathbf{G} \times \mathcal{S}) \times_{\mathfrak{g}_{\text{reg}}} (\mathbf{G} \times \mathcal{S})$  with  $\mathbf{G} \times \mathbf{I}_{\mathcal{S}}$ , so that the two projections to  $\mathbf{G} \times \mathcal{S}$  are given by  $(g, (h, s)) \mapsto (g, s)$  and  $(g, (h, s)) \mapsto (gh, s)$  respectively. With this identification, the descent datum is equivalent to the datum of an  $\mathbf{I}_{\mathcal{S}}$ -equivariant structure on  $\mathcal{F}'$ , where  $\mathbf{I}_{\mathcal{S}}$  acts on  $\mathbf{G} \times \mathcal{S}$  via  $(h, s) \cdot (g, s) = (gh, s)$ . However  $\mathcal{G}'$  can be considered as an  $\mathbf{I}_{\mathcal{S}}$ -equivariant coherent sheaf on  $\mathcal{S}$ , and this structure is obtained by considering  $\mathcal{F}'$  as the inverse image of  $\mathcal{G}'$  under the ( $\mathbf{I}_{\mathcal{S}}$ -equivariant) projection  $\mathbf{G} \times \mathcal{S} \rightarrow \mathcal{S}$ .  $\square$

**3.4. Lie algebras.** We set

$$\begin{aligned} \mathfrak{J} &:= \text{Lie}(\mathbf{I}/\mathfrak{g}), & \mathfrak{J}_{\text{reg}} &:= \text{Lie}(\mathbf{I}_{\text{reg}}/\mathfrak{g}_{\text{reg}}), & \mathfrak{J}_{\text{rs}} &:= \text{Lie}(\mathbf{I}_{\text{rs}}/\mathfrak{g}_{\text{rs}}), \\ \mathfrak{J}_{\mathcal{S}} &:= \text{Lie}(\mathbf{I}_{\mathcal{S}}/\mathcal{S}), & \mathfrak{J} &:= \text{Lie}(\mathbf{J}/(\mathfrak{t}/W)). \end{aligned}$$

It follows from Lemma 2.2.1(2) that  $\mathfrak{J}_{\text{reg}}$ , resp.  $\mathfrak{J}_{\text{rs}}$ , is the restriction of  $\mathfrak{J}$  to  $\mathfrak{g}_{\text{reg}}$ , resp.  $\mathfrak{g}_{\text{rs}}$ . We also set

$$\begin{aligned} \mathbb{I} &:= \text{Lie}(\mathbf{I}/\mathfrak{g}), & \mathbb{I}_{\text{reg}} &:= \text{Lie}(\mathbf{I}_{\text{reg}}/\mathfrak{g}_{\text{reg}}), & \mathbb{I}_{\text{rs}} &:= \text{Lie}(\mathbf{I}_{\text{rs}}/\mathfrak{g}_{\text{rs}}), \\ \mathbb{I}_{\mathcal{S}} &:= \text{Lie}(\mathbf{I}_{\mathcal{S}}/\mathcal{S}), & \mathbb{J} &:= \text{Lie}(\mathbf{J}/(\mathfrak{t}/W)). \end{aligned}$$

The following properties are easy consequences of the results of §3.3.

**Proposition 3.4.1.** *Assume that (C3) holds. Then:*

- (1) *the coherent sheaf  $\mathfrak{J}_{\text{reg}}$ , resp.  $\mathfrak{J}_{\text{rs}}$ , resp.  $\mathfrak{J}_{\mathcal{S}}$ , resp.  $\mathfrak{J}$ , on  $\mathfrak{g}_{\text{reg}}$ , resp.  $\mathfrak{g}_{\text{rs}}$ , resp.  $\mathcal{S}$  resp.  $\mathfrak{t}/W$ , is locally free of finite rank;*
- (2) *the Lie algebra  $\mathbb{I}_{\text{reg}}$ , resp.  $\mathbb{I}_{\text{rs}}$ , resp.  $\mathbb{I}_{\mathcal{S}}$ , resp.  $\mathbb{J}$ , is a vector bundle over  $\mathfrak{g}_{\text{reg}}$ , resp.  $\mathfrak{g}_{\text{rs}}$ , resp.  $\mathcal{S}$ , resp.  $\mathfrak{t}/W$ ;*
- (3) *the Lie algebras  $\mathfrak{J}_{\text{reg}}$ ,  $\mathfrak{J}_{\text{rs}}$ ,  $\mathfrak{J}_{\mathcal{S}}$  and  $\mathfrak{J}$  are commutative;*
- (4) *the restriction of  $\mathfrak{J}_{\text{reg}}$  to  $\mathcal{S}$  is canonically isomorphic to  $\mathfrak{J}_{\mathcal{S}}$ ; in other words there exists a canonical Cartesian diagram*

$$\begin{array}{ccc} \mathbb{I}_{\mathcal{S}} & \longrightarrow & \mathbb{I}_{\text{reg}} \\ \downarrow & & \downarrow \\ \mathcal{S} & \longrightarrow & \mathfrak{g}_{\text{reg}}; \end{array}$$

- (5) *there exists a canonical isomorphism  $\chi_{\text{reg}}^*(\mathfrak{J}) \cong \mathfrak{J}_{\text{reg}}$ ; in other words there exists a canonical Cartesian diagram*

$$\begin{array}{ccc} \mathbb{I}_{\text{reg}} & \longrightarrow & \mathbb{J} \\ \downarrow & & \downarrow \\ \mathfrak{g}_{\text{reg}} & \xrightarrow{\chi_{\text{reg}}} & \mathfrak{t}/W; \end{array}$$

- (6) *the coherent sheaf  $\mathfrak{J}_{\mathcal{S}}$  is canonically isomorphic to the pullback of  $\mathfrak{J}$  under the isomorphism  $\mathcal{S} \xrightarrow{\sim} \mathfrak{t}/W$ ; in other words  $\mathbb{I}_{\mathcal{S}}$  is canonically isomorphic to the pullback of  $\mathbb{J}$  under the isomorphism  $\mathcal{S} \xrightarrow{\sim} \mathfrak{t}/W$ .*

*Proof.* (1) and (2) follow from Lemma 2.2.1(1) together with Proposition 3.3.5, Corollary 3.3.6 and Proposition 3.3.9. (3) follows from Corollary 3.3.8 and Proposition 3.3.9. (4) follows from Lemma 2.2.1(2) and Corollary 3.3.6. (5) follows from Lemma 2.2.1(2) and Proposition 3.3.9. Finally, (6) follows from Remark 3.3.10.  $\square$

The main result of this subsection gives a more concrete description of the Lie algebras  $\mathbb{J}$ ,  $\mathbb{I}_{\text{reg}}$  and  $\mathbb{I}_{\mathcal{S}}$ . The proof will require that (C4) holds. In fact, from now on we fix a  $\mathbf{G}$ -equivariant isomorphism  $\kappa: \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$ .

**Theorem 3.4.2.** *Assume that (C4) holds. Then there exist canonical isomorphisms*

$$\mathbb{J} \cong \mathbb{T}^*(\mathfrak{t}/W), \quad \mathbb{I}_{\text{reg}} \cong \mathfrak{g}_{\text{reg}} \times_{\mathfrak{t}/W} \mathbb{T}^*(\mathfrak{t}/W), \quad \mathbb{I}_{\mathcal{S}} \cong \mathbb{T}^*(\mathcal{S})$$

*of commutative Lie algebras over  $\mathfrak{t}/W$ ,  $\mathfrak{g}_{\text{reg}}$  and  $\mathcal{S}$ , respectively.*

*Proof.* It is enough to construct the third isomorphism; then the other two isomorphisms follow, using Proposition 3.4.1(5)-(6).

First we construct a morphism of coherent sheaves  $\mathfrak{J}_{\mathcal{S}} \rightarrow \Omega_{\mathcal{S}}$ . Since  $\mathbf{I}_{\mathcal{S}}$  is a closed subgroup of  $\mathbf{G} \times \mathcal{S}$ , the Lie algebra  $\mathfrak{J}_{\mathcal{S}}$  embeds naturally in  $\mathcal{L}ie(\mathbf{G} \times \mathcal{S}/\mathcal{S}) = \mathfrak{g} \otimes_{\mathbb{F}} \mathcal{O}_{\mathcal{S}}$ . Now since  $\mathcal{S}$  is a smooth closed subvariety in  $\mathfrak{g}$ , the tangent sheaf  $\mathcal{T}_{\mathcal{S}}$  embeds in

the restriction of  $\mathcal{T}_{\mathfrak{g}}$  to  $\mathcal{S}$ , i.e. in  $\mathfrak{g} \otimes_{\mathbb{F}} \mathcal{O}_{\mathcal{S}}$ , and the cokernel of this embedding is a locally free sheaf. We deduce a canonical surjection  $\mathfrak{g}^* \otimes_{\mathbb{F}} \mathcal{O}_{\mathcal{S}} \rightarrow \Omega_{\mathcal{S}}$ . Then we define our morphism as the composition

$$(3.4.1) \quad \mathcal{I}_{\mathcal{S}} \hookrightarrow \mathfrak{g} \otimes_{\mathbb{F}} \mathcal{O}_{\mathcal{S}} \xrightarrow[\sim]{\kappa \otimes \mathcal{O}_{\mathcal{S}}} \mathfrak{g}^* \otimes_{\mathbb{F}} \mathcal{O}_{\mathcal{S}} \rightarrow \Omega_{\mathcal{S}}.$$

To finish the proof it suffices to prove that this morphism is an isomorphism.

Recall the contracting  $\mathbb{G}_m$ -action on  $\mathcal{S}$  considered in §3.1. One can “extend” this action to a compatible action on  $\mathbf{G} \times \mathcal{S}$  by group automorphisms, setting  $t \cdot (g, x) = (\check{\lambda}_o(t)g\check{\lambda}_o(t)^{-1}, t \cdot x)$ . This action stabilizes  $\mathbf{I}_{\mathcal{S}}$ , and we deduce a  $\mathbb{G}_m$ -equivariant structure on  $\mathcal{I}_{\mathcal{S}}$ . Clearly  $\Omega_{\mathcal{S}}$  is also  $\mathbb{G}_m$ -equivariant, and (3.4.1) is a morphism of  $\mathbb{G}_m$ -equivariant coherent sheaves. Since  $\Omega_{\mathcal{S}}$  is locally free (in particular, has no torsion), using some version of the graded Nakayama lemma we deduce that to prove that (3.4.1) is an isomorphism it suffices to prove that the induced morphism

$$(3.4.2) \quad i_e^*(\mathcal{I}_{\mathcal{S}}) \rightarrow i_e^*(\Omega_{\mathcal{S}})$$

is an isomorphism, where  $i_e: \{e\} \hookrightarrow \mathcal{S}$  is the inclusion.

We claim that there exists a canonical isomorphism  $i_e^*(\mathcal{I}_{\mathcal{S}}) \xrightarrow{\sim} \mathfrak{g}_e$ . Indeed, using Lemma 2.2.1(2) and Proposition 3.3.5 there exists a canonical isomorphism  $i_e^*(\mathcal{I}_{\mathcal{S}}) \xrightarrow{\sim} \mathcal{L}ie(\mathbf{G}_e)$ , and by Remark 3.3.7 the right-hand side identifies with  $\mathfrak{g}_e$  canonically. Using this claim and the obvious isomorphism  $i_e^*(\Omega_{\mathcal{S}}) \cong \mathfrak{s}^*$ , one can identify (3.4.2) with the composition

$$\mathfrak{g}_e \hookrightarrow \mathfrak{g} \xrightarrow[\sim]{\kappa} \mathfrak{g}^* \rightarrow \mathfrak{s}^*.$$

The fact that this morphism is an isomorphism follows from Lemma 2.4.4 and Lemma 3.1.3.  $\square$

**3.5. Variant for the Grothendieck resolution.** In this subsection we consider analogues of the objects studied above for the Grothendieck resolution  $\tilde{\mathfrak{g}}$ . Recall that this variety is the vector bundle on the flag variety  $\mathcal{B}$  of  $\mathbf{G}$  (considered as the variety of Borel subgroups in  $\mathbf{G}$ ) defined as

$$\tilde{\mathfrak{g}} = \{(x, \mathbf{B}') \in \mathfrak{g} \times \mathcal{B} \mid x \in \mathcal{L}ie(\mathbf{B}')\}.$$

There exists a natural projective morphism  $\pi: \tilde{\mathfrak{g}} \rightarrow \mathfrak{g}$  defined by  $\pi(x, \mathbf{B}') = x$ . We denote by  $\tilde{\mathfrak{g}}_{\text{reg}}$ , resp.  $\tilde{\mathfrak{g}}_{\text{rs}}$ , the inverse image of  $\mathfrak{g}_{\text{reg}}$ , resp.  $\mathfrak{g}_{\text{rs}}$ , in  $\tilde{\mathfrak{g}}$ , and by  $\pi_{\text{reg}}: \tilde{\mathfrak{g}}_{\text{reg}} \rightarrow \mathfrak{g}_{\text{reg}}$ , resp.  $\pi_{\text{rs}}: \tilde{\mathfrak{g}}_{\text{rs}} \rightarrow \mathfrak{g}_{\text{rs}}$ , the restriction of  $\pi$ .

We will also use the morphism  $\nu: \tilde{\mathfrak{g}} \rightarrow \mathfrak{t}$  defined as follows: for  $g \in \mathbf{G}$  and  $x \in \mathcal{L}ie(g\mathbf{B}g^{-1})$ ,  $\nu(x, g\mathbf{B}g^{-1})$  is the inverse image under the isomorphism  $\mathfrak{t} \hookrightarrow \mathfrak{b} \rightarrow \mathfrak{b}/\mathfrak{n}$  of the image of  $g^{-1} \cdot x$  in  $\mathfrak{b}/\mathfrak{n}$ . (One can easily check that this morphism is well defined, and is a smooth morphism which satisfies  $\tilde{\mathfrak{g}}_{\text{rs}} = \nu^{-1}(\mathfrak{t}_{\text{rs}})$ , see e.g. [J3, §13.3].) We denote by  $\nu_{\text{reg}}: \tilde{\mathfrak{g}}_{\text{reg}} \rightarrow \mathfrak{t}$  the restriction of  $\nu$  to  $\tilde{\mathfrak{g}}_{\text{reg}}$ . Combining  $\pi$  and  $\nu$  we obtain a morphism

$$\varphi: \tilde{\mathfrak{g}} \rightarrow \mathfrak{g} \times_{\mathfrak{t}/W} \mathfrak{t}.$$

We denote by  $\varphi_{\text{reg}}: \tilde{\mathfrak{g}}_{\text{reg}} \rightarrow \mathfrak{g}_{\text{reg}} \times_{\mathfrak{t}/W} \mathfrak{t}$ , resp.  $\varphi_{\text{rs}}: \tilde{\mathfrak{g}}_{\text{rs}} \rightarrow \mathfrak{g}_{\text{rs}} \times_{\mathfrak{t}_{\text{rs}}/W} \mathfrak{t}_{\text{rs}}$ , the restriction of  $\varphi$  to  $\tilde{\mathfrak{g}}_{\text{reg}}$ , resp.  $\tilde{\mathfrak{g}}_{\text{rs}}$ .

We will consider the schemes

$$\tilde{\mathcal{S}} := \mathcal{S} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}, \quad \tilde{\Upsilon} := \Upsilon \times_{\mathfrak{g}} \tilde{\mathfrak{g}}.$$

These schemes are also endowed with an action of  $\mathbb{G}_m$ , obtained by restricting the action on  $\tilde{\mathfrak{g}}$  defined by

$$t \cdot (x, \mathbf{B}') := (t^{-2}\tilde{\lambda}_o(t) \cdot x, \tilde{\lambda}_o(t)\mathbf{B}'\tilde{\lambda}_o(t)^{-1})$$

for  $t \in \mathbb{G}_m$  and  $(x, \mathbf{B}') \in \tilde{\mathfrak{g}}$ . The natural morphisms  $\tilde{\mathcal{S}} \rightarrow \mathcal{S}$  and  $\tilde{\Upsilon} \rightarrow \Upsilon$  are  $\mathbb{G}_m$ -equivariant. Note that the set-theoretic fiber of  $\pi$  over  $e$  is reduced to  $(e, \mathbf{B}^+)$  by [Sp, Lemma 5.3]. Hence this  $\mathbb{G}_m$ -action on  $\tilde{\Upsilon}$  is contracting to  $(e, \mathbf{B}^+)$ .

One can also consider the universal stabilizer  $\tilde{\mathbf{I}}$  over  $\tilde{\mathfrak{g}}$ , defined as the fiber product

$$\tilde{\mathbf{I}} := \tilde{\mathfrak{g}} \times_{\tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}}} (\mathbf{G} \times \tilde{\mathfrak{g}}).$$

If  $(x, \mathbf{B}')$  is a point in  $\tilde{\mathfrak{g}}$ , then the fiber of  $\tilde{\mathbf{I}}$  over  $(x, \mathbf{B}')$  identifies with  $(\mathbf{B}')_x$ . We will denote by  $\tilde{\mathbf{I}}_{\text{reg}}$  the restriction of  $\tilde{\mathbf{I}}$  to  $\tilde{\mathfrak{g}}_{\text{reg}}$ , by  $\tilde{\mathbf{I}}_{\text{rs}}$  its restriction to  $\tilde{\mathfrak{g}}_{\text{rs}}$ , and by  $\tilde{\mathbf{I}}_{\mathcal{S}}$  its restriction to  $\tilde{\mathcal{S}}$ . If (C3) holds, then it follows from Corollary 3.3.8 that these group schemes are commutative.

**Lemma 3.5.1.** *Assume that (C3) holds. Then the morphism*

$$\tilde{a}: \mathbf{G} \times \tilde{\mathcal{S}} \rightarrow \tilde{\mathfrak{g}}_{\text{reg}}$$

*induced by the  $\mathbf{G}$ -action on  $\tilde{\mathfrak{g}}$  is smooth and surjective.*

*Proof.* By  $\mathbf{G}$ -equivariance the following diagram is Cartesian:

$$\begin{array}{ccc} \mathbf{G} \times \tilde{\mathcal{S}} & \xrightarrow{\tilde{a}} & \tilde{\mathfrak{g}}_{\text{reg}} \\ \downarrow & & \downarrow \pi_{\text{reg}} \\ \mathbf{G} \times \mathcal{S} & \xrightarrow{a} & \mathfrak{g}_{\text{reg}}, \end{array}$$

where the left vertical map is the product of  $\text{id}_{\mathbf{G}}$  with the morphism induced by  $\pi$ . Hence smoothness and surjectivity of  $\tilde{a}$  follow from the same properties for  $a$ , which we proved in Lemma 3.3.1.  $\square$

*Remark 3.5.2.* Using similar arguments one can prove that the morphism  $\mathbf{U} \times \tilde{\mathcal{S}} \rightarrow \tilde{\Upsilon}$  induced by the  $\mathbf{U}$ -action on  $\tilde{\mathfrak{g}}$  is an isomorphism, assuming that (C2) holds.

**Lemma 3.5.3.** *Assume that (C3) holds. Then the morphism*

$$\varphi_{\text{reg}}: \tilde{\mathfrak{g}}_{\text{reg}} \rightarrow \mathfrak{g}_{\text{reg}} \times_{\mathfrak{t}/W} \mathfrak{t}$$

*is an isomorphism.*

*Proof.* Following the arguments in [G2, Remark 4.2.4(i)], we only have to prove the following properties:

- (1) the morphism  $\varphi_{\text{reg}}$  is finite;
- (2) the scheme  $\mathfrak{g}_{\text{reg}} \times_{\mathfrak{t}/W} \mathfrak{t}$  is a smooth variety;
- (3)  $\varphi_{\text{rs}}$  is an isomorphism.

Indeed, then the claim follows from the general result that if  $f: X \rightarrow Y$  is a finite, birational morphism of integral Noetherian schemes such that  $Y$  is normal, then  $f$  is an isomorphism.

First, we claim that the projection  $\pi_{\text{reg}}: \tilde{\mathfrak{g}}_{\text{reg}} \rightarrow \mathfrak{g}_{\text{reg}}$  is quasi-finite. In fact, since the locus where  $\pi_{\text{reg}}$  is quasi-finite is open (see [SP, Tag 01TI]), it is enough to prove that this morphism is quasi-finite at all closed points of  $\tilde{\mathfrak{g}}_{\text{reg}}$ . Since any closed point in  $\tilde{\mathfrak{g}}_{\text{reg}}$  is  $\mathbf{G}$ -conjugate to a closed point in  $\tilde{\mathcal{S}}$  (see Lemma 3.5.1), it is enough to prove

that  $\pi_{\text{reg}}$  is quasi-finite at all closed point of  $\tilde{\mathcal{S}}$ . Using the contracting  $\mathbb{G}_m$ -action on  $\tilde{\mathcal{S}}$  (and again the fact that the quasi-finite locus is open), it is enough to prove that  $\pi_{\text{reg}}$  is quasi-finite at  $(e, \mathbf{B}^+)$ . This property follows from [SP, Tag 02NG] and the fact that the set-theoretic fiber of  $\pi_{\text{reg}}$  over  $e$  is reduced to  $(e, \mathbf{B}^+)$ .

Since  $\pi_{\text{reg}}$  is quasi-finite, a fortiori  $\varphi_{\text{reg}}$  is quasi-finite, see [SP, Tag 03WR]. Since this morphism is also projective, it is finite, see [SP, Tags 02NH and 02LS], which proves (1).

Property (2) is a consequence of Proposition 3.3.3.

Finally we turn to (3). We remark that  $\pi_{\text{rs}}$  is étale by [J3, Lemma 13.4]. On the other hand, the quotient morphism  $\mathfrak{t}_{\text{rs}} \rightarrow \mathfrak{t}_{\text{rs}}/W$  is étale by Lemma 2.4.3(1) and [J3, Remark 12.8], hence the morphism  $\mathfrak{g}_{\text{rs}} \times_{\mathfrak{t}_{\text{rs}}/W} \mathfrak{t}_{\text{rs}} \rightarrow \mathfrak{g}_{\text{rs}}$  is also étale. We deduce that  $\varphi_{\text{rs}}$  is étale. Moreover, comparing the fibers of  $\pi_{\text{rs}}$  (see [J3, Lemma 13.3]) and of the projection  $\mathfrak{g}_{\text{rs}} \times_{\mathfrak{t}_{\text{rs}}/W} \mathfrak{t}_{\text{rs}} \rightarrow \mathfrak{g}_{\text{rs}}$  (see Lemma 2.4.3(1)) over closed points, we obtain that  $\varphi_{\text{rs}}$  induces a bijection at the level of closed points. Using [SP, Tag 04DH], we deduce that  $\varphi_{\text{rs}}$  is an isomorphism.  $\square$

**Proposition 3.5.4.** *Assume that (C3) holds.*

(1) *The morphisms*

$$\tilde{\mathcal{S}} \rightarrow \mathcal{S} \times_{\mathfrak{t}/W} \mathfrak{t}, \quad \text{and} \quad \tilde{\Upsilon} \rightarrow \Upsilon \times_{\mathfrak{t}/W} \mathfrak{t}$$

*induced by  $\varphi$  are isomorphisms. In particular,  $\tilde{\mathcal{S}}$  and  $\tilde{\Upsilon}$  are affine schemes.*

(2) *The morphism  $\nu_{\text{reg}}$  restricts to an isomorphism  $\tilde{\mathcal{S}} \xrightarrow{\sim} \mathfrak{t}$ .*

*Proof.* (1) follows from (3.1.1) and Lemma 3.5.3. Then (2) follows from (1) and Theorem 3.2.2.  $\square$

**Proposition 3.5.5.** *Assume that (C3) holds.*

(1) *The group scheme  $\tilde{\mathbf{I}}_{\mathcal{S}}$  is smooth over  $\tilde{\mathcal{S}}$ .*

(2) *The natural commutative diagram*

$$\begin{array}{ccc} \tilde{\mathbf{I}}_{\mathcal{S}} & \longrightarrow & \mathbf{I}_{\mathcal{S}} \\ \downarrow & & \downarrow \\ \tilde{\mathcal{S}} & \longrightarrow & \mathcal{S} \end{array}$$

*is Cartesian.*

*Proof.* The proof of (1) is similar to the proof of Proposition 3.3.5, namely we use isomorphisms

$$\tilde{\mathbf{I}}_{\mathcal{S}} = \tilde{\mathcal{S}} \times_{\tilde{\mathfrak{g}} \times \tilde{\mathcal{S}}} (\mathbf{G} \times \tilde{\mathcal{S}}) \cong \tilde{\mathcal{S}} \times_{\tilde{\mathfrak{g}} \times \mathfrak{t}} (\mathbf{G} \times \tilde{\mathcal{S}}) \cong \tilde{\mathcal{S}} \times_{\tilde{\mathfrak{g}}} (\mathbf{G} \times \tilde{\mathcal{S}})$$

(where the last isomorphism uses Proposition 3.5.4(2)), and then the result follows from Lemma 3.5.1.

To prove (2) we use the isomorphisms

$$\mathbf{I}_{\mathcal{S}} \cong \mathcal{S} \times_{\mathfrak{g}_{\text{reg}}} (\mathbf{G} \times \mathcal{S}) \quad \text{and} \quad \tilde{\mathbf{I}}_{\mathcal{S}} \cong \tilde{\mathcal{S}} \times_{\tilde{\mathfrak{g}}_{\text{reg}}} (\mathbf{G} \times \tilde{\mathcal{S}})$$

constructed in the proof of Proposition 3.3.5 and of (1), respectively. By definition there exists a natural closed embedding  $\tilde{\mathbf{I}}_{\mathcal{S}} \rightarrow \tilde{\mathcal{S}} \times_{\mathcal{S}} \mathbf{I}_{\mathcal{S}}$ , which we interpret as a morphism

$$(3.5.1) \quad \tilde{\mathcal{S}} \times_{\tilde{\mathfrak{g}}_{\text{reg}}} (\mathbf{G} \times \tilde{\mathcal{S}}) \rightarrow \tilde{\mathcal{S}} \times_{\mathcal{S}} (\mathcal{S} \times_{\mathfrak{g}_{\text{reg}}} (\mathbf{G} \times \mathcal{S})).$$

Using the isomorphisms  $\tilde{\mathcal{S}} \xrightarrow{\sim} \mathfrak{t}$  and  $\mathcal{S} \xrightarrow{\sim} \mathfrak{t}/W$  induced by  $\nu$  and  $\chi$  respectively, the right-hand side in (3.5.1) is isomorphic to  $\mathfrak{t} \times_{\mathfrak{t}/W} (\mathcal{S} \times_{\mathfrak{g}_{\text{reg}}} (\mathbf{G} \times \mathcal{S}))$ . Then, using the isomorphisms  $\tilde{\mathcal{S}} \xrightarrow{\sim} \mathfrak{t}$  (again induced by  $\nu$ ) and  $\varphi_{\text{reg}}: \tilde{\mathfrak{g}}_{\text{reg}} \xrightarrow{\sim} \mathfrak{g}_{\text{reg}} \times_{\mathfrak{t}/W} \mathfrak{t}$  (see Lemma 3.5.3), the morphism (3.5.1) identifies with the natural morphism

$$(\mathfrak{t} \times_{\mathfrak{t}/W} \mathcal{S}) \times_{\mathfrak{t} \times_{\mathfrak{t}/W} \mathfrak{g}_{\text{reg}}} (\mathfrak{t} \times_{\mathfrak{t}/W} (\mathbf{G} \times \mathcal{S})) \rightarrow \mathfrak{t} \times_{\mathfrak{t}/W} (\mathcal{S} \times_{\mathfrak{g}_{\text{reg}}} (\mathbf{G} \times \mathcal{S})),$$

which is clearly an isomorphism.  $\square$

**Corollary 3.5.6.** *Assume that (C3) holds. Then the group scheme  $\tilde{\mathbf{I}}_{\text{reg}}$  is smooth over  $\tilde{\mathfrak{g}}_{\text{reg}}$ , and the natural commutative diagram*

$$\begin{array}{ccc} \tilde{\mathbf{I}}_{\text{reg}} & \longrightarrow & \mathbf{I}_{\text{reg}} \\ \downarrow & & \downarrow \\ \tilde{\mathfrak{g}}_{\text{reg}} & \xrightarrow{\pi_{\text{reg}}} & \mathfrak{g}_{\text{reg}} \end{array}$$

is Cartesian.

*Proof.* The smoothness of  $\tilde{\mathbf{I}}_{\text{reg}}$  can be proved by the same arguments as for Corollary 3.3.6 (or can be deduced from the second claim). To prove that the diagram is Cartesian it suffices (by descent) to do so after pullback under the smooth and surjective morphism  $\tilde{a}$  (see Lemma 3.5.1). Then the claim follows from Proposition 3.5.5(2) (see the Cartesian diagram in the proof of Corollary 3.3.6).  $\square$

*Remark 3.5.7.* Assume that (C3) holds. Then the second claim of Corollary 3.5.6 implies that for  $(x, \mathbf{B}') \in \tilde{\mathfrak{g}}_{\text{reg}}$  the inclusion  $(\mathbf{B}')_x \hookrightarrow \mathbf{G}_x$  is an isomorphism.

From Proposition 3.3.9 and Corollary 3.5.6 we deduce that, if (C3) holds, there exists a smooth, affine and commutative group scheme  $\tilde{\mathbf{J}}$  on  $\mathfrak{t}$  whose pullback under  $\nu_{\text{reg}}: \tilde{\mathfrak{g}}_{\text{reg}} \rightarrow \mathfrak{t}$  is  $\tilde{\mathbf{I}}_{\text{reg}}$ . In fact we have Cartesian diagrams

$$(3.5.2) \quad \begin{array}{ccc} \tilde{\mathbf{J}} & \longrightarrow & \mathbf{J} \\ \downarrow & & \downarrow \\ \mathfrak{t} & \longrightarrow & \mathfrak{t}/W \end{array} \quad \text{and} \quad \begin{array}{ccc} \tilde{\mathbf{I}}_{\mathcal{S}} & \xrightarrow{\sim} & \tilde{\mathbf{J}} \\ \downarrow & & \downarrow \\ \tilde{\mathcal{S}} & \xrightarrow{\sim} & \mathfrak{t}. \end{array}$$

*Remark 3.5.8.* Assume that (C3) holds. Then, using the same arguments as for Proposition 3.3.11, one can check that there exists a canonical equivalence of categories  $\text{Coh}^{\mathbf{G}}(\tilde{\mathfrak{g}}_{\text{reg}}) \cong \text{Rep}(\tilde{\mathbf{J}})$ .

We set

$$\begin{aligned} \tilde{\mathfrak{J}} &:= \text{Lie}(\tilde{\mathbf{I}}/\tilde{\mathfrak{g}}), & \tilde{\mathfrak{J}}_{\text{reg}} &:= \text{Lie}(\tilde{\mathbf{I}}_{\text{reg}}/\tilde{\mathfrak{g}}_{\text{reg}}), & \tilde{\mathfrak{J}}_{\text{rs}} &:= \text{Lie}(\tilde{\mathbf{I}}_{\text{rs}}/\tilde{\mathfrak{g}}_{\text{rs}}), \\ \tilde{\mathfrak{J}}_{\mathcal{S}} &:= \text{Lie}(\tilde{\mathbf{I}}_{\mathcal{S}}/\tilde{\mathcal{S}}), & \tilde{\mathfrak{J}} &:= \text{Lie}(\tilde{\mathbf{J}}/\mathfrak{t}) \end{aligned}$$

and

$$\begin{aligned} \tilde{\mathbb{I}} &:= \text{Lie}(\tilde{\mathbf{I}}/\tilde{\mathfrak{g}}), & \tilde{\mathbb{I}}_{\text{reg}} &:= \text{Lie}(\tilde{\mathbf{I}}_{\text{reg}}/\tilde{\mathfrak{g}}_{\text{reg}}), & \tilde{\mathbb{I}}_{\text{rs}} &:= \text{Lie}(\tilde{\mathbf{I}}_{\text{rs}}/\tilde{\mathfrak{g}}_{\text{rs}}), \\ \tilde{\mathbb{I}}_{\mathcal{S}} &:= \text{Lie}(\tilde{\mathbf{I}}_{\mathcal{S}}/\tilde{\mathcal{S}}), & \tilde{\mathbb{J}} &:= \text{Lie}(\tilde{\mathbf{J}}/\mathfrak{t}). \end{aligned}$$

The following proposition is analogous to Proposition 3.4.1, and its proof is similar (hence left to the reader).

**Proposition 3.5.9.** *Assume that (C3) holds. Then:*

- (1) the coherent sheaf  $\tilde{\mathfrak{I}}_{\text{reg}}$ , resp.  $\tilde{\mathfrak{I}}_{\text{rs}}$ , resp.  $\tilde{\mathfrak{I}}_{\mathcal{S}}$ , resp.  $\tilde{\mathfrak{J}}$  on  $\tilde{\mathfrak{g}}_{\text{reg}}$ , resp.  $\tilde{\mathfrak{g}}_{\text{rs}}$ , resp.  $\tilde{\mathcal{S}}$ , resp.  $\mathfrak{t}$ , is locally free of finite rank;
- (2) the Lie algebra  $\tilde{\mathbb{I}}_{\text{reg}}$ , resp.  $\tilde{\mathbb{I}}_{\text{rs}}$ , resp.  $\tilde{\mathbb{I}}_{\mathcal{S}}$ , resp.  $\tilde{\mathbb{J}}$ , is a vector bundle over  $\tilde{\mathfrak{g}}_{\text{reg}}$ , resp.  $\tilde{\mathfrak{g}}_{\text{rs}}$ , resp.  $\tilde{\mathcal{S}}$ , resp.  $\mathfrak{t}$ ;
- (3) the Lie algebras  $\tilde{\mathfrak{I}}_{\text{reg}}$ ,  $\tilde{\mathfrak{I}}_{\text{rs}}$ ,  $\tilde{\mathfrak{I}}_{\mathcal{S}}$  and  $\tilde{\mathfrak{J}}$  are commutative;
- (4) the restriction of  $\tilde{\mathfrak{I}}_{\text{reg}}$  to  $\tilde{\mathcal{S}}$  is canonically isomorphic to  $\tilde{\mathfrak{I}}_{\mathcal{S}}$ ; in other words there exists a canonical Cartesian diagram

$$\begin{array}{ccc} \tilde{\mathbb{I}}_{\mathcal{S}} & \longrightarrow & \tilde{\mathbb{I}}_{\text{reg}} \\ \downarrow & & \downarrow \\ \tilde{\mathcal{S}} & \longrightarrow & \tilde{\mathfrak{g}}_{\text{reg}}; \end{array}$$

- (5) there exists a canonical isomorphism  $\nu_{\text{reg}}^*(\tilde{\mathfrak{J}}) \cong \tilde{\mathfrak{I}}_{\text{reg}}$ ; in other words there exists a canonical Cartesian diagram

$$\begin{array}{ccc} \tilde{\mathbb{I}}_{\text{reg}} & \longrightarrow & \tilde{\mathbb{J}} \\ \downarrow & & \downarrow \\ \tilde{\mathfrak{g}}_{\text{reg}} & \xrightarrow{\nu_{\text{reg}}} & \mathfrak{t}; \end{array}$$

- (6) the coherent sheaf  $\tilde{\mathfrak{I}}_{\mathcal{S}}$  is canonically isomorphic to the pullback of  $\tilde{\mathfrak{J}}$  under the isomorphism  $\tilde{\mathcal{S}} \xrightarrow{\sim} \mathfrak{t}$ ; in other words  $\tilde{\mathbb{I}}_{\mathcal{S}}$  is canonically isomorphic to the pullback of  $\tilde{\mathbb{J}}$  under the isomorphism  $\tilde{\mathcal{S}} \xrightarrow{\sim} \mathfrak{t}$ .  $\square$

**Lemma 3.5.10.** *Assume that (C3) holds. Then there exist canonical Cartesian diagrams*

$$\begin{array}{ccc} \tilde{\mathbb{I}}_{\mathcal{S}} & \longrightarrow & \mathbb{I}_{\mathcal{S}} \\ \downarrow & & \downarrow \\ \tilde{\mathcal{S}} & \longrightarrow & \mathcal{S}, \end{array} \quad \begin{array}{ccc} \tilde{\mathbb{I}}_{\text{reg}} & \longrightarrow & \mathbb{I}_{\text{reg}} \\ \downarrow & & \downarrow \\ \tilde{\mathfrak{g}}_{\text{reg}} & \longrightarrow & \mathfrak{g}_{\text{reg}} \end{array} \quad \text{and} \quad \begin{array}{ccc} \tilde{\mathbb{J}} & \longrightarrow & \mathbb{J} \\ \downarrow & & \downarrow \\ \mathfrak{t} & \longrightarrow & \mathfrak{t}/W. \end{array}$$

*Proof.* This follows from Lemma 2.2.1(2), Corollary 3.3.6, Proposition 3.5.5, Corollary 3.5.6, and the left diagram in (3.5.2).  $\square$

Combining Theorem 3.4.2 and Lemma 3.5.10 we obtain the following.

**Theorem 3.5.11.** *Assume that (C4) holds. Then there exist canonical isomorphisms*

$$\tilde{\mathfrak{J}} \cong \mathfrak{t} \times_{\mathfrak{t}/W} \mathbb{T}^*(\mathfrak{t}/W), \quad \tilde{\mathbb{I}}_{\text{reg}} \cong \tilde{\mathfrak{g}}_{\text{reg}} \times_{\mathfrak{t}/W} \mathbb{T}^*(\mathfrak{t}/W), \quad \tilde{\mathbb{I}}_{\mathcal{S}} \cong \tilde{\mathcal{S}} \times_{\mathcal{S}} \mathbb{T}^*(\mathcal{S})$$

of commutative Lie algebras over  $\mathfrak{t}$ ,  $\tilde{\mathfrak{g}}_{\text{reg}}$  and  $\tilde{\mathcal{S}}$ , respectively.  $\square$

**3.6. Comparison with the regular semisimple locus.** In this subsection we assume that (C4) holds.

In Theorem 3.5.11 we have obtained a description of  $\tilde{\mathbb{I}}_{\text{reg}}$ . Restricting to  $\tilde{\mathfrak{g}}_{\text{rs}}$ , we deduce a canonical isomorphism

$$(3.6.1) \quad \tilde{\mathbb{I}}_{\text{rs}} \cong \tilde{\mathfrak{g}}_{\text{rs}} \times_{\mathfrak{t}_{\text{rs}}/W} \mathbb{T}^*(\mathfrak{t}_{\text{rs}}/W).$$

On the other hand one can obtain a simpler description of  $\tilde{\mathbb{I}}_{\mathfrak{r}_s}$  (which only requires (C1) to hold) as follows. By [J3, Lemma 13.4], there exists a canonical isomorphism

$$\mathbf{G}/\mathbf{T} \times \mathfrak{t}_{\mathfrak{r}_s} \xrightarrow{\sim} \tilde{\mathfrak{g}}_{\mathfrak{r}_s},$$

which sends  $(g\mathbf{T}, x)$  to  $(g \cdot x, g\mathbf{B}g^{-1})$ . The universal stabilizer for the  $\mathbf{G}$ -action on  $\mathbf{G}/\mathbf{T} \times \mathfrak{t}_{\mathfrak{r}_s}$  is  $\mathbf{G}/\mathbf{T} \times \mathfrak{t}_{\mathfrak{r}_s} \times \mathbf{T}$ , considered as a closed subgroup of  $\mathbf{G}/\mathbf{T} \times \mathfrak{t}_{\mathfrak{r}_s} \times \mathbf{G}$  via the morphism  $(g\mathbf{T}, x, t) \mapsto (g\mathbf{T}, x, gtg^{-1})$ . The Lie algebra of  $\mathbf{G}/\mathbf{T} \times \mathfrak{t}_{\mathfrak{r}_s} \times \mathbf{T}$  is  $\mathbf{G}/\mathbf{T} \times \mathfrak{t}_{\mathfrak{r}_s} \times \mathfrak{t}$ . Hence we obtain a commutative diagram

$$\begin{array}{ccc} \mathbf{G}/\mathbf{T} \times \mathfrak{t}_{\mathfrak{r}_s} \times \mathfrak{t} & \xrightarrow{\sim} & \tilde{\mathbb{I}}_{\mathfrak{r}_s} \\ \downarrow & & \downarrow \\ \mathbf{G}/\mathbf{T} \times \mathfrak{t}_{\mathfrak{r}_s} & \xrightarrow{\sim} & \tilde{\mathfrak{g}}_{\mathfrak{r}_s}. \end{array}$$

Comparing with (3.6.1), we obtain an isomorphism

$$(3.6.2) \quad \mathbf{G}/\mathbf{T} \times \mathfrak{t}_{\mathfrak{r}_s} \times \mathfrak{t} \xrightarrow{\sim} (\mathbf{G}/\mathbf{T} \times \mathfrak{t}_{\mathfrak{r}_s}) \times_{\mathfrak{t}_{\mathfrak{r}_s}/W} \mathbb{T}^*(\mathfrak{t}_{\mathfrak{r}_s}/W)$$

of commutative Lie algebras over  $\mathbf{G}/\mathbf{T} \times \mathfrak{t}_{\mathfrak{r}_s}$ .

To describe this isomorphism more explicitly, recall that we have fixed a  $\mathbf{G}$ -equivariant isomorphism  $\kappa: \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$ , and that  $\kappa$  induces an isomorphism  $\mathfrak{t} \xrightarrow{\sim} \mathfrak{t}^*$ . This allows to identify  $\mathfrak{t}_{\mathfrak{r}_s} \times \mathfrak{t}$  with the cotangent bundle  $\mathbb{T}^*(\mathfrak{t}_{\mathfrak{r}_s})$ . The quotient morphism  $\mathfrak{t}_{\mathfrak{r}_s} \rightarrow \mathfrak{t}_{\mathfrak{r}_s}/W$  is étale (see the proof of Lemma 3.5.3); therefore it induces an isomorphism  $\mathbb{T}(\mathfrak{t}_{\mathfrak{r}_s}) \xrightarrow{\sim} \mathfrak{t}_{\mathfrak{r}_s} \times_{\mathfrak{t}_{\mathfrak{r}_s}/W} \mathbb{T}(\mathfrak{t}_{\mathfrak{r}_s}/W)$  between tangent bundles, hence an isomorphism  $\mathbb{T}^*(\mathfrak{t}_{\mathfrak{r}_s}) \xrightarrow{\sim} \mathfrak{t}_{\mathfrak{r}_s} \times_{\mathfrak{t}_{\mathfrak{r}_s}/W} \mathbb{T}^*(\mathfrak{t}_{\mathfrak{r}_s}/W)$ . Combining these remarks we finally obtain an isomorphism of vector bundles over  $\mathfrak{t}_{\mathfrak{r}_s}$

$$\mathfrak{t}_{\mathfrak{r}_s} \times \mathfrak{t} \xrightarrow{\sim} \mathfrak{t}_{\mathfrak{r}_s} \times_{\mathfrak{t}_{\mathfrak{r}_s}/W} \mathbb{T}^*(\mathfrak{t}_{\mathfrak{r}_s}/W).$$

We claim that the pullback of this isomorphism under the projection  $\mathbf{G}/\mathbf{T} \times \mathfrak{t}_{\mathfrak{r}_s} \rightarrow \mathfrak{t}_{\mathfrak{r}_s}$  coincides with (3.6.2). Indeed, this claim amounts to the following observation. Both of the compositions

$$\mathfrak{t}_{\mathfrak{r}_s} \xrightarrow{x \mapsto (x, \mathbf{B})} \tilde{\mathfrak{g}}_{\mathfrak{r}_s} \xrightarrow{\nu} \mathfrak{t}_{\mathfrak{r}_s}$$

and

$$\tilde{\mathcal{S}}_{\mathfrak{r}_s} \hookrightarrow \tilde{\mathfrak{g}}_{\mathfrak{r}_s} \xrightarrow{\nu} \mathfrak{t}_{\mathfrak{r}_s}$$

(where  $\tilde{\mathcal{S}}_{\mathfrak{r}_s} := \tilde{\mathcal{S}} \cap \tilde{\mathfrak{g}}_{\mathfrak{r}_s}$ ) are isomorphisms, which provides a canonical isomorphism  $\sigma: \mathfrak{t}_{\mathfrak{r}_s} \xrightarrow{\sim} \tilde{\mathcal{S}}_{\mathfrak{r}_s}$ . Now if  $(y, g\mathbf{B}) = \sigma(x)$ , we have to check that the diagram

$$\begin{array}{ccc} \mathfrak{t} = \mathfrak{g}_x & \xrightarrow{\sim} & \mathfrak{g}_y \\ \downarrow \wr & & \downarrow \wr \\ \mathfrak{t}^* = \mathbb{T}_x^*(\mathfrak{t}_{\mathfrak{r}_s}) & \xrightarrow[\sim]{d_x \sigma} & \mathbb{T}_{(y, g\mathbf{B})}^*(\tilde{\mathcal{S}}_{\mathfrak{r}_s}) \end{array}$$

is commutative. (Here the upper horizontal arrow is the canonical isomorphism, induced by any  $h \in \mathbf{G}$  such that  $y = h \cdot x$ , and the vertical arrows are the isomorphisms constructed above.) Using the fact that the morphisms  $\tilde{\mathcal{S}}_{\mathfrak{r}_s} \rightarrow \mathcal{S}_{\mathfrak{r}_s}$  and

$\mathfrak{t}_{rs} \rightarrow \mathfrak{t}_{rs}/W$  are étale and Remark 3.3.4, one can rewrite this diagram as

$$\begin{array}{ccc} \mathfrak{g}_x & \xrightarrow{\sim} & \mathfrak{g}_y \\ \downarrow \wr & & \downarrow \wr \\ (\mathfrak{g}/[x, \mathfrak{g}])^* & \xrightarrow{\sim} & (\mathfrak{g}/[y, \mathfrak{g}])^*, \end{array}$$

and the commutativity follows from the  $\mathbf{G}$ -equivariance of  $\kappa$ .

#### 4. THE CASE OF $\mathfrak{R}$

From now on, for simplicity we restrict to the case of semisimple, simply connected groups.

**4.1. Definitions and preliminary results.** Let  $\mathbf{G}_{\mathbb{Z}}$  be a split connected, simply-connected, semi-simple algebraic group over  $\mathbb{Z}$ . We let  $\mathbf{B}_{\mathbb{Z}} \subset \mathbf{G}_{\mathbb{Z}}$  be a Borel subgroup, and  $\mathbf{T}_{\mathbb{Z}} \subset \mathbf{B}_{\mathbb{Z}}$  be a (split) maximal torus. We denote by  $\mathfrak{g}_{\mathbb{Z}}$ ,  $\mathfrak{b}_{\mathbb{Z}}$ ,  $\mathfrak{t}_{\mathbb{Z}}$  the Lie algebras of  $\mathbf{G}_{\mathbb{Z}}$ ,  $\mathbf{B}_{\mathbb{Z}}$ ,  $\mathbf{T}_{\mathbb{Z}}$ .

We let  $N$  be the product of all the prime numbers which are not very good for  $\mathbf{G}_{\mathbb{Z}}$ , and set  $\mathfrak{R} := \mathbb{Z}[1/N]$ . We let  $\mathbf{G}_{\mathfrak{R}}$ ,  $\mathbf{B}_{\mathfrak{R}}$ ,  $\mathbf{T}_{\mathfrak{R}}$  be the groups obtained from  $\mathbf{G}_{\mathbb{Z}}$ ,  $\mathbf{B}_{\mathbb{Z}}$ ,  $\mathbf{T}_{\mathbb{Z}}$  by base change to  $\mathfrak{R}$ , and  $\mathfrak{g}_{\mathfrak{R}}$ ,  $\mathfrak{b}_{\mathfrak{R}}$ ,  $\mathfrak{t}_{\mathfrak{R}}$  be their respective Lie algebras. We have  $\mathfrak{g}_{\mathfrak{R}} = \mathfrak{R} \otimes_{\mathbb{Z}} \mathfrak{g}_{\mathbb{Z}}$ . We also denote by  $\mathbf{U}_{\mathfrak{R}}$  the unipotent radical of  $\mathbf{B}_{\mathfrak{R}}$ , by  $\mathfrak{n}_{\mathfrak{R}}$  its Lie algebras, and by  $W$  the Weyl group of  $(\mathbf{G}_{\mathfrak{R}}, \mathbf{T}_{\mathfrak{R}})$ .

We denote by  $\Phi \subset X^*(\mathbf{T}_{\mathfrak{R}})$  the root system of  $\mathbf{G}_{\mathfrak{R}}$ , by  $\Phi^+ \subset \Phi$  the set of roots which are opposite to the  $\mathbf{T}_{\mathfrak{R}}$ -weights in  $\mathfrak{n}_{\mathfrak{R}}$ , by  $\Delta \subset \Phi^+$  the corresponding basis, by  $\check{\Phi} \subset X_*(\mathbf{T}_{\mathfrak{R}})$  the coroots, and by  $\check{\Phi}^+ \subset \check{\Phi}$  the coroots corresponding to  $\Phi^+$ .

For any geometric point  $\mathbb{F}$  of  $\mathfrak{R}$  we set

$$\mathbf{G}_{\mathbb{F}} := \mathrm{Spec}(\mathbb{F}) \times_{\mathrm{Spec}(\mathfrak{R})} \mathbf{G}_{\mathfrak{R}}.$$

We also denote by  $\mathbf{B}_{\mathbb{F}}$ ,  $\mathbf{T}_{\mathbb{F}}$  the base change of  $\mathbf{B}_{\mathfrak{R}}$ ,  $\mathbf{T}_{\mathfrak{R}}$ , and by  $\mathfrak{t}_{\mathbb{F}} \subset \mathfrak{b}_{\mathbb{F}} \subset \mathfrak{g}_{\mathbb{F}}$  the Lie algebras of  $\mathbf{T}_{\mathbb{F}} \subset \mathbf{B}_{\mathbb{F}} \subset \mathbf{G}_{\mathbb{F}}$ .

**Proposition 4.1.1.** *The inclusion  $\mathfrak{t}_{\mathfrak{R}} \hookrightarrow \mathfrak{g}_{\mathfrak{R}}$  induces an isomorphism of  $\mathfrak{R}$ -schemes*

$$\mathfrak{t}_{\mathfrak{R}}/W \xrightarrow{\sim} \mathfrak{g}_{\mathfrak{R}}/\mathbf{G}_{\mathfrak{R}}.$$

*Moreover, these  $\mathfrak{R}$ -schemes are smooth (in fact they are affine spaces).*

*Proof.* Consider the first claim. Under our assumptions, the group  $\mathbf{G}_{\mathfrak{R}}$  is a product of quasi-simple groups. Hence it is enough to prove the claim in the case  $\mathbf{G}_{\mathfrak{R}}$  is quasi-simple. In this case, it follows from [CR, Theorem 1].

The second claim follows from [De, Théorème 3].  $\square$

Using Proposition 4.1.1, we will freely identify  $\mathfrak{g}_{\mathfrak{R}}/\mathbf{G}_{\mathfrak{R}}$  with  $\mathfrak{t}_{\mathfrak{R}}/W$ . Let us also note the following corollary.

**Corollary 4.1.2.** *For any geometric point  $\mathbb{F}$  of  $\mathfrak{R}$ , the natural morphism*

$$\mathfrak{g}_{\mathbb{F}}/\mathbf{G}_{\mathbb{F}} \rightarrow \mathrm{Spec}(\mathbb{F}) \times_{\mathrm{Spec}(\mathfrak{R})} (\mathfrak{g}_{\mathfrak{R}}/\mathbf{G}_{\mathfrak{R}})$$

*is an isomorphism.*

*Proof.* By Proposition 2.4.2 and Proposition 4.1.1 it is enough to prove that the natural morphism

$$\mathfrak{t}_{\mathbb{F}}/W \rightarrow \mathrm{Spec}(\mathbb{F}) \times_{\mathrm{Spec}(\mathfrak{R})} (\mathfrak{t}_{\mathfrak{R}}/W)$$

is an isomorphism. The latter property follows from [De, Corollary on p. 296].  $\square$

**Lemma 4.1.3.** *There exists a symmetric  $\mathbf{G}_{\mathfrak{R}}$ -invariant bilinear form on  $\mathfrak{g}_{\mathfrak{R}}$  which is a perfect pairing.*

*Proof.* As in the proof of Proposition 4.1.1, we can assume that  $\mathbf{G}_{\mathfrak{R}}$  is quasi-simple (and simply-connected). If  $\mathbf{G}_{\mathfrak{R}}$  is exceptional then by [SS, Corollary I.4.9] the discriminant of the Killing form of  $\mathfrak{g}_{\mathbb{Z}}$  is invertible in  $\mathfrak{R}$ , hence the Killing form of  $\mathfrak{g}_{\mathfrak{R}}$  satisfies the conditions of the lemma. If  $\mathbf{G}_{\mathfrak{R}}$  is classical then  $\mathfrak{g}_{\mathfrak{R}}$  can be realized naturally as a subalgebra of  $\mathfrak{gl}(m, \mathfrak{R})$  for some  $m$ , and one can check that the restriction of the bilinear form

$$(4.1.1) \quad \begin{cases} \mathfrak{gl}(m, \mathfrak{R}) \otimes_{\mathfrak{R}} \mathfrak{gl}(m, \mathfrak{R}) & \rightarrow & \mathfrak{R} \\ X \otimes Y & \mapsto & \mathrm{tr}(XY) \end{cases}$$

is non-degenerate, see the considerations in [SS, p. 184]. For instance, let us explain this argument in types  $B$  or  $D$ : in this case there exists a natural quotient morphism

$$\mathbf{G}_{\mathfrak{R}} \rightarrow \mathrm{SO}(m, \mathfrak{R}) = \{M \in \mathrm{GL}(m, \mathfrak{R}) \mid {}^t M \cdot M = 1\}$$

for some  $m \geq 3$ . The induced morphism on Lie algebras is an isomorphism since 2 is invertible in  $\mathfrak{R}$ , so that we can identify  $\mathfrak{g}_{\mathfrak{R}}$  with

$$\mathfrak{so}(m, \mathfrak{R}) = \{x \in \mathfrak{gl}(m, \mathfrak{R}) \mid {}^t x = -x\}.$$

Then the submodule

$$\{x \in \mathfrak{gl}(m, \mathfrak{R}) \mid {}^t x = x\}$$

is a complement to  $\mathfrak{so}(m, \mathfrak{R})$  in  $\mathfrak{gl}(m, \mathfrak{R})$  which is orthogonal to  $\mathfrak{so}(m, \mathfrak{R})$  for (4.1.1), which proves that the restriction to  $\mathfrak{so}(m, \mathfrak{R})$  is a perfect pairing.  $\square$

Lemma 4.1.3 implies in particular that for any geometric point  $\mathbb{F}$  of  $\mathfrak{R}$ , the group  $\mathbf{G}_{\mathbb{F}}$  satisfies condition (C4) of §2.3. It follows that all the results of Section 3 are applicable to this group.

From now on, we fix a bilinear form as in Lemma 4.1.3, and denote by  $\kappa: \mathfrak{g}_{\mathfrak{R}} \xrightarrow{\sim} \mathfrak{g}_{\mathfrak{R}}^*$  the induced isomorphism of  $\mathbf{G}_{\mathfrak{R}}$ -modules. (Here  $\mathfrak{g}_{\mathfrak{R}}^* := \mathrm{Hom}_{\mathfrak{R}}(\mathfrak{g}_{\mathfrak{R}}, \mathfrak{R})$ .)

**4.2. Kostant section.** We now explain how to define the Kostant section over  $\mathfrak{R}$ . For any  $\alpha \in \Delta$  we choose a vector  $e_{\alpha} \in \mathfrak{g}_{\mathbb{Z}}$  which forms a  $\mathbb{Z}$ -basis of the  $\alpha$ -weight space in  $\mathfrak{g}_{\mathbb{Z}}$  (with respect to the action of  $\mathbf{T}_{\mathbb{Z}}$ ), and set

$$e = \sum_{\alpha \in \Delta} e_{\alpha}.$$

We denote similarly the image of this vector in  $\mathfrak{g}_{\mathfrak{R}}$ .

**Lemma 4.2.1.** *The  $\mathfrak{R}$ -module  $\mathfrak{b}_{\mathfrak{R}}/[e, \mathfrak{n}_{\mathfrak{R}}]$  is free of rank  $r$ .*

*Proof.* This follows from the results of [Sp] as in the proof of Lemma 3.1.2.  $\square$

Now we consider the cocharacter  $\check{\lambda}_o := \sum_{\check{\alpha} \in \check{\Phi}^+} \check{\alpha} \in X_*(\mathbf{T}_{\mathfrak{R}})$ , and the  $(\mathbb{G}_m)_{\mathfrak{R}}$ -action on  $\mathfrak{g}_{\mathfrak{R}}$  defined by

$$z \cdot x := z^{-2\check{\lambda}_o}(z) \cdot x.$$

Then  $e$  is fixed under this action, and the subalgebras  $\mathfrak{b}_{\mathfrak{R}}$  and  $\mathfrak{n}_{\mathfrak{R}}$  are  $(\mathbb{G}_m)_{\mathfrak{R}}$ -stable. Using Lemma 4.2.1 we can choose a  $(\mathbb{G}_m)_{\mathfrak{R}}$ -stable free  $\mathfrak{R}$ -submodule  $\mathfrak{s}_{\mathfrak{R}} \subset \mathfrak{b}_{\mathfrak{R}}$  such that  $\mathfrak{b}_{\mathfrak{R}} = \mathfrak{s}_{\mathfrak{R}} \oplus [e, \mathfrak{n}_{\mathfrak{R}}]$ . Then we set

$$\mathcal{S}_{\mathfrak{R}} := e + \mathfrak{s}_{\mathfrak{R}}, \quad \Upsilon_{\mathfrak{R}} := e + \mathfrak{b}_{\mathfrak{R}}.$$

It is clear that  $\mathcal{S}_{\mathfrak{R}}$  and  $\Upsilon_{\mathfrak{R}}$  are  $(\mathbb{G}_m)_{\mathfrak{R}}$ -stable.

By construction, if  $\mathbb{F}$  is a geometric point of  $\mathfrak{X}$ , the base change of  $\Upsilon_{\mathfrak{X}}$  to  $\mathbb{F}$  is the scheme  $\Upsilon$  studied in Section 3 for the group  $\mathbf{G}_{\mathbb{F}}$ , and the base change of  $\mathcal{S}_{\mathfrak{X}}$  is the scheme  $\mathcal{S}$  studied in Section 3 for  $\mathbf{G}_{\mathbb{F}}$ , for the choice  $\mathfrak{s} = \mathbb{F} \otimes_{\mathfrak{X}} \mathfrak{s}_{\mathfrak{X}}$ .

**Proposition 4.2.2.** *The morphism*

$$\mathbf{U}_{\mathfrak{X}} \times_{\mathrm{Spec}(\mathfrak{X})} \mathcal{S}_{\mathfrak{X}} \rightarrow \Upsilon_{\mathfrak{X}}$$

*induced by the adjoint action is an isomorphism of  $\mathfrak{X}$ -schemes.*

*Proof.* We have to prove that the induced morphism  $\mathcal{O}(\Upsilon_{\mathfrak{X}}) \rightarrow \mathcal{O}(\mathbf{U}_{\mathfrak{X}} \times \mathcal{S}_{\mathfrak{X}})$  is an isomorphism of  $\mathfrak{X}$ -modules. Now, as in the proof of Proposition 3.2.1, each of these modules is endowed with a natural  $(\mathbb{G}_m)_{\mathfrak{X}}$ -action (equivalently, a  $\mathbb{Z}$ -grading), and one can easily check that the weight spaces are finitely generated free  $\mathfrak{X}$ -modules. Moreover, our morphism is  $(\mathbb{G}_m)_{\mathfrak{X}}$ -equivariant. Hence it suffices to prove that for any prime  $\ell$  not invertible in  $\mathfrak{X}$ , the induced morphism  $\mathbb{F}_{\ell} \otimes_{\mathfrak{X}} \mathcal{O}(\Upsilon_{\mathfrak{X}}) \rightarrow \mathbb{F}_{\ell} \otimes_{\mathfrak{X}} \mathcal{O}(\mathbf{U}_{\mathfrak{X}} \times \mathcal{S}_{\mathfrak{X}})$  is an isomorphism. The latter result follows from the similar claim for an algebraic closure of  $\mathbb{F}_{\ell}$ , which is a consequence of Proposition 3.2.1.  $\square$

**Theorem 4.2.3** (Kostant's theorem over  $\mathfrak{X}$ ). *The natural morphisms*

$$\mathcal{S}_{\mathfrak{X}} \rightarrow \Upsilon_{\mathfrak{X}}/\mathbf{U}_{\mathfrak{X}} \rightarrow \mathfrak{g}_{\mathfrak{X}}/\mathbf{G}_{\mathfrak{X}}$$

*are isomorphisms of  $\mathfrak{X}$ -schemes.*

*Proof.* The fact that the morphism  $\mathcal{S}_{\mathfrak{X}} \rightarrow \Upsilon_{\mathfrak{X}}/\mathbf{U}_{\mathfrak{X}}$  is an isomorphism is a consequence of Proposition 4.2.2. Hence what remains is to prove that the morphism  $\mathcal{S}_{\mathfrak{X}} \rightarrow \mathfrak{g}_{\mathfrak{X}}/\mathbf{G}_{\mathfrak{X}}$  is an isomorphism. By the same arguments as in the proof of Proposition 4.2.2 it suffices to observe that for any geometric point  $\mathbb{F}$  of  $\mathfrak{X}$ , the induced morphism

$$\mathrm{Spec}(\mathbb{F}) \times_{\mathrm{Spec}(\mathfrak{X})} \mathcal{S}_{\mathfrak{X}} \rightarrow \mathrm{Spec}(\mathbb{F}) \times_{\mathrm{Spec}(\mathfrak{X})} (\mathfrak{g}_{\mathfrak{X}}/\mathbf{G}_{\mathfrak{X}})$$

is an isomorphism (by Corollary 4.1.2 and Theorem 3.2.2 applied to  $\mathbf{G}_{\mathbb{F}}$ ).  $\square$

**Proposition 4.2.4.** *The morphism*

$$a: \mathbf{G}_{\mathfrak{X}} \times_{\mathrm{Spec}(\mathfrak{X})} \mathcal{S}_{\mathfrak{X}} \rightarrow \mathfrak{g}_{\mathfrak{X}}$$

*induced by the  $\mathbf{G}_{\mathfrak{X}}$ -action on  $\mathfrak{g}_{\mathfrak{X}}$  is smooth.*

*Proof.* This follows from Proposition 2.1.3 and Lemma 3.3.1.  $\square$

*Remark 4.2.5.* Since  $a$  is in particular a flat morphism, its image is an open subscheme in  $\mathfrak{g}_{\mathfrak{X}}$ . One can define the regular locus  $\mathfrak{g}_{\mathfrak{X}}^{\mathrm{reg}} \subset \mathfrak{g}_{\mathfrak{X}}$  as this image. Then the morphism  $a: \mathbf{G}_{\mathfrak{X}} \times \mathcal{S}_{\mathfrak{X}} \rightarrow \mathfrak{g}_{\mathfrak{X}}^{\mathrm{reg}}$  is smooth and surjective. Moreover, the same argument as in the proof of Proposition 3.3.3 shows that the restriction  $\mathfrak{g}_{\mathfrak{X}}^{\mathrm{reg}} \rightarrow \mathfrak{t}_{\mathfrak{X}}/W$  of the adjoint quotient is smooth and surjective.

**4.3. The universal centralizer and its Lie algebra.** We define the (affine) group scheme  $\mathbf{I}_{\mathfrak{X}}$  over  $\mathfrak{g}_{\mathfrak{X}}$  as the fiber product

$$\mathbf{I}_{\mathfrak{X}} := \mathfrak{g}_{\mathfrak{X}} \times_{\mathfrak{g}_{\mathfrak{X}} \times \mathfrak{g}_{\mathfrak{X}}} (\mathbf{G}_{\mathfrak{X}} \times \mathfrak{g}_{\mathfrak{X}}),$$

where the morphisms are similar to those considered in §3.3. We also denote by  $\mathbf{I}_{\mathcal{S}}^{\mathfrak{X}}$  the restriction of  $\mathbf{I}_{\mathfrak{X}}$  to  $\mathcal{S}_{\mathfrak{X}}$ . It is clear that for any geometric point  $\mathbb{F}$  of  $\mathfrak{X}$ , the base change of  $\mathbf{I}_{\mathfrak{X}}$ , resp.  $\mathbf{I}_{\mathcal{S}}^{\mathfrak{X}}$ , to  $\mathbb{F}$  is the corresponding group scheme defined and studied in §3.3 for the group  $\mathbf{G}_{\mathbb{F}}$ .

**Proposition 4.3.1.** *The group scheme  $\mathbf{I}_{\mathcal{S}}^{\mathfrak{X}}$  is smooth over  $\mathcal{S}_{\mathfrak{X}}$ , and commutative.*

*Proof.* Smoothness follows from the same arguments as in the proof of Proposition 3.3.5, using Theorem 4.2.3 and Proposition 4.2.4.

To prove commutativity, we have to prove that the comultiplication morphism  $\mathcal{O}(\mathbf{I}_S^{\mathfrak{A}}) \rightarrow \mathcal{O}(\mathbf{I}_S^{\mathfrak{A}}) \otimes_{\mathcal{O}(\mathcal{S}_{\mathfrak{A}})} \mathcal{O}(\mathbf{I}_S^{\mathfrak{A}})$  is cocommutative. However using flatness of  $\mathbf{I}_S^{\mathfrak{A}}$  over  $\mathcal{S}_{\mathfrak{A}}$  (hence over  $\mathfrak{A}$ ) we obtain that the vertical arrows in the natural commutative diagram

$$\begin{array}{ccc} \mathcal{O}(\mathbf{I}_S^{\mathfrak{A}}) & \longrightarrow & \mathcal{O}(\mathbf{I}_S^{\mathfrak{A}}) \otimes_{\mathcal{O}(\mathcal{S}_{\mathfrak{A}})} \mathcal{O}(\mathbf{I}_S^{\mathfrak{A}}) \\ \downarrow & & \downarrow \\ \mathcal{O}(\mathbf{I}_S^{\mathbb{C}}) & \longrightarrow & \mathcal{O}(\mathbf{I}_S^{\mathbb{C}}) \otimes_{\mathcal{O}(\mathcal{S}_{\mathbb{C}})} \mathcal{O}(\mathbf{I}_S^{\mathbb{C}}) \end{array}$$

are injective, where  $\mathbf{I}_S^{\mathbb{C}} := \mathrm{Spec}(\mathbb{C}) \times_{\mathrm{Spec}(\mathfrak{A})} \mathbf{I}_S^{\mathfrak{A}}$  and  $\mathcal{S}_{\mathbb{C}} := \mathrm{Spec}(\mathbb{C}) \times_{\mathrm{Spec}(\mathfrak{A})} \mathcal{S}_{\mathfrak{A}}$ . Hence the desired cocommutativity follows from the case  $\mathbb{F} = \mathbb{C}$  of Corollary 3.3.8.  $\square$

*Remark 4.3.2.* If one defines  $\mathfrak{g}_{\mathfrak{A}}^{\mathrm{reg}}$  as in Remark 4.2.5, then the same arguments as in the proof of Corollary 3.3.6 imply that the restriction of  $\mathbf{I}_{\mathfrak{A}}$  to the open subscheme  $\mathfrak{g}_{\mathfrak{A}}^{\mathrm{reg}}$  is smooth (and commutative). Moreover, one can construct an affine smooth group scheme  $\mathbf{J}_{\mathfrak{A}}$  over  $\mathfrak{t}_{\mathfrak{A}}/W$  as in Proposition 3.3.9.

We set

$$\mathfrak{J}_S^{\mathfrak{A}} := \mathcal{L}ie(\mathbf{I}_S^{\mathfrak{A}}/\mathcal{S}_{\mathfrak{A}}), \quad \mathbb{I}_S^{\mathfrak{A}} := \mathrm{Lie}(\mathbf{I}_S^{\mathfrak{A}}/\mathcal{S}_{\mathfrak{A}}).$$

- Lemma 4.3.3.** (1) *The coherent sheaf  $\mathfrak{J}_S^{\mathfrak{A}}$  on  $\mathcal{S}_{\mathfrak{A}}$  is locally free of finite rank, and the Lie algebra  $\mathbb{I}_S^{\mathfrak{A}}$  is a vector bundle over  $\mathcal{S}_{\mathfrak{A}}$ .*  
(2) *The Lie algebra  $\mathfrak{J}_S^{\mathfrak{A}}$  is commutative.*  
(3) *For any geometric point  $\mathbb{F}$  of  $\mathfrak{A}$ , the Lie algebra  $\mathrm{Spec}(\mathbb{F}) \times_{\mathrm{Spec}(\mathfrak{A})} \mathbb{I}_S^{\mathfrak{A}}$  coincides with the Lie algebra  $\mathbb{I}_{\mathbb{F}}$  of §3.4 for the group  $\mathbf{G}_{\mathbb{F}}$ .*

*Proof.* These properties follows from Lemma 2.2.1 and Proposition 4.3.1.  $\square$

**Theorem 4.3.4.** *There exists a canonical isomorphism*

$$\mathbb{I}_S^{\mathfrak{A}} \xrightarrow{\sim} \mathbb{T}^*(\mathcal{S}_{\mathfrak{A}})$$

*of commutative Lie algebras over  $\mathcal{S}_{\mathfrak{A}}$ . In other words, if one identifies  $\mathcal{S}_{\mathfrak{A}}$  with  $\mathfrak{t}_{\mathfrak{A}}/W$  via the isomorphism of Theorem 4.2.3, there exists a canonical isomorphism*

$$\mathbb{I}_S^{\mathfrak{A}} \xrightarrow{\sim} \mathbb{T}^*(\mathfrak{t}_{\mathfrak{A}}/W)$$

*of commutative Lie algebras over  $\mathfrak{t}_{\mathfrak{A}}/W$ .*

*Proof.* As in the proof of Theorem 3.4.2 one can construct a canonical morphism  $\mathfrak{J}_S^{\mathfrak{A}} \rightarrow \Omega_{\mathcal{S}_{\mathfrak{A}}}$  of coherent sheaves on  $\mathcal{S}_{\mathfrak{A}}$ . To prove that this morphism is an isomorphism it suffices to prove that its cone is isomorphic to 0. Since both  $\mathfrak{J}_S^{\mathfrak{A}}$  and  $\Omega_{\mathcal{S}_{\mathfrak{A}}}$  are flat over  $\mathfrak{A}$ , to prove this it suffices to prove that for any geometric point  $\mathbb{F}$  of  $\mathfrak{A}$ , the induced morphism  $\mathbb{F} \otimes_{\mathfrak{A}} \mathfrak{J}_S^{\mathfrak{A}} \rightarrow \mathbb{F} \otimes_{\mathfrak{A}} \Omega_{\mathcal{S}_{\mathfrak{A}}}$  is an isomorphism (see [BR2, Lemma 1.4.1(1)]). However, by Lemma 4.3.3(3),  $\mathbb{F} \otimes_{\mathfrak{A}} \mathfrak{J}_S^{\mathfrak{A}}$  is the Lie algebra  $\mathfrak{J}_{\mathbb{F}}$  of §3.4 for the group  $\mathbf{G}_{\mathbb{F}}$ , hence the desired claim follows from Theorem 3.4.2.  $\square$

*Remark 4.3.5.* If one defines  $\mathfrak{g}_{\mathfrak{A}}^{\mathrm{reg}}$  as in Remark 4.2.5, and  $\mathbf{J}_{\mathfrak{A}}$  as in Remark 4.3.2, then one can also obtain analogues over  $\mathfrak{A}$  of the first and second isomorphisms in Theorem 3.4.2.

**4.4. Variant for the Grothendieck resolution.** In this subsection we consider the Grothendieck resolution

$$\tilde{\mathfrak{g}}_{\mathfrak{R}} := \mathbf{G}_{\mathfrak{R}} \times^{\mathbf{B}_{\mathfrak{R}}} \mathfrak{b}_{\mathfrak{R}}.$$

It is clear that, for any geometric point  $\mathbb{F}$  of  $\mathfrak{R}$ , the base change of  $\tilde{\mathfrak{g}}_{\mathfrak{R}}$  to  $\mathbb{F}$  is the variety  $\tilde{\mathfrak{g}}$  studied in §3.5, for the group  $\mathbf{G}_{\mathbb{F}}$ . There is a natural projective morphism  $\pi: \tilde{\mathfrak{g}}_{\mathfrak{R}} \rightarrow \mathfrak{g}_{\mathfrak{R}}$  induced by the adjoint action, and we set

$$\tilde{\mathcal{S}}_{\mathfrak{R}} := \mathcal{S}_{\mathfrak{R}} \times_{\mathfrak{g}_{\mathfrak{R}}} \tilde{\mathfrak{g}}_{\mathfrak{R}}.$$

We will also consider the natural morphism  $\nu: \tilde{\mathfrak{g}}_{\mathfrak{R}} \rightarrow \mathfrak{t}_{\mathfrak{R}}$ .

**Proposition 4.4.1.** *The scheme  $\tilde{\mathcal{S}}_{\mathfrak{R}}$  is affine, and the natural morphisms*

$$(4.4.1) \quad \tilde{\mathcal{S}}_{\mathfrak{R}} \rightarrow \mathcal{S}_{\mathfrak{R}} \times_{\mathfrak{t}_{\mathfrak{R}}/W} \mathfrak{t}_{\mathfrak{R}} \rightarrow \mathfrak{t}_{\mathfrak{R}}$$

*are isomorphisms.*

*Proof.* First we claim that  $\tilde{\mathcal{S}}_{\mathfrak{R}}$  is flat over  $\mathfrak{R}$ . In fact, by  $\mathbf{G}_{\mathfrak{R}}$ -equivariance the following diagram is Cartesian:

$$\begin{array}{ccc} \mathbf{G}_{\mathfrak{R}} \times \tilde{\mathcal{S}}_{\mathfrak{R}} & \longrightarrow & \tilde{\mathfrak{g}}_{\mathfrak{R}} \\ \downarrow & & \downarrow \\ \mathbf{G}_{\mathfrak{R}} \times \mathcal{S}_{\mathfrak{R}} & \xrightarrow{a} & \mathfrak{g}_{\mathfrak{R}}. \end{array}$$

Since  $a$  is smooth (see Proposition 4.2.4), the upper arrow is also smooth, and we deduce that  $\mathbf{G}_{\mathfrak{R}} \times \tilde{\mathcal{S}}_{\mathfrak{R}}$  is flat over  $\mathfrak{R}$ . Since  $\mathfrak{R}$  is a direct factor in  $\mathcal{O}(\mathbf{G}_{\mathfrak{R}})$ , this implies that  $\tilde{\mathcal{S}}_{\mathfrak{R}}$  is also flat over  $\mathfrak{R}$ . Then, since  $\tilde{\mathcal{S}}_{\mathfrak{R}}$  is  $\mathfrak{R}$ -flat and projective over  $\mathcal{S}_{\mathfrak{R}}$ , the first assertion follows from Proposition 3.5.4(1) and Lemma 2.1.4.

The fact that the second morphism in (4.4.1) is an isomorphism follows from Theorem 4.2.3. To prove that the first one is also an isomorphism, consider the induced morphism

$$(4.4.2) \quad \mathcal{O}(\mathcal{S}_{\mathfrak{R}} \times_{\mathfrak{t}_{\mathfrak{R}}/W} \mathfrak{t}_{\mathfrak{R}}) \rightarrow \mathcal{O}(\tilde{\mathcal{S}}_{\mathfrak{R}}).$$

Then both sides are finite modules over  $\mathcal{O}(\mathfrak{t}_{\mathfrak{R}}/W)$  (by [De, Théorème 2] and [Ha, Theorem III.5.2], respectively), and are  $\mathfrak{R}$ -flat. Hence, using [BR2, Lemma 1.4.1(1)], to prove that (4.4.2) is an isomorphism it suffices to prove that it becomes an isomorphism after applying  $\mathbb{F} \otimes_{\mathfrak{R}} (-)$  for all geometric points  $\mathbb{F}$  of  $\mathfrak{R}$ . The latter fact was proved in Proposition 3.5.4(1).  $\square$

We define the universal stabilizer  $\tilde{\mathbf{I}}_{\mathfrak{R}}$  over  $\tilde{\mathfrak{g}}_{\mathfrak{R}}$  as the fiber product

$$\tilde{\mathbf{I}}_{\mathfrak{R}} := \tilde{\mathfrak{g}}_{\mathfrak{R}} \times_{\tilde{\mathfrak{g}}_{\mathfrak{R}} \times \tilde{\mathfrak{g}}_{\mathfrak{R}}} (\mathbf{G}_{\mathfrak{R}} \times \tilde{\mathfrak{g}}_{\mathfrak{R}}).$$

We denote by  $\tilde{\mathbf{I}}_{\mathcal{S}}^{\mathfrak{R}}$  the restriction of  $\tilde{\mathbf{I}}_{\mathfrak{R}}$  to  $\tilde{\mathcal{S}}_{\mathfrak{R}}$ .

**Proposition 4.4.2.** *The group scheme  $\tilde{\mathbf{I}}_{\mathcal{S}}^{\mathfrak{R}}$  is smooth over  $\tilde{\mathcal{S}}_{\mathfrak{R}}$ , and the following natural commutative diagram is Cartesian:*

$$\begin{array}{ccc} \tilde{\mathbf{I}}_{\mathcal{S}}^{\mathfrak{R}} & \longrightarrow & \mathbf{I}_{\mathcal{S}}^{\mathfrak{R}} \\ \downarrow & & \downarrow \\ \tilde{\mathcal{S}}_{\mathfrak{R}} & \longrightarrow & \mathcal{S}_{\mathfrak{R}}. \end{array}$$

*Proof.* The proof of the first claim is similar to the proof of Proposition 3.5.5(1).

To prove the second claim, we observe that the commutative square of the statement induces a canonical morphism of  $\tilde{\mathcal{S}}_{\mathfrak{R}}$ -group schemes

$$(4.4.3) \quad \tilde{\mathbf{I}}_{\mathcal{S}}^{\mathfrak{R}} \rightarrow \tilde{\mathcal{S}}_{\mathfrak{R}} \times_{\mathcal{S}_{\mathfrak{R}}} \mathbf{I}_{\mathcal{S}}^{\mathfrak{R}}.$$

Since both group schemes are closed subschemes of  $\mathbf{G}_{\mathfrak{R}} \times_{\mathrm{Spec}(\mathfrak{R})} \tilde{\mathcal{S}}_{\mathfrak{R}}$ , and since this morphism is compatible with their inclusion in  $\mathbf{G}_{\mathfrak{R}} \times_{\mathrm{Spec}(\mathfrak{R})} \tilde{\mathcal{S}}_{\mathfrak{R}}$ , (4.4.3) must be a closed embedding. Since the  $\mathfrak{R}$ -algebra  $\mathcal{O}(\tilde{\mathcal{S}}_{\mathfrak{R}} \times_{\mathcal{S}_{\mathfrak{R}}} \mathbf{I}_{\mathcal{S}}^{\mathfrak{R}})$  is of finite type and flat over  $\mathfrak{R}$ , and since  $\tilde{\mathbf{I}}_{\mathcal{S}}^{\mathfrak{R}}$  is also  $\mathfrak{R}$ -flat, by [BR2, Lemma 1.4.1(1)], to prove that (4.4.3) is an isomorphism it suffices to prove that, for any geometric point  $\mathbb{F}$  of  $\mathfrak{R}$ , the morphism

$$\mathrm{Spec}(\mathbb{F}) \times_{\mathrm{Spec}(\mathfrak{R})} \tilde{\mathbf{I}}_{\mathcal{S}}^{\mathfrak{R}} \rightarrow \mathrm{Spec}(\mathbb{F}) \times_{\mathrm{Spec}(\mathfrak{R})} (\tilde{\mathcal{S}}_{\mathfrak{R}} \times_{\mathcal{S}_{\mathfrak{R}}} \mathbf{I}_{\mathcal{S}}^{\mathfrak{R}})$$

is an isomorphism, which follows from Proposition 3.5.5(2).  $\square$

Now we set

$$\tilde{\mathcal{J}}_{\mathcal{S}}^{\mathfrak{R}} := \mathcal{L}ie(\tilde{\mathbf{I}}_{\mathcal{S}}^{\mathfrak{R}}/\tilde{\mathcal{S}}_{\mathfrak{R}}), \quad \tilde{\mathbb{I}}_{\mathcal{S}}^{\mathfrak{R}} := \mathrm{Lie}(\tilde{\mathbf{I}}_{\mathcal{S}}^{\mathfrak{R}}/\tilde{\mathcal{S}}_{\mathfrak{R}}).$$

**Lemma 4.4.3.** *There exists a canonical Cartesian diagram*

$$\begin{array}{ccc} \tilde{\mathbb{I}}_{\mathcal{S}}^{\mathfrak{R}} & \longrightarrow & \mathbb{I}_{\mathcal{S}}^{\mathfrak{R}} \\ \downarrow & & \downarrow \\ \tilde{\mathcal{S}}_{\mathfrak{R}} & \longrightarrow & \mathcal{S}_{\mathfrak{R}}. \end{array}$$

*Proof.* This follows from Lemma 2.2.1(2), Proposition 4.3.1 and Proposition 4.4.2.  $\square$

Combining Lemma 4.4.3 and Theorem 4.3.4 we deduce the following result.

**Theorem 4.4.4.** *There exists a canonical isomorphism*

$$\tilde{\mathbb{I}}_{\mathcal{S}}^{\mathfrak{R}} \xrightarrow{\sim} \tilde{\mathcal{S}}_{\mathfrak{R}} \times_{\mathcal{S}_{\mathfrak{R}}} \mathbb{T}^*(\mathcal{S}_{\mathfrak{R}})$$

*of commutative Lie algebras over  $\tilde{\mathcal{S}}_{\mathfrak{R}}$ . In other words, if one identifies  $\tilde{\mathcal{S}}_{\mathfrak{R}}$  with  $\mathfrak{t}_{\mathfrak{R}}$  via the isomorphism of Proposition 4.4.1, there exists a canonical isomorphism*

$$\tilde{\mathbb{I}}_{\mathcal{S}}^{\mathfrak{R}} \xrightarrow{\sim} \mathfrak{t}_{\mathfrak{R}} \times_{\mathfrak{t}_{\mathfrak{R}}/W} \mathbb{T}^*(\mathfrak{t}_{\mathfrak{R}}/W)$$

*of commutative Lie algebras over  $\mathfrak{t}_{\mathfrak{R}}$ .*  $\square$

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