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Finite speed of propagation for a non-local porous medium equation

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Abstract

This note is concerned with proving the finite speed of propagation for some non-local porous medium equation by adapting arguments developed by Caffarelli and Vázquez (2010).

AMS Classification: 35K55, 35B30

Keywords: Non-local porous medium equation, Non-local non-linear pressure law, finite speed of propagation, fractional Laplacian

1 Introduction

Caffarelli and Vázquez [3] proved finite speed of propagation for non-negative weak solutions of

$$\partial_t u = \nabla \cdot (u \nabla^{\alpha-1} u), \quad t > 0, x \in \mathbb{R}^d \quad (1)$$

with $\alpha \in (0, 2)$ and $\nabla^{\alpha-1}$ stands for $\nabla(-\Delta)^{\frac{\alpha}{2}-1}$. We adapt here their proof in order to treat the more general case

$$\partial_t u = \nabla \cdot (u \nabla^{\alpha-1} u^{m-1}), \quad t > 0, x \in \mathbb{R}^d \quad (2)$$

for $m > m_\alpha := 1 + d^{-1}(1 - \alpha)_+ + 2(1 - \alpha^{-1})_+$. Equation (2) is supplemented with the following initial condition

$$u(0, x) = u_0(x), \quad x \in \mathbb{R}^d. \quad (3)$$

The result contained in this note gives a positive answer to a question posed in [4] where finite or infinite speed of propagation is studied for another generalization of (1). We recall that weak solutions of (2)-(3) are constructed in [2] for $m > m_\alpha$ (see also [1]).

In the following statement (and the remaining of the note), B_R denotes the ball of radius $R > 0$ centered at the origin.

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Theorem 1.1 (Finite speed of propagation). *Let $m > m_\alpha$ and assume that $u_0 \geq 0$ is supported in B_{R_0} . Then the non-negative weak solution u of (2)-(3) is supported in $B_{R(t)}$ where*

$$R(t) = R_0 + Ct^{\frac{1}{\alpha}} \quad \text{with} \quad C = C_0 \|u_0\|_\infty^{\frac{m-1}{\alpha}}$$

for some constant $C_0 > 0$ only depending on dimension, α and m .

Remark 1.2. The technical assumption $m > m_\alpha$ is imposed to ensure the existence of weak solutions; see [2].

Remark 1.3. In view of the Barenblatt solutions constructed in [2], the previous estimate of the speed of propagation is optimal.

The remaining of the note is organized as follows. In preliminary Section 2, the equation is written in non-divergence form, non-local operators appearing in it are written as singular integrals, invariant scalings are exhibited and an approximation procedure is recalled. Section 3 is devoted to the contact analysis. A first lemma for a general barrier is derived in Subsection 3.1. The barrier to be used in the proof of the theorem is constructed in Subsection 3.2. The main error estimate is obtained in Subsection 3.3. Theorem 1.1 is finally proved in Section 4.

Notation. For $a \in \mathbb{R}$, a_+ denotes $\max(0, a)$. An inequality written as $A \lesssim B$ means that there exists a constant C only depending on dimension, α and m such that $A \leq CB$. If $\alpha \in (0, 1)$, a function u is in \mathcal{C}^α means that it is α -Hölder continuous. If $\alpha \in (1, 2)$, it means that ∇u is $(\alpha - 1)$ -Hölder continuous. For $\alpha \in (0, 2)$, a function u is in $\mathcal{C}^{\alpha+0}$ if it is in $\mathcal{C}^{\alpha+\varepsilon}$ for some $\varepsilon > 0$ and $\alpha + \varepsilon \neq 1$.

2 Preliminaries

The contact analysis relies on writing Eq. (2) into the following non-divergence form

$$\partial_t u = \nabla u \cdot \nabla p + u \Delta p \tag{4}$$

where p stands for the pressure term and is defined as

$$p = (-\Delta)^{\frac{\alpha}{2}-1} u^{m-1}.$$

It is also convenient to write $v = u^{m-1} = G(u)$.

We recall that for a smooth and bounded function v , the non-local operators appearing in (4) have the following singular integral representations,

$$\begin{aligned} \nabla(-\Delta)^{\frac{\alpha}{2}-1} v &= c_\alpha \int (v(x+z) - v(x)) z \frac{dz}{|z|^{d+\alpha}}, \\ -(-\Delta)^{\frac{\alpha}{2}} v &= \bar{c}_\alpha \int (v(x+z) + v(x-z) - 2v(x)) \frac{dz}{|z|^{d+\alpha}}. \end{aligned}$$

The following elementary lemma makes the scaling of the equation precise.

Lemma 2.1 (Scaling). *If u satisfies (2) then $U(t, x) = Au(Tt, Bx)$ satisfies (2) as soon as*

$$T = A^{m-1}B^\alpha.$$

Consider non-negative solutions of the viscous approximation of (2), i.e.

$$\partial_t u = \nabla \cdot (u \nabla^{\alpha-1} G(u)) + \delta \Delta u, \quad t > 0, x \in \mathbb{R}^d.$$

For sufficiently smooth initial data u_0 , solutions are at least \mathcal{C}^2 with respect to x and \mathcal{C}^1 with respect to t .

3 Contact analysis

3.1 The contact analysis lemma

Lemma 3.1 (Contact analysis). *Let $U(t, x)$ be \mathcal{C}^2 , radially symmetric w.r.t. x , non-increasing w.r.t. $|x|$. If*

$$\begin{cases} u \leq U \text{ for } (t, x) \text{ in } [0, t_c] \times \mathbb{R}^d, \\ u(t_c, x_c) = U(t_c, x_c), \end{cases}$$

then

$$\partial_t U \leq \nabla U \cdot \nabla P + U \Delta P + \delta \Delta U + e$$

holds at (t_c, x_c) where

$$\begin{cases} P = -(-\Delta)^{\frac{\alpha}{2}-1} G(U) \\ e = |\nabla U| (I_{\text{out},+}(V) - I_{\text{out},+}(v)) \end{cases}$$

with

$$I_{\text{out},+}(w) = \begin{cases} \int_{\substack{|y| \geq \gamma \\ y \cdot \hat{x}_c \geq 0}} (w(x_c + y) - w(x_c)) (y \cdot \hat{x}_c) \frac{dy}{|y|^{d+\alpha}} & \text{if } \alpha \geq 1 \\ \int_{\substack{|y| \geq \gamma \\ y \cdot \hat{x}_c \geq 0}} w(x_c + y) (y \cdot \hat{x}_c) \frac{dy}{|y|^{d+\alpha}} & \text{if } \alpha \in (0, 1) \end{cases}$$

for γ such that

$$c_\alpha \gamma |\nabla U(x_c)| \leq \bar{c}_\alpha U(x_c)$$

where c_α and \bar{c}_α are the constants appearing in the definitions of the two non-local operators.

Proof. At the contact point (t_c, x_c) , the following holds true

$$\begin{aligned} \partial_t u &\geq \partial_t U \\ \nabla u &= \nabla U = -|\nabla U| \hat{x}_c \\ \Delta u &\leq \Delta U. \end{aligned}$$

This implies that

$$\partial_t U \leq \nabla U \cdot \nabla p + U \Delta p + \delta \Delta U. \quad (5)$$

We next turn our attention to ∇p and Δp . We drop the time dependence of functions since it plays no role in the remaining of the analysis.

We first remark that

$$\nabla U \cdot \nabla p = -|\nabla U| I(v) \quad (6)$$

where

$$I(v) = \begin{cases} \int (v(x_c + y) - v(x_c))(y \cdot \hat{x}_c) \frac{dy}{|y|^{d+\alpha}} & \text{if } \alpha \in [1, 2), \\ \int v(x_c + y)(y \cdot \hat{x}_c) \frac{dy}{|y|^{d+\alpha}} & \text{if } \alpha \in (0, 1). \end{cases}$$

We now split I into several pieces by splitting the domain of integration \mathbb{R}^d into $B_\gamma^{\text{in},\pm} = \{y \in B_\gamma : \pm y \cdot \hat{x}_c \geq 0\}$ and $B_\gamma^{\text{out},\pm} = \mathbb{R}^d \setminus B_\gamma^{\text{in},\pm}$ for some parameter $\gamma > 0$ to be fixed later. We thus can write

$$I(v) = I_{\text{in},+}(v) + I_{\text{in},-}(v) + I_{\text{out},+}(v) + I_{\text{out},-}(v)$$

where

$$I_{\text{in/out},\pm}(v) = \begin{cases} c_\alpha \int_{B_\gamma^{\text{in/out},\pm}} (v(x_c + y) - v(x_c))(y \cdot \hat{x}_c) \frac{dy}{|y|^{d+\alpha}} & \text{if } \alpha \in [1, 2), \\ c_\alpha \int_{B_\gamma^{\text{in/out},\pm}} v(x_c + y)(y \cdot \hat{x}_c) \frac{dy}{|y|^{d+\alpha}} & \text{if } \alpha \in (0, 1). \end{cases}$$

We can proceed similarly for Δp . Remark that

$$\Delta p = J(v) \quad (7)$$

where

$$J(v) = \bar{c}_\alpha \int (v(x+y) + v(x-y) - 2v(x)) \frac{dy}{|y|^{d+\alpha}}.$$

We can introduce $J_{\text{in/out},\pm}(v)$ analogously.

We first remark that,

$$\begin{cases} -I_{\text{in/out},-}(v) \leq -I_{\text{in/out},-}(V) \\ J_{\text{in/out},\pm}(v) \leq J_{\text{in/out},\pm}(V) \end{cases} \quad (8)$$

holds at x_c where $V = G(U)$.

We next remark that since G is non-decreasing and vanishes at 0 and $w = v - V$ reaches a zero maximum at $x = x_c$,

$$-I_{\text{in},+}(v - V) \leq -\tilde{c}_\alpha \gamma J_{\text{in},+}(v - V) \quad (9)$$

holds at x_c .

Combining (5)-(9), we get (at x_c),

$$\begin{aligned} \partial_t U \leq & |\nabla U|(-I_{\text{in},+}(V) - I_{\text{in},-}(V) - I_{\text{out},+}(v) - I_{\text{out},-}(V) + \tilde{c}_\alpha \gamma J_{\text{in},+}(V)) \\ & + (U - \tilde{c}_\alpha \gamma |\nabla U|) J_{\text{in},+}(v) \\ & + U(J_{\text{in},-}(V) + J_{\text{out},+}(V) + J_{\text{out},-}(V)) + \delta \Delta U. \end{aligned}$$

In view of the choice of γ , we get

$$\partial_t U \leq -|\nabla U|I(V) + UJ(V) + |\nabla U|(-I_{\text{out},+}(v) + I_{\text{out},+}(V)) + \delta \Delta U.$$

We now remark that $-|\nabla U|I(V) = \nabla U \cdot \nabla P$ and $J(V) = \Delta P$ we get the desired inequality. \square

3.2 Construction of the barrier

Lemma 3.2 (Construction of a barrier). *Assume that*

$$\|u\|_\infty \leq 1 \quad \text{and} \quad 0 \leq u_0(x) \leq (R_0 - |x|)_+^\beta \quad \text{with} \quad R_0 \geq 2$$

for some $\beta > \max(2, \alpha(m-1)^{-1})$. There then exist $C > 0$ and $T > 0$ (only depending on d, m, α, β) and $U \in \mathcal{C}^2((0, +\infty) \times \mathbb{R}^d \setminus B_1)$ defined as follows,

$$U(t, x) = \omega^\beta + (R(t) - |x|)_+^\beta + \omega^\beta \frac{t}{T} \quad (10)$$

where $R(t) = R_0 + Ct$, such that

i) the following holds true

$$\nabla P, \Delta P, J_{\text{in},+}(V), I_{\text{out},+}(V), \Delta U \text{ are bounded}; \quad (11)$$

ii) u and U cannot touch at a time $t < T$ and a point $x_c \in B_1$ or $x_c \notin \bar{B}_{R(t)}$;

iii) if U touches u from above at (t_c, x_c) with $t_c < T$ and $x_c \in B_{R(t)}$, then

$$C \lesssim 1 - I_{\text{out},+}(v) + \frac{\delta}{\omega}. \quad (12)$$

Proof. We first remark that the condition $R_0 \geq 2$ ensures that the contact point is out of B_1 since $\|u\|_\infty \leq 1$.

The fact that U is \mathcal{C}^2 and $V = U^{m-1}$ is $\mathcal{C}^{\alpha+0}$ in $\mathbb{R}^d \setminus B_1$ ensures that (11) holds true. Notice that the condition: $\beta(m-1) > \alpha$ is used here.

We should now justify that the contact point cannot be outside $B_{R(t)}$ at a time $t \in (0, T)$ for some small time T under control. If $|x_c| > R(t)$ and $t_c < T$ then

$$\begin{cases} 0 \leq U \leq 2\omega^\beta \\ \partial_t U = \frac{\omega^\beta}{T} \\ |\nabla U| = 0 \\ \Delta U = 0. \end{cases}$$

The contact analysis lemma 3.1 (with $\gamma = 1$, say) and (11) then implies that

$$\frac{\omega^\beta}{T} \leq |\Delta P|U \lesssim \omega^\beta$$

and choosing T small enough (but under control) yields a contradiction.

It remains to study what happens if $t_c < T$ and $x_c \in B_{R(t)} \setminus B_1$. In order to do so, we first define h and H as follows:

$$U = h^\beta + H^\beta \leq 1$$

with $H^\beta = \omega^\beta t T^{-1} \leq \omega^\beta$ for $t \in (0, T)$. Remark that $h \geq \omega \geq H$. In the contact analysis lemma 3.1, we choose γ such that

$$\beta c_\alpha \gamma \leq \bar{c}_\alpha h.$$

If $x_c \in B_{R(t)} \setminus B_1$,

$$\begin{cases} \partial_t U = \beta C h^{\beta-1} + \frac{\omega^\beta}{T} \geq \beta C h^{\beta-1} \\ |\nabla U| = \beta h^{\beta-1} \end{cases} \quad (13)$$

Combining Lemma 3.1 with (11)-(13), we get (12). \square

3.3 Estimate of the error term

Lemma 3.3. *The following estimate holds true at x_c ,*

$$-I_{\text{out},+}(v) \lesssim \begin{cases} G(2h^\beta)h^{1-\alpha} & \text{if } \alpha > 1 \\ R_0^{1-\alpha+\varepsilon} & \text{if } \alpha \leq 1 \end{cases} \quad (14)$$

for all $\varepsilon > 0$.

Proof. We begin with the easy case $\alpha > 1$. In this case, we simply write

$$\begin{aligned} -I_{\text{out},+}(v) &= \int_{\substack{|y| \geq \gamma \\ y \cdot \hat{x}_c \geq 0}} (v(x_c) - v(x_c + y))(y \cdot \hat{x}_c) \frac{dy}{|y|^{d+\alpha}} \\ &\leq v(x_c) \int_{\substack{|y| \geq \gamma \\ y \cdot \hat{x}_c \geq 0}} (y \cdot \hat{x}_c) \frac{dy}{|y|^{d+\alpha}} \\ &\leq v(x_c) \int_{|y| \geq \gamma} \frac{dy}{|y|^{d+\alpha-1}} \end{aligned}$$

where we used the fact that $v \geq 0$. By remarking that

$$v = G(u) = G(h^\beta + H^\beta) \leq G(2h^\beta)$$

at the contact point and through an easy and standard computation, we get the desired estimate in the case $\alpha > 1$.

We now turn to the more subtle case $\alpha \in (0, 1]$. In this case,

$$I_{\text{out},+}(v) = I^*(v) + K \star v$$

where

$$\begin{cases} -I^*[v] = \int_{\substack{\gamma \leq |y| \leq 1 \\ y \cdot \hat{x}_c \geq 0}} -v(x_c + y)(y \cdot \hat{x}_c) \frac{dy}{|y|^{d+\alpha}} \\ K = \frac{y \cdot \hat{x}_c}{|y|^{d+\alpha}} \mathbf{1}_{|y| \geq 1, y \cdot \hat{x}_c \geq 0}. \end{cases}$$

We first remark that

$$|I^*[v]| \leq \|v\|_\infty \int_{B_1} \frac{dy}{|y|^{d+\alpha-1}} \lesssim 1.$$

We next remark that $K \in L^p(\mathbb{R}^d)$ for all $p > \frac{d}{d-(1-\alpha)} \geq 1$. Hence,

$$|K \star v| \leq \|K\|_p \|v\|_q = \|K\|_p \|u\|_{(m-1)q}^{m-1}$$

with p as above and $q^{-1} = 1 - p^{-1}$.

We next estimate $\|u\|_{(m-1)q}$. Interpolation leads

$$\|u\|_{(m-1)q}^{m-1} \leq \|u\|_1^{\frac{1}{q}} \|u\|_\infty^{(m-1) - \frac{1}{q}} \leq \|u\|_1^{\frac{1}{q}}$$

since $\|u\|_\infty \leq 1$. Finally, we use mass conservation in order to get

$$\|u\|_1 = \|u_0\|_1 \leq \int \min(1, (R_0 - |x|)_+^\beta) dx \leq \omega_d R_0^d.$$

Finally, we have

$$|I_{\text{out},+}(v)| \lesssim R_0^{\frac{d}{q}}$$

for all $q < \frac{d}{1-\alpha}$ which yields the desired result. \square

Combining now Lemmas 3.2 and 3.3, we get the following one.

Lemma 3.4 (Estimate of the speed of propagation). *Assume that*

$$\|u\|_\infty \leq 1 \quad \text{and} \quad 0 \leq u_0(x) \leq (R_0 - |x|)_+^\beta \quad \text{with} \quad R_0 \geq 2$$

for some $\beta > \max(2, \alpha(m-1)^{-1})$. Then there exists $T > 0$ and $C_0 > 0$ only depending on dimension, m , α and β (and ε for $\alpha \leq 1$) such that, for $t \in (0, T)$, u is supported in B_{R_0+Ct} with

$$C = \begin{cases} C_0 & \text{if } \alpha > 1, \\ C_0 R_0^{1-\alpha-\varepsilon} & \text{if } \alpha \leq 1 \end{cases} \quad (15)$$

(for $\varepsilon > 0$ arbitrarily small).

Proof. In view of Lemma 3.2, the parameter ω is chosen so that $\omega \gg \delta$, say $\omega = \sqrt{\delta}$. Now Lemmas 3.2 and 3.3 imply that if C is chosen as indicated in (15), then u remains below U at least up to time T . Letting (ω, δ) go to 0 yields the desired result. \square

4 Proof of Theorem 1.1

We can now prove Theorem 1.1.

Proof of Theorem 1.1. We treat successively the $\alpha > 1$ and $\alpha \leq 1$.

First case. In the case $\alpha > 1$, if $\|u\|_\infty = \|u_0\|_\infty \leq 1$ and

$$u_0(x) \leq (R_0 - |x|)_+^\beta,$$

then Lemma 3.4 implies that the support of u is contained in $B_{R(t)}$ with

$$R(t) = R_0 + C_0 t$$

for some constant C_0 only depending on dimension, m and α . Rescaling the solution (see Lemma 2.1), we get

$$R(t) = R_0 + C_0 L^{m-1-\frac{\alpha-1}{\beta}} a^{\frac{\alpha-1}{\beta}} t$$

as soon as

$$u_0(x) \leq a(R_0 - |x|)_+^\beta \quad \text{and} \quad L = \|u\|_\infty = \|u_0\|_\infty.$$

If we simply know that u_0 is supported in B_{R_0} and $\|u\|_\infty = \|u_0\|_\infty = L$, then we can pick any $a > 0$ and $r_1 > 0$ such that $ar_1^\beta = L$ and get

$$u_0(x) \leq a(r_1 + R_0 - |x|)_+^\beta.$$

By the previous reasoning, we get that

$$R(t) \leq R_0 + r_1 + C_0 L^{m-1-\frac{\alpha-1}{\beta}} a^{\frac{\alpha-1}{\beta}} t = R_0 + r_1 + C_0 L^{m-1} r_1^{1-\alpha} t.$$

Minimizing with respect to r_1 yields the desired result in the case $\alpha > 1$.

Second case. We now turn to the case $\alpha \in (0, 1]$. Lemma 3.4 yields for $t \in [0, T_1]$ with $T_1 = \frac{R_0}{C_1}$

$$C_1 \lesssim R_0^{1-\alpha+\varepsilon}$$

(recall that $R_0 \geq 2$).

We now start with $R_1 = R_0 + C_1 T_1 = 2R_0$ and we get

$$C_2 \lesssim (3R_0)^{1-\alpha+\varepsilon}$$

for $t \in [T_1, T_2]$ with

$$T_2 - T_1 = \frac{R_0}{C_2}.$$

More generally, for $t \in [T_k, T_{k+1}]$,

$$C_k \simeq ((k+1)R_0)^{1-\alpha+\varepsilon} \simeq (kR_0)^{1-\alpha+\varepsilon}$$

with

$$T_{k+1} - T_k = \frac{R_0}{C_k} \simeq \frac{R_0^{\alpha-\varepsilon}}{(k+1)^{1-\alpha+\varepsilon}}.$$

We readily see that the series $\sum_k (T_{k+1} - T_k)$ diverges. More precisely,

$$T_k \simeq (kR_0)^{\alpha-\varepsilon}.$$

Moreover, we get that the function u is supported in $B_{R(t)}$ with

$$R(t) - R_0 \lesssim kR_0 + C_k(t - T_k) \lesssim (T_k)^{\frac{1}{\alpha-\varepsilon}} + (T_k)^{\frac{1-\alpha+\varepsilon}{\alpha-\varepsilon}} t \lesssim t^{\frac{1}{\alpha-\varepsilon}}$$

for $t \in [T_k, T_{k+1}]$. Hence, we get the result but not with the right power. Precisely, for $L = 1$ and

$$0 \leq u_0(x) \leq (R_0 - |x|)_+^\beta$$

we get

$$R(t) = R_0 + C_0 t^\beta$$

with $\beta > \frac{1}{\alpha}$. Rescaling and playing again with r_1 and a such that $ar_1^\beta = L$ yields the desired result in the case $\alpha < 1$. The proof of the theorem is now complete. \square

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