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Large deviations for clocks of self-similar processes

Nizar Demni, Alain Rouault and Marguerite Zani

In memoriam, Marc Yor

Abstract The Lamperti correspondence gives a prominent role to two random time changes: the exponential functional of a Lévy process drifting to ∞ and its inverse, the clock of the corresponding positive self-similar process. We describe here asymptotical properties of these clocks in large time, extending the results of [27].

1 Introduction

This problem is an extension of a question raised by Marc Yor during the defense of the thesis of Marguerite, under the supervision of Alain, long time ago in 2000. The last part of this thesis was dedicated to the study of large deviations principles (LDP) for Maximum Likelihood Estimates of diffusion coefficients (for squared-radial Ornstein–Uhlenbeck processes, squared Bessel processes and Jacobi processes). The main tool there was a convenient Girsanov change of probability. This method allowed to convert the computation of Laplace transform of some additive functionals into the computation of Laplace transform of a single variable. This trick was used before in [25] page 26, or [19], page 30, where Marc Yor called

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it "reduction method". In those times, Marc was interested in exponential functionals of Brownian motion, Lamperti transform and Asian options and he guessed that this LDP could be applied to the Bessel clock and solved effectively the problem with Marguerite in [27] a couple of weeks later. Marc suggested to extend it to the Cauchy clock and gave a sketch of proof, but a technical difficulty stopped the project. Meanwhile, Marguerite and Nizar published a paper [12] on Jacobi diffusions where the reduction method is again crucial. Recently, the three of us felt the need to revisit the problem of Cauchy clock with the hope of new ideas.

We planned to discuss with Marc, and promised him we would keep him informed of our progress. We did not have the time...

In this paper, we extend the methods and results of [27] to a large class of clocks issued from positive self-similar Markov processes. In Section 2 we recall some basic results about the Lamperti correspondence between these processes and Lévy processes and we give the definition of the clocks. We also define a generalized Ornstein-Uhlenbeck process which will be useful in the sequel. In Section 3 and Section 4, we present the main results: Law of Large Numbers and Large Deviations for the clocks. In Section 5 we show some examples illustrating our main theorems.

2 Positive self-similar Markov processes and Lamperti transformation

In [17] Lamperti defined (positive) semi-stable process which are nowadays called positive self-similar Markov process.

Definition 1. For $\alpha > 0$, a positive self-similar Markov process (pssMp) of index α , is a $[0, \infty)$ -valued strong Markov process (X, \mathbb{Q}_a) , $a > 0$ with càdlàg paths, fulfilling the scaling property

$$(\{bX_{b^{-\alpha}t}, t \geq 0\}, \mathbb{Q}_a) \stackrel{(law)}{=} (\{X_t, t \geq 0\}, \mathbb{Q}_{ba}) \quad (1)$$

for every $a, b > 0$.

Lamperti [17] has shown that these processes can be connected to Lévy processes by a one-to-one correspondence, that we develop below. We refer to Kyprianou [15] especially Chapter 13 for properties of Lévy processes and pssMp. One can also see [4] for the Lamperti's correspondence. One can notice that there is a little confusion in the notion of index of these processes. In [17], [3] and [2], the index is $1/\alpha$, and in [4], [5] and [15], the index is α . We take this latter convention. These processes have a natural application in the theory of self-similar fragmentations : see [1], references therein, and [6]. For other areas of application, such as diffusions in random environments, see Section 6 of [4].

2.1 From X to ξ

Any pssMp X which never reaches the boundary state 0 may be expressed as the exponential of a Lévy process not drifting to $-\infty$, time changed by the inverse of its exponential functional. More formally, if $(X, (\mathbb{Q}_a)_{a>0})$ is a pssMp of index α which never reaches 0, set

$$T^{(X)}(t) = \int_0^t \frac{ds}{X_s^\alpha}, \quad (t \geq 0) \quad (2)$$

and let $A^{(X)}$ be its inverse, defined by

$$A^{(X)}(t) = \inf\{u \geq 0 : T^{(X)}(u) \geq t\}, \quad (3)$$

and let ξ be the process defined by

$$\xi_t = \log X_{A^{(X)}(t)} - \log X_0, \quad (t \geq 0). \quad (4)$$

Then, for every $a > 0$, the distribution of $(\xi_t, t \geq 0)$ under \mathbb{Q}_a does not depend on a and is the distribution of a Lévy process starting from 0.

Moreover, if we set

$$\mathcal{A}^{(\xi)}(t) = \int_0^t e^{\alpha \xi_s} ds \quad (5)$$

and $\tau^{(\xi)}$ its inverse defined by

$$\tau^{(\xi)}(t) = \inf\{u \geq 0 : \mathcal{A}^{(\xi)}(u) \geq t\} \quad (6)$$

we have

$$\tau^{(\xi)}(t) = T^{(X)}(tX_0^\alpha). \quad (7)$$

Let us remark that the self-similarity property (1) leads to the following relation:

$$\text{the law of } T^{(X)}(\cdot b^{-\alpha}) \text{ under } \mathbb{Q}_a \text{ is the law of } T^{(X)}(\cdot) \text{ under } \mathbb{Q}_{ba}. \quad (8)$$

2.2 From ξ to X

Let ξ_t be a Lévy process starting from 0 and let \mathbb{P} and \mathbb{E} denote the underlying probability and expectation, respectively. Fix $\alpha > 0$. Let $\mathcal{A}^{(\xi)}$ be its exponential functional defined by (5). When ξ drifts to $-\infty$, this functional is very popular in mathematical finance (see [26]), with important properties of the perpetuity $\mathcal{A}^{(\xi)}(\infty)$. Here, we rather assume that ξ does not drift to $-\infty$ i.e. satisfies, $\limsup_{t \uparrow \infty} \xi_t = \infty$. We define the inverse process $\tau^{(\xi)}$ of $\mathcal{A}^{(\xi)}$ by (6).

For every $a > 0$, let \mathbb{Q}_a be the law under \mathbb{P} of the time-changed process

$$X_t = a \exp \xi_{\tau^{(\xi)}(ta^{-\alpha})}, \quad (t \geq 0), \quad (9)$$

then $(X, (\mathbb{Q}_a)_{a>0})$ is a pssMp of index α which never reaches 0 and we have the fundamental relation (7).

2.3 Index and starting point

If $(X, (\mathbb{Q}_a)_{a>0})$ is an pssMa of index α , then the process $Y = (X^\alpha, (\mathbb{Q}_{a^\alpha})_{a>0})$, is a pssMp of index 1. Conversely if $(Y, (\mathbb{Q}_a)_{a>0})$ is a pssMp of index 1 then, for any $\alpha > 0$, the process $(X = Y^{1/\alpha}, (\mathbb{Q}_{a^{1/\alpha}})_{a>0})$ is a pssMp of index α . Hence we may and will assume without loss of generality, that $\alpha = 1$, except some examples in Section 5.

Let us stress also the fact that the distribution of $T^{(X)}$ is considered under \mathbb{Q}_a for $a > 0$, and the distribution of $\tau^{(\xi)}$ is considered under \mathbb{P} .

2.4 Ornstein-Uhlenbeck process

Having defined the above pair of associated processes (ξ, X) it is usual to consider a third process, very useful in the sequel. To introduce it, we need additional properties on the pssMp X . In Theorem 1 of [3], Bertoin and Yor have shown that whenever the support of ξ is not arithmetic and $\mathbb{E}(\xi_1) > 0$, then as $a \downarrow 0$, the probability measures \mathbb{Q}_a (defined in section 2.2) converge in the sense of finitely dimensional marginals to a probability measure denoted by \mathbb{Q}_0 . This latter measure is an entrance law for the semigroup $Q_t f(x) = \mathbb{E}_x f(X_t)$ and satisfies

$$\mathbb{Q}_0(f(X_t)) = \frac{1}{\mathbb{E}\xi_1} \mathbb{E}[I_\infty^{-1} f(t/I_\infty)] \quad (10)$$

where

$$I_\infty = \int_0^\infty e^{-\xi_s} ds.$$

Now we define the associated generalized Ornstein-Uhlenbeck (OU) process by

$$U(t) := e^{-t} X(e^t).$$

This process has been studied in [8], and further on by [24] and [6]. It is strictly stationary, Markovian and ergodic under \mathbb{E}_0 , and its invariant measure is the law of X_1 under \mathbb{Q}_0 , i.e.

$$\mu(f) = \frac{1}{\mathbb{E}\xi_1} \mathbb{E}[I_\infty^{-1} f(I_\infty)].$$

The purpose of this paper is the study of the asymptotic behavior of the process $T^{(X)}$, called the clock of the pssMp X , as $t \rightarrow \infty$ under \mathbb{Q}_a for any $a > 0$, or equivalently the asymptotic behavior of the process $\tau^{(\xi)}$, as $t \rightarrow \infty$ under \mathbb{P} . The two points

of view are complementary: the clock as a functional of the pssMp or the clock as a functional of the Lévy process. We will restrict us to the case of $\mathbb{E}\xi_1 \in (0, \infty)$, which ensures a Law of Large Numbers (LLN) and a Large Deviation Principle (LDP).

3 Law of Large Numbers

The first step of our study is the a.s. convergence of our functionals.

Theorem 1. *If $\mathbb{E}\xi_1$ is finite positive (hence $\psi'(0) > 0$), then for every $a > 0$, we have*

$$\frac{T^{(X)}(t)}{\log t} \rightarrow (\mathbb{E}\xi_1)^{-1}, \quad \mathbb{Q}_a - a.s. \quad (11)$$

or equivalently

$$\frac{\tau^{(\xi)}(t)}{\log t} \rightarrow (\mathbb{E}\xi_1)^{-1}, \quad \mathbb{P} - a.s. \quad (12)$$

Proof. From (7) it is enough to prove (11). Let us now repeat verbatim the trick from [6] Section 2. By the ergodic theorem, we have that for $f \in L^1(\mu)$:

$$\frac{1}{t} \int_0^t f(U_s) ds \rightarrow \mu(f), \quad \mathbb{Q}_0 - a.s.$$

or, by scaling

$$\frac{1}{\log t} \int_1^t f(u^{-1}X_u) \frac{du}{u} \rightarrow \mu(f), \quad \mathbb{Q}_0 - a.s. \quad (13)$$

For $f(x) = x^{-1}$ this yields

$$\frac{1}{\log(1+t)} \int_1^{1+t} \frac{du}{X_u} \rightarrow (\mathbb{E}\xi_1)^{-1}, \quad \mathbb{Q}_0 - a.s.$$

If \mathcal{H} denotes the set where the convergence holds, the Markov property yields:

$$\mathbb{E}_0 \left[\mathbb{Q}_{X_1} \left(\frac{T^{(X)}(t)}{\log(1+t)} \rightarrow (\mathbb{E}\xi_1)^{-1} \right) \right] = \mathbb{P}_0(\mathcal{H}) = 1.$$

If we remember that μ is the law of X_1 under \mathbb{Q}_0 , we have

$$\mathbb{Q}_x \left(\frac{T^{(X)}(t)}{\log(1+t)} \rightarrow (\mathbb{E}\xi_1)^{-1} \right) = 1, \quad \mu - a.e..$$

Fixing such an x and using the scaling property (8), we get (11). \square

It is then natural to look for a Large Deviation Principle (LDP) to characterize the speed of this convergence.

4 LDP

We present a LDP for the clock $T^{(X)}(t)$ (Theorem 2), whose proof relies on the reduction method (Lemma 1). The remaining of the section is devoted to some illustrative remarks and the proof of a technical lemma. In view of (7), the following statements are valid when $T^{(X)}$ and \mathbb{Q}_a are replaced by $\tau^{(\xi)}$ and \mathbb{P} , respectively.

4.1 Main result

Theorem 2. *Assume¹ that $\text{dom } \psi = (m_-, m_+) \ni 0$ and that $\psi'(0) > 0$. Set*

$$m_0 = \inf\{\theta : \psi'(\theta) > 0\}, \tau_+ = \frac{1}{\psi'(m_+)}, \tau_0 = \frac{1}{\psi'(m_0)} \text{ and } \Delta = (\tau_+, \tau_0),$$

where $1/\psi'(\pm\infty) := \lim_{m \rightarrow \pm\infty} m/\psi(m)$. Let $a > 0$.

1. Then, for every $x \in \bar{\Delta}$

$$\lim_{\varepsilon \rightarrow 0} \lim_{t \rightarrow \infty} \frac{1}{\log t} \log \mathbb{Q}_a \left(\frac{T^{(X)}(t)}{\log t} \in [x - \varepsilon, x + \varepsilon] \right) = -\mathcal{I}(x) \quad (14)$$

where

$$\mathcal{I}(x) = \sup_{m \in (m_0, m_+)} \{m - x\psi(m)\}. \quad (15)$$

2. Moreover

$$\lim_{A \rightarrow \infty} \limsup_{t \rightarrow \infty} \frac{1}{\log t} \log \mathbb{Q}_a \left(\frac{T^{(X)}(t)}{\log t} > A \right) = -\infty. \quad (16)$$

3. If either $\Delta = (0, \infty)$ or the complement of $\bar{\Delta}$ is exponentially negligible, i.e. if

$$\forall \varepsilon > 0 \quad \limsup_{t \rightarrow \infty} \frac{1}{\log t} \log \mathbb{Q}_a \left(\inf_{x \in \bar{\Delta}} \left\{ \left| x - \frac{T^{(X)}(t)}{\log t} \right| \right\} > \varepsilon \right) = -\infty \quad (17)$$

then the family of distributions of $T^{(X)}(t)/\log t$ under \mathbb{Q}_a satisfies the LDP on $[0, \infty)$ at scale $\log t$ with good rate function :

$$\tilde{\mathcal{I}}(x) = \begin{cases} \mathcal{I}(x) & \text{if } x \in \bar{\Delta}, \\ \infty & \text{otherwise.} \end{cases}$$

The following proposition describes some properties of \mathcal{I} which are direct consequences of (15). The proof is left to the reader.

¹ Domains are taken open par convention

Proposition 1. 1. If ψ^* denotes the Fenchel-Legendre dual of ψ defined by

$$\psi^*(x) = \sup_{m \in \mathbb{R}} mx - \psi(m), \quad (x \in \mathbb{R}),$$

then

$$\mathcal{I}(x) = x\psi^*(x^{-1}), \quad (x \in \Delta). \quad (18)$$

2. Let $\tau_e = 1/\mathbb{E}\xi_1 \in (\tau_+, \tau_0)$. The function \mathcal{I} is convex on Δ , decreasing on (τ_+, τ_e) , has a minimum 0 at τ_e , and is increasing on (τ_e, τ_0) .
3. a. If m_0 is a true minimum, i.e. if $m_0 > -\infty$ and $\psi'(m_0) = 0$, then $\tau_0 = \infty$, and as $x \rightarrow \infty$, \mathcal{I} admits the asymptote $y = -x\psi(m_0) + m_0$.
b. If $m_0 = -\infty$ and $\psi'(m_0) \in (0, \infty)$, set $\lim_{m \rightarrow m_0} \psi(m) - m\psi'(m_0) = -b \in [-\infty, 0)$. Then $\tau_0 < \infty$ and $\mathcal{I}(\tau_0) = b\tau_0$ and $\mathcal{I}'(\tau_0) = -\psi(m_0) = \infty$.
c. If $m_0 = -\infty$ and $-\infty < \psi(m_0) < 0$, hence $\psi'(m_0) = 0$, then $\tau_0 = \infty$ and $\mathcal{I}(x)/x \rightarrow -\psi(m_0)$ and $\mathcal{I}(x) + \psi(m_0)x \rightarrow -\infty$, as $x \rightarrow \infty$.
4. a. If $m_+ < \infty$ and $\psi(m_+) = \infty$, then $\psi'(m_+) = \infty$. We get $\tau_+ = 0$ and $\mathcal{I}(0) = m_+$ with $\mathcal{I}'(0) = \infty$.
b. If $m_+ = \infty$ and $\psi'(m_+) < \infty$, set $\lim_{m \rightarrow m_+} \psi(m) - m\psi'(m_+) = -b \in [-\infty, 0)$. Then $\tau_+ > 0$, $\mathcal{I}(\tau_+) = b\tau_+ \leq \infty$ and $\mathcal{I}'(\tau_+) = -\psi(m_+) = -\infty$.
c. If $m_+ = \infty$, with $\psi(m_+) = \infty$ and $\psi'(m_+) = \infty$ then $\tau_+ = 0$ and $\mathcal{I}(\tau_+) = m_+ = \infty$.

Proof. of Theorem 2

1) A slight adaptation of the Gärtner-Ellis method ([11] Th. 2.3.6) allows to deduce (14) from the asymptotic behaviour of the normalized log-Laplace transform of $T^{(X)}(t)$, given by the following lemma. Recall that ψ is increasing on (m_0, m_+) .

Lemma 1. For $\theta \in (-\psi(m_+), -\psi(m_0))$, set $L_t(\theta) = \log \mathbb{E}_a \exp(\theta T^{(X)}(t))$. Then as $t \rightarrow \infty$ we have

$$\frac{1}{\log t} L_t(\theta) \rightarrow L(\theta) \quad (19)$$

where

$$L(\theta) = -m \iff \theta = -\psi(m). \quad (20)$$

The function L is differentiable on $(-\psi(m_+), -\psi(m_0))$ and satisfies $L'(\theta) = 1/\psi'(m)$, so that the range of L' is precisely Δ . Then the left-hand-side of (14) admits the limit $\mathcal{I}(x) = x\theta - L(\theta)$ where θ is the unique solution of $L'(\theta) = x$ i.e., thanks to (20), $\mathcal{I}(x) = -x\psi(m) + m$ which is exactly the right-hand-side of (14).

Proof. of Lemma 1

To compute the Laplace transform of $T^{(X)}(t)$, we use a Girsanov type change of probability.

For $m \in (m_-, m_+)$ let

$$\psi_m(\theta) = \psi(m + \theta) - \psi(m),$$

and let $\{(\xi_t, t \geq 0); \mathbb{P}^{(m)}\}$ be a Lévy process starting from 0 whose exponent is ψ_m (Esscher transform). Finally let $\{(X_t, t \geq 0); (\mathbb{Q}_a^m)_{a>0}\}$ be the associated pssMp.

Besides, from [10] the following relation between the infinitesimal generators L^ξ of ξ and L^X of X under \mathbb{Q}_a :

$$L^X f(x) = \frac{1}{x} L^\xi (f \circ \exp)(\log x),$$

implies (see (2.7) in [10]),

$$L^X f_m = \psi(m) f_{m-1}$$

for $f_m(x) = x^m$ so that

$$\mathbb{Q}_a^m |_{\mathcal{F}_t} = \left(\frac{X_t}{a} \right)^m \exp \left(-\psi(m) \int_0^t \frac{ds}{X_s} \right) \cdot \mathbb{Q}_a |_{\mathcal{F}_t}, \quad (t \geq 0). \quad (21)$$

We deduce that

$$\mathbb{E}_a \exp \left(-\psi(m) \int_0^t \frac{ds}{X_s} \right) = a^m \mathbb{E}_a^m (X_t)^{-m} \quad (22)$$

and, owing to the scaling property

$$\mathbb{E}_a \exp \left(-\psi(m) \int_0^t \frac{ds}{X_s} \right) = a^m t^{-m} \mathbb{E}_{a/t}^m (X_1)^{-m}. \quad (23)$$

Let us choose m such that $\psi'(m) > 0$.

We can now use the results on entrance boundary, for example [3]. Since the new Lévy process $\xi^{(m)}$ has a positive mean

$$\mathbb{E} \xi^{(m)} = \psi'_m(0) = \psi'(m) > 0$$

then Theorem 1 i) therein entails that

$$\lim_{t \uparrow \infty} \mathbb{E}_{a/t}^m (X_1)^{-m} = \mathbb{E}_0^m (X_1)^{-m}. \quad (24)$$

Lemma 2. *With the notations of Theorem 2, the quantity $F(m) := \mathbb{E}_0^m (X_1)^{-m}$ is finite for $m \in (m_0, m^+)$.*

From (23)

$$L_t(\theta) = -m \log t + m \log a + \log \mathbb{E}_{a/t}^m (X_1)^{-m}$$

and then, as $t \rightarrow \infty$, thanks to Lemma 2:

$$\frac{L_t(\theta)}{\log t} \rightarrow L(\theta) = -m. \quad (25)$$

This ends the proof of Lemma 1 and consequently the proof of part 1) in Theorem 2, up to the result of Lemma 2 whose proof is postponed.

2) The statement is a consequence of the Chernov inequality. Indeed for fixed $\theta \in (0, -\psi(m_0))$

$$\log \mathbb{Q}_a \left(\frac{T^{(X)}(t)}{\log t} > -A \right) \leq -\theta A + L(\theta)$$

for every $\theta \in (0, -\psi(m_0))$.

3) Let us assume that $\Delta = (0, \infty)$. To get a full LDP, it is enough to prove that

$$\lim_{A \rightarrow \infty} \lim_{t \rightarrow \infty} \frac{1}{\log t} \log \mathbb{P} \left(\frac{\tau(t)}{\log t} > A \right) = -\infty,$$

(exponential tightness) but it is a consequence of the Chernov inequality:

$$\log \mathbb{Q}_a \left(\frac{T^{(X)}(t)}{\log t} > -A \right) \leq -\theta A + L(\theta)$$

for every $\theta \in (0, -\psi(m_0))$.

Assume now that the condition (17) is satisfied. For every x in the complement of $\bar{\Delta}$ we have

$$\lim_{\delta \rightarrow 0} \lim_{t \rightarrow \infty} \frac{1}{\log t} \log \mathbb{Q}_a \left(\frac{T^{(X)}(t)}{\log t} \in [x - \delta, x + \delta] \right) = -\infty,$$

so that a weak LDP is satisfied on $[0, \infty)$. To strenghten it is enough to apply the above reasoning. \square

4.2 Remarks

Remark 1. (Reciprocal pairs) The relation (18) between two rate functions is known to hold for pairs of inverse processes. This problem arose historically when people deduced a LDP for renewal processes whose interarrival times have sums satisfying a LDP. It was extended to more general processes, see [13] and the bibliography therein.

If (ξ_t) is a spectrally negative Lévy process with Laplace exponent ψ , then $t^{-1}\xi_t$ satisfies the LDP at scale t with rate function ψ^* . The subordinator

$$\hat{\tau}(u) = \inf\{t > 0 : \xi_t > u\} \tag{26}$$

has a Laplace exponent which is exactly L defined by he relation (20)

$$\psi(m) = -\theta, L(\theta) = -m. \tag{27}$$

Then $t^{-1}\hat{\tau}(t)$ satisfies the LDP at scala t with rate function $x \rightarrow x\psi^*\left(\frac{1}{x}\right)$. This can be seen as an application of Theorem 1 of [13]).

The relation (6) between $\mathcal{A}_u^{(\xi)}$ and $\tau^{(\xi)}(t)$:

$$\tau^{(\xi)}(t) = \inf\{u \geq 0 : \mathcal{A}_u^{(\xi)} = \int_0^u e^{\xi s} ds \geq t\}, \quad (t \geq 0)$$

is more involved than (26). An alternative proof of our Theorem 2 about LDP for $(\log t)^{-1} \tau^{(\xi)}(t)$, in the spirit of Theorem 1 of [13]), would require an LDP for $t^{-1} \log \mathcal{A}_t^{(\xi)}$ which is far from obvious. At first glance, the lower bound of large deviations seems accessible since we have for convenient x and δ ,

$$(\xi_s/s \in [x - \delta, x + \delta] \forall s \in [0, u]) \Rightarrow \left(\mathcal{A}_u^{(\xi)} \in \left[\frac{e^{(x-\delta)u} - 1}{(x-\delta)}, \frac{e^{(x+\delta)u} - 1}{(x+\delta)} \right] \right).$$

Conversely, we can apply Theorem 1 of [13] to get

Proposition 2. *The family of distributions of $t^{-1} \log \mathcal{A}_t^{(\xi)}$ (under \mathbb{P}) satisfies the LDP at scale t , with good rate function ψ^* .*

To end this first remark, let us add two more comments:

- The statement of Lemma 1, viewed in terms of ξ can be rephrased as the striking relation

$$\lim_{t \rightarrow \infty} \frac{1}{\log t} \mathbb{E} \exp \theta \tau^{(\xi)}(t) = \log \mathbb{E} \exp \theta \hat{\tau}(1).$$

- The relation (27) is also well known in the study of exponential families (inverse families or reciprocal pairs), see [14] Section 5.4 and [18] Section 5A.

Remark 2. (Another approach for the LDP) We saw in a previous section that the core of the proof of the LLN is the convergence of the occupation measure

$$\frac{1}{t} \int_0^t \delta_{U_s} ds \rightarrow \mu$$

followed by a scaling and by the choose of $f(x) = x^{-1}$ as a test function. It could be natural to look for a LDP according the same line of reasoning. Such a proof would consist of three steps. First establish an LDP for the law of the occupation measure under \mathbb{Q}_0 . The rate functional is expressed with the help of the infinitesimal generator of U . Then apply the contraction by the (non-continuous) mapping $v \mapsto v(f)$, hence solve a variational problem. This would give an LDP for the law of

$$\frac{1}{t} \int_0^t \frac{ds}{U_s} = \frac{1}{t} \int_1^{e^t} \frac{ds}{X_s},$$

under \mathbb{Q}_0 . It would remain to convert it into an LDP under \mathbb{Q}_a . In the Appendix of [27], the authors give the first rate functional and solved the variational problem, but with the lack of justification for the contraction principle. The result fits with the rate function of Theorem 2.

Remark 3. (Functional LDP) In the Bessel clock, a functional LDP was stated ([27] Th. 4.1). Here it is possible to consider the same problem, i.e. the study of the LDP for the sequence of processes

$$u \in [0, 1] \mapsto \frac{1}{n} T^{(X)}(e^{nu})$$

under \mathbb{Q}_a . The rate function involves an action functional built on \mathcal{S} and a recession function which will be $x \mapsto rx$ where $r = -\psi(m_0)$ (when it is finite).

Remark 4. (Central Limit Theorem) It is known for the Bessel clock (Theorem 1.1 in [27]) but seems unknown otherwise. We can conjecture that

$$\sqrt{\log t} \left(\frac{T^{(X)}(t)}{\log t} - \frac{1}{\mathbb{E}\xi_1} \right) \Longrightarrow \mathcal{N}(0; \psi''(0)/\psi'(0)^3) \quad (28)$$

as soon as $\mathbb{E}(\xi_1)^2 < \infty$ (or $\psi''(0) < \infty$).

4.3 Proof of Lemma 2

From (10) applied to the Esscher transform we have for $m > m_0$

$$F(m) = \mathbb{E}^{(m)}(I_\infty^{m-1}).$$

The finiteness of $F(m)$ is a consequence of the following Lemma summarizing properties of the moments of the exponential functionals of Lévy processes which we detail for completeness.

Lemma 3. *Let ζ be a Lévy process of Laplace exponent φ given by*

$$\mathbb{E} \exp(\lambda \zeta_t) = \exp(t\varphi(\lambda))$$

such that $\varphi'(0) \in (0, \infty)$ and let

$$I_\infty = \int_0^\infty e^{-\zeta_s} ds.$$

Then

1)

$$\mathbb{E} I_\infty^s < \infty \text{ for all } s \in [-1, 0] \text{ and all } s > 0 : \varphi(-s) < 0. \quad (29)$$

2) *For $r > 0$, if $\varphi(r) < \infty$ and $\mathbb{E} I_\infty^{-r} < \infty$, then*

$$\mathbb{E} I_\infty^{-r-1} = \frac{\varphi(r)}{r} \mathbb{E} I_\infty^{-r}. \quad (30)$$

Proof. of Lemma 3

The statement 1) comes from a rephrased part of a lemma in Maulik and Zwart on the existence of moments of exponential functionals (see [20] Lemma 2.1).

The statement 2) comes from a recursive argument due to Bertoin and Yor (see [4], Theorem 2 (i) and Theorem 3), which we detail here for the sake of completeness. Set, for $t \geq 0$

$$J_t = \int_t^\infty e^{-\zeta_s} ds \quad (t \geq 0) ; J_0 = I_\infty = \int_0^\infty e^{-\zeta_s} ds$$

For all $r > 0$ we have

$$J_t^{-r} - J_0^{-r} = r \int_0^t e^{-\zeta_s} J_s^{-r-1} ds \quad (31)$$

Besides, from the properties of Lévy processes, we deduce

$$J_s = e^{-\zeta_s} \hat{I}(s)$$

and

$$\hat{I}(s) = \int_0^\infty e^{-(\zeta_{s+u} - \zeta_s)} du \stackrel{(law)}{=} I_\infty$$

with $\hat{I}(s)$ independent of J_s . Plugging into (31) we get

$$e^{r\zeta_t} \hat{I}(t)^{-r} - I_\infty^{-r} = r \int_0^t e^{r\zeta_s} \hat{I}(s)^{-r-1} ds \quad (32)$$

Assume that $\varphi(r) < \infty$ and $\mathbb{E}I^{-r-1} = \infty$, taking expectations on both sides of the inequality

$$e^{r\zeta_t} \hat{I}(t)^{-r} > r \int_0^t e^{r\zeta_s} \hat{I}(s)^{-r-1} ds$$

we would get $\mathbb{E}I^{-r} = \infty$. Consequently, if $\mathbb{E}I^{-r} < \infty$, we have $\mathbb{E}I^{-r-1} < \infty$. Moreover, taking expectations on both sides of (32) we get (30). \square

Now, to end the proof of Lemma 2, we apply Lemma 3 with $\varphi = \psi_m$.

For $m > 0$, we choose $s = m - 1$ hence $\varphi(-s) = \psi(1) - \psi(m)$ which is negative if $m > 1$. So from 1) of Lemma 3, $F(m)$ is finite for $m > 0$.

Now, if $m \in [m_0, 0]$, set $k = \lfloor -m \rfloor$. Then $m + k \in [-1, 0]$, and from 1) of Lemma 3, $\mathbb{E}^{(m)} I_\infty^{m+k}$ is finite. Besides, we have $\psi(m) < \psi(-j)$ for all integers j such that $0 \leq j \leq k$, hence $\varphi(-m - j) = \psi(-j) - \psi(m) > 0$ and applying (30) recursively, we have

$$F(m) = \frac{\varphi(-m)\varphi(-m-1)\cdots\varphi(-m-k)}{(-m)(-m-1)\cdots(-m-k)} \mathbb{E}^{(m)} I_\infty^{m+k} \quad (33)$$

and this quantity is finite. \square

5 Examples

For all the following examples, we give the Laplace exponent ψ and the parameters τ_+ , τ_e . In Sections 5.1, and 5.2 we obtain an explicit expression for τ_0 and \mathcal{I} . In Section 5.3 we do not have an explicit expression for \mathcal{I} (and sometimes for τ_0). For all examples (but one) we identify the behaviour at the boundary, according to the classification of Proposition 1.

5.1 The Brownian process with drift (Bessel clock [27])

For $\nu > 0$, we consider the Lévy process

$$\xi_t = 2B_t + 2\nu t.$$

The pssMp (X_t) is the squared Bessel process of dimension $d = 2(1 + \nu)$. Its index is $\alpha = 1$ and it is the only continuous pssMp (of index 1). The clock is related to the scale function of a diffusion in random environment (see [26] chap. 10).

We have

$$\psi(m) = 2m(m + \nu), \quad m_{\pm} = \pm\infty, \quad \psi(m_{\pm}) = +\infty.$$

Here $m_0 = -\nu/2$ and $\psi(m_0) = -\nu^2/4$, so that

$$\tau_+ = 0, \quad \tau_0 = \infty, \quad \tau_e = \frac{1}{2\nu}, \quad \Delta = (0, \infty).$$

The rate function is

$$\mathcal{I}(x) = \frac{(1 - 2\nu x)^2}{8x}, \quad (x > 0).$$

Notice that here the function L may be obtained by explicit inversion

$$L(\theta) = \frac{-\nu + \sqrt{\nu^2 - 2\theta}}{2}, \quad (\theta < \nu^2/2).$$

The boundary $\tau_+ = 0$ is in the situation 4c) of Proposition 1 and the boundary τ_0 is in situation 3a, with asymptote $y = \frac{\nu^2}{2}x - \frac{\nu}{2}$. The minimum of \mathcal{I} , reached in τ_e , corresponds to the LLN (Theorem 1) which is in [23] Exercise (4.23) Chap. IV and in [27] Theorem 1.1.

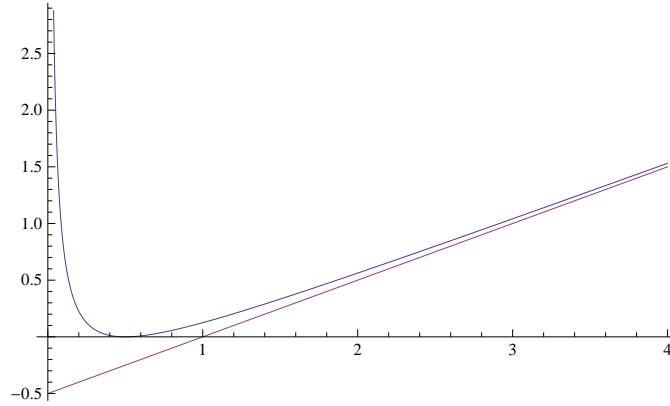


Fig. 1 Example 5.1: The rate function \mathcal{S} for $\nu = 1$.

Let us now comment the change of probability. Under \mathbb{P}^m , the processes (ξ_t) and (X_t) remain in the same family as under \mathbb{P} : they are Brownian motion with drift and squared Bessel, respectively, but with ν replaced by $\nu + 2m$. Moreover, it is known from [26] Chap. 8 (or equivalently [7]), that under \mathbb{P} ,

$$I_\infty \stackrel{(law)}{=} (2Z_\nu)^{-1}$$

where Z_ν is a gamma variable of parameter ν . This allows to see directly that the assertion of Lemma 2 holds.

5.2 Lévy processes of the form $\xi_t = \pm dt \pm \text{Pois}(\beta, \gamma)_t$

Here $\text{Pois}(a, b)_t$ is the compound Poisson process of parameter a whose jumps are exponential r.v. of parameter b . They are studied in Section 8.4.3 of [26] (or equivalently [7]), with very informative Tables. In particular, these families are invariant by changes of probability and the law of the exponential functional I_∞ is known, which allows again to see directly that the assertion of Lemma 2 holds.

5.2.1 $\xi_t = dt + \text{Pois}(\beta, \gamma)_t$ with $d \geq 0$

(The particular case of the compound Poisson process corresponds to the case $d = 0$). We have

$$\psi(m) = m \left(d + \frac{\beta}{\gamma - m} \right), \quad m_- = -\infty, \quad m_+ = \gamma, \quad \psi(m_+) = \infty,$$

Here $m_0 = -\infty$ and $\psi(m_0) = -\infty$ so that

$$\tau_+ = 0, \tau_0 = d^{-1}, \tau_e = \frac{\gamma}{\gamma d + \beta}, \Delta = (0, d^{-1}).$$

The rate function is

$$\mathcal{I}(x) = \left(\sqrt{\gamma(1-dx)} - \sqrt{\beta x} \right)^2, \quad (0 \leq x \leq d^{-1}).$$

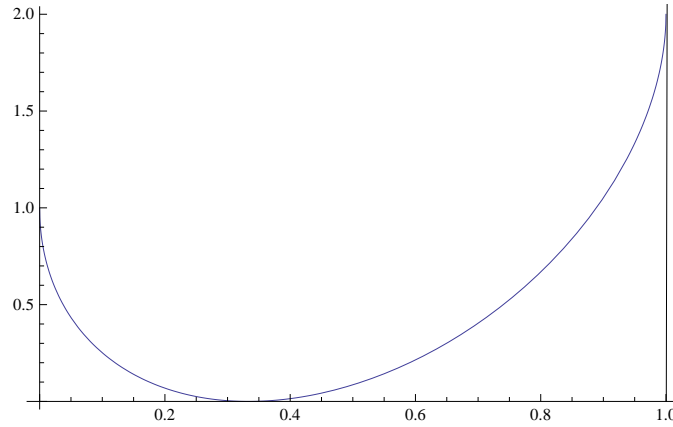


Fig. 2 Example 5.2.1: The rate function \mathcal{I} for $\beta = 2, \gamma = 1, d = 1$.

To check the assumption (17) we go back to the process ξ . Notice that $\xi_s \geq ds$ for every $s \geq 0$, so that $\tau^{(\xi)}(t) \leq d^{-1} \log(1+dt)$, whence, for every $\varepsilon > 0$

$$\mathbb{P} \left(\frac{\tau^{(\xi)}(t)}{\log t} > \tau_0 + \varepsilon \right) = 0$$

for t large enough, which is equivalent to (17) thanks to (7).

The boundary τ_+ is in situation 4a and τ_0 in situation 3b, i.e. vertical tangents.

In the following examples, with no loss of generality we may and will assume that $d = 1$, since (with obvious notations)

$$\tau^{(\xi;d,\beta)}(t) \stackrel{(law)}{=} d^{-1} \tau^{(\xi;1,\beta/d)}(dt).$$

5.2.2 $\xi_t = -t + \text{Pois}(\beta, \gamma)_t$ with $0 < \gamma < \beta$

We have

$$\psi(m) = m \left(-1 + \frac{\beta}{\gamma - m} \right), m_- = -\infty, m_+ = \gamma, \psi(m_+) = \infty.$$

Here

$$m_0 = \gamma - \sqrt{\beta\gamma}, \psi(m_0) = -(\sqrt{\gamma} - \sqrt{\beta})^2,$$

so that

$$\tau_+ = 0, \tau_0 = \infty, \tau_e = \frac{\gamma}{\beta - \gamma}, \Delta = (0, \infty).$$

The rate function is

$$\mathcal{I}(x) = \left(\sqrt{\gamma(1+x)} - \sqrt{\beta x} \right)^2 \quad (x > 0).$$

The boundary τ_+ is in situation 4a and τ_0 in situation 3a with asymptote $y = (\sqrt{\beta} - \sqrt{\gamma})^2 x + \gamma - \sqrt{\beta\gamma}$.

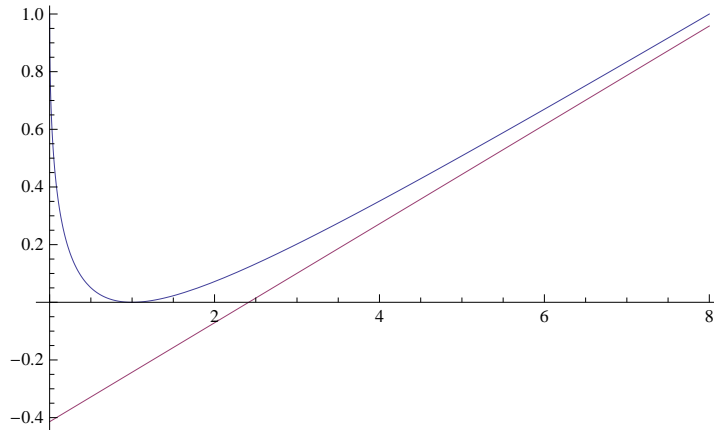


Fig. 3 Example 5.2.2: The rate function \mathcal{I} for $\beta = 2, \gamma = 1$.

5.2.3 The saw-tooth process

It is a particular case of the Cramer-Lundberg risk process ([15] Section 1.3.1). $0 < \beta < \gamma$. The Lévy process ξ is defined by

$$\xi_t = t - \text{Pois}(\beta, \gamma)_t. \quad (34)$$

The self-similar process X_t is described precisely in ([9] p. 327).

The Lévy exponent of (ξ_t) is given by:

$$\psi(m) := m \frac{\gamma - \beta + m}{\gamma + m}, \quad m_- = -\gamma, \quad m_+ = \infty, \quad \psi(m_+) = \infty, \quad \psi'(m_+) = 1. \quad (35)$$

We have easily

$$m_0 = -\gamma + \sqrt{\beta\gamma}, \quad \psi(m_0) = -\left(\sqrt{\gamma} - \sqrt{\beta}\right)^2$$

so that

$$\tau_+ = 1, \quad \tau_0 = \infty, \quad \tau_e = \frac{\gamma}{\gamma - \beta}, \quad \Delta = (1, \infty).$$

The rate function is

$$\mathcal{I}(x) = \left(\sqrt{\gamma(x-1)} - \sqrt{\beta x}\right)^2, \quad (x \geq 1).$$

The condition (17) is fulfilled since by definition $\xi_t \leq t$ for all $t > 0$, so we have $\tau^{(\xi)}(t) \geq \log t$ for all $t > 0$. The minimum of the rate function \mathcal{I} is reached at $x = \frac{\gamma}{\gamma - \beta}$ in accordance with the law of large numbers ([9], Theorem 4.7 ii)).

The boundary $\tau_+ = 1$ is in situation 4b and the boundary τ_0 in situation 3a, with an asymptote of equation $y = (\sqrt{\gamma} - \sqrt{\beta})^2 x + \sqrt{\beta\gamma} - \gamma$.

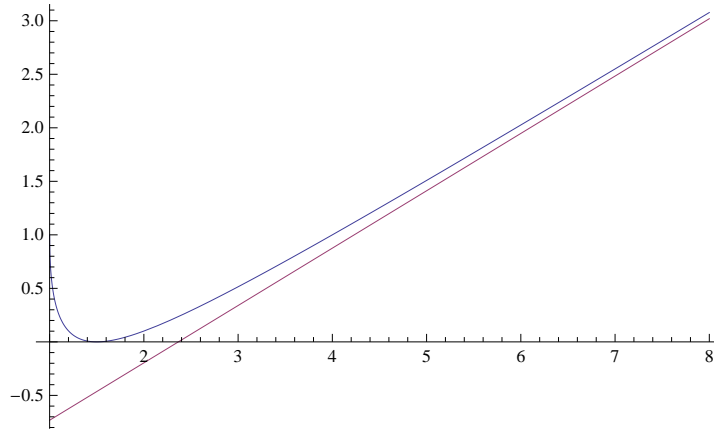


Fig. 4 Example 5.2.3: The rate function \mathcal{I} for $\beta = 1, \gamma = 3$.

5.3 Three more examples

The first and the second may be seen for instance in [21]. The third one is a general class (see [16]).

5.3.1 First

For $\alpha \in (1, 2)$, let X^\uparrow be the spectrally negative regular α -stable Lévy process conditioned to stay positive. It is a pssMp of index α and its associated Lévy process ξ^\uparrow has Laplace exponent

$$\psi(m) = c \frac{\Gamma(m + \alpha)}{\Gamma(m)}$$

and

$$m_- = -\alpha, \quad m_+ = \infty, \quad \psi(m_+) = \infty, \quad \psi'(m_+) = \infty$$

where c is a positive constant. We have

$$\frac{\psi'(m)}{\psi(m)} = \Psi(m + \alpha) - \Psi(m)$$

where Ψ is the Digamma function. Then $m_0 = \gamma_\alpha$ where γ_α is the (unique) solution in $(-1, 0)$ of the equation

$$\Psi(\gamma + \alpha) = \Psi(\gamma),$$

and then $\psi(\gamma_\alpha) = c \frac{\Gamma(\gamma_\alpha + \alpha)}{\Gamma(\gamma_\alpha)} < 0$. We have

$$\tau_+ = 0, \quad \tau_0 = \infty, \quad \tau_e = \frac{1}{c\Gamma(\alpha)}, \quad \Delta = (0, \infty).$$

The boundary τ_+ is in situation 4c and τ_0 in situation 3a with asymptote.

Notice that when $\alpha \uparrow 2$ we end up with the Laplace exponent of a Brownian motion with drift $1/2$.

5.3.2 Second

Let $\kappa \in (0, 1]$ and $\delta > \kappa/(1 + \kappa)$. Let X be the continuous state branching process with immigration whose branching mechanism (see [21] Lemma 4.8.) is

$$\varphi(u) = -\frac{c}{\kappa} u^{\kappa+1}$$

and immigration mechanism is

$$\chi(u) = c\delta \frac{\kappa+1}{\kappa} u^\kappa.$$

It is a pssMp of index κ and the associated Lévy process has Laplace exponent

$$\psi(m) = c(\kappa - (\kappa + 1)\delta - m) \frac{\Gamma(-m + \kappa)}{\Gamma(-m)}, \quad m_- = -\infty, \quad m_+ = \kappa, \quad \psi(m_+) = \infty.$$

First, we see that $\psi'(0) = \lim \psi(m)/m = c((\kappa + 1)\delta - \kappa)\Gamma(\kappa) > 0$ and $\tau_e = 1/\psi'(0)$. Since as $m \rightarrow -\infty$ we have $\psi(m) \sim c(-m)^\kappa \rightarrow \infty$, we deduce the existence of $m_0 > -\infty$ such that $\psi'(m_0) = 0$ (and $\psi(m_0) < 0$). Besides, when $m \uparrow \kappa$ or $h = \kappa - m \downarrow 0$ then

$$\psi'(m) \sim \frac{(\kappa + 1)\delta\Gamma'(h)}{\Gamma(-\kappa)} \uparrow \infty.$$

We have then

$$\tau_+ = 0, \tau_0 = \infty, \Delta = (0, \infty).$$

The boundary $\tau_+ = 0$ is in situation 4c and τ_0 is in situation 3a with asymptote.

5.3.3 Hypergeometric-stable process

The modulus of a Cauchy process in \mathbb{R}^d for $d > 1$ is a pssMp of index 1 with infinite lifetime. Actually the associated Lévy process is a particular case of hypergeometric-stable process of index α as defined in [5], with $\alpha < d$.

The characteristic exponent is given therein by Theorem 7, hence the Laplace exponent is :

$$\psi(m) = -2^\alpha \frac{\Gamma((-m + \alpha)/2)}{\Gamma(-m/2)} \frac{\Gamma((m + d)/2)}{\Gamma((m + d - \alpha)/2)}$$

and

$$m_- = -d, m_+ = \alpha, \psi(m_+) = \infty.$$

Since $m\Gamma(-m/2) = -2\Gamma((2 - m)/2)$ we have

$$\psi'(0) = \lim_{m \rightarrow 0} \psi(m)/m = 2^{\alpha-1} \frac{\Gamma(\alpha/2)\Gamma(d/2)}{\Gamma((d - \alpha)/2)} > 0.$$

Moreover, for $m \neq 0$

$$\frac{\psi'(m)}{\psi(m)} = -\Psi((-m + \alpha)/2) + \Psi((m + d)/2) + \Psi(-m/2) - \Psi(m + d - \alpha)/2),$$

where Ψ is the digamma function.

It is then easy to see that ψ' vanishes at $m = m_0 := (\alpha - d)/2$ and

$$\psi(m_0) = -2^\alpha \left(\frac{\Gamma((d + \alpha)/4)}{\Gamma((d - \alpha)/4)} \right)^2.$$

We have then

$$\tau_+ = 0, \tau_0 = \infty, \Delta = (0, \infty).$$

In the particular case $\alpha = 1$ and $d = 3$, applying the identity

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}$$

we have

$$\psi(m) = (m+1) \tan \frac{\pi m}{2} \quad (36)$$

with $m_- = -3$, $m_+ = 1$ and $\psi(-1) = -2/\pi$ by continuity. A direct computation gives

$$\psi'(m) = \frac{\sin \pi m + \pi m + \pi}{2 \cos^2 \frac{\pi m}{2}}$$

so that

$$\psi'(0) = \pi/2, \quad m_0 = -1, \quad \theta_0 = -\psi(m_0) = 2/\pi \text{ and } \Delta = (0, \infty)$$

We have no close expression for the rate function. The asymptote as $x \rightarrow \infty$ is the line $y = \frac{2x}{\pi} + 1$.

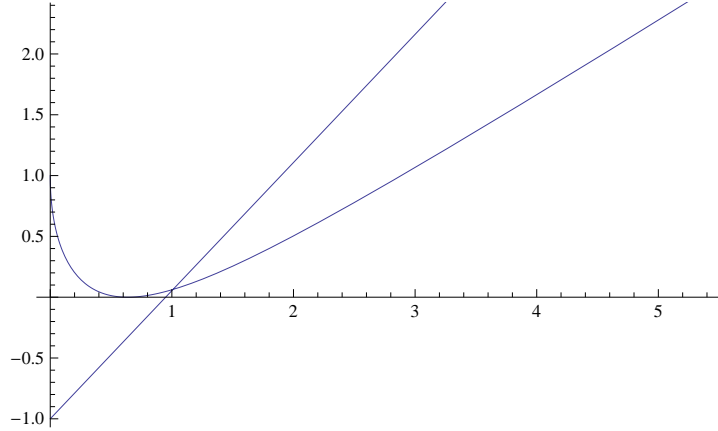


Fig. 5 Example 5.3.3: The rate function $\mathcal{S}(x)$ for $\alpha = 1$, $d = 3$.

Notice that in [10], the authors announced a study of the modulus of a multidimensional Cauchy process. They found the expression (36) but never published it ([22]).

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