# Inverse problem on a tree-shaped network

Lucie Baudouin, Masahiro Yamamoto

# Inverse problem on a tree-shaped network: unified approach for uniqueness.

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**Abstract**

In this article, we prove uniqueness results for coefficient inverse problems regarding wave, heat or Schrödinger equation on a tree-shaped network, as well as the corresponding stability result of the inverse problem for the wave equation. The objective is the determination of the potential on each edge of the network from the additional measurement of the solution at all but one external end points. Several results have already been obtained in this precise setting or in similar cases, and our main goal is to propose a unified and simpler method of proof of some of these results. The idea which we will develop for proving the uniqueness is to use a more traditional approach in coefficient inverse problems by Carleman estimates. Afterwards, using an observability estimate on the whole network, we apply a compactness-uniqueness argument and prove the stability for the wave inverse problem.

**Keywords:** networks, inverse problem, Carleman estimate.

**AMS subject classifications:** 35R30, 93C20, 34B45

## 1 Introduction and main results

Systems known as *multi-link structures* have large applicative relevance and their mathematical study attracts a growing interest. The modeling of the evolution of these structures is generally quite complex and can be often described with systems of partial differential equations on networks or graphs. As large as the literature on this subject can be (see e.g. [12] and the references therein), from a mathematical and scientific point of view, there are still many necessary works needed to complete the theory. Regarding more specific topics such as the control and inverse problems on networks, a full theory cannot be developed by simply superposing the existing results for partial differential equations on one-dimensional domains. Indeed, the interaction between the different components of a multi-link structure may generate unexpected phenomena, as [12] pointed out mainly for vibrations on networks.

Concerning other operators on metric graphs, they can arise in many applications, e.g., chemistry or engineering, and we refer to high-temperature granular superconductors [1] as one example. One can look deeper in the review papers [13], [22] (or even in the introduction of [15]) for more information on this topic.

Besides, inverse problems have been studied only recently on networks, and we refer to the articles [7] and [2] for uniqueness results with the boundary control approach, or in [15] and [3] for uniqueness and stability results with Carleman estimates.

Here we discuss the same type of inverse problems for the wave and the Schrödinger equations as in [15] and [3], while for the heat equation, our inverse problem concerns the initial-boundary value problem and is different from the case of the heat equation treated in [15] where initial values are also unknown in the inverse problem. Our purpose is to propose a unified and simpler method to prove some of these results by using previously known results for the inverse problem

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of determining a coefficient.

Let us now give the detailed mathematical framework within which we will work. Let $\Lambda$ be a tree-shaped network composed of $N+1$ open segments $(e_j)_{j=0,1,...,N}$ of length $\ell_j$, linked by $N_1$ internal node points belonging to the set $\Pi_1$ and let us denote by $\Pi_2$ the set of $N_2$ exterior end points where only one segment starts. By a "tree-shaped network", we mean that $\Lambda$ does not contain any closed loops, and and one can see for instance the network pictured in Figure 1. Moreover we assume that at each interior node point, at least three segments $e_j$ meet.

We define any function on the network $f : \Lambda \to \mathbb{R}$ by $f = (f_0, ..., f_N)$, where $f_j = f|_{e_j}$ denotes the restriction of $f$ to the edge $e_j$. Moreover, for any internal node $P \in \Pi_1$ where $n_P$ segments, say $e_1, ..., e_{n_P}$, meet, we set

$$[f]_P := \sum_{j=1}^{n_P} f_j(P).$$

We consider on this plane 1-d tree-shaped network $\Lambda$ either wave or heat or even Schrödinger equations, with a different potential term $x \mapsto p_j(x)$ in $L^\infty(e_j)$ on each segment.

Our first system of interest is the following 1-d wave equation on the network $\Lambda$:

$$\begin{cases} \partial_t^2 u_j - \partial_x^2 u_j + p_j(x) u_j = 0 & \forall j \in \{0, 1, ..., N\}, (x,t) \in e_j \times (0,T), \\ u(Q,t) = h(t), & \forall Q \in \Pi_2, t \in (0,T), \\ u(x,0) = u^0(x), \quad \partial_t u(x,0) = u^1(x), & x \in \Lambda, \end{cases} \tag{1}$$

assuming some compatibility condition between the boundary and initial data. Moreover we assume the continuity and what is called the Kirchhoff law at any internal node $P \in \Pi_1$, which are given by

$$u_j(P,t) = u_k(P,t) =: u(P,t), \quad \forall j,k \in \{1, ..., n_P\}, \, 0 < t < T, \tag{2}$$

$$[u_x(t)]_P := \sum_{j=1}^{n_P} \partial_{n_e} u_j(P,t) = 0, \quad 0 < t < T. \tag{3}$$

Here and henceforth we choose an orientation of $\Lambda$ such that to two endpoints of each segment $e$, correspond an initial node $I(e)$ and a terminal node $T(e)$, and further define the outward normal derivative $\partial_{n_e} u_j$ at a node $P$ of $e_j$ by

$$\partial_{n_e} u_j(P,t) = \begin{cases} -\partial_x u_j(P,T), & \text{if } P \in I(e_j), \\ \partial_x u_j(P,T), & \text{if } P \in T(e_j). \end{cases}$$

Since one can prove the unique existence of solution to (1) - (3) in a suitable function space (from Lions and Magenes [26] or more explicitly in [23, Chapter 3], and see also Lemma 1 below), we denote the solution by $u[p](x,t)$, and we set $u[p] = (u[p]_0, ..., u[p]_N)$.

Moreover we consider the following heat system on the same network $\Lambda$

$$\begin{cases} \partial_t u_j - \partial_x^2 u_j + p_j(x) u_j = 0 & \forall j \in \{0, 1, ..., N\}, \forall (x,t) \in e_j \times (0,T), \\ \partial_x u(Q,t) = 0, & \forall Q \in \Pi_2, \forall t \in (0,T), \\ u(x,0) = u^0(x), & \forall x \in \Lambda, \end{cases} \tag{4}$$

and the Schrödinger system on the network $\Lambda$

$$\begin{cases} i\partial_t u_j - \partial_x^2 u_j + p_j(x) u_j = 0 & \forall j \in \{0, 1, ..., N\}, \forall (x,t) \in e_j \times (0,T), \\ u(Q,t) = h(t), & \forall Q \in \Pi_2, \forall t \in (0,T), \\ u(x,0) = u^0(x), & \forall x \in \Lambda, \end{cases} \tag{5}$$

both under the same node conditions (2) and (3). Here and henceforth we set $i = \sqrt{-1}$. If there is no possible confusion, by the same notation $u[p]$ we denote the solution to (4) or (5) under (2) and (3). Again the uniqueness, existence and regularity of solutions to these systems can be deduced by a standard method (see e.g. [26]).

This article focuses on results on networks concerning the following question.

**Inverse Problem**: Is it possible to retrieve the potential $p$ everywhere in the whole network $\Lambda$ from measurements at all external nodes except one?

We will address the following two fundamental theoretical questions concerning this coefficient inverse problem. Let $Q_{N_2} \in \Pi_2$ be arbitrarily fixed.

**Uniqueness**: Do the equalities of the measurements $\partial_x u[p](Q,t) = \partial_x u[q](Q,t)$ for all $t \in (0,T)$ and $Q \in \Pi_2 \setminus \{Q_{N_2}\}$ imply $p = q$ on $\Lambda$?

**Stability**: Can we estimate, in appropriate norms, the difference of two potentials $p - q$ on $\Lambda$ by the difference of the corresponding measurements $\partial_x u[p](Q,t) - \partial_x u[q](Q,t)$ for all $t \in (0,T)$ and $Q \in \Pi_2 \setminus \{Q_{N_2}\}$ ?

As it will be clear when reading the proof, we cannot expect the uniqueness by data $\partial_x u[p](Q,t)$ at external nodes $Q$ except more than one.

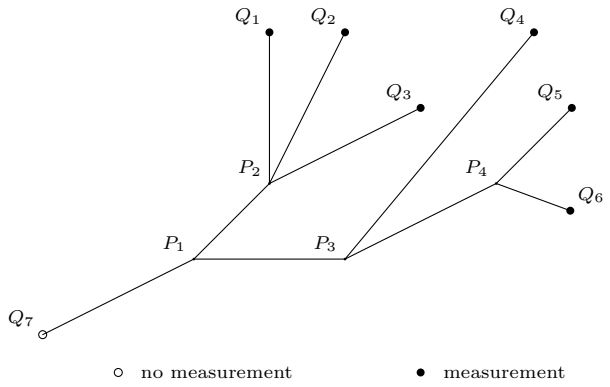


Figure 1: A star-shaped network with 10 edges ($N = 9$, $N_1 = 4$, $N_2 = 7$).

This inverse problem is nonlinear and we will give here the proof of the uniqueness of the solution with an argument which do not use a global Carleman estimate. Very recent papers on coefficient inverse problems on networks, as Baudouin, Crépeau and Valein [3] for the wave equation, and Ignat, Pazoto and Rosier [15] for the heat and the Schrödinger equations, give indeed the *stability* and therefore the *uniqueness* in the determination of the potential from boundary measurements using appropriate global Carleman estimates.

For the wave, the heat and the Schrödinger equations, our method yields the same uniqueness results as in the existing works, and we basically want to give a technically different approach of the same questions. However, our method is widely applicable to other equations such as a system of beam equations on networks and can yields novel uniqueness results. The question of the proof of the Lipschitz stability in the case of the wave equation will be addressed afterwards, using a compacteness-uniqueness argument, and relies on the observability estimate on the whole network which was already proved in the literature in several situations. However, this Lipschitz stability result given in Theorem 4 will require more regularity of the solutions than [3], as a consequence of the use of the compactness-uniqueness argument.

Concerning the precise topic we are considering, the bibliography lies in two different domains, namely coefficient inverse problems for partial differential equation on the one hand and control and stabilization in networks on the other hand.

Actually, as the first answer to the uniqueness for a coefficient inverse problem with a single measurement, we refer to Bukhgeim and Klibanov [9]. Later on, the works of Klibanov [19] and Yamamoto [33] are also important references related to our topic. Here we do not intend to give an exhaustive list of references but still, we can mention a recent comprehensive survey [20]. After the proof of uniqueness using the basic 1-d result on the basis of local Carleman estimates, the idea beneath this article is to take advantage of an observability estimate to obtain the Lipschitz stability of the inverse problem with a compactness-uniqueness argument (see [33]). We can refer to Isakov [18] which addresses other techniques linked to the study of inverse problems for several partial differential equations.

Nowadays, many results on the stability of inverse problems are derived directly from global Carleman estimates (e.g., [3] and [15]) for networks, or also in [5] for the related case of a hyperbolic transmission equation. We should underline that this "global" method goes back to [16] and [17] for the wave equation. One should also know that studies on inverse problems and controllability of partial differential equations share some technical materials such as Carleman estimates and observability inequalities. In the particular network setting, we would like to make use of classical results such as well-known 1-d local Carleman estimates and observability estimates on the network, borrowed from control studies, in order to obtain first uniqueness and then stability results, using a compactness-uniqueness argument (e.g. [34], [29]).

Besides, the control, observation and stabilization problems of networks have been objects of recent and intensive researches such as e.g., Dáger and Zuazua [12], Lagnese, Leugering and Schmidt [23], Zuazua [37]. More specifically, the control being only applied at one single end of the network, the articles Dáger [10], Dáger and Zuazua [11, 12] prove controllability results for the wave equation on networks, using observability inequalities under assumptions about the irrationality properties of the ratios of the lengths of the strings. We can also underline that many results of controllability on networks concern only the wave equation without lower order terms (see [23] and [31] for instance). However, it is difficult to consider such measurements at a more limited number of nodes for the inverse problem and we do not consider the measurements at less external nodes than $\Pi_2 \setminus \{Q_{N_2}\}$.

In the sequel, we shall use the following notations:

$$\begin{aligned} L^\gamma(\Lambda) &= \{f;\, f_j \in L^\gamma(e_j),\, \forall j \in \{0,1,...,N\}\}, \quad \gamma \geq 1, \\ H_0^1(\Lambda) &= \Big\{f;\, f_j \in H^1(e_j),\, \forall j \in \{0,1,...,N\},\, f_j(P) = f_k(P) \text{ if } e_j \text{ and } e_k \text{ meet at } P, \\ &\quad \forall P \in \Pi_1, \text{ and } f(Q) = 0,\, \forall Q \in \Pi_2\Big\}. \end{aligned}$$

For shortness, for $f \in L^1(\Lambda)$, we often write,

$$\int_\Lambda f dx = \sum_{j=0}^N \int_{e_j} f_j(x) dx,$$

where the integral on $e_j$ is oriented from $I(e_j)$ to $T(e_j)$. Then the norms of the Hilbert spaces $L^2(\Lambda)$ and $H_0^1(\Lambda)$ are defined by

$$\|f\|^2_{L^2(\Lambda)} = \int_\Lambda |f|^2\, dx \text{ and } \|f\|^2_{H_0^1(\Lambda)} = \int_\Lambda |\partial_x f|^2\, dx.$$

We are ready to state our first main result:

**Theorem 1 (Uniqueness)** *Let $r > 0$ be an arbitrary constant. Assume that $p, q \in L^\infty(\Lambda)$ and the initial value $u^0$ satisfies*

$$|u^0(x)| \geq r > 0, \quad a.e.\ in\ \Lambda.$$

*Assume further that the solutions $u[p], u[q]$ of (1)-(2)-(3) belong to*

$$C^1([0,T]; H^1(\Lambda)) \cap C^2([0,T]; L^2(\Lambda)).$$

*Then there exists sufficiently large $T_0 > 0$ such that for all $T \geq T_0$, if*

$$\partial_x u[p](Q,t) = \partial_x u[q](Q,t) \quad \text{for each } t \in (0,T) \text{ and } Q \in \Pi_2 \setminus \{Q_{N_2}\},$$

*then we have $p = q$ in $\Lambda$.*

The proof of this result in Section 2 relies on the uniqueness for the determination of potential in the one-dimensional wave equation and a "removing" argument.

Our argument gives the uniqueness for the inverse problems of determination of potentials on tree-shaped networks also for the heat and the Schrödinger equations using only measurements at $N_2 - 1$ exterior end points. In fact, our arguments in proving the uniqueness for the wave and the Schrödinger equations are essentially the same and are based on local Carleman estimates, while the uniqueness for the inverse heat problem is reduced to the uniqueness for the corresponding inverse wave problem (in a sense to be detailed later).

**Theorem 2 (Uniqueness for the heat inverse problem)** *Assume that $p, q \in L^\infty(\Lambda)$, the initial value $u^0$ satisfies*

$$|u^0(x)| \geq r > 0, \quad a.e. \text{ in } \Lambda$$

*for some constant $r$, and the solutions $u[p]$ and $u[q]$ to (4)-(2)-(3) satisfy*

$$\partial_t^j u[p], \partial_t^j u[q] \in H^1(0,T;L^2(\Lambda)) \cap L^2(0,T;H^2(\Lambda)), \quad j = 0, 1.$$

*Then there exists $T > 0$ such that if*

$$u[p](Q,t) = u[q](Q,t) \quad \text{for each } t \in (0,T) \text{ and } Q \in \Pi_2 \setminus \{Q_{N_2}\},$$

*then we have $p = q$ in $\Lambda$.*

We note that our inverse heat problem is an inverse problem for the initial-boundary value problem and is different from Ignat, Pazoto and Rosier [15] where an initial value is known but value at positive time is given. In general dimensions, the uniqueness for the inverse problem of determining coefficients in a parabolic equation by overdetermining boundary data on arbitrary subboundary is an open problem if we are not given $u(x, t_0)$ over the whole domain at $t_0 > 0$. In other words, even the uniqueness the inverse heat problem for the classical initial-boundary value problem is not solved except for the measurement subboundary is a sufficienly large part of the whole boundary, because one cannot have relevant Carleman estimates for the heat equation for $t \geq 0$.

**Theorem 3 (Uniqueness for the Schrödinger inverse problem)** *Assume that $p, q \in L^\infty(\Lambda)$, the initial value $u^0$ satisfies*

$$|u^0(x)| \geq r > 0, \quad a.e. \text{ in } \Lambda$$

*for some constant $r$, and the solutions $u[p]$ and $u[q]$ to (5)-(2)-(3) satisfy*

$$\partial_t^j u[p], \partial_t^j u[q] \in H^1(0,T;L^2(\Lambda)) \cap L^2(0,T;H^2(\Lambda)), \quad j = 0, 1.$$

*Then there exists $T > 0$ such that*

$$\partial_x u[p](Q,t) = \partial_x u[q](Q,t) \quad \text{for each } t \in (0,T) \text{ and } Q \in \Pi_2 \setminus \{Q_{N_2}\},$$

*then we have $p = q$ in $\Lambda$.*

In Theorems 2 and 3, we may be able to relax the regularity condition for $u[p]$ and $u[q]$, but we do not want to go into further details in this article.

One can refer to [3] for the same inverse problem in the wave equation on a network where the proof is detailed in a star-shaped network but is actually generalizable to tree-shaped networks. The paper [15] treats the Schrödinger case in a star-shaped network and needs measurements at all external nodes. We do not know any uniqueness result for non-tree graphs, which are graphs containing a closed cycle.

For $M \geq 0$, we introduce the set

$$L^\infty_M(\Lambda) = \left\{ q = (q_0, ..., q_N);\ q_j \in L^\infty(e_j),\ \forall j \in \{0, 1, ..., N\} \text{ such that } \|q\|_{L^\infty(\Lambda)} \leq M \right\}.$$

For the inverse problem in the wave equation case, we state

**Theorem 4 (Stability)** *Let $M > 0$, $r > 0$ and $K > 0$ be arbitrarily fixed constants. Assume that $p \in L^\infty_M(\Lambda)$ and the solutions $u[p]$ and $u[q]$ to (1)-(2)-(3) satisfy $u[p], u[q] \in C^1([0,T];H^1(\Lambda)) \cap C^2([0,T];L^\infty(\Lambda))$ with*

$$\|u\|_{C^1([0,T];H^1(\Lambda)) \cap C^2([0,T];L^\infty(\Lambda))} \leq K. \tag{6}$$

*Assume also that the initial data $u^0$ satisfies*

$$|u^0(x)| \geq r > 0, \quad a.e. \text{ in } \Lambda.$$

*Then there exists sufficiently large $T_0 > 0$ such that for all $T \geq T_0$, there exists a constant $C = C(T, M, r, K, \Lambda) > 0$ such that*

$$||q - p||_{L^2(\Lambda)} \leq C \sum_{j=1}^{N_2 - 1} \|\partial_x u[p]_j(Q_j) - \partial_x u[q]_j(Q_j)\|_{H^1(0,T)}. \tag{7}$$

One can notice that for our proof based on a compactness-uniqueness argument, we assume a little more regularity than in Theorem 1.

This paper is composed of five sections. The proof of uniqueness in the inverse problem in the wave equation case (Theorem 1) is presented in Section 2. Section 3 is devoted to the proofs of Theorems 2 and 3. Theorem 4 is finally proven in Section 5 by a compactness-uniqueness argument and an observability estimate on the whole network.

We conclude this section with a classical result on the existence and regularity of solutions of the wave system and provide the corresponding energy estimates for the solution which we will need later.

**Lemma 1** *Let $\Lambda$ be a tree-shaped network and assume that $p \in L^\infty_M(\Lambda)$, $g \in L^1(0,T;L^2(\Lambda))$, $u^0 \in H^1_0(\Lambda)$ and $u^1 \in L^2(\Lambda)$. We consider the 1-d wave equation on the network with the conditions (2) and (3):*

$$\begin{cases} \partial_t^2 u - \partial_x^2 u + p(x)u = g(x,t), & \text{in } \Lambda \times (0,T), \\ u(Q,t) = 0, & \text{in } (0,T),\ Q \in \Pi_2, \\ u_j(P,t) = u_k(P,t), & \text{in } (0,T),\ P \in \Pi_1,\ j,k \in \{1,...,n_P\}, \\ [\partial_x u(t)]_P = 0, & \text{in } (0,T),\ P \in \Pi_1, \\ u(0) = u^0, \quad \partial_t u(0) = u^1, & \text{in } \Lambda. \end{cases} \tag{8}$$

*The Cauchy problem is well-posed and equation (8) admits a unique weak solution*

$$u \in C([0,T];H^1_0(\Lambda)) \cap C^1([0,T];L^2(\Lambda)).$$

*Moreover there exists a constant $C = C(\Lambda,T,M) > 0$ such that for all $t \in (0,T)$, the energy*

$$E(t) = ||\partial_t u(t)||^2_{L^2(\Lambda)} + ||\partial_x u(t)||^2_{L^2(\Lambda)}, \quad 0 \le t \le T$$

*of the system (8) satisfies*

$$E(t) \le C\left(||u^0||^2_{H^1_0(\Lambda)} + ||u^1||^2_{L^2(\Lambda)} + \|g\|^2_{L^1(0,T,L^2(\Lambda))}\right), \quad 0 \le t \le T \tag{9}$$

*and we also have the following trace estimate*

$$\sum_{j=1}^{N_2} \|\partial_x u_j(Q_j)\|^2_{L^2(0,T)} \le C\left(||u^0||^2_{H^1_0(\Lambda)} + ||u^1||^2_{L^2(\Lambda)} + ||g||^2_{L^1(0,T,L^2(\Lambda))}\right). \tag{10}$$

The proof of the unique existence of solution to equation (8) can be deduced from [26], vol.1, Chapter 3 or [23, Chapter 3].

Estimate (9) is a classical result which can be formally obtained by multiplying the main equation in (8) by $\partial_t u_j$, summing up for $j \in \{0,...,N\}$ the integral of this equality on $(0,T) \times e_j$ and using some integrations by parts. Estimate (10) is a hidden regularity result which can be obtained by multipliers technique (we refer to [25, Chapter 1]). Formally, for the particular case of a star-shaped network of vertex $P = 0$ for example, it comes from the multiplication of (8) by $m(x)\partial_x u_j$, where $m \in C^1(\bar{\Lambda})$ with $m(0) = 0$ and $m_j(l_j) = 1$, summing up the integrals of this equality on $(0,T) \times (0,l_j)$ over $j \in \{0,...,N\}$ and using integrations by parts. Here we omit the details and we concentrate on the inverse problems.

# 2 Uniqueness of the inverse problem - wave network case

As already evoked in the introduction, the proof of Theorem 1 will use a well-known uniqueness result for the inverse problem by a local Carleman estimate. We state it in the following lemma.

**Lemma 2** *Let $r > 0$, $p \in L^\infty(0,\ell)$ and $T > \ell$. Let $f \in L^2(0,\ell)$, $R \in H^1(0,T;L^\infty(0,\ell))$ and $\partial_t R \in L^\infty((0,\ell) \times (0,T))$ and assume that $|R(x,0)| \ge r > 0$ a.e. in $(0,\ell)$.*
*Let $y$ satisfy*

$$\begin{cases} \partial_t^2 y - \partial_x^2 y + p(x)y = f(x)R(x,t), & (x,t) \in (0,\ell) \times (0,T), \\ y(x,0) = 0, \quad \partial_t y(x,0) = 0, & x \in (0,\ell), \end{cases} \tag{11}$$

*Assume further that*

$$y \in C^1([0,T];H^1(0,\ell)) \cap C^2([0,T];L^2(0,\ell)). \tag{12}$$

*Therefore, if* $y(0,t) = \partial_x y(0,t) = 0$ *for all* $t \in (0,T)$*, then we have* $f \equiv 0$ *in* $(0,\ell)$*.*
*Moreover if* $0 < T_1 < T - \ell$*, then we obtain* $y(x,t) = 0$ *for* $0 < x < \ell$ *and* $0 < t < T_1$*.*

This lemma is a classical uniqueness result for the inverse source problem in a wave equation and the proof can be done by the method in [9] on the basis of a Carleman estimate and the even extension of $y$ to negative times $t$. We further refer to Imanuvilov and Yamamoto [16], [17], Klibanov [19], Klibanov and Timonov [21] for example, and for completeness, we prove it in Appendix A.

**Proof of Theorem 1.** We define the following operation of "removing" segments from the tree-shaped network $\Lambda$, starting from all the external nodes except for one where we make measurements. We divide the proof into three steps.

**Step 1.** From Lemma 2, we can easily prove that if $e_j$ is a segment of $\Lambda$ which ends at an external node $Q_j \in \Pi_2$, and if the solutions $u[p]$ and $u[q]$ to (1) satisfy $\partial_x u[p](Q_j,t) = \partial_x u[q](Q_j,t)$ for all $t \in (0,T)$, then $p = q$ on the segment $e_j$ and $u[p](x,t) = u[q](x,t)$ for all $x \in e_j$ and all $t \in (0,T_1)$, for $T_1 \in (0, T-\ell)$.
Indeed, if we set $y = u[p]_j - u[q]_j$, then

$$\begin{cases} \partial_t^2 y - \partial_x^2 y + p_j(x) y = (q_j - p_j)(x) u[q]_j(x,t) & (x,t) \in (0,\ell) \times (0,T), \\ y(Q_j,t) = 0, & t \in (0,T), \\ y(x,0) = 0, \quad \partial_t y(x,0) = 0, & x \in (0,\ell), \end{cases} \tag{13}$$

and noting that $T > 0$ is sufficiently large, we can apply Lemma 2 on $e_j$ since $\partial_x y(Q_j,t) = 0$ for all $t \in (0,T)$. Moreover we note that $u[q]_j \in C^1([0,T];H^1(e_j)) \subset H^1(0,T;L^\infty(e_j))$ and $\partial_t u[q]_j \in C([0,T];H^1(e_j)) \subset L^\infty(e_j \times (0,T))$ by the Sobolev embdedding, and we also note $|u_j^0(x)| \geq r > 0$ on $e_j$, so that $R := u[q]_j$ satisfies the required assumptions. Thus, Lemma 2 allows to obtain $p_j \equiv q_j$ on $e_j$ and consequently $u[p]_j(x,t) = u[q]_j(x,t)$ in $e_j \times (0,T_1)$.

Therefore, for any segment $e$ with the end points $P$ and $Q$ such that $Q \in \Pi_2 \setminus \{Q_{N_2}\}$, we see that $p = q$ on $e$ and $(u[p]|_e)(P,t) = (u[q]|_e)(P,t)$, $(\partial_x u[p]|_e)(P,t) = (\partial_x u[q]|_e)(P,t)$ for $0 < t < T_1$.

Let $\Pi_1^2$ be all the interior node points $P$ of segments of $\Lambda$ having their other end point in $\Pi_2 \setminus \{Q_{N_2}\}$. We note that $\Pi_1^2 \subset \Pi_1$. Applying the above argument to all the exterior end points except for $Q_{N_2}$, we have

$$u[p]_j(P,t) = u[q]_j(P,t), \quad \partial_x u[p]_j(P,t) = \partial_x u[q]_j(P,t)$$

for each $P \in \Pi_1^2$, $0 < t < T_1$ and $j \in \{1,...,N_3\}$. Here by $e_1,...,e_{N_3}$, we enumerate the segments connecting a point in $\Pi_1^2$ and a point in $\Pi_2 \setminus \{Q_{N_2}\}$.

**Step 2.** Let $P \in \Pi_1$ be a given node such that $n_P$ segments, say, $e_1,...,e_{n_P}$ meet at $P$ and $e_1,...,e_{n_P-1}$ connect $P$ with exterior end points, say, $Q_1,...,Q_{n_P-1} \in \Pi_2$ and

$$\begin{aligned} u[p]_j(P,t) &= u[q]_j(P,t), \\ \partial_x u[p]_j(P,t) &= \partial_x u[q]_j(P,t), \quad j \in \{1,...,n_P - 1\},\ 0 < t < T_1. \end{aligned} \tag{14}$$

Using the continuity (2) and the Kirchhoff law (3) at node $P$, we can deduce that

$$\begin{aligned} u[p]_{n_P}(P,t) &= u[q]_{n_P}(P,t), \\ \partial_x u[p]_{n_P}(P,t) &= \partial_x u[q]_{n_P}(P,t), \quad 0 < t < T_1. \end{aligned}$$

**Step 3.** Let $\Lambda^2$ be the graph generated from $\Lambda$ by removing $e_1,...,e_{N_3}$. Since $T_0 > 0$ is sufficiently large by assumption in Theorem 1, when applying the same argument as in Step 1 to the graph $\Lambda^2$, the corresponding time $T_1 > 0$ just adds up to the largest one induced by Step 2. Thus we can repeat this operation to obtain the sets $\Lambda^3$, and then $\Lambda^4$,..., $\Lambda^n$. Hence, let $L^k$ be the set of all the open segments of $\Lambda_k$, $\Pi_1^k$ the set of the interior node points of $\Lambda_k$, $\Pi_2^k$ the set of external endpoints of $\Lambda_k$. Setting $\Lambda^1 = \Lambda$, we note that $L^1 = \{e_0,...,e_N\}$, $\Pi_1^1 = \{P_1,...,P_{N_1}\}$, $\Pi_2^1 = \{Q_1,...,Q_{N_2}\}$.

By (2) and (3), we see that

$$\Pi_1^{k-1} \supset \Pi_1^k, \qquad \forall k \in \mathbb{N}$$

and

$$\Lambda_k = L^k \cup \Pi_1^k \cup \Pi_2^k, \quad L^k \cap \Pi_1^k = L^k \cap \Pi_2^k = \Pi_1^k \cap \Pi_2^k = \emptyset, \quad \forall k \in \mathbb{N}.$$

In order to complete the proof, it is sufficient to prove there exists $n \in \mathbb{N}$ such that

$$\Lambda_n = \emptyset. \tag{15}$$

Assume contrarily that $\Lambda_n \neq \emptyset$ for all $n \in \mathbb{N}$. Since every segment with exterior end point in $\Pi_2 \setminus \{Q_{N_2}\}$, can be removed (meaning that $u[p] = u[q]$ on the segment) by the above operation, we obtain that there exists $n_0 \in N$ such that $\Lambda_{n_0} = L^{n_0} \cup \Pi_1^{n_0}$, *i.e.,* $\Pi_2^{n_0} = \emptyset$. Then $\Lambda_{n_0}$ must be a closed loop since it possesses no external end points. By the assumption, there exist no closed loops in a tree-shape network. This is a contradiction and thus the proof of (15), and therefore, the one of Theorem 1 is completed. □

# 3 Uniqueness for the inverse problem - Schrödinger and heat network cases

## 3.1 Proof of Theorem 2 - Heat case.

We apply an argument similar to the proof of Theorem 4.7 in [19] which is based on the reduction of the inverse heat problem to an inverse wave problem by a kind of Laplace transform called the Reznitzkaya transform (e.g., [18], [24], [30]).

First we define an operator $\Delta_\Lambda$ in $L^2(\Lambda)$ by $\Delta_\Lambda u = \partial_x^2 u_j$ in $e_j$, for all $j \in \{0, 1, ...., N\}$ with

$$\mathcal{D}(\Delta_\Lambda) = \big\{u = (u_0, ..., u_N);\, u_j \in H^2(e_j),\, \partial_x u(Q) = 0 \text{ for } Q \in \Pi_2,\, u_j \text{ satisfying (2) and (3)}\big\}.$$

Here, $e_j$ is oriented from $I(e_j)$ to $T(e_j)$ when defining $\partial_x^2$. Then, similarly to [15], we can prove that $\Delta_\Lambda$ is self-adjoint and $(\Delta_\Lambda u, u)_{L^2(\Lambda)} := \sum_{j=0}^N (\partial_x^2 u_j, u_j)_{L^2(e_j)} \leq 0$. Therefore $\Delta_\Lambda$ generates an analytic semigroup $e^{t\Delta_\Lambda}$, $t > 0$ (e.g., Pazy [28], Tanabe [32]).

Since $p \in L^\infty(\Lambda)$, the perturbed operator $\Delta_\Lambda - p$ generates an analytic semigroup (e.g., Theorem 2.1 in [28], p.80). Therefore by the semigroup theory (e.g., [28], [32]), we know that the solutions $u[p](x,t)$ and $u[q](x,t)$ of equation (4) are analytic in $t$ for any fixed $x \in \Lambda$. More precisely, $u[p], u[q] : (0, \infty) \longrightarrow H^2(\Lambda)$ are analytic in $t > 0$.

By $u^H[p]$ we denote the solution of the heat system (4) and by $u^H[q]$ the corresponding solution when the potential is $q$. By the analyticity in $t$ and the assumption in the theorem, we have

$$u^H[p](Q,t) = u^H[q](Q,t), \quad \forall Q \in \Pi_2 \setminus \{Q_{N_2}\},\, \forall t > 0. \tag{16}$$

On the other hand, we denote by $\widetilde{u}[p]$ the solution of the wave system

$$\begin{cases} \partial_t^2 u_j - \partial_x^2 u_j + p_j(x) u_j = 0, & \forall j \in \{0, 1, ..., N\}, \forall (x,t) \in e_j \times (0, \infty), \\ \partial_x u[p](Q,t) = 0, & \forall Q \in \Pi_2, \forall t \in (0, \infty), \\ u[p](x,0) = 0,\, \partial_t u(x,0) = u^0(x), & \forall x \in \Lambda \end{cases} \tag{17}$$

and by $\widetilde{u}[q]$ the corresponding solution when the potential is $q$. Then we obtain (e.g., [24, pp.251-252]) that

$$\frac{1}{2\sqrt{\pi t^3}} \int_0^\infty \tau e^{-\frac{\tau^2}{4t}} \widetilde{u}[p](x,\tau) d\tau$$

satisfies (4). The uniqueness of solution to equation (4) implies

$$u^H[p](x,t) = \frac{1}{2\sqrt{\pi t^3}} \int_0^\infty \tau e^{-\frac{\tau^2}{4t}} \widetilde{u}[p](x,\tau) d\tau, \quad \forall x \in \Lambda, \forall t > 0$$

and the same equality with $q$. By assumption (16), we obtain

$$\frac{1}{2\sqrt{\pi t^3}} \int_0^\infty \tau e^{-\frac{\tau^2}{4t}} (\widetilde{u}[p] - \widetilde{u}[q])(Q,\tau) d\tau = 0, \quad \forall Q \in \Pi_2 \setminus \{Q_{N_2}\}, \forall t > 0.$$

By the change of variables $s = \frac{1}{4t}$ and $\tau^2 = \eta$, we obtain

$$\int_0^\infty e^{-s\eta} (\widetilde{u}[p] - \widetilde{u}[q])(Q, \sqrt{\eta}) d\eta = 0, \quad \forall Q \in \Pi_2 \setminus \{Q_{N_2}\}, \forall s > 0$$

and the injectivity of the Laplace transform yields

$$(\widetilde{u}[p] - \widetilde{u}[q])(Q, \sqrt{\eta}) = 0, \quad \forall Q \in \Pi_2 \setminus \{Q_{N_2}\}, \forall \eta > 0. \tag{18}$$

Applying the same argument as in Section 2 for the wave system, we prove $p = q$ in $\Lambda$. Thus the proof of Theorem 2 is completed. □

### 3.2 Proof of Theorem 3 - Schrödinger case.

It is sufficient to prove the following lemma.

**Lemma 3** *Let $r > 0$ and $p \in L^\infty(0,\ell)$, $f \in L^2(0,\ell)$ be real-valued, and $T > 0$ be arbitrarily fixed. We consider a 1-d Schrödinger equation:*

$$\begin{cases} i\partial_t y - \partial_x^2 y + p(x)y = f(x)R(x,t), & \forall (x,t) \in (0,\ell)\times(0,T), \\ y(0,t) = 0, & \forall t \in (0,T), \\ y(x,0) = 0, & \forall x \in (0,\ell), \end{cases}$$

*where $R \in H^1(0,T;L^\infty(0,\ell))$, $\partial_t R \in L^\infty((0,\ell)\times(0,T))$ and $|R(x,0)| \geq r > 0$ a.e. in $(0,\ell)$. If $\partial_x y(0,t) = 0$ for all $t \in (0,T)$, then we have $f = 0$ in $(0,\ell)$ and $y = 0$ in $(0,\ell)\times(0,T)$.*

Using the same method as the one for the proof of Lemma 2, this lemma is proved by means of the following Carleman estimate:

**Lemma 4** *For $x_0 \notin [0,\ell]$ and $\beta > 0$ arbitrarily fixed, we set*

$$Sv = i\partial_t v - \partial_x^2 v, \quad \varphi(x,t) = e^{\lambda(|x-x_0|^2 - \beta t^2)}, \quad (x,t) \in (0,\ell)\times(0,T),$$

*where $\lambda > 0$ is chosen sufficiently large. Then, there exists $s_0 > 0$ and a constant $C > 0$ such that*

$$\int_0^T \int_0^\ell (s|\partial_x v|^2 + s^3|v|^2)e^{2s\varphi}dxdt \leq C\int_0^T \int_0^\ell |Sv|^2 e^{2s\varphi}dxdt$$

*for all $s > s_0$ and all $v \in L^2(0,T;H_0^2(0,\ell)) \cap H_0^1(0,T;L^2(0,\ell))$.*

This is a Carleman estimate with regular weight function $\lambda(|x-x_0|^2 - \beta t^2)$ and for the proof, we refer to e.g., [35, Lemma 2.1], [36]. Concerning a Carleman estimate for the Schrödinger equation in a bounded domain $\Omega \subset \mathbb{R}^n$ with singular weight function $\varphi$, we can refer for example to [6, 27].

On the basis of this lemma, the proof of Lemma 3 is done by a usual method by Bukhgeim and Klibanov [9] by using the extension of $y$ to $-T < t < 0$ by $y(\cdot,t) = \overline{y(\cdot,-t)}$ and a cut-off argument and the proof is similar to the proof of Lemma 2 in Appendix. We omit the details.

## 4 Observability in the wave network

The proof of the stability result will rely strongly on the classical result of observability that we are now presenting and proving. One should specifically mention a survey [37] and books [12], [23], where the question of observability in networks of strings (or wave equations) is widely explored in different cases.

We concentrate here on the case where the available observation comes from all but one external nodes, in a setting with a system of wave equations with potential. Since most of the literature on string networks focus only on the wave equation without lower order terms (see [23] or [12] for instance), we detail here how to obtain the observability result for the wave equation with potential. In some other cases, we can prove the observability inequality directly by a global Carleman estimate (e.g., [3]).

**Theorem 5 (Observability inequality)** *On the tree-shaped network $\Lambda$, assuming $p \in L^\infty(\Lambda)$, let us consider the system of 1-d wave equations under the continuity and Kirchhoff law assumptions (2) and (3):*

$$\begin{cases} \partial_t^2 u - \partial_x^2 u + p(x)u = 0, & \text{in } \Lambda\times(0,T), \\ u(Q,t) = 0, & \text{in } (0,T),\ \forall Q \in \Pi_2, \\ u_j(P,t) = u_k(P,t), & \text{in } (0,T),\ \forall P \in \Pi_1,\ \forall j,k \in \{1,...,n_P\}, \\ [\partial_x u(t)]_P = 0, & \text{in } (0,T), \forall P \in \Pi_1, \\ u(x,0) = 0, \quad \partial_t u(x,0) = a(x), & \text{in } \Lambda, \end{cases} \tag{19}$$

*Then there exists a constant $\widetilde{T}$ such that for all $T > \widetilde{T}$, the following observability estimate holds for $u$ solution of (19):*

$$\int_\Lambda |a(x)|^2 dx \leq C\sum_{j=1}^{N_2-1} \int_0^T |\partial_x u_j(Q_j,t)|^2 dt. \tag{20}$$

**Proof of Theorem 5.** Let $v$ be the solution of the system

$$\begin{cases} \partial_t^2 v - \partial_x^2 v = -pu & \forall (x,t) \in \Lambda \times (0,T), \\ v(Q,t) = 0, & \forall Q \in \Pi_2, t \in (0,T), \\ v_j(x,0) = 0, \quad \partial_t v_j(x,0) = 0, & \forall j \in \{0,1,...,N\}, \quad x \in e_j, \end{cases}$$

under conditions (2) and (3). Then (10) in Lemma 1 and $p \in L^\infty(\Lambda)$ yields

$$\sum_{j=1}^{N_2} \int_0^T |\partial_x v_j(Q_j,t)|^2 dt \leq C \int_0^T \int_\Lambda |pu|^2 dx dt \leq C \int_0^T \int_\Lambda |u|^2 dx dt. \tag{21}$$

Setting $w = u - v$, we still have (2), (3) and the following equation

$$\begin{cases} \partial_t^2 w - \partial_x^2 w = 0 & \forall (x,t) \in \Lambda \times (0,T), \\ w(Q,t) = 0, & \forall Q \in \Pi_2, t \in (0,T), \\ w_j(x,0) = 0, \quad \partial_t w_j(x,0) = a(x), & \forall x \in \Lambda. \end{cases}$$

Therefore, using the existing observability inequality in the case where $p = 0$ (see e.g., [12, 23]), we have

$$\int_\Lambda |a(x)|^2 dx \leq C \sum_{j=1}^{N_2-1} \int_0^T |\partial_x w_j(Q_j,t)|^2 dt.$$

Hence, by (21), we have

$$\begin{aligned} \int_\Lambda |a(x)|^2 dx &\leq C \sum_{j=1}^{N_2-1} \int_0^T |\partial_x u_j(Q_j,t)|^2 dt + C \sum_{j=1}^{N_2-1} \int_0^T |\partial_x v_j(Q_j,t)|^2 dt \\ &\leq C \sum_{j=1}^{N_2-1} \int_0^T |\partial_x u_j(Q_j,t)|^2 dt + C \int_0^T \int_\Lambda |u|^2 dx dt. \end{aligned} \tag{22}$$

Therefore a usual compactness-uniqueness argument yields the observability inequality (20). Indeed, if (20) is not satisfied, then we can assume that there exists $a^n \in L^2(\Lambda), n \in \mathbb{N}$ such that

$$\|a^n\|_{L^2(\Lambda)} = 1, \, \forall n \in \mathbb{N} \quad \text{and} \quad \lim_{n \to +\infty} \sum_{j=1}^{N_2-1} \int_0^T |\partial_x u_j^n(Q_j,t)|^2 dt = 0. \tag{23}$$

Here $u_j^n$ is the solution to (19) with initial value $a^n$. Using the energy estimate (9) of Lemma 1 to $u^n$, we obtain

$$||u^n(t)||^2_{H_0^1(\Lambda)} + \|\partial_t u^n(t)\|^2_{L^2(\Lambda)} \leq C||a^n||^2_{L^2(\Lambda)} \leq C.$$

Since the embedding $C([0,T]; H_0^1(\Lambda)) \cap C^1([0,T]; L^2(\Lambda)) \subset L^2(\Lambda \times (0,T))$ is compact, we can extract a subsequence, denoted again by the same notation, such that $(u^n)_{n \in \mathbb{N}^*}$ is convergent in $L^2(\Lambda)$. Therefore, using (22), we obtain

$$\begin{aligned} \int_\Lambda |a^n - a^m|^2 dx \leq\ & C \sum_{j=1}^{N_2-1} \int_0^T |\partial_x u_j^n(Q_j,t)|^2 dt + C \sum_{j=1}^{N_2-1} \int_0^T |\partial_x u_j^m(Q_j,t)|^2 dt \\ & + \ C \int_0^T \int_\Lambda |u^n - u^m|^2 dx dt \end{aligned}$$

so that (23) and $\lim_{n,m \to \infty} \|u^n - u^m\|_{L^2(\Lambda \times (0,T))} = 0$ imply $\lim_{n,m \to \infty} ||a^n - a^m||^2_{L^2(\Lambda)} = 0$. Consequently, there exists a limit $a_0$ such that $\lim_{n \to +\infty} a^n = a_0$ in $L^2(\Lambda)$ and from (23), we have $\|a_0\|_{L^2(\Lambda)} = 1$ and the solution $u[a_0]$ of system (19) with initial data $a_0$ that satisfies

$$\partial_x u_j[a_0](Q,t) = 0, \quad \forall t \in (0,T), \forall Q \in \Pi_2.$$

The goal is now to deduce $u[a_0] \equiv 0$ in $\Lambda \times (0,T)$, and for it we will need a classical unique continuation result for a 1-d wave equation and a "removing" argument similar to the one in the proof of Theorem 1.

**Lemma 5** *We choose $T > \ell$. Let $y \in C([0,T];H^1(0,\ell)) \cap C^1([0,T];L^2(0,\ell))$ satisfy*

$$\begin{cases} \partial_t^2 y(x,t) - \partial_x^2 y(x,t) + p(x)y = 0, & (x,t) \in (0,\ell) \times (0,T), \\ y(0,t) = 0, & t \in (0,T), \\ y(x,0) = 0, & x \in (0,\ell). \end{cases}$$

*If $\partial_x y(0,\cdot) \in L^2(0,T)$ and $\partial_x y(0,t) = 0$ for $0 < t < T$, then for $0 < T_1 < T - \ell$ we have $y(x,t) = 0$ for $0 < x < \ell$ and $0 < t < T_1$.*

This is a type of unique continuation and for completeness, the proof is given in Appendix B. On the basis of Lemma 5, with $(0,\ell)$ replaced by $e_j$ for $j \in \{1,\dots,N_2-1\}$, we follow the ideas of Steps 1 - 3 of the proof of Theorem 1 to obtain that $u[a_0] \equiv 0$ everywhere in $\Lambda \times (0,T)$, which implies $\partial_t u[a_0](x,0) = a_0(x) = 0$ for all $x \in \Lambda$. It finally yields a contradiction with $\|a_0\|_{L^2(\Lambda)} = 1$, so that this ends the proof of Theorem 5. □

# 5 Proof of the stability for the wave network inverse problem

This section is devoted to the proof of Theorem 4. The proof relies on a compactness-uniqueness argument and the observability estimate (Theorem 5) on the whole network.

Let us denote by $u[p]$ the solution of (1) under the assumptions (2) and (3). Henceforth we always assume the conditions (2) and (3). We consider $y = \partial_t (u[p] - u[q])$ that satisfy

$$\begin{cases} \partial_t^2 y - \partial_x^2 y + q(x)y = (q-p)\partial_t u[p] & \forall (x,t) \in \Lambda \times (0,T), \\ y(Q,t) = 0, & \forall Q \in \Pi_2, t \in (0,T), \\ y(x,0) = 0, \quad \partial_t y(x,0) = (q-p)u^0(x), & \forall x \in \Lambda, \end{cases} \tag{24}$$

We set $f = q - p$. We define $\psi$ and $\phi$ as the solutions of

$$\begin{cases} \partial_t^2 \psi - \partial_x^2 \psi + q(x)\psi = f\partial_t u[p] & \forall (x,t) \in \Lambda \times (0,T), \\ \psi(Q,t) = 0, & \forall Q \in \Pi_2, t \in (0,T), \\ \psi(x,0) = 0, \quad \partial_t \psi(x,0) = 0, & \forall x \in \Lambda, \end{cases} \tag{25}$$

and

$$\begin{cases} \partial_t^2 \phi - \partial_x^2 \phi + q(x)\phi = 0 & \forall (x,t) \in \Lambda \times (0,T), \\ \phi(Q,t) = 0, & \forall Q \in \Pi_2, t \in (0,T), \\ \phi(x,0) = 0, \quad \partial_t \phi(x,0) = fu^0(x), & \forall x \in \Lambda. \end{cases} \tag{26}$$

Then $y = \psi + \phi$. We can apply Theorem 5 to equation (26) so that

$$\int_\Lambda |fu^0|^2 dx \le C \sum_{j=1}^{N_2-1} \int_0^T |\partial_x \phi_j(Q_j,t)|^2 dt. \tag{27}$$

On the other hand, taking the time derivative of equation (25) and applying (10) from Lemma 1, by (6) we have

$$\begin{aligned} \sum_{j=1}^{N_2} \|\partial_x \psi_j(Q_j)\|^2_{H^1(0,T)} &\le C\left(||fu^1||^2_{L^2(\Lambda)} + ||f\partial_t u[p]||^2_{W^{1,1}(0,T,L^2(\Lambda))}\right) \\ &\le 2CK^2||f||^2_{L^2(\Lambda)}, \end{aligned} \tag{28}$$

where we used that $u[p] \in C^1([0,T],H^1(\Lambda)) \cap C^2([0,T],L^\infty(\Lambda))$ yields $\partial_t u[p] \in C([0,T];L^\infty(\Lambda))$ so that $u^1 \in L^\infty(\Lambda)$ and (6) gives $\|u^1\|_{L^\infty(\Lambda)} \le K$.
By the compact embedding $H^1(0,T) \subset L^2(0,T)$, we see that the operator $\Psi : L^2(\Lambda) \to L^2(0,T)$ defined by

$$(\Psi f)(t) = \sum_{j=1}^{N_2} \partial_x \psi_j(Q_j,t), \qquad 0 < t < T$$

is compact.

Therefore, since we have $|u^0(x)| \geq r > 0$ almost everywhere in $\Lambda$, by (27) and (28), we obtain

$$\begin{aligned} ||f||^2_{L^2(\Lambda)} &\leq C\int_\Lambda |fu^0|^2dx \leq C\sum_{j=1}^{N_2-1}\int_0^T |\partial_x\phi_j(Q_j,t)|^2dt \\ &\leq C\sum_{j=1}^{N_2-1}\int_0^T |\partial_x y_j(Q_j,t)|^2dt + C\sum_{j=1}^{N_2}\int_0^T |\partial_x\psi_j(Q_j,t)|^2dt \\ &\leq C\sum_{j=1}^{N_2-1}\int_0^T |\partial_x y_j(Q_j,t)|^2dt + C||\Psi f||^2_{L^2(0,T)} \qquad (29) \\ &\leq C\sum_{j=1}^{N_2-1} \|\partial_x u[p]_j(Q_j) - \partial_x u[q]_j(Q_j)\|^2_{H^1(0,T)} + C||\Psi f||^2_{L^2(0,T)}. \end{aligned}$$

We aim at proving that we can get rid of the second term on the right-hand side of the last estimate in order to obtain (7). Again, a compactness-uniqueness argument will be the key and it relies here on the compactness of $\Psi$ and the uniqueness result of Theorem 1.

In fact, by contradiction, we assume that

$$||f||_{L^2(\Lambda)} \leq C\sum_{j=1}^{N_2-1} \|\partial_x y_j(Q_j)\|_{L^2(0,T)},$$

which is equivalent to (7), does not hold. Then one can assume that there exists $f^n \in L^2(\Lambda), n \in \mathbb{N}$ such that

$$\|f^n\|_{L^2(\Lambda)} = 1, \ \forall n \in \mathbb{N} \quad \text{and} \quad \lim_{n\to+\infty}\sum_{j=1}^{N_2-1} \left\|\partial_x y_j^n(Q_j)\right\|_{L^2(0,T)} = 0. \qquad (30)$$

Here $y^n = (y_0^n, ..., y_N^n)$ is the solution to

$$\begin{cases} \partial_t^2 y^n - \partial_x^2 y^n + q(x)y^n = f^n\partial_t u[p] & \forall (x,t) \in \Lambda\times(0,T), \\ y^n(Q,t) = 0, & \forall Q \in \Pi_2, t \in (0,T), \\ y^n(x,0) = 0, \quad \partial_t y^n(x,0) = (f^n u^0)(x), & \forall x \in \Lambda. \end{cases}$$

First, since the sequence $(f^n)_{n\in\mathbb{N}}$ is bounded in $L^2(\Lambda)$, we can extract a subsequence denoted again by $(f^n)_{n\in\mathbb{N}}$ such that it converges towards some $f^0 \in L^2(\Lambda)$ weakly in $L^2(\Lambda)$. Since $\Psi$ is a compact operator, we obtain therefore the strong convergence result

$$\lim_{n,m\to\infty} \|\Psi f^n - \Psi f^m\|_{L^2(0,T)} = 0. \qquad (31)$$

Then, from (29) we can write

$$||f^n - f^m||^2_{L^2(\Lambda)} \leq C\sum_{j=1}^{N_2-1} \left\|\partial_x y_j^n(Q_j)\right\|^2_{L^2(0,T)} + C\sum_{j=1}^{N_2-1} \left\|\partial_x y_j^m(Q_j)\right\|^2_{L^2(0,T)} + C||\Psi f^n - \Psi f^m||^2_{L^2(\Lambda)}$$

and deduce from (30) and (31) that $\lim_{n,m\to\infty} \|f^n - f^m\|_{L^2(\Lambda)} = 0$, so that there exists $f^0 \in L^2(\Lambda)$ such that $\lim_{n\to\infty} \|f^n - f^0\|_{L^2(\Lambda)} = 0$ with

$$\|f^0\|_{L^2(\Lambda)} = 1. \qquad (32)$$

By Lemma 1, there exists a solution $y^0 \in C([0,T]; H_0^1(\Lambda)) \cap C^1([0,T]; L^2(\Lambda))$ to the system (24) with initial data $(0, f^0u^0)$ and source term $f^0\partial_t u[p]$. Moreover, using the trace estimate (10) of Lemma 1 for the solution $y^n$ of system (24) with initial data $f^n u^0$ and source term $f^n\partial_t u[p]$, we obtain (in the same way we proved (28))

$$\sum_{j=1}^{N_2-1} \left\|\partial_x y_j^n(Q_j)\right\|^2_{L^2(0,T)} \leq C\left(||f^n u^0||^2_{L^2(\Lambda)} + ||f^n\partial_t u[p]||^2_{L^1(0,T,L^2(\Lambda))}\right) \leq 2CK^2\|f^n\|^2_{L^2(\Lambda)}.$$

Thus we can write

$$\lim_{n\to\infty}\sum_{j=1}^{N_2-1} \left\|\partial_x y_j^n(Q_j) - \partial_x y_j^0(Q_j)\right\|^2_{L^2(0,T)} \leq 2CK^2 \lim_{n\to\infty}\|f^n - f^0\|^2_{L^2(\Lambda)} = 0,$$

which, combined with (30), gives

$$\partial_x y_j^0(Q_j,t)=0, \qquad \forall Q_j\in\Pi_2\setminus\{Q_{N_2}\},\forall t\in(0,T).$$

Therefore we have

$$\begin{cases} \partial_t^2 y^0-\partial_x^2 y^0+q(x)y^0=f^0\partial_t u[p] & \forall (x,t)\in\Lambda\times(0,T),\\ y^0(Q,t)=0, & \forall Q\in\Pi_2, t\in(0,T),\\ y^0(x,0)=0, \quad \partial_t y^0(x,0)=(f^0u^0)(x), & \forall x\in\Lambda \end{cases}$$

and $\partial_x y^0(Q,t)=0$, for all $Q\in\Pi_2\setminus\{Q_{N_2}\}$ and $t\in(0,T)$. The last objective is now to infer that $f_0\equiv 0$ in $\Lambda$, and in order to be able to apply the unique continuation given by Lemma 2 and the "removing" argument of the proof of Theorem 1, we need to work on a primitive of $y^0$.

Therefore, we set $\widetilde{y}(x,t)=\displaystyle\int_0^t y^0(x,\xi)d\xi$ for $(x,t)\in\Lambda\times(0,T)$. In view of the initial data $y^0(x,0)=0$ and $\partial_t y^0(x,0)=(f^0u^0)(x)$ for all $x\in\Lambda$, we can verify that $\widetilde{y}\in C^1([0,T];H_0^1(\Lambda))\cap C^2([0,T];L^2(\Lambda))$ and $\widetilde{y}$ satisfies

$$\begin{cases} \partial_t^2\widetilde{y}-\partial_x^2\widetilde{y}+q(x)\widetilde{y}=f^0u[p] & \forall (x,t)\in\Lambda\times(0,T),\\ \widetilde{y}(Q,t)=0, & \forall Q\in\Pi_2, t\in(0,T),\\ \partial_x\widetilde{y}(Q,t)=0, & \forall Q\in\Pi_2\setminus\{Q_{N_2}\}, t\in(0,T),\\ \widetilde{y}(x,0)=\partial_t\widetilde{y}(x,0)=0, & x\in\Lambda. \end{cases}$$

Since $T>T_0$, noting that $|u[p](x,0)|=|u^0(x)|>0$ for $x\in\Lambda$ and using Lemma 2, we repeat the arguments of the proof of Theorem 1 to conclude that $f^0=0$ everywhere in $\Lambda$. This contradicts (32) and thus the proof of Theorem 4 is complete. □

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# A Proof of Lemma 2

We begin this appendix by recalling the following known Carleman estimate:

**Lemma 6** *We assume that $p_0 \in L^\infty((0,\ell)\times(-T,T))$, $x_0 > \ell$, $\beta \in (0,1)$, and we set*

$$\psi(x,t) = |x-x_0|^2 - \beta t^2, \quad \varphi(x,t) = e^{\lambda\psi(x,t)},$$

*where we fix $\lambda > 0$ sufficiently large. Then there exist contants $s_0 > 0$ and $C > 0$ independent of $s$, such that*

$$\int_{-T}^{T}\int_0^\ell (s|\partial_x v|^2 + s|\partial_t v|^2 + s^3 v^2)e^{2s\varphi}dxdt \le C\int_{-T}^{T}\int_0^\ell \left|\partial_t^2 v - \partial_x^2 v + p_0 v\right|^2 e^{2s\varphi}dxdt \tag{33}$$

*for all $s \ge s_0$ and $v \in L^2(-T,T;H_0^1(\Omega))$ satisfying $\partial_t^2 v - \partial_x^2 v \in L^2((0,\ell)\times(-T,T))$, $v(\cdot,\pm T) = \partial_t v(\cdot,\pm T) = 0$, in $(0,\ell)$, and $\partial_x v(0,\cdot) = \partial_x u(\ell,\cdot) = 0$ in $(-T,T)$.*

**Proof.** This result is not new and the proof of this Lemma can be done in several ways. For convenience of the reader, we refer to [4] where the detailed proof of a more general Carleman estimate, given in Theorem 2.1, can be read.

Lemma 6 is a particular case of [4, Theorem 2.1]. On the one hand, by choosing large $s_0 > 0$, a Carleman estimate for the operator $\partial_t^2 - \partial_x^2 + p_0$ with $p_0 \in L^\infty((0,\ell)\times(-T,T))$ is very easily deduced from a Carleman estimate in the case of $p_0 = 0$ which is given by Theorem 2.1. Moreover we notice that the weight function used in Theorem 2.1 is $\psi_0(x,t) = |x-x_0|^2 - \beta t^2 + C_0$ with some constant $C_0$, but, since $\lambda$ is fixed large enough in our lemma, we do not need $C_0$ as we can write $2s\varphi = 2se^{\lambda\psi} = 2se^{\lambda\psi_0}e^{-\lambda C_0}$ and set $s := se^{-\lambda C_0}$.

The Carleman estimate for a hyperbolic equation is well-known for functions with compact supports, i.e. $u \in H_0^2((0,\ell)\times(0,T))$, and we can refer to several works for the proof , e.g., Hörmander [14] for a general theory, Klibanov and Timonov [21], and Chapter 1 in Bellassoued and Yamamoto [8] where a direct proof is given. In view of such a classical Carleman estimates, Lemma 6 can also be proved by approximating the right-hand side by suitable smooth functions. □

**Proof of Lemma 2.**
Our proof is based on Bukhgeim and Klibanov [9] but here we use the argument from Imanuvilov and Yamamoto [16].

**Step 1.** First we extend $y$ in $(0,T)$ to $(-T,T)$ by setting $y(x,t) = y(x,-t)$ for $t < 0$. We also set $R(x,t) = R(x,-t)$ for $t < 0$. By the initial conditions in (11), we can directly verify that $y \in C^1([-T,T];H_0^1(0,\ell)) \cap C^2([-T,T];L^2(0,\ell))$ and

$$\begin{cases} \partial_t^2 y - \partial_x^2 y + p(x)y = f(x)R(x,t) & (x,t) \in (0,\ell)\times(-T,T), \\ y(0,t) = 0, \quad \partial_x y(0,t) = 0, & t \in (-T,T), \\ y(x,0) = 0, \quad \partial_t y(x,0) = 0, & x \in (0,\ell). \end{cases}$$

In order to apply Lemma 6, we need to design and use a cut-off function $\chi$ because $y$ does not satisfy all the expected zero boundary conditions.

Let $0 < \delta_0 < \delta$ and set $x_0 = \ell + \delta_0$. Since $T > \ell$, we can choose $\delta > 0$ small enough such that $\ell^2 + 2\ell\delta < T^2$. Therefore we can choose $0 < \beta < 1$ such that $\frac{\ell^2+2\ell\delta}{T^2} < \beta < 1$, i.e.,

$$\sqrt{\frac{x_0^2 - \delta^2}{\beta}} < T. \tag{34}$$

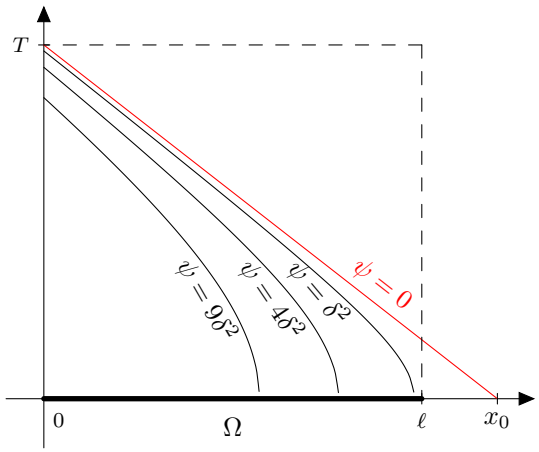


Figure 2: Isovalues of the weight function $\psi(x,t) = |x-x_0|^2 - \beta t^2$, $\beta \in (0,1)$, for $\Omega = (0,\ell)$, $x_0 = \ell + \delta_0$, $0 < \delta_0 < \delta$ and $T = x_0/\sqrt{\beta}$.

Indeed, we easily have: $x_0^2 - \delta^2 = (\ell+\delta_0)^2 - \delta^2 = \ell^2 + 2\ell\delta_0 + \delta_0^2 - \delta^2 < \ell^2 + 2\ell\delta < \beta T^2$.
With this $\beta$ and $x_0$, we set

$$\psi(x,t) = |x-x_0|^2 - \beta t^2, \quad \varphi(x,t) = e^{\lambda\psi(x,t)}$$

with large fixed $\lambda > 0$ (see Figure 2). We further set

$$Q_\mu = \{(x,t);\, 0 < x < \ell, \quad \psi(x,t) > \mu^2\}.$$

Then we can easily verify $Q_\delta \subset (0, x_0-\delta) \times \left(-\sqrt{\frac{x_0^2-\delta^2}{\beta}}, \sqrt{\frac{x_0^2-\delta^2}{\beta}}\right)$ so that, by (34), we have $Q_\delta \subset (0,\ell)\times(-T,T)$.
Thus, we define a cut-off function $\chi \in C^\infty(\mathbb{R}_+ \times \mathbb{R})$ such that $0 \le \chi \le 1$ and

$$\chi(x,t) = \begin{cases} 1, & (x,t) \in Q_{3\delta}, \\ 0, & (x,t) \in ((0,\ell)\times(-T,T)) \setminus Q_{2\delta} \end{cases} \tag{35}$$

and we set

$$z = \chi(\partial_t y)e^{s\varphi}.$$

Then, in the sense of distribution in $(0,\ell)\times(-T,T)$, we easily have

$$\begin{array}{rcl} \partial_t z &=& (\partial_t\chi)(\partial_t y)e^{s\varphi} + \chi(\partial_t^2 y)e^{s\varphi} + s(\partial_t\varphi)z, \\ \partial_x z &=& (\partial_x\chi)(\partial_t y)e^{s\varphi} + \chi(\partial_x\partial_t y)e^{s\varphi} + s\chi(\partial_x\varphi)(\partial_t y)e^{s\varphi}, \\ \partial_t^2 z &=& \chi(\partial_t^3 y)e^{s\varphi} + s\chi(\partial_t\varphi)(\partial_t^2 y)e^{s\varphi} + s(\partial_t^2\varphi)z + s(\partial_t\varphi)\partial_t z + Q_1(s)e^{s\varphi}, \\ \partial_x^2 z &=& \chi(\partial_x^2\partial_t y)e^{s\varphi} + s\chi(\partial_x\varphi)(\partial_x\partial_t y)e^{s\varphi} + s(\partial_x^2\varphi)z + s(\partial_x\varphi)\partial_x z + Q_2(s)e^{s\varphi}, \end{array} \tag{36}$$

where $Q_1(s)$ denotes a linear combination of $\partial_t^j y$ with $j = 0,1,2$ whose coefficients include the time derivatives of $\chi$ and the power $s^k$ with $k = 0,1$ as factors, and $Q_2(s)$ denotes a linear combination of $\partial_x^j\partial_t y$ with $j = 0,1$ whose coefficients include the space derivatives of $\chi$ and the power $s^k$ with $k = 0,1$ as factors. Therefore we obtain

$$\begin{array}{rcl} \partial_t^2 z - \partial_x^2 z + pz &=& \chi\partial_t(\partial_t^2 y - \partial_x^2 y + py)e^{s\varphi} + s\chi((\partial_t\varphi)\partial_t^2 y - (\partial_x\varphi)(\partial_x\partial_t y))e^{s\varphi} \\ && + s(\partial_t^2\varphi - \partial_x^2\varphi)z + s((\partial_t\varphi)\partial_t z - (\partial_x\varphi)\partial_x z) + (Q_1(s) - Q_2(s))e^{s\varphi} \\ &=& \chi f(x)(\partial_t R)e^{s\varphi} + s\chi((\partial_t\varphi)\partial_t^2 y - (\partial_x\varphi)(\partial_x\partial_t y))e^{s\varphi} \\ && + s(\partial_t^2\varphi - \partial_x^2\varphi)z + s((\partial_t\varphi)\partial_t z - (\partial_x\varphi)\partial_x z) + (Q_1(s) - Q_2(s))e^{s\varphi}. \end{array} \tag{37}$$

Now if we set $w = \chi\partial_t y$ (such that $z = e^{s\phi}w$), then, by (37) with $s = 0$ and the regularity assumptions of $y$ and $R$, we have

$$\partial_t^2 w - \partial_x^2 w + pw = \chi f\partial_t R + (Q_1(0) - Q_2(0)) \quad \text{in } L^2((0,\ell)\times(-T,T)).$$

Since $y(0,t)=\partial_x y(0,t)=0$, $-T<t<T$, we see by (35) that $w\in C([-T,T];H_0^1(0,\ell))\cap C^1([-T,T];L^2(0,\ell))$, $\partial_t^j w(x,\pm T)=0$ for $0<x<\ell$ and $j=0,1$, $\partial_x w(0,t)=\partial_x w(\ell,t)=0$ for $-T<t<T$ and $\partial_t^2 w-\partial_x^2 w=\chi f\partial_t R+Q_1(0)-Q_2(0)-pw\in L^2((0,\ell)\times(-T,T))$. Hence we can apply the Carleman estimate of Lemma 6 to $w$:

$$\int_{-T}^{T}\int_0^\ell (s|\partial_x w|^2+s|\partial_t w|^2+s^3w^2)e^{2s\varphi}dxdt \leq C\int_{-T}^{T}\int_0^\ell \chi^2|f|^2|\partial_t R|^2e^{2s\varphi}dxdt+C\int_{-T}^{T}\int_0^\ell |Q_1(0)-Q_2(0)|^2e^{2s\varphi}dxdt. \quad (38)$$

Now, since

$$\chi\partial_x\partial_t y=\partial_x w-(\partial_t y)\partial_x\chi \ \text{ and } \ \chi\partial_t^2 y=\partial_t w-(\partial_t y)\partial_t\chi,$$

we have

$$\begin{aligned}&\int_{Q_{2\delta}}(s|\chi\partial_x\partial_t y|^2+s|\chi\partial_t^2 y|^2+s^3|\chi\partial_t y|^2)e^{2s\varphi}dxdt\\ \leq\ & 2\int_{Q_{2\delta}}(s|\partial_x w|^2+s|\partial_t w|^2+s^3w^2)e^{2s\varphi}dxdt+2\int_{Q_{2\delta}}(s|\partial_t y|^2|\partial_x\chi|^2+s|\partial_t y|^2|\partial_t\chi|^2)e^{2s\varphi}dxdt.\end{aligned}$$

Therefore, from the definition (35) of the cut-off function, and using (38), the definition of $Q_{3\delta}$ and the regularity assumptions (12) on $y$, we obtain

$$\begin{aligned}&\int_{Q_{2\delta}}(s|\chi\partial_x\partial_t y|^2+s|\chi\partial_t^2 w|^2+s^3|\chi\partial_t y|^2)e^{2s\varphi}dxdt\\ &\leq C\int_{Q_{2\delta}}|f|^2e^{2s\varphi}dxdt+C\int_{Q_{2\delta}\setminus Q_{3\delta}}|Q_1(0)-Q_2(0)|^2e^{2s\varphi}dxdt+Cs\int_{Q_{2\delta}}(|\partial_x\chi|^2+|\partial_t\chi|^2)|\partial_t y|^2e^{2s\varphi}dxdt\\ &\leq C\int_{Q_{2\delta}}|f|^2e^{2s\varphi}dxdt+Cs^2e^{2se^{9\lambda\delta^2}} \quad (39)\end{aligned}$$

since derivatives of $\chi$, as well as $Q_1(0)-Q_2(0)$, are vanishing everywhere outside $Q_{2\delta}\setminus Q_{3\delta}$.

**Step 2.** Next we set $Q_\delta^+=\{(x,t)\in Q_\delta;\, t>0\}$. Then

$$Q_\delta^+=\left\{(x,t);0<x<x_0-\delta,\,0<t<\sqrt{\frac{|x-x_0|^2-\delta^2}{\beta}}\right\}.$$

Multiplying (37) by $\partial_t z$ and integrating over $Q_\delta^+$, we have

$$\begin{aligned}&\int_{Q_\delta^+}(\partial_t^2 z-\partial_x^2 z+pz)\partial_t z\,dxdt=\sum_{k=1}^4 I_k\\ :=\ &\int_{Q_\delta^+}\chi f(\partial_t R)e^{s\varphi}(\partial_t z)\,dxdt+s\int_{Q_\delta^+}\chi((\partial_t\varphi)\partial_t^2 y-(\partial_x\varphi)(\partial_x\partial_t y))(\partial_t z)e^{s\varphi}\,dxdt\\ &+s\int_{Q_\delta^+}\{(\partial_t^2\varphi-\partial_x^2\varphi)z+((\partial_t\varphi)\partial_t z-(\partial_x\varphi)\partial_x z)\}\partial_t z\,dxdt\\ &+\int_{Q_\delta^+}(Q_1(s)-Q_2(s))(\partial_t z)e^{s\varphi}\,dxdt.\end{aligned}$$

On the one hand, by integration by parts and using $z=\partial_x z=\partial_t z=0$ on $\partial Q_\delta^+\setminus\{t=0\}$ and $z=\partial_x z=0$ on $\partial Q_\delta^+\cap\{t=0\}$, we have

$$\left|\int_{Q_\delta^+}(\partial_t^2 z-\partial_x^2 z+pz)\partial_t z\,dxdt\right|=\frac12\int_0^{x_0-\delta}|\partial_t z(x,0)|^2dx.$$

Moreover, by $\partial_t y(x,0)=0$ for $0<x<\ell$ and $z=\chi(\partial_t y)e^{s\varphi}$, we have $\partial_t z(x,0)=\chi(x,0)\partial_t^2 y(x,0)e^{s\varphi(x,0)}$. Therefore, using (11) and $|R(x,0)|\geq r$ on $[0,\ell]$, we obtain

$$\begin{aligned}&\left|\int_{Q_\delta^+}(\partial_t^2 z-\partial_x^2 z+pz)\partial_t z\,dxdt\right|=\frac12\int_0^{x_0-\delta}|\chi(x,0)|^2|\partial_t^2 y(x,0)|^2e^{2s\varphi(x,0)}dx\\ &=\frac12\int_0^{x_0-\delta}|\chi(x,0)|^2|f(x)|^2|R(x,0)|^2e^{2s\varphi(x,0)}dx\geq C\int_0^{x_0-\delta}|\chi(x,0)|^2|f(x)|^2e^{2s\varphi(x,0)}dx. \quad (40)\end{aligned}$$

On the other hand, by $\partial_t R \in L^\infty((0,\ell)\times(0,T))$ and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
I_1 &\leq C\int_{Q_\delta^+}\chi^2|f(x)|^2e^{2s\varphi}dxdt + C\int_{Q_\delta^+}|\partial_t z|^2dxdt,\\
I_2 &\leq Cs\int_{Q_\delta^+}(\chi|\partial_t^2 y| + \chi|\partial_t\partial_x y|)|\partial_t z|e^{s\varphi}dxdt\\
&\leq Cs\int_{Q_\delta}(|\chi\partial_t^2 y|^2 + |\chi\partial_t\partial_x y|^2)e^{2s\varphi}dxdt + C\int_{Q_\delta}|\partial_t z|^2dxdt,\\
I_3 &\leq Cs\int_{Q_\delta^+}(|z| + |\partial_t z| + |\partial_x z|)|\partial_t z|dxdt \;\leq\; Cs\int_{Q_\delta}(z^2 + |\partial_t z|^2 + |\partial_x z|^2)\\
I_4 &\leq Cs^2e^{2se^{9\lambda\delta^2}} + C\int_{Q_\delta}|\partial_t z|^2dxdt.dxdt,
\end{aligned}$$

using also that $Q_1(s) - Q_2(s)$ vanishes everywhere outside $Q_{2\delta}\setminus Q_{3\delta}$.

Now, analyzing the terms in $z$ on the right-hand side of these estimates, from (36), we have that in $Q_\delta$,

$$|\partial_t z|^2 + |\partial_x z|^2 \leq C(|\chi\partial_t^2 y|^2 + |\chi\partial_t\partial_x y|^2)e^{2s\varphi} + Cs^2|\chi\partial_t y|^2e^{2s\varphi} + (|\partial_t\chi|^2 + |\partial_x\chi|^2)|\partial_t y|^2e^{2s\varphi}.$$

Therefore, since $|\partial_t\chi|^2$ and $|\partial_x\chi|^2$ vanish everywhere outside $Q_{2\delta}\setminus Q_{3\delta}$ and $\chi = 0$ outside of $Q_{2\delta}$, using the definition of $Q_{3\delta}$ and the regularity assumptions (12) on $y$, we have

$$\begin{aligned}
&\int_{Q_\delta}(s|\partial_t z|^2 + s|\partial_x z|^2 + s^3z^2)dxdt\\
\leq\;& C\int_{Q_{2\delta}}(s|\chi\partial_t^2 y|^2 + s|\chi\partial_x\partial_t y|^2 + s^3|\chi\partial_t y|^2)e^{2s\varphi}dxdt + Cs^2e^{2se^{9\lambda\delta^2}}.
\end{aligned}$$

Hence, using also (39), we finally obtain

$$\begin{aligned}
\sum_{k=1}^4|I_k| \leq C\int_{Q_{2\delta}}|f|^2e^{2s\varphi}dxdt + C\int_{Q_{2\delta}}(s|\chi\partial_t^2 y|^2 + s|\chi\partial_x\partial_t y|^2 + s^3|\chi\partial_t y|^2)e^{2s\varphi}dxdt + Cs^2e^{2se^{9\lambda\delta^2}}\\
\leq C\int_{Q_{2\delta}}|f|^2e^{2s\varphi}dxdt + Cs^2e^{2se^{9\lambda\delta^2}}. \quad (41)
\end{aligned}$$

**Step 3.** From (40) and (41), we obtain

$$\int_0^{x_0-\delta}|\chi(x,0)|^2|f(x)|^2e^{2s\varphi(x,0)}dx \leq C\int_{Q_{2\delta}}|f|^2e^{2s\varphi}dxdt + Cs^2e^{2se^{9\lambda\delta^2}}.$$

Noting that $(0, x_0 - m\delta)\times\{0\}\subset Q_{m\delta}$ and $Q_{m\delta}\subset Q_{2\delta}$ for $m\geq 3$ and $\chi = 1$ in $Q_{3\delta}$, setting $\mu_m = e^{m^2\lambda\delta^2}$, we have

$$\int_0^{x_0-3\delta}|f(x)|^2e^{2s\varphi(x,0)}dx \leq C\int_{Q_{3\delta}}|f|^2e^{2s\varphi}dxdt + C\int_{Q_{2\delta}\setminus Q_{3\delta}}|f|^2e^{2s\varphi}dxdt + Cs^2e^{2s\mu_3}.$$

Besides, we have

$$\int_{Q_{3\delta}}|f|^2e^{2s\varphi}dxdt \leq \int_0^{x_0-3\delta}|f(x)|^2e^{2s\varphi(x,0)}\left(\int_{-T}^T e^{2s(\varphi(x,t)-\varphi(x,0))}dt\right)dx.$$

Here

$$\varphi(x,t) - \varphi(x,0) = e^{\lambda(|x-x_0|^2-\beta t^2)} - e^{\lambda|x-x_0|^2} \leq \left(\min_{0\leq x\leq\ell}e^{\lambda|x-x_0|^2}\right)(e^{-\lambda\beta t^2} - 1) < 0$$

for $t\neq 0$ and $0\leq x\leq\ell$, and the Lebesgue theorem yields

$$\max_{0\leq x\leq\ell}\int_{-T}^T e^{2s(\varphi(x,t)-\varphi(x,0))}dt \leq C\int_{-T}^T\exp(2sC(e^{-\lambda\beta t^2} - 1))dt = o(1)$$

as $s \to \infty$. Consequently, $\displaystyle\int_{Q_{3\delta}} |f|^2 e^{2s\varphi} dxdt = o(1) \int_0^{x_0-3\delta} |f(x)|^2 e^{2s\varphi(x,0)} dx$. Hence

$$\begin{aligned}(1-o(1)) \int_0^{x_0-4\delta} |f(x)|^2 e^{2s\varphi(x,0)} dx &\leq (1-o(1)) \int_0^{x_0-3\delta} |f(x)|^2 e^{2s\varphi(x,0)} dx \\ &\leq C \int_{Q_{2\delta}\setminus Q_{3\delta}} |f|^2 e^{2s\varphi} dxdt + Cs^2 e^{2s\mu_3}.\end{aligned}$$

Therefore, since $(0, x_0 - 4\delta) \times \{0\} \subset Q_{4\delta}$ implies $e^{2s\varphi(x,0)} \geq e^{2s\mu_4}$ for $0 < x < x_0 - 4\delta$ and $|e^{2s\varphi}| \leq e^{2s\mu_3}$ in $Q_{2\delta} \setminus Q_{3\delta}$, we have

$$(1-o(1))e^{2s\mu_4} \int_0^{x_0-4\delta} |f|^2 dx \leq Ce^{2s\mu_3} \|f\|^2_{L^2(0,\ell)} + Cs^2 e^{2s\mu_3}.$$

Choosing $s > 0$ large and dividing by $e^{2s\mu_4}$, we obtain

$$\int_0^{\ell+\delta_0-4\delta} |f(x)|^2 dx \leq C(s^2 + \|f\|^2_{L^2(0,\ell)}) e^{-2s(\mu_4-\mu_3)}$$

By $\mu_4 - \mu_3 > 0$, letting $s \to \infty$, we see that the right-hand side tends to 0, and $f(x) = 0$ for $0 \leq x \leq \ell + \delta_0 - 4\delta$. We can choose $\delta, \delta_0 > 0$ arbitrarily small, provided that $0 < \delta_0 < \delta$, so that $f = 0$ in $(0, \ell)$.

**Step 4.** It remains to prove the latter part of the Lemma 2. Since $f = 0$ in $(0, \ell)$, estimate (39) yields

$$\int_{Q_{2\delta}} s^3 |\chi(x,t)|^2 |\partial_t y|^2 e^{2s\varphi} dxdt \leq Cs^2 e^{2s\mu_3} \tag{42}$$

for all $s \geq s_0$. Since

$$\int_{Q_{4\delta}} s^3 |\chi(x,t)|^2 |\partial_t y|^2 e^{2s\varphi} dxdt \leq \int_{Q_{2\delta}} s^3 |\chi(x,t)|^2 |\partial_t y|^2 e^{2s\varphi} dxdt,$$

$e^{2s\varphi} \geq e^{2s\mu_4}$ in $Q_{4\delta}$ and $\chi = 1$ in $Q_{4\delta}$, from (42) we obtain

$$\int_{Q_{4\delta}} |\partial_t y|^2 dxdt \leq \frac{C}{s} e^{-2s(\mu_4-\mu_3)}$$

for all $s \geq s_0$. Letting $s \to \infty$, we obtain $\int_{Q_{4\delta}} |\partial_t y|^2 dxdt = 0$. One can directly verify that we have the inclusion

$$(0, x_0 - 5\delta) \times \left(-\frac{\delta}{\sqrt{\beta}}, \frac{\delta}{\sqrt{\beta}}\delta\right) \subset Q_{4\delta}, \tag{43}$$

so that

$$\partial_t y(x,t) = 0 \quad \text{in } (0, x_0 - 5\delta) \times \left(-\frac{\delta}{\sqrt{\beta}}, \frac{\delta}{\sqrt{\beta}}\right).$$

We fix $\delta > 0$ arbitrarily small and choose the weight function $\psi_\theta(x,t) = |x - x_0|^2 - \beta(t-\theta)^2$ for each $\theta \in (-T_1, T_1)$. Noting that $|T - \theta| > \ell$ for $|\theta| < T_1$, we can apply the above arguments from (39) and obtain

$$\partial_t y(x,t) = 0 \quad \text{in } (0, \ell + \delta_0 - 5\delta) \times (-T_1, T_1).$$

Since $y(x,0) = 0$ for $0 < x < \ell$, we have $y(x,t) = 0$ in $(0, \ell + \delta_0 - 5\delta) \times (-T_1, T_1)$. Since $\delta_0, \delta > 0$ can be chosen arbitrarily small, we can obtain $y = 0$ in $(0, \ell) \times (-T_1, T_1)$. Thus the proof of Lemma 2 is completed. □

# B Proof of Lemma 5

We choose the same $x_0, \delta_0, \delta, \beta$ and use the same notations as in the proof of Lemma 2. We extend $y(x,t)$ to $t < 0$ by setting $y(x,-t) = -y(x,t)$ for $t > 0$. Then by $y(\cdot, 0) = 0$ in $(0, \ell)$, we can verify that $y \in C([-T,T]; H^1(0,\ell)) \cap C^1([-T,T]; L^2(0,\ell))$ and $y$ satisfies by assumption

$$\begin{cases} \partial_t^2 y - \partial_x^2 y + p(x)y = 0, & (x,t) \in (0,\ell) \times (-T,T), \\ y(0,t) = 0, \quad \partial_x y(0,t) = 0, & t \in (-T,T). \end{cases}$$

Defining the cut-off function $\chi$ by (35), we apply Lemma 6 to $y_1 := \chi y$. Similar but simpler arguments to the proof of Lemma yields $y = 0$ in $Q_{4\delta}$. We also repeat the proof of the final part of Lemma 2, in view of (43), so that the proof of Lemma 5 is completed. □