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ON LOW TREE-DEPTH DECOMPOSITIONS

JAROSLAV NEŠETŘIL AND PATRICE OSSONA DE MENDEZ

ABSTRACT. The theory of sparse structures usually uses tree like structures as building blocks. In the context of sparse/dense dichotomy this role is played by graphs with bounded tree depth. In this paper we survey results related to this concept and particularly explain how these graphs are used to decompose and construct more complex graphs and structures. In more technical terms we survey some of the properties and applications of low tree depth decomposition of graphs.

1. TREE-DEPTH

The *tree-depth* of a graph is a minor monotone graph invariant that has been defined in [47], and which is equivalent or similar to the *rank function* (used for the analysis of countable graphs, see e.g. [56]), the *vertex ranking number* [12, 61], and the minimum height of an elimination tree [6]. Tree-depth can also be seen as an analog for undirected graphs of the cycle rank defined by Eggen [18], which is a parameter relating digraph complexity to other areas such as regular language complexity and asymmetric matrix factorization. The notion of tree-depth found a wide range of applications, from the study of non-repetitive coloring [25] to the proof of the homomorphism preservation theorem for finite structures [59]. Recall the definition of tree-depth:

Definition 1. The *tree-depth* $\text{td}(G)$ of a graph G is defined as the minimum height¹ of a rooted forest Y such that G is a subgraph of the closure of Y (that is of the graph obtained by adding edges between a vertex and all its ancestors). In particular, the tree-depth of a disconnected graph is the maximum of the tree-depths of its connected components.

Several characterizations of tree-depth have been given, which can be seen as possible alternative definitions. Let us mention:

TD1. The tree-depth of a graph is the order of the largest clique in a trivially perfect supergraph of G [65]. Recall that a graph is *trivially perfect* if it has the property that in each of its induced subgraphs the size of the maximum independent set equals the number of maximal cliques [31]. This characterization follows directly from the property that a connected graph is trivially perfect if and only if it is the comparability graph of a rooted tree [31].

TD2. The tree-depth of a graph is the minimum number of colors in a *centered coloring* of G , that is in a vertex coloring of G such that in every connected subgraph of G some color appears exactly once [47].

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¹Here the height is defined as the maximum number of vertices in a chain from a root to a leaf

TD3. A strongly related notion is vertex ranking, which has been investigated in [12, 61]. The *vertex ranking* (or *ordered coloring*) of a graph is a vertex coloring by a linearly ordered set of colors such that for every path in the graph with end vertices of the same color there is a vertex on this path with a higher color. The equality of the minimum number of colors in a vertex ranking and the tree-depth is proved in [47].

TD4. The tree-depth of a graph G with connected components G_1, \dots, G_p , is recursively defined by:

$$\text{td}(G) = \begin{cases} 1 & \text{if } G \simeq K_1 \\ \max_{i=1}^p \text{td}(G_i) & \text{if } G \text{ is disconnected} \\ 1 + \min_{v \in V(G)} \text{td}(G - v) & \text{if } G \text{ is connected and } G \not\simeq K_1 \end{cases}$$

The equivalence between the value given by this recursive definition and minimum height of an elimination tree, as well as the equality of this value with the tree-depth are proved in [47].

TD5. The tree-depth can also be defined by means of games, see [30, 33, 36]. In particular, this leads to a min-max formula for tree-depth in the spirit of the min-max formula relating tree-width and bramble size [62]. Precisely, a *shelter* in a graph G is a family \mathcal{S} of non-empty connected subgraphs of G partially ordered by inclusion such that for every subgraph $H \in \mathcal{S}$ not minimal in \mathcal{S} and for every $x \in H$ there exists $H' \in \mathcal{S}$ covered by H (in the partial order) such that $x \notin H'$. The *thickness* of a shelter \mathcal{S} is the minimal length of a maximal chain of \mathcal{S} . Then the tree-depth of a graph G equals the maximum thickness of a shelter in G [30].

TD6. Also, graphs with tree-depth at most t can be theoretically characterized by means of a finite set of forbidden minors, subgraphs, or even induced subgraphs. But in each case, the number of obstruction grows at least like a double (and at most a triple) exponential in t [16].

More generally, classes with bounded tree-depth can be characterized by several properties:

TD7. A class of graphs \mathcal{C} has bounded tree-depth if and only if there is some integer k such that graphs in \mathcal{C} exclude P_k as a subgraph. More precisely, while computing the tree-depth of a graph G is a hard problem, it can be (very roughly) approximated by considering the height h of a Depth-First Search tree of G , as $\lceil \log_2(h+2) \rceil \leq \text{td}(G) \leq h$ [53].

TD8. A class of graphs \mathcal{C} has bounded tree-depth if and only if there is some integers s, t, q such that graphs in \mathcal{C} exclude P_s, K_t , and $K_{q,q}$ as induced subgraphs (this follows from the previous item and [5, Theorem 3], which states that for every s, t , and q , there is a number $Z = Z(s, t, q)$ such that every graph with a path of length at least Z contains either P_s or K_t or $K_{q,q}$ as an induced subgraph.

TD9. A monotone class of graphs has bounded tree-depth if and only if it is well quasi-ordered for the induced-subgraph relation (with vertices possibly colored using $k \geq 2$ colors) (follows from [14]).

TD10. A monotone class of graphs has bounded tree-depth if and only if First-order logic (FO) and monadic second-order (MSO) logic have the same expressive power on the class [19].

Classes of graphs with tree-depth at most t are computationally very simple, as witnessed by the following properties:

It follows from **TD9** that every hereditary property can be tested in polynomial time when restricted to graphs with tree-depth at most t . Let us emphasize how one can combine **TD8** and **TD9** to get complexity results for P_s -free graphs. Recall that a graph G is k -choosable if for every assignment of a set $S(v)$ of k colors to every vertex v of G , there is a proper coloring of G that assigns to each vertex v a color from $S(v)$ [64, 23]. Note that in general, for $k > 2$, deciding k -choosability for bipartite graphs is Π_2^P -complete, hence more difficult than both NP and co-NP problems. It was proved in [34] that for P_5 -free graphs, that is, graphs excluding P_5 as an induced subgraph, k -choosability is fixed-parameter tractable. For general P_s -free graphs we prove:

Theorem 2. *For every integers s and k , there is a polynomial time algorithm to decide whether a P_s -free graph G is k -choosable.*

Proof. Assume G is P_s -free. We can decide in polynomial time whether G includes K_{k+1} or K_{k,k^k} as an induced subgraph. In the affirmative, G is not k -choosable. Otherwise, the tree-depth of G is bounded by some constant $C(s, k)$. As the property to be k -choosable is hereditary, we can use a polynomial time algorithm deciding whether a graph with tree-depth at most $C(s, k)$ is k -choosable \square

Graphs with tree-depth at most t have a (homomorphism) core of order bounded by a function of t [47]. In other words, every graph G with tree-depth at most t has an induced subgraph H of order at most $F(t)$ such that there exists an adjacency preserving map (that is: a *homomorphism*) from $V(G)$ to $V(H)$.

The complexity of checking the satisfaction of an MSO_2 property ϕ on a class with tree-depth at most t is in time $O(f(\phi, t) \cdot |G|)$, where f has an elementary dependence on ϕ [28]. This is in contrast with the dependence arising for MSO_2 -model checking in classes with bounded treewidth using Courcelle's algorithm [10], where f involves a tower of exponents of height growing with ϕ (what is generally unavoidable [26]). These properties led to the study of classes with bounded shrub-depth, generalizing classes with bounded tree-depth, and enjoying similar properties for MSO_1 -logic [28, 29]. Concerning the dependency on the tree-depth t , note that the $(t + 1)$ -fold exponential algorithm for MSO model-checking given by Gajarský and Hliněný in [27] is essentially optimal [42].

Graphs with bounded tree depth form the building blocks for more complicated graphs, with which we deal in the next section.

2. LOW TREE-DEPTH DECOMPOSITION OF GRAPHS

Several extensions of chromatic number have been proposed and studied in the literature. For instance, the *acyclic chromatic number* is the minimum number of colors in a proper vertex-coloring such that any two colors induce an acyclic graph (see e.g. [3, 7]). More generally, for a fixed parameter p , one can ask what is the minimum number of colors in a proper vertex-coloring of a graph G , such that any subset I of at most p colors induce a subgraph with treewidth at most $|I| - 1$. In this setting, the value obtained for $p = 1$ is the chromatic number, while the value obtained for $p = 2$ is the acyclic chromatic number.

In this setting, the following result has been proved by Devos, Oporowski, Sanders, Reed, Seymour and Vertigan using the structure theorem for graphs excluding a minor:

Theorem 3 ([13]). *For every proper minor closed class \mathcal{K} and integer $k \geq 1$, there is an integer $N = N(\mathcal{K}, k)$, such that every graph $G \in \mathcal{K}$ has a vertex partition into N graphs such that any $j \leq k$ parts form a graph with tree-width at most $j - 1$.*

The stronger concept of low tree-depth decomposition has been introduced by the authors in [47].

Definition 4. A *low tree-depth decomposition* with parameter p of a graph G is a coloring of the vertices of G , such that any subset I of at most p colors induce a subgraph with tree-depth at least $|I|$. The minimum number of colors in a low tree-depth decomposition with parameter p of G is denoted by $\chi_p(G)$.

For instance, $\chi_1(G)$ is the (standard) chromatic number of G , while $\chi_2(G)$ is the *star chromatic number* of G , that is the minimum number of colors in a proper vertex-coloring of G such that any two colors induce a star forest (see e.g. [2, 45]).

The authors were able to extend Theorem 3 to low tree-depth decomposition in [47]. Then, using the concept of transitive fraternal augmentation [48], the authors extended further existence of low tree-depth decomposition (with bounded number of colors) to classes with bounded expansion, the definition of which we recall now:

Definition 5. A class \mathcal{C} has *bounded expansion* if there exists a function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that every topological minor H of a graph $G \in \mathcal{C}$ has an average degree bounded by $f(p)$, where p is the maximum number of subdivisions per edge needed to turn H into a subgraph of G .

Extending low tree-depth decomposition to classes with bounded expansion in is the best possible:

Theorem 6 ([48]). *Let \mathcal{C} be a class of graphs, then the following are equivalent:*

- (1) *for every integer p it holds $\sup_{G \in \mathcal{C}} \chi_p(G) < \infty$;*
- (2) *the class \mathcal{C} has bounded expansion.*

Properties and characterizations of classes with bounded expansion will be discussed in more details in Section 5 (we refer the reader to [53] for a thorough analysis). Let us mention that classes with bounded expansion in particular include proper minor closed classes (as for instance planar graphs or graphs embeddable on some fixed surface), classes with bounded degree, and more generally classes excluding a topological minor. Thus on the one side the classes of graphs with bounded expansion include most of the sparse classes of structural graph theory, yet on the other side they have pleasant algorithmic and extremal properties.

On the other hand, one could ask whether for proper minor-closed classes one could ask there exists a stronger coloring than the one given by low tree-depth decompositions. Precisely, one can ask what is the minimum number of colors required for a vertex coloring of a graph G , so that any subgraph H of G gets at least $f(H)$ colors. (For instance that the star coloring corresponds to the graph function where any P_4 gets at least 3 colors.) Define the *upper chromatic number* $\bar{\chi}(H)$ of a graph H as the greatest integer, such that for any proper minor closed class of graph \mathcal{C} , there exists a constant $N = N(\mathcal{C}, H)$, such that any graph $G \in \mathcal{C}$ has a vertex coloring by at N colors so that any subgraph of G isomorphic to H gets at least $\bar{\chi}(H)$ colors. The authors proved in [47] that $\bar{\chi}(H) = \text{td}(H)$, showing that low tree-depth decomposition is the best we can achieve for proper minor closed classes. Note that the tree-depth of a graph G is also related to the chromatic numbers $\chi_p(G)$ by $\text{td}(G) = \max_p \chi_p(G)$ [47].

3. LOW TREE-DEPTH DECOMPOSITION AND RESTRICTED DUALITIES

The original motivation of low tree-depth decomposition was to prove the existence of a triangle free graph H such that every triangle-free planar G admits a

homomorphism to H , thus providing a structural strengthening of Grötzsch’s theorem [45]. Recall that a *homomorphism* of a graph G to a graph H is a mapping from the vertex set $V(G)$ of G to the vertex set $V(H)$ of H that preserves adjacency. The existence (resp. non-existence) of a homomorphism of G to H will be denoted by $G \rightarrow H$ (resp. by $G \not\rightarrow H$). We refer the interested reader to the monograph [35] for a detailed study of graph homomorphisms.

Thus the above planar triangle-free problem can be restated as follows: Prove that there exists a graph H such that $K_3 \not\rightarrow H$ and such that for every planar graph G it holds

$$K_3 \not\rightarrow G \iff G \rightarrow H.$$

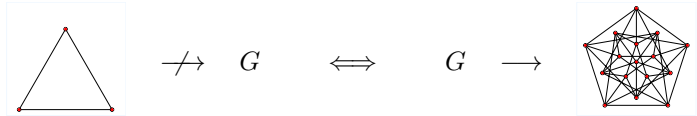
More generally, we are interested in the following problem: given a class of graphs \mathcal{C} and a connected graph F , find a graph $D_{\mathcal{C}}(F)$ for \mathcal{C} (which we shall refer to as a *dual* of F for \mathcal{C}), such that $F \not\rightarrow D_{\mathcal{C}}(F)$ and such that for every $G \in \mathcal{C}$ it holds

$$F \not\rightarrow G \iff G \rightarrow D_{\mathcal{C}}(F).$$

(Note that $D_{\mathcal{C}}(F)$ is not uniquely determined by the above equivalence.) A couple $(F, D_{\mathcal{C}}(F))$ with the above property is called a *restricted duality* of \mathcal{C} .

Example 7. For the special case of triangle-free planar graphs, the existence of a dual was proved by the authors in [47] and the minimum order dual has been proved to be the Clebsch graph by Naserasr [43].

\forall planar G :



Note that this restricted homomorphism duality extends to the class of all graphs excluding K_5 as a minor [44].

Example 8. A restricted homomorphism duality for toroidal graphs follows from the existence of a finite set of obstructions for 5-coloring proved by Thomassen in [63]: Noticing that all the obstructions shown Fig. 1 are homomorphic images of one of them, namely C_1^3 .

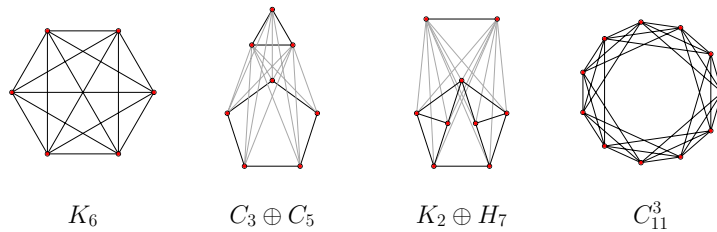
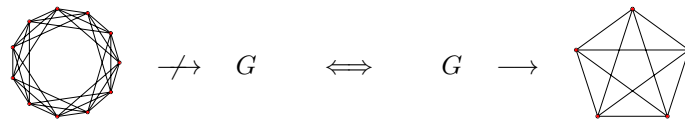


FIGURE 1. The 6-critical graphs for the torus.

Thus we get the following restricted homomorphism duality.

\forall toroidal G :



Definition 9. A class \mathcal{C} with the property that every connected graph F has a dual for \mathcal{C} is said to have *all restricted dualities*.

In [47] we proved, using low tree-depth decomposition, that for every proper minor closed class \mathcal{C} has all restricted dualities. We generalized in [50] this result to classes with bounded expansions. We briefly outline this.

In the study of restricted homomorphism dualities, a main tool appeared to be notion of t -approximation:

Definition 10. Let G be a graph and let t be a positive integer. A graph H is a t -approximation of G if G is homomorphic to H (i.e. $G \rightarrow H$) and every subgraph of H of order at most t is homomorphic to G .

Indeed the following theorem is proved in [54]:

Theorem 11. *Let \mathcal{C} be a class of graphs. Then the following are equivalent:*

- (1) *The class \mathcal{C} is bounded and has all restricted dualities (i.e. every connected graph F has a dual for \mathcal{C});*
- (2) *For every integer t there is a constant $N(t)$ such that every graph $G \in \mathcal{C}$ has a t -approximation of order at most $N(t)$.*

The following lemma stresses the connection existing between t -approximation and low tree-depth decomposition:

Theorem 12 ([54]). *For every integer t there exists a constant C_t such that every graph G has a t -approximation H with order*

$$|H| \leq C_t^{\chi_t(G)}.$$

Hence we have the following corollary of Theorems 11, 12, and 6, which was originally proved in [50]:

Corollary 1. *Every class with bounded expansion has all restricted dualities.*

The connection between classes with bounded expansion and restricted dualities appears to be even stronger, as witnessed by the following (partial) characterization theorem.

Theorem 13 ([54]). *Let \mathcal{C} be a topologically closed class of graphs (that is a class closed by the operation of graph subdivision). Then the following are equivalent:*

- (1) *the class \mathcal{C} has all restricted dualities;*
- (2) *the class \mathcal{C} has bounded expansion.*

This theorem has also a variant in the context of directed graphs:

Theorem 14 ([54]). *Let \mathcal{C} be a class of directed graphs closed by reorientation. Then the following are equivalent:*

- (1) *the class \mathcal{C} has all restricted dualities;*
- (2) *the class \mathcal{C} has bounded expansion.*

4. INTERMEZZO: LOW TREE-DEPTH DECOMPOSITION AND ODD-DISTANCE COLORING

Let n be an odd integer and let G be a graph. The problem of finding a coloring of the vertices of G with minimum number of colors such that two vertices at distance n are colored differently, called D_n -coloring of G , was introduced in 1977 in Graph Theory Newsletter by E. Sampathkumar [60] (see also [37]). In [60], Sampathkumar claimed that every planar graph has a D_n -coloring for every odd integer n with 5 colors, and conjectured that 4 colors suffice. Unfortunately, the claimed result was flawed, as witnessed by the graph depicted on Figure 2, which needs 6 colors for a D_3 -coloring [53].

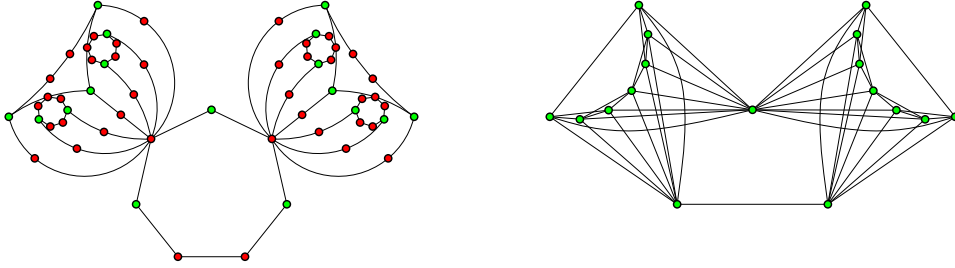


FIGURE 2. On the left, a planar graph G needing 6-colors for a D_3 -coloring. On the right, a witness: this a graph with vertex set $A \subset V(G)$ in which adjacent vertices are at distance 3 in G , thus should get distinct colors in a D_3 -coloring of G .

Low tree-depth decomposition allows to prove that for any odd integer n , a fixed number of colors is sufficient for D_n -coloring planar graphs, and this results extends to all classes with bounded expansion.

Theorem 15 ([53]). *For every class with bounded expansion \mathcal{C} and every odd integer n there exists a constant N such that every graph $G \in \mathcal{C}$ has a D_n -coloring with at most N colors.*

The proof of Theorem 15 relies on low tree-depth decomposition, and the bound N given in [53] for the number of colors sufficient for a D_n -coloring of a graph G is double exponential in $\chi_n(G)$. Hence it is still not clear whether a uniform bound could exist for D_n -coloring of planar graphs.

Problem 1 (van den Heuvel and Naserasr). Does there exist a constant C such that for every odd integer n , it holds that every planar graph has a D_n -coloring with at most C colors?

Note that, however, there exists no bound for the *odd-distance coloring* of planar graphs, which requires that two vertices at odd distance get different colors. Indeed, one can construct outerplanar graphs having an arbitrarily large subset of vertices pairwise at odd distance (see Fig. 3).

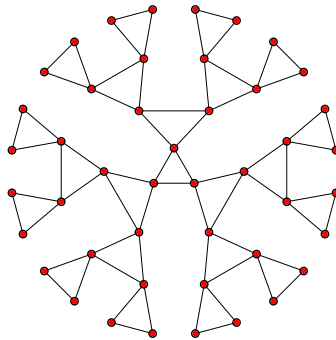


FIGURE 3. There exist outerplanar graphs with arbitrarily large subset of vertices pairwise at odd distance. (In the figure, the vertices in the periphery are pairwise at distance 1, 3, 5, or 7.)

However, no construction requiring a large number of colors without having a large set of vertices pairwise at odd-distance is known. Hence the following problem.

Problem 2 (Thomassé). Does there exist a function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that every planar graph without k vertices pairwise at odd distance has an odd-distance coloring with at most $f(k)$ colors?

5. LOW TREE-DEPTH DECOMPOSITION AND DENSITY OF SHALLOW MINORS, SHALLOW TOPOLOGICAL MINORS, AND SHALLOW IMMERSIONS

Classes with bounded expansion, which have been introduced in [48], may be viewed as a relaxation of the notion of proper minor closed class. The original definition of classes with bounded expansion relates to the notion of shallow minor, as introduced by Plotkin, Rao, and Smith [58].

Definition 16. Let G, H be graphs with $V(H) = \{v_1, \dots, v_h\}$ and let r be an integer. A graph H is a *shallow minor* of a graph G at depth r , if there exists disjoint subsets A_1, \dots, A_h of $V(G)$ such that (see Fig. 4)

- the subgraph of G induced by A_i is connected and has radius at most r ,
- if v_i is adjacent to v_j in H , then some vertex in A_i is adjacent in G to some vertex in A_j .

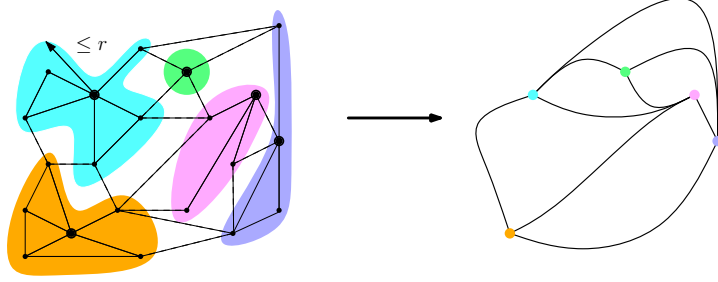


FIGURE 4. A shallow minor

We denote [48, 53] by $G \nabla r$ the class of the (simple) graphs which are shallow minors of G at depth r , and we denote by $\nabla_r(G)$ the maximum density of a graph in $G \nabla r$, that is:

$$\nabla_r(G) = \max_{H \in G \nabla r} \frac{\|H\|}{|H|}$$

A class \mathcal{C} has *bounded expansion* if $\sup_{G \in \mathcal{C}} \nabla_r(G) < \infty$ for each value of r .

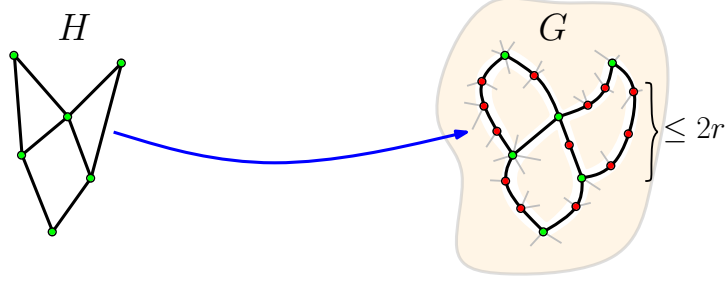
Considering shallow minors may, at first glance, look arbitrary. Indeed one can define as well the notions of shallow topological minors and shallow immersions:

Definition 17. A graph H is a *shallow topological minor* at depth r of a graph G if some subgraph of G is isomorphic to a subdivision of H in which every edge has been subdivided at most $2r$ times (see Fig. 5).

We denote [48, 53] by $G \tilde{\nabla} r$ the class of the (simple) graphs which are shallow topological minors of G at depth r , and we denote by $\tilde{\nabla}_r(G)$ the maximum density of a graph in $G \tilde{\nabla} r$, that is:

$$\tilde{\nabla}_r(G) = \max_{H \in G \tilde{\nabla} r} \frac{\|H\|}{|H|}$$

Note that shallow topological minors can be alternatively defined by considering how a graph H can be topologically embedded in a graph G : a graph H with vertex set $V(H) = \{a_1, \dots, a_k\}$ is a shallow topological minor of a graph G at depth r if there exists vertices v_1, \dots, v_k in G and a family \mathcal{P} of paths of G such that

FIGURE 5. H is a shallow topological minor of G at depth r

- two vertices a_i and a_j are adjacent in H if and only if there is a path in \mathcal{P} linking v_i and v_j ;
- no vertex v_i is interior to a path in \mathcal{P} ;
- the paths in \mathcal{P} are internally vertex disjoint;
- every path in \mathcal{P} has length at most $2r + 1$.

We can similarly define the notion of shallow immersion:

Definition 18. A graph H with vertex set $V(H) = \{a_1, \dots, a_k\}$ is a *shallow immersion* of a graph G at depth r if there exists vertices v_1, \dots, v_k in G and a family \mathcal{P} of paths of G such that

- two vertices a_i and a_j are adjacent in H if and only if there is a path in \mathcal{P} linking v_i and v_j ;
- the paths in \mathcal{P} are edge disjoint;
- every path in \mathcal{P} has length at most $2r + 1$;
- no vertex of G is internal to more than r paths in \mathcal{P} .

We denote [48, 53] by $G \overset{\sim}{\nabla}_r$ the class of the (simple) graphs which are shallow immersions of G at depth r , and we denote by $\overset{\sim}{\nabla}_r(G)$ the maximum density of a graph in $G \overset{\sim}{\nabla}_r$, that is:

$$\overset{\sim}{\nabla}_r(G) = \max_{H \in G \overset{\sim}{\nabla}_r} \frac{\|H\|}{|H|}$$

It appears that although minors, topological minors, and immersions behave very differently, their shallow versions are deeply related, as witnessed by the following theorem:

Theorem 19 ([53]). *Let \mathcal{C} be a class of graphs. Then the following are equivalent:*

- (1) *the class \mathcal{C} has bounded expansion;*
- (2) *for every integer r it holds $\sup_{G \in \mathcal{C}} \nabla_r(G) < \infty$;*
- (3) *for every integer r it holds $\sup_{G \in \mathcal{C}} \overset{\sim}{\nabla}_r(G) < \infty$;*
- (4) *for every integer r it holds $\sup_{G \in \mathcal{C}} \overset{\sim}{\nabla}_r(G) < \infty$;*
- (5) *for every integer r it holds $\sup_{H \in \mathcal{C}_{\nabla_r}} \chi(H) < \infty$;*
- (6) *for every integer r it holds $\sup_{H \in \mathcal{C}_{\overset{\sim}{\nabla}_r}} \chi(H) < \infty$;*
- (7) *for every integer r it holds $\sup_{H \in \mathcal{C}_{\overset{\sim}{\nabla}_r}} \chi(H) < \infty$.*

In the above theorem, we see that not only shallow minors, shallow topological minors, and shallow immersions behave closely, but that the (sparse) graph density $\|G\|/|G|$ and the chromatic number $\chi(G)$ of a graph G are also related. This last relation is intimately related to the following result of Dvořák [15].

Lemma 20. *Let $c \geq 4$ be an integer and let G be a graph with average degree $d > 56(c-1)^2 \frac{\log(c-1)}{\log c - \log(c-1)}$. Then the graph G contains a subgraph G' that is the 1-subdivision of a graph with chromatic number c .*

It follows from Theorem 19 that the notion of class with bounded expansion is quite robust. Not only classes with bounded expansion can be defined by edge densities and chromatic number, but also by virtually all common combinatorial parameters [53].

If one considers the clique number instead of the density or the chromatic number, then a different type of classes is defined:

Definition 21. A class of graph \mathcal{C} is *somewhere dense* if there exists an integer p such that every clique is a shallow topological minor at depth p of some graph in \mathcal{C} (in other words, $\mathcal{C} \tilde{\nabla} p$ contain all graphs); the class \mathcal{C} is *nowhere dense* if it is not somewhere dense.

Similarly that Theorem 19, we have several characterizations of nowhere dense classes.

Theorem 22 ([53]). *Let \mathcal{C} be a class of graphs. Then the following are equivalent:*

- (1) *the class \mathcal{C} is nowhere dense;*
- (2) *for every integer r it holds $\limsup_{G \in \mathcal{C}} \frac{\log \nabla_r(G)}{\log |G|} = 0$;*
- (3) *for every integer r it holds $\limsup_{G \in \mathcal{C}} \frac{\log \tilde{\nabla}_r(G)}{\log |G|} = 0$;*
- (4) *for every integer r it holds $\limsup_{G \in \mathcal{C}} \frac{\log \tilde{\tilde{\nabla}}_r(G)}{\log |G|} = 0$;*
- (5) *for every integer r it holds $\sup_{H \in \mathcal{C} \nabla_r} \omega(H) < \infty$;*
- (6) *for every integer r it holds $\sup_{H \in \mathcal{C} \tilde{\nabla}_r} \omega(H) < \infty$;*
- (7) *for every integer r it holds $\sup_{H \in \mathcal{C} \tilde{\tilde{\nabla}}_r} \omega(H) < \infty$.*

Note that every class with bounded expansion is nowhere dense. As mentioned in Theorem 6, classes with bounded expansion are also characterized by the fact that they allow low tree-depth decompositions with bounded number of colors. A similar statement holds for nowhere dense classes:

Precisely, we have the following:

Theorem 23. *Let \mathcal{C} be a class of graphs, then the following are equivalent:*

- (1) *for every integer p it holds $\limsup_{G \in \mathcal{C}} \frac{\chi_p(G)}{\log |G|} = 0$;*
- (2) *the class \mathcal{C} is nowhere dense.*

The direction bounding $\chi_p(G)$ of both Theorem 6 and 23 follow from the next more precise result:

Theorem 24 ([53]). *For every integer p there is a polynomial P_p ($\deg P_p \approx 2^{2^p}$) such that for every graph G it holds*

$$\chi_p(G) \leq P_p(\tilde{\tilde{\nabla}}_{2^{p-2}+1}(G)).$$

Note that the original proof given in [48] gave a slightly weaker bound, and that an alternative proof of this result has been obtained by Zhu [66], in a paper relating low tree-depth decomposition with the generalized coloring numbers introduced by Kierstead and Yang [40].

6. LOW TREE-DEPTH DECOMPOSITION AND COVERING

In a low treedepth decomposition of a graph G by N colors and for parameter t , the subsets of t colors define a disjoint union of clusters that cover the graph, such that each cluster has tree-depth at most t , every vertex belongs to at most $\binom{N}{t}$ clusters, and every connected subgraph of order t is included in at least one cluster.

It is natural to ask whether the condition that such a covering comes from a coloring could be dropped.

Theorem 25. *Let \mathcal{C} be a monotone class.*

Then \mathcal{C} has bounded expansion if and only if there exists a function f such that for every integer t , every graph $G \in \mathcal{C}$ has a covering C_1, \dots, C_k of its vertex set such that

- *each C_i induces a connected subgraph with tree-depth at most t ;*
- *every vertex belongs to at most $f(t)$ clusters;*
- *every connected subgraph of order at most t is included in at least one cluster.*

Proof. One direction is a direct consequence of Theorem 6. Conversely, assume that the class \mathcal{C} does not have bounded expansion. Then there exists an integer p such that for every integer d the class \mathcal{C} contains the p -th subdivision of a graph H_d with average degree at least d . Moreover, it is a standard argument that we can require H_d to be bipartite (as every graph with average degree $2d$ contains a bipartite subgraph with average degree at least d).

Let $t = 2(p+1)$ and let $d = 2f(t) + 1$. Assume for contradiction that there exist clusters C_1, \dots, C_k as required, then we can cover H_d by clusters C'_1, \dots, C'_k such that each C'_i induces a star (possibly reduced to an edge), every vertex belongs to at most $f(t)$ clusters, and every edge is included in at least one cluster. If an edge $\{u, v\}$ of H_d is included in more than two clusters, it is easily checked that (at least) one of u and v can be safely removed from one of the cluster. Hence we can assume that each edge of H_d is covered exactly once. To each cluster C'_i associates the center of the star induced by C'_i (or an arbitrary vertex of C'_i if C'_i has cardinality 2) and orient the edges of the star induced by C'_i away from the center. This way, every edge is oriented once and every vertex gets indegree at most $f(t)$. However, summing the indegrees we get $f(t) \geq d/2$, a contradiction. \square

It is natural to ask whether similar statements would hold, if we weaken the condition that each cluster has tree-depth at most t while we strengthen the condition that every connected subgraph of order at most t is included in some cluster. Namely, we consider the question whether a similar statement holds if we allow each cluster to have radius at most $2t$ while requiring that every t -neighborhood is included in some cluster. In the context of their solution of model checking problem for nowhere dense classes, Grohe, Kreutzer and Siebertz introduced in [32] the notion of r -neighborhood cover and proved that nowhere dense classes admit such cover with small maximum degree, and proved that nowhere dense classes and bounded expansion classes admit such nice covering.

Precisely, for $r \in \mathbb{N}$, an r -neighborhood cover \mathcal{X} of a graph G is a set of connected subgraphs of G called *clusters*, such that for every vertex $v \in V(G)$ there is some $X \in \mathcal{X}$ with $N_r(v) \subseteq X$. The *radius* $\text{rad}(\mathcal{X})$ of a cover \mathcal{X} is the maximum radius of its clusters. The *degree* $d^{\mathcal{X}}(v)$ of v in \mathcal{X} is the number of clusters that contain v . The *maximum degree* $\Delta(\mathcal{X}) = \max_{v \in V(G)} d^{\mathcal{X}}(v)$. For a graph G and $r \in \mathbb{N}$ we define $\tau_r(G)$ as the minimum maximum degree of an r -neighborhood cover of radius at most $2r$ of G .

The following theorem is proved in [32].

Theorem 26. *Let \mathcal{C} be a class of graphs with bounded expansion. Then there is a function f such that for all $r \in \mathbb{N}$ and all graphs $G \in \mathcal{C}$, it holds $\tau_r(G) \leq f(r)$.*

In order to prove the converse statement, we shall need the following result of Kühn and Osthus [41]:

Theorem 27. *For every k there exists $d = d(k)$ such that every graph of average degree at least d contains a subgraph of average degree at least k whose girth is at least six.*

We are now ready to turn Theorem 26 into a characterization theorem of classes with bounded expansion.

Theorem 28. *Let \mathcal{C} be an infinite monotone class of graphs. Then \mathcal{C} has bounded expansion if and only if, for every integer r it holds*

$$\sup_{G \in \mathcal{C}} \tau_r(G) < \infty.$$

Proof. One direction follows from Theorem 26. For the other direction, assume that the class \mathcal{C} does not have bounded expansion. Then there exists an integer p such that for every integer n , \mathcal{C} contains the p -th subdivision of a graph G_n with average degree at least n .

Let $d \in \mathbb{N}$. According to Theorem 27, there exists $N(d)$ such that every graph with average degree at least $N(d)$ contains a subgraph of girth 6 and average degree at least d . We deduce that \mathcal{C} contains the p -th subdivision H'_d of a graph H_d with girth at least 6 and average degree at least d . As in the proof of Theorem 31, we get

$$\sup_{G \in \mathcal{C}} \tau_{p+1}(G) \geq \sup_d \tau_{p+1}(H'_d) \geq \sup_d \tau_1(H_d) \geq \sup_d \frac{\|H_d\|}{|H_d|} = \infty.$$

□

Also, similar statements exist for nowhere dense classes:

Theorem 29. *A hereditary class \mathcal{C} is nowhere dense if there exists a function f such that for every integer t and every $\epsilon > 0$, every graph $G \in \mathcal{C}$ of order $n \geq f(t, \epsilon)$ has a covering C_1, \dots, C_k of its vertex set such that*

- each C_i induces a connected subgraph with tree-depth at most t ;
- every vertex belongs to at most n^ϵ clusters;
- every connected subgraph of order at most t is included in at least one cluster.

Proof. One direction directly follows from Theorem 23. For the reverse direction, assume that \mathcal{C} is not nowhere dense. Then there exists p such that for every $n \in \mathbb{N}$, the class \mathcal{C} contains a graph G_n having the p -th subdivision of K_n as the spanning subgraph. Assume that a covering exists for $t = 3p + 3$. Then every p -subdivided triangle of K_n is included in some cluster. As the p -subdivided K_n includes $\binom{n}{3}$ triangles, and as there are at most $n^{1+\epsilon}$ clusters including some principal vertex of the subdivided K_n (which is necessary to include some subdivided triangle), some cluster C includes at least $n^{2-\epsilon}$ triangles. It follows that the subgraph induced by C has a minor H of order at most n with at least $n^{2-\epsilon}$ triangles. However, as tree-depth is minor monotone, the graph H has tree-depth at most t hence is t -degenerate thus cannot contain more than $\binom{t}{2}n$ triangles. Whence we are led to a contradiction if $n > \binom{t}{2}^{\frac{1}{1-\epsilon}}$. □

Theorem 30 ([32]). *Let \mathcal{C} be a nowhere dense class of graphs. Then there is a function f such that for all $r \in \mathbb{N}$ and $\epsilon > 0$ and all graphs $G \in \mathcal{C}$ with $n \geq f(r, \epsilon)$ vertices, it holds $\tau_r(G) \leq n^\epsilon$.*

In other words, every infinite nowhere dense class of graphs \mathcal{C} is such that

$$\sup_{r \in \mathbb{N}} \limsup_{G \in \mathcal{C}} \frac{\log \tau_r(G)}{\log |G|} = 0.$$

We shall deduce from this theorem the following characterization of nowhere dense classes of graphs.

Theorem 31. *Let \mathcal{C} be an infinite monotone class of graphs. Then*

$$\sup_{r \in \mathbb{N}} \limsup_{G \in \mathcal{C}} \frac{\log \tau_r(G)}{\log |G|}$$

is either 0 if \mathcal{C} is nowhere dense, at least $1/3$ if \mathcal{C} is somewhere dense.

This theorem will directly follow from Theorem 30 and the following two lemmas.

Lemma 32. *Let G be a graph of girth at least 5. Then it holds*

$$\tau_1(G) \geq \nabla_0(G),$$

where

$$\nabla_0(G) = \max_{H \subseteq G} \frac{\|H\|}{|H|}.$$

Proof. Let \mathcal{X} be a 1-neighborhood cover of radius at most 2 of G with maximum degree $\tau_1(G)$. Let X_1, \dots, X_k be the clusters of \mathcal{X} . For an edge $e = \{u, v\}$, let $i \leq k$ be the minimum integer such that $N_1(u)$ or $N_1(v)$ is included in X_i . Let c_i be a center of X_i . Then e belongs to a path of length at most 2 with endpoint c_i . We orient e according to the orientation of this path away from c_i . Note that by the process, we orient every edge, and that every vertex v gets at most one incoming edge by cluster that contains v . Hence we constructed an orientation of G with maximum degree at most $\tau_1(G)$. As the maximum indegree of an orientation of G is at least $\nabla_0(G)$, we get $\tau_1(G) \geq \nabla_0(G)$. \square

We deduce the following

Lemma 33. *Let \mathcal{C} be a monotone somewhere dense class of graphs. Then*

$$\sup_{r \in \mathbb{N}} \limsup_{G \in \mathcal{C}} \frac{\log \tau_r(G)}{\log |G|} \geq \frac{1}{3}.$$

Proof. A \mathcal{C} is monotone and somewhere dense, there exists integer $p \geq 0$ such that for every $n \in \mathbb{N}$, the p -th subdivision $\text{Sub}_p(K_n)$ of K_n belongs to \mathcal{C} . For $n \in \mathbb{N}$, let H_n be a graph of girth at least 5, with order $|H_n| \sim n$ and size $\|H_n\| \sim n^{3/2}$. If $p = 0$, then according to Lemma 32 it holds

$$\begin{aligned} \sup_{r \in \mathbb{N}} \limsup_{G \in \mathcal{C}} \frac{\log \tau_r(G)}{\log |G|} &\geq \limsup_{G \in \mathcal{C}} \frac{\log \tau_1(G)}{\log |G|} \\ &\geq \lim_{n \rightarrow \infty} \frac{\log \nabla_0(H_n)}{\log |H_n|} \\ &\geq \lim_{n \rightarrow \infty} \frac{\log \|H_n\| - \log |H_n|}{\log |H_n|} = \frac{1}{2}. \end{aligned}$$

Thus assume $p \geq 1$. Denote by H'_n the p -th subdivision of H_n , where we identify $V(H_n)$ with a subset of $V(H'_n)$ for convenience. Then $|H_n| \sim pn^{3/2}$. Let $\mathcal{X} = \{X_1, \dots, X_k\}$ be a $(p+1)$ -neighborhood cover of radius at most $2(p+1)$ of H'_n with maximum degree $\tau_{p+1}(H'_n)$. Let c_i be a center of cluster X_i , and let d_i be a vertex of H_n at minimal distance of c_i in H'_n . It is easily checked that there exists a cluster X'_i with center d_i and radius $2(p+1)$ such that $X_i \cap V(H_n) = X'_i \cap V(H_n)$. Define $Y_i = X'_i \cap V(H_n)$. As \mathcal{X} is a $(p+1)$ -neighborhood cover of radius at most $2(p+1)$ of H'_n with maximum degree $\tau_{p+1}(H'_n)$, the cover $\mathcal{Y} = \{Y_i\}$ is a 1-neighborhood cover of radius 2 of H_n with maximum degree $\tau_{p+1}(H'_n)$. Hence $\tau_1(H_n) \leq \tau_{p+1}(H'_n)$. Thus

it holds

$$\begin{aligned} \sup_{r \in \mathbb{N}} \limsup_{G \in \mathcal{C}} \frac{\log \tau_r(G)}{\log |G|} &\geq \lim_{n \rightarrow \infty} \frac{\log \tau_{p+1}(H'_n)}{\log |H'_n|} \\ &\geq \lim_{n \rightarrow \infty} \frac{\log \tau_1(H_n)}{\log |H'_n|} \\ &\geq \lim_{n \rightarrow \infty} \frac{\log \|H_n\| - \log |H_n|}{\log |H'_n|} = \frac{1}{3}. \end{aligned}$$

□

7. ALGORITHMIC APPLICATIONS OF LOW TREE-DEPTH DECOMPOSITION

Theorem 24 has the following algorithmic version.

Theorem 34 ([53]). *There exist polynomials P_p ($\deg P_p \approx 2^{2^p}$) and an algorithm that computes, for input graph G and integer p , a low tree-depth decomposition of G with parameter p using $N_p(G)$ colors in time $O(N_p(G) |G|)$, where*

$$\chi_p(G) \leq N_p(G) \leq P_p(\tilde{\nabla}_{2^{p-2}+1}(G)).$$

It is not surprising that low tree-depth decompositions have immediately found several algorithmic applications [46, 49].

As noticed in [8], the existence of an orientation of planar graphs with bounded out-degree allows for a planar graph G (once such an orientation has been computed for G) an easy $O(1)$ adjacency test, and an enumeration of all the triangles of G in linear time.

For a fixed pattern H , the problem is to check whether an input graph G has an induced subgraph isomorphic to H is called the *subgraph isomorphism problem*. This problem is known to have complexity at most $O(n^{\omega l/3})$ where l is the order of H and where ω is the exponent of square matrix fast multiplication algorithm [55] (hence $O(n^{0.792 l})$ using the fast matrix algorithm of [9]). The particular case of subgraph isomorphism in planar graphs have been studied by Plehn and Voigt [57], Alon [4] with super-linear bounds and then by Eppstein [20, 21] who gave the first linear time algorithm for fixed pattern H and G planar. This was extended to graphs with bounded genus in [22]. We further generalized this result to classes with bounded expansion [49]:

Theorem 35. *There is a function f and an algorithm such that for every input graphs G and H , counts the number of occurrences of H in G in time*

$$O(f(H) (N_{|H|}(G))^{|H|} |G|),$$

where $N_p(G)$ is the number of colors computed by the algorithm in Theorem 34.

In particular, for every fixed bounded expansion class (resp. nowhere dense class) \mathcal{C} and every fixed pattern H , the number of occurrences of H in a graph $G \in \mathcal{C}$ can be computed in linear time (resp. in time $O(|G|^{1+\epsilon})$ for any fixed $\epsilon > 0$).

Theorem 35 can be extended from the subgraph isomorphism problem to first-order model checking.

Theorem 36 ([17], see also [11]). *Let \mathcal{C} be a class of graphs with bounded expansion, and let ϕ be a first-order sentence (on the natural language of graphs). There exists a linear time algorithm that decides whether a graph $G \in \mathcal{C}$ satisfies ϕ .*

The above theorem relies on low tree-depth decomposition. However, the next result, due to Kazana and Segoufin, is based on the notion of transitive fraternal augmentation, which was introduced in [48] to prove Theorem 24.

Theorem 37 ([39]). *Let \mathcal{C} be a class of graphs with bounded expansion and let ϕ be a first-order formula. Then, for all $G \in \mathcal{C}$, we can compute the number $|\phi(G)|$ of satisfying assignments for ϕ in G in time $O(|G|)$.*

Moreover, the set $\phi(G)$ can be enumerated in lexicographic order in constant time between consecutive outputs and linear time preprocessing time.

Eventually, the existence of efficient model checking algorithm has been extended to nowhere dense classes by Grohe, Kreutzer, and Siebertz [32] using the notion of r -neighborhood cover we already mentioned:

Theorem 38. *For every nowhere dense class \mathcal{C} and every $\epsilon > 0$, every property of graphs definable in first-order logic can be decided in time $O(n^{1+\epsilon})$ on \mathcal{C} .*

However, it is still open whether a counting version of Theorem 38 (in the spirit of Theorem 37) holds.

8. LOW TREE-DEPTH DECOMPOSITION AND LOGARITHMIC DENSITY OF PATTERNS

We have seen in the Section 7 that low tree-depth decomposition allows an easy counting of patterns. It appears that they also allow to prove some “extremal” results. A typical problem studied in extremal graph theory is to determine the maximum number of edges $\text{ex}(n, H)$ a graph on n vertices can contain without containing a subgraph isomorphic to H . For non-bipartite graph H , the seminal result of Erdős and Stone [24] gives a tight bound:

Theorem 39.

$$\text{ex}(n, H) = \left(1 - \frac{1}{\chi(H) - 1}\right) \binom{n}{2} + o(n^2).$$

In the case of bipartite graphs, less is known. Let us mention the following result of Alon, Krivelevich and Sudakov [1]

Theorem 40. *Let H be a bipartite graph with maximum degree r on one side.*

$$\text{ex}(n, H) = O(n^{2-\frac{1}{r}}).$$

The special case where H is a subdivision of a complete graph will be of prime interest in the study of nowhere dense classes. Precisely, denoting $\text{ex}(n, K_t^{(\leq p)})$ the maximum number of edges a graph on n vertices can contain without containing a subdivision of K_t in which every edge is subdivided at most p times, Jiang [38] proved the following bound:

Theorem 41. *For every integers k, p it holds*

$$\text{ex}(n, K_k^{(\leq p)}) = O(n^{1+\frac{10}{p}}).$$

From this theorem follows that if a class \mathcal{C} is such that $\limsup_{G \in \mathcal{C}} \tilde{\nu}_t \frac{\log \|G\|}{\log |G|} > 1 + \epsilon$ then $\mathcal{C} \tilde{\nu}_{\frac{10t}{\epsilon}}$ contains graphs with unbounded clique number. This property is a main ingredient in the proof of the following classification “trichotomy” theorem.

Theorem 42 ([52]). *Let \mathcal{C} be an infinite class of graphs. Then*

$$\sup_t \limsup_{G \in \mathcal{C}} \tilde{\nu}_t \frac{\log \|G\|}{\log |G|} \in \{-\infty, 0, 1, 2\}.$$

Moreover, \mathcal{C} is nowhere dense if and only if $\sup_t \limsup_{G \in \mathcal{C}} \tilde{\nu}_t \frac{\log \|G\|}{\log |G|} \leq 1$.

Note that the property that the logarithmic density of edges is integral needs to consider all the classes $\mathcal{C} \tilde{\vee} t$. For instance, the class \mathcal{D} of graphs with no C_4 has a bounding logarithmic edge density of $3/2$, which jumps to 2 when one considers $\mathcal{D} \tilde{\vee} 1$.

Using low tree-depth decomposition, it is possible to extend Theorem 42 to other pattern graphs:

Theorem 43 ([51]). *For every infinite class of graphs \mathcal{C} and every graph F*

$$\lim_{i \rightarrow \infty} \limsup_{G \in \mathcal{C} \tilde{\vee} i} \frac{\log(\#F \subseteq G)}{\log |G|} \in \{-\infty, 0, 1, \dots, \alpha(F), |F|\},$$

where $\alpha(F)$ is the stability number of F .

Moreover, if F has at least one edge, then \mathcal{C} is nowhere dense if and only if $\lim_{i \rightarrow \infty} \limsup_{G \in \mathcal{C} \tilde{\vee} i} \frac{\log(\#F \subseteq G)}{\log |G|} \leq \alpha(F)$.

The main ingredient in the proof of this theorem is the analysis of local configurations, called (k, F) -sunflowers (see Fig. 6). Precisely, for graphs F and G , a (k, F) -sunflower in G is a $(k+1)$ -tuple $(C, \mathcal{F}_1, \dots, \mathcal{F}_k)$, such that $C \subseteq V(G)$, $\mathcal{F}_i \subseteq \mathcal{P}(V(G))$, the sets in $\{C\} \cup \mathcal{F}_i$ are pairwise disjoint and there exists a partition (K, Y_1, \dots, Y_k) of $V(F)$ so that

- $\forall i \neq j, \omega(Y_i, Y_j) = \emptyset$,
- $G[C] \approx F[K]$,
- $\forall X_i \in \mathcal{F}_i, G[X_i] \approx F[Y_i]$,
- $\forall (X_1, \dots, X_k) \in \mathcal{F}_1 \times \dots \times \mathcal{F}_k$, the subgraph of G induced by $C \cup X_1 \cup \dots \cup X_k$ is isomorphic to F .

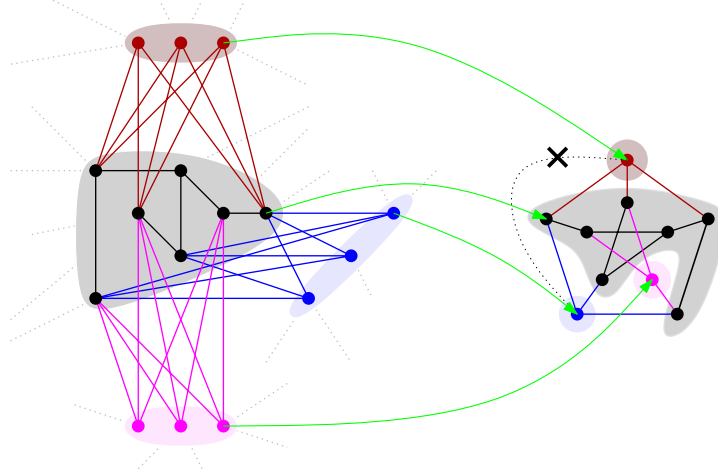


FIGURE 6. A (3, Petersen)-sunflower

The following stepping up lemma gives some indication on how low tree-depth decomposition is related to the proof of Theorem 43:

Lemma 44 ([51]). *There exists a function τ such that for every integers p, k , every graph F of order p , every $0 < \epsilon < 1$, the following property holds:*

Every graph G such that $(\#F \subseteq G) > |G|^{k+\epsilon}$ contains a $(k+1, F)$ -sunflower $(C, \mathcal{F}_1, \dots, \mathcal{F}_{k+1})$ with

$$\min_i |\mathcal{F}_i| \geq \left(\frac{|G|}{\binom{\chi_p(G)}{p}^{1/\epsilon}} \right)^{\tau(\epsilon, p)}$$

In particular, G contains a subgraph G' such that

$$|G'| \geq (k+1) \left(\frac{|G|}{\binom{\chi_p(G)}{p}^{1/\epsilon}} \right)^{\tau(\epsilon, p)}$$

and $(\#F \subseteq G') \geq \left(\frac{|G'| - |F|}{k+1} \right)^{k+1}$.

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JAROSLAV NEŠETRIL, COMPUTER SCIENCE INSTITUTE OF CHARLES UNIVERSITY (IUUK AND ITI), MALOSTRANSKÉ NÁM.25, 11800 PRAHA 1, CZECH REPUBLIC
E-mail address: nesetril@iuuk.mff.cuni.cz

PATRICE OSSONA DE MENDEZ, CENTRE D’ANALYSE ET DE MATHÉMATIQUES SOCIALES (CNRS, UMR 8557), 190-198 AVENUE DE FRANCE, 75013 PARIS, FRANCE — AND — COMPUTER SCIENCE INSTITUTE OF CHARLES UNIVERSITY (IUUK), MALOSTRANSKÉ NÁM.25, 11800 PRAHA 1, CZECH REPUBLIC
E-mail address: pom@ehess.fr