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ON THE BERTOZZI-ESEDOGLU-GILLETTE-CAHN-HILLIARD EQUATION WITH LOGARITHMIC NONLINEAR TERMS

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Abstract. Our aim in this paper is to study the existence of local (in time) solutions for the Bertozzi-Esedoglu-Gillette-Cahn-Hilliard equation with logarithmic nonlinear terms. This equation was proposed in view of applications to binary image inpainting. We also give some numerical simulations which show the efficiency of the model.

1. Introduction

The Cahn-Hilliard equation plays an important role in materials science and describes phase separation processes. This can be observed, e.g., when a binary alloy is cooled down sufficiently. One then observes a partial nucleation (i.e., the apparition of nucleides in the material) or a total nucleation, the so-called spinodal decomposition: the material quickly becomes inhomogeneous, forming a fine-grained structure in which each of the two components appears more or less alternatively. In a second stage, which is called coarsening and occurs at a slower time scale, these microstructures coarsen. Such phenomena play an essential role in the mechanical properties of the material, e.g., strength. We refer the reader to, e.g., [5], [6], [9], [15], [23], [24], [26], [27], [35] and [36] for more details.

It is also interesting to note that the Cahn-Hilliard equation, or some of its variants, is relevant in other contexts, in which phase separation and coarsening/clustering processes can be observed or come into play. We can mention, for instance, population dynamics (see [12]), bacterial films (see [22]), wound healing and tumor growth (see [10], [21] and [29]), thin films (see [38] and [39]), image processing and inpainting (see [2], [3], [4], [7], [8] and [14]) and even the rings of Saturn (see [40]) and the clustering of mussels (see [25]).

In particular, in [2] and [3], the authors proposed the following variant of the Cahn-Hilliard equation:

$$\frac{\partial u}{\partial t} + \epsilon\Delta^2 u - \frac{1}{\epsilon}\Delta f(u) + \lambda_0\chi_{\Omega\setminus D}(u-h) = 0,\ \epsilon > 0,\ \lambda_0 > 0, \tag{1.1}$$

in view of applications to binary image inpainting. Here, $h = h(x)$ is a given (damaged) image and $D \subset \Omega$ is the inpainting region (Ω is the total region). Furthermore, the term $\lambda_0\chi_{\Omega\setminus D}(u-h)$, called fidelity term, is added in order to keep the solution u close to the image h outside the damaged region (χ denotes the indicator function). Finally, the nonlinear term f is regular and cubic, typically, $f(s) = 4s^3 - 6s^2 + 2s$. The idea in this

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model is to solve (1.1) up to steady state in order to obtain an inpainted version $u(x)$ of $h(x)$.

This equation was studied, endowed with Neumann boundary conditions, in [2], [4] and [8]. In particular, one has well-posedness and regularity results, as well as the existence of finite-dimensional attractors. Furthermore, numerical simulations were given. In particular, the simulations in [8] show that, in some situations, a dynamic one step scheme with threshold involving the diffuse interface thickness ϵ (we note that, in [2] and [3], the authors first considered a large value of ϵ and then a smaller one in order to obtain their numerical simulations) allows to connect regions across large inpainting domains.

Our aim in this paper is to consider (1.1) now with logarithmic nonlinear terms f (note indeed that the original Cahn-Hilliard equation was actually proposed with thermodynamically relevant logarithmic nonlinear terms which follow from a mean-field model; regular (and, in particular, cubic) nonlinear terms are approximations of such logarithmic nonlinear terms).

The Bertozzi-Esedoglu-Gillette-Cahn-Hilliard equation, with logarithmic nonlinearities and Neumann boundary conditions, appears to be much more complicated, from a mathematical point of view, than the Cahn-Hilliard equation. Consequently, we are only able to prove the local (in time) existence of solutions.

We also give some numerical simulations which confirm that the one step algorithm with threshold proposed in [8] is efficient. Actually, in that case, we can obtain better results, when using logarithmic nonlinear terms, than those obtained with polynomial nonlinear terms, as far as the convergence time is concerned. Furthermore, we give an example for which the one step algorithm gives much better results when considering a logarithmic nonlinearity.

Notation. We denote by $((\cdot,\cdot))$ the usual L^2-scalar product, with associated norm $\|\cdot\|$. We further set $\|\cdot\|_{-1} = \|(-\Delta)^{-\frac{1}{2}}\cdot\|$, where $(-\Delta)^{-1}$ denotes the inverse minus Laplace operator associated with Neumann boundary conditions and acting on functions with null spatial average. More generally, $\|\cdot\|_X$ denotes the norm on the Banach space X.

Throughout the paper, the same letters c and c' denote (generally positive) constants which may vary from line to line.

2. Setting of the problem

We consider the following initial and boundary value problem in a bounded and regular domain $\Omega \subset \mathbb{R}^n$, $n = 1$, 2 or 3, with boundary Γ:

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) + \chi_{\Omega\setminus D}(x)(u-h) = 0, \tag{2.1}$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0 \text{ on } \Gamma, \tag{2.2}$$

$$u|_{t=0} = u_0, \tag{2.3}$$

where $D \Subset \Omega$ (for simplicity, we have set all constants in (1.1) equal to one).

We assume that $h \in L^2(\Omega)$ and

$$\int_{\Omega\setminus D} h\,dx = 0. \tag{2.4}$$

Remark 2.1. We need (2.4) in view of the mathematical analysis of the problem. However, this condition is not necessary for the numerical simulations below.

Furthermore, as far as the nonlinear term f is concerned, we assume that

$$f = F',\ \text{where } F(s) = \frac{\lambda_1}{2}(1-s^2) + \frac{\lambda_2}{2}((1-s)\ln\frac{1-s}{2} + (1+s)\ln\frac{1+s}{2}), \tag{2.5}$$

$$0 < \lambda_2 < \lambda_1,\ s \in (-1,1),$$

$$\text{hence } f(s) = -\lambda_1 s + \frac{\lambda_2}{2}\ln\frac{1+s}{1-s},\ s \in (-1,1).$$

Moreover, there holds

$$f' \geq -\lambda_1. \tag{2.6}$$

Writing $F(s) = \frac{\lambda_1}{2}(1-s^2) + F_1(s)$ and $f_1 = F_1'$, we introduce, following [20] and for $N \in \mathbb{N}$, the approximated function $F_{1,N} \in C^4(\mathbb{R})$ defined by

$$F_{1,N}^{(4)}(s) = \begin{cases} F_1^{(4)}(1-\frac{1}{N}),\ s \geq 1-\frac{1}{N}, \\ F_1^{(4)}(s),\ |s| \leq 1-\frac{1}{N}, \\ F_1^{(4)}(-1+\frac{1}{N}),\ s \leq -1+\frac{1}{N}, \end{cases} \tag{2.7}$$

$$F_{1,N}^{(k)}(0) = F_1^{(k)}(0),\ k = 0,\ 1,\ 2,\ 3, \tag{2.8}$$

so that

$$F_{1,N}(s) = \begin{cases} \sum_{k=0}^4 \frac{1}{k!}F_1^{(k)}(1-\frac{1}{N})(s-1+\frac{1}{N})^k,\ s \geq 1-\frac{1}{N}, \\ F_1(s),\ |s| \leq 1-\frac{1}{N}, \\ \sum_{k=0}^4 \frac{1}{k!}F_1^{(k)}(-1+\frac{1}{N})(s+1-\frac{1}{N})^k,\ s \leq -1+\frac{1}{N}. \end{cases} \tag{2.9}$$

Setting $F_N(s) = \frac{\lambda_1}{2}(1-s^2) + F_{1,N}(s)$, $f_{1,N} = F_{1,N}'$ and $f_N = F_N'$, there holds

$$f_{1,N}' \geq 0,\ f_N' \geq -\lambda_1, \tag{2.10}$$

$$F_N \geq -c_1,\ c_1 \geq 0, \tag{2.11}$$

and (see [20] and [33])

$$f_N(s)s \geq c_2(F_N(s) + |f_N(s)|) - c_3,\ c_2 > 0,\ c_3 \geq 0,\ s \in \mathbb{R}, \tag{2.12}$$

where the constants c_i, $i = 1$, 2 and 3, are independent of N, for N large enough. We further have the

Proposition 2.2. *There holds, for N large enough,*

$$(f_N(s+a) - f_N(a))s \geq c_4(s^4 + a^2s^2) - c_5,\ c_4 > 0,\ c_5 \geq 0,\ s,\ a \in \mathbb{R}, \tag{2.13}$$

where the constants c_4 and c_5 are independent of N.

Proof. Note that it suffices to prove (2.13) for $f_{1,N}$. Furthermore, all constants below are independent of N.

Case 1: $s + a \geq 1 - \frac{1}{N}$, $a \geq 1 - \frac{1}{N}$.

We have, in that case and for N large enough,

$$(f_{1,N}(s+a) - f_{1,N}(a))s = s\sum_{k=0}^{3} \frac{1}{k!} f_1^{(k)}(1 - \frac{1}{N})((s+a-1+\frac{1}{N})^k - (a-1+\frac{1}{N})^k)$$

$$\geq \frac{1}{6}((s+a-1+\frac{1}{N})^3 - (a-1+\frac{1}{N})^3)s.$$

Here and below, we use the facts that $f_1^{(k)}(1 - \frac{1}{N}) \geq 0$, $k = 0$, ..., 3, and $\lim_{N\to+\infty} f_1'''(1 - \frac{1}{N}) = +\infty$. It then follows from [8] that

$$(f_{1,N}(s+a) - f_{1,N}(a))s \geq c(s^4 + s^2(a-1+\frac{1}{N})^2) - c'$$

$$\geq c(s^4 + a^2s^2) - c',\ c > 0,$$

noting that $|1 - \frac{1}{N}| \leq 1$.

Case 2: $s + a \geq 1 - \frac{1}{N}$, $|a| \leq 1 - \frac{1}{N}$.

Note that, in that case, $s \geq 1 - \frac{1}{N} - a \geq 0$. Furthermore,

$$(f_{1,N}(s+a) - f_{1,N}(a))s = (\sum_{k=0}^{3} \frac{1}{k!} f_1^{(k)}(1 - \frac{1}{N})(s+a-1+\frac{1}{N})^k - f_1(a))s.$$

Noting that $f_1(1 - \frac{1}{N}) \geq f_1(a)$ and that $|a| \leq 1$, we obtain, for N large enough,

$$(f_{1,N}(s+a) - f_{1,N}(a))s \geq \frac{1}{6}(s+a-1+\frac{1}{N})^3 s$$

$$\geq \frac{1}{6}(s-2)^3 s \geq cs^4 - c'$$

$$\geq c(s^4 + a^2s^2) - c',\ c > 0.$$

Case 3: $s + a \geq 1 - \frac{1}{N}$, $a \leq -1 + \frac{1}{N}$.

Noting that $f_1(-s) = -f_1(s)$, we have

$$(f_{1,N}(s+a)-f_{1,N}(a))s=(2f_1(1-\frac{1}{N})+f_1'(1-\frac{1}{N})(s-2+\frac{2}{N})$$

$$+\frac{1}{2}f_1''(1-\frac{1}{N})((s+a-1+\frac{1}{N})^2+(a+1-\frac{1}{N})^2)+\frac{1}{6}f_1'''(1-\frac{1}{N})(s+a-1+\frac{1}{N})^3-(a+1-\frac{1}{N})^3)s.$$

Therefore, noting that $s\geq 1-\frac{1}{N}-a\geq 2-\frac{2}{N}$ and $a\leq 0$, we find, owing again to [8] and for N large enough,

$$(f_{1,N}(s+a)-f_{1,N}(a))s\geq\frac{1}{6}((s+a-1+\frac{1}{N})^3-(a+1-\frac{1}{N})^3)s$$

$$\geq\frac{1}{12}((s+a)^3-a^3)s-cs$$

$$\geq c(s^4+a^2s^2)-c',\ c>0.$$

Case 4: $|s+a|\leq 1-\frac{1}{N}$, $|a|\leq 1-\frac{1}{N}$.

In that case, we have, noting that $f_1'\geq 0$,

$$(f_{1,N}(s+a)-f_{1,N}(a))s\geq 0\geq K(s^4+a^2s^2)-c_K,\ \forall K>0,$$

since $|a|\leq 1$ and $|s|\leq 2$.

Case 5: $|s+a|\leq 1-\frac{1}{N}$, $a\geq 1-\frac{1}{N}$.

Note that, in that case, $s\leq 1-\frac{1}{N}-a\leq 0$ and, as $s\to-\infty$, $a\sim -s$. Furthermore, for N large enough,

$$(f_{1,N}(s+a)-f_{1,N}(a))s=(f_1(s+a)-f_1(1-\frac{1}{N})-f_1'(1-\frac{1}{N})(a-1+\frac{1}{N})$$

$$-\frac{1}{2}f_1''(1-\frac{1}{N})(a-1+\frac{1}{N})^2-\frac{1}{6}f_1'''(1-\frac{1}{N})(a-1+\frac{1}{N})^3)s$$

$$\geq-\frac{1}{6}(a-1+\frac{1}{N})^3s\geq-\frac{1}{12}a^3s+cs.$$

Therefore, since, as $s\to-\infty$,

$$-\frac{1}{12}a^3s+cs\sim\frac{1}{12}s^4\sim\frac{1}{24}(s^4+a^2s^2),$$

we deduce that, for $s\leq -s_0$ and $a\geq s_0$ (with $|s+a|\leq 1-\frac{1}{N}$), $s_0>0$ independent of N,

$$(f_{1,N}(s+a)-f_{1,N}(a))s\geq\frac{1}{48}(s^4+a^2s^2)$$

and the result follows.

The remaining cases can be treated in a similar way.

□

3. A priori estimates

In this section, all constants are independent of N.

We consider, for $N \in \mathbb{N}$, the approximated problem

$$\frac{\partial u^N}{\partial t} + \Delta^2 u^N - \Delta f_N(u^N) + \chi_{\Omega\setminus D}(x)(u^N - h) = 0, \tag{3.1}$$

$$\frac{\partial u^N}{\partial \nu} = \frac{\partial \Delta u^N}{\partial \nu} = 0 \text{ on } \Gamma, \tag{3.2}$$

$$u^N|_{t=0} = u_0. \tag{3.3}$$

First, integrating (3.1) over Ω, we have, owing to (2.4),

$$\frac{d\langle u^N\rangle}{dt} + \frac{1}{\mathrm{Vol}(\Omega)} \int_{\Omega\setminus D} u^N \, dx = 0, \tag{3.4}$$

where $\langle\cdot\rangle = \frac{1}{\mathrm{Vol}(\Omega)} \int_\Omega \cdot \, dx$. Setting $u^N = \langle u^N\rangle + v^N$ (so that $\langle v^N\rangle = 0$), we can rewrite (3.4) as

$$\frac{d\langle u^N\rangle}{dt} + c_0\langle u^N\rangle = -\frac{1}{\mathrm{Vol}(\Omega)} \int_{\Omega\setminus D} v^N \, dx, \tag{3.5}$$

where $c_0 = \frac{\mathrm{Vol}(\Omega\setminus D)}{\mathrm{Vol}(\Omega)}$ and v^N is solution to

$$\frac{\partial v^N}{\partial t} + \Delta^2 v^N - \Delta(f_N(u^N) - \langle f_N(u^N)\rangle) + \chi_{\Omega\setminus D}(x)(u^N - h) - \langle \chi_{\Omega\setminus D}(x)(u^N - h)\rangle = 0, \tag{3.6}$$

$$\frac{\partial v^N}{\partial \nu} = \frac{\partial \Delta v^N}{\partial \nu} = 0 \text{ on } \Gamma, \tag{3.7}$$

$$v^N|_{t=0} = v_0 = u_0 - \langle u_0\rangle. \tag{3.8}$$

We rewrite (3.6)-(3.7) in the equivalent form

$$(-\Delta)^{-1}\frac{\partial v^N}{\partial t} - \Delta v^N + f_N(u^N) - \langle f_N(u^N)\rangle \tag{3.9}$$

$$+(-\Delta)^{-1}(\chi_{\Omega\setminus D}(x)(u^N - h) - \langle \chi_{\Omega\setminus D}(x)(u^N - h)\rangle) = 0,$$

$$\frac{\partial v^N}{\partial \nu} = 0 \text{ on } \Gamma. \tag{3.10}$$

We multiply (3.9) by v^N to obtain

$$\frac{1}{2}\frac{d}{dt}\|v^N\|_{-1}^2 + \|\nabla v^N\|^2 \tag{3.11}$$

$$+((f_N(u^N) - \langle f_N(u^N)\rangle, v^N)) + ((\chi_{\Omega\setminus D}(x)(u^N - h), (-\Delta)^{-1}v^N)) = 0.$$

Noting that

$$((f_N(u^N) - \langle f_N(u^N)\rangle, v^N)) = ((f_N(u^N) - f_N(\langle u^N\rangle), v^N)),$$

it follows from (2.13) that

$$((f_N(u^N) - \langle f_N(u^N)\rangle, v^N)) \geq c_4(\|v^N\|_{L^4(\Omega)}^4 + \langle u^N\rangle^2\|v^N\|^2) - c. \tag{3.12}$$

Furthermore,

$$|((\chi_{\Omega\setminus D}(x)(u^N - h), (-\Delta)^{-1}v^N))| \leq c(\|v^N\|^2 + |\langle u^N\rangle|\|v^N\| + \|h\|^2) \tag{3.13}$$

$$\leq \frac{c_4}{2}(\|v^N\|_{L^4(\Omega)}^4 + \langle u^N\rangle^2\|v^N\|^2) + c(\|h\|^2 + 1).$$

We thus deduce from (3.11)-(3.13) that

$$\frac{d}{dt}\|v^N\|_{-1}^2 + \|\nabla v^N\|^2 + c_4(\|v^N\|_{L^4(\Omega)}^4 + \langle u^N\rangle^2\|v^N\|^2) \leq c(\|h\|^2 + 1). \tag{3.14}$$

Next, it follows from (3.5) that

$$\frac{d\langle u^N\rangle^2}{dt} + c_0\langle u^N\rangle^2 \leq c\|v^N\|^2,$$

hence

$$\frac{d\langle u^N\rangle^2}{dt} + c_0\langle u^N\rangle^2 \leq \frac{c_4}{2}(\|v^N\|_{L^4(\Omega)}^4 + \langle u^N\rangle^2\|v^N\|^2) + c. \tag{3.15}$$

Summing (3.14) and (3.15), we find a differential inequality of the form

$$\frac{dE_{1,N}}{dt} + c(\|u^N\|_{H^1(\Omega)}^2 + \|v^N\|_{L^4(\Omega)}^4 + \langle u^N\rangle^2\|v^N\|^2) \leq c(\|h\|^2 + 1),\ c > 0, \tag{3.16}$$

where

$$E_{1,N} = \langle u^N\rangle^2 + \|v^N\|_{-1}^2$$

satisfies

$$E_{1,N} \geq c\|u^N\|_{H^{-1}(\Omega)}^2,\ c > 0, \tag{3.17}$$

where $H^{-1}(\Omega)$ is the topological dual of $H^1(\Omega)$. Here, we have used the fact that $v \mapsto (\langle v\rangle^2 + \|v - \langle v\rangle\|_{-1}^2)^{\frac{1}{2}}$ (resp., $v \mapsto (\langle v\rangle^2 + \|\nabla v\|^2)^{\frac{1}{2}}$) is a norm on $H^{-1}(\Omega)$ (resp., $H^1(\Omega)$)

which is equivalent to the usual one (being understood that, for $v \in H^{-1}(\Omega)$, then $\langle v \rangle = \frac{1}{\mathrm{Vol}(\Omega)} \langle v, 1 \rangle_{H^{-1}(\Omega), H^1(\Omega)}$).

We then multiply (3.1) by u^N and have, owing to (2.10),

$$\frac{d}{dt}\|u^N\|^2 + \|\Delta u^N\|^2 \leq 2\lambda_1 \|\nabla u^N\|^2 + c(\|u^N\|^2 + \|h\|^2). \tag{3.18}$$

Summing (3.16) and (3.18) multiplied by δ_1, where $\delta_1 > 0$ is chosen small enough, we obtain a differential inequality of the form

$$\frac{dE_{2,N}}{dt} + c(\|u^N\|^2_{H^2(\Omega)} + \|v^N\|^4_{L^4(\Omega)} + \langle u^N \rangle^2 \|v^N\|^2) \leq c(\|h\|^2 + 1),\ c > 0, \tag{3.19}$$

where

$$E_{2,N} = \delta_1 \|u^N\|^2 + E_{1,N}$$

satisfies

$$E_{2,N} \geq c\|u^N\|^2,\ c > 0. \tag{3.20}$$

We now rewrite (3.1)-(3.2) in the equivalent form

$$\frac{\partial u^N}{\partial t} + \chi_{\Omega \setminus D}(x)(u_N - h) = \Delta \mu^N, \tag{3.21}$$

$$\mu^N = -\Delta u^N + f_N(u^N), \tag{3.22}$$

$$\frac{\partial u^N}{\partial \nu} = \frac{\partial \mu^N}{\partial \nu} = 0 \text{ on } \Gamma, \tag{3.23}$$

where, by analogy with the original Cahn-Hilliard equation, μ^N is called chemical potential.

We multiply (3.21) by μ^N and (3.22) by $\frac{\partial u^N}{\partial t}$ to find

$$\frac{1}{2}\frac{d}{dt}(\|\nabla u^N\|^2 + 2\int_\Omega F_N(u^N)\,dx) + \|\nabla \mu^N\|^2 = -((u^N - h, \chi_{\Omega \setminus D}(x)\mu^N)). \tag{3.24}$$

Furthermore, multiplying (3.22) by $\chi_{\Omega \setminus D}(x)u^N$, we have

$$((u^N, \chi_{\Omega \setminus D}(x)\mu^N)) = -((\Delta u^N, \chi_{\Omega \setminus D}(x)u^N)) + \int_{\Omega \setminus D} f_N(u^N)u^N\,dx. \tag{3.25}$$

Finally, it follows from (2.4) that

$$((h, \chi_{\Omega \setminus D}(x)\mu^N)) = ((\chi_{\Omega \setminus D}(x)h, \mu^N - \langle \mu^N \rangle)),$$

hence

$$|((h, \chi_{\Omega\setminus D}(x)\mu^N))| \le c\|h\|\|\nabla\mu^N\|. \tag{3.26}$$

We deduce from (2.12) and (3.24)-(3.26) that

$$\frac{d}{dt}(\|\nabla u^N\|^2 + 2\int_\Omega F_N(u^N)\,dx) \tag{3.27}$$

$$+c(\|\nabla\mu^N\|^2 + \int_{\Omega\setminus D}|f_N(u^N)|\,dx + \int_{\Omega\setminus D}F_N(u^N)\,dx) \le c'(\|u^N\|^2_{H^2(\Omega)} + \|h\|^2),\ c>0.$$

Summing (3.19) and (3.27) multiplied by δ_2, where $\delta_2 > 0$ is chosen small enough, we obtain a differential inequality of the form

$$\frac{dE_{3,N}}{dt} + c(\|u^N\|^2_{H^2(\Omega)} + \|v^N\|^4_{L^4(\Omega)} + \langle u^N\rangle^2\|v^N\|^2 \tag{3.28}$$

$$+\int_{\Omega\setminus D}|f_N(u^N)|\,dx + \int_{\Omega\setminus D}F_N(u^N)\,dx + \|\nabla\mu^N\|^2) \le c(\|h\|^2+1),\ c>0,$$

where

$$E_{3,N} = \delta_2(\|\nabla u^N\|^2 + 2\int_\Omega F_N(u^N)\,dx) + E_{2,N}$$

satisfies

$$E_{3,N} \ge c\|u^N\|^2_{H^1(\Omega)} - c',\ c>0. \tag{3.29}$$

Rewriting (3.21)-(3.22) in the equivalent form

$$(-\Delta)^{-1}\frac{\partial v^N}{\partial t} + (-\Delta)^{-1}(\chi_{\Omega\setminus D}(x)(u_N - h) - \langle\chi_{\Omega\setminus D}(x)(u_N-h)\rangle) = -(\mu^N - \langle\mu^N\rangle), \tag{3.30}$$

$$\mu^N - \langle\mu^N\rangle = -\Delta v^N + f_N(u^N) - \langle f_N(u^N)\rangle, \tag{3.31}$$

we deduce from (3.30) that

$$\|\frac{\partial v^N}{\partial t}\|_{-1} \le c(\|u^N\| + \|\nabla\mu^N\| + \|h\|),$$

hence, owing to (3.5),

$$\|\frac{\partial u^N}{\partial t}\|_{H^{-1}(\Omega)} \le c(\|u^N\| + \|\nabla\mu^N\| + \|h\|). \tag{3.32}$$

Furthermore, (3.31) yields

$$\|f_N(u^N) - \langle f_N(u^N)\rangle\| \le c(\|u^N\|_{H^2(\Omega)} + \|\nabla\mu^N\|). \tag{3.33}$$

It thus follows from (3.28) and (3.32)-(3.33) that

$$\frac{dE_{3,N}}{dt} + c(\|u^N\|^2_{H^2(\Omega)} + \|v^N\|^4_{L^4(\Omega)} + \langle u^N\rangle^2\|v^N\|^2 \tag{3.34}$$

$$+\|\frac{\partial u^N}{\partial t}\|^2_{H^{-1}(\Omega)} + \|f_N(u^N) - \langle f_N(u^N)\rangle\|^2$$

$$+\int_{\Omega\setminus D} |f_N(u^N)|\,dx + \int_{\Omega\setminus D} F_N(u^N)\,dx + \|\nabla\mu^N\|^2) \leq c(\|h\|^2 + 1),\ c > 0.$$

We can note that (3.34) is not sufficient to pass to the limit in the nonlinear term $f_N(u^N)$ (say, in a variational formulation). To do so, we also need an estimate on $|\langle f_N(u^N)\rangle|$ (in order to have an estimate on $\|f_N(u^N)\|$). This could be done if we were able to prove that $|\langle u^N(t)\rangle| \leq 1-\delta$, $t \geq 0$, $\delta \in (0,1)$ (see [32]; see also below). Unfortunately, we are not able to prove such a result and, therefore, we will only be able to obtain a local (in time) result.

We now assume that $|\langle u_0\rangle| < 1$. Then, there exists $\delta \in (0,1)$ such that $|\langle u_0\rangle| \leq 1-2\delta$. Therefore, since the function $t \mapsto \langle u^N(t)\rangle$ is continuous, there exists $T_0 = T_0(\delta, N)$ such that, if $t \in [0, T_0]$, then $|\langle u^N(t)\rangle| \leq 1-\delta$.

Actually, we can note that it follows from (3.5) that

$$\langle u^N(t)\rangle = e^{-c_0 t}\langle u_0\rangle - e^{-c_0 t}\int_0^t e^{c_0 s}\,ds \int_{\Omega\setminus D} v^N\,dx,$$

so that

$$|\langle u^N(t)\rangle| \leq |\langle u_0\rangle| + ce^{-c_0 t}\int_0^t e^{c_0 s}\|u^N\|\,ds \tag{3.35}$$

$$\leq 1 - 2\delta + c(1 - e^{-c_0 t}),$$

where we emphasize that $c = c(u_0)$ is independent of N (note indeed that it follows from (3.19)-(3.20) and Gronwall's lemma that $\|u^N\|$ is bounded uniformly with respect to time and N). We can thus find $T_0 = T_0(\delta, u_0)$ independent of N such that, if $t \in [0, T_0]$, then $|\langle u^N(t)\rangle| \leq 1-\delta$.

Then, noting that we have a similar result for f (see [32]), it is not difficult to prove that, for N large enough,

$$f_N(s+m)s \geq c_m|f_N(s+m)| - c'_m,\ c_m > 0,\ c'_m \geq 0,\ s \in \mathbb{R},\ m \in (-1,1), \tag{3.36}$$

where the constants c_m and c'_m depend continuously on m (see also [33]).

Remark 3.1. When $|m| > 1$, then we cannot expect to have such a result. Indeed, if, e.g., $m = 3$, then, as $s \to -2^-$, $f(s+3)s$ tends to $-\infty$, while $|f(s+3)|$ tends to $+\infty$.

Having (3.36), we obtain, proceeding as in [32], Proposition A.2,

$$|\langle f_N(v)\rangle| \leq c_\delta \|v - \langle v\rangle\| \|f_N(v) - \langle f_N(v)\rangle\| + c'_\delta, \tag{3.37}$$

$$c_\delta > 0,\ c'_\delta \geq 0,\ v \in L^2(\Omega),\ |\langle v\rangle| \leq 1-\delta,\ \delta \in (0,1),$$

for $N \geq N_0 = N_0(\delta)$.

It then follows from (3.37) (taking $v = u^N$) that

$$|\langle f_N(u^N)\rangle| \leq c_\delta \|v^N\| \|f_N(u^N) - \langle f_N(u^N)\rangle\| + c'_\delta,\ t \in [0,T_0],$$

hence

$$\int_0^{T_0} |\langle f_N(u^N)\rangle|^2\, ds \leq c_\delta \|v^N\|^2_{L^\infty(0,T_0;L^2(\Omega))} \|f_N(u^N) - \langle f_N(u^N)\rangle\|^2_{L^2((0,T_0)\times\Omega)} + c'_\delta. \tag{3.38}$$

Therefore, noting that $v \mapsto (|\langle v\rangle|^2 + \|v - \langle v\rangle\|^2)^{\frac{1}{2}}$ is a norm on $L^2(\Omega)$ which is equivalent to the usual L^2-norm, (3.33) and (3.38) yield that

$$\|f_N(u^N)\|_{L^2((0,T_0)\times\Omega)} \tag{3.39}$$

$$\leq c_\delta(\|u^N\|_{L^\infty(0,T_0;L^2(\Omega))} + 1)(\|u^N\|_{L^2(0,T_0;H^2(\Omega))} + \|\nabla \mu^N\|_{L^2((0,T_0)\times\Omega)^n}) + c'_\delta.$$

Noting finally that $\langle \mu^N\rangle = \langle f_N(u^N)\rangle$, we deduce that

$$\|\mu^N\|_{L^2(0,T_0;H^1(\Omega))} \tag{3.40}$$

$$\leq c_\delta(\|u^N\|_{L^\infty(0,T_0;L^2(\Omega))} + 1)(\|u^N\|_{L^2(0,T_0;H^2(\Omega))} + \|\nabla \mu^N\|_{L^2((0,T_0)\times\Omega)^n}) + c'_\delta.$$

4. A local existence result

We have the

Theorem 4.1. *We assume that $u_0 \in H^1(\Omega)$, $\int_\Omega F(u_0)\, dx < +\infty$, $|\langle u_0\rangle| < 1$ and $-1 < u_0(x) < 1$ a.e. $x \in \Omega$. Then, there exists $T_0 = T_0(u_0)$ and a solution to (2.1)-(2.3) on $[0,T_0]$ such that $u \in \mathcal{C}([0,T_0];H^{-1}(\Omega)) \cap L^\infty(0,T_0;H^1(\Omega)) \cap L^2(0,T_0;H^2(\Omega))$ and $\frac{\partial u}{\partial t} \in L^2(0,T_0;H^{-1}(\Omega))$. Furthermore, $-1 < u(t,x) < 1$ a.e. $(t,x) \in (0,T_0)\times\Omega$.*

Proof. We consider the solution u^N to the approximated problem (3.1)-(3.3) (the proof of existence, uniqueness and regularity of such a solution can be adapted from the results in [8], owing to (2.13)). Then, it follows from the a priori estimates derived in the previous section that, up to a subsequence, this solution converges to a limit function u such that

$$u^N \to u \text{ in } L^\infty(0,T_0;H^1(\Omega)) \text{ weak}-\star \text{ and in } L^2(0,T_0;H^2(\Omega)) \text{ weak},$$

$$u^N \to u \text{ a.e. } (t,x) \in (0,T_0)\times\Omega,$$

$$\frac{\partial u^N}{\partial t} \to \frac{\partial u}{\partial t} \text{ in } L^2(0,T_0;H^{-1}(\Omega)) \text{ weak.}$$

The only difficulty here is to pass to the limit in the nonlinear term $f_N(u^N)$.

First, it follows from (3.39) that $f_N(u^N)$ is bounded, independently of N, in $L^1((0,T_0)\times\Omega)$. Then, it follows from the explicit expression of f_N that

$$\text{meas}(E_{N,M}) \leq c\varphi(\frac{1}{N}),\ N \leq M,$$

where

$$E_{N,M} = \{(t,x) \in (0,T_0)\times\Omega,\ |u^M(t,x)| > 1 - \frac{1}{N}\}$$

and

$$\varphi(s) = \frac{1}{|f(1-s)|},$$

the constant c being independent of N and M. Note indeed that there holds

$$\int_0^{T_0}\int_\Omega |f_M(u^M)|\,dx\,dt \geq \int_{E_{N,M}} |f_M(u^M)|\,dx\,dt \geq c'\text{meas}(E_{N,M})|f(1-\frac{1}{N})|, \tag{4.1}$$

where the constant c' is independent of N and M (recall that $f(-s) = -f(s)$). We can pass to the limit $M \to +\infty$ (employing Fatou's lemma, see (4.1)) and then $N \to +\infty$ (noting that $\lim_{s\to 0}\varphi(s) = 0$) to find

$$\text{meas}\{(t,x) \in (0,T_0)\times\Omega,\ |u(t,x)| \geq 1\} = 0,$$

so that

$$-1 < u(t,x) < 1 \text{ a.e. } (t,x) \in (0,T_0)\times\Omega.$$

Next, it follows from the above almost everywhere convergence of u^N to u (and also from the explicit expression of f_N) that

$$f_N(u^N) \to f(u) \text{ a.e. } (t,x) \in (0,T_0)\times\Omega. \tag{4.2}$$

Finally, since, owing to (3.39), $f_N(u^N)$ is bounded, independently of N, in $L^2((0,T_0)\times\Omega)$, it follows from (4.2) that $f_N(u^N) \to f(u)$ in $L^2((0,T_0)\times\Omega)$ weak, which finishes the proof of the passage to the limit.

□

Remark 4.2. We assume that $|u_0(x)| \leq 1-\delta$ a.e., $\delta \in (0,1)$. Then, recalling that $f_N(s) = f(s)$ when $|s| \leq 1 - \frac{1}{N}$, we can also obtain a local (in time) existence result. However, the local existence given by Theorem 4.1 is more general.

Remark 4.3. a) When the positive constant c (which depends on Ω, D, h and, for the more general equation (1.1), on ϵ and λ_0) in (3.35) is small (namely, $c \leq \delta$), then one actually has a global (in time) existence result. Note however that, in concrete applications, λ_0 is large, while ϵ can be small, so that this condition should be too restrictive.

b) When $D = \emptyset$, in which case one obtains the so-called Oono equation (for $h = 0$; see [28], [37] and [41]), one has a global well-posedness result (see [31]). One also has a global well-posedness result, for the Bertozzi-Esedoglu-Gillette-Cahn-Hilliard equation, when considering Dirichlet boundary conditions (see [30]).

5. Numerical simulations

As far as the numerical simulations are concerned, we rewrite the problem in the form

$$\frac{\partial u}{\partial t} + \Delta\mu + \lambda_0 \chi_{\Omega\setminus D}(x)(u-h) = 0, \tag{5.1}$$

$$\mu = \epsilon\Delta u - \frac{1}{\epsilon} f(u), \tag{5.2}$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial \mu}{\partial \nu} = 0 \text{ on } \Gamma, \tag{5.3}$$

$$u_{|t=0} = u_0, \tag{5.4}$$

which has the advantage of splitting the fourth-order (in space) equation into a system of two second-order ones (see [16], [18] and [19]). Consequently, we use a P1-finite element for the space discretization, together with a semi-implicit Euler time discretization (i.e., implicit for the linear terms and explicit for the nonlinear ones). The numerical simulations are performed with the software Freefem++ (see [17]).

In the numerical results presented below, Ω is a $(0, 0.5) \times (0, 0.5)$-square. The triangulation is obtained by dividing Ω into 200×200 rectangles and by dividing each rectangle along the same diagonal.

In order to obtain the final inpainting results, we use a dynamic one step algorithm with threshold involving the diffuse interface thickness ϵ (see [8]).

5.1. **Inpainting of a triangle.**

The gray region in Figure 1(a) corresponds to the inpainting region. We run the modified Cahn-Hilliard equation with $f(s) = -2\ln(3)s + \ln\left(\frac{1+s}{1-s}\right)$ (note that f vanishes at -0.5 and 0.5), $\epsilon = 0.03$, $\lambda_0 = 900000$ and $\Delta t = 0.05$. Furthermore, we take the initial datum u_0 in $[0, 0.5]$ (random in the inpainting region), so that $|\langle u_0 \rangle| < 1$. We observe that the solution remains in $(-1, 1)$ when considering an intermediate value of the diffuse interface thickness ϵ. We are close to a steady state at $t = 0.45$, as shown in Figure 1(b), and we replace all values larger than $\frac{1}{4}$ by $\frac{1}{2}$ and all those smaller than $\frac{1}{4}$ by 0 to obtain the final inpainting in Figure 1(c).

Then, we run again the modified Cahn-Hilliard equation with the same $\epsilon = 0.03$, $\Delta t = 0.05$ and $\lambda_0 = 900000$, but we now take $f(s) = 4s^3 - 6s^2 + 2s$ and the initial datum in $[0, 1]$ instead of $[0, 0.5]$ in order to have comparable inpainting results (note that the solution does not remain in the relevant interval $[0, 1]$). We are close to a steady state

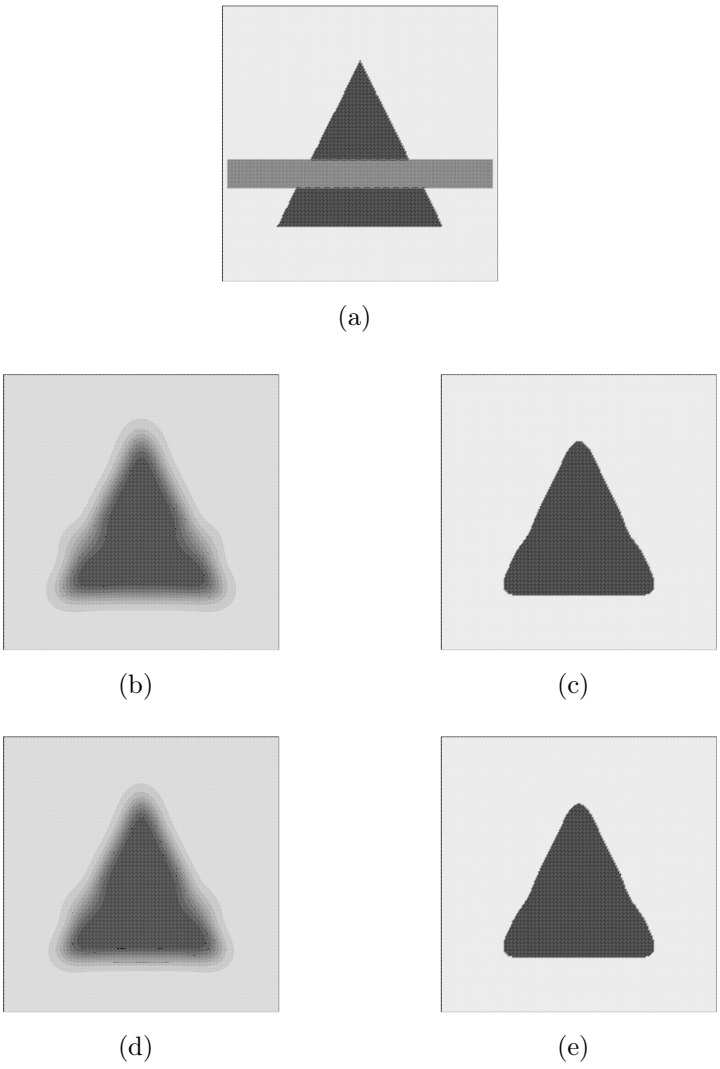


FIGURE 1. (a) Inpainting region in gray, $\epsilon = 0.03$. (b) Solution at $t = 0.45$, $f(s) = -2\ln(3)s + \ln\left(\frac{1+s}{1-s}\right)$. (c) Replacing the values larger than $\frac{1}{4}$ by $\frac{1}{2}$ and those smaller than $\frac{1}{4}$ by 0. (d) Solution at $t = 1.2$, $f(s) = 4s^3 - 6s^2 + 2s$. (e) Replacing the values larger than $\frac{1}{2}$ by 1 and those smaller than $\frac{1}{2}$ by 0.

at $t = 1.2$, as shown in Figure 1(d), and we replace all values larger than $\frac{1}{2}$ by 1 and all those smaller than $\frac{1}{2}$ by 0 to obtain the final inpainting in Figure 1(e).

5.2. **Inpainting of a bar.** Here and in the other simulations below, the initial datum is taken as above.

In Figure 2(a), the gray region corresponds to the inpainting region. We run the modified Cahn-Hilliard equation with $f(s) = -2\ln(3)s + \ln\left(\frac{1+s}{1-s}\right)$, $\epsilon = 0.05$, $\Delta t = 0.05$ and $\lambda_0 = 900000$. We are close to a steady state at $t = 0.4$, as shown in Figure 2(b), and

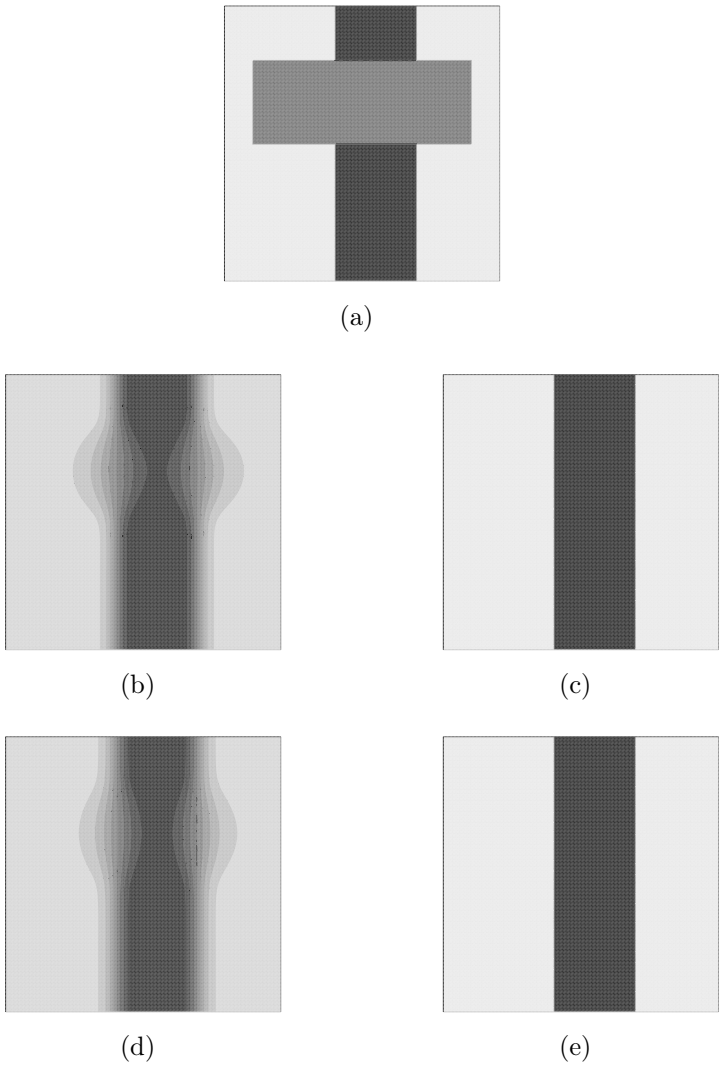


FIGURE 2. (a) Inpainting region in gray, $\epsilon = 0.05$. (b) Solution at $t = 0.4$, $f(s) = -2\ln(3)s + \ln\left(\frac{1+s}{1-s}\right)$. (c)Replacing the values larger than $\frac{1}{4}$ by $\frac{1}{2}$ and those smaller than $\frac{1}{4}$ by 0. (d) Solution at $t = 0.75$, $f(s) = 4s^3 - 6s^2 + 2s$. (e) Replacing the values larger than $\frac{1}{2}$ by 1 and those smaller than $\frac{1}{2}$ by 0.

we replace all values larger than $\frac{1}{4}$ by $\frac{1}{2}$ and all those smaller than $\frac{1}{4}$ by 0 to obtain the final inpainting in Figure 2(c).

Furthermore, we run again the modified Cahn-Hilliard equation with the same $\epsilon = 0.05$ and $\Delta t = 0.05$, but we now take $f(s) = 4s^3 - 6s^2 + 2s$ and $\lambda_0 = 500000$ (note that, contrary to the next example below, a smaller value of λ_0 allows to have comparable inpainting results). We are close to a steady state at $t = 0.75$, as shown in Figure 2(d), and we replace all values larger than $\frac{1}{2}$ by 1 and all those smaller than $\frac{1}{2}$ by 0 to obtain the final inpainting in Figure 2(e).

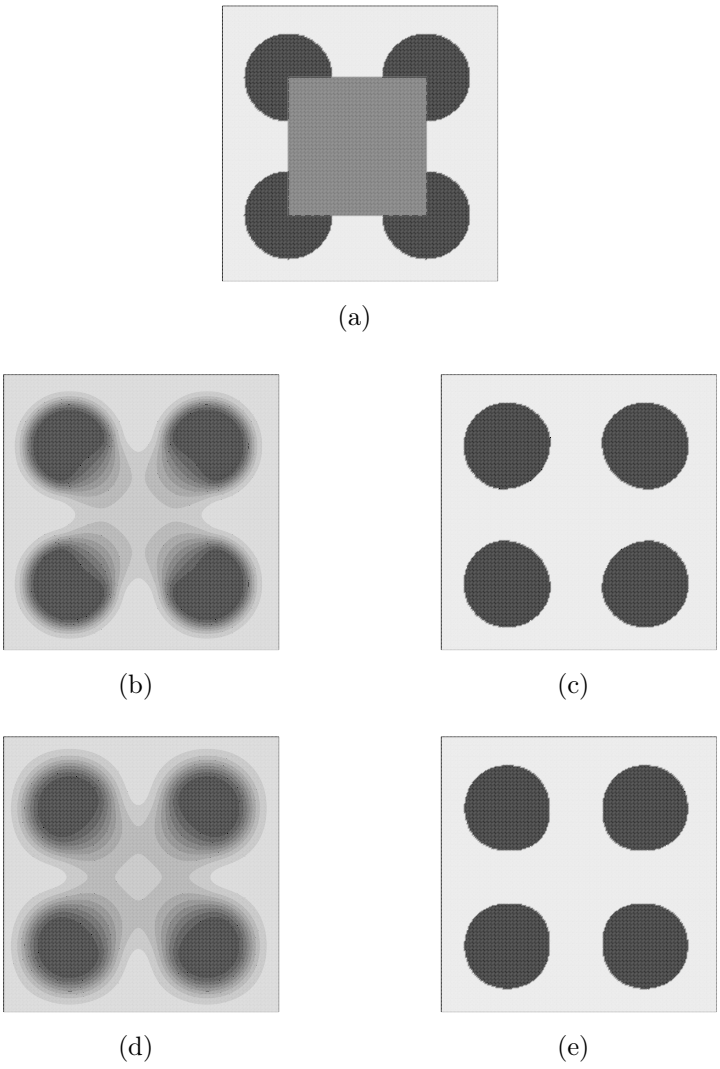


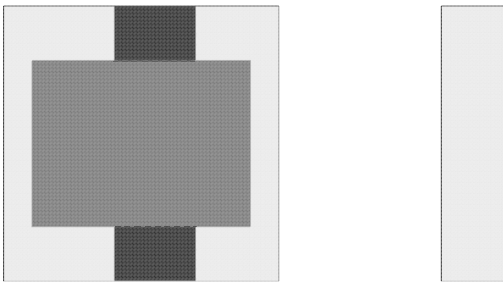
FIGURE 3. (a) Inpainting region in gray, $\epsilon = 0.05$. (b) Solution at $t = 0.4$, $f(s) = -2\ln(3)s + \ln\left(\frac{1+s}{1-s}\right)$. (c) Replacing the values larger than $\frac{1}{4}$ by $\frac{1}{2}$ and those smaller than $\frac{1}{4}$ by 0. (d) Solution at $t = 0.65$, $f(s) = 4s^3 - 6s^2 + 2s$. (e) Replacing the values larger than $\frac{1}{2}$ by 1 and those smaller than $\frac{1}{2}$ by 0.

5.3. **Inpainting of a four circles.** The gray region in Figure 3(a) corresponds to the inpainting region. We run the modified Cahn-Hilliard equation with $f(s) = -2\ln(3)s + \ln\left(\frac{1+s}{1-s}\right)$, $\epsilon = 0.05$, $\Delta t = 0.05$ and $\lambda_0 = 300000$. We are close to a steady state at $t = 0.4$, as shown in Figure 3(b), and we replace all values larger than $\frac{1}{4}$ by $\frac{1}{2}$ and all those smaller than $\frac{1}{4}$ by 0 to obtain the final inpainting in Figure 3(c).

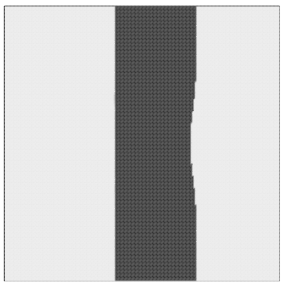
Furthermore, we run again the modified Cahn-Hilliard equation with the same $\epsilon = 0.05$ and $\Delta t = 0.05$, but we now take $f(s) = 4s^3 - 6s^2 + 2s$ and $\lambda_0 = 900000$. We are close to

a steady state at $t = 1.2$, as shown in Figure 3(d), and we replace all values larger than $\frac{1}{2}$ by 1 and all those smaller than $\frac{1}{2}$ by 0 to obtain the final inpainting in Figure 3(e).

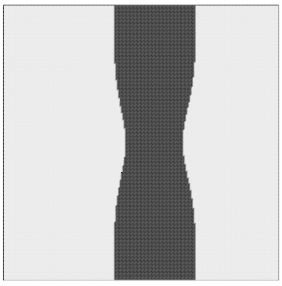
Remark 5.1. We noticed in [8] that the one step algorithm does not always work when the inpainting region is too large. We take here the same counterexample as in [8] of the broken bar illustrated in Figure 4(a). We run the modified Cahn-Hilliard equation with $f(s) = -2\ln(3)s + \ln\left(\frac{1+s}{1-s}\right)$, $\Delta t = 0.05$, $\lambda_0 = 300000$ and $\epsilon = 0.05$. We are close to a steady state at $t = 0.4$ and we replace all values larger than $\frac{1}{4}$ by $\frac{1}{2}$ and all those smaller than $\frac{1}{4}$ by 0 to obtain the final inpainting in Figure 4(b). Furthermore, we run again the modified Cahn-Hilliard equation with $f(s) = 4s^3 - 6s^2 + 2s$, $\Delta t = 0.05$, $\lambda_0 = 300000$ and $\epsilon = 0.05$. We are close to a steady state at $t = 5.6$ and we replace all values larger than $\frac{1}{2}$ by 1 and all those smaller than $\frac{1}{2}$ by 0 to obtain the final inpainting in Figure 4(c). We thus observe that the one step algorithm gives better results for this example when taking a logarithmic nonlinear term. Actually, for a smaller value of λ_0 (namely, $\lambda_0 = 100000$), we observed in [8] that the algorithm fails for the polynomial nonlinear term; however, we again obtain good inpainting results for the logarithmic one.



(a)



(b)



(c)

FIGURE 4. (a) Larger inpainting region in gray, $\epsilon = 0.05$. (b) Final inpainting result (solution close to a steady state at $t = 0.4$ with $\Delta t = 0.05$). Here, $f(s) = -2\ln(3)s + \ln\left(\frac{1+s}{1-s}\right)$. (c) Final inpainting result (solution close to a steady state at $t = 5.6$ with $\Delta t = 0.05$). Here, $f(s) = 4s^3 - 6s^2 + 2s$.

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