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ON THE REPRESENTATION OF FRIABLE INTEGERS BY LINEAR FORMS

ARMAND LACHAND

ABSTRACT. Let $P^+(n)$ denote the largest prime of the integer n . Using the so-called nilpotent Hardy-Littlewood method developed by Green and Tao, we give an asymptotic formula for the cardinal

$$\Psi_{F_1 \dots F_t}(\mathcal{K} \cap [-N, N]^d, N^{1/u}) := \#\{n \in \mathcal{K} \cap [-N, N]^d : P^+(F_1(n) \dots F_t(n)) \leq N^{1/u}\}$$

where (F_1, \dots, F_t) is a system of affine-linear forms of $\mathbf{Z}[X_1, \dots, X_d]$ no two of which are affinely related and \mathcal{K} is a convex body. This improves a work of Balog, Blomer, Dartyge and Tenenbaum [BBDT12] in the case of product of linear forms.

1. INTRODUCTION AND STATEMENT OF THE RESULT

Given a real number $y > 1$, an integer n is said to be y -friable if its greatest prime factor, denoted by $P^+(n)$, satisfies $P^+(n) \leq y$, with the conventions $P^+(\pm 1) = 1$ and $P^+(0) = 0$. Due to the duality between sifted integers and friable integers, these occur in several places in number theory (see [HT93] and [Gra08] for interesting surveys). In particular, a theorem of Hildebrand [Hil86], related to the number $\Psi(N, y)$ of y -friable integers $n \leq N$, asserts that, for any $\varepsilon > 0$ and uniformly in the domain

$$(H_\varepsilon) \quad N \geq 3 \quad \text{and} \quad 1 \leq u \leq \frac{\log N}{(\log \log N)^{5/3+\varepsilon}},$$

we have the asymptotic formula

$$(1.1) \quad \Psi(N, N^{1/u}) := \#\{1 \leq n \leq N : P^+(n) \leq N^{1/u}\} = N\rho(u) \left(1 + O\left(\frac{u \log(u+1)}{\log N}\right)\right)$$

where ρ is the Dickman function.

A practical example of the incidence of integers without large prime factors comes from the factorization algorithm Number Field Sieve (NFS). The interested reader may find a description of this algorithm in the chapter 6 of the book of Crandall and Pomerance [CP05]. Given a binary form $F \in \mathbf{Z}[X_1, X_2]$ without square factors, an important step of NFS rests over the assumption that the cardinality

$$(1.2) \quad \Psi_F(\mathcal{K} \cap [-N, N]^2, N^{1/u}) := \#\{(n_1, n_2) \in \mathcal{K} \cap [-N, N]^2 : P^+(F(n_1, n_2)) \leq N^{1/u}\}$$

is sufficiently large for some large u and \mathcal{K} a sufficiently regular compact.

Let $F_1, \dots, F_t \in \mathbf{Z}[X_1, X_2]$ be the distinct irreducible factors of F and $d_1 \geq \dots \geq d_t \geq 1$ their degree respectively. If we assume that the y -friability of the values of F_1, \dots, F_t are independent events, each described by the probabilistic model suggested by (1.1), we can conjecture the following asymptotic formula

$$(1.3) \quad \Psi_F(\mathcal{K} \cap [-N, N]^2, N^{1/u}) \underset{N \rightarrow +\infty}{\sim} \text{Vol}(\mathcal{K} \cap [-N, N]^2) \rho(d_1 u) \cdots \rho(d_t u)$$

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in a range for u to be determined and for a sufficiently regular compact \mathcal{K} .

Such a result for general forms F seems beyond reach but there exists some partial results for estimating $\Psi_F([0, N]^2, N^{1/u})$ when u is sufficiently small. In [BBDT12], Balog Blomer, Dartyge and Tenenbaum established the existence of a constant $\alpha_F > 1/d_1$ such that, for any $\varepsilon > 0$ and uniformly for $N \geq 2$, we have

$$(1.4) \quad \Psi_F\left([0, N]^2, N^{1/\alpha_F + \varepsilon}\right) \gg_\varepsilon N^2.$$

We now focus on binary forms F of type $F = F_1 \cdots F_t$ where F_1, \dots, F_t are some linear forms in $\mathbf{Z}[X_1, X_2]$. The cases $t = 1$ and $t = 2$ can be deduced from results of [FT91] related to the distribution of friable integers in arithmetic progressions. The case $t = 3$ was essentially considered by a succession of articles of various authors ([dlB99, LS12, dlBG14, Dra13, Dra, Har]). In Corollary 1 of [Har], Harper used the Hardy-Littlewood circle method to show the existence of $c > 0$ such that, for any $\varepsilon > 0$ and uniformly for $N \geq 2$ and $y \geq (\log N)^c$, we have

$$(1.5) \quad \#\left(1 \leq n_1, n_2, n_1 + n_2 \leq N : P^+(n_1 n_2 (n_1 + n_2)) \leq y\right) \underset{N \rightarrow +\infty}{\sim} \frac{\Psi(N, y)^3}{2N}.$$

The study of $\Psi_F(\mathcal{K} \cap [-N, N]^2, N^{1/u})$ when F is a product of linear forms also belongs to the area of linear correlation. As a result of their impressive work on the generalized Hardy-Littlewood conjecture, Green and Tao show that, if $N \geq 2$, $\mathbf{F} := (F_1, \dots, F_t) : \mathbf{Z}^d \rightarrow \mathbf{Z}^t$ is a system of affine-linear forms such that any two forms F_i and F_j are affinely independent over \mathbf{Q} and $|F_i(0)| \leq N$, and $\mathcal{K} \subset [-N, N]^d$ is a convex body such that $\mathbf{F}(\mathcal{K}) \subset [0, N]^t$, then we have

$$(1.6) \quad \sum_{\mathbf{n} \in \mathcal{K} \cap \mathbf{Z}^d} \Lambda(F_1(\mathbf{n})) \cdots \Lambda(F_t(\mathbf{n})) = \text{Vol}(\mathcal{K}) \prod_p \beta_p + o_{\mathcal{K}, F}(N^d)$$

where Λ denotes the von Mangoldt function,

$$\beta_p := \frac{p^{t-d}}{(p-1)^t} \#\{\mathbf{n} \pmod{p} : p \nmid F_i(\mathbf{n}) \text{ for any } 1 \in \{1, \dots, t\}\}$$

and the implicit constant depends only on \mathcal{K} and nonconstant coefficients of F_1, \dots, F_t .

This result was proved in [GT10] under the assumption of two important conjectures, namely the inverse Gowers-norm conjecture and the Möbius and nilsequence conjecture. The first conjecture was settled in [GTZ12] and its resolution is the climax of a serie of papers [GT08, GTZ11, GTZ12]. On the other hand, the Möbius and nilsequence conjecture was proved in [GT12a] where Green and Tao have extended the standard analysis of Type I/II sums, in analogy to the classical Hardy-Littlewood circle method.

With the development of the Green-Tao method, it seems possible to get an asymptotic estimation of the average value

$$(1.7) \quad \sum_{\mathbf{n} \in \mathcal{K} \cap \mathbf{Z}^d} h(F_1(\mathbf{n})) \cdots h(F_t(\mathbf{n}))$$

for a wide class of arithmetic functions h . For example, Green and Tao have considered in [GT10] the case of the Möbius function $h = \mu$ or the Liouville function $h = \lambda$. In this direction, Matthesen obtained some results when $h = \tau$ is the divisor function [Mat12a] or $h = r_G$ is the number of representation of a binary quadratic form G [Mat12b, Mat13]. More recently, Frantzikinakis and Host [FH] have used the Green-Tao method to show that

$$\sum_{1 \leq n_1, n_2 \leq N} h(G(n_1, n_2)^r F_1(n_1, n_2) \cdots F_t(n_1, n_2)) = o(N^2)$$

where h is a 1-bounded multiplicative function satisfying $\sum_{n \leq N} h(a + qn) = o(N)$ for any a and q , $F_1, \dots, F_t \in \mathbf{Z}[X_1, X_2]$ are primitive linear forms such that any two forms are affinely independent, G is a quadratic form of a prescribed shape and $r \geq 2$.

In this note, we study how the nilpotent Hardy-Littlewood method may be directly applied to get a straightforward asymptotic formula of (1.7) when $h = 1_{S(y)}$ is the indicator of the set $S(y)$ of the y -friable integers. Our main result is the following theorem.

Théorème 1.1. *Let $d, t, N, u_0 \geq 1$ be some integers, and let $(F_1, \dots, F_t) : \mathbf{Z}^d \rightarrow \mathbf{Z}^t$ be a system of affine-linear forms such that any two forms F_i and F_j are affinely independent over \mathbf{Q} and $|F_i(0)| \leq N$. Let $\mathcal{K} \subset [-N, N]^d$ be a convex body such that $F(\mathcal{K}) \subset [0, N]^t$. Then, for any $u_1, \dots, u_t \in [0, u_0]$, we have*

$$(1.8) \quad \begin{aligned} \Psi_{F_1 \dots F_t}(\mathcal{K}, N^{1/u}) &:= \#\left\{ \mathbf{n} \in \mathcal{K} \cap \mathbf{Z}^d : P^+(F_i(\mathbf{n})) \leq N^{1/u_i} \text{ for any } 1 \leq i \leq t \right\} \\ &= \text{Vol}(\mathcal{K}) \prod_{i=1}^t \rho(u_i) + o(N^d) \end{aligned}$$

where the implicit constant depends only on \mathcal{K} , u_0 and nonconstant coefficients of F_1, \dots, F_t .

As regards the result of Balog *et al* [BBDT12], we get essentially two major improvements on their works in the case of linear forms :

- the formula (1.8) is an asymptotic equivalent which is consistent with the conjectural formula (1.3) whereas (1.4) only gives a lower bound,
- when $t \geq 4$, the formula (1.4) is valid with the parameter $\alpha_F = 1 + \frac{2}{t-2}$ while Theorem 1.1 shows that we can choose any positive real for α_F .

As stated previously, our result is far to be optimal, as regards the range of validity of the formula for u . Our main goal is to obtain a result with the nilpotent Hardy-Littlewood method and some straightforward combinatoric formula such as the inclusion-exclusion principle and no more effort. In particular, when u is bounded, the indicator of the set $S(N^{1/u})$ have asymptotic density $\rho(u) > 0$ and is uniform in arithmetic progressions of small common difference unlike the previously mentioned works of Green and Tao and Matthiesen. This is why it is not necessary to introduce a pseudorandom measure and to use the transference principle in this work (see [GT10] for a presentation of these two tools).

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2. A BRIEF DESCRIPTION OF THE NILPOTENT HARDY-LITTLEWOOD METHOD

For any parameter of friability N^{1/u_i} , we define the balanced function

$$\begin{aligned} f_i : \mathbf{N} &\rightarrow [-1, 1] \\ n &\mapsto 1_{S(N^{1/u_i})}(n) - \rho(u_i). \end{aligned}$$

By writing $1_{S(N^{1/u_i})}(n) = f_i(n) + \rho(u_i)$ and using the bound $\rho(u_i) \leq 1$, it follows that

$$(2.1) \quad \left| \Psi_{F_1 \dots F_t}(\mathcal{K}, N^{1/u}) - \text{Vol}(\mathcal{K}) \prod_{i=1}^t \rho(u_i) \right| \leq \sum_{\substack{I \subset \{1, \dots, t\} \\ I \neq \emptyset}} \left| \sum_{\mathbf{n} \in \mathcal{K} \cap \mathbf{Z}^d} \prod_{i \in I} f_i(F_i(\mathbf{n})) \right| + O_d(N^{d-1}).$$

In [[GT10], Definition 1.5], Green and Tao introduced the Cauchy-Schwarz complexity of a system of affine-linear forms. With this new concept, they proved that an upper bound of (2.1) may be deduced by some estimations of the Gowers norms $\|f_i\|_{U^s(N)}$ of f_i defined by the relation

$$\|f_i\|_{U^s(N)}^{2^s} := \left(\sum_{\substack{n, \mathbf{h}: \\ 1 \leq n + \omega \cdot \mathbf{h} \leq N \\ \forall \omega \in \{0,1\}^s}} \prod_{\omega \in \{0,1\}^s} f_i(n + \omega \cdot \mathbf{h}) \right) \# \{n, \mathbf{h} : 1 \leq n + \omega \cdot \mathbf{h} \leq N \text{ for any } \omega \in \{0,1\}^s\}^{-1}$$

where $\omega \cdot \mathbf{h} = \omega_1 h_1 + \dots + \omega_s h_s$.

In particular, by an ingenious succession of Cauchy-Schwarz inequalities already appearing in the work of Gowers [Gow01], Green and Tao proved the following generalized von Neumann theorem.

Proposition 2.1 ([GT10], Proposition 7.1). *Let $s, t, d, N \geq 1$ be some integers. Suppose that $f_1, \dots, f_t : [0, N] \cap \mathbf{Z} \rightarrow [-1, 1]$, $(F_1, \dots, F_t) : \mathbf{Z}^d \rightarrow \mathbf{Z}^t$ is a system of affine-linear forms of complexity s such that $|F_i(0)| \leq N$ and $\mathcal{K} \subset [-N, N]^d$ be a convex body such that $F(\mathcal{K}) \subset [0, N]^t$. Suppose also that*

$$(2.2) \quad \min_{1 \leq i \leq t} \|f_i\|_{U^{s+1}[N]} \leq \delta$$

for some $\delta > 0$. Then we have

$$(2.3) \quad \sum_{\mathbf{n} \in \mathcal{K}} \prod_{i=1}^t f_i(F_i(\mathbf{n})) = o_\delta(N^d) + \kappa(\delta)N^d$$

where $\kappa(\delta)$ tends to 0 as $\delta \rightarrow 0$.

One could think that the estimation of the Gowers norms of a function h is as difficult as the majoration of the mean value (1.7) in the case of the system of affine-linear forms defined by $(F_\omega)_{\omega \in \{0,1\}^{s+1}}$ with $F_\omega(X_0, X_1, \dots, X_{s+1}) = X_0 + \sum_{i=1}^{s+1} \omega_i X_i$. At first glance, it is not easy to see what simplification gives the generalized von Neumann theorem. The inverse theorem for the Gowers norms, proved by Green, Tao and Ziegler, exhibits the link between linear correlations and polynomial nilsequences. The reader is referred to [GT12b] for definitions and properties of filtered nilmanifolds and polynomial nilsequences.

Proposition 2.2 ([GTZ12], Theorem 1.3). *Let $s \geq 0$ be an integer and let $\delta \in]0, 1]$. Then there exists a finite collection $\mathcal{M}_{s,\delta}$ of filtered nilmanifolds $(G/\Gamma, G_\bullet)$, each equipped with some smooth Riemannian metric $d_{G/\Gamma}$, as well as constants $C(s, \delta), c(s, \delta) > 0$ with the following property. Whenever $N \geq 1$ and $f : [0, N] \cap \mathbf{Z} \rightarrow [-1, 1]$ is a function such that*

$$\|f\|_{U^{s+1}[N]} \geq \delta,$$

there exists a filtered nilmanifold $G/\Gamma \in \mathcal{M}_{s,\delta}$, a 1-bounded function $F : G/\Gamma \rightarrow \mathbf{C}$ with Lipschitz constant at most $C(s, \delta)$ with respect to the metric $d_{G/\Gamma}$ and a polynomial nilsequence $g : \mathbf{Z} \rightarrow G$ adapted to G_\bullet such that

$$\left| \sum_{0 \leq n \leq N} f(n) F(g(n)\Gamma) \right| \geq c(s, \delta)N.$$

3. NON-CORRELATION WITH NILSEQUENCES

Let $(G/\Gamma, G_\bullet)$ be a filtered nilmanifold and $u_0 \geq 1$. In this part, we show that for any 1-bounded Lipschitz function $F : G/\Gamma \rightarrow \mathbf{C}$, any polynomial nilsequence $g : \mathbf{Z} \rightarrow G$ adapted to G_\bullet , $1 \leq u \leq u_0$ and $N \geq 1$, we have

$$(3.1) \quad \sum_{n \leq N} f(n) F(g(n)\Gamma) = o(N(1 + \|F\|_{\text{Lip}, d_{G/\Gamma}}))$$

where $f(n) := 1_{S(N^{1/u})}(n) - \rho(u)$,

$$\|F\|_{\text{Lip}, d_{G/\Gamma}} := \sup_{g \in G/\Gamma} |F(g)| + \sup_{\substack{g_1, g_2 \in G/\Gamma \\ g_1 \neq g_2}} \frac{|F(g_1) - F(g_2)|}{d_{G/\Gamma}(g_1, g_2)}$$

and the implicit term $o(\cdot)$ depends only on G/Γ and u_0 . In view of the results of the previous section, this will imply Theorem 1.1.

The starting point is the Möbius inversion formula in the following form

$$(3.2) \quad 1_{S(N^{1/u})}(n) = \sum_{P^-(k) > N^{1/u}} \mu(k) 1_{k|n}.$$

We approximate the indicator $1_{k|n}$ by its mean value $\frac{1}{k}$ for $k \leq N^{1-\tau}$ where the parameter $\tau = o(1) \in]1/\log N, 1[$ will be chosen later :

$$\sum_{1 \leq n \leq N} (1_{S(N^{1/u})}(n) - \rho(u)) F(g(n)\Gamma) = \Sigma_1(F, g) + \Sigma_2(F, g)$$

where

$$(3.3) \quad \Sigma_1(F, g) := \sum_{1 \leq n \leq N} f_\tau(n) F(g(n)\Gamma) \quad \text{with} \quad f_\tau(n) = \sum_{\substack{k \leq N^{1-\tau} \\ P^-(k) > N^{1/u}}} \mu(k) \left(1_{k|n} - \frac{1}{k} \right) \leq 2^u$$

and

$$\Sigma_2(F, g) := \sum_{1 \leq n \leq N} \left(\sum_{\substack{k > N^{1-\tau} \\ P^-(k) > N^{1/u}}} \mu(k) 1_{k|n} + \sum_{\substack{k \leq N^{1-\tau} \\ P^-(k) > N^{1/u}}} \frac{\mu(k)}{k} - \rho(u) \right) F(g(n)\Gamma).$$

First, we focus on $\Sigma_2(F, g)$. By Hildebrand's formula (1.1) and the inclusion-exclusion principle (3.2), one gets that

$$(3.4) \quad \sum_{1 \leq n \leq N} \left(\sum_{P^-(k) > N^{1/u}} \mu(k) 1_{k|n} - \rho(u) \right) = \Psi(N, N^{1/u}) - N\rho(u) \ll N\rho(u) \frac{u \log(u+1)}{\log N}.$$

Moreover, the standard estimation of the cardinal of the set of $N^{1/u}$ -sifted integers

$$\#\left\{ n \leq N : P^-(n) > N^{1/u} \right\} \ll u \frac{N}{\log N}$$

implies that

$$(3.5) \quad \sum_{\substack{k \leq N^{1-\tau} \\ P^-(k) > N^{1/u}}} \left| \frac{N}{k} - \sum_{n \leq N} 1_{k|n} \right| \ll u \frac{N^{1-\tau}}{\log N}.$$

In view to the following series of estimations, valid whenever $y > N^\tau$,

$$\begin{aligned} \sum_{\substack{N^{1-\tau} < k \leq N \\ P^-(k) > N^{1/u}}} \frac{\mu^2(k)}{k} &\ll \sum_j \sum_{N^{1/u} < p_2 < \dots < p_j \leq N} \frac{1}{p_2 \cdots p_j} \sum_{\max\left(N^{1/u}, \frac{N^{1-\tau}}{p_2 \cdots p_j}\right) \leq p_1 \leq \frac{N}{p_2 \cdots p_j}} \frac{1}{p_1} \\ &\ll \tau u \sum_j \frac{1}{j!} \left(\sum_{N^{1/u} < p \leq N} \frac{1}{p} \right)^j \\ &\ll \tau u \sum_j \frac{1}{j!} (\log(u) + O(1))^{j-1} \ll \tau u^2, \end{aligned}$$

we also have the upper bound

$$(3.6) \quad \sum_{1 \leq n \leq N} \sum_{\substack{k > N^{1-\tau} \\ P^-(k) > N^{1/u}}} \mu^2(k) 1_{k|n} \ll \tau u^2 N.$$

Finally, it follows from (3.4), (3.5) and (3.6) that

$$\begin{aligned} \Sigma_2(F, g) &\ll \sum_{1 \leq n \leq N} \sum_{\substack{k > N^{1-\tau} \\ P^-(k) > N^{1/u}}} \mu^2(k) 1_{k|n} + \sum_{\substack{k \leq N^{1-\tau} \\ P^-(k) > N^{1/u}}} \left(\frac{N}{k} - \sum_{n \leq N} 1_{k|n} \right) \\ &\quad + \left| \sum_{\substack{k \leq N \\ P^-(k) > N^{1/u}}} \sum_{n \leq N} 1_{k|n} - N \rho(u) \right| = o(N). \end{aligned}$$

In view of the foregoing, it remains to obtain an upper bound of $\Sigma_1(F, g)$. This is the subject of the following proposition.

Proposition 3.1. *Let $m, s \geq 1$ be some integers and $A > 0$. There exists a constant $c(m, s, A) > 0$ with the following property. Whenever $Q, N \geq 2$ are integers such that $N^\tau \geq (\log N)^{c(m, s, A)}$, $(G/\Gamma, G_\bullet)$ is a filtered nilmanifold of degree s and dimension m , \mathcal{X} is a Q -rational Mal'cev basis of $(G/\Gamma, G_\bullet)$, $g : \mathbf{Z} \rightarrow G/\Gamma$ is a polynomial nilsequence adapted to G_\bullet and $F : G/\Gamma \rightarrow [-1, 1]$ is a Lipschitz function, then we have*

$$(3.7) \quad \sum_{1 \leq n \leq N} f_\tau(n) F(g(n)\Gamma) \leq N Q^{c(m, s, A)} (1 + \|F\|_{\text{Lip}, \mathcal{X}}) 2^u (\log N)^{-A}.$$

Since any smooth Riemannian metric $d_{G/\Gamma}$ is equivalent to the metric $d_{\mathcal{X}}$ (see the 4th footnote and Definition 2.2 of [GT12b]), we can deduce from this proposition that (3.1) holds for any choice of $d_{G/\Gamma}$ by fixing a Mal'cev basis \mathcal{X} and choosing $\tau = o(1)$ such that $N^\tau \geq (\log N)^{c(m, s, 1)}$.

The rest of the article is devoted to the proof of Proposition 3.1. The argument follows essentially the proofs of [[GT12a], Proposition 2.1] and [Mat12a], Theorem 9.1] and we only outline the major differences. A key point of the proof consists to reduce the problem to establish the formula (3.7) for a polynomial nilsequence g totally equidistributed, i.e. such that $|P|^{-1} \sum_{n \in P} F(g(n)\Gamma)$ tends to $\int_{G/\Gamma} F$ as P is a subprogression such that $|P| \rightarrow +\infty$.

After this reduction, it will be possible to use the following analogue of [[GT12a], Proposition 2.1] and [[Mat12a], Proposition 9.3].

Proposition 3.2. *Let $m, s \geq 1$ be some integers. There exist some constants $c_0(m, s), c_1(m, s) > 0$ with the following property. Whenever $\delta \in]0, 1/2[$ and $Q \geq 2$ are real, $N \geq 2$ is an integer*

such that $\delta^{-c_0(m,s)} \leq N^\tau$, $P \subset \{1, \dots, N\}$ is an arithmetic progression of size at least N/Q , $(G/\Gamma, G_\bullet)$ is a filtered nilmanifold of degree s and dimension m , \mathcal{X} is a Q -rational Mal'cev basis of $(G/\Gamma, G_\bullet)$, $g : \mathbf{Z} \rightarrow G/\Gamma$ is a polynomial and δ -totally equidistributed nilsequence adapted to G_\bullet and $F : G/\Gamma \rightarrow [-1, 1]$ is a Lipschitz function such that $\int_{G/\Gamma} F = 0$, we have

$$(3.8) \quad \left| \sum_{n \leq N} f_\tau(n) 1_P(n) F(g(n)\Gamma) \right| \leq \delta^{c_1(m,s)} \|F\|_{\text{Lip}, \mathcal{X}} Q N (2^u + \log N).$$

Proof that Proposition 3.2 implies Proposition 3.1. Following the ideas of [GT12a], we can suppose without loss of generality that $\|F\|_{\text{Lip}, \mathcal{X}} = 1$ and $Q \leq \log N$. Let $B > 0$ a parameter which will be explicit at the end of the proof. Applying Theorem 1.19 of [GT12b], there exists an integer M satisfying $\log N \leq M \leq (\log N)^{c(m,s,B)}$ such that we can write the decomposition $g = \varepsilon g' \gamma$ where

- (1) $\varepsilon \in \text{poly}(\mathbf{Z}, G_\bullet)$ is (M, N) -smooth (see [[GT12b], Definition 1.18]),
- (2) $\tilde{g} \in \text{poly}(\mathbf{Z}, G_\bullet)$ takes values in \tilde{G} and $(\tilde{g}(n))_{n \leq N}$ is M^{-B} -totally equidistributed in $\tilde{G}/(\tilde{G} \cap \Gamma)$ for the metric $d_{\tilde{\chi}}$ (see [[GT12b], Definition 1.2]),
- (3) $\gamma \in \text{poly}(\mathbf{Z}, G_\bullet)$ is periodic of period $q \leq M$ and $\gamma(n)$ is M -rational for any $n \in \mathbf{Z}$ (see [[GT12b], Definition 1.17]).

Next, we reproduce the arguments based on partitions and some pigeonhole principle of Green and Tao and use the properties of periodicity and smoothness of γ and ε . In this way, we can reduce the problem to show that

$$(3.9) \quad \sum_{1 \leq n \leq N} f_\tau(n) 1_P(n) F'(g'(n)\Gamma') \leq 2^{u-1} N / (M^2 (\log N)^{2A})$$

where P is a subprogression such that $|P| \geq \frac{N}{2M^2(\log N)^A}$, $(G'/\Gamma', G'_\bullet)$ is a m -dimensional nilmanifold of degree s with $M^{C_1(m,s)}$ -rational Mal'cev basis \mathcal{X}' , $F' : G'/\Gamma' \rightarrow [-1, 1]$ is a Lipschitz function such that $\|F'\|_{\text{Lip}, \mathcal{X}'} \leq M^{C_1(m,s)}$ and $g' \in \text{poly}(\mathbf{Z}, G'_\bullet)$ is $M^{-C_2(m,s)B+C_1(m,s)}$ -totally equidistributed, for some constants $C_1(m,s), C_2(m,s) > 0$.

If we suppose that $\int_{G'/\Gamma'} F' = 0$, then we apply Proposition 3.2 with the choice ' $Q' := M^{C_1(m,s)}$ ' and ' $\delta' := M^{-C_2(m,s)B+C_1(m,s)}$ '. Taking $B, C_1(m,s)$ and $c(m,s,A)$ sufficiently large, the hypothesis on the size of P and δ are satisfied and we get (3.9).

We can reduce to the last case by writing $F' = (F' - \int_{G'/\Gamma'} F') + \int_{G'/\Gamma'} F'$. Indeed, we can observe that $\left| \int_{G'/\Gamma'} F' \right| \leq 1$ and then we have, under the hypothesis $q \leq N^\tau \leq N^{1/u}$, the majoration

$$\begin{aligned} \left| \sum_{1 \leq n \leq N} f_\tau(n) 1_P(n) \int_{G'/\Gamma'} F' \right| &\leq \sum_{\substack{k \leq N^{1-\tau} \\ P^-(k) > N^{1/u}}} \left| \sum_{1 \leq n \leq N} \left(1_{k|n} - \frac{1}{k} \right) 1_P(n) \right| \\ &\leq \# \left\{ k \leq N^{1-\tau} : P^-(k) > N^{1/u} \right\} \\ &\ll u \frac{N^{1-\tau}}{\log N} \end{aligned}$$

which implies (3.9) with the condition $N^\tau \geq (\log N)^{c(m,s,A)}$. \square

Proof of Proposition 3.2. We essentially follow the proof of Proposition 9.2 of [Mat12a] and we suppose that $\|F\|_{\text{Lip}, \mathcal{X}} = 1$ and $Q \leq \delta^{-c_1(m,s)}$. For $\mathcal{T} \in]0, 1/2[$ and $j \geq 1$, we define $S_j(\mathcal{T})$ as

the set of the integers k satisfying

$$\left| \sum_{2^j/k < n \leq 2^{j+1}/k} 1_P(kn)F(g(kn)\Gamma) \right| > \mathcal{T} \frac{2^j}{k}.$$

In view to the trivial upper bound (see formula (3.3))

$$\left| \sum_{n \leq N^{1-\tau/2}} f_\tau(n) 1_P(n) F(g(n)\Gamma) \right| \leq N^{1-\tau/2} 2^u,$$

we concentrate on the integers $n > X^{1-\tau/2}$. Following the proof of Proposition 9.2 of [Mat12a], we make a dyadic splitting over the variables k and n :

$$\begin{aligned} & \sum_{\substack{k \leq N^{1-\tau} \\ P^-(k) > Y}} \left| \sum_{N^{1-\tau/2}/k < n \leq N/k} 1_P(kn)F(g(kn)\Gamma) \right| \\ & \ll \sum_{2^i \leq N^{1-\tau}} \sum_{\frac{N^{1-\tau/2}}{2} \leq 2^j \leq N} 2^j \left(\sum_{\substack{2^i \leq k < 2^{i+1} \\ P^-(k) > N^{1/u}}} \frac{\mathcal{T}}{k} + \sum_{\substack{2^i \leq k < 2^{i+1} \\ P^-(k) > N^{1/u} \\ k \in S(\mathcal{T})}} \frac{1}{k} \right) \\ & \ll \sum_{\frac{N^{1-\tau/2}}{2} \leq 2^j \leq N} 2^j \left(\mathcal{T} \log N + \sum_{2^i \leq N^{1-\tau}} \frac{1}{2^i} \#(S_j(\mathcal{T}) \cap [2^i, 2^{i+1}]) \right). \end{aligned}$$

The rest of the proof consists then to show that, if $K \leq N^{1-\tau}$ and $\mathcal{T} := \delta^{c_1(m,s)} \leq Q^{-1}$ for a constant $c_1(m,s) > 0$ sufficiently small, then we have

$$(3.10) \quad \#(S(\mathcal{T}) \cap [K, 2K]) \leq \mathcal{T}K$$

whenever $\frac{N^{1-\tau/2}}{2} \leq 2^j \leq N$.

To achieve this we follow the discussion of Type I case of [[GT12a], Part3] and the proof of [[Mat12a], Lemma 9.3] and we suppose for contradiction that (3.10) does not hold for some $K \leq N^{1-\tau}$ and $\frac{N^{1-\tau/2}}{2} \leq 2^j \leq N$. By reproducing their arguments, we observe the existence of a non-trivial horizontal character ψ with magnitude $0 < |\psi| \leq \mathcal{T}^{-c_2(m,s)}$ such that, for any $r \geq 1$ and for at least $\mathcal{T}^{c_2(m,s)}K$ values of k , we have

$$(3.11) \quad \|\partial^r(\psi \circ g_k)(0)\|_{\mathbf{R}/\mathbf{Z}} \leq \mathcal{T}^{-c_2(m,s)} (K/2^j)^r$$

where $g_k(n) = g(kn)$, which is the analogous of the formula (3.7) of [GT12a].

By Lemma 2 and 3 of [GT12a], it follows that there exists an integer $q \ll_s 1$ and at least $\mathcal{T}^{c_3(m,s)}K^r$ integers $l \leq 10^d K^r$ such that

$$(3.12) \quad \|ql\beta_r\|_{\mathbf{R}/\mathbf{Z}} \leq \mathcal{T}^{-c_3(m,s)}(K/2^j)^r$$

where the β_r 's are defined by

$$(3.13) \quad \psi \circ g(n) = \beta_d n^d + \dots + \beta_0.$$

To deduce some diophantine information about the β_r 's, we invoke Lemma 3.2 of [GT12b] by an analogue way as [GT12a] after checking that the hypothesis are satisfied. It suffices for it to

see that $r \geq 1$ and $\frac{\mathcal{T}^{2c_3(m,s)}}{10^d} \gg N^{-\tau} \geq \left(\frac{K}{2j}\right)^r$ if the constant $c_1(m, s)$ is chosen sufficiently small. It results that there exists $q' \leq \mathcal{T}^{-c_4(m,s)}$ such that

$$\|q'\beta_r\|_{\mathbf{R}/\mathbf{Z}} \leq \mathcal{T}^{-c_4(m,s)}2^{-rj}$$

for any integer $r \geq 1$. By the definition (3.13), we get the existence of $c_5(m, s) > 0$ sufficiently large such that $q' \leq \mathcal{T}^{c_5(m,s)}$ and

$$(3.14) \quad \|q'(\psi \circ g)(n)\|_{\mathbf{R}/\mathbf{Z}} \leq 1/10$$

for any $n \leq \mathcal{T}^{c_5(m,s)}N$.

Let $\eta : \mathbf{R}/\mathbf{Z} \rightarrow [-1, 1]$ be a Lipschitz function of norm $O(1)$ and mean zero, which is equal to 1 on $[-1/10, 1/10]$ so that

$$\int_{G/\Gamma} \eta \circ (q'\psi) = 0 \quad \text{and} \quad \|\eta \circ (q'\psi)\|_{\text{Lip}, \mathcal{X}} \leq \mathcal{T}^{-c_5(m,s)}.$$

It follows from (3.14) that we have

$$\left| \sum_{n \leq \mathcal{T}^{c_5(m,s)}N} \eta(q'\psi(g(n)\Gamma)) \right| \geq \mathcal{T}^{c_5(m,s)}N - 1 > \delta \|\eta \circ (q'\psi)\|_{\text{Lip}, \mathcal{X}} \mathcal{T}^{-c_5(m,s)}N$$

whenever $c_1(m, s)$ is sufficiently small, which contradicts the hypothesis that the $(g(n))_{n \leq N}$ is δ -totally equidistributed by considering as arithmetic progression the set of integers less than $\mathcal{T}^{c_5(m,s)}N$. \square

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