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# An analytic approach for the evolution of the static-flowing interface in viscoplastic granular flows

François Bouchut\*, Ioan R. Ionescu†, Anne Mangeney‡§

## Abstract

Observed avalanche flows of dense granular material have the property to present two possible behaviours: static (solid) or flowing (fluid). In such situation, an important challenge is to describe mathematically the evolution of the physical interface between the two phases. In this work we derive analytically a set of equations that is able to manage the dynamics of such interface, in the so-called shallow regime where the flow is supposed to be thin compared to its downslope extension. It is obtained via an asymptotics starting from an incompressible viscoplastic model with Drucker-Prager yield stress, in which we have to make several assumptions. Additionally to the classical ones that are that the curvature of the topography, the width of the layer, and the viscosity are small, we assume that the internal friction angle is close to the slope angle, the velocity is small, and the pressure is convex with respect to the normal variable. This last assumption is for the stability of the double layer static-flowing configuration. The resulting model takes the form of a formally overdetermined initial-boundary problem in the variable normal to the topography. It handles arbitrary velocity profiles, and is therefore more general than depth-averaged models. It includes a new non-hydrostatic nonlinear coupling term. It has the property to be numerically solvable, at least in the uncoupled case, with or without viscosity.

**Keywords:** Granular flows, viscoplastic flows, Drucker-Prager yield stress, static-flowing transition, interface dynamics

## 1 Introduction

One of the key features of dense granular materials, like sand or dust, is to flow like a fluid or behave like a solid, depending on the balance between on one hand the driving forces related for instance to gravity, free surface slope or inflow velocity, and on the other hand a threshold related to the frictional properties of the material and to its interaction with the substrate or container. Describing the transition between the so-called flowing and static states is critical for industrial and geophysical purposes. In particular, the static-flowing transition plays a key role in erosion-deposition processes within geophysical flows such as debris flows or debris and snow avalanches [28, 48, 41, 42, 32, 21].

The static-flowing transition is closely related to the rheology of dense granular materials. While this rheology has been very much studied, *e.g.* [2, 43], the accurate description of the static-flowing transition is still an open issue. In particular, depending on the scale of description, this transition can be viewed as a sharp or as a more continuous change of states. Indeed, it has been shown that for surface flows over an erodible substrate, the velocity decreases exponentially from the flowing to the static state, with creeping and intermittent

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motion of the grains near the static state [45]. At the “large scale” (i.e. the scale of the flow), this transition has been described by introducing a Drucker-Prager [34, 36, 17, 31] or a Coulomb [18] yield stress or by describing explicitly the transition from the flowing to the static states through an order parameter that varies continuously between these two states [4, 5, 41].

As proposed by [34], dense granular flows can be described by an incompressible viscoplastic law with Drucker-Prager yield stress. The so-called  $\mu(I)$  rheology, where  $I$  is the inertial number, involves a viscosity that depends on the pressure and the strain rate experienced by the granular material. It enables to quantitatively reproduce granular column collapse over horizontal and inclined slopes and granular flow experiments over erodible substrate [34, 31]. Furthermore, this continuum approach is in good agreement with discrete element simulations of granular column collapse [36]. Reference [31] showed that using a constant viscosity gives very similar results to using the  $\mu(I)$  rheology for granular column collapses of small aspect ratios over horizontal and inclined planes. In all these regimes, the basic ingredients, required to reproduce dry granular flows at the “large scale” from the destabilization phase to the arrest phase, seem to be present in the proposed viscoplastic rheologies. However, the precise description of the yield stress, of the viscosity as well as the possible role of compressibility are still open and challenging questions, e.g. [27, 31, 37]. Furthermore, recent studies show that these models, and in particular the  $\mu(I)$  rheology, are ill-posed for small and large values of the inertial number [8]. This is the case near the static-flowing transition and in particular in some regions within the granular mass during granular column collapse simulated by [36] and [31]. Reference [8] suggests that the good results obtained by [36] despite the ill-posedness of the problem may be due to the use of their regularization method and to the finite pressure they imposed at the free surface. However, [31] also obtained very good results by using an augmented Lagrangian method with no specific condition on the surface pressure. As suggested by [8], these good results may instead be due to the use of a coarse mesh such that simulations avoid the effect of the ill-posedness by the damping of the faster growing high wavenumbers.

Based on these results, we propose to start here from the most simple viscoplastic model, based on incompressibility, constant viscosity and Drucker-Prager yield stress, as tested in [31]. We thus consider the dynamics of an incompressible non-Newtonian material described by the equations:

$$\operatorname{div} \mathbf{U} = 0, \quad (1.1)$$

$$\partial_t \mathbf{U} + \mathbf{U} \cdot \nabla \mathbf{U} = \operatorname{div} \boldsymbol{\sigma} - \mathbf{g}, \quad (1.2)$$

where  $\mathbf{U}$  is the velocity vector,  $-\mathbf{g}$  is the gravity force, and  $\boldsymbol{\sigma}$  is the stress tensor normalized by the density, that is assumed to be constant. The tensor  $\boldsymbol{\sigma}$  is symmetric, and we assume a viscoplastic behaviour described by the relation

$$\boldsymbol{\sigma} = -p \mathbf{Id} + 2\nu D\mathbf{U} + \kappa \sqrt{2} \Lambda, \quad (1.3)$$

where  $p$  is the scalar pressure,  $D\mathbf{U}$  is the strain rate tensor,

$$D\mathbf{U} = \frac{\nabla \mathbf{U} + (\nabla \mathbf{U})^t}{2}, \quad (1.4)$$

and  $\Lambda$  is the normalized strain rate,

$$\begin{cases} \|\Lambda\| \leq 1, & \Lambda^t = \Lambda, & \operatorname{Tr} \Lambda = 0, \\ \Lambda = \frac{D\mathbf{U}}{\|D\mathbf{U}\|} & \text{whenever } D\mathbf{U} \neq 0. \end{cases} \quad (1.5)$$

Here the norm of a matrix  $A = (A_{ij})$  is the Frobenius norm  $\|A\| = (\sum_{ij} A_{ij}^2)^{1/2}$ . We do not put a factor 1/2 in the definition of this norm, as many authors do, and it leads to the factor  $\sqrt{2}$  in (1.3). The coefficient  $\nu \geq 0$  is the kinematic viscosity, and  $\kappa \geq 0$  is the yield stress. In general, in (1.3),  $\nu$  and  $\kappa$  can be rate and pressure dependent, i.e. they can depend on

$\|D\mathbf{U}\|$  and  $p$ . Note however that then different couples  $(\nu, \kappa)$  can correspond to the same law (1.3), since only the sum  $2\nu\|D\mathbf{U}\| + \kappa\sqrt{2}$  is involved. With these notations, the constant yield stress relation  $\kappa(\|D\mathbf{U}\| = 0, p) = \text{cst}$  characterizes the Bingham model. On the contrary, assuming that a free surface separates the material from air at the reference zero pressure, the Drucker–Prager yield stress [19] can be written

$$\kappa(\|D\mathbf{U}\| = 0, p) = \mu_s p, \quad (1.6)$$

where  $\mu_s \geq 0$  is the static internal Coulomb friction coefficient, a constant depending only on the material ( $\mu_s = \tan \delta$ , where  $\delta$  is the internal friction angle of the material). In order to get uniqueness of  $\nu$  and  $\kappa$  in the decomposition (1.3), we adopt the convention of [31] that  $\kappa \equiv \kappa(p)$  and  $\nu(\|D\mathbf{U}\|, p)\|D\mathbf{U}\| \rightarrow 0$  as  $\|D\mathbf{U}\| \rightarrow 0$ . This means that the last term in (1.3) is the rate independent (pure plastic) part of the law. Taking into account an eventually negative pressure, the Drucker–Prager yield stress is thus given by

$$\kappa(p) = \mu_s \max(p, 0). \quad (1.7)$$

As commented above, the viscosity  $\nu$  is taken constant in this work, although the value

$$\nu = (\mu(I) - \mu_s) \frac{\max(p, 0)}{\sqrt{2}\|D\mathbf{U}\|} \quad (1.8)$$

should be preferred, corresponding to the  $\mu(I)$  rheology of [34], with  $I \sim \|D\mathbf{U}\|/\sqrt{p}$ . Taking into account  $\mu(I)$  may be very important in some situations, but would lead here to some complications hiding the main features of our approach. Note that [34] proposes the choice  $\nu = 0$ ,  $\kappa = \mu(I) \max(p, 0)$ , which is equivalent to the choice (1.8), (1.7). The Drucker–Prager yield stress (1.6) provides a better approximation of the behaviour of granular flows than the Bingham yield stress [34]. In particular, while in Bingham fluids, a plug zone may develop on top of the flowing layer [6], for Drucker–Prager or Coulomb fluids, the static zone is generally found near the base of the flow due to the specific form of the  $\kappa$  law. However, contrary to the Bingham model (studied mathematically by [20] in the viscous case and by [38] in the two-dimensional viscous or inviscid cases) that is always well-posed even without viscosity [12], the Drucker–Prager yield stress with viscosity is ill-posed in some situations, as discussed above.

The system (1.1)-(1.5) describes a flowing behaviour where  $D\mathbf{U} \neq 0$ , and a static behaviour where  $D\mathbf{U} = 0$ . However, the interface between these two domains is not written explicitly, it is embedded in the whole formulation. The aim of this work is to derive a more explicit description of the evolution of this interface in the context of a shallow approximation.

Solving the complete set of equations for an incompressible viscoplastic material requires high computational time, preventing applications to natural geophysical flows. As a result, several attempts have been made to reduce the computational cost, essentially based on so-called shallow water, or depth-averaged, models. They have been derived rather precisely for Newtonian flows, and studies for non-Newtonian flows have been proposed in [46, 6, 7, 9, 49, 24, 16, 23, 44, 11, 29, 30, 26, 15]. In these works, the approach is most of the time to average the equations (1.1), (1.2) with respect to a variable normal to the bottom topography, and to close the equations by an assumption on the dependency on this normal variable, motivated by observations from natural or experimental flows. However, doing this, the information contained in (1.2) on the evolution of the full dependence in the normal variable is lost, and is replaced by the external input on the profile. Thus by this method, the obtained depth-averaged system can have no relation to solutions of the original system (1.1), (1.2). There are however cases where the profile can be shown compatible with the original system. Numerical methods for non-Newtonian shallow water models are described in [16, 1, 22].

No static-flowing interface exists in the above cited works on depth-averaged models, meaning that either the whole layer of material flows, either it is fully static. The key issue for describing a static-flowing interface in this context is the knowledge of an equation on the static-flowing interface. Defining an equation for this interface in shallow water models

has been done based on phenomenological models or on strong assumptions such as specified velocity profile or reducing the flow to a sliding block [3, 10, 14, 25, 35, 4, 41, 5, 13, 32]. A review is proposed in [33], together with a discussion on the dependency of the interface equation in terms of averaged quantities.

In this work we propose a new approach for describing the static-flowing interface dynamics. It is analytic, in the sense that no phenomenology is put in the interface equation. Rather, the information on the profile and on the static-flowing interface is deduced from the viscoplastic modelling, by asymptotic expansion. Several assumptions are necessary, in particular a shallow assumption. Moreover, the viscosity has to be small (meaning that the plastic effects are dominant), the slope of the topography has to be nonzero, the internal friction coefficient has to be close to the slope, the velocity has to be moderate, and the pressure has to be convex with respect to the normal variable, which is a kind of static-flowing stability condition. As in [11], we have to push the expansions to higher order than usually in shallow water models in order to close the system. The obtained model is not a depth-averaged model, since the normal variable is still present, and cannot be avoided. It is written as a formally overdetermined boundary problem, for which the extra boundary condition (with respect to a standard parabolic problem) drives the evolution of the interface between the two phases. This closed mathematical formulation replaces a formulation by an ordinary differential equation on the interface, frequently proposed in previous works, based on external input or heuristics. A simplified system is compared to laboratory experiments in [40], confirming the relevance of our approach.

## 2 Two-dimensional model and topography-based coordinates

### 2.1 Flow domain and boundary conditions

We consider a viscoplastic material described by the system (1.1)-(1.7), set in the spatial domain  $x \in \Omega_t$  between a fixed bottom topography and a free surface. The system is completed with the following boundary conditions. At the bottom we set the no slip condition

$$\mathbf{U} = 0 \text{ at the bottom,} \quad (2.1)$$

at the free surface we set the stress free condition

$$\boldsymbol{\sigma} \mathbf{N} = 0 \text{ at the free surface,} \quad (2.2)$$

where  $\mathbf{N}$  is the unit external normal to the free surface. Moreover, the free surface evolves with the material, and thus satisfies the kinematic condition

$$N_t + \mathbf{N} \cdot \mathbf{U} = 0 \text{ at the free surface,} \quad (2.3)$$

where  $(N_t, \mathbf{N})$  is the time-space normal to the free surface.

It is not clear if (2.1) leads to a formally well-posed problem if  $\nu = 0$ . Nevertheless, we shall look for solutions for which  $\mathbf{U}$  vanishes identically in a neighborhood of the topography (static part), thus this condition will be automatically satisfied.

### 2.2 Two-dimensional formulation

We consider the two-dimensional problem. Following [13], the topography can be described by the relation  $z = B(x)$  in horizontal/vertical coordinates  $(x, z)$ , where  $B(x)$  is a smooth function. We denote by  $\theta$  the angle between the horizontal and the tangent to the topography, and by  $X$  the curvilinear coordinate along the topography, so that

$$\tan \theta = \frac{dB}{dx}, \quad \frac{dX}{dx} = \frac{1}{\cos \theta}, \quad \frac{dB}{dX} = \sin \theta. \quad (2.4)$$

We shall consider that  $B$ ,  $\theta$ ,  $x$  are functions of  $X$ , instead of  $B$ ,  $\theta$ ,  $X$  being functions of  $x$ . The case of affine topography  $B$  in terms of  $x$ , i.e. slope angle  $\theta = cst$ , is possible in particular.

A point  $M$  within the material layer  $\Omega_t$  can be described by its distance  $Z$  to the topography and the coordinate  $X$  of its projection on the topography, or equivalently by the relation

$$M = (x - Z \sin \theta, B + Z \cos \theta), \quad (2.5)$$

where  $x$ ,  $X$ ,  $B$ ,  $\theta$  are related by the relations (2.4), as illustrated on Figure 1. The value of

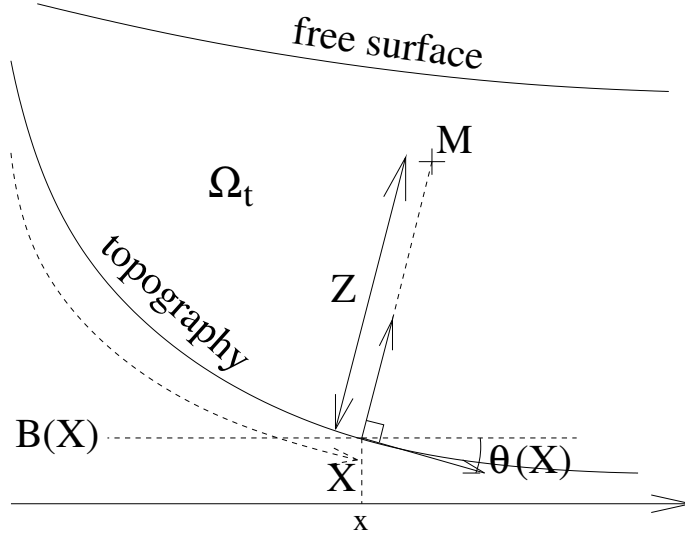


Figure 1: The material layer between topography and free surface.

$Z$  is less than the height  $h(t, X)$  of the material in the normal direction,

$$0 < Z < h(t, X). \quad (2.6)$$

The change of variable between  $M$  and  $(X, Z)$  is a diffeomorphism provided that  $h d_X \theta < 1$ , where  $d_X$  denotes the derivative with respect to the single variable  $X$ . Note that  $d_X \theta \equiv d\theta/dX$  is the curvature of the topography. We define the velocity components  $(U, W)$  at  $M$  in the coordinates tangent and normal to the topography

$$\begin{pmatrix} U \\ W \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \mathbf{U}, \quad (2.7)$$

and the new stress tensor

$$\Sigma = \begin{pmatrix} \Sigma_{XX} & \Sigma_{XZ} \\ \Sigma_{ZX} & \Sigma_{ZZ} \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \sigma \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad (2.8)$$

with  $\Sigma_{XZ} = \Sigma_{ZX}$ . Then, according to the change of variables computations of [13] (that were done in the more general case of time-dependent topography), we can reformulate the equations (1.1)-(1.2) and (2.1)-(2.3) in the coordinates  $(t, X, Z)$ . Denoting by

$$J = 1 - Z d_X \theta, \quad (2.9)$$

the system (1.1)-(1.2) becomes in the new coordinates

$$\partial_X U + \partial_Z (JW) = 0, \quad (2.10)$$

$$\begin{aligned} & \partial_t(JU) + U\partial_X U + JW\partial_Z U + \partial_X(g(B + Z \cos \theta) - \Sigma_{XX}) - \partial_Z(J\Sigma_{XZ}) \\ & = (-\Sigma_{XZ} + UW)d_X\theta, \end{aligned} \quad (2.11)$$

$$\begin{aligned} & \partial_t(JW) + U\partial_X W + JW\partial_Z W - \partial_X\Sigma_{XZ} + J\partial_Z(gZ \cos \theta - \Sigma_{ZZ}) \\ & = (\Sigma_{XX} - \Sigma_{ZZ} - U^2)d_X\theta, \end{aligned} \quad (2.12)$$

where  $g > 0$  is the gravity constant. The kinematic condition (2.3) becomes

$$J_h \partial_t h + U_h \partial_X h = J_h W_h, \quad (2.13)$$

where the index  $h$  means that the quantity is evaluated at  $Z = h(t, X)$ . It can be written equivalently, integrating (2.10) for  $Z$  between 0 and  $h$ , as

$$\partial_t \left( h - \frac{h^2}{2} d_X \theta \right) + \partial_X \left( \int_0^h U dZ \right) = 0. \quad (2.14)$$

The boundary conditions (2.1), (2.2) become at the free surface

$$J\Sigma_{XZ} - \Sigma_{XX}\partial_X h = 0, \quad J\Sigma_{ZZ} - \Sigma_{XZ}\partial_X h = 0, \quad \text{at } Z = h, \quad (2.15)$$

and at the bottom

$$W = 0, \quad U = 0, \quad \text{at } Z = 0. \quad (2.16)$$

Finally, one can check that the Bingham type relations (1.3)-(1.5) become

$$\Sigma_{XX} = -p - \left( 2\nu + \frac{\kappa}{\|D\mathbf{U}\|/\sqrt{2}} \right) \partial_Z W, \quad \Sigma_{ZZ} = -p + \left( 2\nu + \frac{\kappa}{\|D\mathbf{U}\|/\sqrt{2}} \right) \partial_Z W, \quad (2.17)$$

$$\Sigma_{XZ} = \left( 2\nu + \frac{\kappa}{\|D\mathbf{U}\|/\sqrt{2}} \right) \frac{1}{2} \left( \partial_Z U + \frac{\partial_X W + U d_X \theta}{J} \right), \quad (2.18)$$

$$\frac{1}{2} \|D\mathbf{U}\|^2 = (\partial_Z W)^2 + \frac{1}{4} \left( \partial_Z U + \frac{\partial_X W + U d_X \theta}{J} \right)^2, \quad (2.19)$$

where we recall that (2.10) relates  $\partial_X U$  and  $\partial_Z W$ . These formulas (2.17)-(2.19) are valid where  $\|D\mathbf{U}\| \neq 0$ , but they can be obviously generalized via a multivalued formulation like (1.5).

We are interested in flows having a transition between a static phase at rest above the bottom ( $\mathbf{U} = 0$ ), and a flowing phase with shear above ( $D\mathbf{U} \neq 0$ ),

$$\begin{aligned} & U(t, X, Z) = 0 \quad \text{for } 0 < Z < b(t, X), \\ & \left( \partial_Z U + \frac{\partial_X W + U d_X \theta}{J} \right) (t, X, Z) \neq 0 \quad \text{for } b(t, X) < Z < h(t, X), \end{aligned} \quad (2.20)$$

for some interface  $b(t, X)$  such that  $0 < b(t, X) < h(t, X)$ . Note that the first line in (2.20) implies that  $W$  also vanishes for  $Z < b$ , because of (2.10) and the bottom boundary condition on  $W$  in (2.16). An illustration is proposed in Figure 2.

### 2.3 Simple shear flows

In this subsection we would like to show that there exist some flows solutions to the above stated viscoplastic model, with a fluid layer flowing above a static layer as stated in (2.20), at least in the simple shear case, *i.e.* when the quantities do not depend on  $X$ .

We assume a constant slope angle  $\theta$ , and we look for solutions  $h, U, W, p$  to (2.10)-(2.19) with (1.7) such that

$$h, U, W, p, \Sigma_{XX}, \Sigma_{ZZ}, \Sigma_{XZ} \text{ do not depend of } X. \quad (2.21)$$

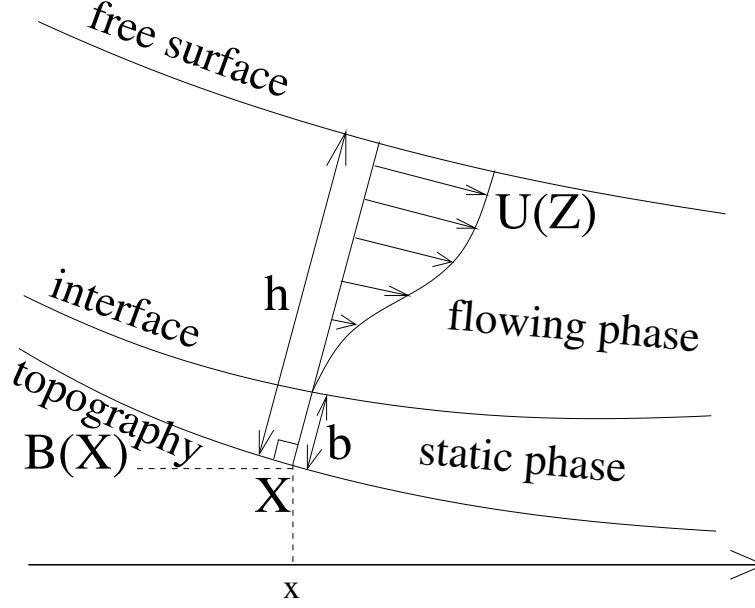


Figure 2: Velocity profile  $U(Z)$  within the granular layer of thickness  $h$ , with the interface  $b$  separating the static and flowing layers, in the case  $\theta < 0$  and  $\partial_Z U > 0$

Then,  $J = 1$ , and (2.10) and the bottom boundary condition gives that  $W \equiv 0$ . The kinematic condition (2.13) (or (2.14)) gives  $\partial_t h = 0$ , thus  $h$  is constant. The normal momentum conservation equation (2.12) gives that  $\partial_Z (gZ \cos \theta - \Sigma_{ZZ}) = 0$ , and together with the free surface boundary condition on  $\Sigma_{ZZ}$  in (2.15) it yields  $\Sigma_{ZZ} = -g \cos \theta (h - Z)$ . Then, (2.17) yields

$$-\Sigma_{XX} = -\Sigma_{ZZ} = p = g \cos \theta (h - Z). \quad (2.22)$$

The remaining relations (2.18)-(2.19) then give

$$\Sigma_{XZ} = \nu \partial_Z U + \kappa \operatorname{sgn}(\partial_Z U), \quad (2.23)$$

where the sign is multivalued, i.e. whenever  $\partial_Z U = 0$ ,  $\operatorname{sgn}(\partial_Z U)$  is any number in  $[-1, 1]$ . It finally remains to write (2.11),

$$\partial_t U + g \sin \theta - \partial_Z \Sigma_{XZ} = 0, \quad (2.24)$$

and the boundary condition on  $\Sigma_{XZ}$  in (2.15),

$$\Sigma_{XZ} = 0 \quad \text{at } Z = h. \quad (2.25)$$

This system is completed by the relation (1.7), which is with (2.22)

$$\kappa = \mu_s g \cos \theta (h - Z). \quad (2.26)$$

The solutions to (2.10)-(2.19), (1.7) satisfying (2.21) are thus obtained by finding a solution  $U(t, Z)$  defined for  $0 < Z < h$  satisfying (2.23)-(2.26).

The solutions to (2.23)-(2.26) satisfying (2.20) for some  $b(t) \in (0, h)$  can then be characterized by a system of equations set in the flowing phase  $Z > b$ , together with a static equilibrium condition, as stated in Appendix (taking  $\Phi = g \sin \theta$  for the source), and we note that here  $\kappa$  is linear, thus convex.

The conclusion is that given  $s_b = \pm 1$  the sign of the shear  $\partial_Z U$  in the flowing phase, and assuming that

$$s_b \theta \leq 0, \quad (2.27)$$



the problem of finding a solution  $U(t, Z)$  defined for  $0 < Z < h$  to (2.23)-(2.26) satisfying (2.20) is equivalent to finding a solution  $U(t, Z)$  defined for  $b(t) < Z < h$  to

$$\partial_t U + g \sin \theta + s_b \mu_s g \cos \theta - \partial_Z(\nu \partial_Z U) = 0 \text{ for } Z > b, \quad (2.28)$$

with

$$s_b \partial_Z U > 0 \text{ for } Z > b, \quad (2.29)$$

satisfying the boundary conditions

$$\nu \partial_Z U = 0 \text{ at } Z = h, \quad (2.30)$$

$$U = 0 \text{ at } Z = b(t), \quad (2.31)$$

$$\nu \partial_Z U = 0 \text{ at } Z = b(t), \quad (2.32)$$

and the friction dominating condition

$$|\tan \theta| \leq \mu_s. \quad (2.33)$$

must hold for the static layer to exist. Note that (2.27) means that the slope and the shear rate  $\partial_Z U$  in the flowing zone have opposite signs, which is very natural, see Figure 2. Here it is important to mention that the boundary condition (2.32) is not imposed a priori, but is a consequence of the analysis provided in Appendix, that states that the shear stress  $\Sigma_{XZ}$ , and indeed  $\nu \partial_Z U$ , must be continuous through the interface. The necessary condition (2.33) is really needed. Indeed, if it does not hold, the static-flowing interface can no longer persist for positive time, and we expect in this case the whole layer of material to flow down immediately.

Under the condition (2.33), numerical simulations performed in [40] show that there is a solution to (2.28)-(2.32). In this system there is one more boundary condition than what is expected in standard parabolic problems (i.e. set in a fixed domain), this is a *formally overdetermined* problem. This extra condition determines the evolution of the interface  $b(t)$ .

Indeed, in the inviscid case  $\nu = 0$ , there remains only the extra condition  $U(t, b(t)) = 0$  to determine the interface. Since, in this inviscid case, (2.28) is just a differential equation with constant right-hand side in the flowing zone, one can check that the solution is given explicitly as follows, in the case  $s_b = 1$ ,  $\theta \leq 0$  to simplify,

$$U(t, Z) = \max(U^0(Z) - g(\sin \theta + \mu_s \cos \theta)t, 0) \text{ for } 0 < Z < h, \quad (2.34)$$

where  $U^0(Z)$  is the initial velocity profile, assumed to be defined for  $0 < Z < h$  satisfying  $\partial_Z U^0 \geq 0$ ,  $U^0 \geq 0$ . The interface  $b(t)$  is given implicitly by the relation

$$U^0(b(t)) = g(\sin \theta + \mu_s \cos \theta)t, \quad (2.35)$$

as long as the right-hand side remains less than  $U^0(h)$ . After that time,  $b$  can be extended by setting  $b(t) = h$ , and  $U = 0$ . Thus this solutions becomes static in finite time, and the interface  $b(t)$  is nondecreasing with respect to time (this is not the case with viscosity, see [40]). Note that the formula (2.34) works for any (nondecreasing) initial profile  $U^0$ . Thus there is no preferred shape for the  $Z$  dependency of the velocity.

### 3 Shallow asymptotics

We consider the two-dimensional viscoplastic model in curvilinear coordinates (2.10)-(2.19) with Drucker-Prager yield stress (1.7), and with the existence of a static-flowing transition (2.20). As is by now common in shallow water asymptotics [11], we consider a small dimensionless parameter  $\varepsilon$  and we assume that

$$h \sim \varepsilon, \quad d_X \theta = O(\varepsilon), \quad (3.1)$$

$$\Sigma = O(\varepsilon), \quad p = O(\varepsilon). \quad (3.2)$$

These assumptions are related to the knowledge of a characteristic length  $L$  and a characteristic time  $\tau$  with  $L/\tau^2 = g$ , and to the introduction of the appropriate scales. Relation (3.1) means that  $h/L$  is of the order of  $\varepsilon$ , and that  $Ld_X\theta$  is at most of the order of  $\varepsilon$ , while (3.2) means that  $\Sigma_{XX}$ ,  $\Sigma_{ZZ}$ ,  $\Sigma_{XZ}$ ,  $p$  are at most of the order of  $(L/\tau)^2\varepsilon$ .

We assume that the derivatives with respect to  $t$  or  $X$  does not induce singularities (they just involve the characteristic scales  $L$  and  $\tau$ , but not  $\varepsilon$ ). However, derivatives with respect to  $Z$  naturally involve, apart from the scale  $L$ , a factor  $1/\varepsilon$  at worse, because  $Z$  lies in the interval  $(0, h)$ , which is of order  $\varepsilon$ . Taking into account the incompressibility (2.10) and the vanishing of  $W$  at the bottom in (2.16), for bounded velocities we get the following natural assumptions of orders of magnitude

$$\begin{aligned} U &= O(1), & W &= O(\varepsilon), \\ \partial_X U &= O(1), & \partial_X W &= O(\varepsilon), \\ \partial_Z U &= O(1/\varepsilon), & \partial_Z W &= O(1). \end{aligned} \quad (3.3)$$

Now and further on, we do not write explicitly the units in terms of  $L$  and  $\tau$ , but only the orders of magnitude in terms of  $\varepsilon$ . Note that another scaling which is used in shallow granular flows consists in taking the free-fall time scale  $\sqrt{h/g}$ , which is in  $\sqrt{\varepsilon}$ , leading to  $U \simeq \sqrt{gh}$ , which is also of order  $\sqrt{\varepsilon}$ .

Additionally to (3.3), we shall also consider the case of slow velocities  $U = O(\varepsilon)$ , which leads to

$$\begin{aligned} U &= O(\varepsilon), & W &= O(\varepsilon^2), \\ \partial_X U &= O(\varepsilon), & \partial_X W &= O(\varepsilon^2), \\ \partial_Z U &= O(1), & \partial_Z W &= O(\varepsilon). \end{aligned} \quad (3.4)$$

In each situation (3.3), respectively (3.4), if  $\partial_Z U$  is really of order  $1/\varepsilon$ , respectively 1 (i.e. it is not smaller than this scale) in the flowing domain, then the expression (2.19) gives

$$\|DU\| = \frac{1}{\sqrt{2}}|\partial_Z U|(1 + O(\varepsilon^2)) \quad \text{for } Z > b(t, X). \quad (3.5)$$

Assuming that the viscosity is small

$$\nu = \begin{cases} O(\varepsilon^2) & \text{in case (3.3),} \\ O(\varepsilon) & \text{in case (3.4),} \end{cases} \quad (3.6)$$

the equations (2.17) give that  $\Sigma_{XX} + p = O(\varepsilon^2)$ ,  $\Sigma_{ZZ} + p = O(\varepsilon^2)$  for  $Z > b$ . We are going to look for solutions that extend these orders of magnitude in the static zone. Thus we assume for  $0 \leq Z \leq h$

$$\Sigma_{XX} + p = O(\varepsilon^2), \quad \Sigma_{ZZ} + p = O(\varepsilon^2). \quad (3.7)$$

These assumptions imply that  $\Sigma_{XX} - \Sigma_{ZZ} = O(\varepsilon^2)$ , and with (3.2) we obtain from (2.15) the approximate boundary conditions at the free surface

$$\Sigma_{XZ} = O(\varepsilon^3) \quad \text{at } Z = h, \quad \Sigma_{ZZ} = O(\varepsilon^4) \quad \text{at } Z = h. \quad (3.8)$$

We are looking for an approximation of (2.11) (momentum along the topography) up to  $O(\varepsilon^2)$ . Therefore we need an approximation of (2.12) (momentum normal to the topography) up to  $O(\varepsilon)$ . This latter equation is therefore expanded as

$$\partial_Z(gZ \cos \theta - \Sigma_{ZZ}) = O(\varepsilon), \quad (3.9)$$

from which we deduce with (3.8)

$$\Sigma_{ZZ} = -g \cos \theta (h - Z) + O(\varepsilon^2). \quad (3.10)$$

Reporting this in (2.11) and using (3.7), we get

$$\partial_t U + U \partial_X U + W \partial_Z U + g \partial_X (B + h \cos \theta) - \partial_Z (\Sigma_{XZ}) = O(\varepsilon^2). \quad (3.11)$$

We have to see that  $W$  is deduced from the knowledge of  $U$  by the incompressibility condition (2.10) and the bottom condition (2.16), that simplify to

$$\partial_X U + \partial_Z W = O(\varepsilon^2), \quad W = 0 \text{ at } Z = 0. \quad (3.12)$$

The equation (3.11) is completed by the asymptotics coming from (2.18), (3.5),

$$\Sigma_{XZ} = \nu \partial_Z U + \kappa \operatorname{sgn}(\partial_Z U) + O(\varepsilon^3), \quad (3.13)$$

where  $\operatorname{sgn}(\partial_Z U)$  has to be interpreted as multivalued. The equations (3.11)-(3.13) are completed by the boundary condition

$$\Sigma_{XZ} = O(\varepsilon^3) \quad \text{at } Z = h, \quad (3.14)$$

coming from (3.8), and by the equation (2.14) giving the evolution of  $h$ .

The system (3.11)-(3.14), (2.14) involves an unknown  $U(t, X, Z)$  defined up to errors in  $\varepsilon^2$ . However, there are two main difficulties that arise. The first is that the value of  $\kappa$  in (3.13) needs to be known up to an error in  $\varepsilon^3$ . According to (1.7) this means to know the pressure  $p$  up to an error in  $\varepsilon^3$ , what we *do not have* from (3.7), (3.10). Thus the system is not closed. The second difficulty is that it is not obvious how to describe the interface dynamics with (3.11), because of the inertial terms  $U \partial_X U$ ,  $W \partial_Z U$ . Note however that the system is coherent with the assumptions (3.2) and (3.3), because the formulas (3.10), (3.7), (3.13) give  $\Sigma = O(\varepsilon)$ , and all the terms in (3.11) are bounded.

## 4 Slow flow model

In order to resolve the difficulties previously stated on the system (3.11)-(3.14), (2.14) under the assumption (3.3), we complete now the analysis in the case of slow velocity, i.e. assumption (3.4).

The first observation to make is that for this assumption (3.4) to remain valid for all time, we need that  $\partial_t U = O(\varepsilon)$ . Looking thus at equation (3.11), we get the compatibility condition

$$g \sin \theta = \partial_Z (\Sigma_{XZ}) + O(\varepsilon). \quad (4.1)$$

In order to simplify, we make the assumption of very small viscosity

$$\nu = O(\varepsilon^2), \quad (4.2)$$

which is stronger than (3.6). Using (3.13) and (1.7), (3.7), (3.10), this leads to the relation in the flowing phase  $g \sin \theta = -g \mu_s \cos \theta \operatorname{sgn}(\partial_Z U) + O(\varepsilon)$ . If  $\mu_s$  is not  $O(\varepsilon)$ , then also  $\theta$  must not be small (not  $O(\varepsilon)$ ), and we get the two conditions

$$\mu_s = |\tan \theta| + O(\varepsilon), \quad (4.3)$$

and

$$\operatorname{sgn}(\partial_Z U) = -\operatorname{sgn}(\theta) \quad \text{for } Z > b(t, X). \quad (4.4)$$

The condition (4.3) means that the effects of gravity and friction compensate, up to a fluctuation of order  $\varepsilon$ . Without this assumption, one of the two forces would dominate the other, with the effect of a strong acceleration, violating our assumption  $\partial_t U = O(\varepsilon)$ .

We therefore make the assumption (3.1), (4.2), (4.3) (note that  $\theta$  can nevertheless be  $X$  dependent). Looking for solutions to (2.10)-(2.19) with (1.7), satisfying (2.20), (3.2), (3.4), we arrive at the system (3.11)-(3.14), (2.14), (1.7), (2.20), (4.4). In this system, according to (3.4) the inertial terms  $U \partial_X U$ ,  $W \partial_Z U$  in (3.11) are  $O(\varepsilon^2)$ , and can be neglected. We can then forget equation (3.12).

We wish now to obtain an expansion of  $p$  up to errors in  $\varepsilon^3$ , so that we can close the equation (3.13) with (1.7). Expanding (2.12) at higher order than in (3.9) yields

$$\partial_Z(gZ \cos \theta - \Sigma_{ZZ}) = \partial_X \Sigma_{XZ} + O(\varepsilon^2). \quad (4.5)$$

But using (4.1) and (3.14) we get

$$\Sigma_{XZ} = -g \sin \theta (h - Z) + O(\varepsilon^2), \quad (4.6)$$

which gives in (4.5) by using (3.1)

$$\partial_Z(gZ \cos \theta - \Sigma_{ZZ}) = -g \sin \theta \partial_X h + O(\varepsilon^2). \quad (4.7)$$

With the boundary condition (3.8) we deduce

$$\Sigma_{ZZ} = -g(\cos \theta + \sin \theta \partial_X h)(h - Z) + O(\varepsilon^3). \quad (4.8)$$

But independently, taking into account (2.17), (3.5), (3.12), one has

$$-\Sigma_{ZZ} - p = 2\kappa \frac{\partial_X U}{|\partial_Z U|} + O(\varepsilon^3), \quad \text{for } Z > b, \quad (4.9)$$

thus with (4.8) we get

$$p = g(\cos \theta + \sin \theta \partial_X h)(h - Z) - 2\kappa \frac{\partial_X U}{|\partial_Z U|} + O(\varepsilon^3), \quad \text{for } Z > b. \quad (4.10)$$

To leading order, this quantity is nonnegative. Taking into account (1.7), (4.3) and the approximation  $g \cos \theta (h - Z)$  of  $p$  up to  $O(\varepsilon^2)$ , we arrive at

$$p = g \left( \cos \theta + \sin \theta \partial_X h - 2|\sin \theta| \frac{\partial_X U}{|\partial_Z U|} \right) (h - Z) + O(\varepsilon^3), \quad \text{for } Z > b. \quad (4.11)$$

This has to be taken into account in (3.11), (3.13), which becomes

$$\partial_t U + g(\sin \theta + \partial_X(h \cos \theta)) - \partial_Z(\nu \partial_Z U + \mu_s p \operatorname{sgn}(\partial_Z U)) = O(\varepsilon^2), \quad (4.12)$$

with the boundary condition (3.14) that simplifies to

$$\nu \partial_Z U = O(\varepsilon^3) \text{ at } Z = h. \quad (4.13)$$

We conclude that under assumptions (3.1), (4.2), (4.3), solutions to the two-dimensional viscoplastic model with static-flowing interface that satisfy (3.2), (3.4) are determined by the system (4.11)-(4.13), (2.14), (4.4). We notice that the pressure in (4.11) is only defined in the flowing phase, because the ratio  $\partial_X U / \partial_Z U$  is not defined in the static phase. In order to go further, we shall assume that

$$\text{the pressure } p \text{ is convex with respect to } Z \text{ in } [0, h]. \quad (4.14)$$

This assumption, including the static phase, seems to be rather valid in viscoplastic flows of interest, see [31], [39, p. 115-120], at least in its weaker form stated after (A.14). In view of (4.11), this condition (4.14) implies a particular shape for the ratio  $\partial_X U / \partial_Z U$  in the flowing phase, this term representing the main non-hydrostatic part of the pressure. The assumption (4.14) expresses a kind of stability of the double layer configuration with flowing on top of static material. According to the Appendix, applied here with  $\Phi = g(\sin \theta + \partial_X(h \cos \theta))$ , it ensures the force balance in the static layer, and without it one would have an immediate flowing of the whole granular material. The assumption (4.14) allows to formulate the problem in the flowing phase only (thus avoiding the knowledge of the values of the pressure in the static phase), together with boundary conditions at the interface.

## 4.1 Formulation of the slow flow model

According to the previous arguments, we obtain the following slow flow model. Under assumptions (3.1), (4.2), (4.3), (4.14), solutions to the two-dimensional viscoplastic model with static-flowing interface that satisfy (3.2), (3.4) are determined by the system (2.14), (4.4),

$$\partial_t U + g(\sin \theta + \partial_X(h \cos \theta)) + \operatorname{sgn}(\theta) \partial_Z(\mu_s p) - \partial_Z(\nu \partial_Z U) = O(\varepsilon^2) \quad \text{for } Z > b, \quad (4.15)$$

with  $p$  given by (4.11), and the boundary conditions

$$\nu \partial_Z U = O(\varepsilon^3) \quad \text{at } Z = h, \quad (4.16)$$

$$U = 0 \quad \text{at } Z = b(t), \quad (4.17)$$

$$\nu \partial_Z U = 0 \quad \text{at } Z = b(t), \quad (4.18)$$

subject to the static equilibrium condition

$$g |\sin \theta + \partial_X(h \cos \theta)| \leq -(\partial_Z(\mu_s p))_b + O(\varepsilon^2). \quad (4.19)$$

A particular solution to this slow flow model (2.14), (4.4), (4.11), (4.15)-(4.19) is the simple shear flow of Subsection 2.3, for which there is no dependency in  $X$ .

We can observe that since  $U(t, X, b(t, X)) = 0$ , differentiating with respect to  $X$  we get

$$\frac{\partial_X U}{\partial_Z U}(t, X, b(t, X)) = -\partial_X b(t, X), \quad (4.20)$$

where the ratio on the left-hand side has to be computed as the limit as  $Z \rightarrow b(t, X)$  by above (since both the numerator and the denominator vanish in the case  $\nu > 0$ ). This gives the limit value at  $Z = b$  in (4.11).

## 4.2 Steady states at rest

An important situation to look at is the case a fully static solution to the slow flow model. We thus look for a solution to (2.14), (4.4), (4.11), (4.15)-(4.19) such that  $U \equiv 0$ . Then  $b \equiv h$ , (4.11), (4.4), (4.15) say nothing, and (4.16)-(4.18) are satisfied. The equation (2.14) gives that  $\partial_t h = 0$ . It remains to write (4.19). Viewing (4.11) as an expansion in  $Z - b$  (recall that  $b = h$ ) and using (4.20) yields the formal relation  $(\partial_Z p)_b = -g(\cos \theta + \sin \theta \partial_X h - 2 \sin \theta \partial_X b) + O(\varepsilon^2)$ . Thus (4.19) reduces to

$$|\sin \theta + \partial_X(h \cos \theta)| \leq \mu_s (\cos \theta + \sin \theta \partial_X h - 2 \sin \theta \partial_X h) + O(\varepsilon^2), \quad (4.21)$$

or equivalently

$$\left| \frac{\sin \theta + \partial_X(h \cos \theta)}{\cos \theta - \partial_X(h \sin \theta)} \right| \leq \mu_s + O(\varepsilon^2). \quad (4.22)$$

The interpretation of this inequality is as follows. Consider the coordinates  $(y(X), z(X))$  of a point at the free surface,

$$y = x - h \sin \theta, \quad z = B + h \cos \theta, \quad (4.23)$$

where  $x$  is the horizontal coordinate, according to (2.4), (2.5), see Figure 1. Then (4.22) says that  $|dz/dy| \leq \mu_s + O(\varepsilon^2)$ , i.e. the slope of the free surface is less than or equal to  $\mu_s$ . We conclude that this natural slope condition (4.22) is the one for the solution to the slow flow model to remain at rest. It is a first order correction taking into account the width  $h$  and the dependency in  $X$  to the simple shear condition (2.33).

### 4.3 Steady flows with static-flowing transition

Some steady flows with static-flowing interface can be obtained for the slow flow model. A solution to (2.14), (4.4), (4.11), (4.15)-(4.19) is built as follows, under the assumption that the viscosity is negligible,

$$\nu = O(\varepsilon^3). \quad (4.24)$$

We assume that  $\theta(X)$  has a constant sign, and we take the velocity of the form

$$U(X, Z) = \tilde{U}(Z - h(X)), \quad (4.25)$$

where  $\tilde{U}(Y)$  is an arbitrary profile verifying

$$\text{sgn}(\theta)d\tilde{U}/dY \leq 0, \quad \tilde{U}(Y) = 0 \quad \text{for } Y \leq -y_0, \quad (4.26)$$

for some  $y_0 \geq 0$ . We consider a spatial domain where  $h(X) \geq y_0$ . Then (2.14) is verified since  $\int_0^h U dZ = \int_{-\infty}^0 \tilde{U} dY$  is independent of  $X$ , and the sign condition (4.4) holds according to (4.26). With (4.17) we have  $b(X) = h(X) + y_1$  with  $y_1$  the largest value for which  $\tilde{U}(y_1)$  vanishes, thus  $\partial_X(h-b) = 0$ ,  $\partial_X U / \partial_Z U = -\partial_X h$ , and (4.11) gives  $p = g(\cos\theta - \sin\theta\partial_X h)(h-Z) + O(\varepsilon^3)$ . Thus (4.15) and (4.19) give the single equation

$$\sin\theta + \partial_X(h \cos\theta) - \mu_s \text{sgn}(\theta)(\cos\theta - \partial_X(h \sin\theta)) = O(\varepsilon^2). \quad (4.27)$$

The boundary conditions (4.16), (4.18) are satisfied up to  $O(\varepsilon^3)$ . Thus there remains only the condition (4.27). With the notations (4.23), this means that  $|dz/dy| = \mu_s + O(\varepsilon^2)$ , i.e. the slope of the free surface is equal to  $\mu_s$ . The steady solution built in this way has free surface and static-flowing interface with slopes  $\mu_s$ , and it generalizes the steady simple shear solution given by (2.34) with  $|\tan\theta| = \mu_s$ . The solution is illustrated in Figure 3.

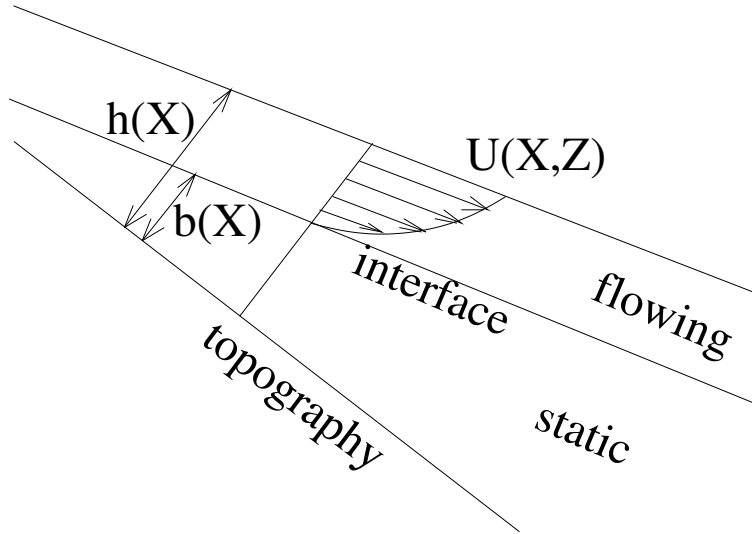


Figure 3: Steady flow with static-flowing transition, in the case when  $\theta$  is constant. The topography has slope angle  $\theta$ . The free surface and the interface have slope  $\mu_s$ , so that gravity balances internal friction.

Observing this picture, the reader may wonder if the situation represented, with  $|\tan\theta| > \mu_s$ , is possible. The answer is yes. Indeed, what the static equilibrium condition (4.19) says here is only that the slope of the free surface is  $\mu_s$ , in contrast with the simple shear solution of Subsection 2.3, where the static equilibrium condition says (2.33), i.e.  $|\tan\theta| \leq \mu_s$ . Thus here there is no condition on  $\theta$ , and  $|\tan\theta|$  can be less or greater than  $\mu_s$ . Note

however that the asymptotic assumption (4.3) imposes that  $|\tan \theta|$  is not far from  $\mu_s$ . The interpretation of the situation depicted here is that the flowing part of the material “does not feel the topography” through the static layer, similarly as in classical shallow water (i.e. without friction) where the rest solution with horizontal free surface exists whatever is the topography.

#### 4.4 Interface dynamics and discussion

The slow flow problem (2.14), (4.4), (4.11), (4.15)-(4.19) can be reformulated as follows. Defining  $S(t, X, Z)$  by

$$S = -g \operatorname{sgn}(\theta)(\sin \theta + \partial_X(h \cos \theta)) - \partial_Z(\mu_s p), \quad (4.28)$$

the equation (4.15) can be written (dropping the  $\varepsilon$ ) as

$$\partial_t U - \operatorname{sgn}(\theta)S - \partial_Z(\nu \partial_Z U) = 0 \quad \text{for } Z > b(t, X), \quad (4.29)$$

with (2.14), (4.4), (4.11), the boundary conditions (4.16)-(4.18), and the static condition (4.19), that becomes

$$S(t, X, b(t, X)) \geq 0. \quad (4.30)$$

Note that in the simple shear case (no dependency in  $X$ ), we recover the system obtained in Subsection 2.3. The pressure convexity assumption (4.14) gives that  $\partial_Z S \leq 0$ . Applying a formal maximum principle [39], this implies that the monotonicity condition (4.4) remains true if it is the case at initial time. Indeed, without (4.14), the solution  $U$  to (4.29) would not remain monotone with respect to  $Z$ , violating the conditions under which it has been derived. Then in the viscous case  $\nu > 0$ , the monotonicity of  $U$  implies the static condition (4.30), see [40]. In the inviscid case  $\nu = 0$ , the static condition (4.30) is not automatically satisfied, and can be interpreted as an entropy condition. As in the simple shear situation, the three boundary conditions (4.16)-(4.18) (reducing to the single condition (4.17) in the inviscid case  $\nu = 0$ ) determine the dynamics of the static-flowing interface  $b(t, X)$ .

If  $\nu = 0$ , differentiating the boundary condition  $U(t, X, b(t, X)) = 0$  with respect to  $t$  enables to get  $\partial_t b(t, X) \partial_Z U(t, X, b(t, X)) = -\partial_t U(t, X, b(t, X))$ , and with (4.29) we obtain

$$\partial_t b(t, X) = -\operatorname{sgn}(\theta) \frac{S(t, X, b(t, X))}{\partial_Z U(t, X, b(t, X))}, \quad \text{if } \partial_Z U(t, X, b(t, X)) \neq 0. \quad (4.31)$$

Using (4.30) and (4.4) we obtain  $\partial_t b \geq 0$  whenever  $\partial_Z U(t, X, b(t, X)) \neq 0$ . However, a precise analysis performed in [39] shows that it is possible to have  $\partial_t U < 0$  together with  $\partial_Z U(t, X, b(t, X)) = 0$  and  $S(t, X, b(t, X)) = 0$ . Hence, the formula (4.31) represents only partly the dynamics, and it is not possible to formulate the evolution of the interface by specifying a simple formula for  $\partial_t b$ . Instead we have to use the whole formulation (4.28), (4.29) with (2.14), (4.4), (4.11), the boundary conditions (4.16)-(4.18), and the static condition (4.30). Specific numerical methods have then to be used, see [40]. Noticing the hydrostatic approximation  $p = g \cos \theta (h - Z) + O(\varepsilon^2)$  from (3.10), and the approximation  $S = -g |\sin \theta| - \mu_s \partial_Z p + O(\varepsilon)$  from (4.28), we get  $S = g \cos \theta (\mu_s - |\tan \theta|) + O(\varepsilon)$ . With (4.3) this is  $O(\varepsilon)$ , but anyway the formula (4.31) with this approximation of  $S$  identifies with the  $\partial_t b$  equation of the models discussed in [13], namely the model equation (4.22) in that reference, the BCRE model ((4.16) in [13]) from [10], and Khakhar’s model ((4.14) in [13]) from [35]. In order to improve the expansion we need here to write down the next terms from (4.11), (4.28). The term in  $\partial_X h$  corresponds in particular to the ones in (4.5) or (4.18) in [13]. However, the term in  $\partial_X U / \partial_Z U$  in (4.11) is specific to our approach, it includes a non-hydrostatic coupling that cannot be expressed by averages in  $Z$ . A description of physically relevant static-flowing interface evolution equations is provided in [33], in the context of depth-averaged models. We remark that in this description, no term in  $\partial_X h$  is present, and no interface evolution is possible if the density is constant, contrarily to our model.

We conclude that on one hand our approach here is quantitatively close to the BCRE and Khakhar models, and on the other hand its formulation (4.28), (4.29) with (2.14), (4.4), (4.11), the boundary conditions (4.16)-(4.18), and the static condition (4.30), is better mathematically, indeed it enables clear numerical schemes [40] (at least in the case of uncoupled source  $S$ , i.e. not depending on  $U$  or its derivatives), including the case of non-zero viscosity. A comparison of the simple shear approximation of Subsection 2.3 with experimental data is performed in [40], showing the relevance of our formulation, including the effect of viscosity.

The effect of the  $Z$  dependency of  $S$  is described in the inviscid case in [39], showing that the eventual zero  $b^*(t, X)$  of  $S$  in the variable  $Z$  (i.e. satisfying  $S(t, X, b^*(t, X)) = 0$ ) merely drives the evolution of  $b(t, X)$ . This property enables to put into motion the static phase (i.e. the decrease of  $b$ ) even without viscosity, which is not possible without dependency in  $Z$ , as seen in (2.34), (2.35). Therefore, in order to describe the fact that the static phase can be put into motion, for example in the case of horizontal space inhomogeneities and initially vanishing velocity, we have to take into account the  $Z$  dependency of the source  $S$ , which means to include the coupling with the pressure expansion (4.11) with the term in  $\partial_X U / \partial_Z U$ , that produces the  $Z$  dependency of  $S$ . The effect of this term is illustrated by the configurations of Subsections 4.2 and 4.3, where  $\partial_X U / \partial_Z U = -\partial_X h$  is non-zero.

When considering the full coupling with (4.28) and the pressure formula (4.11), we have first to remark that the model is only valid as long as the pressure remains convex with respect to  $Z$ , a condition that could eventually break up in finite time. The coupled problem (4.29), (4.28) with  $p$  given by (4.11) looks difficult to solve. It is nonlinear in the two space derivatives of  $U$ , because of the ratio  $\partial_X U / \partial_Z U$  in (4.11). It is possible to remove the infinite values of this ratio, by replacing  $\partial_X U / |\partial_Z U|$  in (4.11) by  $\partial_X U / ((\partial_Z U)^2 + 4(\partial_X U)^2)^{1/2}$ , without affecting the accuracy of the approximation. Indeed the denominator  $|\partial_Z U|$  has been obtained as an expansion of  $\|DU\|$  in (2.19). However by doing this we loose the nice relation (4.20). The equation anyway contains second-order terms linear in  $\partial_{XZ}^2 U$  (and  $\partial_{ZZ}^2 U$ ), that make it ill-posed unless adding sufficient viscosity in  $X$  (and eventually in  $Z$ ). The formulation and simulation of a well-posed suitably modified system is an important issue that will be explored in future works.

## 5 Conclusion

The description of avalanche flows of granular materials involves the key feature of the dynamics of the static-flowing interface. In contrast with the usual approach which is to formulate a phenomenological differential equation on the interface, we have been able to derive analytically a set of equations for the velocity, the width of the domain, and the interface, from a shallow layer asymptotics starting from an incompressible viscoplastic model with Drucker-Prager yield stress. In contrast to the so called shallow water models, our model keeps the normal to the flow variable, and is set in the flowing phase as a parabolic-like model with formally overdetermined boundary conditions. The extra condition determines the dynamics of the interface, but cannot be replaced by a simple ordinary differential equation on the position of the interface. The model is quantitatively close to the BCRE and Khakhar models, but at the same time its formulation is mathematically attractive since it can be solved numerically, and can take into account small, but non-zero, viscosity. In contrast to the BCRE and Khakhar models, it is formulated for arbitrary velocity shape, and it is therefore more general. A specific property is that it involves a non-hydrostatic nonlinear coupling term, that could have the effect of displacing the static-flowing interface according to the space inhomogeneities. The model is described by the equations (2.14), (4.4), (4.11), (4.15)-(4.19), or equivalently by (2.14), (4.4), (4.28), (4.29), (4.11), (4.16)-(4.18), (4.30). It is established under the conditions that the curvature of the topography is small (3.1) the internal friction is close to the slope (4.3), the velocity is small (3.4), the viscosity is small (4.2), and the pressure is convex (4.14).

The evaluation of the model without pressure coupling is performed in [40, 39], and shows promising results. The case of full coupling with dependency in the horizontal variable needs



to be explored.

## Appendix: formulation in the flowing phase with extra boundary condition

In this appendix we would like to justify conditions under which we have the mathematical equivalence between multivalued equations set in the whole material and equations set in the flowing phase with extra boundary condition.

We consider an unknown  $U(t, X, Z)$  defined for  $0 < Z < h(t, X)$ , where  $h(t, X) > 0$  is assumed to be known. Given a source  $\Phi(t, X)$  independent of  $Z$ , the problem is set as

$$\partial_t U + \Phi - \partial_Z(\nu \partial_Z U + \kappa \operatorname{sgn}(\partial_Z U)) = 0, \quad \text{for } 0 < Z < h(t, X), \quad (\text{A.1})$$

where the sign is understood as multivalued,  $\nu \geq 0$  and  $\kappa(t, X, Z)$  are given. We complete the problem with the boundary condition

$$\nu \partial_Z U + \kappa \operatorname{sgn}(\partial_Z U) = 0 \quad \text{at } Z = h(t, X), \quad (\text{A.2})$$

and the no slip condition at the bottom  $U = 0$  at  $Z = 0$ . This condition will be however unnecessary since we are looking for solutions to (A.1) that have a static-flowing interface  $0 < b(t, X) < h(t, X)$  such that

$$\begin{aligned} U(t, X, Z) &= 0 & \text{for } 0 < Z < b(t, X), \\ \partial_Z U(t, X, Z) &\neq 0 & \text{for } b(t, X) < Z < h(t, X). \end{aligned} \quad (\text{A.3})$$

Note that in this problem, the variable  $X$  is only a parameter, and the dynamics is in the variables  $t, Z$ . We shall therefore often omit the variable  $X$ .

The aim of this appendix is to examine under which conditions the problem (A.1)-(A.3) can be formulated in the flowing domain  $Z > b$ , eliminating the static part where  $Z < b$ . We shall assume that  $\kappa$  is continuous with respect to  $Z$  and satisfies

$$\kappa(t, X, h(t, X)) = 0, \quad \kappa(t, X, Z) > 0 \text{ for } 0 < Z < h, \quad (\text{A.4})$$

$$\kappa \text{ is convex with respect to } Z \text{ in } [0, h]. \quad (\text{A.5})$$

We shall use the notation

$$\Sigma_{XZ} = \nu \partial_Z U + \kappa \operatorname{sgn}(\partial_Z U). \quad (\text{A.6})$$

The solutions to (A.1)-(A.3) for some  $b \in (0, h)$  can be characterized by three types of equations: equations in the static phase  $Z < b$ , equations in the flowing phase  $Z > b$ , and relations through the interface  $Z = b$ .

1. Interface jump relations. Assuming that  $U$  does not jump through the interface (this is mandatory when viscosity is present), i.e.

$$U(t, Z) \rightarrow 0, \text{ as } Z \rightarrow b(t)+, \quad (\text{A.7})$$

the "+" meaning that the limit is taken by above, the equation (A.1) yields the property that the shear stress  $\Sigma_{XZ}$  does not jump through the interface  $Z = b(t)$ . We shall denote its value by  $(\Sigma_{XZ})_{b(t)}$ , or simply  $(\Sigma_{XZ})_b$ . Then the relation (A.6) and the assumption (A.4) imply that  $\kappa \operatorname{sgn}(\partial_Z U)$  does not jump through the interface (if  $\nu = 0$  it is obvious, and in the case  $\nu > 0$ , use that the relation (A.6) gives a Lipschitz continuous dependency of  $\partial_Z U$  in terms of  $\Sigma_{XZ}$ ). We shall call its value at the interface  $\kappa_b s_b$ . Taking the limit from the flowing zone yields indeed that  $s_b = \pm 1$  is the sign of the shear  $\partial_Z U$  in the flowing zone. From (A.6) again, we get by difference that  $\nu \partial_Z U$  is also continuous through the interface. We conclude that

$$\nu \partial_Z U \rightarrow 0 \text{ as } Z \rightarrow b(t)+. \quad (\text{A.8})$$

Moreover, we have the relation  $(\Sigma_{XZ})_b = \kappa_b s_b$ .

2. Equation in the static phase. There, call  $s(t, Z) = \text{sgn}(\partial_Z U)$ . Then (A.6) gives that

$$\Sigma_{XZ} = \kappa s \text{ for } 0 < Z < b. \quad (\text{A.9})$$

But (A.1) with the value of  $\Sigma_{XZ}$  at  $b$  given by  $(\Sigma_{XZ})_b = \kappa_b s_b$  yields

$$\Sigma_{XZ} = \kappa_b s_b - (b - Z)\Phi \text{ for } 0 < Z < b. \quad (\text{A.10})$$

It remains then to state with (A.9), (A.10) that  $|s| \leq 1$ . With (A.9) it is equivalent to write  $\kappa - s_b \Sigma_{XZ} \geq 0$  and  $\kappa + s_b \Sigma_{XZ} \geq 0$ . Using the expression (A.10) of  $\Sigma_{XZ}$ , we are led to

$$\begin{aligned} \kappa(t, Z) - s_b (\kappa_b s_b - (b - Z)\Phi) &\geq 0 \text{ for } 0 < Z < b, \\ \kappa(t, Z) + s_b (\kappa_b s_b - (b - Z)\Phi) &\geq 0 \text{ for } 0 < Z < b. \end{aligned} \quad (\text{A.11})$$

Since  $\kappa \geq 0$ , the second inequality holds as soon as

$$s_b \Phi \leq 0, \quad (\text{A.12})$$

a condition that we shall assume to hold. With (A.10) it implies that  $s_b \Sigma_{XZ} > 0$  for  $Z < b$ , and with (A.9) that  $s_b s > 0$  for  $Z < b$ . According to the assumption (A.5), the left-hand side of the first line of (A.11) is convex with respect to  $Z$ . Since it vanishes at  $Z = b$ , we have to write that the derivative with respect to  $Z$  at  $b$  is nonpositive, which leads to

$$(\partial_Z \kappa)_b - s_b \Phi \leq 0, \quad (\text{A.13})$$

or equivalently with (A.12),

$$-(\partial_Z \kappa)_b \geq |\Phi|. \quad (\text{A.14})$$

Note that the convexity assumption (A.5) could be replaced by the weaker one which is that the function  $\kappa(Z)$  is above its tangent at  $b$ .

3. Equation in the flowing phase. Knowing that there

$$s_b \partial_Z U > 0 \text{ for } Z > b, \quad (\text{A.15})$$

the relation (A.6) in the flowing zone  $Z > b$  gives  $\Sigma_{XZ} = \nu \partial_Z U + \kappa s_b$ . Plugging this in (A.1) yields

$$\partial_t U + \Phi - s_b \partial_Z \kappa - \partial_Z (\nu \partial_Z U) = 0 \text{ for } Z > b, \quad (\text{A.16})$$

while the boundary condition (A.2) becomes with (A.4)

$$\nu \partial_Z U = 0 \text{ at } Z = h. \quad (\text{A.17})$$

The system is completed with the boundary conditions at the interface (A.7), (A.8).

The conclusion of the previous analysis is that given  $s_b = \pm 1$  the sign of the shear, and assuming (A.12), (A.4), (A.5), the problem of finding a solution  $U(t, X, Z)$  defined for  $0 < Z < h(t, X)$  to (A.1)-(A.3) is equivalent to finding a solution  $U(t, X, Z)$  defined for  $b(t, X) < Z < h(t, X)$  to (A.16) with (A.15) satisfying the boundary conditions (A.17), (A.7), (A.8), and the static equilibrium condition (A.14) must hold.

Note that as is proved above, if (A.14) is violated, one cannot find a stress satisfying the equations in the static zone. The interpretation is that in this situation, the solution to (A.1)-(A.2), if it exists, has no static phase.

The interesting property in the final formulation in the flowing phase is that we have three boundary conditions (A.17), (A.7), (A.8) (respectively one condition if  $\nu = 0$ ) for a parabolic (respectively hyperbolic) problem, thus one extra condition than expected in problems with fixed boundary. This extra condition determines implicitly the dynamics of the static-flowing interface  $b(t, X)$ .

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