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ON FARRELL-TATE COHOMOLOGY OF SL_2 OVER S -INTEGERS

ALEXANDER D. RAHM AND MATTHIAS WENDT

ABSTRACT. In this paper, we provide number-theoretic formulas for Farrell-Tate cohomology for SL_2 over rings of S -integers in number fields. These formulas describe group cohomology above the virtual cohomological dimension, and can be used to study some questions in homology of linear groups.

We expose three applications, to (I) detection questions, (II) the existence of transfers, (III) cohomology of SL_2 over number fields.

1. INTRODUCTION

The cohomology of arithmetic groups like $\mathrm{SL}_2(\mathcal{O}_{K,S})$ for $\mathcal{O}_{K,S}$ a ring of S -integers in a number field K has been long and intensively studied. In principle, the cohomology groups can be computed from the action of SL_2 on its associated symmetric space, but actually carrying out this program involves a lot of questions and difficulties from algebraic number theory.

In the case of a group Γ of finite virtual cohomological dimension (vcd), Farrell-Tate cohomology provides a modification of group cohomology, which in a sense describes the obstruction for Γ to be a Poincaré-duality group. The Farrell-Tate cohomology of Γ can be described in terms of finite subgroups of Γ and their normalizers, and hence is rather amenable to computation.

The goal of the present paper is to provide explicit formulas for the Farrell-Tate cohomology of SL_2 over rings of S -integers in number fields, with coefficients in \mathbb{F}_ℓ , ℓ odd. In the case $\mathrm{SL}_2(\mathcal{O}_{K,S})$, the conjugacy classification of finite subgroups is directly related to number-theoretic questions about (relative) class groups and unit groups of S -integers in K and some of its cyclotomic extensions. In the special cases of SL_2 of imaginary quadratic number rings, such formulas have previously been obtained by Rahm, building on work of Krämer, cf. [Rah13, Rah14], [Krä80]. Our general formulas still have the same structure as those in [Rah13] - the Farrell-Tate cohomology decomposes as a direct sum over conjugacy classes of subgroups, but the formulas for the direct summands are more complicated due to the fact that the torsion subcomplexes can have arbitrarily large dimension. The main result is the following, cf. Section 6. Let ζ_ℓ be some primitive ℓ -th root of unity.

Theorem 1. *Let K be a global field, let S be a non-empty finite set of places of K containing the infinite places, and denote by $\mathcal{O}_{K,S}$ the ring of S -integers in K . Let ℓ be an odd prime different from the characteristic of K and assume $\zeta_\ell + \zeta_\ell^{-1} \in K$.*

We denote by $\Psi_\ell(T) = T^2 - (\zeta_\ell + \zeta_\ell^{-1})T + 1$ the relevant quadratic factor of the cyclotomic polynomial $\Phi_\ell(T)$, $K(\Psi_\ell) = K[T]/\Psi_\ell(T)$ the corresponding K -algebra, by $R_{K,S,\ell} = \mathcal{O}_{K,S}[T]/(\Psi_\ell(T))$ the corresponding order in $K(\Psi_\ell)$, and by

$$\mathrm{Nm}_1 : R_{K,S,\ell}^\times \rightarrow \mathcal{O}_{K,S}^\times \quad \text{and} \quad \mathrm{Nm}_0 : \widetilde{K}_0(R_{K,S,\ell}) \rightarrow \widetilde{K}_0(\mathcal{O}_{K,S}).$$

the norm maps on class groups and unit groups for the finite extension $R_{K,S,\ell}/\mathcal{O}_{K,S}$.

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- (1) $\widehat{H}^\bullet(\mathrm{SL}_2(\mathcal{O}_{K,S}), \mathbb{F}_\ell) \neq 0$ if and only if $\zeta_\ell + \zeta_\ell^{-1} \in K$ and the Steinitz class $\det_{\mathcal{O}_{K,S}}(R_{K,S,\ell})$ is contained in the image of the norm map Nm_0 .
- (2) Assume the condition in (1) is satisfied. The set \mathcal{C}_ℓ of conjugacy classes of order ℓ elements in $\mathrm{SL}_2(\mathcal{O}_{K,S})$ sits in an extension

$$1 \rightarrow \mathrm{coker} \mathrm{Nm}_1 \rightarrow \mathcal{C}_\ell \rightarrow \ker \mathrm{Nm}_0 \rightarrow 0.$$

The set \mathcal{K}_ℓ of conjugacy classes of order ℓ subgroups of $\mathrm{SL}_2(\mathcal{O}_{K,S})$ can be identified with the quotient $\mathcal{K}_\ell = \mathcal{C}_\ell / \mathrm{Gal}(K(\Psi_\ell)/K)$. There is a direct sum decomposition

$$\widehat{H}^\bullet(\mathrm{SL}_2(\mathcal{O}_{K,S}), \mathbb{F}_\ell) \cong \bigoplus_{[\Gamma] \in \mathcal{K}_\ell} \widehat{H}^\bullet(N_{\mathrm{SL}_2(\mathcal{O}_{K,S})}(\Gamma), \mathbb{F}_\ell).$$

- (3) If the class of Γ is not $\mathrm{Gal}(K(\Psi_\ell)/K)$ -invariant, then

$$N_{\mathrm{SL}_2(\mathcal{O}_{K,S})}(\Gamma) \cong \ker \mathrm{Nm}_1.$$

- (4) If the class of Γ is $\mathrm{Gal}(K(\Psi_\ell)/K)$ -invariant, then there is an extension

$$0 \rightarrow \ker \mathrm{Nm}_1 \rightarrow N_{\mathrm{SL}_2(\mathcal{O}_{K,S})}(\Gamma) \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 1.$$

We recover as special cases the earlier results of Busch [Bus06]. For function fields of curves over algebraically closed fields, there are similar formulas, cf. [Wen14a].

The restriction to odd torsion coefficients is necessary. Only under this assumption, the formula has such a simple structure; for 2-torsion one has to take care of the possible subgroups A_4 , S_4 and A_5 in PSL_2 . Moreover, for \mathbb{F}_2 -coefficients, there is a huge difference between the cohomology of SL_2 and PSL_2 - only the cohomology of PSL_2 has an easy description and the computation of cohomology of SL_2 depends on a lot more than just the finite subgroups. Some results can be achieved, but the additional complications would only obscure the presentation of the results.

The result is proved by following the obvious strategy: we let $\mathrm{SL}_2(\mathcal{O}_{K,S})$ act on the associated symmetric space $\mathfrak{X}_{K,S}$ and determine the subspace of the quotient $\mathrm{SL}_2 \backslash \mathfrak{X}_{K,S}$ consisting of the points having a non-trivial ℓ -torsion stabilizer. Using a little representation theory and a little number theory, this subcomplex can be very explicitly described. The resulting information is precise enough to work out the resulting isotropy spectral sequence.

The explicit formulas obtained allow to discuss a couple of questions concerning cohomology of linear groups. For instance, the above result provides a computation of group cohomology of $\mathrm{SL}_2(\mathcal{O}_{K,S})$ above the virtual cohomological dimension.

As a first application, we get explicit formulas for the ring structure on group cohomology above the virtual cohomological dimension. This allows to discuss a conjecture of Quillen, cf. [Qui71], recalled as Conjecture 6 in Section 7 below. The following two results are proved in Sections 6 and 7, a further discussion of the consequences for Quillen's conjecture can be found in [RW14].

Theorem 2. *Let K be a global field, let S be a non-empty finite set of places of K containing the infinite places, and let ℓ be an odd prime different from the characteristic of K . With the notation of Theorem 1, we have the following:*

- (1) *The decomposition*

$$\widehat{H}^\bullet(\mathrm{SL}_2(\mathcal{O}_{K,S}), \mathbb{F}_\ell) \cong \bigoplus_{[\Gamma] \in \mathcal{K}} \widehat{H}^\bullet(N_{\mathrm{SL}_2(\mathcal{O}_{K,S})}(\Gamma), \mathbb{F}_\ell)$$

is compatible with the ring structure, i.e., the Farrell-Tate cohomology ring of $\mathrm{SL}_2(\mathcal{O}_{K,S})$ is a restricted sum of the sub-rings for the normalizer subgroups $N_{\mathrm{SL}_2(\mathcal{O}_{K,S})}(\Gamma)$.

(2) If the class of Γ is not $\mathrm{Gal}(K(\Psi_\ell)/K)$ -invariant, there is an isomorphism of graded rings

$$\widehat{H}^\bullet(N_{\mathrm{SL}_2(\mathcal{O}_{K,S})}(\Gamma), \mathbb{Z})_{(\ell)} \cong \mathbb{F}_\ell[a_2, a_2^{-1}] \otimes_{\mathbb{F}_\ell} \bigwedge (\ker \mathrm{Nm}_1).$$

In particular, this is a free module over the subring $\mathbb{F}_\ell[a_2^2, a_2^{-2}]$.

(3) If the class of Γ is $\mathrm{Gal}(K(\Psi_\ell)/K)$ -invariant, there is an isomorphism of graded rings

$$\widehat{H}^\bullet(N_{\mathrm{SL}_2(\mathcal{O}_{K,S})}(\Gamma), \mathbb{Z})_{(\ell)} \cong \left(\mathbb{F}_\ell[a_2, a_2^{-1}] \otimes_{\mathbb{F}_\ell} \bigwedge (\ker \mathrm{Nm}_1) \right)^{\mathbb{Z}/2},$$

with the $\mathbb{Z}/2$ -action given by multiplication with -1 on a_2 and $\ker \mathrm{Nm}_1$. In particular, this is a free module over the subring $\mathbb{F}_\ell[a_2^2, a_2^{-2}] \cong \widehat{H}^\bullet(D_{2\ell}, \mathbb{Z})_{(\ell)}$.

We elaborate on this in Section 7 by proving the following.

Theorem 3. *With the notation of Theorem 3, the restriction map induced from the inclusion $\mathrm{SL}_2(\mathcal{O}_{K,S}) \rightarrow \mathrm{SL}_2(\mathbb{C})$ maps the second Chern class c_2 to the sum of the elements a_2^2 in all the components.*

As a consequence, Quillen's conjecture for $\mathrm{SL}_2(\mathcal{O}_{K,S})$ is true for Farrell-Tate cohomology with \mathbb{F}_ℓ -coefficients for all K and S . This also implies that Quillen's conjecture is true for group cohomology with \mathbb{F}_ℓ -coefficients above the virtual cohomological dimension.

However, if the number of conjugacy classes of order ℓ -subgroups is greater than two, the restriction map

$$H^\bullet(\mathrm{SL}_2(\mathcal{O}_{K,S}), \mathbb{F}_\ell) \rightarrow H^\bullet(\mathrm{T}_2(\mathcal{O}_{K,S}), \mathbb{F}_\ell)$$

from $\mathrm{SL}_2(\mathcal{O}_{K,S})$ to the group $\mathrm{T}_2(\mathcal{O}_{K,S})$ of diagonal matrices is not injective.

It is also worthwhile pointing out that the failure of detection as in the previous theorem, i.e., the non-injectivity of the restriction map

$$H^\bullet(\mathrm{SL}_2(\mathcal{O}_{K,S}), \mathbb{F}_\ell) \rightarrow H^\bullet(\mathrm{T}_2(\mathcal{O}_{K,S}), \mathbb{F}_\ell)$$

implies the failure of the unstable Quillen-Lichtenbaum conjecture as formulated in [AR13]. This follows from the work of Dwyer and Friedlander [DF94]. Function field analogues of these statements will be discussed in [Wen14b].

Our results of Section 8 also allow to find examples for non-existence of transfers:

Theorem 4. *Let L/K be a finite separable extension of global fields, let S be a non-empty finite set of places of K containing the infinite places and let \tilde{S} be a set of places of L containing those lying over S . Let ℓ be an odd prime different from the characteristic of K . The restriction map*

$$\widehat{H}^\bullet(\mathrm{SL}_2(\mathcal{O}_{L,\tilde{S}}), \mathbb{F}_\ell) \rightarrow \widehat{H}^\bullet(\mathrm{SL}_2(\mathcal{O}_{K,S}), \mathbb{F}_\ell)$$

induced from the natural ring homomorphism $\mathcal{O}_{K,S} \rightarrow \mathcal{O}_{L,\tilde{S}}$ is compatible with the decomposition of Theorem 1 and is completely described by

(1) *the induced map on class groups*

$$\widetilde{K}_0(R_{K,S,\ell}) \rightarrow \widetilde{K}_0(R_{L,\tilde{S},\ell})$$

(2) *the induced map on unit groups $R_{K,S,\ell}^\times \rightarrow R_{L,\tilde{S},\ell}^\times$.*

Let K be a number field with $\zeta_\ell \in K$, and assume that the class number of K is prime to ℓ . Denoting by H the Hilbert class field of K , the restriction map

$$\widehat{H}^\bullet(\mathrm{SL}_2(\mathcal{O}_H), \mathbb{F}_\ell) \rightarrow \widehat{H}^\bullet(\mathrm{SL}_2(\mathcal{O}_K), \mathbb{F}_\ell)$$

is not surjective. Therefore, this is an example of a finite étale morphism for which it is not possible to define a transfer in the usual K -theoretic sense.

The question of existence of transfers was raised implicitly in [Knu01, section 5.3]; other sources of obstructions to the existence of transfers can be found in [Wen14b].

Finally, we want to note that the precise description of Farrell-Tate cohomology and the relevant restriction maps allows to consider the colimit of the groups $\widehat{H}^\bullet(\mathrm{SL}_2(\mathcal{O}_{K,S}), \mathbb{F}_\ell)$, where S runs through the finite sets of places of K . Using this, we investigate in Section 9 the behaviour of Mislin's extension of Farrell-Tate cohomology with respect to directed colimits:

Theorem 5.

- (1) *There are cases where Mislin's extension of Farrell-Tate cohomology does not commute with directed colimits. A simple example is given by the directed system of groups $\mathrm{SL}_2(\mathbb{Z}[1/n])$ with $n \in \mathbb{N}$.*
- (2) *The Friedlander-Milnor conjecture for $\mathrm{SL}_2(\overline{\mathbb{Q}})$ is equivalent to the question if Mislin's extension of Farrell-Tate cohomology commutes with the directed colimit of $\mathrm{SL}_2(\mathcal{O}_{K,S})$, where K runs through all number fields and S runs through all finite sets of places.*

Similar results can be obtained for higher rank groups, although there are more and more complications coming from the classification of finite subgroups. Away from the order of the Weyl group, the subgroup classification is easier. A discussion of the case SL_3 will be done in the forthcoming paper [RW15].

Structure of the paper: We first recall group cohomology preliminaries in Section 2. Then we discuss the conjugacy classification of finite subgroups as well as their normalizers in Section 3. Section 4 then discusses the structure of the action of SL_2 on its associated symmetric space. This allows to completely describe the ℓ -torsion subcomplex in Section 5. All this information is combined into formulas for Farrell-Tate cohomology in Section 6. Then we discuss three applications of the results, to (I) non-detection in Section 7, (II) the existence of transfers in Section 8 and (III) cohomology of SL_2 over number fields in Section 9.

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2. PRELIMINARIES ON GROUP COHOMOLOGY AND FARRELL-TATE COHOMOLOGY

In this section, we recall the necessary definitions of group cohomology and Farrell-Tate cohomology, introducing notations and results we will need in the sequel. One of the basic references is [Bro94].

2.1. Group cohomology. Group cohomology is defined as the right derived functor of invariants $\mathbb{Z}[G]\text{-mod} \rightarrow \mathbb{Z}\text{-mod} : M \mapsto M^G$. It can be defined algebraically by taking a resolution $P_\bullet \rightarrow \mathbb{Z}$ of \mathbb{Z} by projective $\mathbb{Z}[G]$ -modules and setting

$$H^\bullet(G, M) := H^\bullet(\mathrm{Hom}_G(P_\bullet, M)).$$

Alternatively, it can be defined topologically as the cohomology of the classifying space BG with coefficients in the local system associated to M .

2.2. Farrell-Tate cohomology. We shortly recall the definition and basic properties of Farrell-Tate cohomology, cf. [Bro94, chapter X]. Farrell-Tate cohomology is a completion of group cohomology defined for groups of finite virtual cohomological dimension (vcd). Note that for K a number field, the groups $\mathrm{SL}_2(\mathcal{O}_{K,S})$ have virtual cohomological dimension $2r_1 + 3r_2 + \#S_f - 1$, where r_1 and $2r_2$ are

the numbers of real and complex embeddings of K , respectively, and $\#S_f$ is the number of finite places in S .

For Γ a group with finite virtual cohomological dimension, a complete resolution is the datum of

- an acyclic chain complex F_\bullet of projective $\mathbb{Z}[\Gamma]$ -modules,
- a projective resolution $\epsilon : P_\bullet \rightarrow \mathbb{Z}$ of \mathbb{Z} over $\mathbb{Z}[\Gamma]$, and
- a chain map $\tau : F_\bullet \rightarrow P_\bullet$ which is the identity in sufficiently high dimensions.

The starting point of Farrell-Tate cohomology is the fact that groups of finite virtual cohomological dimensions have complete resolutions.

Definition 2.1. Given a group Γ of finite virtual cohomological dimension, a complete resolution $(F_\bullet, P_\bullet, \epsilon)$ and a $\mathbb{Z}[\Gamma]$ -module M , *Farrell-Tate cohomology of Γ with coefficients in M* is defined by

$$\widehat{H}^\bullet(\Gamma, M) := H^\bullet(\text{Hom}_\Gamma(F_\bullet, M)).$$

The functors \widehat{H}^\bullet satisfy all the usual cohomological properties, cf. [Bro94, X.3.2], and in fact Farrell-Tate cohomology can be seen as projective completion of group cohomology, cf. [Mis94]. Important for our considerations is the fact [Bro94, X.3.4] that there is a canonical map

$$H^\bullet(\Gamma, M) \rightarrow \widehat{H}^\bullet(\Gamma, M)$$

which is an isomorphism above the virtual cohomological dimension.

2.3. Equivariant cohomology and the isotropy spectral sequence. We recall some basic properties of equivariant cohomology and associated spectral sequences, cf. [Bro94, section VII and X.4].

For a group G and a G -CW-complex X , one can define equivariant cohomology by setting

$$H_G^\bullet(X, M) := H^\bullet(\text{Hom}_G(P_\bullet, C^\bullet(X, M))).$$

where $C^\bullet(X, M)$ is the complex of cochains in X with coefficients in M . If X is contractible, then $H_G^\bullet(X, M) \cong H^\bullet(G, M)$. In that case, there is a spectral sequence, cf. [Bro94, VII.7],

$$E_1^{p,q} = \prod_{\sigma \in \Sigma_p} H^q(G_\sigma, M_\sigma) \Rightarrow H_G^{p+q}(X, M) \cong H^{p+q}(G, M),$$

where Σ_p denotes a set of representatives for the G -action on $C^p(X)$. The first differential is induced from the boundary map of the complex X , cf. [Bro94, VII.8]:

$$d_1|_{H^q(G_\sigma, M_\sigma)} : \bigoplus_{\tau \subseteq \sigma} H^q(G_\tau, M_\tau) \rightarrow H^q(G_\sigma, M_\sigma).$$

If G is a group of finite virtual cohomological dimension and X is a finite-dimensional G -CW-complex, one can define equivariant Farrell-Tate cohomology analogously by setting

$$\widehat{H}_G^\bullet(X, M) := H^\bullet(\text{Hom}_G(F_\bullet, C^\bullet(X, M))),$$

where as above $C^\bullet(X, M)$ is the complex of cochains in X with coefficients in M and F_\bullet is a complete resolution of G . If X is contractible and proper, then $\widehat{H}_G^\bullet(X, M) \cong \widehat{H}^\bullet(G, M)$ and there is a spectral sequence as above, cf. [Bro94, X.4],

$$E_1^{p,q} = \prod_{\sigma \in \Sigma_p} \widehat{H}^q(G_\sigma, M_\sigma) \Rightarrow \widehat{H}_G^{p+q}(X, M),$$

We will apply these statements in the case where the group $G = \mathrm{SL}_2(\mathcal{O}_{K,S})$ acts on its associated symmetric space $X = \mathfrak{X}_{K,S}$. In this situation, it is well-known that X is a finite-dimensional G -CW-complex which is contractible and proper.

Finally, we recall a result that can be used to compute Farrell-Tate cohomology at the prime ℓ for groups all whose elementary abelian ℓ -subgroups have rank ≤ 1 . Most of our paper is about making this formula more precise in the case of $G = \mathrm{SL}_2(\mathcal{O}_{K,S})$, and comparing this formula to the spectral sequence for the G -action on its associated symmetric space, cf. [Bro94, corollary X.7.4]:

Proposition 2.2. *Let G be a group such that every elementary abelian ℓ -subgroup of G has rank ≤ 1 . Then*

$$\widehat{H}^\bullet(G, \mathbb{F}_\ell) \cong \prod_{E \in \mathcal{E}} \widehat{H}^\bullet(N_G(E), \mathbb{F}_\ell),$$

where \mathcal{E} is a set of representatives for conjugacy classes of subgroups of G of order ℓ .

2.4. Farrell-Tate cohomology of finite subgroups of $\mathrm{SL}_2(\mathbb{C})$. We recall the well-known formulas for group and Tate cohomology of cyclic and dihedral groups. We restrict to the cohomology with odd torsion coefficients, as our main results only use that case. The formulas below as well as corresponding formulas for the cohomology with \mathbb{F}_2 -coefficients can be found in [AM04]. Here, classes in square brackets are polynomial generators and classes in parentheses are exterior classes; the index of a class specifies its degree in the graded \mathbb{F}_ℓ -algebra.

- The cohomology ring for a cyclic group of order n with $\ell \mid n$ and ℓ odd is given by the formula

$$H^\bullet(\mathbb{Z}/n\mathbb{Z}, \mathbb{F}_\ell) \cong \mathbb{F}_\ell[a_2](b_1).$$

The corresponding Tate cohomology ring is $\widehat{H}^\bullet(\mathbb{Z}/n\mathbb{Z}, \mathbb{F}_\ell) \cong \mathbb{F}_\ell[a_2, a_2^{-1}](b_1)$. Note that cohomology with integral coefficients gets rid of the exterior algebra contribution which come from the universal coefficient formula.

- The cohomology ring for a dihedral group of order $2n$ with $\ell \mid n$ and ℓ odd is given by the formula

$$H^\bullet(D_{2n}, \mathbb{F}_\ell) \cong \mathbb{F}_\ell[a_4](b_3).$$

The corresponding Tate cohomology ring is $\widehat{H}^\bullet(D_{2n}, \mathbb{F}_\ell) \cong \mathbb{F}_\ell[a_4, a_4^{-1}](b_3)$.

- The inclusions $D_6 \hookrightarrow A_4$, $D_6 \hookrightarrow S_4$ and $D_6 \hookrightarrow A_5$ induce isomorphisms in group cohomology with \mathbb{F}_3 -coefficients.
- The inclusion $D_{10} \rightarrow A_5$ induces an isomorphism in group cohomology with \mathbb{F}_5 -coefficients.

3. CYCLIC SUBGROUPS AND THEIR NORMALIZERS

In the following section, we recall the conjugacy classification of finite cyclic subgroups in SL_2 over rings of S -integers in global fields, with some necessary augmentations. For a number field K , the general conjugacy classification of finite subgroups of $\mathrm{SL}_2(\mathcal{O}_K)$ is due to Kramer [Kr80]. Special cases for totally real fields appeared before in the study of Hilbert modular groups, cf. [Pre68] and [Sch75]. A more recent account of this can be found in [Mac06].

We will discuss here a generalization of these results to rings of S -integers in number fields. Some of the necessary modifications for this have been considered in [Bus06]. Similar results can be formulated for function fields in terms of vector bundles, cf. [Wen14b].

We give a short outline of the classification results that will be discussed in the section. Our exposition can be seen as a geometric formulation of the classification result of Latimer-MacDuffee [LM33].

3.1. Finite subgroups in $PSL_2(K)$. We first recall the classification of finite subgroups of $PSL_2(K)$, K any field. *We implicitly assume that the order of the finite subgroup is prime to the characteristic of K .* The classification over algebraically closed fields is due to Klein. In the (slightly different) case of $PGL_2(K)$, K a general field, the classification of finite subgroups can be found in [Bea10].

For K an algebraically closed field, in particular for $K = \mathbb{C}$, Klein's classification provides an exact list of isomorphism types (as well as conjugacy classes) of finite subgroups in $PSL_2(K)$: any finite subgroup of $PSL_2(\mathbb{C})$ is isomorphic to a cyclic group $\mathbb{Z}/n\mathbb{Z}$, a dihedral group D_{2n} , the tetrahedral group A_4 , the octahedral group S_4 or the icosahedral group A_5 .

Over an arbitrary field K , we have the following classification, cf. [Ser72, 2.5], [Krä80, Teil II] or [Bea10, proposition 1.1 and theorem 4.2]. Denote by ζ_n some primitive n -th root of unity.

Proposition 3.1. (i) $PSL_2(K)$ contains a cyclic group $\mathbb{Z}/n\mathbb{Z}$ if and only if

$$\zeta_n + \zeta_n^{-1} \in K.$$

(ii) $PGL_2(K)$ contains a dihedral group D_{2n} if and only if $\zeta_n + \zeta_n^{-1} \in K$. If additionally the symbol $((\zeta_{2n} - \zeta_{2n}^{-1})^2, -1)$ is split, then also $PSL_2(K)$ contains a dihedral group D_{2n} .

(iii) $PSL_2(K)$ contains A_4 if and only if -1 is a sum of squares, i.e., if the symbol $(-1, -1)$ is split.

(iv) $PGL_2(K)$ contains S_4 if and only if -1 is a sum of two squares. If additionally $\sqrt{2} \in K$, then $PSL_2(K)$ contains S_4 .

(v) $PSL_2(K)$ contains A_5 if and only if -1 is a sum of two squares and 5 is a square.

This result is best proved by considering the Wedderburn decomposition of $K[G]$ and checking for (determinant one) two-dimensional representations among the factors.

With the exception of $\mathbb{Z}/2\mathbb{Z}$ and the dihedral groups, all finite subgroups in $PGL_2(K)$ are conjugate whenever they are isomorphic. For the dihedral groups D_r , there is a bijection between $PGL_2(K)$ -conjugacy classes and $K^\times / ((K^\times)^2 \cdot \mu_r(K))$ if $\zeta_r \in K$, cf. [Bea10, theorem 4.2].

3.2. The basic constructions. We describe the basic correspondence between ideal classes in extensions and conjugacy classes of finite-order elements. This correspondence was already described by Latimer-MacDuffee [LM33], and later generalized by Taussky and Bender. The following is a variation on [Ben67, theorem 1].

Proposition 3.2. *Let $A \rightarrow B$ be an integral extension of integral domains, and assume that $[\text{Frac}(B) : \text{Frac}(A)] = 2$. There is a bijective correspondence between conjugacy classes of representations $B \rightarrow M_2(A)$ and ideal classes of B having an A -basis.*

Proof. We only describe the two maps in the correspondence, for the rest of the proof cf. [LM33], or [Con12] for a more modern exposition.

Let $I \subseteq B$ be an ideal; we want to construct a representation $\rho : B \rightarrow M_2(A)$. Assume that I has an A -basis, and let $\{x_1, x_2\}$ be one such. Any element $b \in B$ induces an endomorphism $b : I \rightarrow I : x \mapsto b \cdot x$, and we set $\rho(b)$ to be the representing matrix of b in the chosen basis $\{x_1, x_2\}$. This clearly defines a ring homomorphism $\rho : B \rightarrow M_2(A)$.

Let $\rho : B \rightarrow M_2(A)$ be a representation. Then ρ acts as endomorphisms of $\text{Frac}(A)^2$, making it a one-dimensional $\text{Frac}(B)$ -vector space. The standard lattice $A^2 \subset \text{Frac}(A)^2$ is then a finitely generated B -submodule of the above one-dimensional $\text{Frac}(B)$ -vector space $\text{Frac}(A)^2$. Choosing an isomorphism to $\text{Frac}(B)$

and taking the inverse of A^2 under this isomorphism yields a fractional ideal in $\text{Frac}(B)$.

It can then be checked that these two assignments descend to a bijection between conjugacy classes and ideal classes. \square

The particular application we have in mind is the following: let A be an integral domain, and let $\chi(T)$ be the characteristic polynomial of a finite-order element in $\text{GL}_2(A)$. Then there is a bijection between conjugacy classes of elements with characteristic polynomial χ and ideal classes in the ring $A[T]/\chi(T)$. For example, this applies to $A[\zeta_n]$ whenever $[\text{Frac}(A)(\zeta_n) : \text{Frac}(A)] \leq 2$.

To deal with conjugacy classes of finite order elements in $\text{SL}_2(\mathcal{O}_{K,S})$, we have to provide a slight modification of the above. The following result is implicit in [Bus06, section 3].

Proposition 3.3. *Let A be an integral domain, let $x \in \text{SL}_2(A)$ be an element of finite order, and let $\chi_x(T)$ be its characteristic polynomial. Set $B = A[T]/(\chi_x(T))$, and denote by \mathcal{C} the set of conjugacy classes of elements of $\text{SL}_2(A)$ with characteristic polynomial $\chi_x(T)$. Then there is an exact sequence*

$$0 \rightarrow \text{coker}(\text{Nm}_1 : B^\times \rightarrow A^\times) \rightarrow \mathcal{C} \rightarrow \ker(\text{Nm}_0 : \widetilde{\text{K}}_0(B) \rightarrow \widetilde{\text{K}}_0(A)) \rightarrow 0.$$

Proof. The proof proceeds as in [Bus06], by defining an oriented class group and then following the argument outlined for GL_2 . This implies that the conjugacy classes sit in an extension of the group of B -ideals with A -basis by the group of A -orientations. The existence of a B -ideal with A -basis is guaranteed by the existence of the element x , and the identification of the B -ideals with A -basis as relative class group is discussed in the next subsection. \square

Remark 3.4. Note that in the above proposition, we are classifying *conjugacy classes of elements* of finite order in SL_2 . In Theorem 1 and our applications to computation of Farrell-Tate cohomology, we want to enumerate *conjugacy classes of finite order subgroups*. The difference is completely controlled by the action of $\text{Gal}(\text{Frac}(B)/\text{Frac}(A)) \cong \mathbb{Z}/2\mathbb{Z}$, as we will see below.

Remark 3.5. The cokernel of the norm-map on units can be described in terms of the inert places of the extension B/A , cf. [Bus06, section 3.2].

3.3. Existence of free bases and relative class groups. Next, we provide information on ideal classes having a free basis, cf. [Frö60].

First, recall that for $A \rightarrow B$ an integral extension of commutative rings, there is a norm map on class groups $\text{Nm}_0 : \widetilde{\text{K}}_0(B) \rightarrow \widetilde{\text{K}}_0(A)$. The kernel of $\text{Nm}_{B/A}$ is called the relative class group $\widetilde{\text{K}}_0(B/A)$ of the extension B/A . An ideal I in B is a projective A -module of rank $n = [\text{Frac}(B) : \text{Frac}(A)]$, and $\bigwedge_A^n I$ is a rank one projective module. Its class in $\widetilde{\text{K}}_0(A)$ is called the Steinitz class $c(I)$ of I . The relation between Steinitz class and norm is given by $c(I) = c(B) \text{Nm}_{B/A}(I)$. The square of the Steinitz class $c(B)$ is the relative discriminant $\delta_{B/A}$.

Proposition 3.6. *Let $A \rightarrow B$ be an integral extension of commutative rings.*

- (1) *There exists an ideal of B which has an A -basis if and only if*

$$c(B) \in \text{Im}(\text{Nm}_{B/A} : \widetilde{\text{K}}_0(B) \rightarrow \widetilde{\text{K}}_0(A)).$$

In other words, the obstruction for the existence of a B -ideal with A -basis lives in $\text{coker}(\text{Nm}_0)$.

- (2) *If there is an ideal of B which has an A -basis, then the ideal classes of B -ideals with A -basis are classified by the relative class group $\widetilde{\text{K}}_0(B/A)$.*

Remark 3.7. The cokernel of the norm map $Nm_0 : \widetilde{K}_0(B) \rightarrow \widetilde{K}_0(A)$ is isomorphic to the Galois group of the maximal unramified intermediate extension. For a degree 2 cyclotomic extension, this is trivial because these extensions ramify. In particular, in these situations, there is always some B -ideal with A -basis.

It is possible to determine explicitly the class numbers of orders in global fields, using Dedekind's formula for class groups of orders, cf. [Ded30] or [Neu92, theorem 12.12]. We give a version adapted to our purposes, i.e., a version for degree 2 extensions which also works for S -class groups of orders in global fields.

Proposition 3.8. *Let K be a global field, let S be a non-empty set of places containing the infinite ones. Assume that $K(\zeta_\ell)/K$ is a degree 2 extension. Then we have*

$$\#\widetilde{K}_0(\mathcal{O}_{K,S}[\zeta_\ell]) = \frac{\#\widetilde{K}_0(\mathcal{O}_{K(\zeta_\ell),\bar{S}}) \cdot \#(\mathcal{O}_{K(\zeta_\ell),\bar{S}}/\mathfrak{f})^\times}{[\mathcal{O}_{K(\zeta_\ell),\bar{S}}^\times : \mathcal{O}_{K,S}[\zeta_\ell]^\times] \cdot \#(\mathcal{O}_{K,S}[\zeta_\ell]/\mathfrak{f})^\times}$$

where \mathfrak{f} is the conductor of the order $\mathcal{O}_{K,S}[\zeta_\ell]$ in $\mathcal{O}_{K(\zeta_\ell),\bar{S}}$.

3.4. Automorphisms and normalizers. We also need to discuss the centralizers and normalizers of elements of finite order in $SL_2(A)$ - the relevant data to compute Farrell-Tate cohomology of $SL_2(A)$ is the Farrell-Tate cohomology of the centralizers C_Γ of finite cyclic subgroups Γ together with its action of the Weyl group N_Γ/C_Γ . This also has been worked out in a special case in [Bus06], and we provide a slight generalization:

Proposition 3.9. *Let A be an integral domain, let $x \in GL_2(A)$ be an element of finite order, and let $\chi_x(T)$ be its characteristic polynomial. Set $B = A[T]/(\chi_x(T))$, and fix a representation $\phi : B \rightarrow M_2(A)$.*

- (1) *If $m \in GL_2(A)$ centralizes $\phi(T)$, then $m = \phi(u)$ for some unit $u \in B^\times$.*
- (2) *If additionally $\phi(T) \in SL_2(A)$, and $m \in SL_2(A)$ centralizes $\phi(T)$, then $m = \phi(u)$ for some unit $u \in \ker(Nm_1 : B^\times \rightarrow A^\times)$ of norm 1.*
- (3) *For the Weyl group, we have*

$$N_{SL_2(A)}(\phi(T))/C_{SL_2(A)}(\phi(T)) \cong \text{Stab}_{\text{Gal}(\text{Frac}(B)/\text{Frac}(A))}(\mathcal{I}),$$

where \mathcal{I} is the ideal class corresponding to $\phi(T)$.

Proof. If m centralizes $\phi(T)$, it necessarily centralizes the whole image $\phi(B)$. As an element in the algebraic group $GL_2(\text{Frac}(A))$, it centralizes a maximal torus, and therefore it has to be multiplication with a unit of B .

If an element normalizes $\phi(T)$, it is contained in the normalizer of a maximal torus when viewed as an element of the algebraic group $GL_2(\text{Frac}(A))$. In particular, it normalizes the representation of B , so up to a central element it acts via the Galois group $\text{Gal}(\text{Frac}(B)/\text{Frac}(A))$. Then necessarily, we have $\mathcal{I} \cong \mathcal{I}^\sigma$, i.e., the ideal class \mathcal{I} is invariant under the action of $\text{Gal}(\text{Frac}(B)/\text{Frac}(A)) \cong \mathbb{Z}/2\mathbb{Z}$. \square

Corollary 3.10. *In the situation of Proposition 3.9, a cyclic subgroup Γ of order ℓ in $SL_2(A)$ is embeddable in a dihedral subgroup of $SL_2(A)$ precisely when the associated ideal $\mathcal{I}(\Gamma)$ is invariant under the action of $\text{Gal}(\text{Frac}(B)/\text{Frac}(A))$.*

The conjugacy classes of dihedral overgroups of Γ are in bijection with the number of orientations of $\mathcal{I}^{-1} \otimes \mathcal{I}^\sigma$, which in turn is in bijection with the group

$$\ker(Nm_1 : B^\times \rightarrow A^\times) \otimes_{\mathbb{Z}} \mathbb{Z}/2$$

of square residues of norm 1 units.

3.5. Explicit formulas. Finally, we can combine the previous results into explicit formulas describing conjugacy classes of finite cyclic subgroups in $\mathrm{SL}_2(A)$.

We start with the classification of finite cyclic subgroups in $\mathrm{GL}_2(\mathcal{O}_{K,S})$ with K a global field. Let ι be the inversion involution on the Picard group.

Theorem 3.11. *Let K be a global field, and fix a prime ℓ different from the characteristic of K .*

- (1) *Assume $\zeta_\ell \in K$. Then the conjugacy classes $[\Gamma]$ of ℓ -order subgroups in $\mathrm{GL}_2(\mathcal{O}_{K,S})$ are in bijection with elements of $\widetilde{\mathbb{K}}_0(\mathcal{O}_{K,S})/\iota$. Such a finite group Γ is contained in a dihedral overgroup if and only if the corresponding element $\mathcal{I}(\Gamma)$ in $\widetilde{\mathbb{K}}_0(\mathcal{O}_{K,S})$ is $\mathbb{Z}/2$ -invariant; in that case, all dihedral overgroups are conjugate. The normalizer of Γ in $\mathrm{GL}_2(\mathcal{O}_{K,S})$ is isomorphic to $(\mathcal{O}_{K,S}^\times)^2 \rtimes \mathbb{Z}/2$ if there is a dihedral overgroup, and isomorphic to $(\mathcal{O}_{K,S}^\times)^2$ otherwise.*
- (2) *Assume $\zeta_\ell \notin K$. Then the conjugacy classes $[\Gamma]$ of ℓ -order subgroups in $\mathrm{GL}_2(\mathcal{O}_{K,S})$ are classified by elements of $\widetilde{\mathbb{K}}_0(\mathcal{O}_{K,S}[\zeta_\ell]/\mathcal{O}_{K,S})^{\mathrm{Gal}(K(\zeta_\ell)/K)}$. Such a finite group Γ is contained in a dihedral overgroup if and only if the corresponding element $\mathcal{I}(\Gamma)$ in $\widetilde{\mathbb{K}}_0(\mathcal{O}_{K,S}[\zeta_\ell]/\mathcal{O}_{K,S})$ is $\mathbb{Z}/2$ -invariant; in that case, all dihedral overgroups are conjugate. The normalizer of Γ in $\mathrm{GL}_2(\mathcal{O}_{K,S})$ is isomorphic to $\mathcal{O}_{K,S}[\zeta_\ell]^\times \rtimes \mathbb{Z}/2$ if there is a dihedral overgroup, and to $\mathcal{O}_{K,S}[\zeta_\ell]^\times$ otherwise.*

Proof. The correspondence between conjugacy classes and ideal classes is the one of Proposition 3.2. In case(1), the norm map on class groups is

$$\widetilde{\mathbb{K}}_0(\mathcal{O}_{K,S})^2 \cong \widetilde{\mathbb{K}}_0(\mathcal{O}_{K,S}^2) \rightarrow \widetilde{\mathbb{K}}_0(\mathcal{O}_{K,S}) : (\mathcal{I}, \mathcal{J}) \mapsto \mathcal{I} \otimes \mathcal{J},$$

hence its kernel is identified with $\widetilde{\mathbb{K}}_0(\mathcal{O}_{K,S})$. Normalizer and centralizer are determined by Proposition 3.9. Passing from conjugacy of finite order elements (classified by $\widetilde{\mathbb{K}}_0(\mathcal{O}_{K,S})$) to conjugacy of finite cyclic subgroups happens by dividing out the action of $\mathrm{Gal}(K(\zeta_\ell)/K) \cong \mathbb{Z}/2$ on the kernel of the norm map; in case (1) this is identified with the involution $\iota : \mathcal{I} \mapsto \mathcal{I}^{-1}$. \square

The following result, established by a similar argument, provides the classification of finite cyclic subgroups in $\mathrm{SL}_2(\mathcal{O}_{K,S})$, with K a global field.

Theorem 3.12. *Let K be a global field, and fix a prime ℓ different from the characteristic of K .*

- (1) *Assume $\zeta_\ell \in K$. The conjugacy classes of ℓ -order subgroups in $\mathrm{SL}_2(\mathcal{O}_{K,S})$ are in bijection with elements of $\widetilde{\mathbb{K}}_0(\mathcal{O}_{K,S})/\iota$. Such a finite group Γ is contained in a dihedral overgroup if and only if the corresponding element $\mathcal{I}(\Gamma)$ in $\widetilde{\mathbb{K}}_0(\mathcal{O}_{K,S})$ is $\mathbb{Z}/2$ -invariant. The normalizer of Γ in $\mathrm{SL}_2(\mathcal{O}_{K,S})$ is isomorphic to $\mathcal{O}_{K,S}^\times \rtimes \mathbb{Z}/2$ if there is a dihedral overgroup, and isomorphic to $\mathcal{O}_{K,S}^\times$ otherwise.*
- (2) *Assume $\zeta_\ell \notin K$. The group \mathcal{C}_ℓ of conjugacy classes of elements of order ℓ is subject to the extension*

$$\begin{aligned} 0 &\rightarrow \mathrm{coker} \left(\mathrm{Nm}_1 : \mathcal{O}_{K,S}[\zeta_\ell]^\times \rightarrow \mathcal{O}_{K,S}^\times \right) \rightarrow \mathcal{C}_\ell \rightarrow \\ &\rightarrow \ker \left(\mathrm{Nm}_0 : \widetilde{\mathbb{K}}_0(\mathcal{O}_{K,S}[\zeta_\ell]) \rightarrow \widetilde{\mathbb{K}}_0(\mathcal{O}_{K,S}) \right) \rightarrow 0. \end{aligned}$$

The group \mathcal{C}_ℓ inherits a $\mathrm{Gal}(K(\zeta_\ell)/K)$ -action from $\widetilde{\mathbb{K}}_0(\mathcal{O}_{K,S}[\zeta_\ell])$. The conjugacy classes \mathcal{K}_ℓ of subgroups of order ℓ in $\mathrm{SL}_2(\mathcal{O}_{K,S})$ are classified by elements of the quotient $\mathcal{K}_\ell = \mathcal{C}_\ell/\mathrm{Gal}(K(\zeta_\ell)/K)$.

A finite group Γ with $[\Gamma] \in \mathcal{K}$ is contained in a dihedral overgroup if and only if the corresponding element $\mathcal{I}(\Gamma)$ in \mathcal{K} is $\mathrm{Gal}(K(\zeta_\ell)/K)$ -invariant. The normalizer of Γ in $\mathrm{SL}_2(\mathcal{O}_{K,S})$ is isomorphic to

$$\ker(\mathrm{Nm}_1 : \mathcal{O}_{K,S}[\zeta_\ell]^\times \rightarrow \mathcal{O}_{K,S}) \rtimes \mathbb{Z}/2$$

if there is a dihedral overgroup, and to $\ker(\mathrm{Nm}_1 : \mathcal{O}_{K,S}[\zeta_\ell]^\times \rightarrow \mathcal{O}_{K,S})$ otherwise.

Proof. In case (1), there are no orientation issues because the norm map $(\mathcal{O}_{K,S}^\times)^2 \rightarrow \mathcal{O}_{K,S}^\times : (a_1, a_2) \mapsto a_1 a_2$ is surjective. The rest is a combination of Proposition 3.3 and Proposition 3.9 as before. In case (2), the Galois action on \mathcal{K} is the natural one, in particular, it is trivial on the cokernel of Nm_1 . \square

Remark 3.13. See also [Wen14a] for similar results concerning $\mathrm{SL}_2(K[C])$ with C a smooth affine curve over an algebraically closed field K .

Remark 3.14. The above results are more or less immediate generalizations of the classification in [Bus06]. The relation to the classification results of Prestel [Pr68], Schneider [Sch75], Krämer [Krä80] or Maclachlan [Mac06] is a bit more subtle to discuss. We restrict ourselves to mention the two major differences: one is due to the fact that the cited works consider the more general situation of possibly non-split quaternion algebras (instead of $M_2(A)$ considered here). The second difference is that the cited works provide much more elaborate formulas for the orders of relative class groups, where we are basically stopping at Proposition 3.3 and Proposition 3.8. For our applications to computations of Farrell-Tate cohomology we do not need the actual numbers, but the more conceptual explanations provided by the results above.

4. RECOLLECTIONS ON THE ACTION OF SL_2 ON THE SYMMETRIC SPACE

We will typically denote by K a global field, by S a non-empty finite set of places of K containing all the infinite places. The corresponding ring of S -integers is denoted by $\mathcal{O}_{K,S}$. We will be interested in the Farrell-Tate cohomology of $\mathrm{SL}_2(\mathcal{O}_{K,S})$. For the later development, we recall some basic facts on symmetric spaces and finite subgroups in SL_2 .

4.1. Recollection on symmetric spaces. For each of the places $v \in S$, we have a corresponding completion K_v which is isomorphic to \mathbb{R} , \mathbb{C} or a non-archimedean local field K/\mathbb{Q}_p or $K/\mathbb{F}_q((T))$.

We recall the usual choices for maximal compact subgroups in the corresponding Lie groups $\mathrm{SL}_2(K_v)$. For v real, a maximal compact subgroup of $\mathrm{SL}_2(\mathbb{R})$ is $\mathrm{SO}(2)$. For v complex, a maximal compact subgroup of $\mathrm{SL}_2(\mathbb{C})$ is $\mathrm{SU}(2)$. For v finite, a maximal compact subgroup of $\mathrm{SL}_2(K_v)$ is $\mathrm{SL}_2(\mathcal{O}_v)$.

The set of maximal compact subgroups in the Lie group $\mathrm{SL}_2(K_v)$ forms a symmetric space of non-compact type. For v real, this is the upper half-plane $\mathbb{H} \cong \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2)$ which is a Riemannian manifold of dimension 2. For v complex, this is the complex upper half-space $\mathrm{SL}_2(\mathbb{C})/\mathrm{SU}(2)$, a Riemannian manifold of dimension 3. For v finite, this is the Bruhat-Tits tree $\mathrm{SL}_2(K_v)/\mathrm{SL}_2(\mathcal{O}_v)$ associated to the non-archimedean valuation v . This fails to be a manifold, it is a branched tree, with branching given as follows. The link of any vertex of this tree can be identified with the discrete set $\mathbb{P}^1(\mathbb{F}_q)$, where \mathbb{F}_q is the residue field at the place v .

We introduce notation for the product of these symmetric spaces, the products being taken over the real, complex and finite places of K , respectively:

$$\mathfrak{X}_{K,S} = \prod_{v \text{ real}} \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2) \times \prod_{v \text{ complex}} \mathrm{SL}_2(\mathbb{C})/\mathrm{SU}(2) \times \prod_{v \text{ finite}} \mathrm{SL}_2(K_v)/\mathrm{SL}_2(\mathcal{O}_v).$$

The diagonal embedding $\mathrm{SL}_2(\mathcal{O}_{K,S}) \hookrightarrow \prod_{v \in S} \mathrm{SL}_2(K_v)$ induces an action of $\mathrm{SL}_2(\mathcal{O}_{K,S})$ on the symmetric space $\mathfrak{X}_{K,S}$. This action is properly discontinuous, in particular with finite stabilizers, and it factors through a faithful action of $\mathrm{PSL}_2(\mathcal{O}_{K,S})$.

The symmetric space $\mathfrak{X}_{K,S}$ is a $\mathrm{CAT}(0)$ -space, in particular it is contractible. Therefore, the Borel isotropy spectral sequence can be used to obtain information on group homology of $\mathrm{SL}_2(\mathcal{O}_{K,S})$ from its action on $\mathfrak{X}_{K,S}$.

4.2. Recollections on linear and projective representations. Now we consider the local structure of the quotient $\mathrm{SL}_2(\mathcal{O}_{K,S}) \backslash \mathfrak{X}_{K,S}$. In particular, we want to know the topological structure of the quotient in the neighbourhood of points which have non-trivial isotropy. Since $\mathrm{SL}_2(\mathcal{O}_{K,S})$ acts isometrically on the $\mathrm{CAT}(0)$ -space $\mathfrak{X}_{K,S}$, it suffices to understand the action of the stabilizer Γ_x on the tangent space $T_x \mathfrak{X}_{K,S}$. The latter can be understood using representation theory.

We make a case distinction, depending on the type of the place.

Proposition 4.1. *Let v be a real place, let Γ be a finite subgroup of $\mathrm{SL}_2(\mathbb{R})$ fixing a point $x \in \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2)$. Then Γ is cyclic.*

The matrix

$$\begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix}, \quad \phi = \frac{2\pi l}{n}, \quad 0 < l < n$$

in $\mathrm{SO}(2)$ acts on the tangent space of $[1] \in \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2)$ as the representation $\chi_{2,2\phi}$ - the 2-dimensional representation rotating with double speed.

Proof. Any finite subgroup Γ of $\mathrm{SL}_2(\mathbb{R})$ has to preserve a quadratic form, and hence is conjugate to a subgroup of $\mathrm{SO}(2)$. But this means that Γ has a 2-dimensional, faithful, orientation-preserving, orthogonal representation. The only possible such groups are cyclic. For the description of the action, we can assume that Γ is a subgroup of the standard $\mathrm{SO}(2)$ inside $\mathrm{SL}_2(\mathbb{R})$, and hence fixes the class of 1 in $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2)$. The tangent space at 1 of $\mathrm{SL}_2(\mathbb{R})$ is the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$. The subspace \mathfrak{k} of the tangent space corresponding to the subgroup $\mathrm{SO}(2)$ can be identified with the skew-symmetric trace-less matrices in $\mathfrak{sl}_2(\mathbb{R})$. We can choose the subspace \mathfrak{p} of symmetric trace-less matrices in $\mathfrak{sl}_2(\mathbb{R})$ as representative for the tangent space of $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2)$. After making these identifications, the group $\mathrm{SO}(2)$ acts on the tangent space of its fixed point $[1] \in \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2)$ as the adjoint representation on \mathfrak{p} .

In the adjoint representation, the action of $\begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \in \mathrm{SO}(2)$ on the symmetric trace-less matrices \mathfrak{p} has trace $2 \cos(2\phi)$, i.e., the adjoint representation of $\mathrm{SO}(2)$ on \mathfrak{p} is the 2-dimensional representation which rotates with double speed. As we are interested in groups Γ of odd order, this is an irreducible representation with trivial fixed space. \square

Proposition 4.2. *Let v be a complex place, let Γ be a finite subgroup of $\mathrm{SU}(2)$.*

- (1) *If Γ is cyclic of order n , it is conjugate to the subgroup generated by the matrix $\mathrm{diag}(\zeta_n, \zeta_n^{-1})$ for some primitive n -th root of unity ζ_n . The corresponding representation of Γ on the tangent space of $1 \in \mathrm{SL}_2(\mathbb{C})/\mathrm{SU}(2)$ has trace $\zeta_n^2 + \zeta_n^{-2} + 1$.*
- (2) *If $\Gamma \cong D_{2n}$ is dihedral, then its representation on the tangent space of $1 \in \mathrm{SL}_2(\mathbb{C})/\mathrm{SU}(2)$ is the direct sum of the sign representation and the two-dimensional representation where the cyclic group acts with trace $\zeta_n^2 + \zeta_n^{-2}$.*

In particular, the fixed space of the subgroup Γ is a line. If Γ is a dihedral group, then the normal cyclic subgroup fixes this line pointwise; and the action of the quotient $\mathbb{Z}/2$ on this line is by reflection at the origin.

Proof. Analogously to the real case, we can assume that Γ is a subgroup of the standard $SU(2)$ inside $SL_2(\mathbb{C})$, and hence fixes the class of 1 in $SL_2(\mathbb{C})/SU(2)$. The tangent space at 1 of $SL_2(\mathbb{C})$ is the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. The subspace \mathfrak{k} of the tangent space corresponding to the subgroup $SU(2)$ can be identified with the skew-hermitian trace-less matrices in $\mathfrak{sl}_2(\mathbb{C})$. We can choose the subspace \mathfrak{p} of hermitian trace-less matrices in $\mathfrak{sl}_2(\mathbb{C})$ as representative for the tangent space of $SL_2(\mathbb{C})/SU(2)$. After making these identifications, the group $SU(2)$ acts on the tangent space of its fixed point $[1] \in SL_2(\mathbb{C})/SU(2)$ as the adjoint representation on \mathfrak{p} .

A diagonal matrix $\text{diag}(\lambda_1, \lambda_2) \in SU(2)$ acts via the adjoint representation of $SU(2)$ on \mathfrak{p} with trace $\frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_1} + 1$. The statements for cyclic and dihedral groups follow. \square

For the finite places where no tangent space is available, we can use the classification of automorphisms of trees to describe the action in a neighbourhood of a fixed point.

Proposition 4.3. *Let K_v be a non-archimedean local field; denote by \mathcal{O}_v its valuation ring and by k_v its residue field. Let g be an element of $SL_2(\mathcal{O}_v)$ of odd order n . Assume that n is coprime to the characteristic of K_v . Then we have the following cases for the action of g on the link of $1 \in SL_2(K_v)/SL_2(\mathcal{O}_v)$, which can be identified with $\mathbb{P}^1(k_v)$:*

- (1) *The real case: if n is prime to $\text{char } k_v$ and $\zeta_n \notin K_v$, then the element g is conjugate to one in the non-split torus and hence does not have a fixed point on $\mathbb{P}^1(k_v)$.*
- (2) *The complex case: if n is prime to $\text{char } k_v$ and $\zeta_n \in K_v$, then the element g is conjugate to one in the split torus and hence has two fixed points on $\mathbb{P}^1(k_v)$.*
- (3) *The bad case: if $\text{char } k_v | n$, then the residue of the matrix in $SL_2(k_v)$ is conjugate to a strictly upper triangular matrix. If this strictly upper triangular matrix is non-trivial, it has exactly one fixed point on $\mathbb{P}^1(k_v)$. It may also be trivial in which case everything is fixed.*

Proof. Assume first that n is prime to $\text{char } k_v$. If $\zeta_n \in K_v$, then there exists a unique conjugacy class of cyclic subgroups of order n in $PSL_2(K_v)$. In particular, g is diagonalizable and fixes exactly the standard apartment. Conversely, if g has a non-isolated fixed point, then it fixes a non-trivial 1-simplex and must be conjugate in $PSL_2(K_v)$ to an upper triangular matrix. Because of the characteristic assumption, it must be diagonalizable, which implies $\zeta_n \in K_v$. This implies cases (1) and (2).

Case (3) is clear from the classification of subgroups of $SL_2(\mathbb{F}_q)$ and the explicit description of the action of $SL_2(\mathcal{O}_v)$ on the link of 1 by Möbius transformations. \square

Corollary 4.4. *Let K_v be a non-archimedean local field, and denote by \mathcal{O}_v its valuation ring and by k_v its residue field. Let g be an element of $SL_2(\mathcal{O}_v)$ of order n . Assume that n is coprime to the characteristic of K_v .*

- (1) *In the real case, g has an isolated fixed point on the Bruhat-Tits tree.*
- (2) *In the complex case, g fixes a whole apartment of the Bruhat-Tits tree.*
- (3) *In the bad case, g fixes a whole subtree containing exactly two ends of the Bruhat-Tits tree.*

Proof. (1) and (2) are immediate from the Proposition 4.3. We only need to discuss the third case. The boundary of the building can be identified with $\mathbb{P}^1(K_v)$; the element g acts via Möbius transformations on this projective line, and an element of finite order has two fixed points.

The fixed set is connected. More generally, let X be a CAT(0)-space and let G be a group acting isometrically on X . If $g \in G$ is an element of finite order, then its fixed set is connected: any two points of X are connected by a unique geodesic because X is CAT(0). If $x, y \in X^g$, then the unique geodesic connecting x and y also has to be fixed set-wise because the group action is isometric. But then the geodesic also has to be fixed point-wise because the distances of points on the geodesic has to be preserved by the action.

This implies the claims. \square

Remark 4.5. The precise structure of the fixed set of the finite order element in the bad case is described in [BC14]. We thank Joël Bellaïche for pointing this out in his answer to MathOverflow question 171589 “Fixed set of order p automorphism of Bruhat-Tits tree”.

5. ON THE STRUCTURE OF ℓ -TORSION SUBCOMPLEXES

This section recalls basic facts about ℓ -torsion subcomplexes and how they can be used to describe Farrell-Tate cohomology, cf. [Rah13, Rah14]. We also give an explicit description of ℓ -torsion subcomplexes for SL_2 over rings of S -integers in global fields. This generalizes the Bianchi group formulas of [Rah13]. We will assume throughout that ℓ is a prime different from the characteristic of the global field K .

We recall the definition of the ℓ -torsion subspace:

Definition 5.1. Let K be a global field, let S be a finite set of places containing the infinite ones. The ℓ -torsion subspace of $\mathfrak{X}_{K,S}$ is the subspace consisting of those $x \in \mathfrak{X}_{K,S}$ which are fixed by some element $g_x \in \mathrm{SL}_2(\mathcal{O}_{K,S})$ of order ℓ . The ℓ -torsion subspace is denoted by $\mathfrak{X}_{K,S}^{(\ell)}$.

As a piece of notation, we will, for each individual place $v \in S$, define $\mathfrak{X}_v^{(\ell)}$ as the set of those $x \in \mathfrak{X}_v$ fixed by some order ℓ element $g \in \mathrm{SL}_2(\mathcal{O}_{K,S})$.

Our main goal is a description of the structure of this subspace, and how it influences the Farrell-Tate cohomology of $\mathrm{SL}_2(\mathcal{O}_{K,S})$. First of all, the following lemma is fairly clear from the definition:

Lemma 5.2. *Let K be a number field, let S be a finite set of places containing the infinite ones. Then the action of $\mathrm{SL}_2(\mathcal{O}_{K,S})$ on $\mathfrak{X}_{K,S}$ restricts to the subspace $\mathfrak{X}_{K,S}^{(\ell)}$.*

We denote by $\mathcal{A}_{K,S}^{(\ell)}$ the complex of elementary abelian ℓ -subgroups of the group $\mathrm{SL}_2(\mathcal{O}_{K,S})$, as in [Bro94]. We will compare the ℓ -torsion subcomplex to the complex of elementary abelian subgroups.

5.1. Connected components. We first investigate the structure and number of connected components of the ℓ -torsion complex resp. its quotient.

Lemma 5.3. *Let K be a global field, let S be a finite set of places containing the infinite ones, and let $g \in \mathrm{SL}_2(\mathcal{O}_{K,S})$ be an element of finite order. The fixed space $(\mathfrak{X}_{K,S})^{(g)}$ is a totally geodesic subspace whose dimension equals the number of complex archimedean places together with the complex and bad non-archimedean places. Moreover, the fixed space has a product decomposition*

$$\mathfrak{X}_{K,S}^{(g)} \cong \prod_{v \in S} \mathfrak{X}_v^{(g)},$$

and the product decomposition is compatible with the action of $\mathrm{SL}_2(\mathcal{O}_{K,S})$.

Proof. For the archimedean places, the dimension of the torsion subcomplex can be determined from the action of $\langle g \rangle$ on the tangent space. If $x \in \mathfrak{X}_{K,S}$ is a fixed point, then the dimension of the ℓ -torsion subcomplex at x equals the maximal dimension of a subspace of $T_x \mathfrak{X}_{K,S}$ fixed by an element $g \in \mathrm{Stab}_x \mathrm{SL}_2(\mathcal{O}_{K,S})$. From the definition of $\mathfrak{X}_{K,S}$ as a product, we have a direct sum decomposition of tangent spaces to the archimedean symmetric spaces

$$\bigoplus_{r_1} V_{\mathrm{real}} \oplus \bigoplus_{r_2} V_{\mathrm{complex}}.$$

We can determine the dimension of the torsion subcomplex by looking at the representations of finite stabilizers on the tangent spaces of fixed points, cf. Proposition 4.1 and Proposition 4.2. The representations of the finite stabilizers are direct sums of representations for the real and the complex places. For each of these representations, every element acts as rotation. The dimension of the corresponding fixed set is then 0 for the real places and 1 for the complex places.

Similarly, we can deal with the non-archimedean places, cf. Proposition 4.3. The real case corresponds to isolated fixed points and thus does not contribute to the dimension of the fixed space $(\mathcal{X}_{K,S})^{\langle g \rangle}$. The complex case contributes one dimension, coming from the fixed apartment. The bad case also contributes one dimension, coming from the tree thickening of the geodesic connecting the two fixed points on $\mathbb{P}^1(K_v)$. This proves the claim. \square

Remark 5.4. We are making strong use of the fact that $\mathrm{SL}_2(\mathcal{O}_{K,S})$ acts isometrically on a CAT(0)-space, which implies a close link between the infinitesimal structure (the representation of the finite stabilizer group on the tangent space) and the global structure (the totally geodesic subspace fixed by the stabilizer group). Because of this, the dimension of $\mathfrak{X}_{K,S}^{(\ell)}$ can be read off from the dimension of the fixed part of the tangent space.

Lemma 5.5. *Let K be a global field, let S be a nonempty finite set of places containing the infinite ones, and let ℓ be an odd prime different from the characteristic of K . There is a bijection*

$$\pi_0 \left(\mathrm{SL}_2(\mathcal{O}_{K,S}) \backslash \mathfrak{X}_{K,S}^{(\ell)} \right) \xrightarrow{\cong} \pi_0 \left(\mathrm{SL}_2(\mathcal{O}_{K,S}) \backslash \mathcal{A}_{K,S}^{(\ell)} \right).$$

Proof. We first define the map. For this, let $x \in \mathfrak{X}_{K,S}^{(\ell)}$. Then there exists a subgroup of order ℓ which fixes x . Any two such subgroups are contained in a finite subgroup, because the stabilizer of x is finite. As ℓ is odd, the classification of finite subgroups of $\mathrm{SL}_2(\mathbb{C})$ implies that all order ℓ subgroups of $\mathrm{SL}_2(\mathcal{O}_{K,S})$ fixing x have to be conjugate, even equal. Call Γ_x the respective ℓ -subgroup. The map then sends the connected component containing x to its stabilizer group Γ_x . As argued in the proof of Corollary 4.4, the fixed point set of Γ_x is connected; and any point y in this set has all of its order ℓ isotropy matrices contained in Γ_x . Therefore, the fixed point set of Γ_x is a whole connected component of $\mathfrak{X}_{K,S}^{(\ell)}$, which makes the map well-defined.

The map is evidently surjective, because a finite group acting on a CAT(0)-space has a fixed point.

For injectivity, consider two subgroups $\langle g_1 \rangle$ and $\langle g_2 \rangle$ of order ℓ . If these groups are conjugate, then the conjugating matrix induces a map $\mathfrak{X}_{K,S}^{\langle g_1 \rangle} \rightarrow \mathfrak{X}_{K,S}^{\langle g_2 \rangle}$, and therefore the two totally geodesic subspaces will lie in the same connected component in $\mathrm{SL}_2(\mathcal{O}_{K,S}) \backslash \mathfrak{X}_{K,S}$. \square

5.2. Actions on connected components and quotients.

Proposition 5.6. *Let K be a global field, let S be a nonempty finite set of places containing the infinite places and let ℓ be an odd prime different from the characteristic of K . Let g be an element in $\mathrm{SL}_2(\mathcal{O}_{K,S})$ of order ℓ , and let \mathcal{W}_g be the totally geodesic subspace fixed by g . If $h \in \mathrm{SL}_2(\mathcal{O}_{K,S})$ sends \mathcal{W}_g to itself, then $h \in N_{\mathrm{SL}_2(\mathcal{O}_{K,S})}(\langle g \rangle)$.*

Proof. As the fixed points of dihedral groups in $\mathrm{SL}_2(\mathcal{O}_{K,S})$ cannot occupy the whole geodesic subspace \mathcal{W}_g , there exists $x \in \mathcal{W}_g$ with only cyclic stabilizer $\langle g \rangle$. Then $h\langle g \rangle h^{-1}$ is the stabilizer of hx ; and $hx \in \mathcal{W}_g$ implies $h\langle g \rangle h^{-1} = \langle g \rangle$. \square

Corollary 5.7. *Let K be a global field, let S be a nonempty finite set of places containing the infinite places and let ℓ be an odd prime different from the characteristic of K . Denote by \mathcal{K}_ℓ the set of conjugacy classes of subgroups of $\mathrm{SL}_2(\mathcal{O}_{K,S})$ of order ℓ . There is a decomposition*

$$\mathfrak{X}_{K,S}^{(\ell)} \cong \bigsqcup_{\Gamma \in \mathcal{K}_\ell} \left(\bigsqcup_{\mathrm{SL}_2(\mathcal{O}_{K,S})/N_{\mathrm{SL}_2(\mathcal{O}_{K,S})}(\Gamma)} \mathfrak{X}_{K,S}^\Gamma \right).$$

and the map $\mathfrak{X}_{K,S}^{(\ell)} \rightarrow \mathcal{A}_{K,S}^{(\ell)}$ sending the fixed space $\mathfrak{X}_{K,S}^\Gamma$ of $\Gamma \in \mathcal{K}_\ell$ to Γ induces an isomorphism on $\mathrm{SL}_2(\mathcal{O}_{K,S})$ -equivariant homology.

Proof. The bijection on connected components of the quotient by the SL_2 -action has been established in Lemma 5.5. The same argument in fact shows that mapping a component of $\mathfrak{X}_{K,S}^{(\ell)}$ to the subgroup fixing it induces a bijection of connected components of $\mathfrak{X}_{K,S}^{(\ell)}$ and $\mathcal{A}_{K,S}^{(\ell)}$. Each component $\mathfrak{X}_{K,S}^\Gamma$ is a totally geodesic subspace, and hence contractible.

Now let $\mathfrak{X}_{K,S}^\Gamma$ be a component fixed by the cyclic subgroup Γ . The normalizer $N_{\mathrm{SL}_2(\mathcal{O}_{K,S})}(\Gamma)$ acts with finite stabilizers on $\mathfrak{X}_{K,S}^\Gamma$, and the stabilizer groups containing Γ are cyclic or dihedral groups. There is a maximal cyclic subgroup stabilizing $\mathfrak{X}_{K,S}^\Gamma$ pointwise, call it $\tilde{\Gamma}$. Then the group $N_{\mathrm{SL}_2(\mathcal{O}_{K,S})}(\Gamma)/\tilde{\Gamma}$ acts on $\mathfrak{X}_{K,S}^\Gamma$, and from our study of the local geometry of the action of finite subgroups, we know that the stabilizers of the latter action have order 2 (coming from the dihedral groups). As 2 is assumed prime to ℓ , the projection $\mathfrak{X}_{K,S}^\Gamma \rightarrow \mathrm{pt}$ then induces an isomorphism of $N_{\mathrm{SL}_2(\mathcal{O}_{K,S})}(\Gamma)/\tilde{\Gamma}$ -equivariant homology with \mathbb{F}_ℓ -coefficients. But since $\mathfrak{X}_{K,S}^\Gamma$ is stabilized pointwise by $\tilde{\Gamma}$, the Hochschild-Serre spectral sequence implies that the projection $\mathfrak{X}_{K,S}^\Gamma \rightarrow \mathrm{pt}$ also induces an isomorphism of $N_{\mathrm{SL}_2(\mathcal{O}_{K,S})}(\Gamma)$ -equivariant homology with \mathbb{F}_ℓ -coefficients. This is what we wanted to prove. \square

Remark 5.8. The above result provides a precise relation between the ℓ -torsion subcomplex and the complex of elementary abelian ℓ -groups, in an ℓ -rank 1 case. As a result, we see that for computation of Farrell-Tate cohomology at the prime ℓ , we can either use the ℓ -torsion subcomplex, or Brown's formula

$$\hat{H}^\bullet(\mathrm{SL}_2(\mathcal{O}_{K,S}), \mathbb{Z})_{(\ell)} \cong \bigoplus_{[\Gamma \leq \mathrm{SL}_2], \Gamma \text{ cyclic}} \hat{H}^\bullet(C_{\mathrm{SL}_2}(\Gamma), \mathbb{Z})_{(\ell)}^{N_{\mathrm{SL}_2}(\Gamma)/C_{\mathrm{SL}_2}(\Gamma)},$$

where the sum on the right is indexed by conjugacy classes of finite cyclic subgroups Γ in $\mathrm{SL}_2(\mathcal{O}_{K,S})$. However, the ℓ -torsion subcomplex still seems a bit nicer to perform the computations because it breaks up the structure of the normalizer a bit further.

6. FORMULAS FOR FARRELL-TATE COHOMOLOGY

This section will deal with a proof of Theorem 1, the number-theoretic formula for Farrell-Tate cohomology of $SL_2(\mathcal{O}_{K,S})$.

The first thing to note is that the results of Section 5 imply that the Farrell-Tate cohomology $\widehat{H}^\bullet(SL_2(\mathcal{O}_{K,S}), \mathbb{F}_\ell)$ is non-trivial if and only if there exists a cyclic subgroup of $SL_2(\mathcal{O}_{K,S})$ of order ℓ . Part (1) of Theorem 1 is then a consequence of Proposition 3.3 and Proposition 3.6.

Part (2) of Theorem 1 follows directly from Lemma 5.5, Theorem 3.12 and Proposition 5.6.

The description of the relevant normalizers for parts (3) and (4) of Theorem 1 can be found in Theorem 3.12 resp. Proposition 3.9.

Then to prove Theorem 2, we provide the actual computation of the Farrell-Tate cohomology of the normalizers.

I) First, we consider the case of cyclic subgroups whose conjugacy class is not Galois invariant. In this case, the normalizer equals the centralizer and is isomorphic to the kernel of the norm map on units. As a group, the normalizer is then of the form $\mathbb{Z}/n \times \mathbb{Z}^r$ for suitable n and r , by Dirichlet's unit theorem. The computation of the relevant Farrell-Tate cohomology is straightforward:

Proposition 6.1. *Let $A = \mathbb{Z}/n \times \mathbb{Z}^r$, and let ℓ be an odd prime with $\ell \mid n$. Then, with b_1, x_1, \dots, x_r denoting classes in degree 1 and a_2 a class of degree 2, we have*

$$\widehat{H}^\bullet(A, \mathbb{F}_\ell) \cong \widehat{H}^\bullet(\mathbb{Z}/n, \mathbb{F}_\ell) \otimes_{\mathbb{F}_\ell} \bigwedge^r \mathbb{F}_\ell \cong \mathbb{F}_\ell[a_2, a_2^{-1}](b_1, x_1, \dots, x_r).$$

Proof. We begin with a computation of group cohomology. In this case, the Künneth formula implies

$$H^\bullet(A, \mathbb{F}_\ell) \cong H^\bullet(\mathbb{Z}/n, \mathbb{F}_\ell) \otimes_{\mathbb{F}_\ell} H^\bullet(\mathbb{Z}^r, \mathbb{F}_\ell).$$

But $H^\bullet(\mathbb{Z}^r, \mathbb{F}_\ell) \cong \bigwedge^r \mathbb{F}_\ell$, again by iterated application of the Künneth formula. Therefore, the first isomorphism claimed above is true with group cohomology instead of Farrell-Tate cohomology.

Now group cohomology and Farrell-Tate cohomology agree above the virtual cohomological dimension, which in this case is r . Moreover, the only finite subgroup of A is \mathbb{Z}/n , and hence by [Bro94, theorem X.6.7] the group A has periodic Farrell-Tate cohomology. The latter in particular means that there is an integer d such that $\widehat{H}^i(A, \mathbb{F}_\ell) \cong \widehat{H}^{i+d}(A, \mathbb{F}_\ell)$ for all i . These two assertions imply that the formula we obtained for group cohomology above is also true for Farrell-Tate cohomology.

The second isomorphism then just combines the first isomorphism with the formula for the cyclic groups discussed earlier. \square

This proves Theorem 2.(2).

II) Secondly, we discuss the case where the cyclic subgroup is Galois invariant. Recall from Theorem 3.12 that the group structure in this case is $\ker \text{Nm}_1 \rtimes \mathbb{Z}/2\mathbb{Z}$, where the action of $\mathbb{Z}/2\mathbb{Z} \cong \text{Gal}(K(\zeta_\ell)/K)$ on $\ker \text{Nm}_1$ is the natural Galois action.

Proposition 6.2. *The Hochschild-Serre spectral sequence associated to the semi-direct product $\ker \text{Nm}_1 \rtimes \mathbb{Z}/2\mathbb{Z}$ degenerates and yields an isomorphism*

$$\widehat{H}^\bullet(\ker \text{Nm}_1 \rtimes \mathbb{Z}/2\mathbb{Z}, \mathbb{F}_\ell) \cong \widehat{H}^\bullet(\ker \text{Nm}_1, \mathbb{F}_\ell)^{\mathbb{Z}/2\mathbb{Z}}.$$

The $\text{Gal}(K(\zeta_\ell)/K)$ -action on $\ker \text{Nm}_1$ is by multiplication with -1 . The invariant classes are then given by $a_2^{\otimes 2i}$ tensor the even exterior powers plus $a_2^{\otimes (2i+1)}$ tensor the odd exterior powers.

This proves Theorem 2.(3); and parts (2) and (3) of Theorem 2 together yield its part (1).

Remark 6.3. Our result provides an SL_2 -analogue of the computations of Anton [Ant99].

7. APPLICATION I: NON-DETECTION

In this section, we want to discuss some consequences of our computations for a conjecture of Quillen, cf. [Qui71, p. 591]. In particular, we are going to prove Theorem 3.

Conjecture 6 (Quillen). *Let ℓ be a prime number. Let K be a number field with $\zeta_\ell \in K$, and S a finite set of places containing the infinite places and the places over ℓ . Then the natural inclusion $\mathcal{O}_{K,S} \hookrightarrow \mathbb{C}$ makes $\mathrm{H}^\bullet(\mathrm{GL}_n(\mathcal{O}_{K,S}), \mathbb{F}_\ell)$ a free module over the cohomology ring $\mathrm{H}_{\mathrm{cts}}^\bullet(\mathrm{GL}_n(\mathbb{C}), \mathbb{F}_\ell)$.*

The range of validity of the conjecture has not yet been decided. Positive cases in which the conjecture has been established are $n = \ell = 2$ by Mitchell [Mit92], $n = 3, \ell = 2$ by Henn [Hen99], and $n = 2, \ell = 3$ by Anton [Ant99].

On the other hand, cases where the Quillen conjecture is known to be false can all be traced to [HLS95, remark on p. 51], which shows that Quillen's conjecture for $\mathrm{GL}_n(\mathbb{Z}[1/2])$ implies that the restriction map

$$\mathrm{H}^\bullet(\mathrm{GL}_n(\mathbb{Z}[1/2]), \mathbb{F}_2) \rightarrow \mathrm{H}^\bullet(\mathrm{T}_n(\mathbb{Z}[1/2]), \mathbb{F}_2)$$

from $\mathrm{GL}_n(\mathbb{Z}[1/2])$ to the subgroup $\mathrm{T}_n(\mathbb{Z}[1/2])$ of diagonal matrices is injective. Non-injectivity of the restriction map has been shown by Dwyer [Dwy98] for $n \geq 32$ and $\ell = 2$. Dwyer's bound was subsequently improved by Henn and Lannes to $n \geq 14$. At the prime $\ell = 3$, Anton proved non-injectivity for $n \geq 27$, cf. [Ant99].

Using our Farrell-Tate cohomology computations, we can discuss these conjectures - or better, weaker versions - in the case $n = 2$. For the Quillen conjecture, we can determine precisely the module structure above the virtual cohomological dimension; this allows to relate the Quillen conjecture for SL_2 to statements about Steinberg homology.

Proposition 7.1. *Let $G = \mathbb{Z}/\ell \times \mathbb{Z}^n$.*

- (1) *Denoting by b_1, x_1, \dots, x_n exterior classes of degree 1, and by a_2 a polynomial class of degree 2, the cohomology ring*

$$\mathrm{H}^\bullet(G, \mathbb{F}_\ell) \cong \mathbb{F}_\ell[a_2](b_1, x_1, \dots, x_n)$$

is a free module of rank 2^{n+2} over the subring $\mathbb{F}_\ell[a_2^2]$.

- (2) *Let $\mathbb{Z}/2$ act via multiplication by -1 on all the generators. The invariant ring $\mathrm{H}^\bullet(G, \mathbb{F}_\ell)^{\mathbb{Z}/2}$ is a free module of rank 2^{n+1} over the subring $\mathbb{F}_\ell[a_2^2]$.*

Proof. (1) is clear from the explicit formula given in Proposition 6.1; and 2^{n+2} is the rank obtained from the basis consisting of all the wedge products of the set $\{a_2, b_1, x_1, \dots, x_n\}$.

(2) follows from this, the invariant ring is additively generated by $a_2^{\otimes(2i+1)}$ tensor the odd degree part of $\bigwedge(b_1, x_1, \dots, x_n)$ and $a_2^{\otimes(2i)}$ tensor the even degree part of $\bigwedge(b_1, x_1, \dots, x_n)$. \square

These statements now allow to formulate the following result, which we would like to see as a version of the Quillen conjecture for SL_2 above the virtual cohomological dimension. Recall that $\mathrm{H}_{\mathrm{cts}}^\bullet(\mathrm{SL}_2(\mathbb{C}), \mathbb{F}_\ell) \cong \mathbb{F}_\ell[c_2]$ is generated by the second Chern class c_2 (which is a class in degree 4). This is the subring over which we have to express $\mathrm{H}^\bullet(\mathrm{SL}_2(\mathcal{O}_{K,S}), \mathbb{F}_\ell)$ as a free module for the Quillen conjecture.

Theorem 7.2. *Let ℓ be an odd prime number. Let K be a number field, and let S be a finite set of places containing the infinite places. Let $G = SL_2(\mathcal{O}_{K,S})$. In the splitting of Theorem 1,*

$$\widehat{H}^\bullet(SL_2(\mathcal{O}_{K,S}), \mathbb{F}_\ell) \cong \bigoplus_{[\Gamma] \in \mathcal{K}} \widehat{H}^\bullet(N_G(\Gamma), \mathbb{F}_\ell),$$

denote by n_1 the number of components where $N_G(\Gamma)$ is abelian, and by n_2 the number of components where it is not. For each component group $N_G(\Gamma)$, denote by a_Γ the second Chern class of the standard representation of Γ (which is a polynomial class in degree 4). Denote by r the rank of the relative unit group of $\mathcal{O}_{K,S}[\zeta_\ell]/\mathcal{O}_{K,S}$. Then the restriction map induced by the natural inclusion $\mathcal{O}_{K,S} \hookrightarrow \mathbb{C}$ is given as follows:

$$\mathbb{F}_\ell[c_2] \rightarrow H^\bullet(SL_2(\mathcal{O}_{K,S}), \mathbb{F}_\ell) \rightarrow \widehat{H}^\bullet(SL_2(\mathcal{O}_{K,S}), \mathbb{F}_\ell) : c_2 \mapsto \sum_{[\Gamma] \in \mathcal{K}} a_\Gamma,$$

and $\widehat{H}^\bullet(SL_2(\mathcal{O}_{K,S}), \mathbb{F}_\ell)$ is a free module of rank $2^{r+1}(2n_1 + n_2)$ over the Laurent polynomial subring generated by the image of c_2 .

Proof. All the subgroups Γ become conjugate in $SL_2(\mathbb{C})$ and the same is true for the centralizers $C_G(\Gamma)$. The restriction map for all groups $N_G(\Gamma)$ then has to map c_2 to the second Chern class of the standard representation of the cyclic or dihedral group, which is the element a_Γ . The image of the restriction map is then the diagonal subring generated by $\sum a_\Gamma$. By Proposition 7.1, the cohomology ring is free as a module over this subring with the specified rank. \square

Corollary 7.3. *Let K be a number field, let S be a finite set of places containing the infinite ones, and let ℓ be an odd prime.*

- (1) *The Quillen conjecture holds for group cohomology $H^\bullet(SL_2(\mathcal{O}_{K,S}), \mathbb{F}_\ell)$ above the virtual cohomological dimension.*
- (2) *(A suitable reformulation of) The Quillen conjecture holds for Farrell-Tate cohomology $\widehat{H}^\bullet(SL_2(\mathcal{O}_{K,S}), \mathbb{F}_\ell)$.*

Remark 7.4. As a result, we can reformulate the Quillen conjecture for group cohomology as a relation between Farrell-Tate cohomology and the Steinberg homology $H_\bullet(SL_2(\mathcal{O}_{K,S}), St^{SL_2(\mathcal{O}_{K,S})} \otimes \mathbb{F}_\ell)$. Vanishing of Steinberg homology guarantees the Quillen conjecture; however, it is possible that the Quillen conjecture is true even with non-vanishing Steinberg homology. The Quillen conjecture fails whenever the map

$$H^\bullet(SL_2(\mathcal{O}_{K,S}), \mathbb{F}_\ell) \rightarrow \widehat{H}^\bullet(SL_2(\mathcal{O}_{K,S}), \mathbb{F}_\ell)$$

is not injective. Then there exist elements in group cohomology which - after multiplication with some power of the second Chern class - become trivial.

Next, we can discuss the detection questions. Using our explicit computations, we can easily find examples where the cohomology of $SL_2(\mathcal{O}_{K,S})$ cannot be detected on the diagonal matrices. The following general result, deducing non-detection from non-triviality of suitable class groups, is very much in the spirit of Dwyer's disproof of detection for $GL_{32}(\mathbb{Z}[1/2])$.

Proposition 7.5. *Let ℓ be an odd prime number. Let K be a number field with $\zeta_\ell \in K$, and let S be a finite set of places containing the infinite places. Assume that the class group $\widetilde{K}_0(\mathcal{O}_{K,S})$ has more than 2 elements. Then the restriction map*

$$\widehat{H}^\bullet(SL_2(\mathcal{O}_{K,S}), \mathbb{F}_\ell) \rightarrow \widehat{H}^\bullet(T_2(\mathcal{O}_{K,S}), \mathbb{F}_\ell)$$

from $SL_2(\mathcal{O}_{K,S})$ to the diagonal matrices $T_2(\mathcal{O}_{K,S})$ is not injective.

Proof. Under the assumptions of the proposition, Theorem 3.12(1) implies that the splitting of Theorem 1 becomes

$$\widehat{H}^\bullet(\mathrm{SL}_2(\mathcal{O}_{K,S}), \mathbb{F}_\ell) \cong \bigoplus_{[\Gamma] \in \widetilde{\mathcal{K}}_0(\mathcal{O}_{K,S})/\iota} \widehat{H}^\bullet(N_G(\Gamma), \mathbb{F}_\ell).$$

The group $N_G(\Gamma)$ is the normalizer in $G = \mathrm{SL}_2(\mathcal{O}_{K,S})$ of the order ℓ subgroup representing $[\Gamma]$. Recall that the Farrell-Tate cohomology $\widehat{H}^\bullet(N_G(\Gamma))$ is obtained by making the cohomology rings from Proposition 7.1 periodic for a_2 ; in particular, all exterior products with an odd number of factors live in odd degree, products with even number of factors live in even degree. In both cases, half of the elements is invariant under multiplication with -1 . For the trivial ideal class, we have a contribution of half the rank of $\widehat{H}^\bullet(\mathrm{T}_2(\mathcal{O}_{K,S}), \mathbb{F}_\ell)$. Under the assumption on the class group, we either (at least) have two further ι -invariant ideal classes or a ι -orbit. In either case, the resulting direct summands in $\bigoplus_{[\Gamma] \in \widetilde{\mathcal{K}}_0(\mathcal{O}_{K,S})/\iota} \widehat{H}^\bullet(N_G(\Gamma), \mathbb{F}_\ell)$ yield a further contribution equal to the rank of $\widehat{H}^\bullet(\mathrm{T}_2(\mathcal{O}_{K,S}), \mathbb{F}_\ell)$. Therefore, the restriction map cannot be injective because the rank of the source is bigger than the rank of the target. \square

Remark 7.6. There are other cases in which non-detection results can be established. The above proposition is one of the easier ones to formulate, the cases where $\zeta_\ell \notin K$ need some more complicated conditions.

Probably, the above result could be improved by showing that the only summand with a non-trivial restriction map is one corresponding to the trivial ideal class.

Example 7.7. Let $K = \mathbb{Q}(\zeta_{23})$ and $S = \{(23)\} \cup S_\infty$. The S -class group of K has order 3. The induced morphism

$$\widehat{H}^\bullet(\mathrm{SL}_2(\mathcal{O}_{K,S}), \mathbb{F}_{23}) \rightarrow \widehat{H}^\bullet(\mathrm{T}_2(\mathcal{O}_{K,S}), \mathbb{F}_{23})$$

is not injective - the source has two copies of the cohomology of a dihedral extension of $\mathcal{O}_{K,S}^\times$, but the target has only one copy of the cohomology of $\mathcal{O}_{K,S}$. \square

Example 7.8. There are infinitely many counterexamples to detection at the prime 3.

Let m be a positive square-free integer such that $m \equiv 1 \pmod{3}$. In this case, the prime 3 is inert in the extension $\mathbb{Q}(\sqrt{-m})/\mathbb{Q}$ and ramified in $\mathbb{Q}(\zeta_3)/\mathbb{Q}$. In particular, there is only one place of $\mathbb{Q}(\sqrt{-m}, \zeta_3)$ lying over the place v_3 of \mathbb{Q} , and this place is ramified in the extension $\mathbb{Q}(\sqrt{-m}, \zeta_3)/\mathbb{Q}(\sqrt{-m})$. The extension $\mathbb{Q}(\sqrt{-m}, \zeta_3)/\mathbb{Q}(\sqrt{-m})$ being ramified, the induced map on class groups must be injective, by class field theory. Moreover, as noted above, (3) remains a prime ideal in $\mathcal{O}_{\mathbb{Q}(\sqrt{-m}, \zeta_3)}$. Therefore, the class group of $\mathcal{O}_{\mathbb{Q}(\sqrt{-m}, \zeta_3)}[1/3]$ is isomorphic to the class group of $\mathcal{O}_{\mathbb{Q}(\sqrt{-m}, \zeta_3)}$.

We conclude that the size of the class group of $\mathcal{O}_{\mathbb{Q}(\sqrt{-m})}$ is a lower bound for the size of the class group of $\mathcal{O}_{\mathbb{Q}(\sqrt{-m}, \zeta_3)}[1/3]$. It is well-known that the only rings $\mathcal{O}_{\mathbb{Q}(\sqrt{-m})}$ with trivial class groups occur for $m = 1, 2, 3, 7, 19, 43, 67, 163$. Moreover, there are infinitely many S -integer rings of the form $\mathcal{O}_{\mathbb{Q}(\sqrt{-m}, \zeta_3)}[1/3]$ whose class group has more than two elements. Each such ring R gives an example where the restriction map

$$H^\bullet(\mathrm{GL}_2(R), \mathbb{F}_3) \rightarrow H^\bullet(\mathrm{T}_2(R), \mathbb{F}_3)$$

fails to be injective, but for which the Quillen conjecture is true above the virtual cohomological dimension. \square

The question for unstable analogues of the Quillen-Lichtenbaum conjecture was implicit in [DF94], and was raised explicitly in [AR13]. The results of [DF94] show

that the unstable Quillen-Lichtenbaum conjecture in the situation of linear groups over S -integers implies detection. The failure of detection as in Proposition 7.5 and Example 7.7 also implies the failure of the unstable Quillen-Lichtenbaum conjecture. The above results and their consequences for the Quillen conjecture are further discussed in [RW14].

8. APPLICATION II: ON THE EXISTENCE OF TRANSFERS

Next, we are interested in the existence of transfer maps in Farrell-Tate cohomology as well as group cohomology. Transfers in the cohomology of linear groups have been suggested as one way of establishing the Friedlander-Milnor conjecture in [Knu01, section 5.3]. In this section, we show examples that demonstrate the impossibility of defining transfers on group (co-)homology with reasonable properties. In particular, we are going to prove Theorem 4. The general setup in this section will be the following:

Let L/K be a degree n extension of global fields, let S be a set of places of K and denote by \tilde{S} the set of places of L lying over S . Let ℓ be a prime different from the characteristic of K .

8.1. Definition of transfer maps. We first recall various relevant notions of transfers. For a finite covering $p : E \rightarrow B$ of CW-complexes, there is a transfer map $\mathrm{tr}_p : H_\bullet(B) \rightarrow H_\bullet(E)$ such that the respective composition is multiplication with the degree: $\mathrm{tr}_p \circ p_\bullet = \deg p$. There are similar transfer maps for cohomology. More generally, the Becker-Gottlieb transfer provides a wrong-way stable map $\Sigma^\infty B \rightarrow \Sigma^\infty E$ which induces the transfer on cohomology theories. This can be applied to group (co-)homology to recover the classical definition of transfer: for $H \subset G$ a finite-index subgroup of a group G and a G -module M , there are transfer maps

$$\mathrm{tr}_H^G : H_\bullet(G, M) \rightarrow H_\bullet(H, M) \quad \text{and} \quad \mathrm{Cor}_H^G : H^\bullet(H, M) \rightarrow H^\bullet(G, M),$$

such that the following respective equalities hold where $i : H \rightarrow G$ denotes the inclusion of the subgroup:

$$i_\bullet \circ \mathrm{tr}_H^G = [G : H] \quad \text{and} \quad \mathrm{Cor}_H^G \circ i^\bullet = [G : H]$$

These can be obtained from the (usual or Becker-Gottlieb) transfer applied to the finite covering $BH \rightarrow BG$.

Similar transfer maps can be considered in algebraic K-theory: for a finite flat map $f : X \rightarrow Y$ of schemes, there is a K-theory transfer $\mathrm{tr}_f : K_\bullet(X) \rightarrow K_\bullet(Y)$ such that the composition is the multiplication with the degree: $\mathrm{tr}_f \circ f^* = \deg f$. There are a number of generalizations of this concept. We use one of the simpler ones, cf. [Knu01, section 5.3]: an abelian group-valued functor $\mathcal{F} : \mathrm{Sch}^{\mathrm{op}} \rightarrow \mathcal{A}b$ is said to admit transfers if for any finite flat morphism $f : X \rightarrow Y$ there is a homomorphism $\mathrm{tr}_f : \mathcal{F}(X) \rightarrow \mathcal{F}(Y)$ such that we have $\mathrm{tr}_f \circ \mathcal{F}(f) = \deg f$.

We can apply this definition to group homology: for fixed $i \in \mathbb{N}$, we have a functor on the category of smooth affine k -schemes

$$H_i(\mathrm{SL}_2(-), \mathbb{Z}/\ell) : \mathrm{Sm}_{\mathrm{aff}}^{\mathrm{op}}/k \rightarrow \mathcal{A}b : \mathrm{Spec} k[X] \mapsto H_i(\mathrm{SL}_2(k[X]), \mathbb{Z}/\ell).$$

One can ask a similar question for cohomology, i.e., if for any finite flat map $f : \mathrm{Spec} k[X] \rightarrow \mathrm{Spec} k[Y]$ of affine schemes there is a transfer homomorphism $\mathrm{tr}^f : H^\bullet(\mathrm{SL}_2(k[Y]), \mathbb{Z}/\ell) \rightarrow H^\bullet(\mathrm{SL}_2(k[X]), \mathbb{Z}/\ell)$ such that the composition is multiplication with the degree: $H^\bullet(f) \circ \mathrm{tr}^f = \deg f$. It is an implicit question in the discussion of [Knu01, section 5.3] if group homology functors as above can be equipped with transfers. To quote Knudson, cf. p.134 of loc.cit.: "Unfortunately, there appears to be no way to equip these functors with transfer maps." In the following section, we want to make this precise: we will compute the restriction

maps in Farrell-Tate cohomology, and from these computations it will be obvious that it is not possible to define transfer maps satisfying the degree condition. Similar counterexamples for function fields can be deduced from the computations of [Wen14a]; counterexamples of a different type are discussed in [Wen14b].

8.2. Description of restriction maps. The next step is an explicit description of the restriction morphisms

$$\widehat{H}^\bullet(\mathrm{SL}_2(\mathcal{O}_{L,\tilde{S}}), \mathbb{F}_\ell) \rightarrow \widehat{H}^\bullet(\mathrm{SL}_2(\mathcal{O}_{K,S}), \mathbb{F}_\ell).$$

Recall from Section 6, that we have a splitting of Farrell-Tate cohomology

$$\widehat{H}^\bullet(\mathrm{SL}_2(\mathcal{O}_{K,S}), \mathbb{F}_\ell) \cong \bigoplus_{[\Gamma] \in \mathcal{K}} \widehat{H}^\bullet(N_G(\Gamma), \mathbb{F}_\ell),$$

where the set \mathcal{K} is given by the quotient of an oriented relative class group modulo the Galois action, and the group $N_G(\Gamma)$ is the normalizer of the order ℓ subgroup $\Gamma < G = \mathrm{SL}_2(\mathcal{O}_{K,S})$. There are two types of components: if Γ is not contained in a dihedral group, then $N_G(\Gamma)$ is abelian, determined by a relative unit group. Otherwise, $N_G(\Gamma)$ is non-abelian and it is a semidirect product of a relative unit group with an $\mathbb{Z}/2\mathbb{Z}$ acting via inversion.

In the extension of number rings $\mathcal{O}_{K,S} \rightarrow \mathcal{O}_{L,\tilde{S}}$, the following things can happen:

- (C 1) Several non-conjugate cyclic subgroups of order ℓ in $\mathrm{SL}_2(\mathcal{O}_{K,S})$ can become conjugate in $\mathrm{SL}_2(\mathcal{O}_{L,\tilde{S}})$.
- (C 2) A cyclic group which is not contained in a dihedral group in $\mathrm{SL}_2(\mathcal{O}_{K,S})$ acquires a dihedral overgroup in $\mathrm{SL}_2(\mathcal{O}_{L,\tilde{S}})$.
- (C 3) There appear several new cyclic subgroups of order ℓ in $\mathrm{SL}_2(\mathcal{O}_{L,\tilde{S}})$ which are not conjugate to subgroups coming from $\mathrm{SL}_2(\mathcal{O}_{K,S})$.

Now we can discuss the restriction maps on Farrell-Tate cohomology in these three cases. The straightforward proof of the following proposition is left to the reader.

Proposition 8.1. *Fix an odd prime ℓ . Let K be a global field of characteristic different from ℓ , let S be a non-empty finite set of places containing the infinite ones. Let L/K be a finite separable extension of K , and let \tilde{S} be a finite set of places containing those places lying over S .*

- (C 1) *Assume that exactly the classes $[\Gamma_1], \dots, [\Gamma_m] \in \mathcal{K}(K)$ become identified to a single component $[\Gamma] \in \mathcal{K}(L)$. Then the restriction map*

$$\widehat{H}^\bullet(N_{\mathrm{SL}_2(\mathcal{O}_{L,\tilde{S}})}(\Gamma), \mathbb{F}_\ell) \rightarrow \bigoplus_{[\Gamma_i]} \widehat{H}^\bullet(N_{\mathrm{SL}_2(\mathcal{O}_{K,S})}(\Gamma_i), \mathbb{F}_\ell)$$

is the sum of the natural maps induced from the inclusions
 $N_{\mathrm{SL}_2(\mathcal{O}_{K,S})}(\Gamma_i) \rightarrow N_{\mathrm{SL}_2(\mathcal{O}_{L,\tilde{S}})}(\Gamma)$.

- (C 2) *Assume that a cyclic group representing the class $[\Gamma] \in \mathcal{K}(K)$ is not contained in a dihedral group over K , but is contained in a dihedral group over L . Then the restriction map*

$$\widehat{H}^\bullet(N_{\mathrm{SL}_2(\mathcal{O}_{L,\tilde{S}})}(\Gamma), \mathbb{F}_\ell) \rightarrow \widehat{H}^\bullet(N_{\mathrm{SL}_2(\mathcal{O}_{K,S})}(\Gamma), \mathbb{F}_\ell)$$

is given by the natural inclusion of fixed points.

- (C 3) *The restriction map is trivial on the new components of $\mathcal{K}(L)$.*

8.3. Non-existence of transfers.

Theorem 8.2. *The functor “Farrell-Tate cohomology of SL_2 ” given by*

$$A \mapsto \widehat{H}^\bullet(SL_2(A), \mathbb{F}_\ell)$$

does not admit transfers. More precisely, there exists a finite flat morphism of commutative rings $\phi : A \rightarrow B$ such that for no morphism

$$\text{tr} : \widehat{H}^\bullet(SL_2(A), \mathbb{F}_\ell) \rightarrow \widehat{H}^\bullet(SL_2(B), \mathbb{F}_\ell)$$

we have $\phi^\bullet \circ \text{tr} = \text{deg } \phi$.

Proof. Let $A \rightarrow B$ be any extension $\mathcal{O}_{K,S} \rightarrow \mathcal{O}_{L,\bar{S}}$ of degree prime to ℓ such that situation (C 1) occurs.

For example, we can take \mathcal{O}_K with non-trivial class group, L the Hilbert class field for \mathcal{O}_K and ℓ prime to $[L : K]$. Let m be the number of classes of order ℓ subgroups which become conjugate to Γ . By Proposition 8.1, the restriction map is

$$\widehat{H}^\bullet(N_{SL_2(\mathcal{O}_{L,\bar{S}})}(\Gamma), \mathbb{F}_\ell) \rightarrow \bigoplus_{i=1}^m \widehat{H}^\bullet(N_{SL_2(\mathcal{O}_{K,S})}(\Gamma_i), \mathbb{F}_\ell).$$

The composition $\phi^\bullet \circ \text{tr}$ cannot be surjective, because the restriction map has its image contained in the diagonal subring. By the previous choice of ℓ and $[L : K]$, multiplication with the degree $[L : K]$ will have full rank. Therefore, the two maps cannot be equal, no matter how we choose tr . \square

Example 8.3. To give a specific example, consider $K = \mathbb{Q}(\zeta_{23})$. Its class number is 3, and its group of units is $\mathbb{Z}^{10} \times \mathbb{Z}/46$. Therefore, we have

$$\widehat{H}^\bullet(SL_2(\mathcal{O}_{\mathbb{Q}(\zeta_{23})}), \mathbb{F}_{23}) \cong \widehat{H}^\bullet(\mathbb{Z}^{10} \times \mathbb{Z}/46, \mathbb{F}_{23}) \oplus \widehat{H}^\bullet(\mathbb{Z}^{10} \times \mathbb{Z}/46, \mathbb{F}_{23})^{(-1)}$$

i.e., the Farrell-Tate cohomology $SL_2(\mathcal{O}_{\mathbb{Q}(\zeta_{23})})$ is given by the restricted sum of one copy of the Farrell-Tate cohomology of the unit group and one copy of the -1 -invariants of Farrell-Tate cohomology. Note that we have 2 summands because this is the cardinality of the quotient of the class group $\mathbb{Z}/3$ modulo the inversion involution. The Hilbert class field of $\mathbb{Q}(\zeta_{23})$ is a degree 3 unramified extension $H/\mathbb{Q}(\zeta_{23})$ such that the restriction map $\widetilde{K}_0(\mathcal{O}_{\mathbb{Q}(\zeta_{23})}) \rightarrow \widetilde{K}_0(H)$ is the trivial map. In particular, the restriction map on Farrell-Tate cohomology will factor through the diagonal map

$$\widehat{H}^\bullet(T_2(\mathcal{O}_H) \rtimes \mathbb{Z}/2, \mathbb{F}_{23}) \rightarrow \widehat{H}^\bullet(\mathbb{Z}^{10} \times \mathbb{Z}/46, \mathbb{F}_{23}) \oplus \widehat{H}^\bullet(\mathbb{Z}^{10} \times \mathbb{Z}/46, \mathbb{F}_{23})^{(-1)}$$

No matter how the transfer map

$$\widehat{H}^\bullet(SL_2(\mathcal{O}_{\mathbb{Q}(\zeta_{23})}), \mathbb{F}_{23}) \rightarrow \widehat{H}^\bullet(SL_2(\mathcal{O}_H), \mathbb{F}_{23})$$

is defined, the composition with the restriction map will not be multiplication with 3 on $\widehat{H}^\bullet(SL_2(\mathcal{O}_{\mathbb{Q}(\zeta_{23})}), \mathbb{F}_{23})$: multiplication with 3 is invertible in \mathbb{F}_{23} , but there are many classes which are not in the image of the composition. \square

Remark 8.4. Similarly, the situation (C 2) obstructs transfers because the inclusion of fixed points is not surjective. The situation (C 3) does not obstruct transfers.

Corollary 8.5. *The functor “group cohomology of SL_2 ”, which is given by $A \mapsto H^\bullet(SL_2(A), \mathbb{F}_\ell)$, does not admit transfers.*

Proof. Farrell-Tate and group cohomology agree above the virtual cohomological dimension, which for SL_2 over S -integers is finite. In the degrees above the vcd, the argument of Theorem 8.2 also applies to group cohomology. \square

9. APPLICATION III: COHOMOLOGY OF SL_2 OVER NUMBER FIELDS

Using the previous formulas for Farrell-Tate cohomology and restriction maps from Section 6, we can now compute the colimit of the Farrell-Tate cohomology groups over all possible finite sets of places. This could be interpreted as “Farrell-Tate cohomology of SL_2 over global fields”. The quotes are necessary as Farrell-Tate cohomology is only defined for groups of finite virtual cohomological dimension. The goal of this section is to prove Theorem 5.

9.1. Recollection on Mislin-Tate cohomology. In [Mis94], Mislin has defined an extension of Farrell-Tate cohomology to arbitrary groups. The basic idea behind the definitions in [Mis94] is to use satellites of group cohomology to kill projectives in the derived category of $\mathbb{Z}[G]$ -modules, and obtain a completed cohomology, cf. [Mis94, section 2]. For groups of finite virtual cohomological dimension, Mislin’s version of Tate cohomology agrees with Farrell-Tate cohomology, cf. [Mis94, lemma 3.1]. Moreover, Mislin shows that for K a number field and $G = \mathrm{GL}_n(K)$, his version of Tate cohomology can be identified with group homology, cf. [Mis94, theorem 3.2].

9.2. Colimit computations.

Theorem 9.1. *Let K be a number field, and let ℓ be an odd prime. We have the following three cases:*

(1) *If $\zeta_\ell + \zeta_\ell^{-1} \notin K$, then*

$$\lim_{S \supseteq S_\infty, \text{finite}} \widehat{H}^\bullet(\mathrm{SL}_2(\mathcal{O}_{K,S}), \mathbb{F}_\ell) = 0.$$

(2) *If $\zeta_\ell + \zeta_\ell^{-1} \in K$ and the symbol $((\zeta_\ell - \zeta_\ell^{-1})^2, -1)$ is non-trivial, then*

$$\lim_{S \supseteq S_\infty, \text{finite}} \widehat{H}^\bullet(\mathrm{SL}_2(\mathcal{O}_{K,S}), \mathbb{F}_\ell) \cong \mathbf{H}^\bullet(\mathbf{T}(K), \mathbb{F}_\ell),$$

where $\mathbf{T}(K)$ denotes the K -points of a maximal torus in SL_2 .

(3) *If $\zeta_\ell + \zeta_\ell^{-1} \in K$ and the symbol $((\zeta_\ell - \zeta_\ell^{-1})^2, -1)$ is trivial, then*

$$\lim_{S \supseteq S_\infty, \text{finite}} \widehat{H}^\bullet(\mathrm{SL}_2(\mathcal{O}_{K,S}), \mathbb{F}_\ell) \cong \mathbf{H}^\bullet(\mathbf{N}(K), \mathbb{F}_\ell),$$

where $\mathbf{N}(K)$ denotes the K -points of the normalizer of a maximal torus in SL_2 .

Corollary 9.2. *There are cases where Mislin’s version of Farrell-Tate cohomology does not commute with filtered colimits.*

Proof. By results of Krämer [Krä80, Satz 13.3.i], the group $\mathrm{PSL}_2(K)$ contains a dihedral group D_{2n} if and only if the symbol $((\zeta_{2n} - \zeta_{2n}^{-1})^2, -1)$ splits.

For $K = \mathbb{Q}$, we have $\zeta_3 + \zeta_3^{-1} \in \mathbb{Q}$, hence $\mathrm{PSL}_2(\mathbb{Q})$ contains a cyclic group of order 3. On the other hand, $((\zeta_6 - \zeta_6^{-1})^2, -1) = (-3, -1) = -1$, hence $\mathrm{PSL}_2(\mathbb{Q})$ does not contain a dihedral group of order 6. By the above results, we have

$$\lim_{S \supseteq S_\infty, \text{finite}} \widehat{H}^\bullet(\mathrm{SL}_2(\mathcal{O}_{\mathbb{Q},S}), \mathbb{F}_3) = \mathbf{H}^\bullet(\mathbb{Q}^\times, \mathbb{F}_3).$$

In particular, the limit of Farrell-Tate cohomology groups in degree 1 is

$$\lim_{S \supseteq S_\infty, \text{finite}} \widehat{H}^1(\mathrm{SL}_2(\mathcal{O}_{\mathbb{Q},S}), \mathbb{F}_3) = \mathbf{H}^1(\mathbb{Q}^\times, \mathbb{F}_3) \cong \mathrm{Hom}_{\mathbb{F}_3}(\mathbb{Q}^\times / (\mathbb{Q}^\times)^3, \mathbb{F}_3),$$

which is an infinite-dimensional \mathbb{F}_3 -vector space. However, by [Mis94, theorem 3.2] there is an isomorphism

$$\widehat{H}^1(\mathrm{SL}_2(\mathbb{Q}), \mathbb{F}_3) \cong \mathbf{H}^1(\mathrm{SL}_2(\mathbb{Q}), \mathbb{F}_3),$$

and the latter is trivial. Therefore, Mislin's version of Farrell-Tate cohomology does not commute with directed colimits. \square

For an extension of number fields L/K , we can precisely describe the induced morphism on the colimit of the Farrell-Tate homology - it is induced by the inclusion $K^\times \hookrightarrow L^\times$. Again taking a colimit, we arrive at the following "Farrell-Tate cohomology of $SL_2(\overline{\mathbb{Q}})$ ":

Corollary 9.3. *Let K be a number field and ℓ be an odd prime. We fix an algebraic closure $\overline{\mathbb{Q}}$ of \mathbb{Q} . Then we have*

$$\lim_{\overline{\mathbb{Q}} \supseteq L \supseteq \mathbb{Q}, S \supseteq S_\infty} \widehat{H}^\bullet(SL_2(\mathcal{O}_{L,S}), \mathbb{F}_\ell) \cong H^\bullet(N(\overline{\mathbb{Q}}), \mathbb{F}_\ell),$$

where S runs through the sets of places of L containing the infinite places, and $N(\overline{\mathbb{Q}})$ denotes the $\overline{\mathbb{Q}}$ -points of the normalizer of a maximal torus of SL_2 .

Finally, we want to explain how these colimits over Farrell-Tate cohomology groups allow to provide a reformulation of the Friedlander-Milnor conjecture. For a discussion of the Friedlander-Milnor conjecture, we refer to [Knu01, chapter 5]. Recall that Milnor's form of what is now called the Friedlander-Milnor conjecture predicts that for a complex Lie group G , the natural change-of-topology map $H_{\text{cts}}^\bullet(G, \mathbb{F}_\ell) \rightarrow H^\bullet(G^\delta, \mathbb{F}_\ell)$ is an isomorphism, where the source is continuous cohomology of the Lie group G with the analytic topology, and G^δ is the group with the discrete topology. On the other hand, Friedlander's generalized isomorphism conjecture predicts that for each algebraically closed field K of characteristic different from ℓ and each linear algebraic group G over K , another natural change-of-topology map $H_{\text{ét}}^\bullet(BG_K, \mathbb{F}_\ell) \rightarrow H^\bullet(BG(K), \mathbb{F}_\ell)$ is an isomorphism. From the rigidity property of étale cohomology, it follows that the Friedlander-Milnor conjecture for SL_2 over $\overline{\mathbb{Q}}$ is equivalent to the natural restriction map

$$H_{\text{cts}}^\bullet(SL_2(\mathbb{C}), \mathbb{F}_\ell) \rightarrow H^\bullet(SL_2(\overline{\mathbb{Q}}), \mathbb{F}_\ell)$$

being an isomorphism. Note that the continuous cohomology

$$H_{\text{cts}}^\bullet(SL_2(\mathbb{C}), \mathbb{F}_\ell) \cong \mathbb{F}_\ell[c_2]$$

is a polynomial ring generated by the second Chern class. Therefore, the Friedlander-Milnor conjecture for SL_2 over $\overline{\mathbb{Q}}$ can be reformulated as the claim that

$$\widehat{H}^\bullet(SL_2(\overline{\mathbb{Q}}), \mathbb{F}_\ell) \cong \mathbb{F}_\ell[c_2].$$

Combined with our computations of Farrell-Tate cohomology, we can reformulate the Friedlander-Milnor conjecture for SL_2 over $\overline{\mathbb{Q}}$ as the requirement that group cohomology and "Farrell-Tate cohomology" of $SL_2(\overline{\mathbb{Q}})$ are isomorphic:

Corollary 9.4. *In particular, the Friedlander-Milnor conjecture for SL_2 over $\overline{\mathbb{Q}}$ with \mathbb{F}_ℓ -coefficients is equivalent to both of the two following statements:*

- (1) *The colimit of homology groups of Steinberg modules vanishes:*

$$\lim_{\overline{\mathbb{Q}} \supseteq L \supseteq \mathbb{Q}, S \supseteq S_\infty} H_\bullet(SL_2(\mathcal{O}_{L,S}), St_2(\mathcal{O}_{L,S}); \mathbb{F}_\ell) = 0.$$

where as above S runs through the finite sets of places of L containing the infinite places.

- (2) *Mislin's version of Farrell-Tate cohomology commutes with the filtered colimit over the intermediate fields $\overline{\mathbb{Q}}/L/\mathbb{Q}$ and their finite sets S of places.*

Proof. We denote by μ_{ℓ^∞} the group of ℓ -power roots of unity in $\overline{\mathbb{Q}}$. The group $\overline{\mathbb{Q}}^\times / \mu_{\ell^\infty}$ is uniquely ℓ -divisible. In particular, the inclusion $\mu_{\ell^\infty} \subseteq \overline{\mathbb{Q}}^\times$ induces an isomorphism

$$H^q(N(\overline{\mathbb{Q}}), \mathbb{F}_\ell) \cong H^q(\mu_{\ell^\infty} \rtimes \mathbb{Z}/2, \mathbb{F}_\ell) \cong \begin{cases} \mathbb{F}_\ell, & q \equiv 0 \pmod{4}, \\ 0, & \text{otherwise,} \end{cases}$$

which is what is predicted by the Friedlander-Milnor conjecture.

Let $G = \mathrm{SL}_2(\mathcal{O}_{L,S})$. To prove Assertion (1), we take the filtered colimit of the long exact sequence

$$\cdots \rightarrow \widehat{H}^{\bullet-1}(G) \rightarrow H_{n-\bullet}(G, \mathrm{St}^G) \rightarrow H^\bullet(G) \rightarrow \widehat{H}^\bullet(G) \rightarrow \cdots$$

Group cohomology commutes with the filtered colimit, and by the above, the Friedlander-Milnor conjecture is equivalent to the fact that the colimit of the morphisms $H^\bullet(\mathrm{SL}_2(\mathcal{O}_{K,S}) \rightarrow \widehat{H}^\bullet(G))$ is an isomorphism. Since filtered colimits are exact for \mathbb{F}_ℓ -vector spaces, the claim follows.

Assertion (2) follows similarly. By [Mis94, theorem 3.2], Mislin-Tate cohomology of $\mathrm{SL}_2(\overline{K})$ agrees with group cohomology, so the Friedlander-Milnor conjecture is equivalent to the commutation of Farrell-Tate cohomology with the filtered colimit. \square

Remark 9.5. Note that our computations of Farrell-Tate cohomology above imply that “Farrell-Tate cohomology of $\mathrm{SL}_2(\overline{\mathbb{Q}})$ ” is in fact detected on the normalizer of the maximal torus. This relates our result above to another reformulation of the Friedlander-Milnor conjecture: [Knu01, corollary 5.2.10] shows that the Friedlander-Milnor conjecture is equivalent to group cohomology being detected on the normalizer, cf. also the related discussion in [Wen14a].

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ALEXANDER D. RAHM, DEPARTMENT OF MATHEMATICS, NATIONAL UNIVERSITY OF IRELAND AT GALWAY, SCHOOL OF MATHEMATICS, STATISTICS AND APPLIED MATHEMATICS
E-mail address: Alexander.Rahm@nuigalway.ie

MATTHIAS WENDT, FAKULTÄT MATHEMATIK, UNIVERSITÄT DUISBURG-ESSEN, THEA-LEYMANN-STRASSE 9, 45127, ESSEN, GERMANY
E-mail address: matthias.wendt@uni-due.de